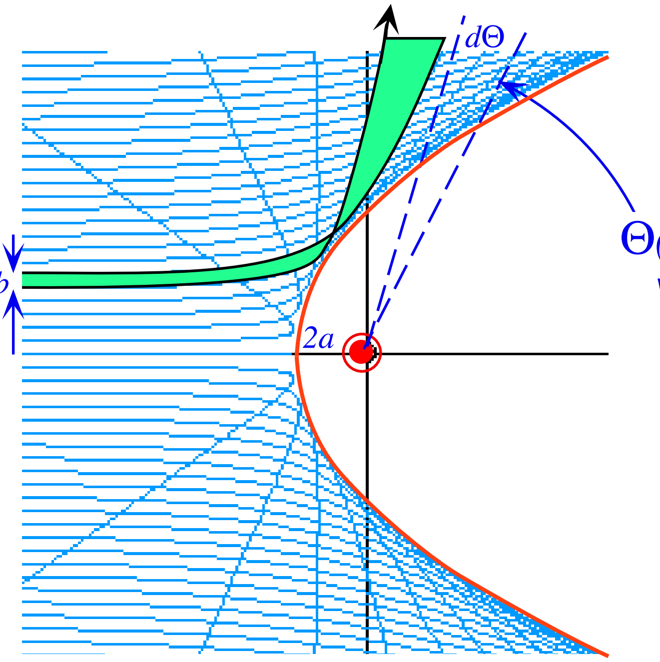
Lecture 26
Mon. 11.25.2019

Geometry and Symmetry of Coulomb Orbital Dynamics

(Ch. 2-4 of Unit 5)

Rutherford scattering and hyperbolic orbit geometry
- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*



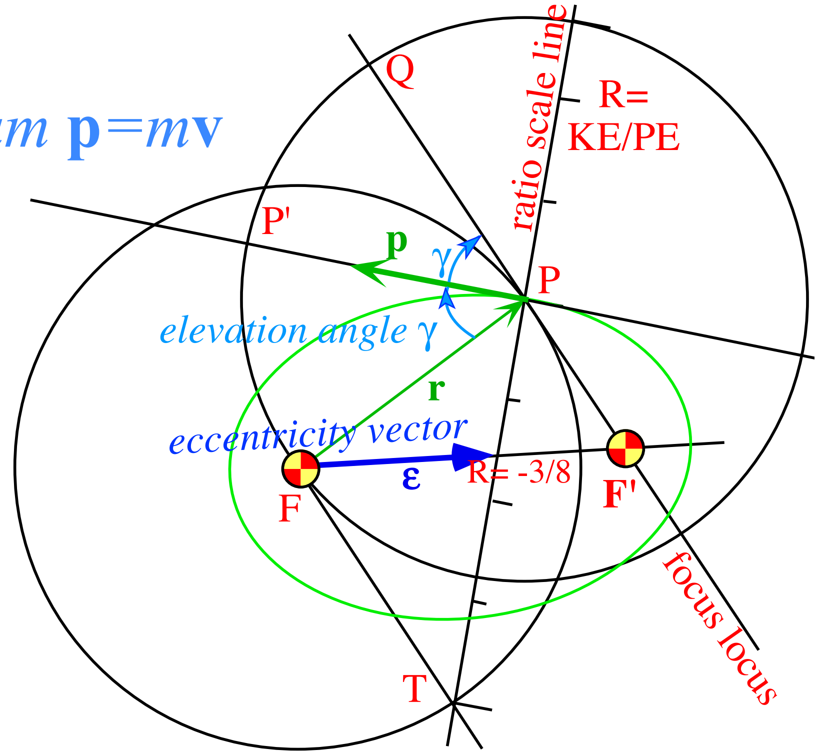
Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*
Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$
- *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*
- *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$
- *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)



Excerpts from Lect. 27

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lecture #22-26

In reverse order

CoulIt Web App Simulations: p19, p32, p72, p73, p92, R=-0.375, R=+0.5, Rutherford

OscillatorPE Web App: IHO Scenario 2, Coulomb Scenario 3
RelaWavity Web App/Simulator/Calculator: Elliptical - IHO orbits

JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap(1)
MolVibes Web App: C3vN3
WaveIt Web App:
  Dim = 3 w/Wave Components;
  Static Char Table: 6, 12, 12(b), 16, 36, 256
  Quantum Carpet with N=20: Gaussian, Boxcar
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015
QTCA Unit_5 Ch14 2013
Lester. R. Ford, Am. Math. Monthly 45,586(1938)
John Farey, Phil. Mag.(1816) Wolfram
Harter, J. Mol. Spec. 210, 166-182 (2001)
Harter, Li IMSS (2013)
Li, Harter, Chem.Phys.Letters (2015)

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
  Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
  Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
  Pure C-Type A,D=4.055, B=0, C=0.1
  Mixed AB-Type w/Cosine
  Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
  Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
  Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
  Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Select, exciting, and/or related Research

This Indestructible NASA Camera Revealed Hidden Patterns on Jupiter - seeker-yt-2019
What did NASA's New Horizons discover around Pluto? - Astrum-yt-2018
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019

Classical Mechanics with a Bang! 2018
  Lectures 8, 9, 23 page 93
  Text Unit 6, page=27
ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
  and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Spectral Decomposition with Repeated Eigenvalues - 2017 GTQM - Lecture 5

Web based 3D & XR ($x\in${A,M,V}, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

Recent In-House draft Articles:

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 1 - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 2 - Reimer-www-2019

Quantum Computing (*QC*) and Geometric Algebra (*GA*) references:

Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Quantum Computing for Computer Scientists - Helwer-mr-yt-2018, Slides
Quantum Computing and Workforce, Curriculum, and App Devel - Roetteler-MS-2019

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Excerpts (Page 44-47 in *Preliminary Draft*) for a GA take on the Complex Numbers
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
GA & QC references (Page 11-16 in Preliminary Draft)

In development, but close to role out.
More Advanced QM and classical references will *soon* be available through our: ***References Page***
Would be great to have our Apache SOLR *Search & Index system up for a bigger* Bang!*)*

Continued for 3 more pages⤵

This Lecture's Reference Link Listing

Web Resources - front page
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Quantum Theory for the Computer Age
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Classical Mechanics with a Bang!
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2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #21

In reverse order

Wiki on Pafnuty Chebyshev
Nobelprize.org
    2005 Physics Award

***BoxIt* Web Simulations:**

A-Type w/Cosine, A-Type w/Freq ratios,
AB-Type w/Cosine, AB-Type 2:1 Freq ratio

***OscillIt* Web Simulations:**

Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
    Way above resonance,Underdamped
    Complex Response Plot

***CoulIt* Web Simulations:**

Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

***Trebuchet* Web Simulations:**

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:

A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for …
    Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

    Using Earth as a clock,
    Tesla's AC Phasors ,
    Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
    Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
    Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
    Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

An assist from *Physics Girl* (YouTube Channel):

How to Make VORTEX RINGS in a Pool
Crazy pool vortex - pg-yt-2014
Fun with Vortex Rings in the Pool - pg-yt-2014

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74, Page=75, Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted, FVPlot*

CMwithBang Lecture 8, page=20

WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019

A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018

*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,
Nuclear_Collisions - Mishustin-PhysRevC-2007, APS Link & Abstract
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceItIt Web Animation - Scenarios:
49:1 y vs t, 49:1 V2 vs V1, 1:500:1 - 1D Gas Model w/ faux restorative force (Cool),
1:500:1 - 1D Gas (Warm), 1:500:1 - 1D Gas Model (Cool, Zoomed in),
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, 1000:1 with t vs x - Minkowski Plot
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, Quantum_Carpet_wMBars,
Quantum_Carpet_BCar, Quantum_Carpet_BCar_wMBars
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot *V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$*
m_1:m_2 = 4:1
v2 vs v1, y2 vs y1
m_1:m_2 = 100:1, *$(v_1, v_2)=(1, 0)$: V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<Under Construction>>

➤ *Rutherford scattering and hyperbolic orbit geometry*

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

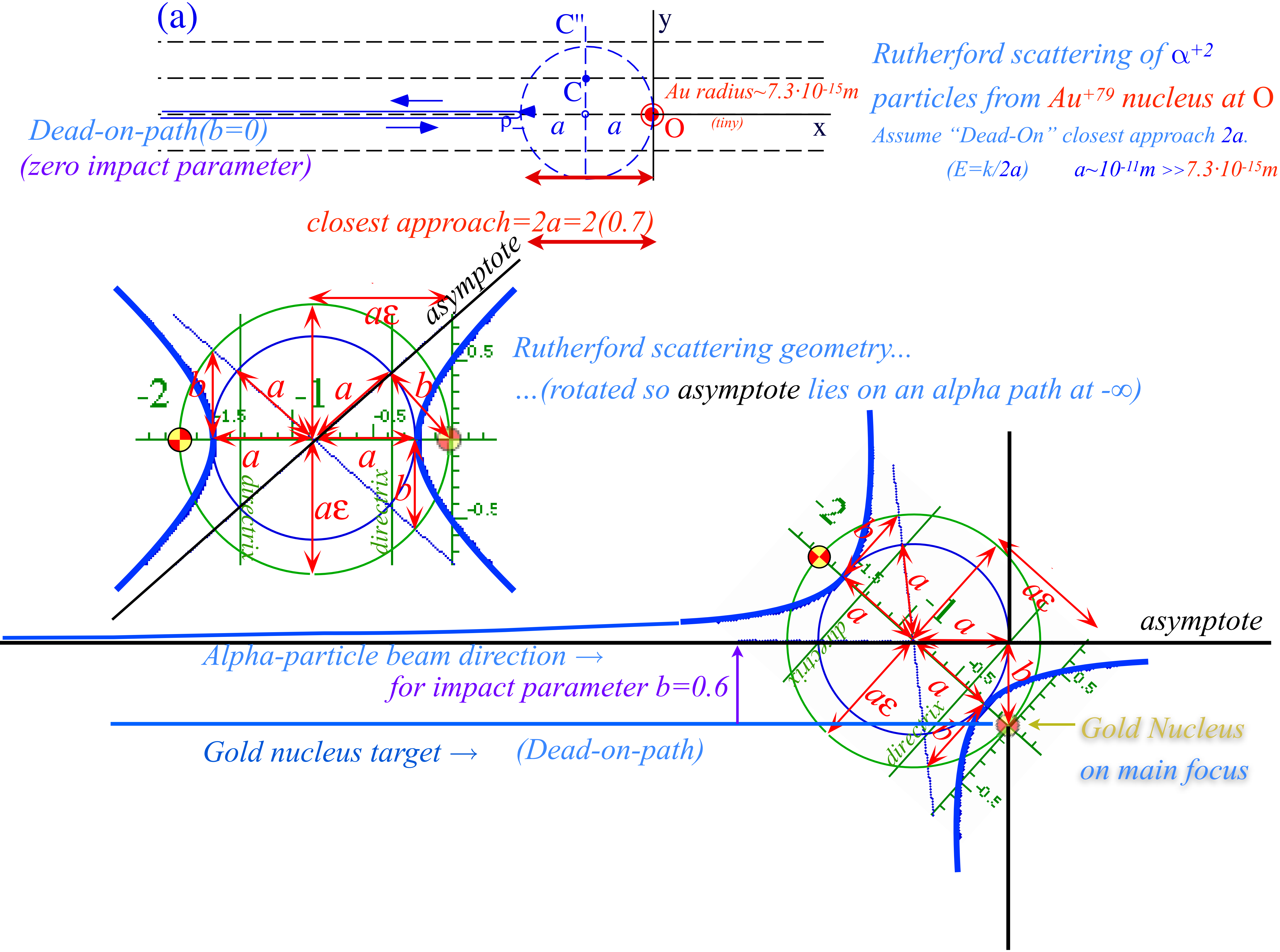
General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

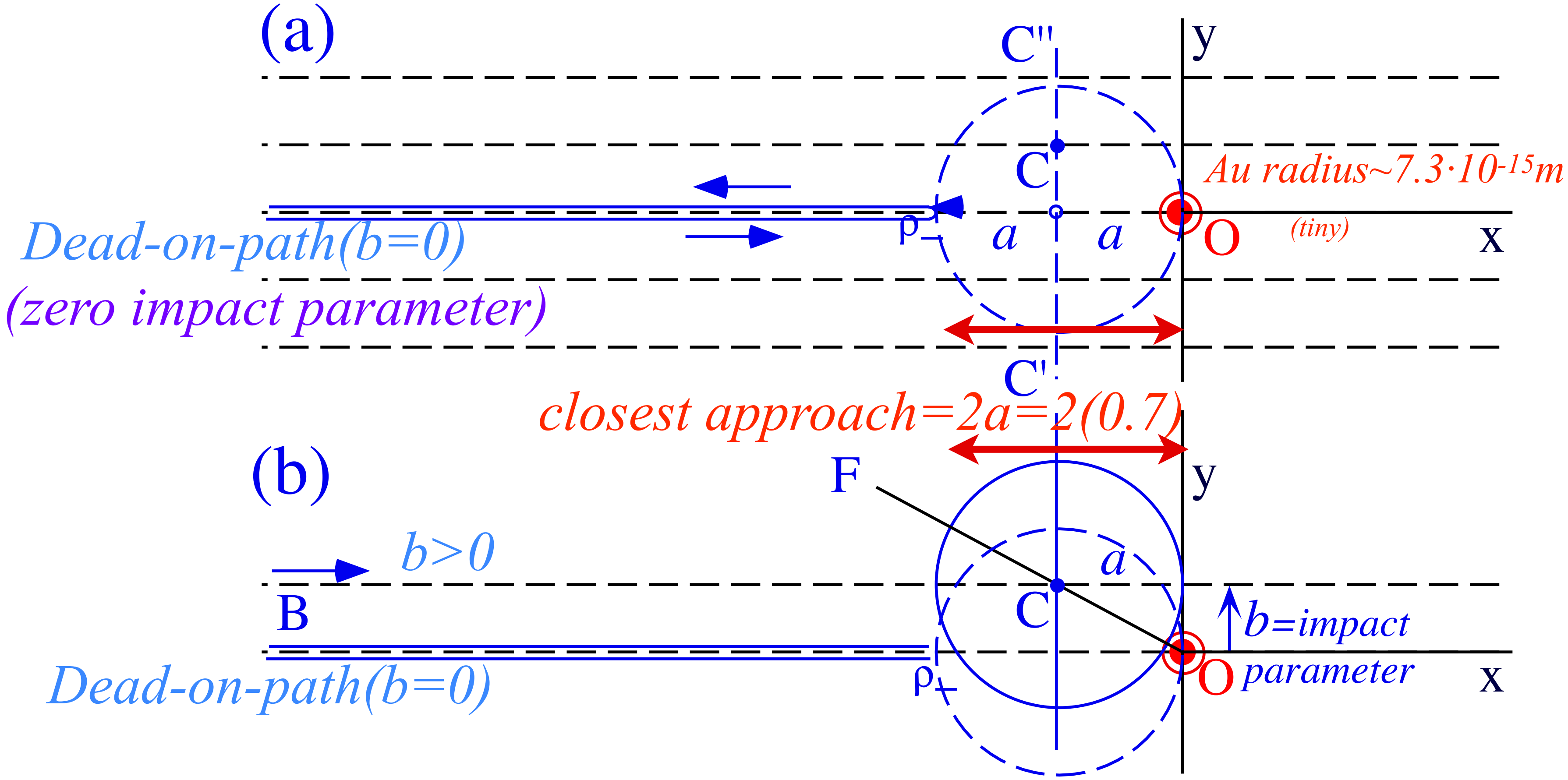
Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)


(a)
C''
y
C
Au radius~7.3·10-15m
Dead-on-path(b=0)
(zero impact parameter)
a
a
O
(tiny)
x
Rutherford scattering of α+2
particles from Au+79 nucleus at O
Assume "Dead-On" closest approach 2a.
(E=k/2a)
a~10-11m >>7.3·10-15m
closest approach=2a=2(0.7)
asymptote
aε
Rutherford scattering geometry...
...(rotated so asymptote lies on an alpha path at -∞)
-2
-1
b
a
-1.5
-0.5
0.5
-0.5
directrix
directrix
asymptote
Alpha-particle beam direction →
for impact parameter b=0.6
Gold nucleus target →
(Dead-on-path)
Gold Nucleus
on main focus



Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O

Assume "Dead-On" closest approach $2a$.

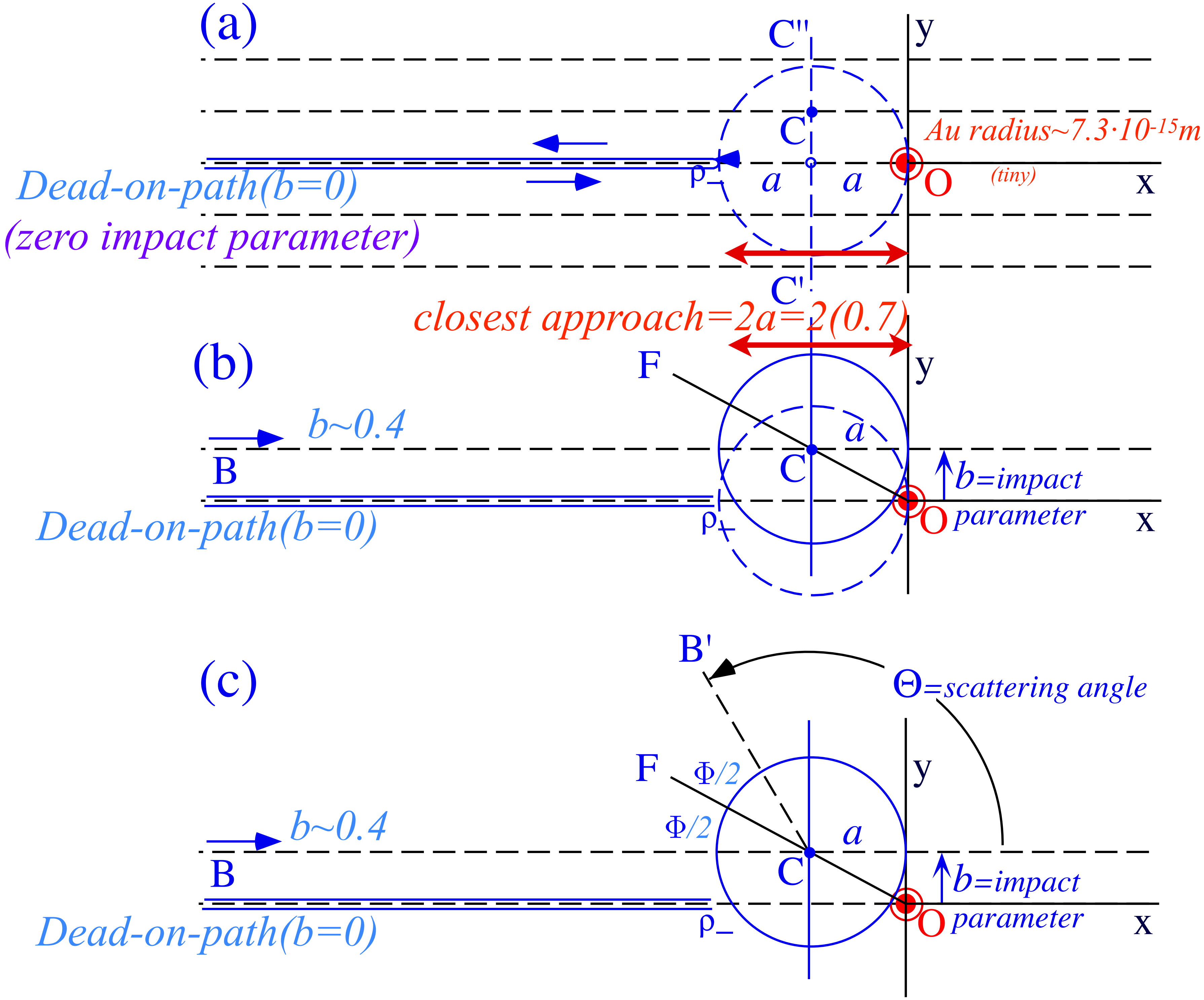
$(E=k/2a)$ $\quad a \sim 10^{-11}m >> 7.3\cdot 10^{-15}m$

Pick an "impact parameter" line $y=b$.

Draw circle of radius a around

center point $C=(-a,b)$ tangent to y-axis.

Draw "focus-locus" line OCF.



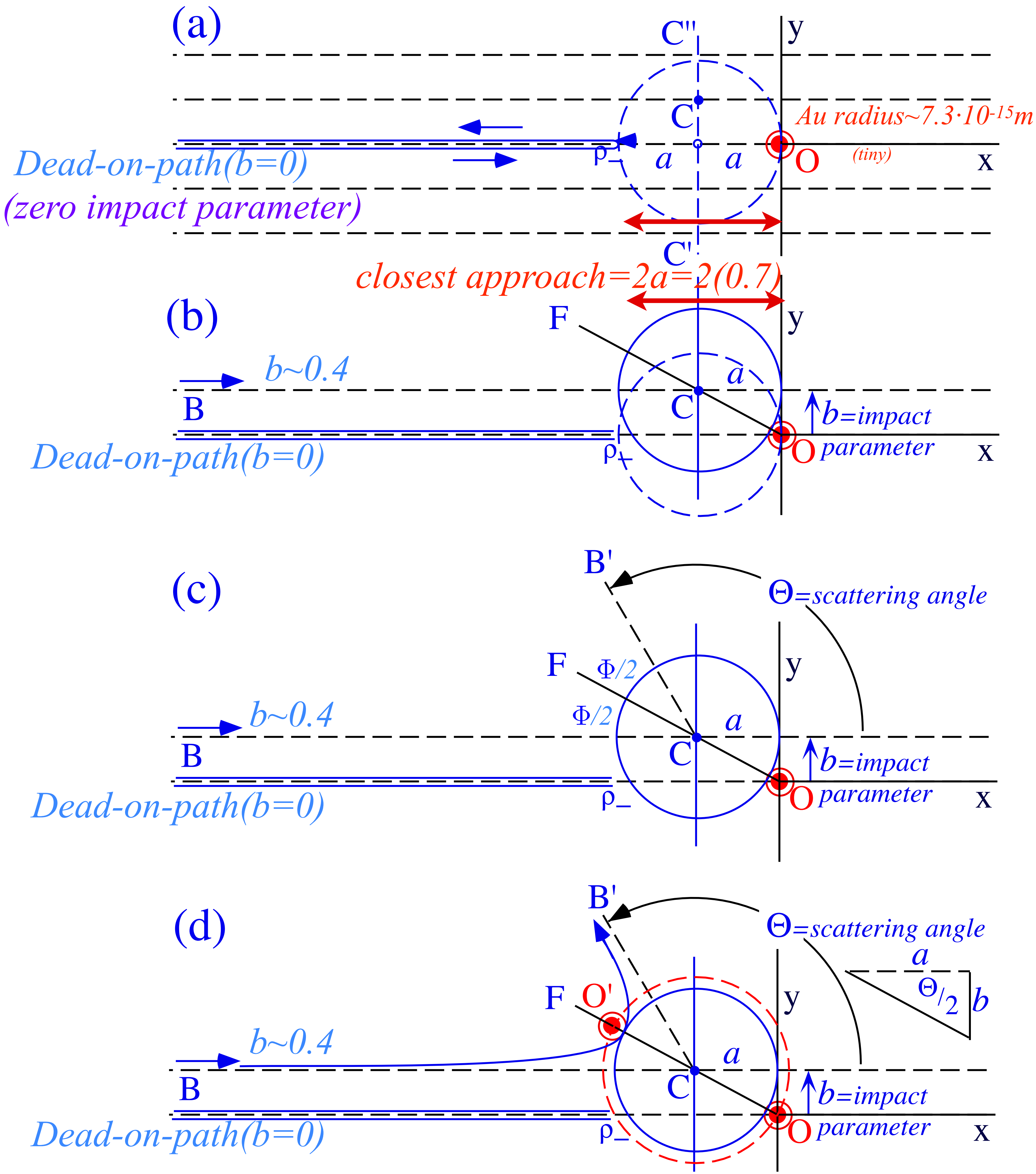
Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach $2a$.

$(E=k/2a)$ $\quad a\sim10^{-11}m >> 7.3\cdot10^{-15}m$

Pick an "impact parameter" line $y=b$.
Draw circle of radius **a** *around center point* C=$(-a,b)$ *tangent to* y*-axis.*
Draw "focus-locus" line OCF.

Copy angle ∠BCF (*equal to* Φ/2)
to make angle ∠FCB′ (*also equal to* Φ/2)
Resulting line CB′ *is outgoing asymptote*
at scattering angle Θ .



Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O

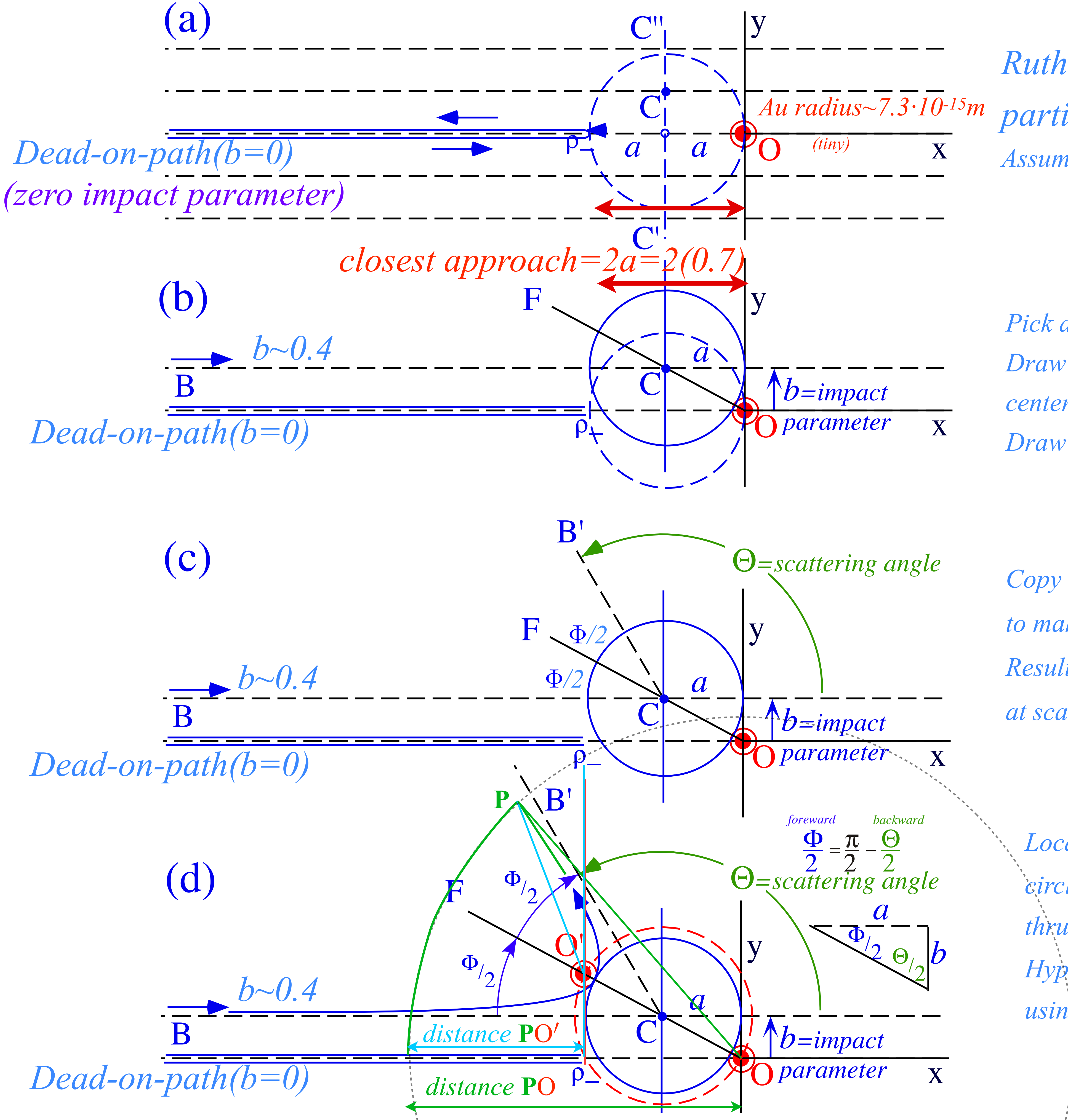
Assume "Dead-On" closest approach $2a$.

($E=k/2a$) $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$

Pick an "impact parameter" line $y=b$.
Draw circle of radius a around center point $C=(-a,b)$ tangent to y-axis.
Draw "focus-locus" line OCF.

Copy angle $\angle$BCF (equal to $\Phi/2$) to make angle $\angle$FCB′ (also equal to $\Phi/2$)
Resulting line CB′ is outgoing asymptote at scattering angle Θ .

Locate secondary focus O′ by drawing circle around point C of diameter CO thru point O. Diameter O′CO is $2a\varepsilon$.
*Hyperbolic orbit points **P** now found using constant $2a$=**P**O-**P**O′*



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach $2a$.

$(E=k/2a)$ $\quad a\sim10^{-11}m >> 7.3\cdot10^{-15}m$

Pick an "impact parameter" line $y=b$.
Draw circle of radius a *around center point* C=$(-a,b)$ *tangent to* y-*axis.*
Draw "focus-locus" line OCF.

Copy angle ∠BCF (*equal to* $\Phi/2$)
to make angle ∠FCB′ (*also equal to* $\Phi/2$)
Resulting line CB′ *is outgoing asymptote at scattering angle* Θ .

Locate secondary focus O′ *by drawing circle around point* C *of diameter* CO *thru point* O. *Diameter* O′CO *is* $2a\varepsilon$.
Hyperbolic orbit points **P** *now found using constant* $2a$=**P**O-**P**O′

Rutherford scattering and hyperbolic orbit geometry

➡ *Backward vs forward scattering angles and orbit construction example*

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

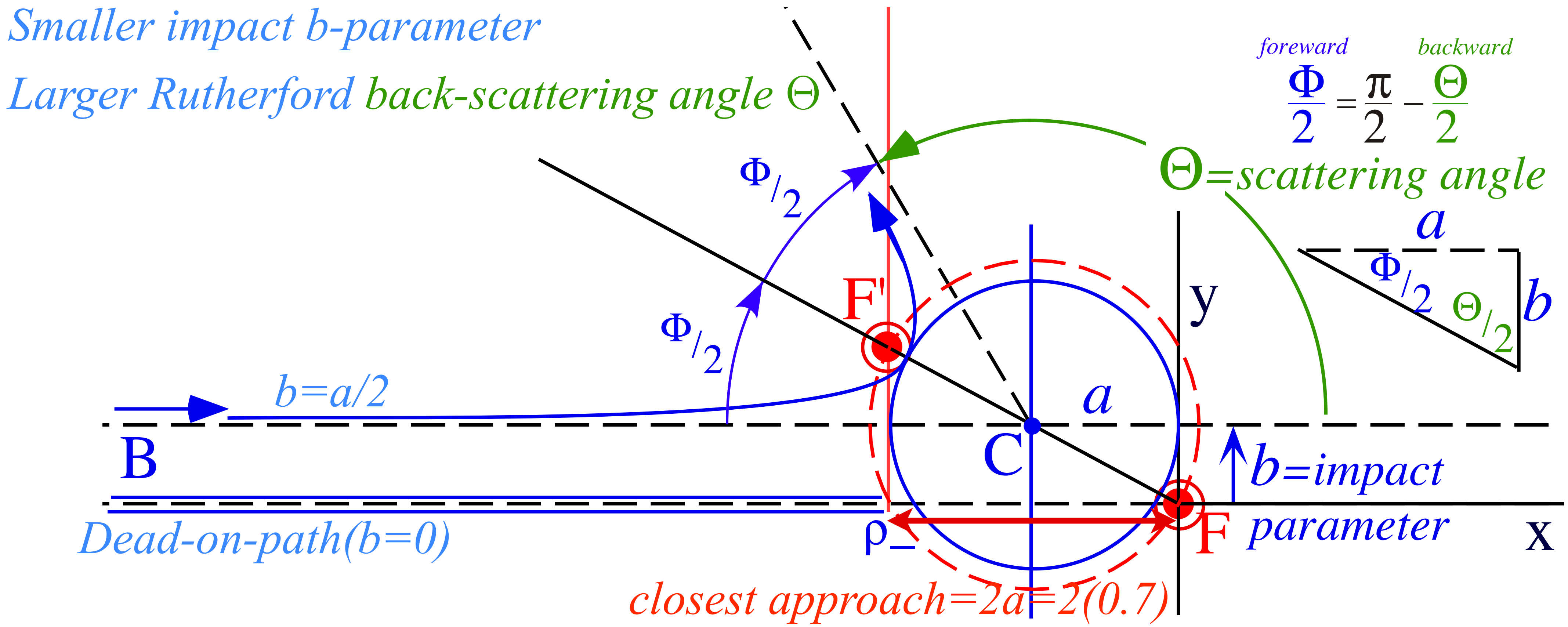
General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

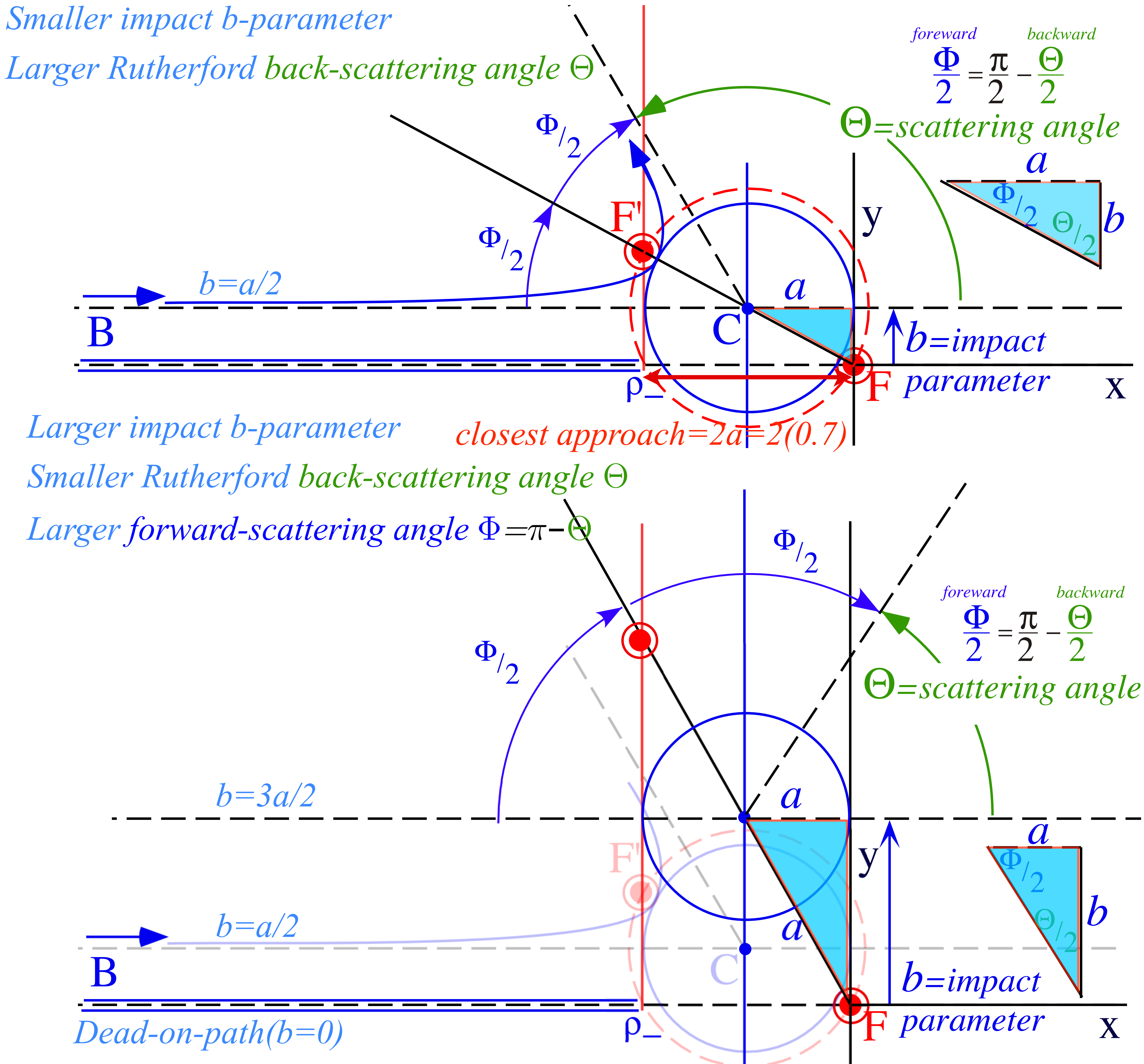
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

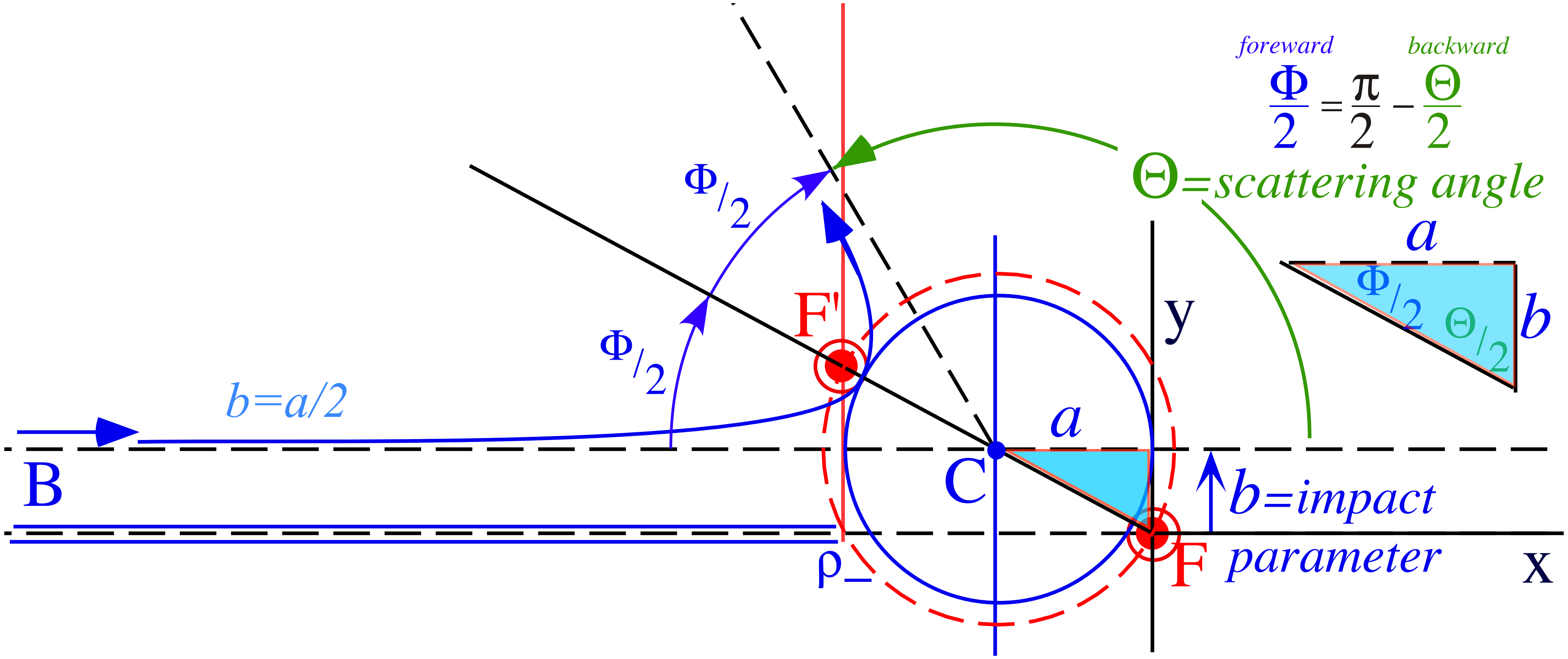
Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)



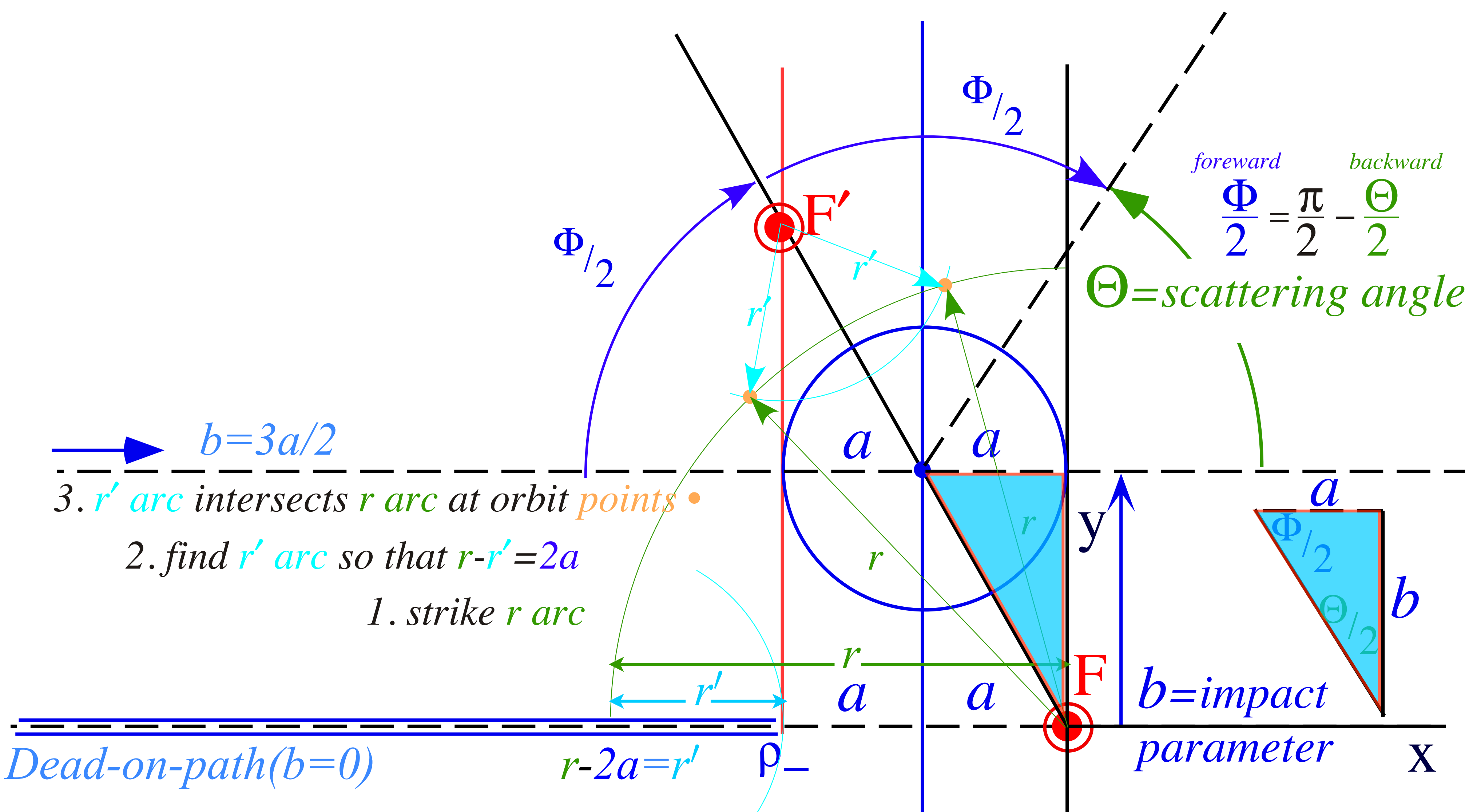
Smaller impact b-parameter
Larger Rutherford back-scattering angle Θ
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
a
Φ/2
Θ/2
b
y
b=a/2
a
B
C
b=impact parameter
Dead-on-path(b=0)
ρ_
F
x
closest approach=2a=2(0.7)



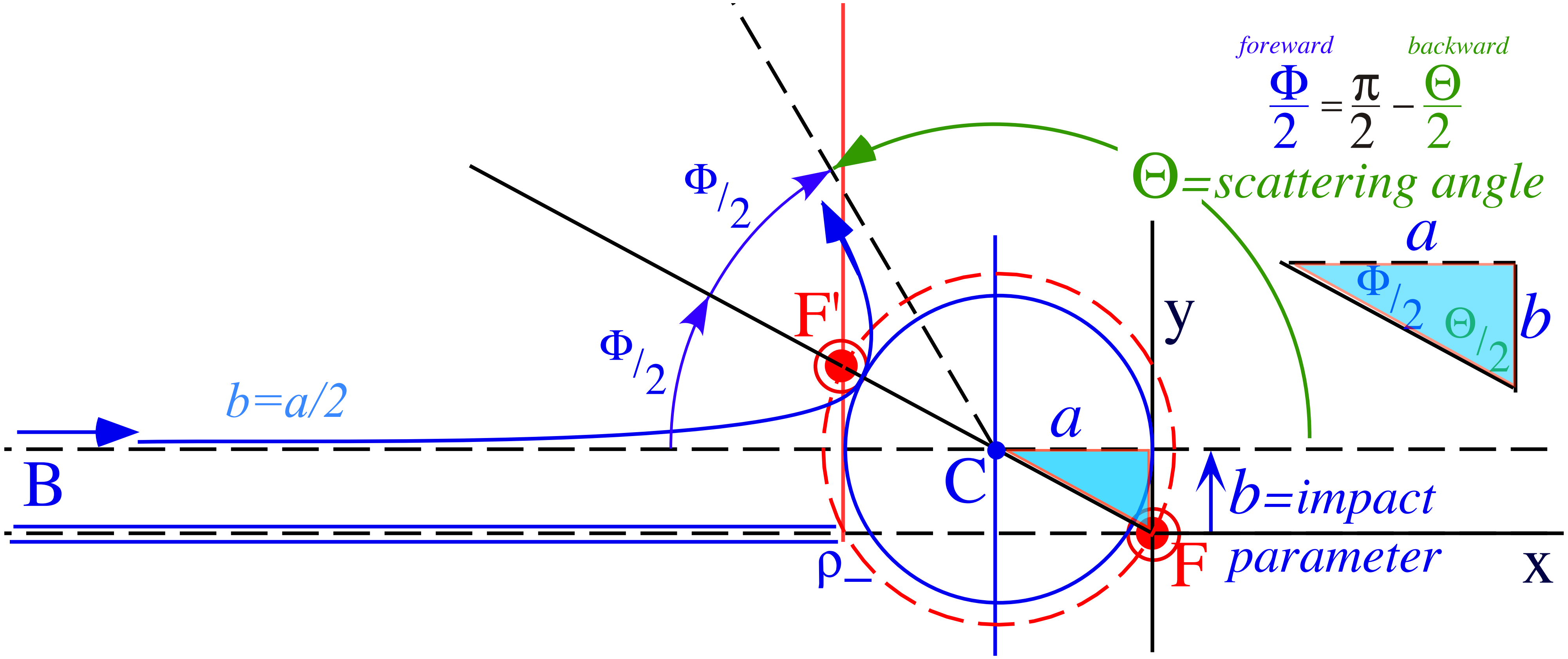
Smaller impact b-parameter
Larger Rutherford back-scattering angle Θ
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Θ/2
y
b=a/2
a
B
C
b=impact
parameter
x
ρ_
F
closest approach=2a=2(0.7)
Larger impact b-parameter
Smaller Rutherford back-scattering angle Θ
Larger forward-scattering angle Φ=π−Θ
Φ/2
Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
b=3a/2
a
a
F'
y
a
Φ/2
b
Θ/2
b=a/2
B
C
a
b=impact
parameter
Dead-on-path(b=0)
ρ_
F
x



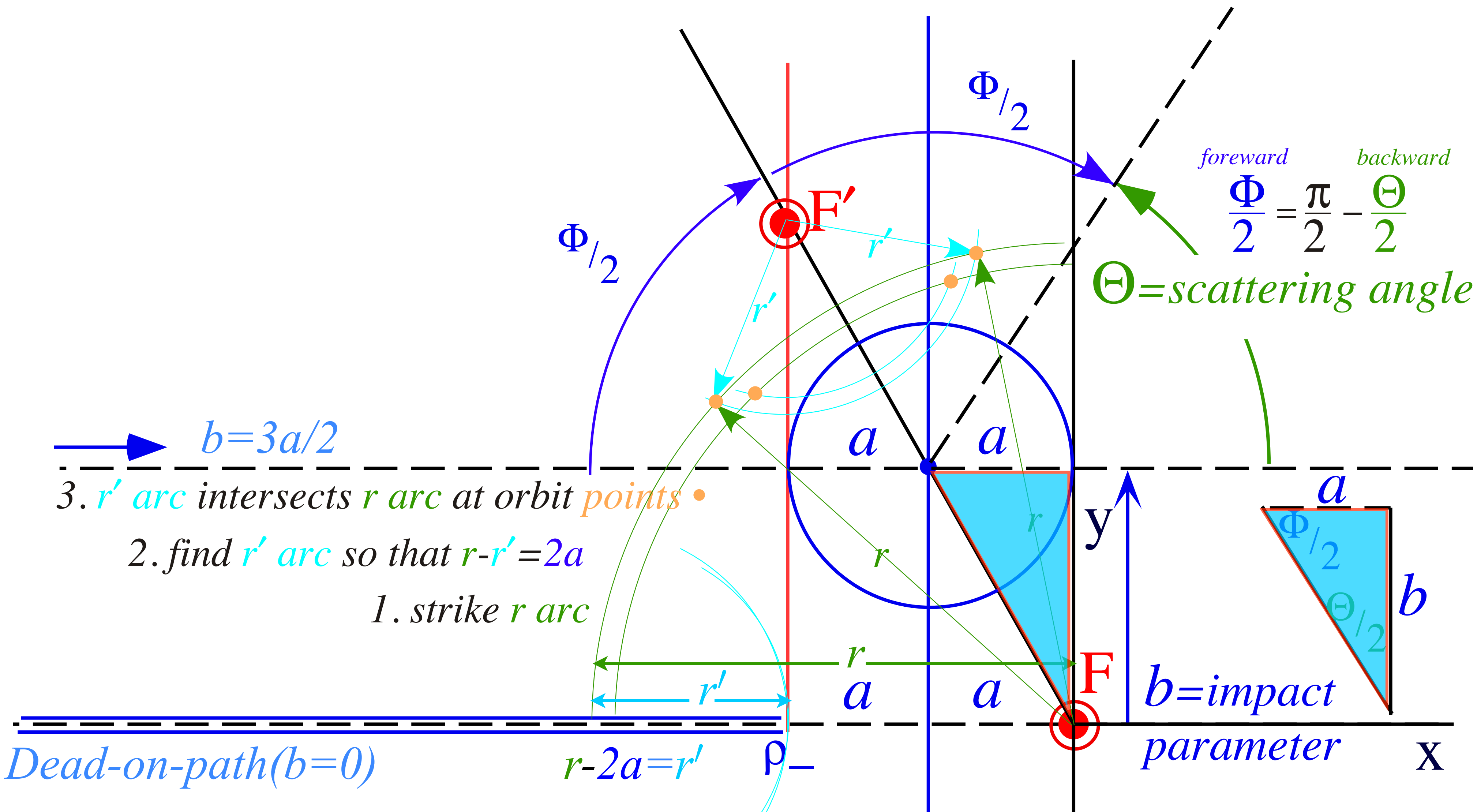
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
a
Φ/2
Θ/2
b
F'
y
b=a/2
a
B
C
b=impact
parameter
ρ_
F
x



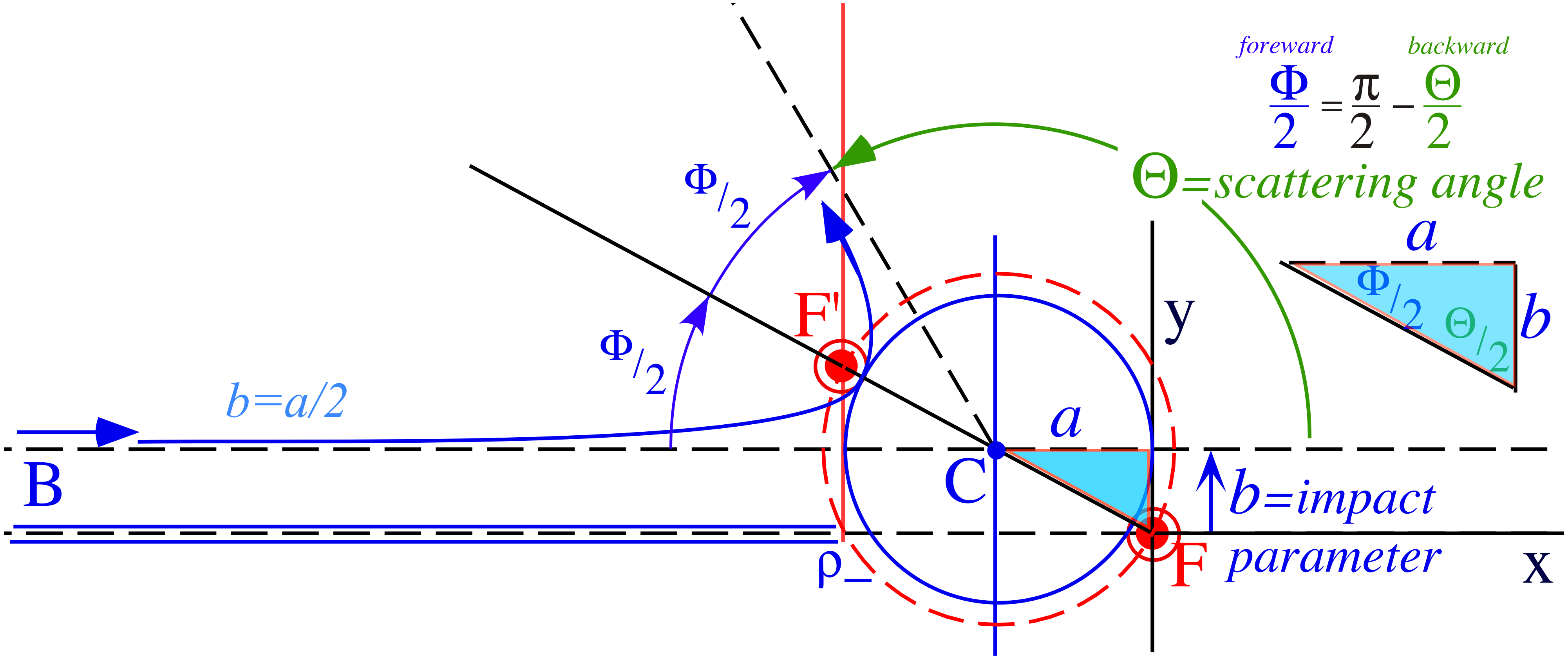
Φ/2
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Φ/2
F′
r′
r′
Θ=scattering angle
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points •
r
y
a
Φ/2
2. find r′ arc so that r-r′=2a
r
b
Θ/2
1. strike r arc
r
F
r′
a
a
b=impact
parameter
Dead-on-path(b=0)
r-2a=r′
ρ_
x



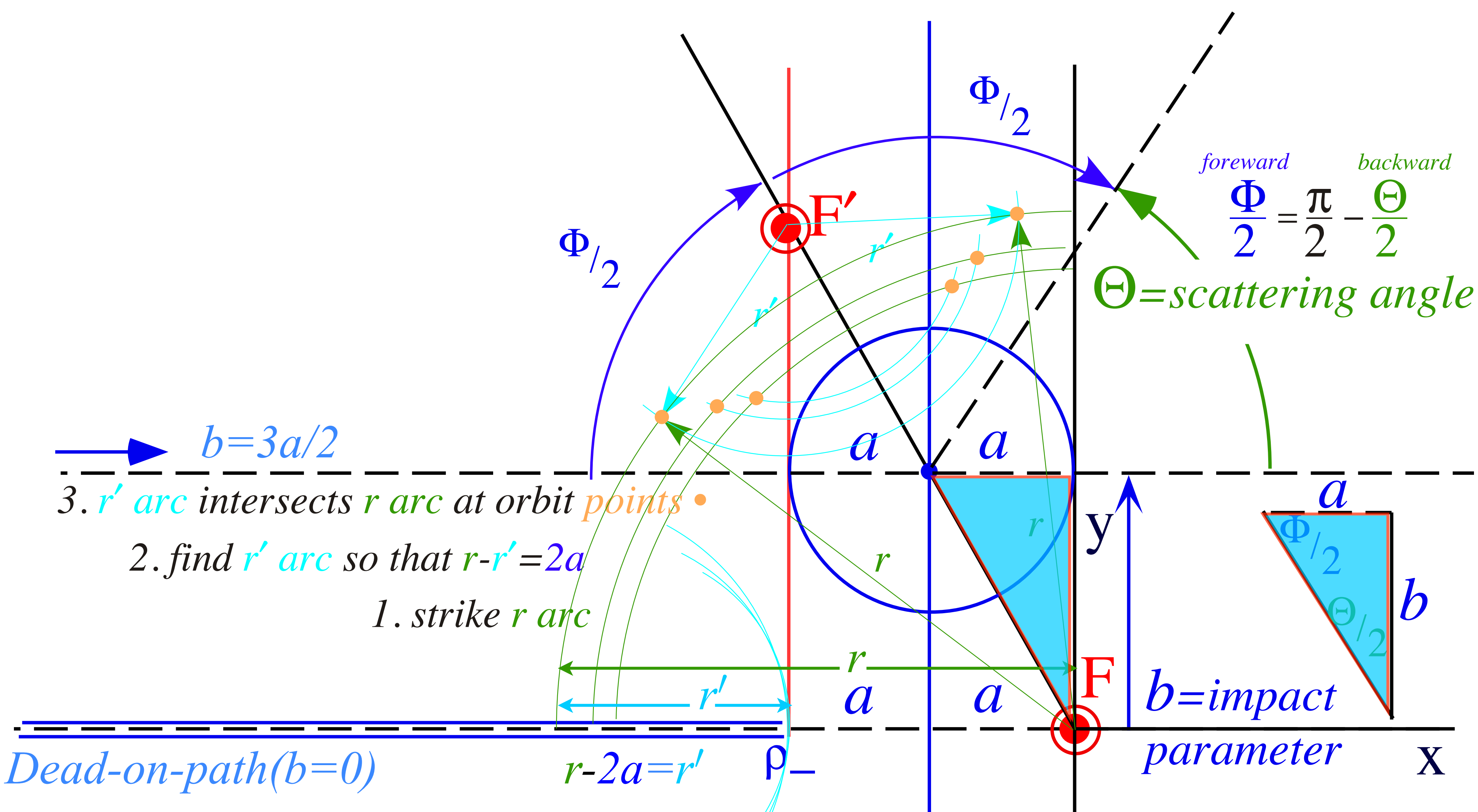
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
a
B
C
a
b
Φ/2
Θ/2
y
b=impact
parameter
ρ_
F
x



Φ/2
Φ/2
F′
r′
r′
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
a
Φ/2
Θ/2
b
r
r′
a
a
F
b=impact
parameter
Dead-on-path(b=0)
r-2a=r′
ρ_
x



foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
C
a
y
b=impact
parameter
F
x
ρ_
a
b
Φ/2
Θ/2



Φ/2
Φ/2
F′
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
r′
r′
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points
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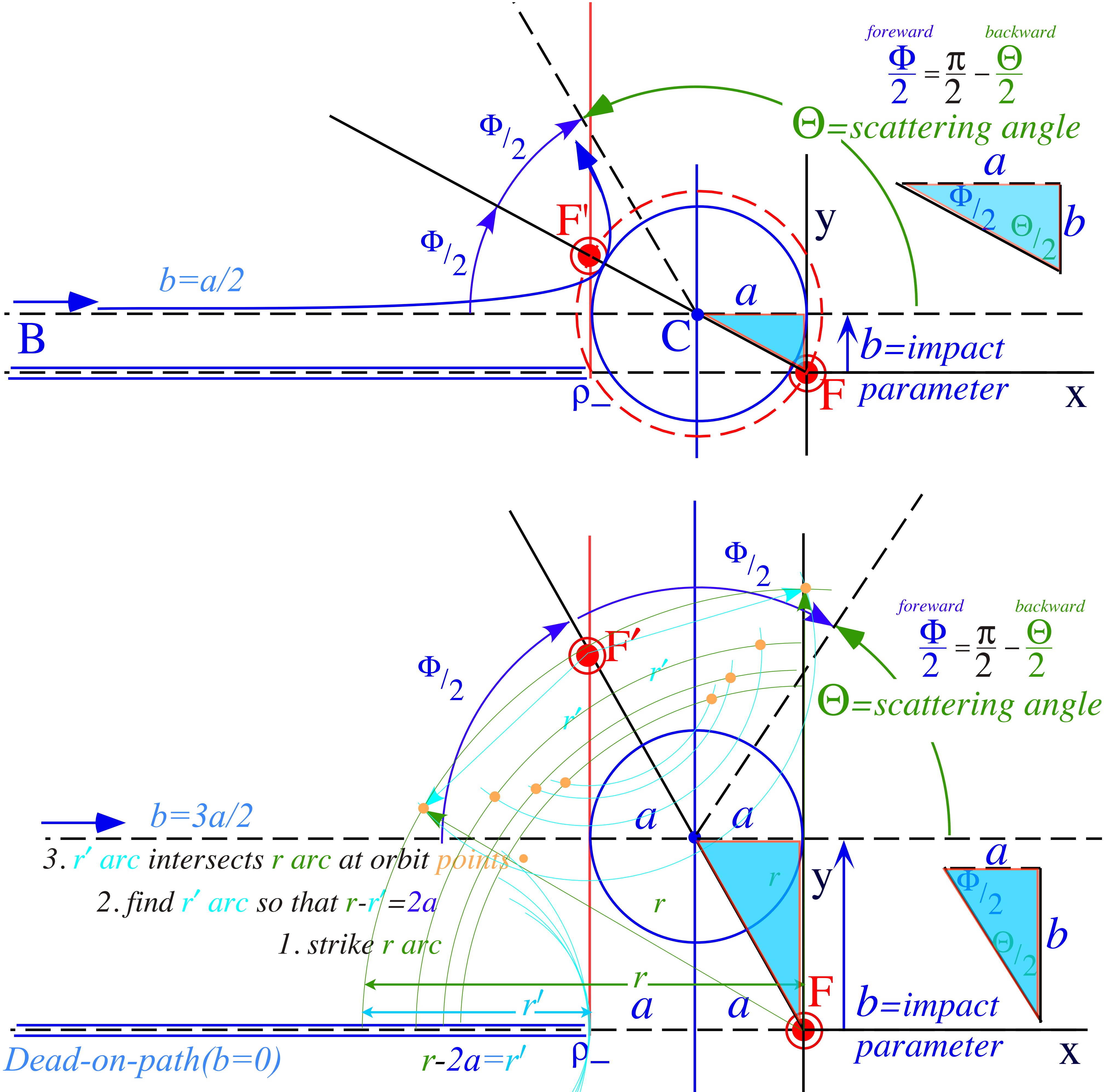
r
r
y
a
Φ/2
Θ/2
b
r
r′
a
a
F
b=impact
parameter
x
Dead-on-path(b=0)
r-2a=r′
ρ_



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backward
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Θ=scattering angle
b=a/2
B
C
F'
F
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x
a
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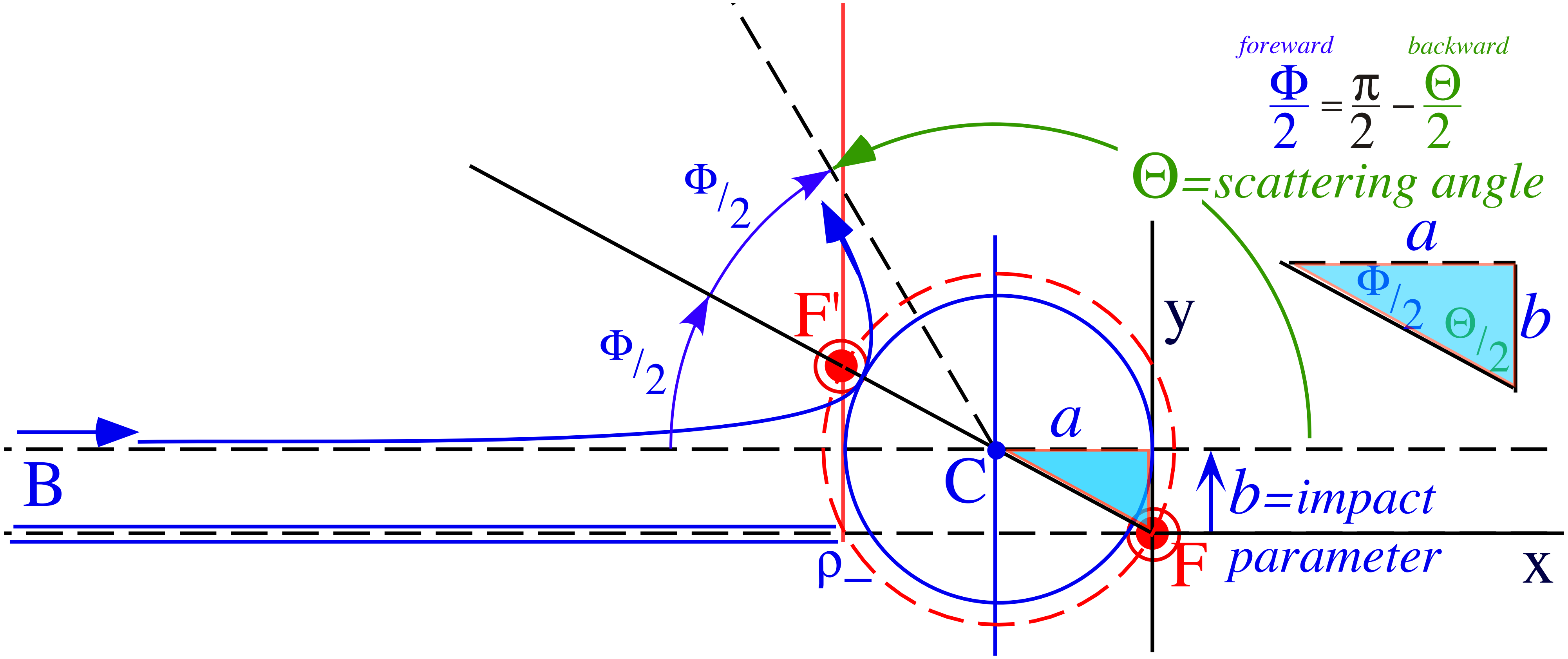
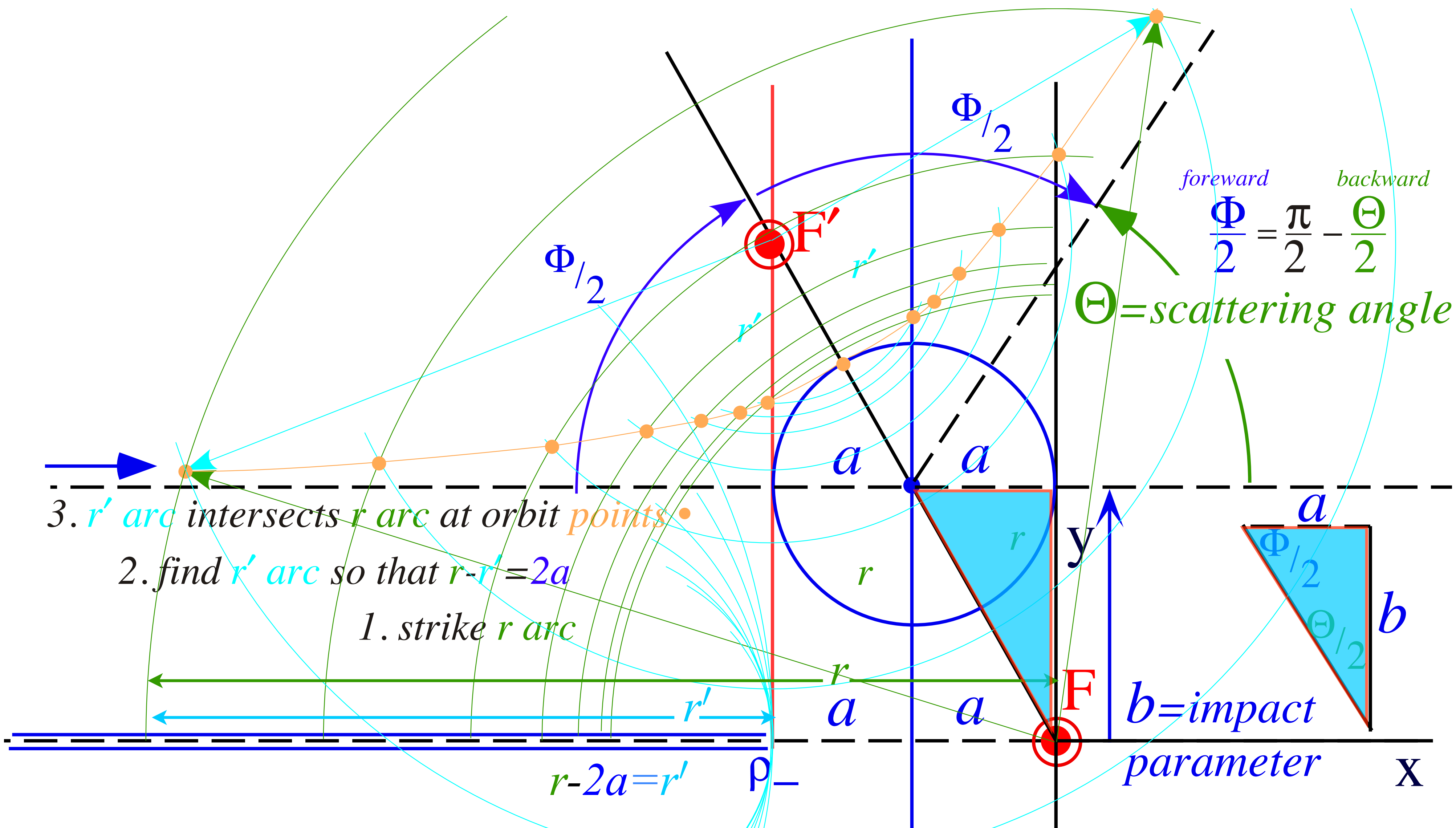
b=impact parameter
ρ_
b=3a/2
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r-2a=r′
Dead-on-path(b=0)


foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
Φ/2
Θ/2
b
y
B
C
a
b=impact
parameter
F
ρ_
x


Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
F′
Φ/2
r′
r′
Θ=scattering angle
a
a
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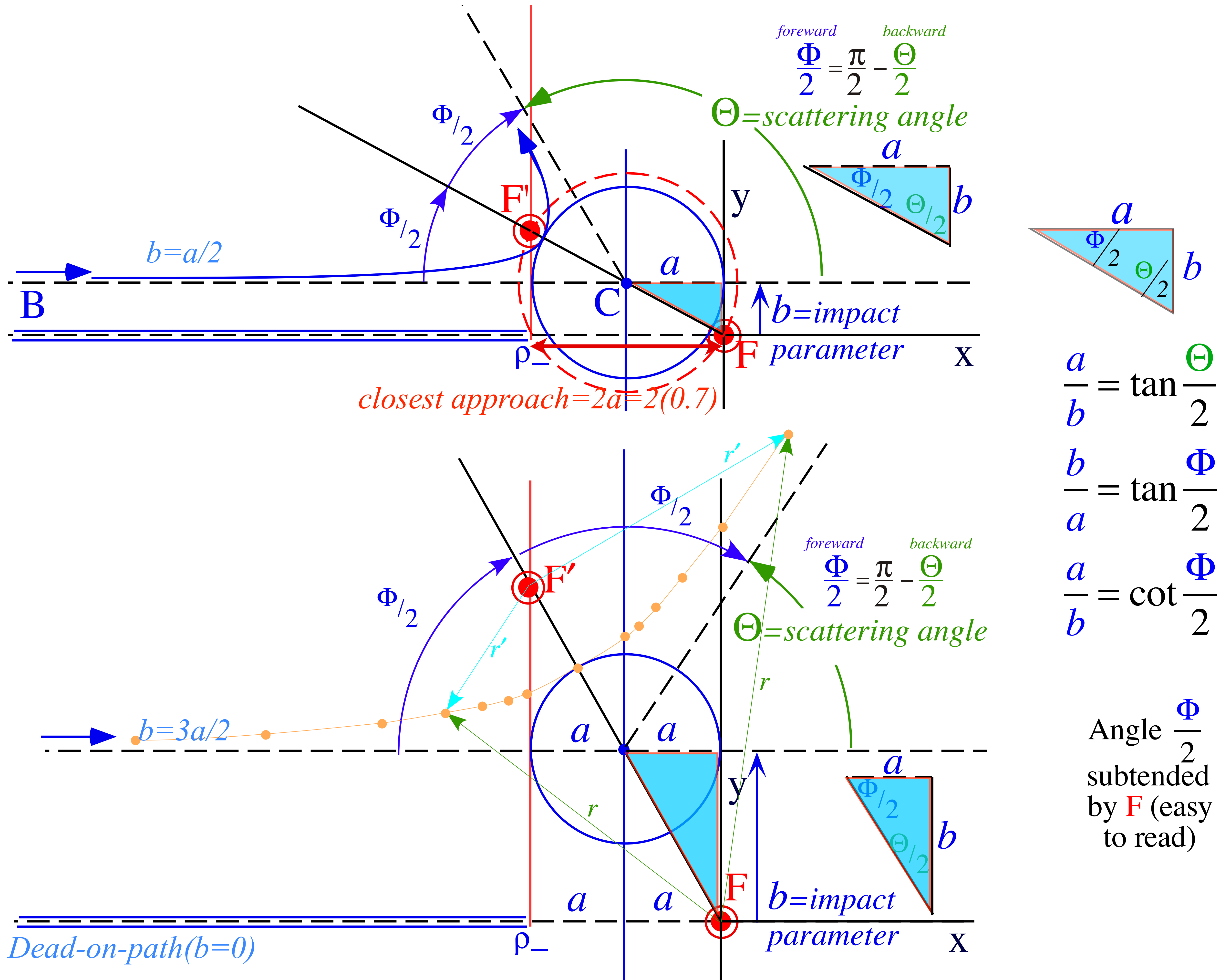
r
r
y
a
Φ/2
Θ/2
b
r
r′
F
a
a
b=impact
parameter
r-2a=r′
ρ_
x



foreward backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
C
a
b
y
b=impact parameter
ρ_
F
x
closest approach=2a=2(0.7)
r'
Φ/2
F′
foreward backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
r'
b=3a/2
r
a
a
r
y
a
b
Θ/2
F
a
a
b=impact parameter
ρ_
Dead-on-path(b=0)
x
a/b = tan Θ/2
b/a = tan Φ/2
a/b = cot Φ/2
Angle Φ/2 subtended by F (easy to read)

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

➔ *Parabolic "kite" and orbital envelope geometry*

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

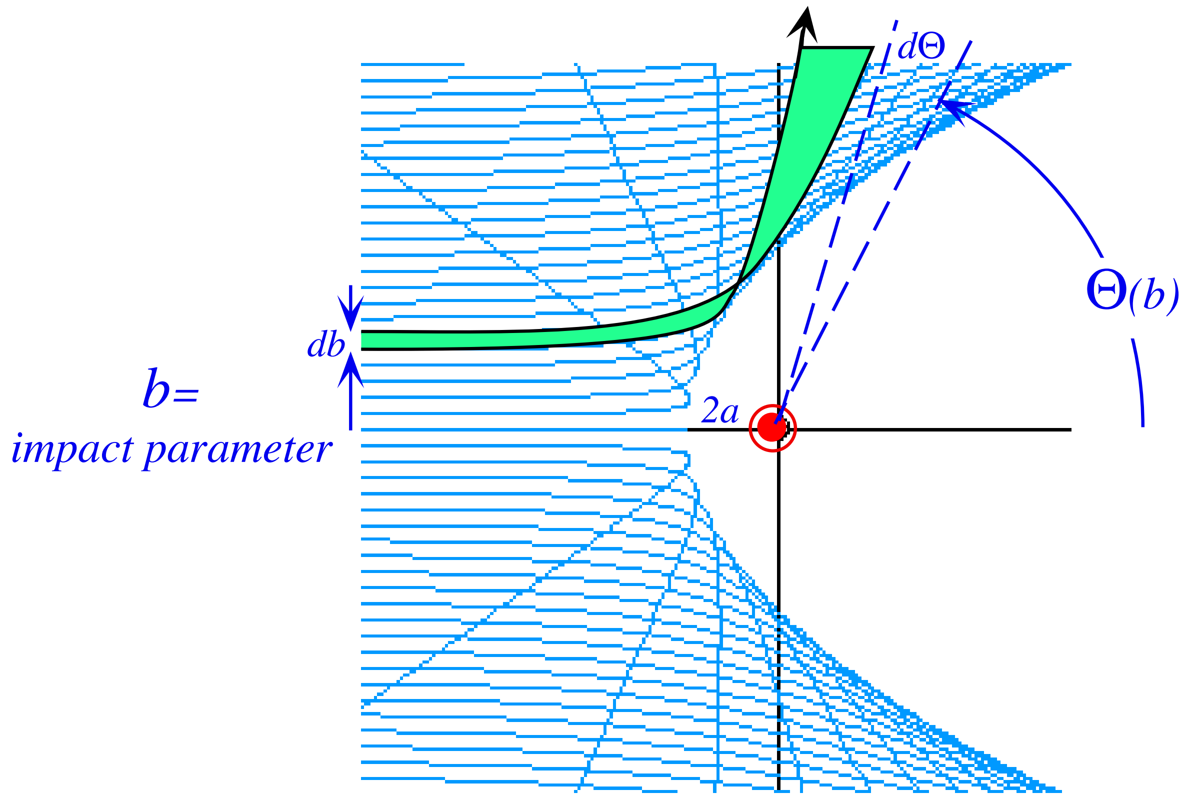
Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

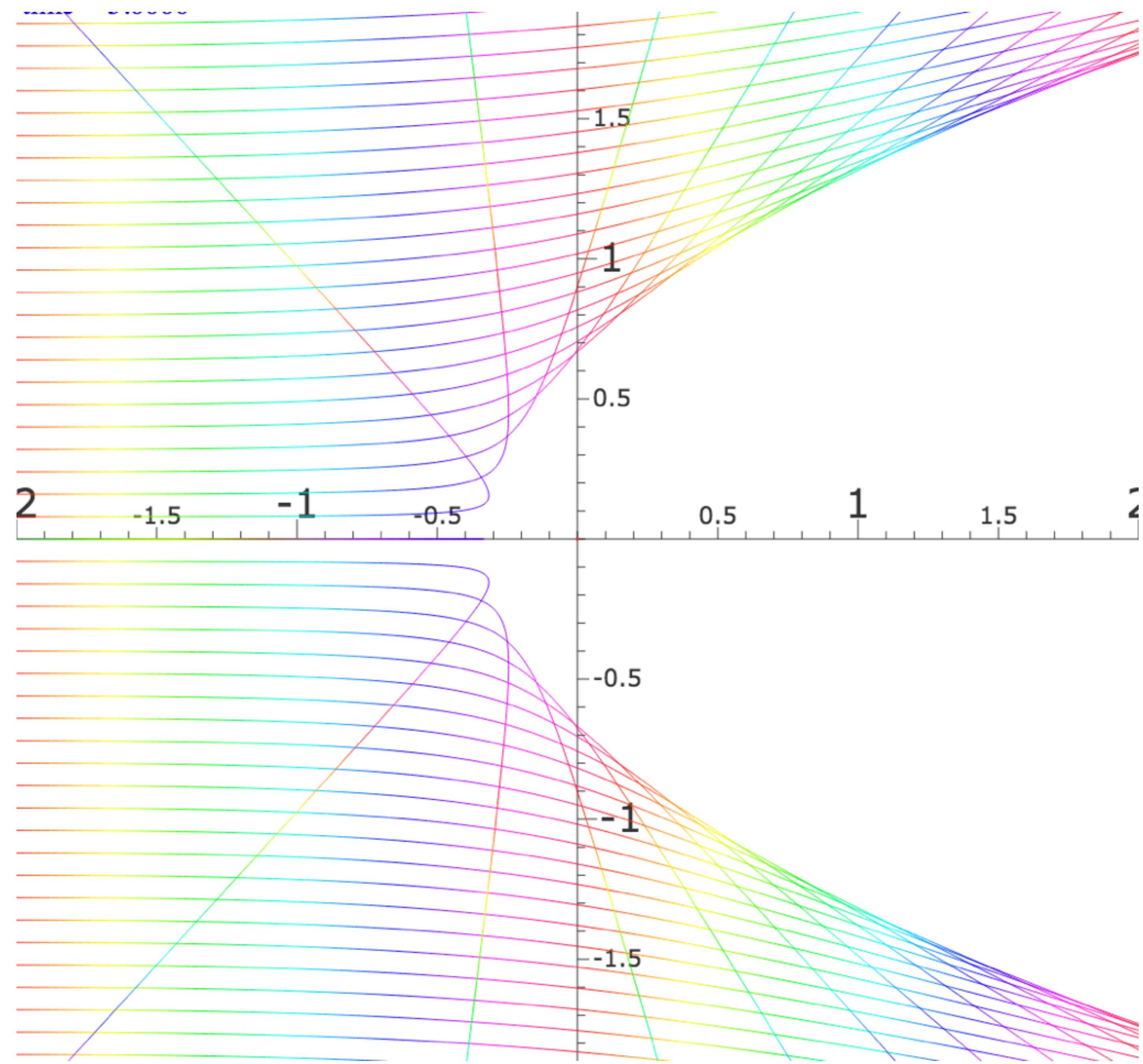
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

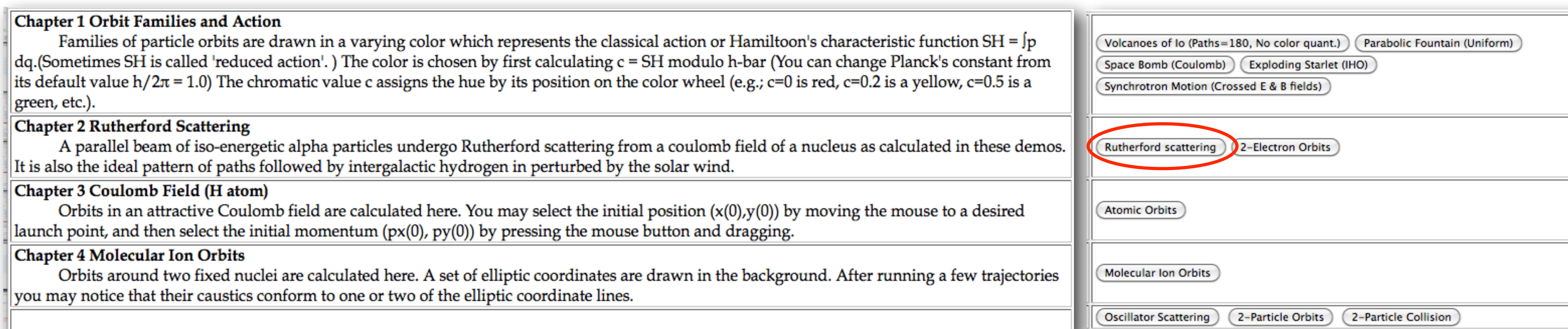
Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Rutherford scattering geometry





https://modphys.hosted.uark.edu/CoulItWeb.html?scenario=Rutherford



Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Volcanoes of Io (Paths=180, No color quant.) | Parabolic Fountain (Uniform) | Space Bomb (Coulomb) | Exploding Starlet (IHO) | Synchrotron Motion (Crossed E & B fields)

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Rutherford scattering | 2-Electron Orbits

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Atomic Orbits

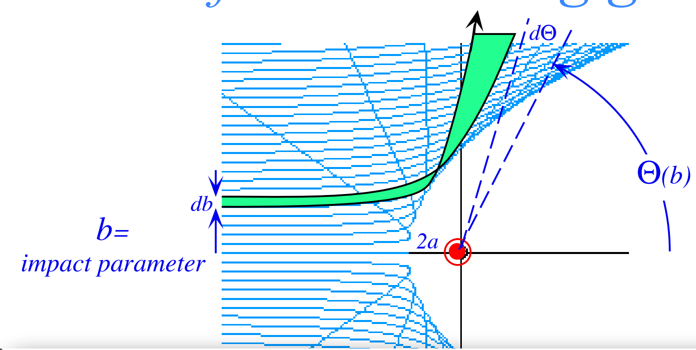
Chapter 4 Molecular Ion Orbits

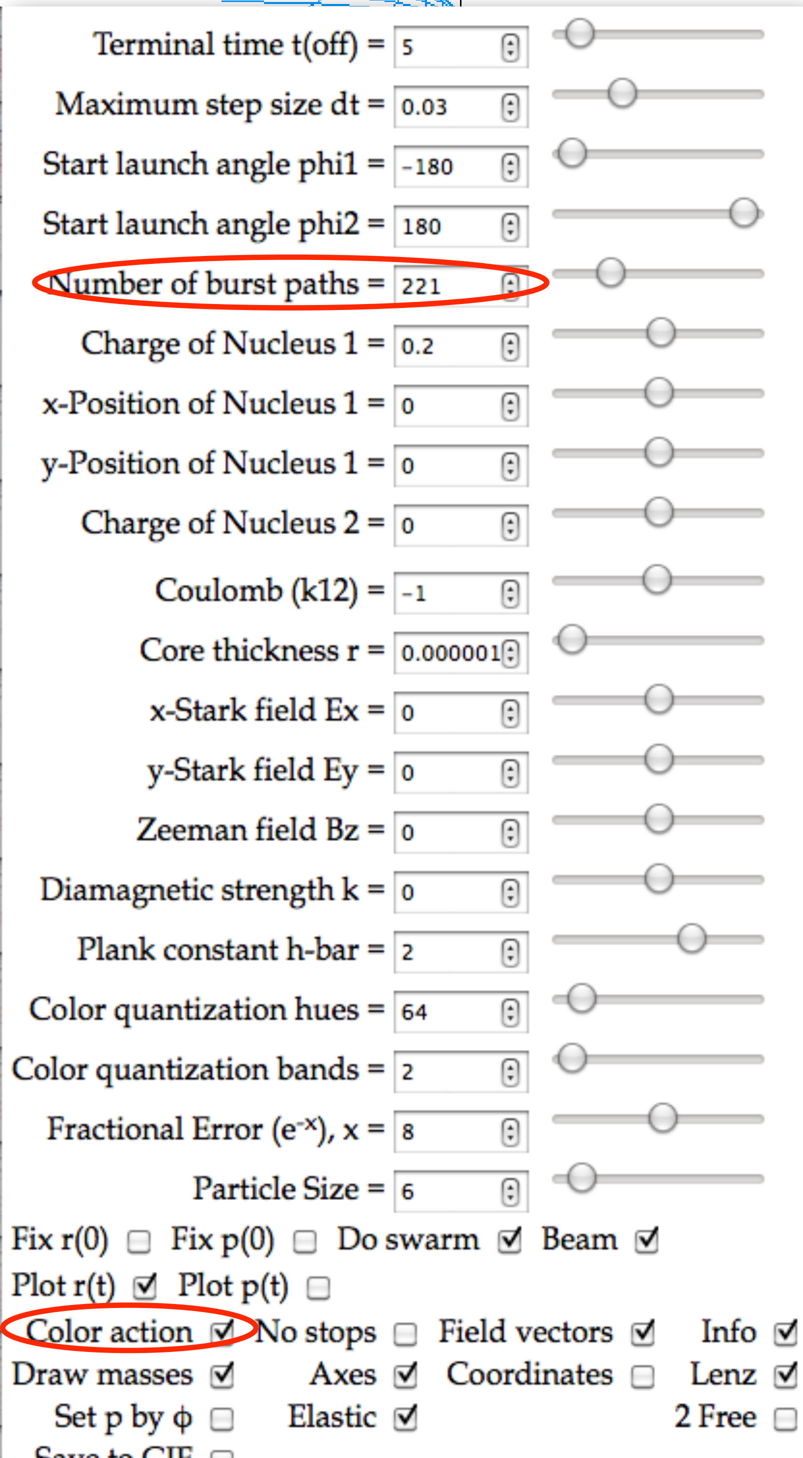
Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

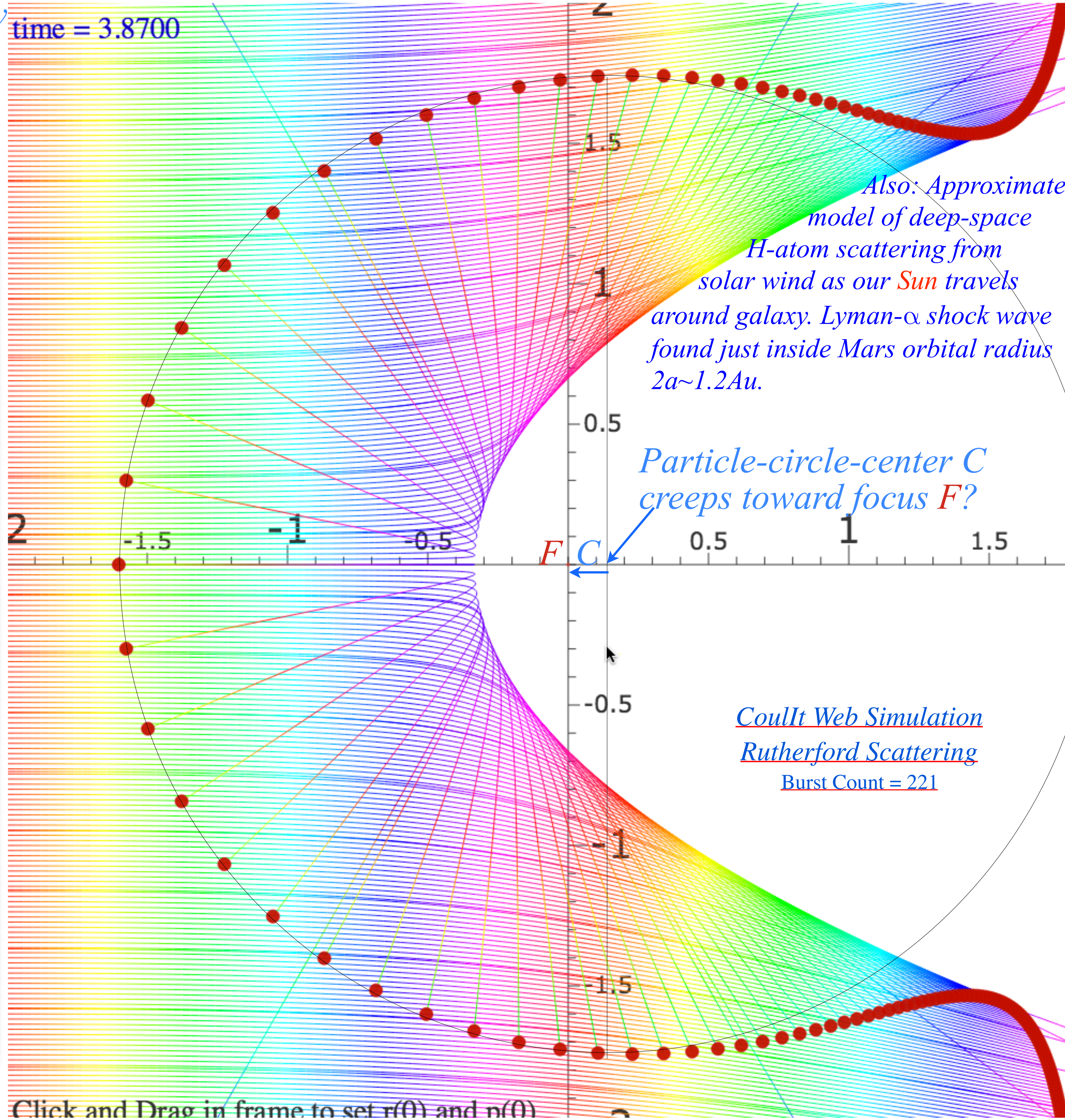
Molecular Ion Orbits

Oscillator Scattering | 2-Particle Orbits | 2-Particle Collision

Rutherford scattering geometry







Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Particle-circle-center C creeps toward focus F?

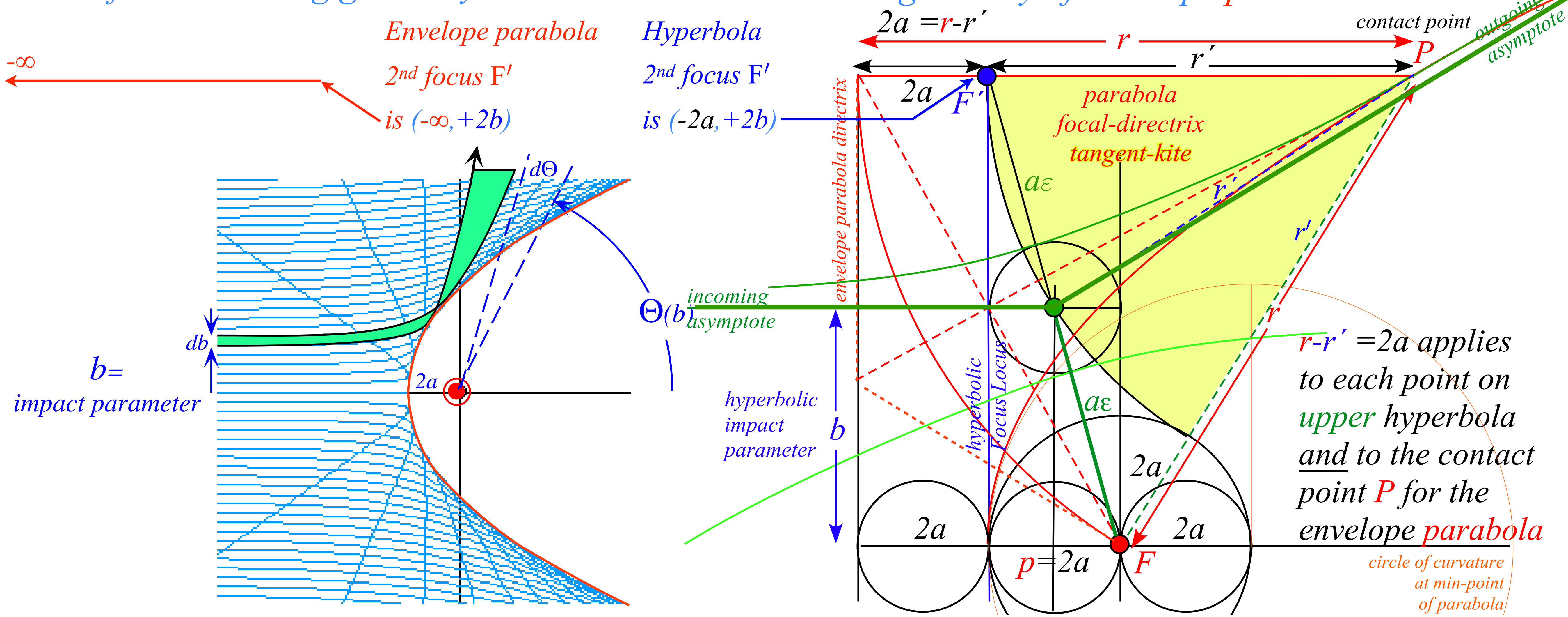
[CoulIt Web Simulation](#)

[Rutherford Scattering](#)

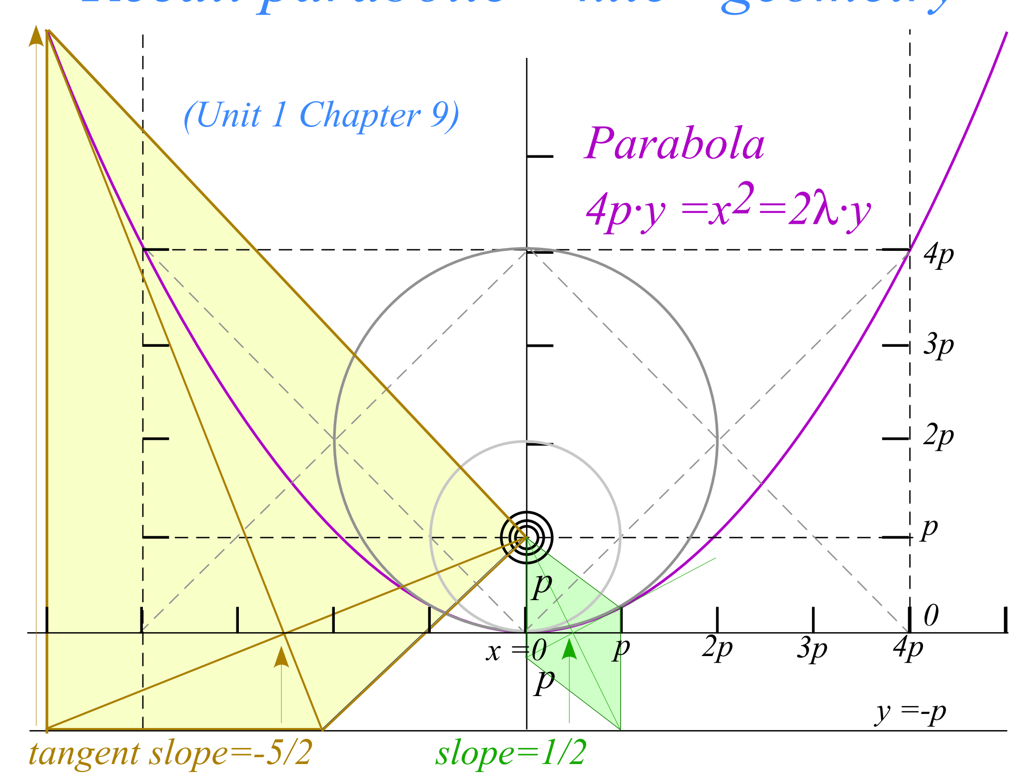
Burst Count = 221

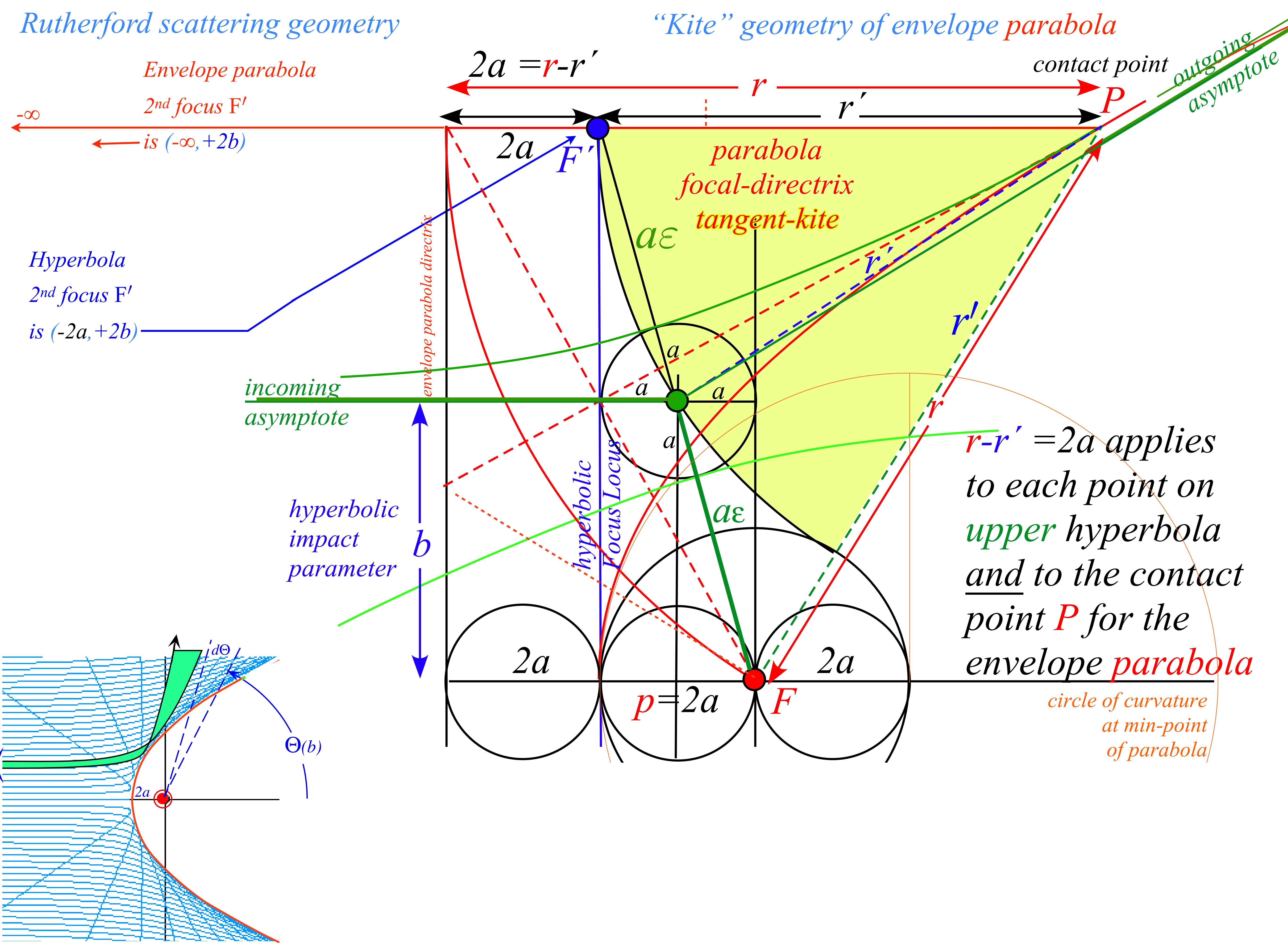
Rutherford scattering geometry

"Kite" geometry of envelope parabola

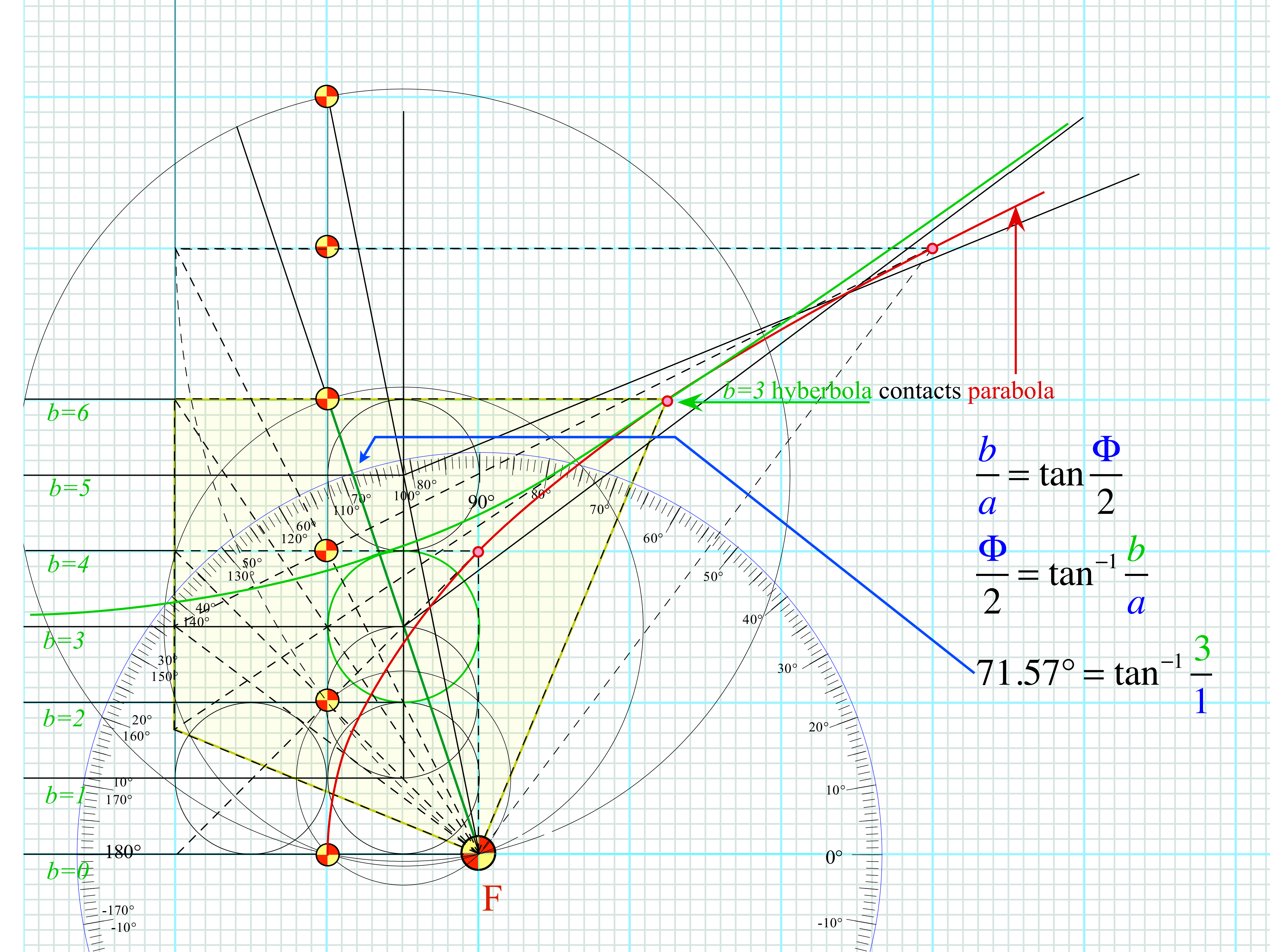


Recall parabolic " kite" geometry



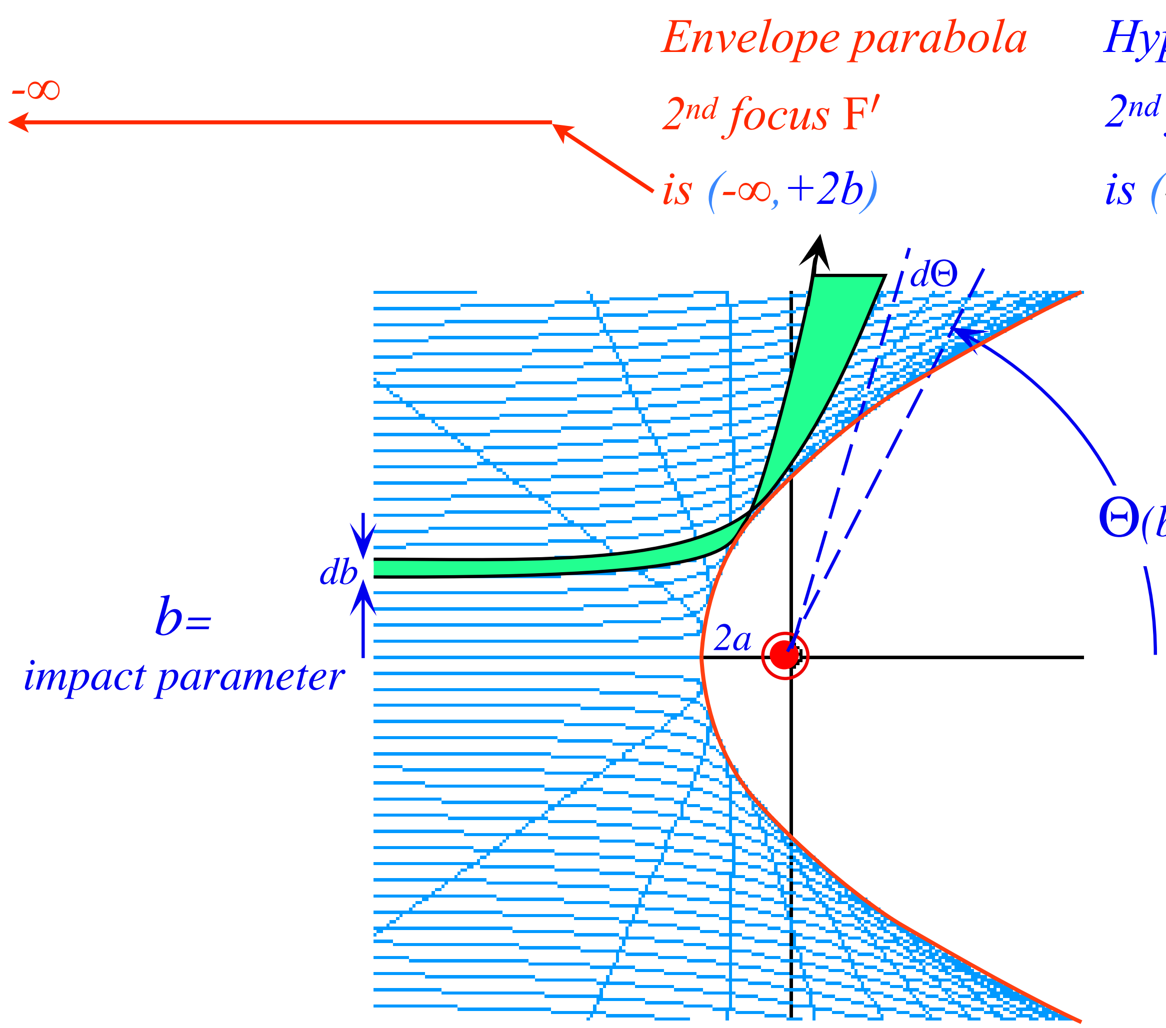


Rutherford scattering geometry
"Kite" geometry of envelope parabola
Envelope parabola
2nd focus F′
is (-∞,+2b)
-∞
2a =r-r′
r
r′
contact point
P
outgoing asymptote
2a
F′
parabola focal-directrix tangent-kite
aε
Hyperbola
2nd focus F′
is (-2a,+2b)
envelope parabola directrix
incoming asymptote
a
a
a
a
r′
r′
r
hyperbolic impact parameter
b
hyperbolic Focus Locus
aε
r-r′ =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola
2a
2a
p=2a
F
circle of curvature at min-point of parabola
dΘ
Θ(b)
2a

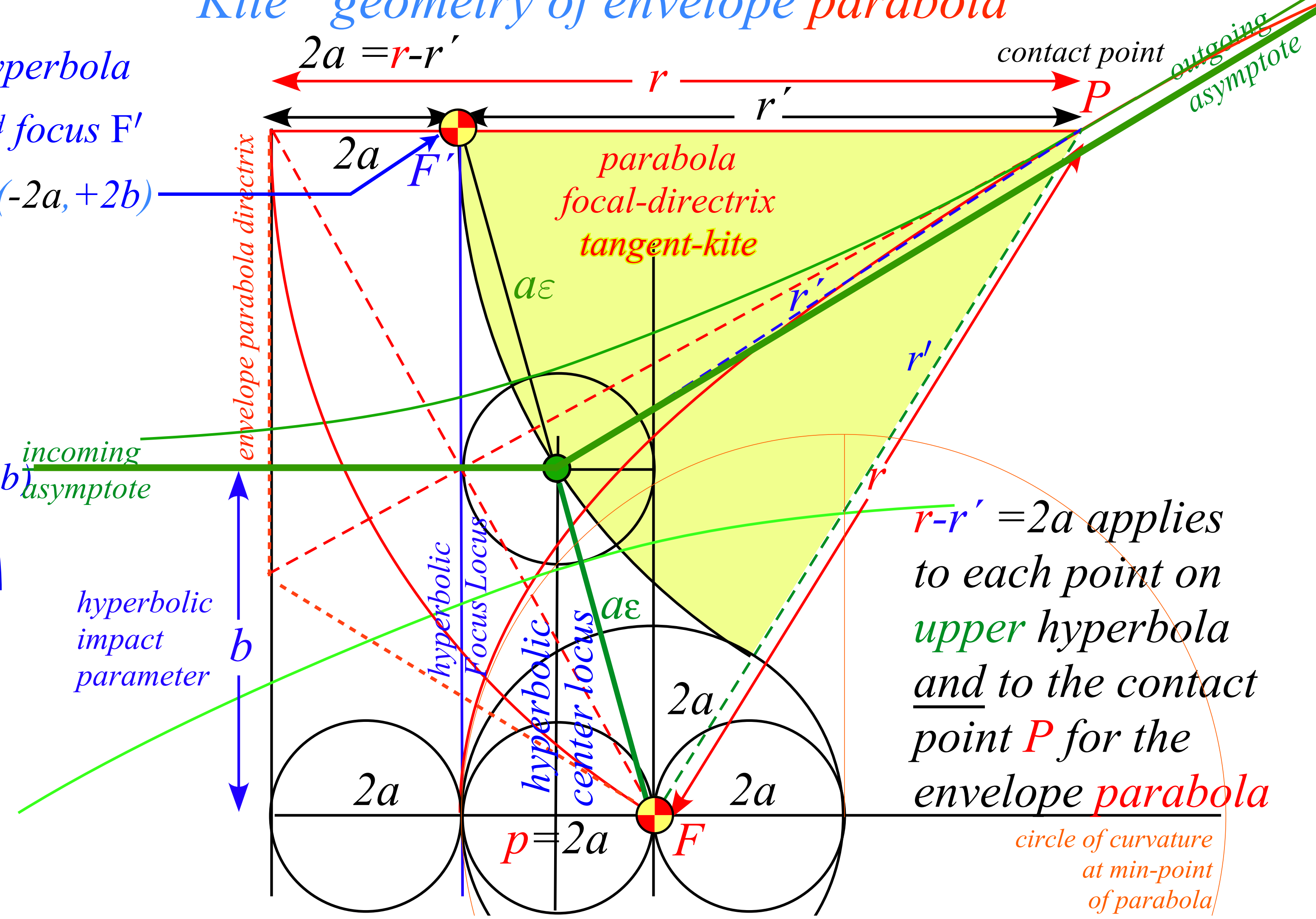


b=6
b=5
b=4
b=3
b=2
b=1
b=0
F
b=3 hyberbola contacts parabola
$\frac{b}{a} = \tan\frac{\Phi}{2}$
$\frac{\Phi}{2} = \tan^{-1}\frac{b}{a}$
$71.57° = \tan^{-1}\frac{3}{1}$
180°
0°
90°
-170°
-10°

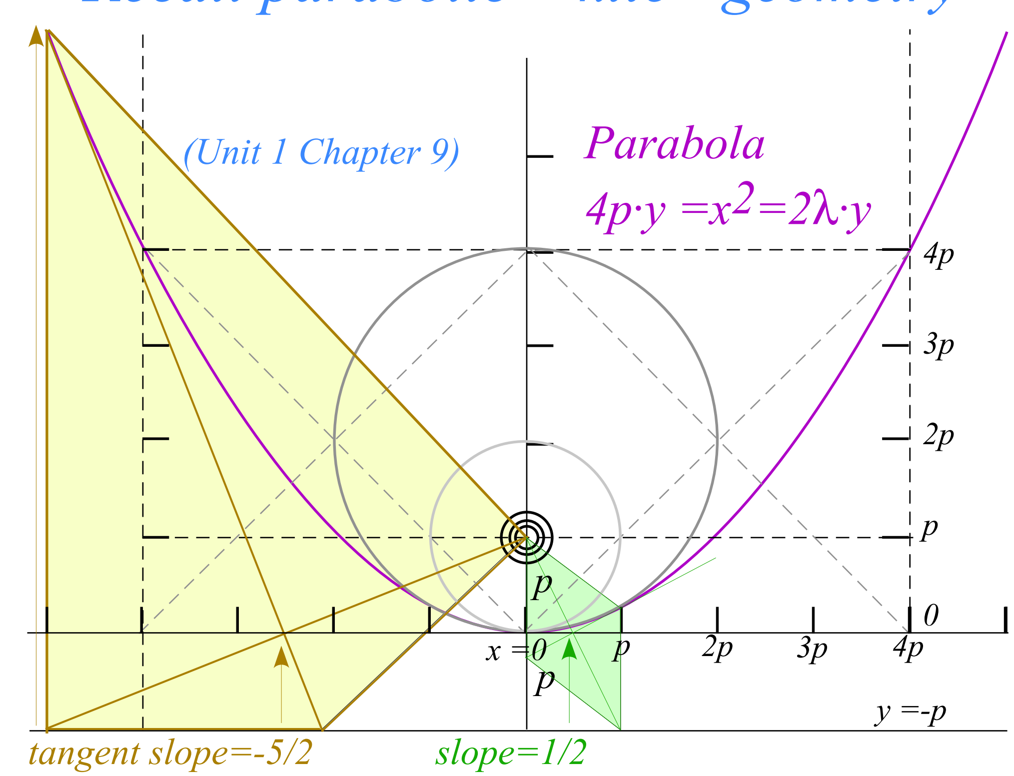
Rutherford scattering geometry



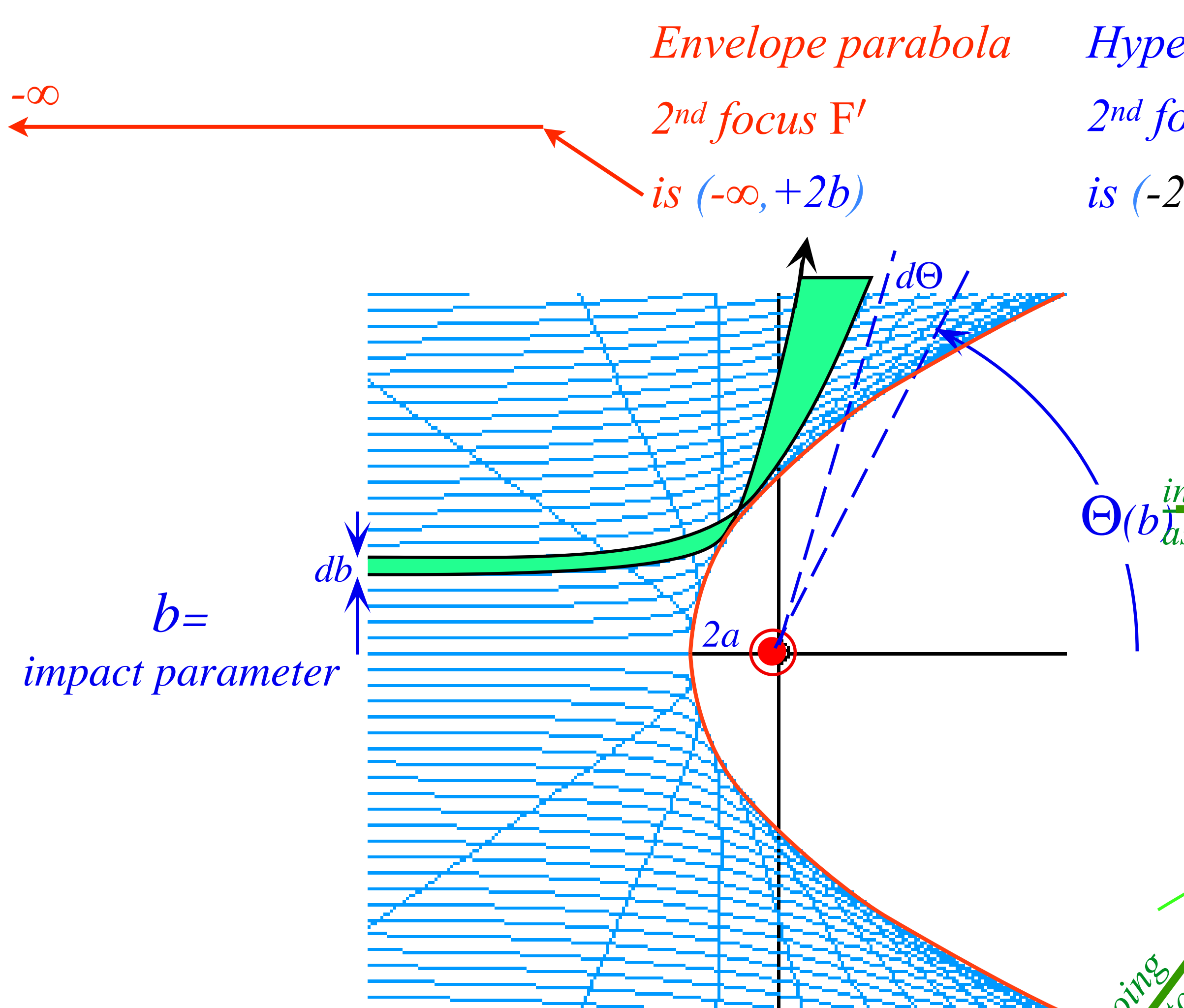
"Kite" geometry of envelope parabola



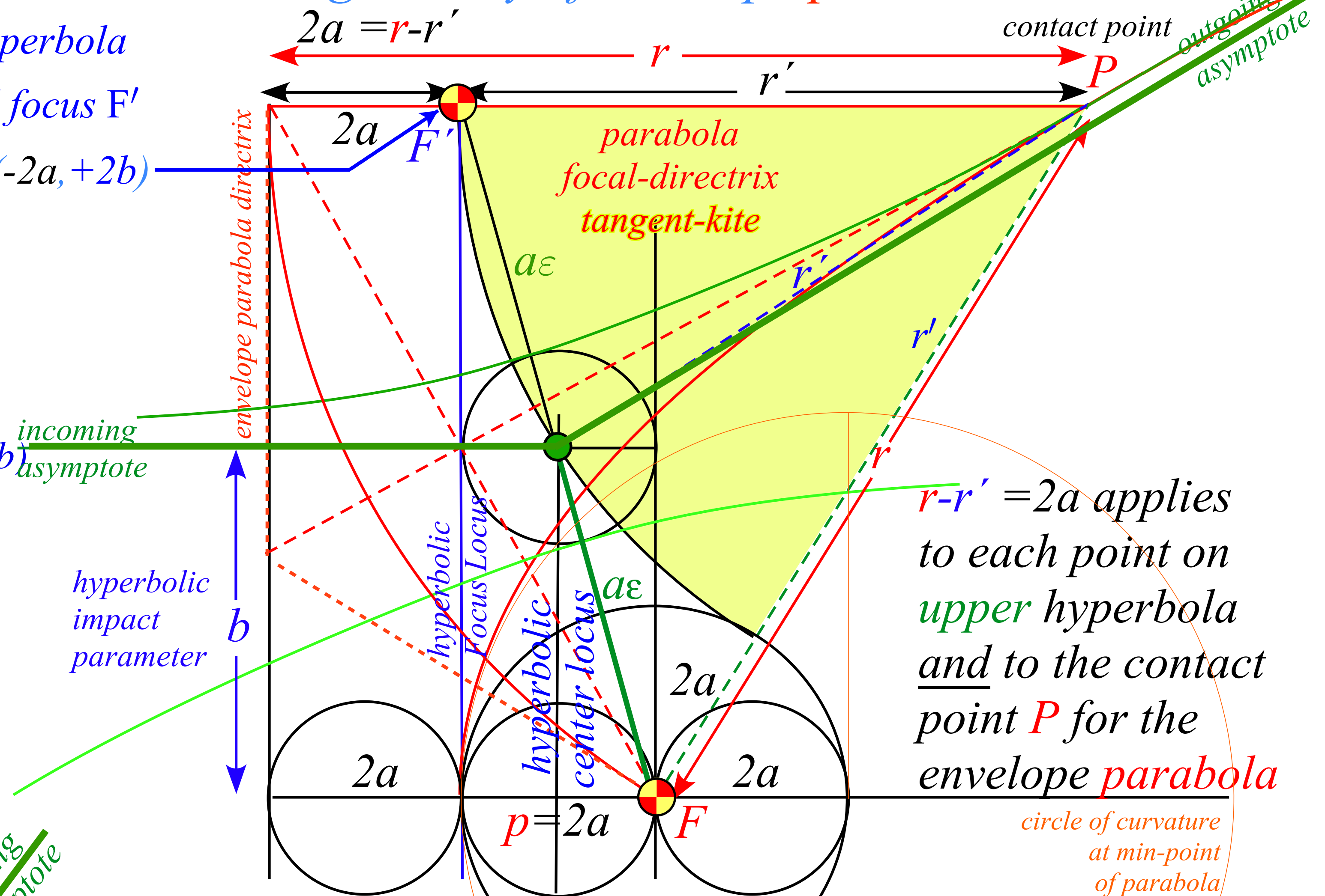
Recall parabolic " kite" geometry



Rutherford scattering geometry

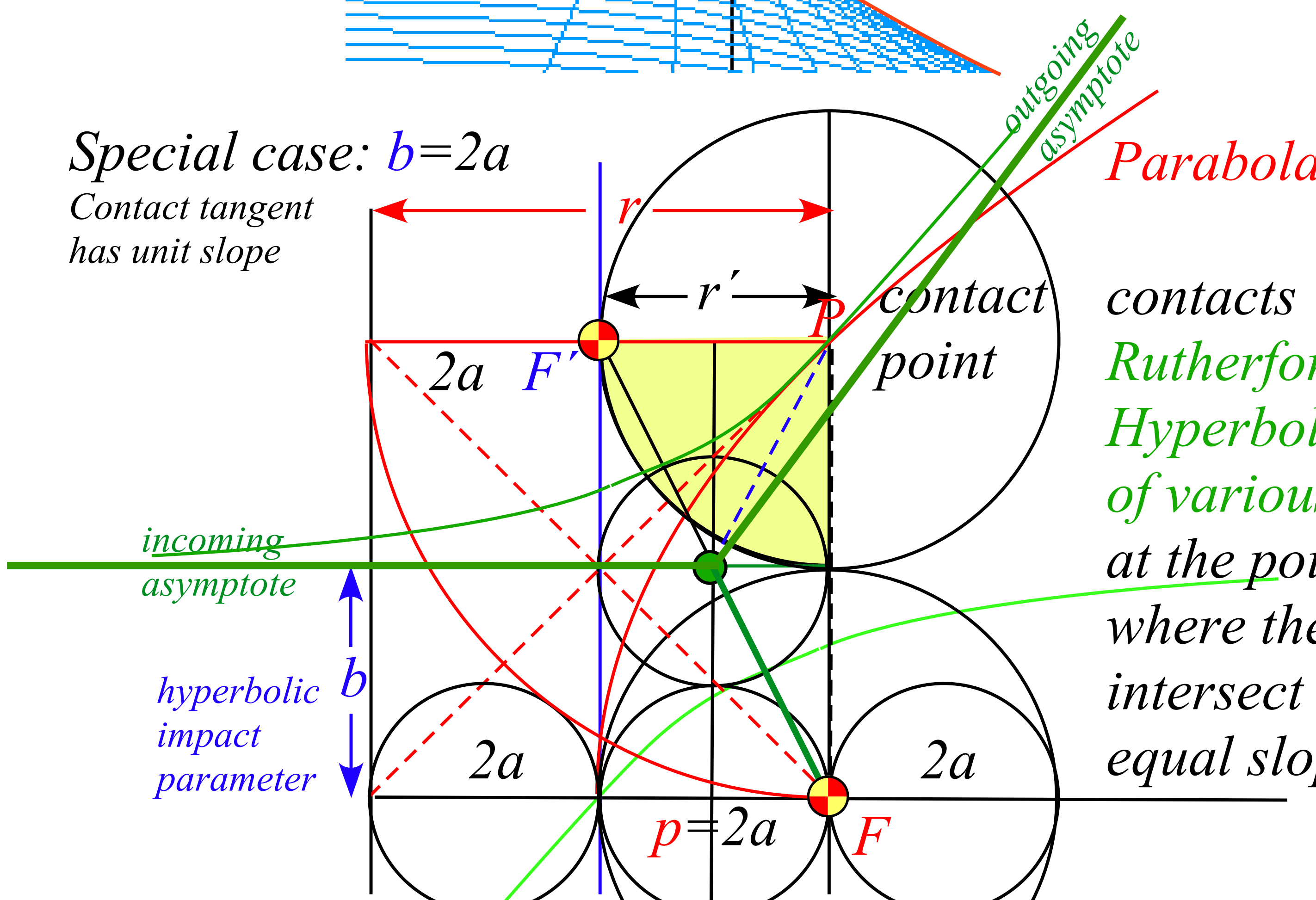


"Kite" geometry of envelope parabola



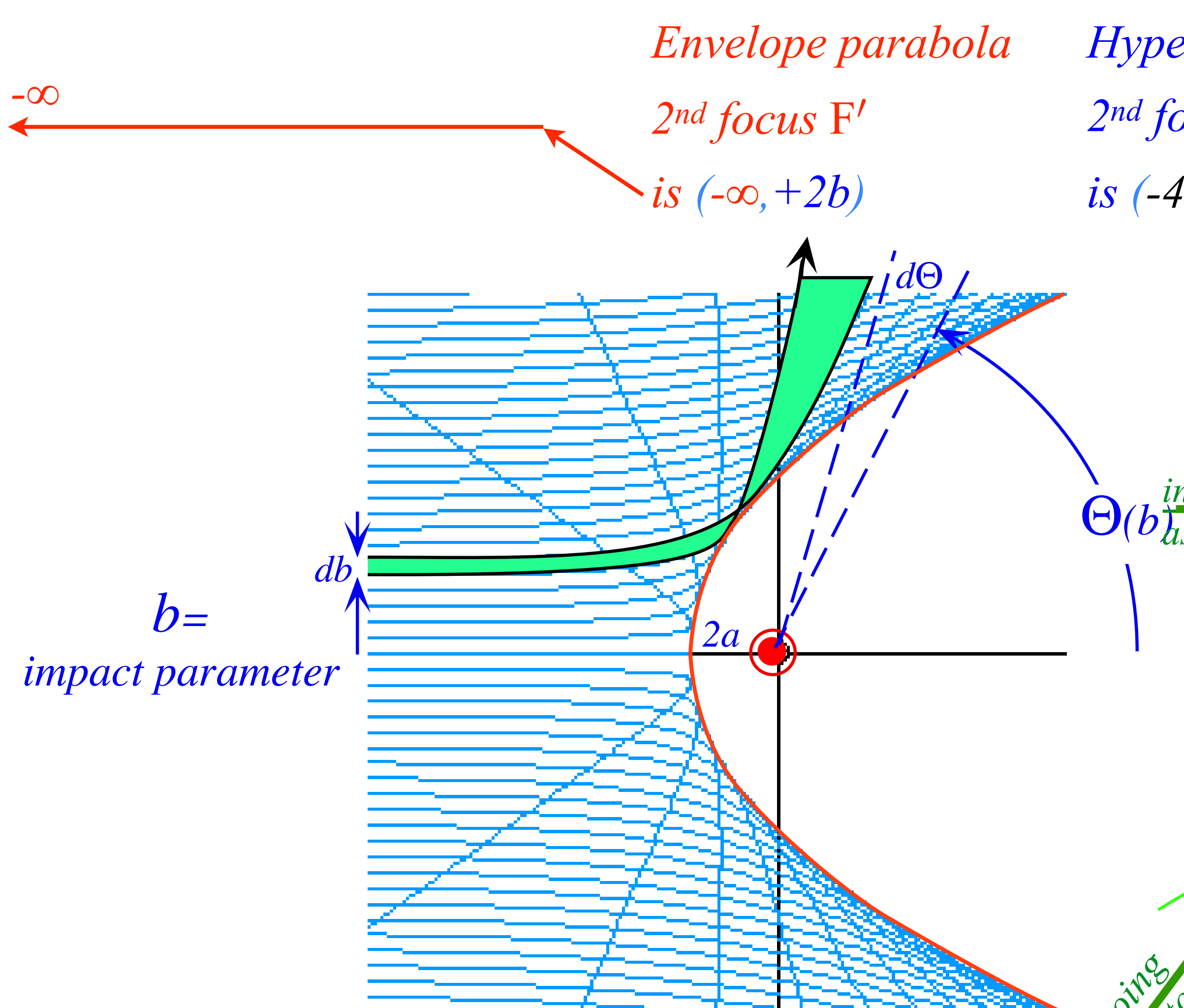
Special case: b=2a

Contact tangent has unit slope

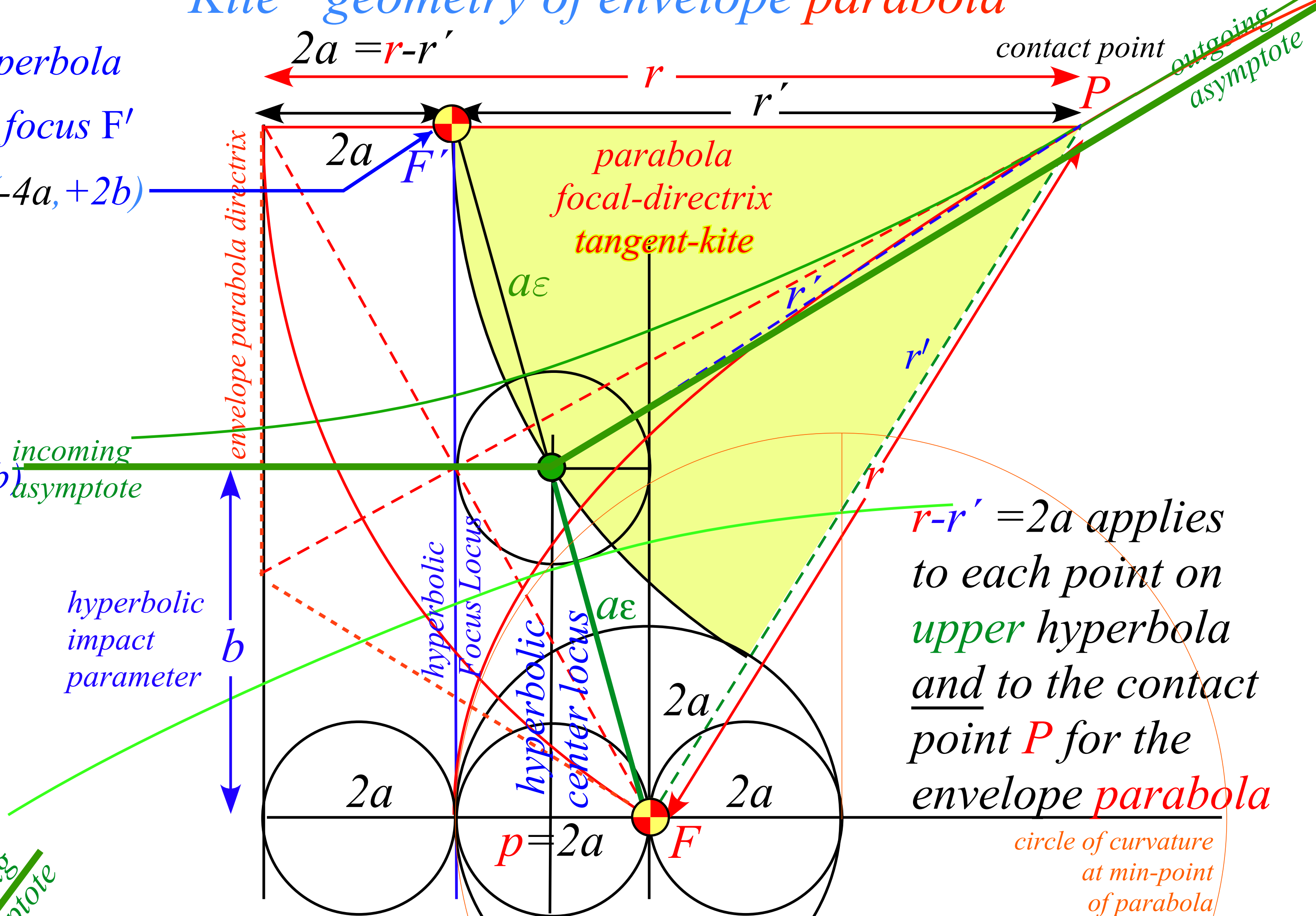


contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Rutherford scattering geometry



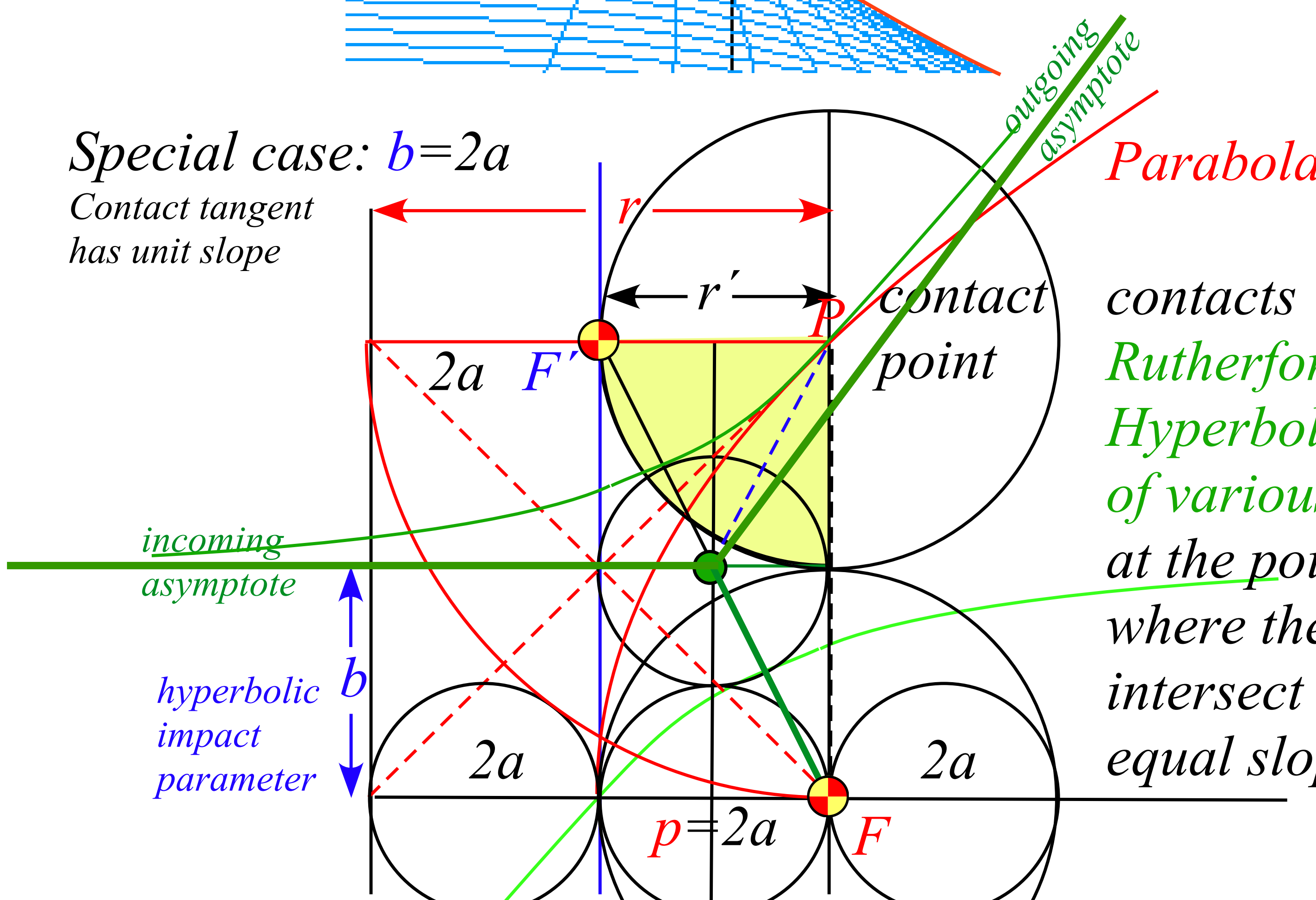
"Kite" geometry of envelope parabola



r-r′ =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola

Special case: b=2a

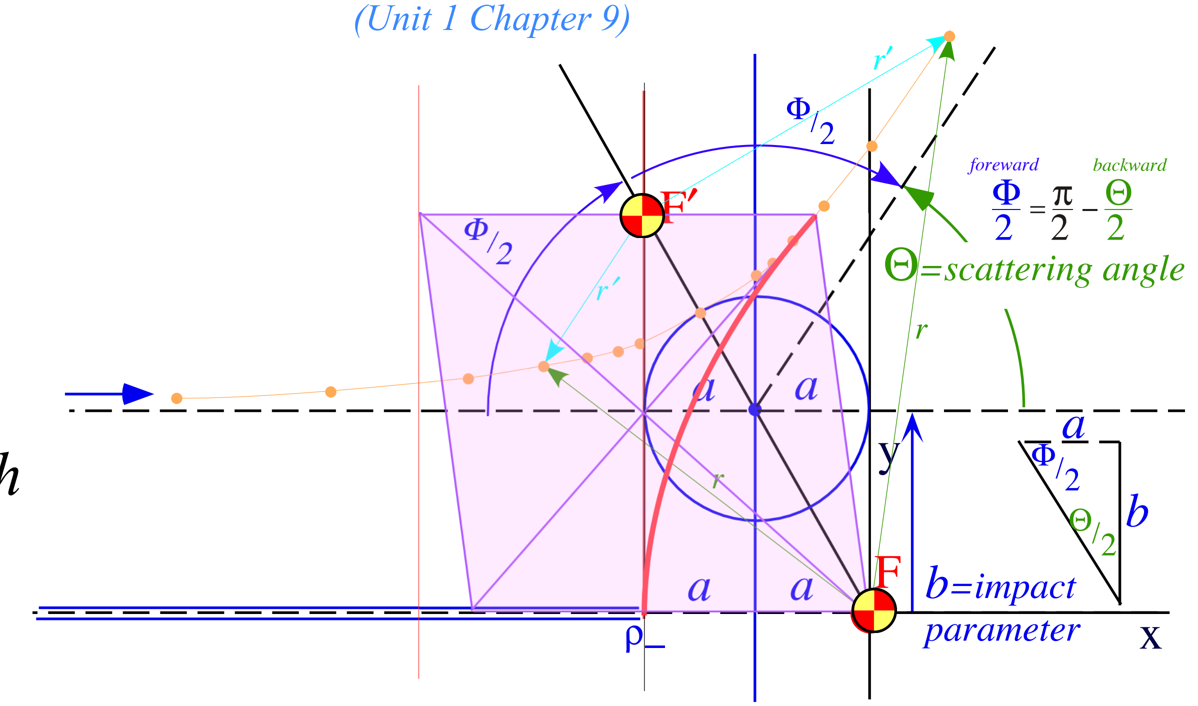
Contact tangent has unit slope



contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Recall parabolic " kite" geometry

(Unit 1 Chapter 9)



Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

➔ *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

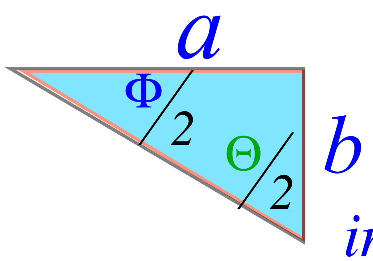
Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

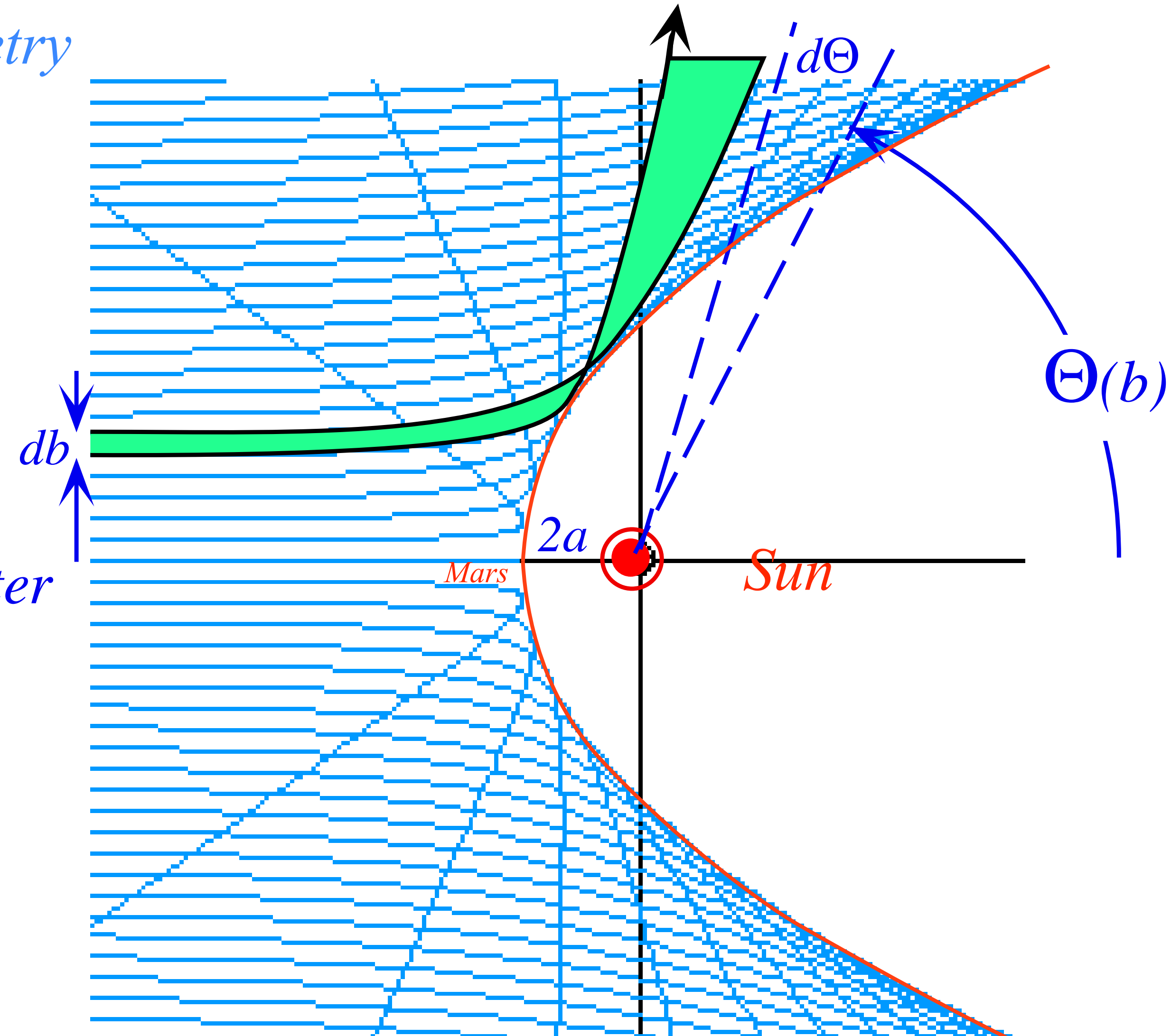
Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Rutherford scattering geometry



b= impact parameter

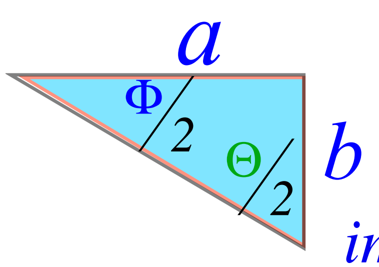


Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

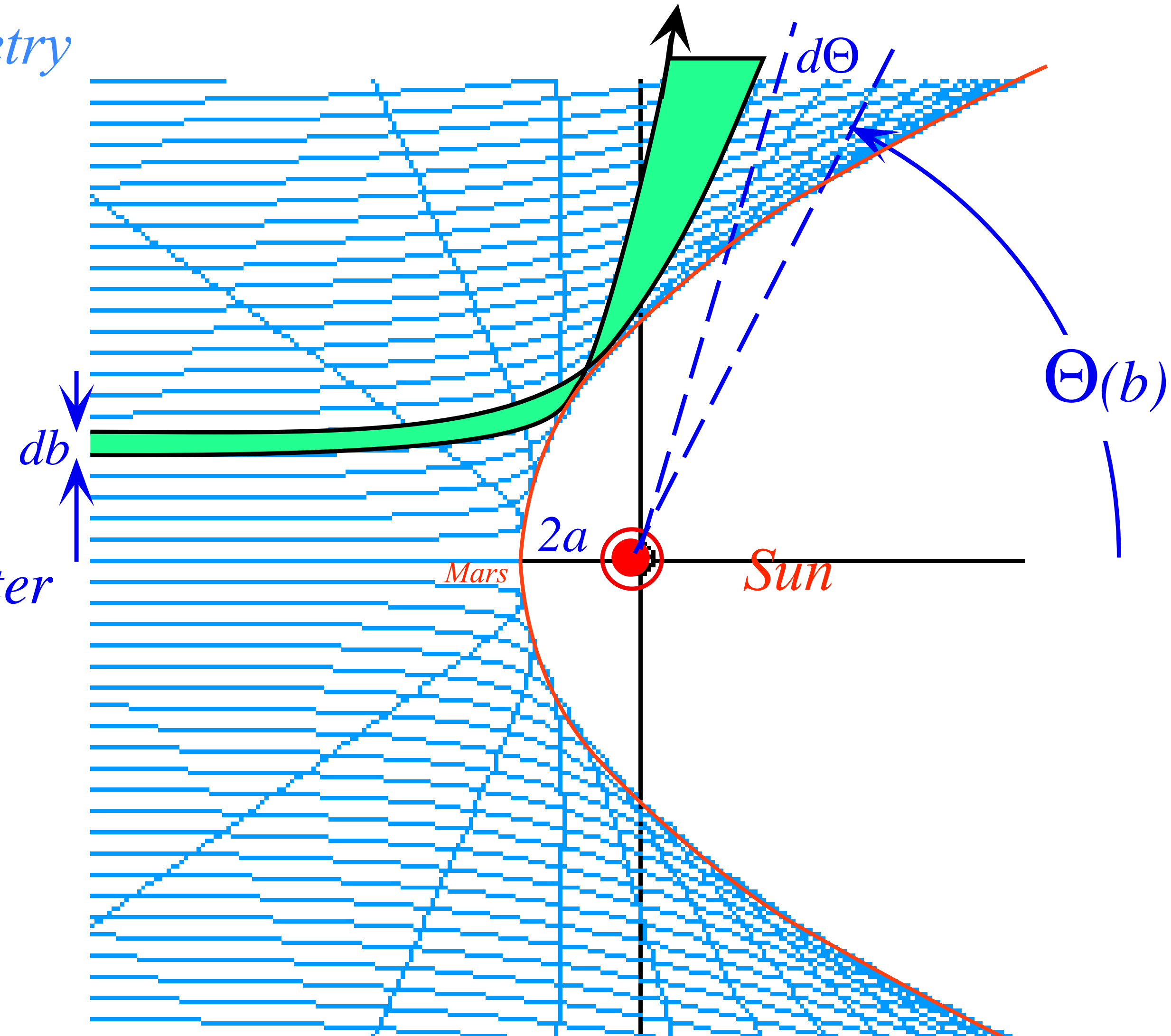
Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma = b \cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Rutherford scattering geometry



b= impact parameter



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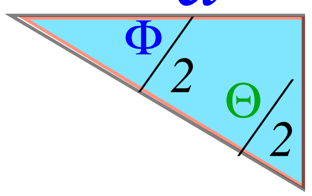
Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

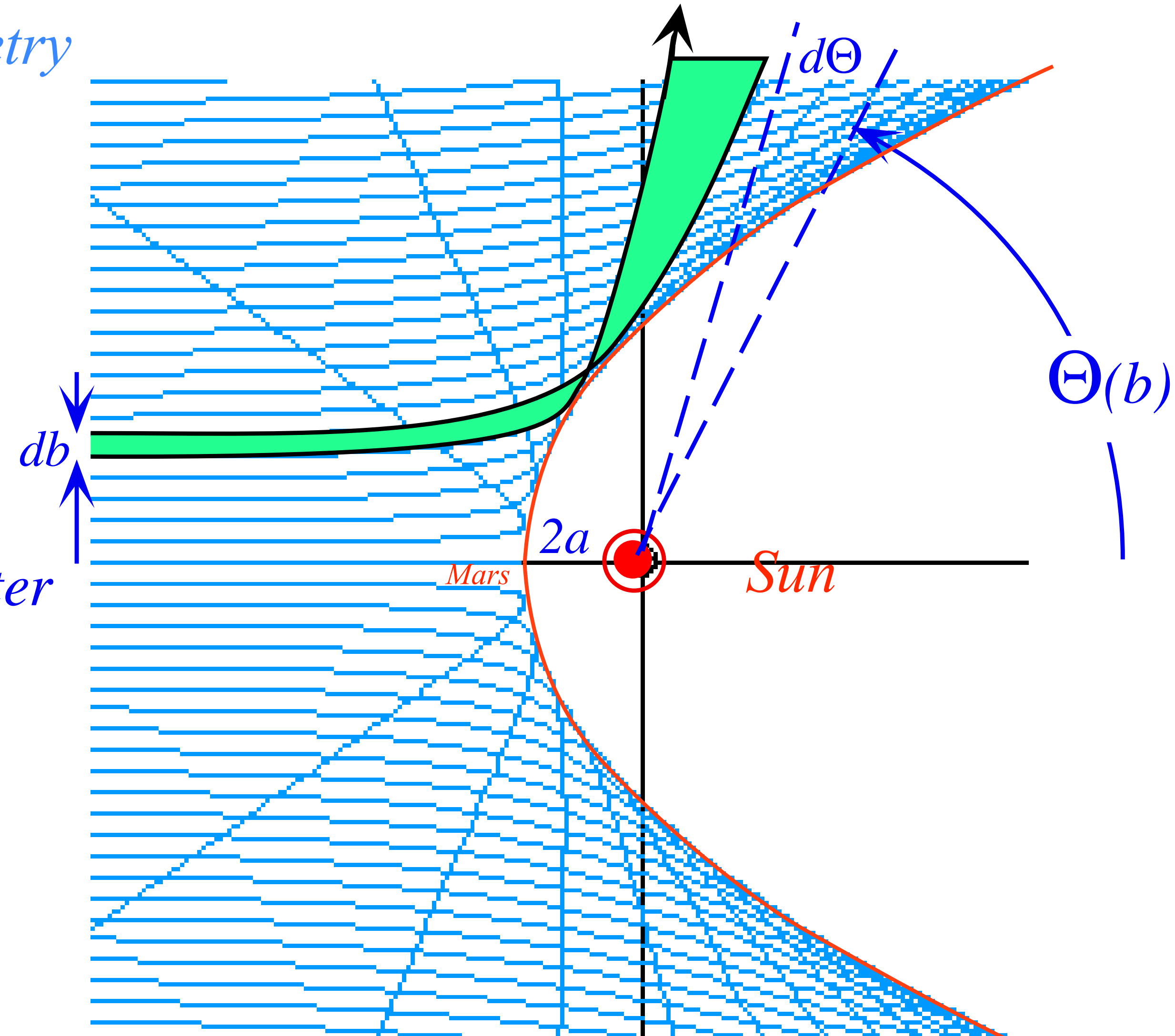
Incremental window $d\sigma=b\cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Ratio $\displaystyle \frac{d\sigma}{d\Omega} = \frac{b\,db\,d\varphi}{\sin\Theta d\Theta d\varphi} = \frac{b}{\sin\Theta}\frac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

Rutherford scattering geometry



b= impact parameter



$$\frac{b}{a} = \cot\frac{\Theta}{2}$$

Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

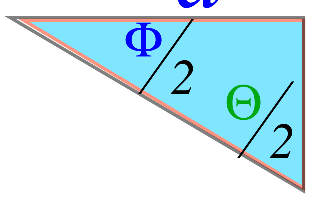
Incremental window $d\sigma = b \cdot db$ normal to beam axis at x=-∞ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at R=+∞ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Ratio $\dfrac{d\sigma}{d\Omega} = \dfrac{b\, db\, d\varphi}{\sin\Theta d\Theta d\varphi} = \dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

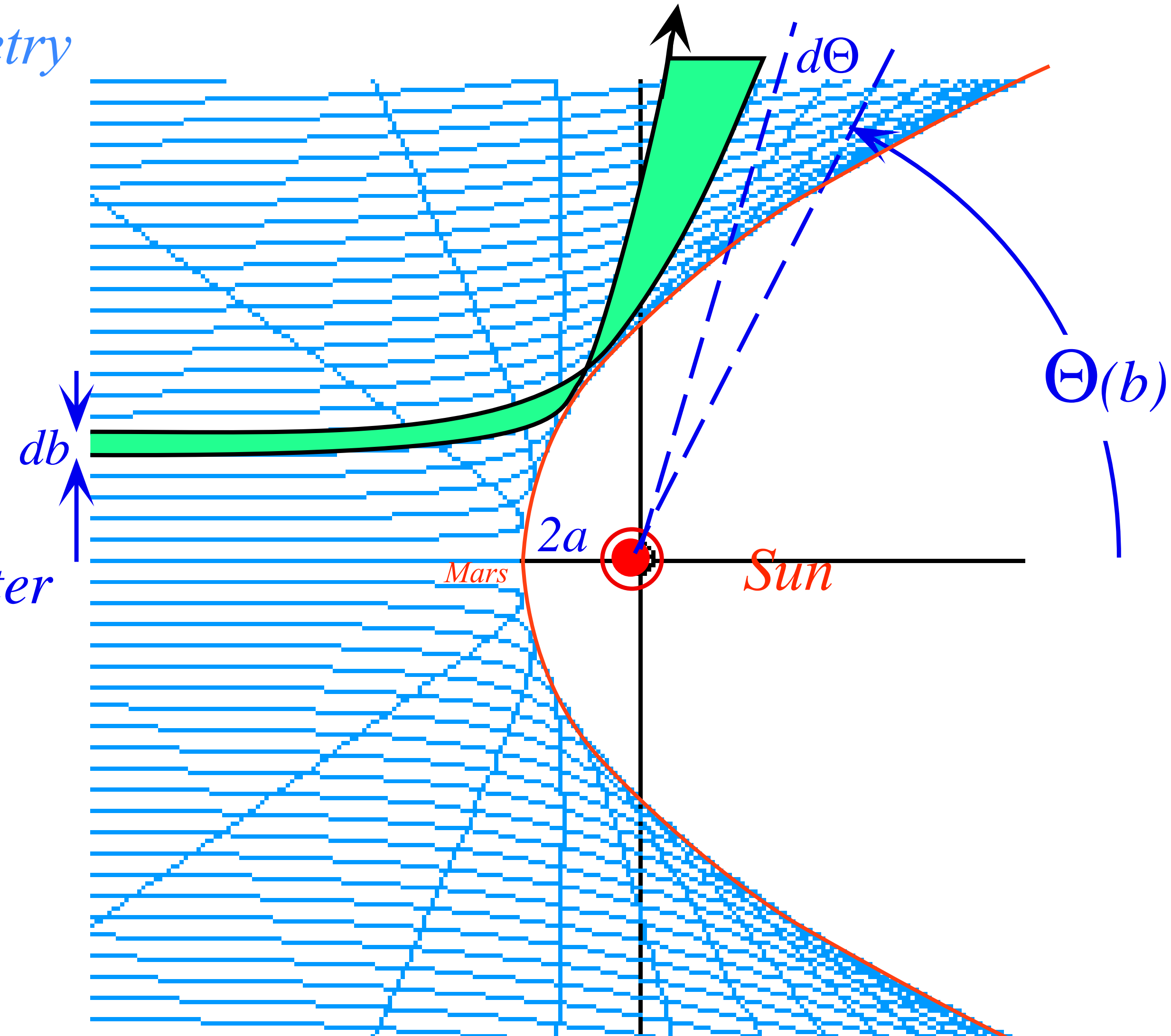
Geometry: $b = a\cot\dfrac{\Theta}{2}$

with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2}$

Rutherford scattering geometry



b= impact parameter



Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

$$\frac{b}{a} = \cot\frac{\Theta}{2}$$

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma = b\cdot db$ normal to beam axis at x=-∞ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at R=+∞ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Ratio $\dfrac{d\sigma}{d\Omega} = \dfrac{b\, db\, d\varphi}{\sin\Theta d\Theta d\varphi} = \dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

Geometry: $b = a\cot\dfrac{\Theta}{2} = \dfrac{k}{2E}\cot\dfrac{\Theta}{2}$

with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2} = \dfrac{-a}{2\sin^2\frac{\Theta}{2}}$

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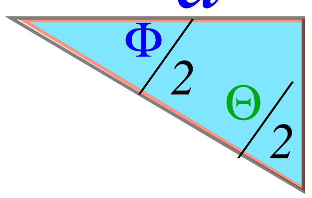
Rutherford scattering geometry



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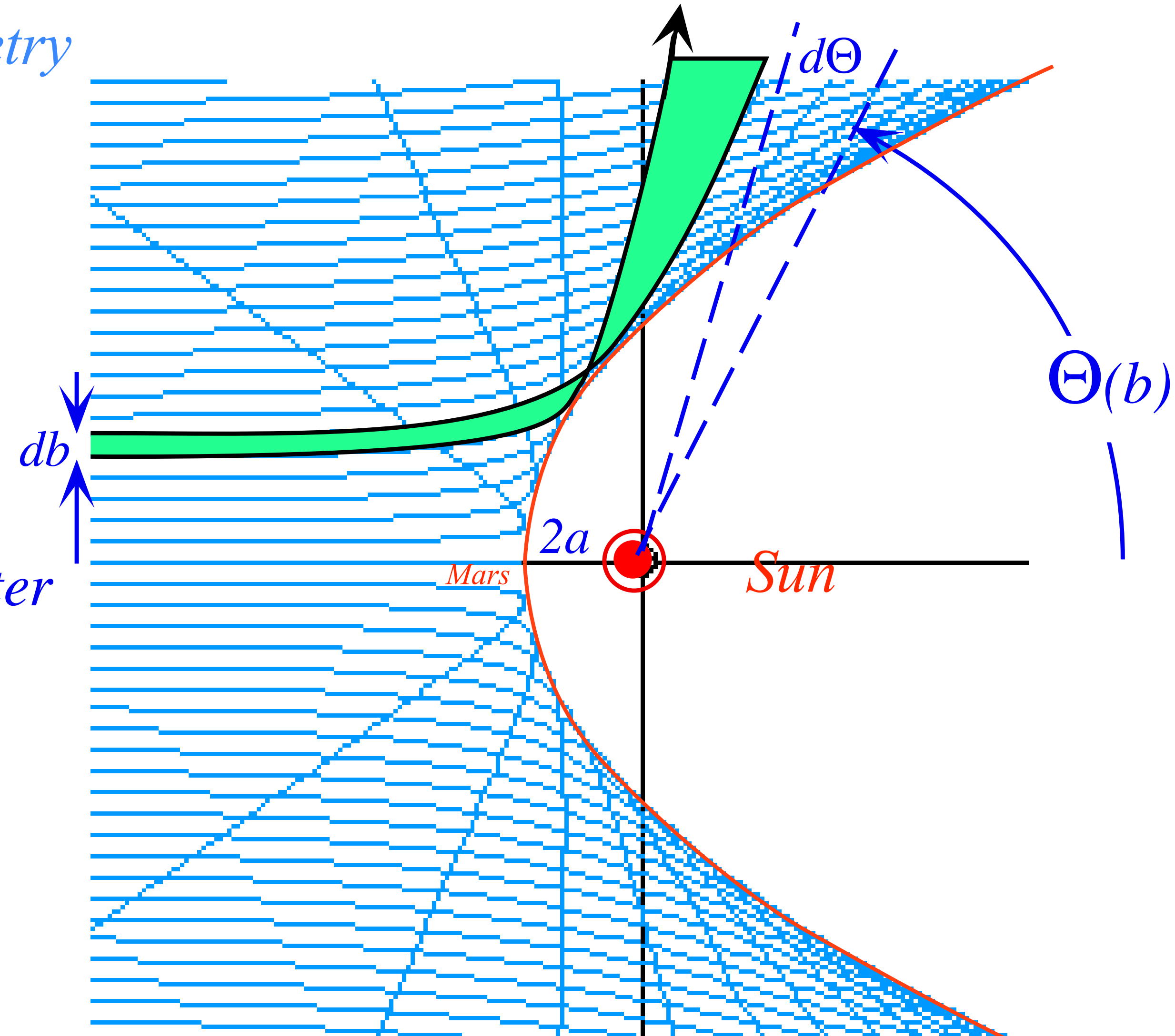

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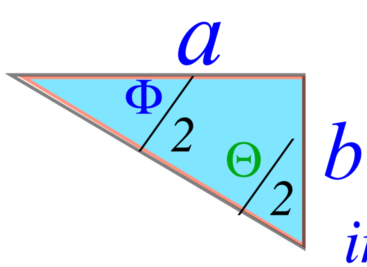

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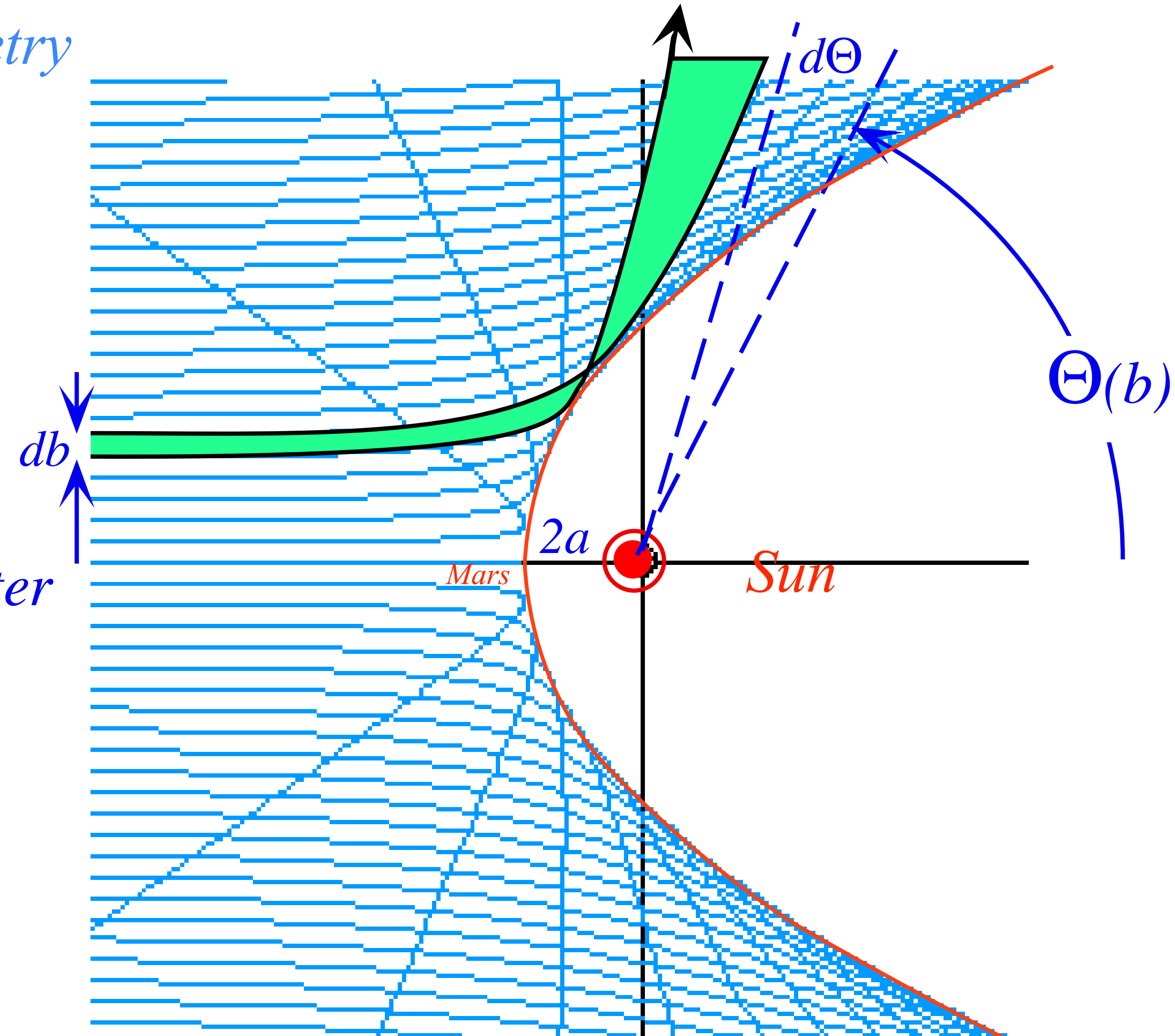

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$$\boxed{\frac{d\sigma}{d\Omega} = \frac{-a^2\cos\frac{\Theta}{2}}{2\sin\Theta\sin^3\frac{\Theta}{2}} = \frac{-k^4}{16E^2}\sin^{-4}\frac{\Theta}{2}}$$

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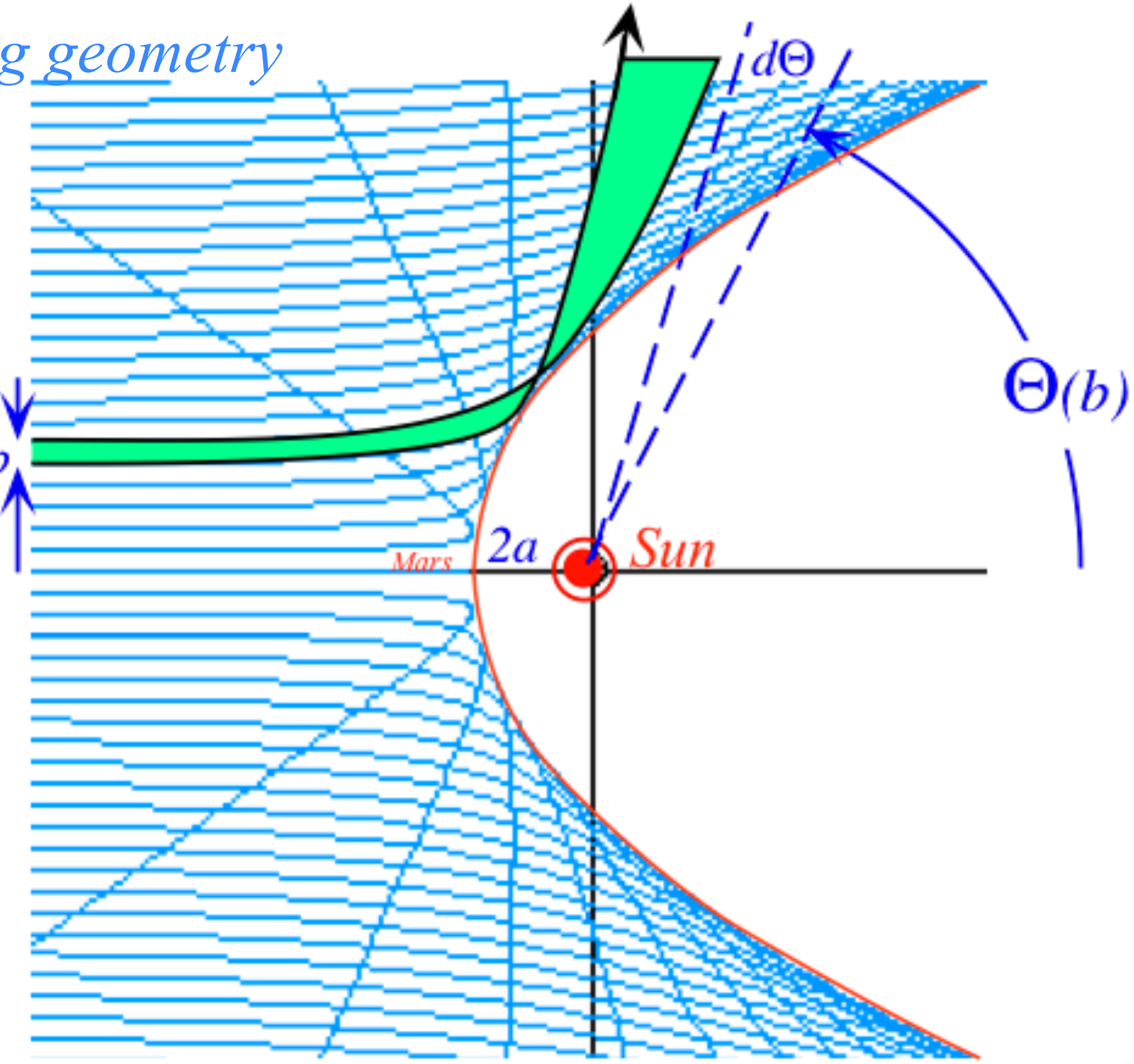
This classical result agrees exactly with 1st Born approximation to *quantum* Coulomb *DSC*!

(*Never forget*!: $a = \dfrac{-k}{2E}$ or: $E = \dfrac{-k}{2a}$)

Rutherford scattering geometry

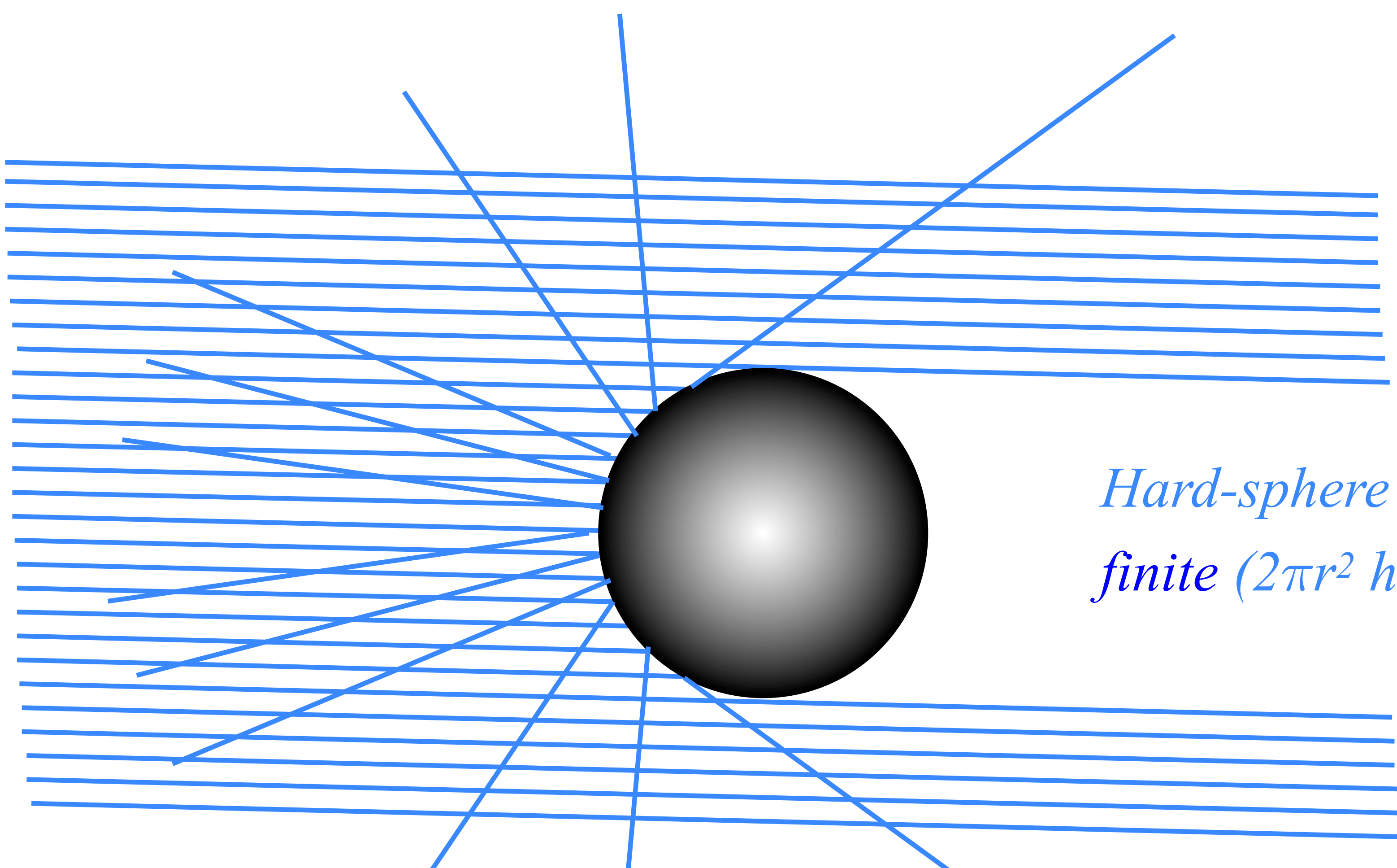


***b*=**
impact parameter

Two Extremes:

Rutherford (Coulomb) scattering has ***infinite*** *(∞) total cross section*

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{k^4}{16E^2} \int d\Omega \sin^{-4}\frac{\Theta}{2} = \infty$$



Hard-sphere scattering has ***finite*** *($2\pi r^2$ here) total cross section*

CoulIt Web Simulation
Hard Sphere - Elastic scattering

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

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Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

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Isotropic field *V=V(r)* guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

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Coulomb *V=-k/r* also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

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(...for sake of comparison...)

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$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Eccentricity vector ε *and (ε,λ) geometry of orbital mechanics*

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Generates symmetry groups: $R(3) \subset R(3) \times R(3) \subset O(4)$

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Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

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Let angle ϕ be angle between ε and radial vector **r**

$$\varepsilon\, r\cos\phi = r - \frac{L^2}{km}$$

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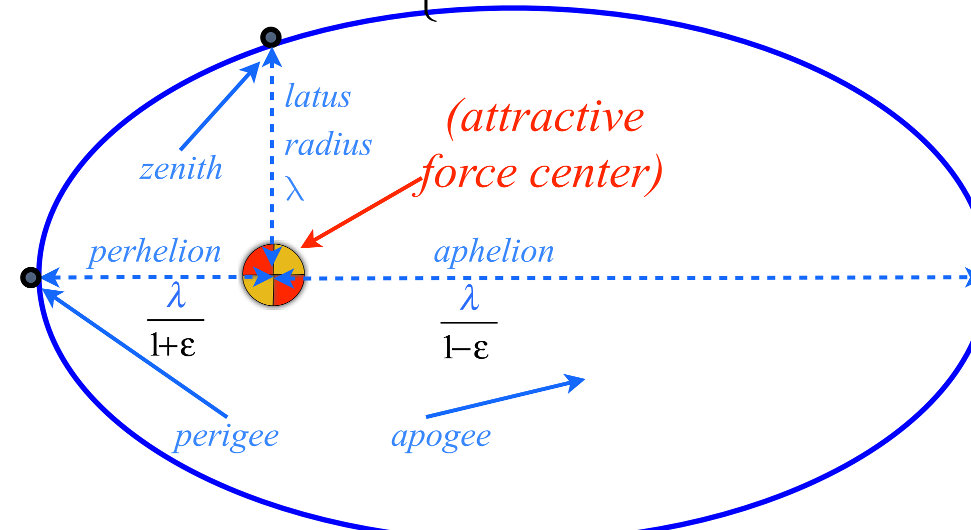
$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km}\qquad\text{or:}\qquad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

For $\lambda=L^2/km$ that matches: $r=\dfrac{\lambda}{1-\varepsilon\cos\phi}=\begin{cases}\dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee}\\[2ex] \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith}\\[2ex] \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee}\end{cases}$



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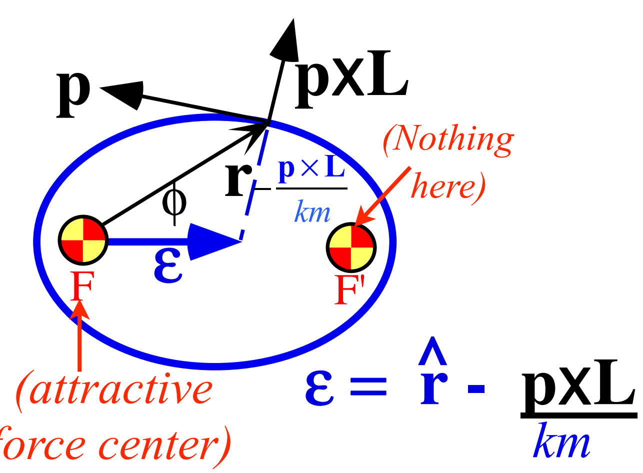
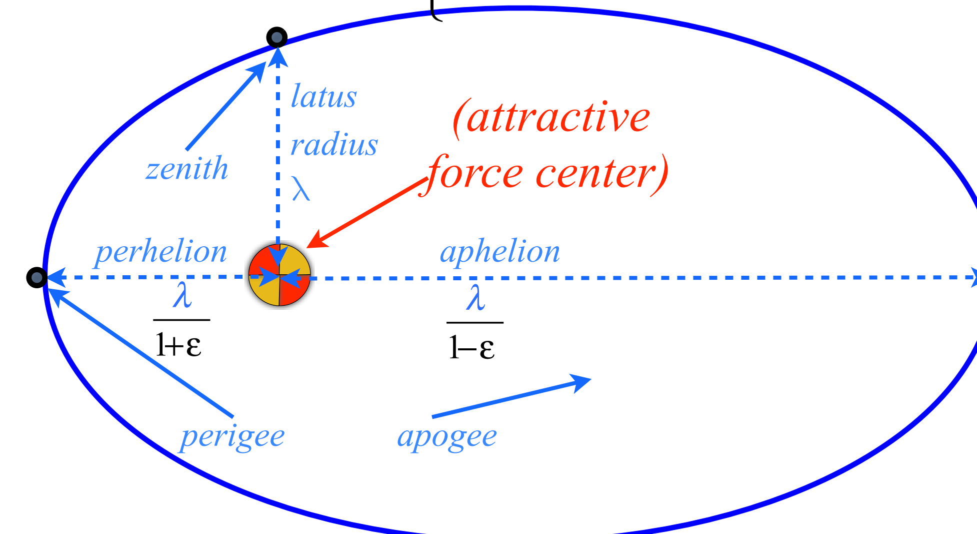
(a) Attractive (k>0)
Elliptic (E<0)

(Rotational momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)





Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

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$$\boxed{\begin{aligned} S_A&=\tfrac{1}{2}(x_1{}^2+p_1{}^2-x_2{}^2-p_2{}^2)\\ S_B&=x_1p_1+x_2p_2\\ S_C&=x_1p_2-x_2p_1\end{aligned}}$$

$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector **p**:

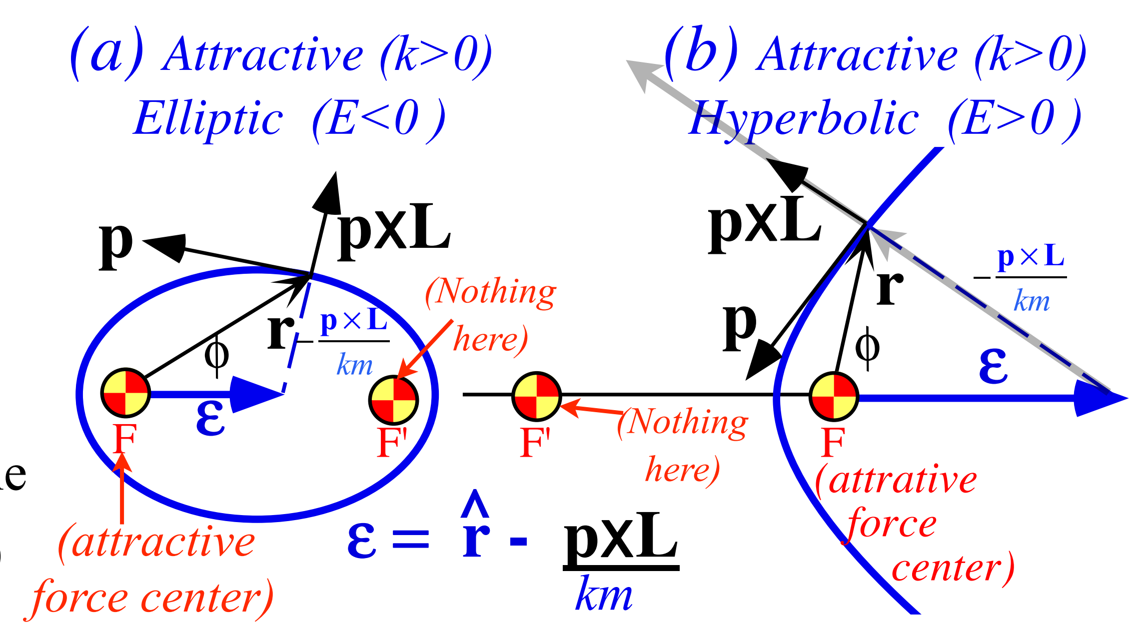
$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\cancel{\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

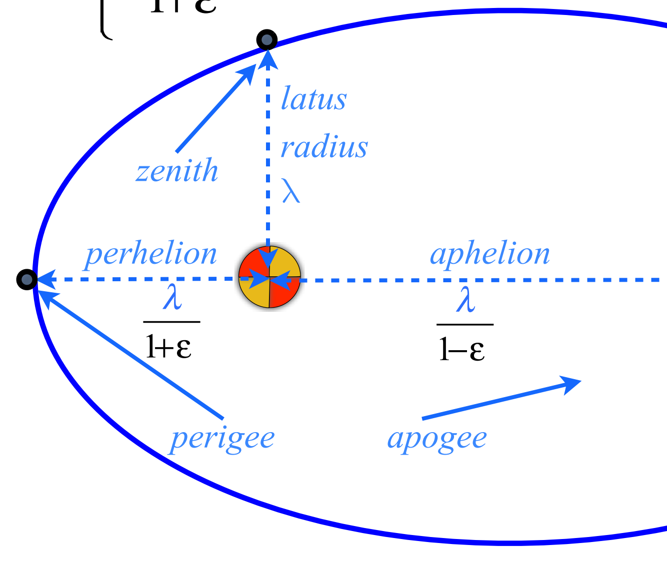
Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector **r**

$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km}\qquad\text{or:}\qquad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

For $\lambda=L^2/km$ that matches: $r=\dfrac{\lambda}{1-\varepsilon\cos\phi}=\begin{cases}\dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee}\\[2ex] \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith}\\[2ex] \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee}\end{cases}$

(Rotational momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)





Eccentricity vector $\boldsymbol{\varepsilon}$ and (ε,λ) geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* $\mathbf{L}$

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boxed{\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$\boxed{\begin{aligned} S_A&=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)\\ S_B&=x_1p_1+x_2p_2\\ S_C&=x_1p_2-x_2p_1\end{aligned}}$$

$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\cancel{\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector $\mathbf{r}$

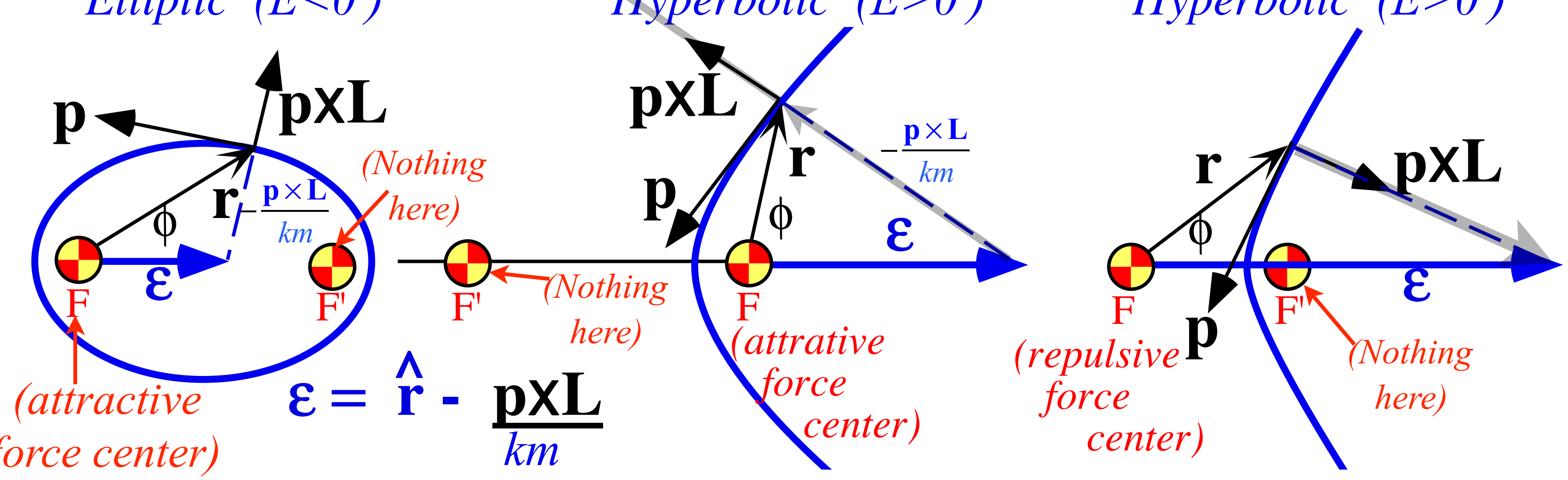
momentum $\mathbf{p}=m\mathbf{v}$*: p.81 to p.104*

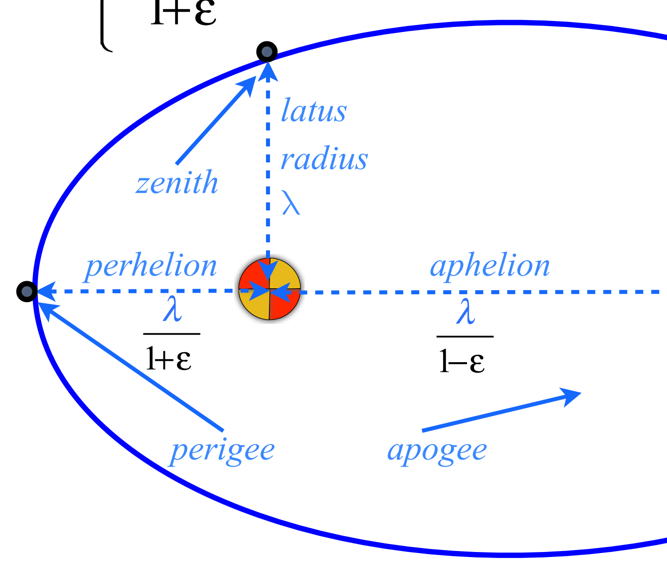
$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km}\qquad\text{or:}\qquad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

For $\lambda=L^2/km$ that matches: $r=\dfrac{\lambda}{1-\varepsilon\cos\phi}=\begin{cases}\dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee}\\[2ex] \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith}\\[2ex] \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee}\end{cases}$

(a) Attractive (k>0) Elliptic (E<0) — *(b) Attractive (k>0) Hyperbolic (E>0)* — *(c) Repulsive (k<0) Hyperbolic (E>0)*

(Rotational momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)





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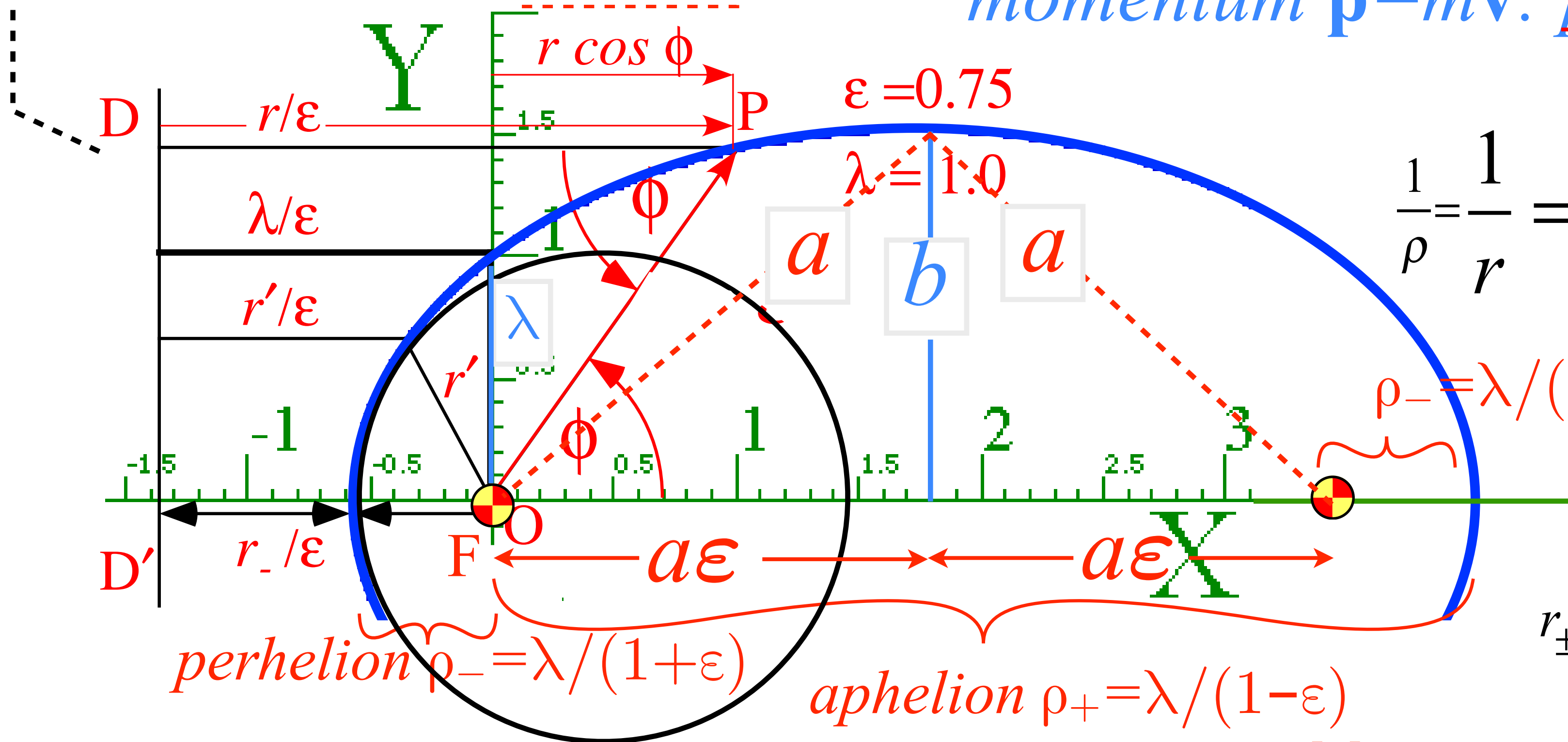
Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

(From Lecture 25 p.78 and 79)

Geometry of Coulomb orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$

momentum $\mathbf{p}=m\mathbf{v}$: *p.81 to p.104*



By geometry:

$$\frac{1}{\rho}=\frac{1}{r}=\frac{1-\varepsilon\cos\phi}{\lambda}=\frac{1}{\lambda}-\frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on p.79

Recall p.66 formula:

$$r_{\pm}=\rho_{\pm}=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = $\varepsilon\cdot$Distance to *Directrix* **DD′**

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon)+\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon)-\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Latus radius: $\lambda = a(1-\varepsilon^2)$

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E=\frac{-k}{2a}}$$

Minor radius:

$b=\sqrt{(a^2-a^2\varepsilon^2)}=\sqrt{(a\lambda)}$ *(ellipse: $\varepsilon<1$)*

$b=\sqrt{(a^2\varepsilon^2-a^2)}=\sqrt{(\lambda a)}$ *(hyperb: $\varepsilon>1$)*

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2+\frac{2\mu^2}{Em}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2E}{k^2m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1+\frac{2\mu^2E}{k^2m}} = \varepsilon \quad \text{implies: } \lambda = a(1-\varepsilon^2) = a\frac{2\mu^2E}{k^2m} = \frac{\mu^2}{km}$$

Also important! $\boxed{\mu=\sqrt{km\lambda}}$

$b/a=\sqrt{(1-\varepsilon^2)}$ *(ellipse: $\varepsilon<1$)*

$b/a=\sqrt{(\varepsilon^2-1)}$ *(hyperb: $\varepsilon>1$)*

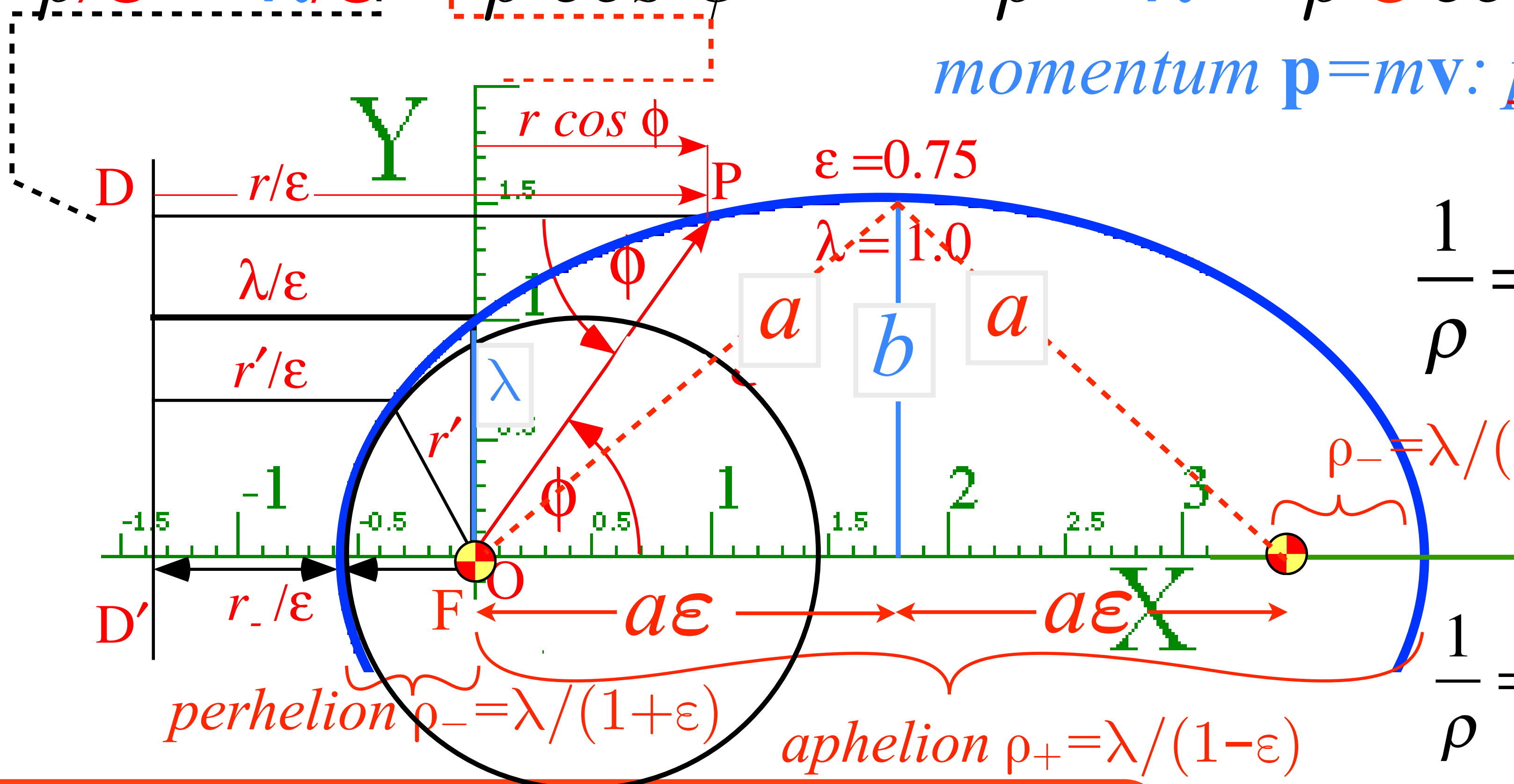
$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon<1$)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb: $\varepsilon>1$)*

(From Lecture 25 p.78 and 79) *Geometry of Coulomb orbits (Let: $r = \rho$ here)*

$$\rho/\varepsilon = \lambda/\varepsilon + \rho\cos\phi \qquad \rho = \lambda + \rho\,\varepsilon\cos\phi \qquad \rho = \frac{\lambda}{1-\varepsilon\cos\phi}$$

momentum $\mathbf{p}=m\mathbf{v}$: *p.81 to p.104*



$$\frac{1}{\rho} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by: *Eccentricity* ε

Defining ***eccentricity*** ε

Distance to F$_{\text{ocal}}$-point $= \varepsilon \cdot$ Distance to D$_{\text{irectrix}}$-line DD′

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon < 1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon > 1$)*

(x,y) *parameters*	*physical constants*	(r,ϕ) *parameters*
major radius $a = \frac{k}{2E}$	*Energy* $E = \frac{k}{2a}$	$\varepsilon = \sqrt{\frac{k^2 m \pm 2\mu^2 E}{k^2 m}} = \sqrt{1 \pm \frac{b^2}{a^2}}$ *eccentricity*
minor radius $b = \frac{\mu}{\sqrt{2m\lvert E\rvert}}$	*Orbital Momentum* $\mu = \sqrt{km\lambda}$	*latus radius* $\lambda = \frac{\mu^2}{km} = \frac{b^2}{a}$

$\varepsilon^2 = 1 - \frac{b^2}{a^2}$ (ellipse: $\varepsilon<1$) $\frac{b}{a} = \sqrt{1-\varepsilon^2}$

$\varepsilon^2 = 1 + \frac{b^2}{a^2}$ (hyperbola: $\varepsilon>1$) $\frac{b}{a} = \sqrt{\varepsilon^2 - 1}$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon<1$)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb: $\varepsilon>1$)*

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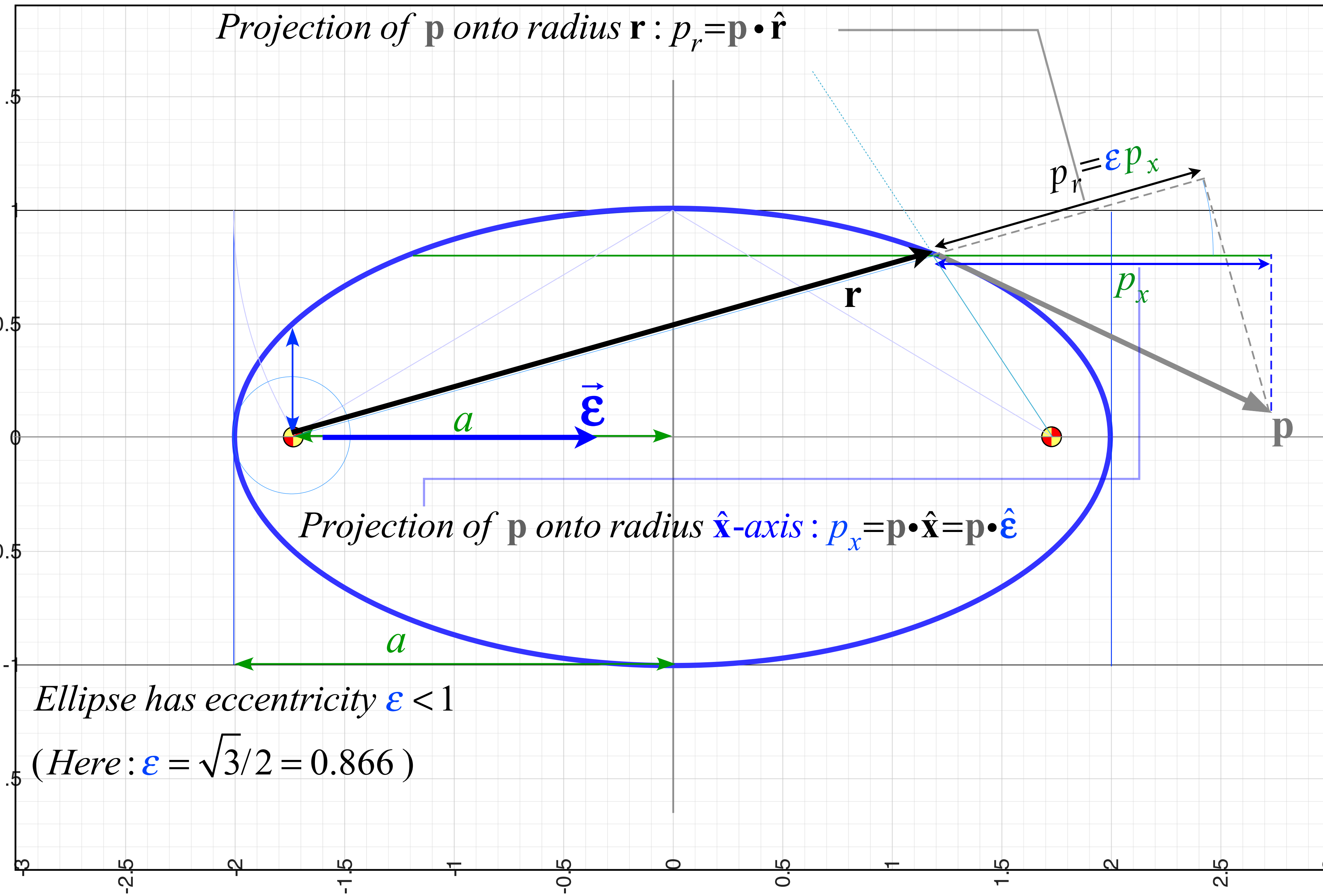
Dot product of $\boldsymbol{\varepsilon}$ with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\cancel{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

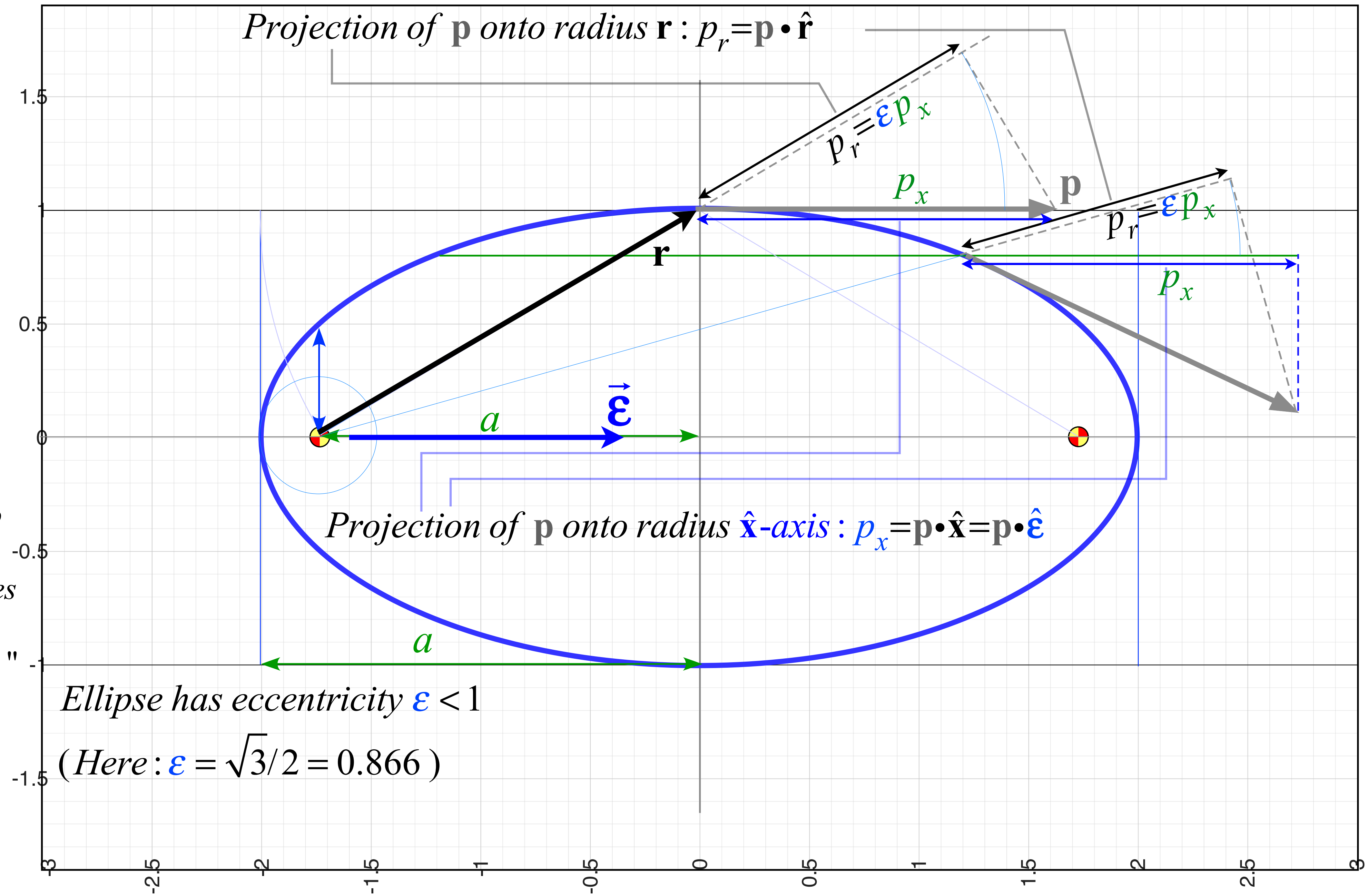
This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection p_x of* $\mathbf{p}$ *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "



Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\cancel{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}}{km}$$
$$= \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon\, p_x$$

This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

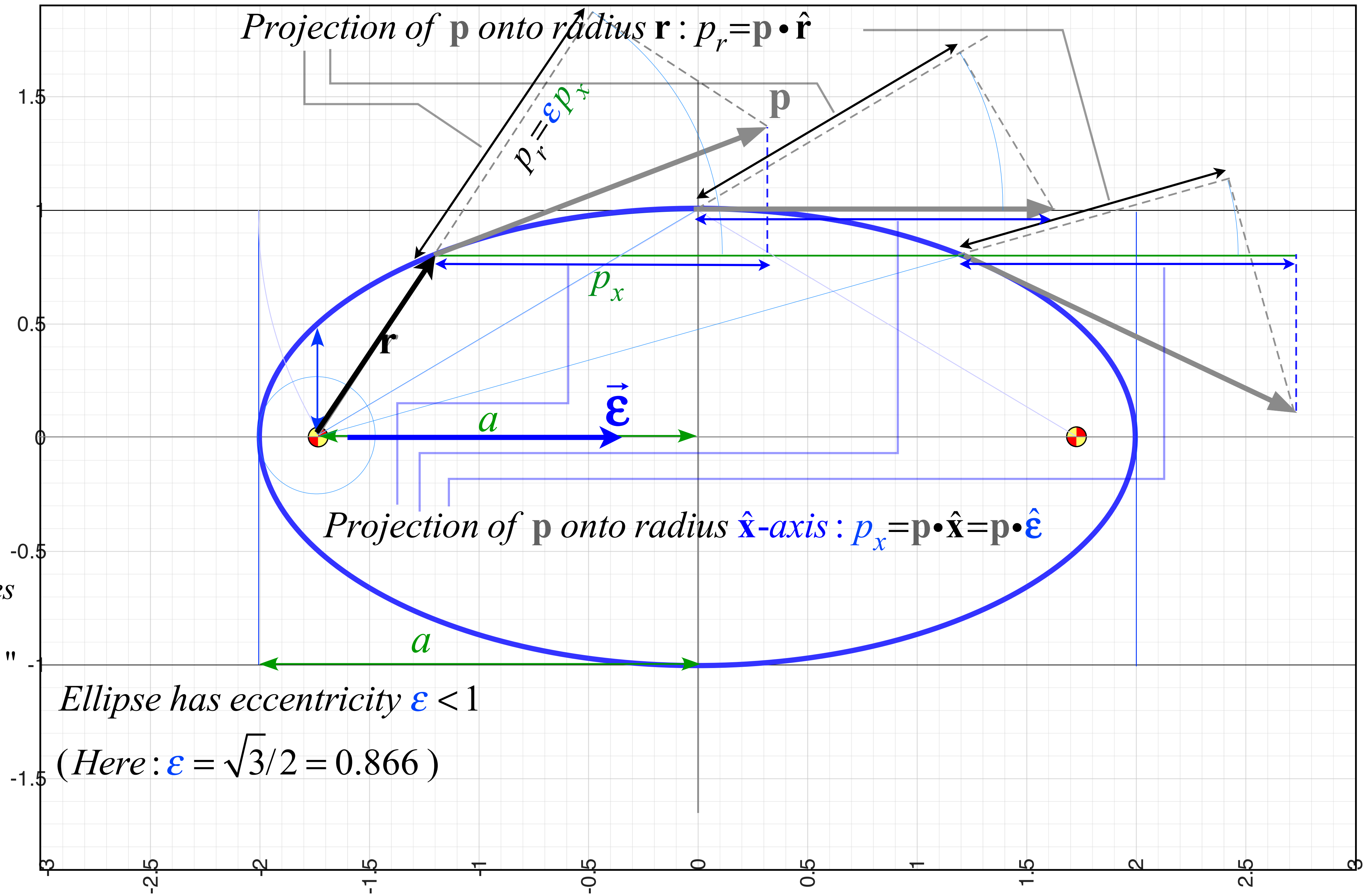


NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn so they correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 95-104)

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\cancel{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}}{km}$$
$$= \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection p_r of* **p** *onto radial* **r** *or* **r′** *lines equals eccentricity ε times projection p_x of* **p** *onto orbit major axis*: ($\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}}$) "



NOTE: Lengths of vectors **p** and -**p** are not drawn so they correctly show that momentum **p**=*m***v** grows as radial distance r=|**r**| falls. (To be shown on p. 95-104)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\cancel{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}}{km}$$
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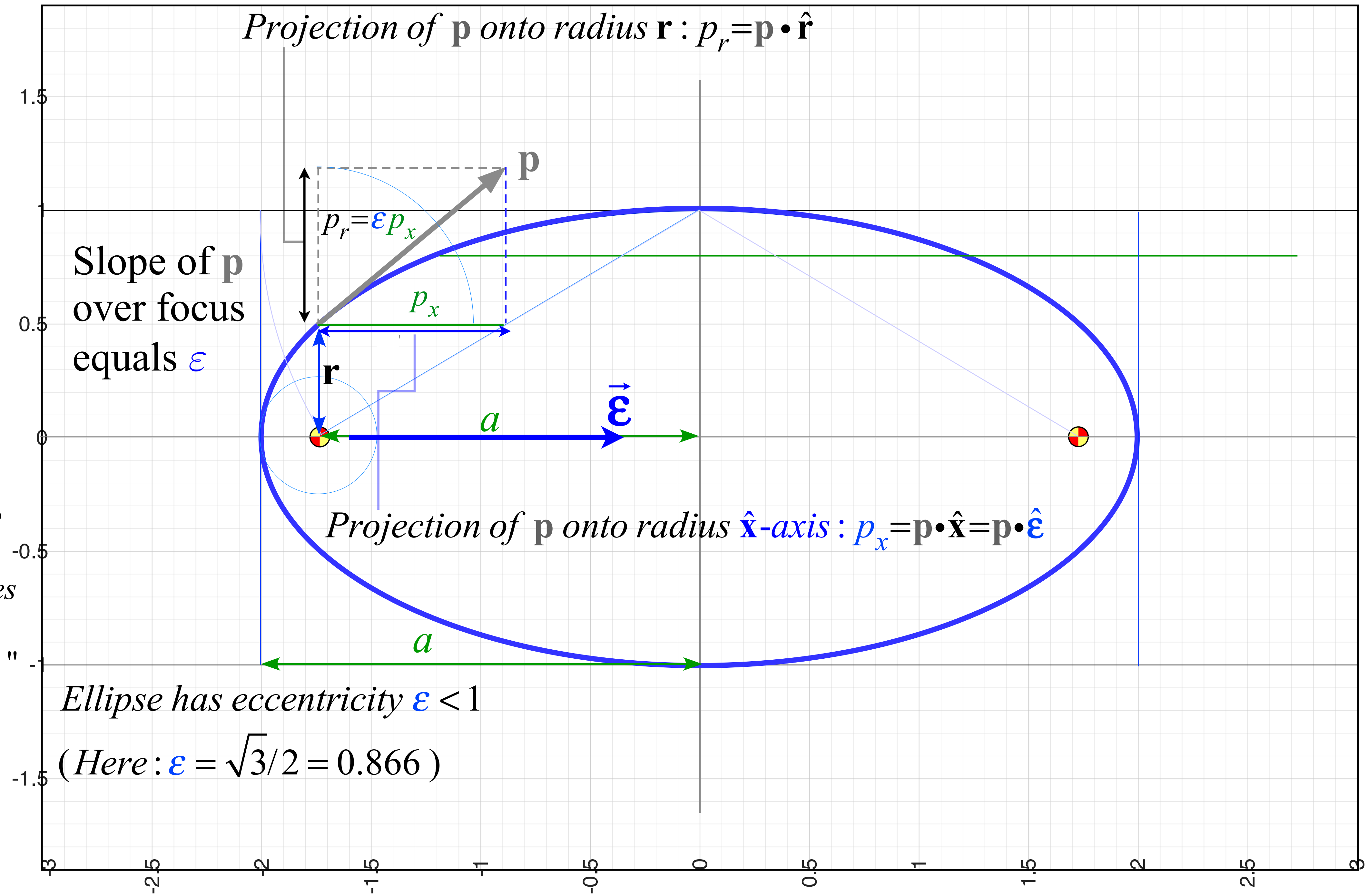

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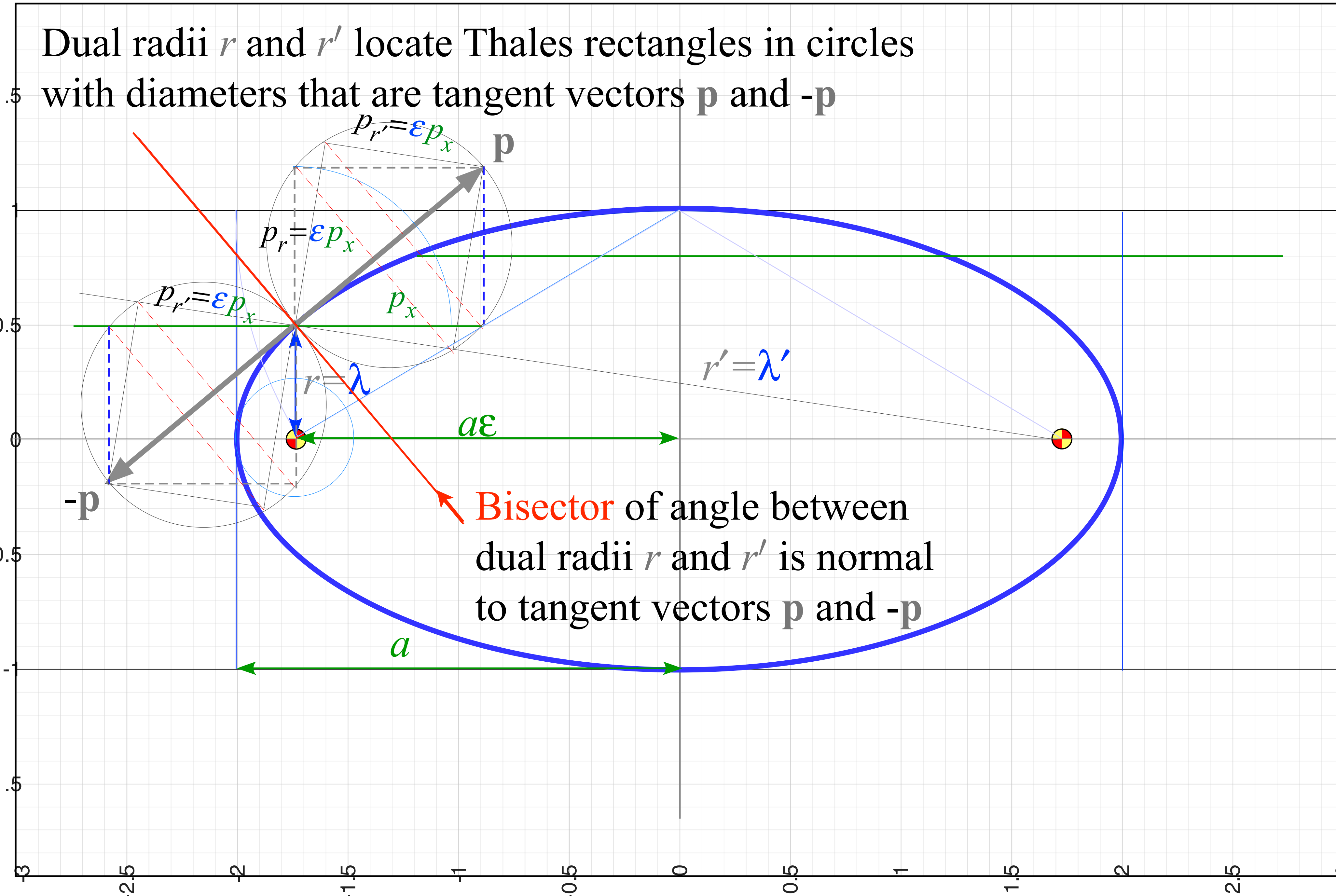
Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"



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Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\cancel{\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}}$$
$$=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r=\varepsilon p_x$$

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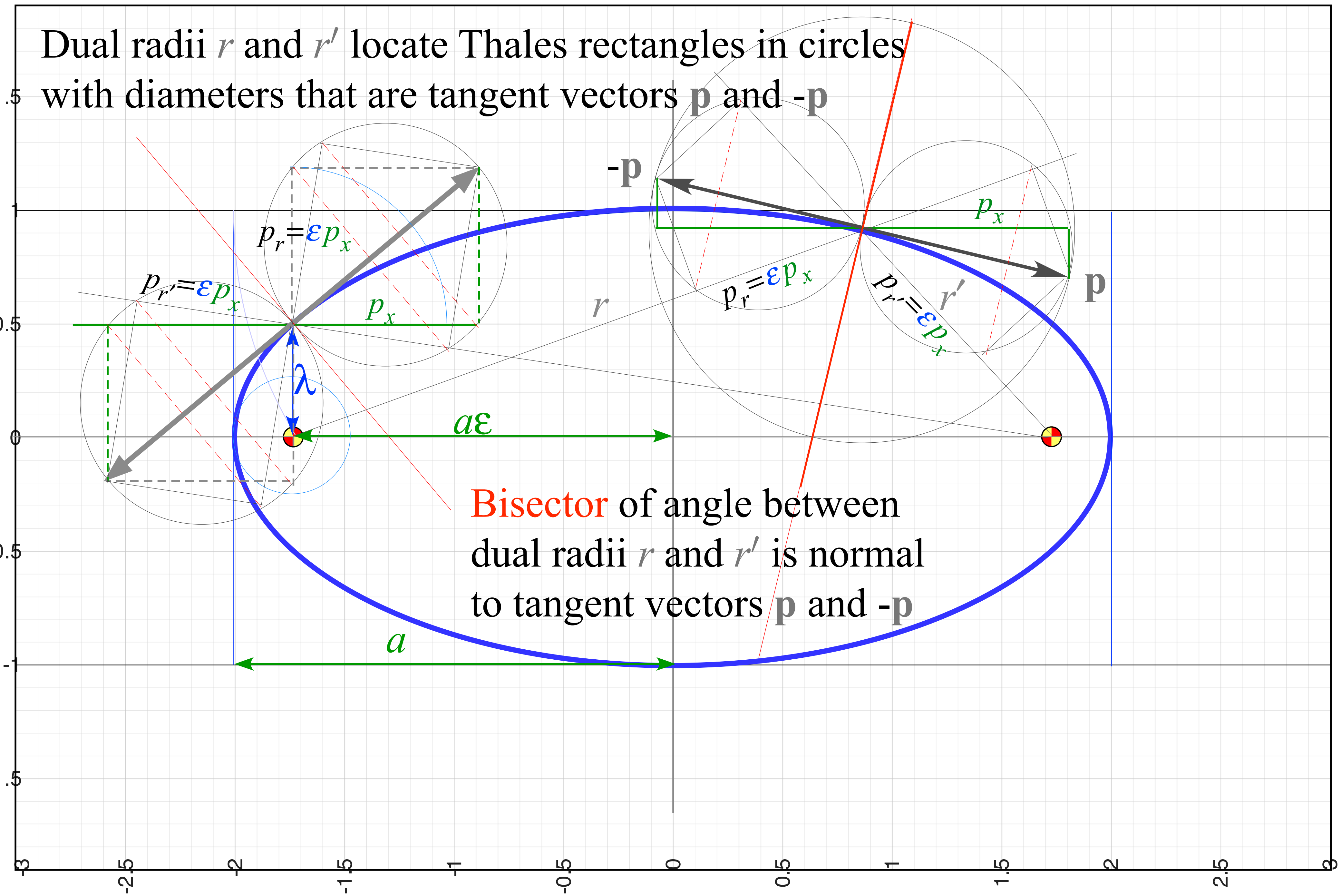

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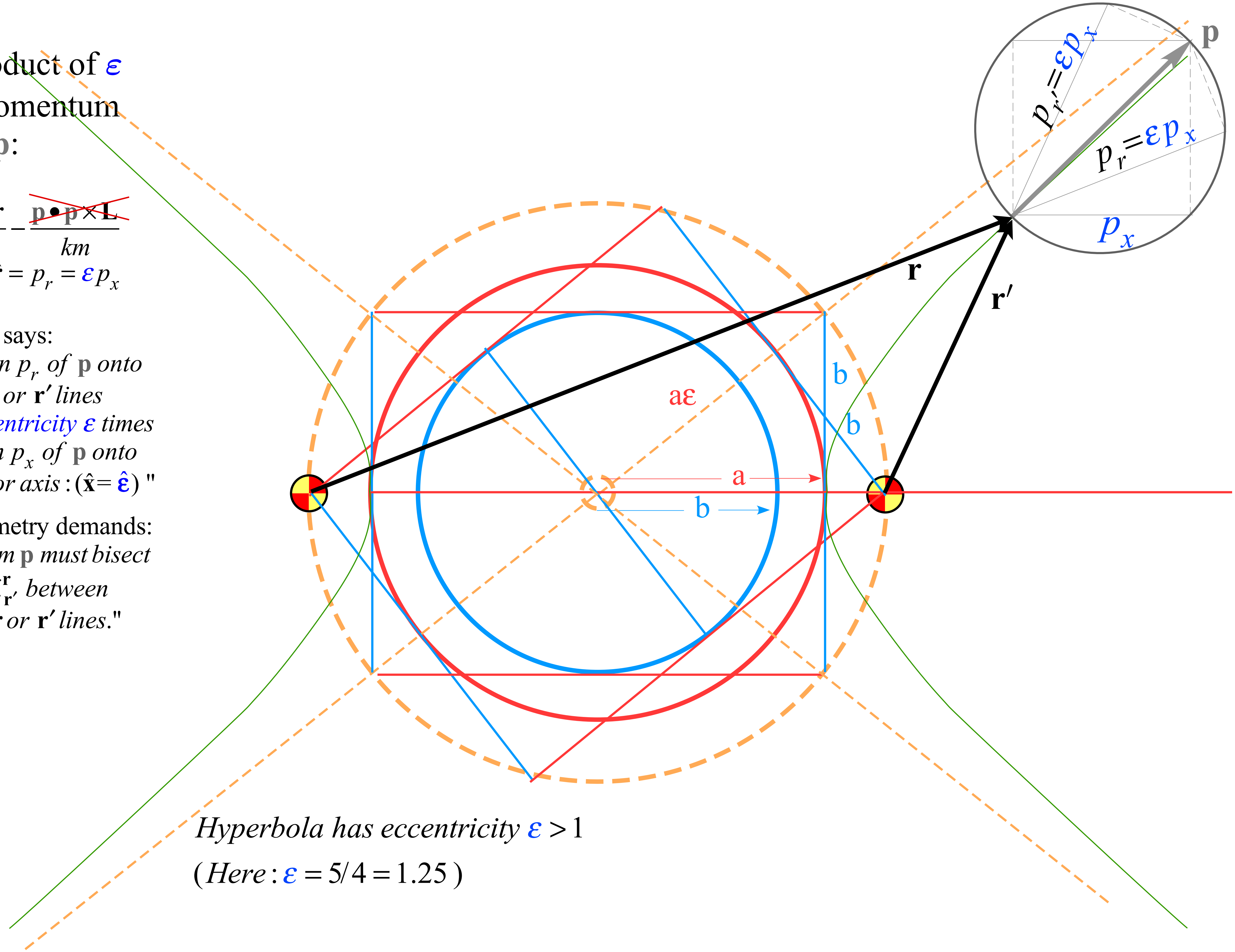

Hyperbola has eccentricity $\varepsilon > 1$

(*Here*: $\varepsilon = 5/4 = 1.25$)

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*General geometric orbit construction using **ε-vector** and (γ,R)-parameters*

*Next several pages give step-by-step constructions of **ε-vector** and Coulomb orbit and trajectory physics*

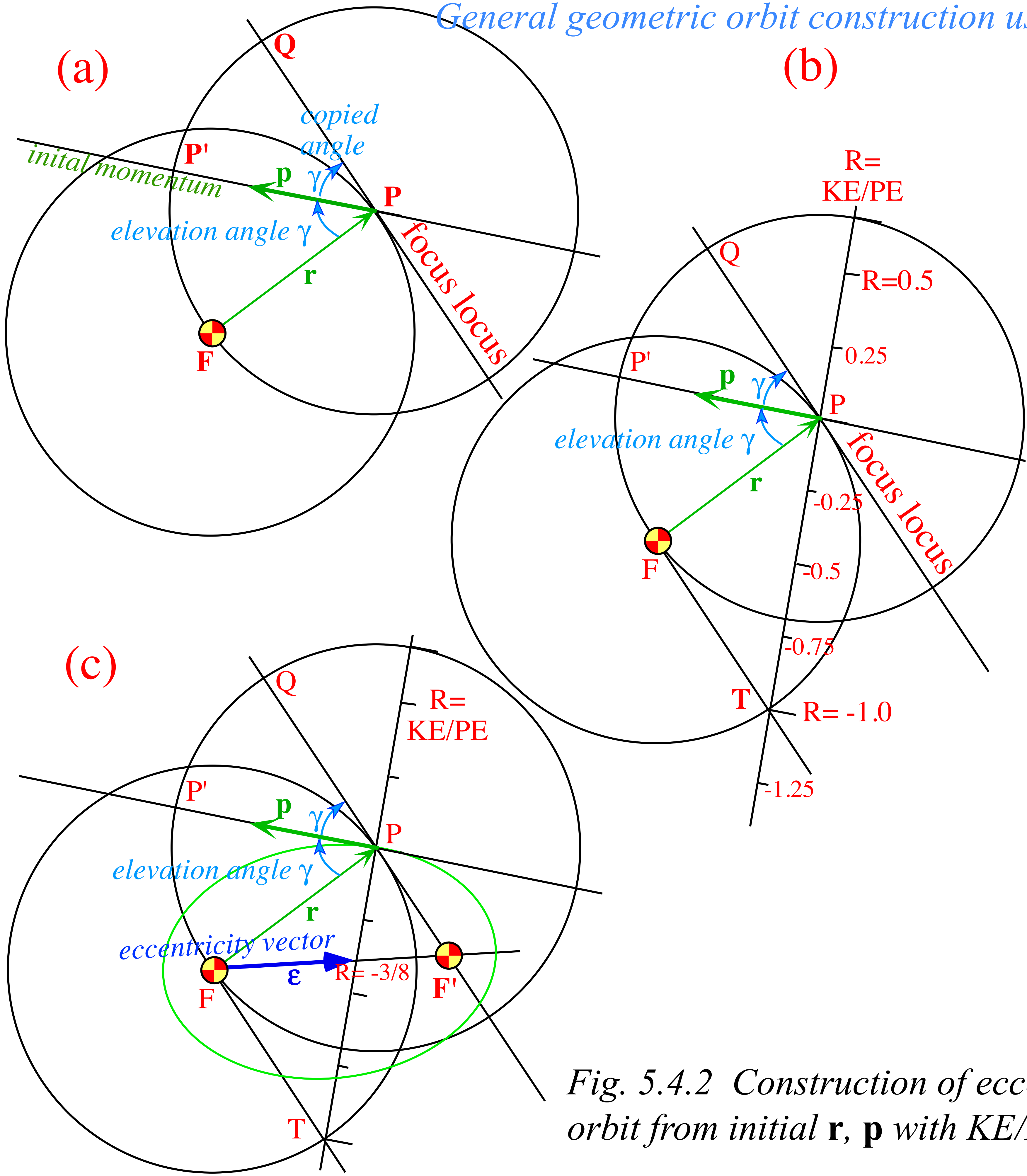


Fig. 5.4.2 Construction of eccentricity vector **ε** *and orbit from initial* **r**, **p** *with KE/PE=-3/8.*

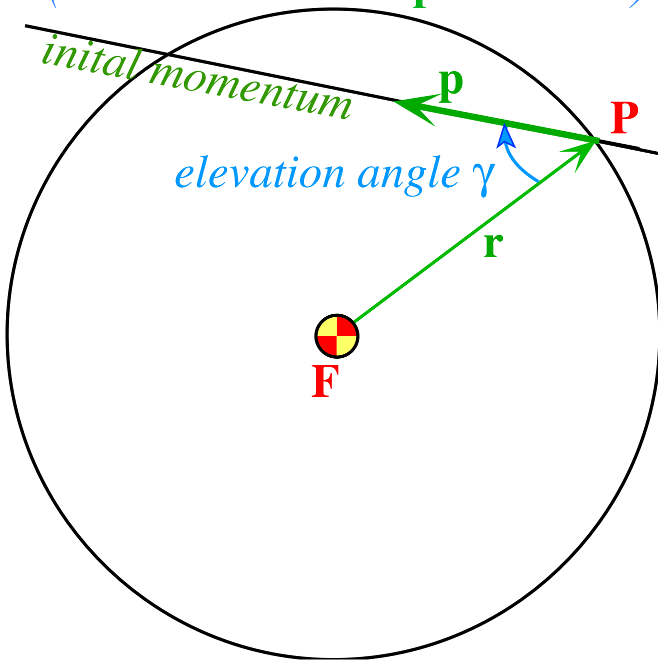
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

(momentum initial **p** *direction)*



General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ from radius

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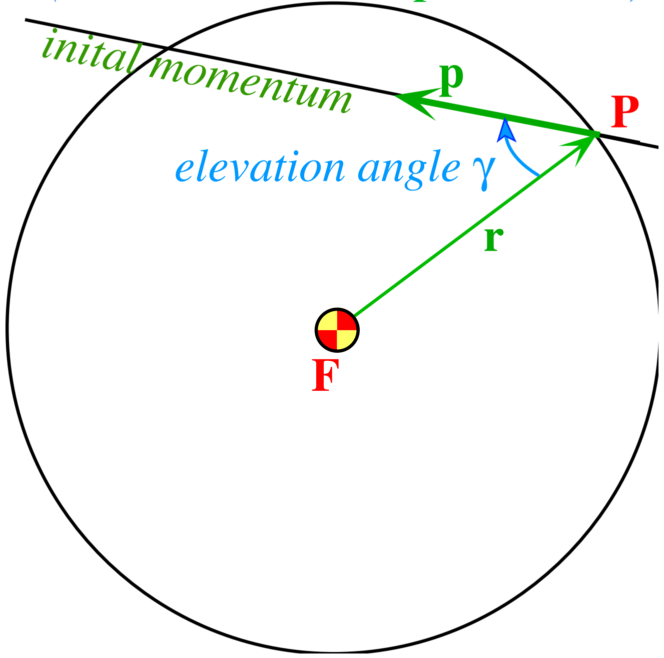
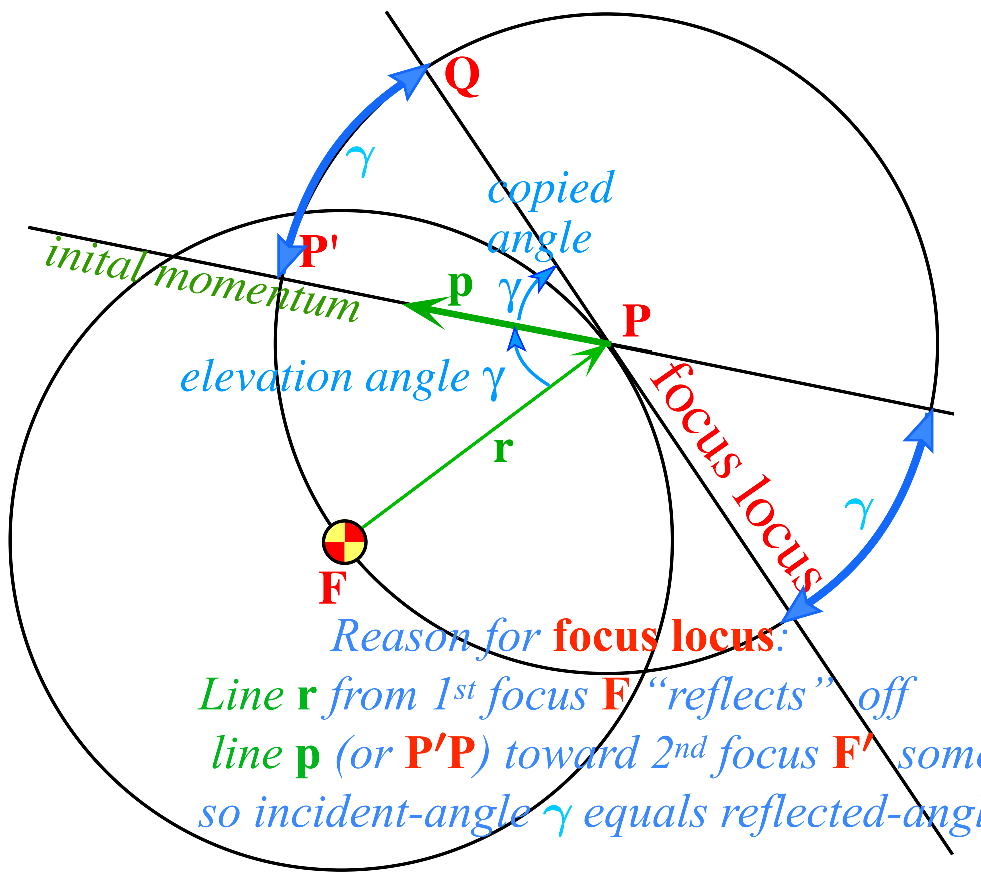


Copy F-center circle around launch point P

Copy elevation angle γ (∠FPP′) onto ∠P′PQ

Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus***:*

Line **r** *from 1st focus* **F** *"reflects" off*

line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*

so incident-angle γ equals reflected-angle γ

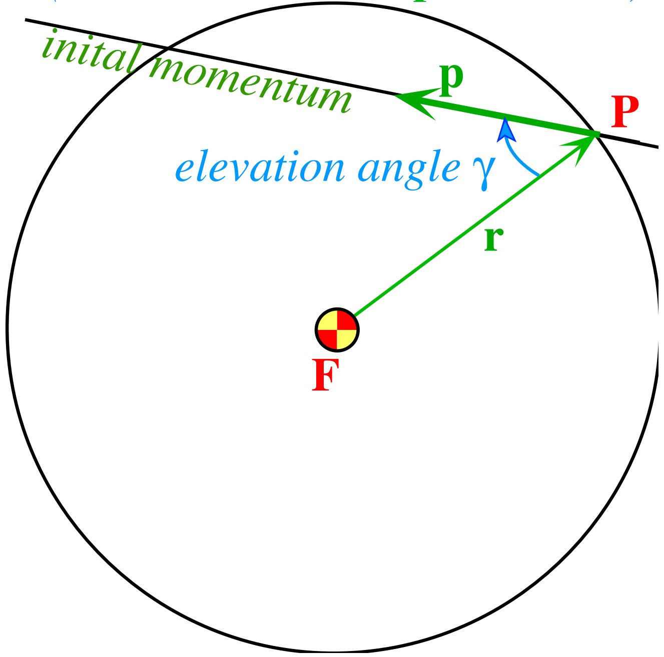
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ from radius

(momentum initial **p** *direction)*



Copy F-center circle around launch point P

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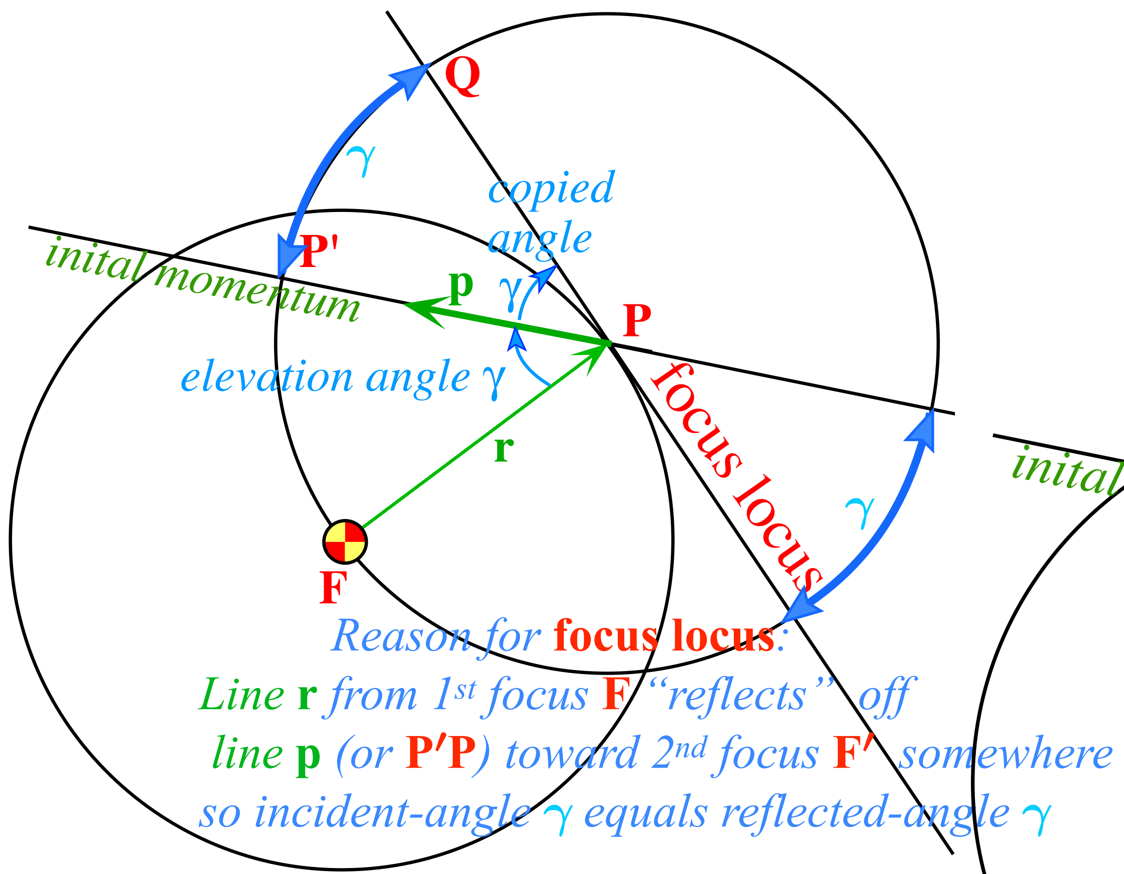
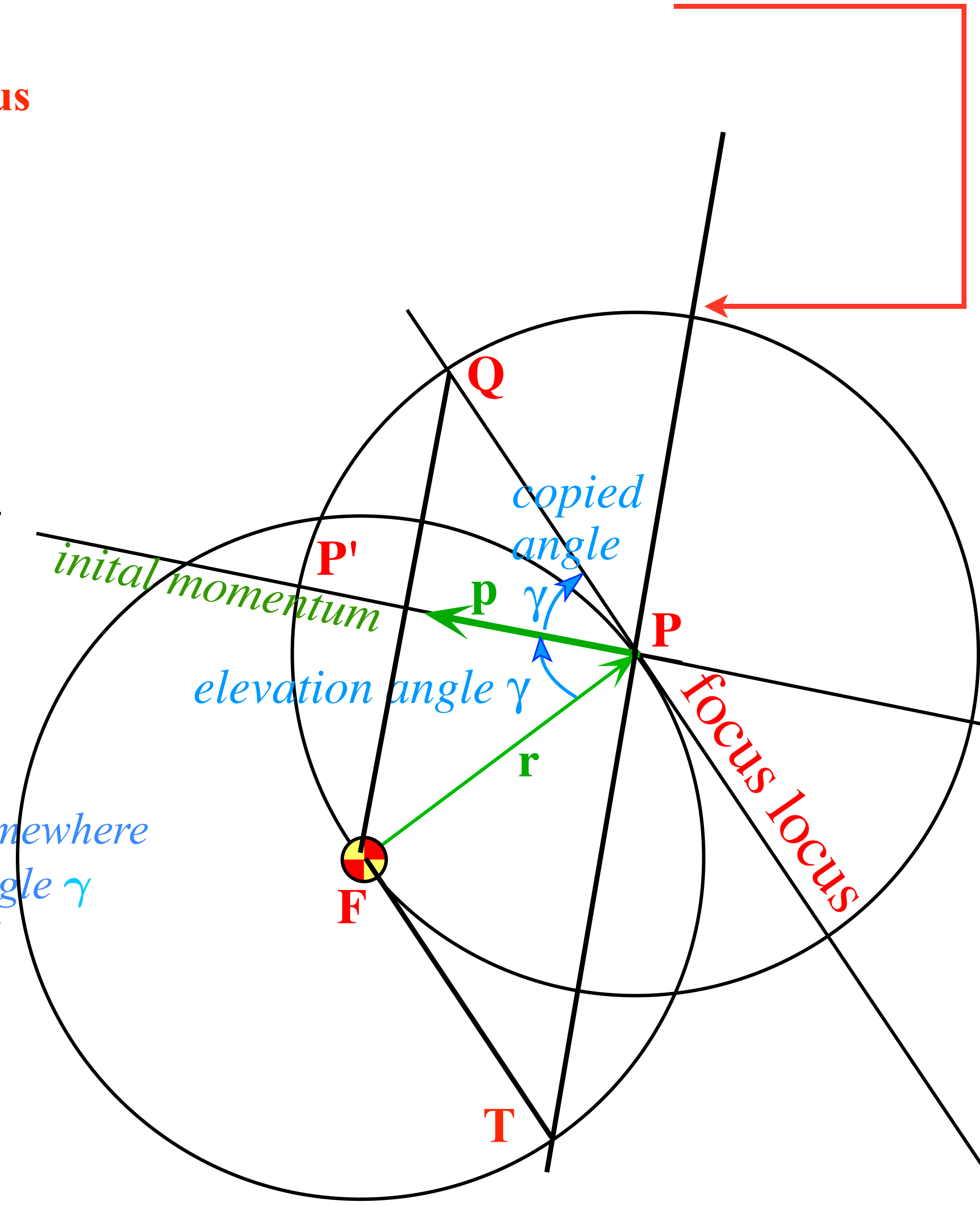
Extend resulting line QPQ′ *to make* **focus locus**



Copy double angle 2γ (∠FPQ) onto ∠PFT

*Extend ∠PFT chord PT to make R-***ratio scale line**

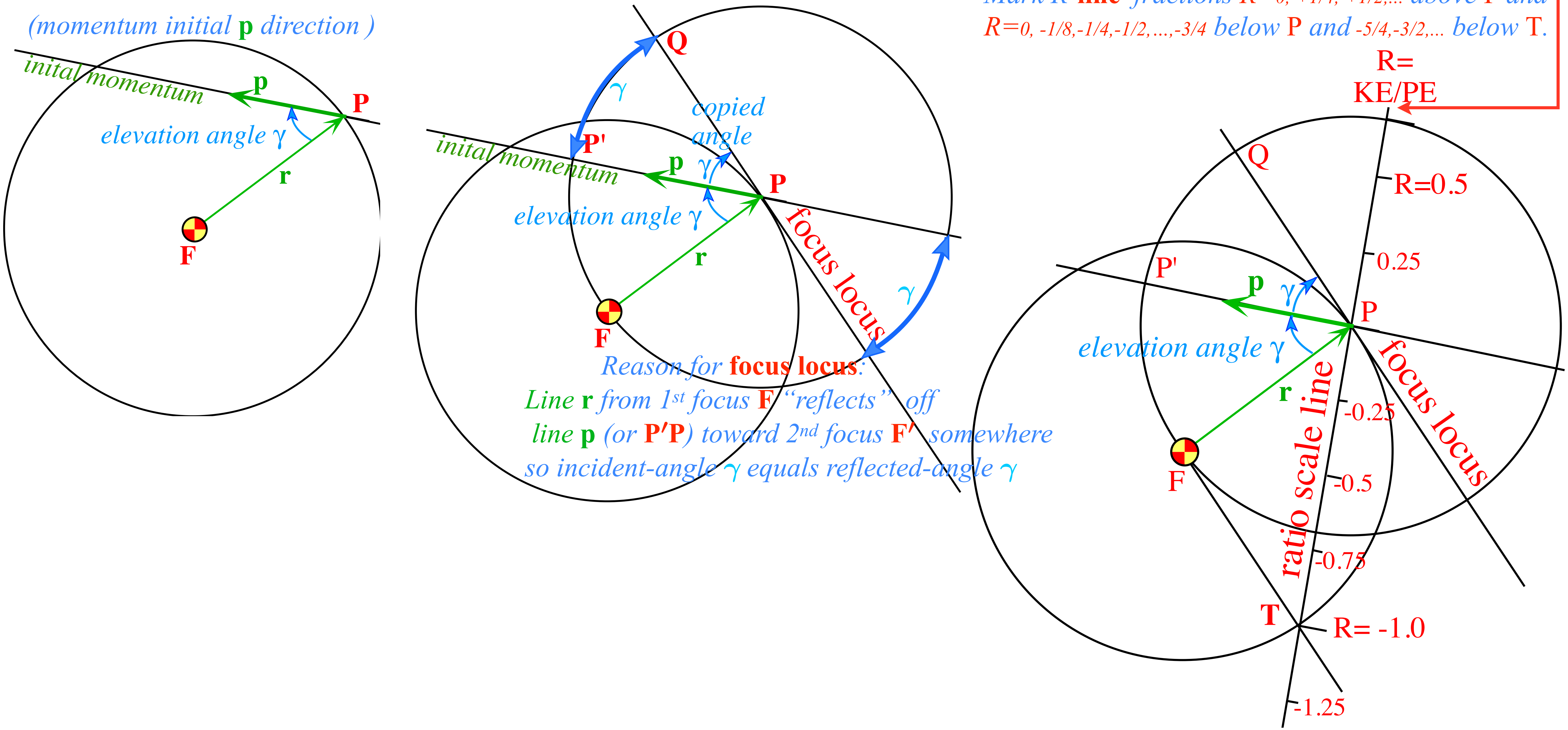


General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**

Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.



Reason for **focus locus**:
Line **r** *from 1st focus* **F** *“reflects” off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ equals reflected-angle γ

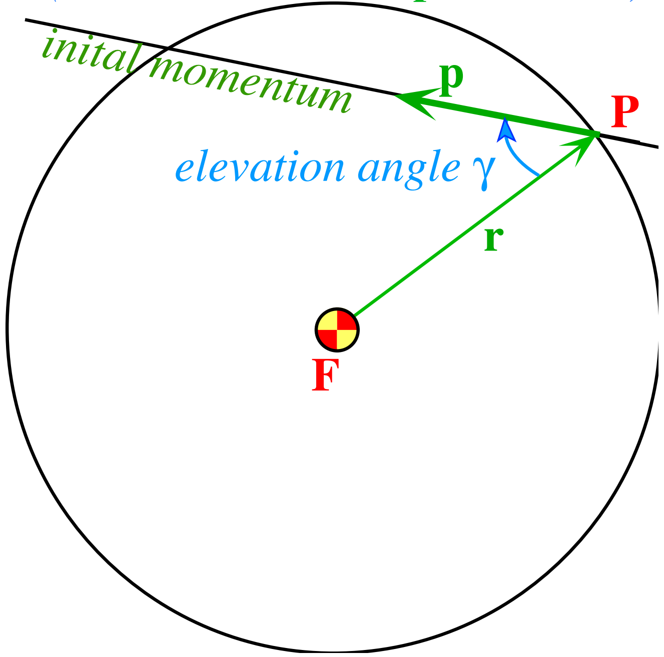
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

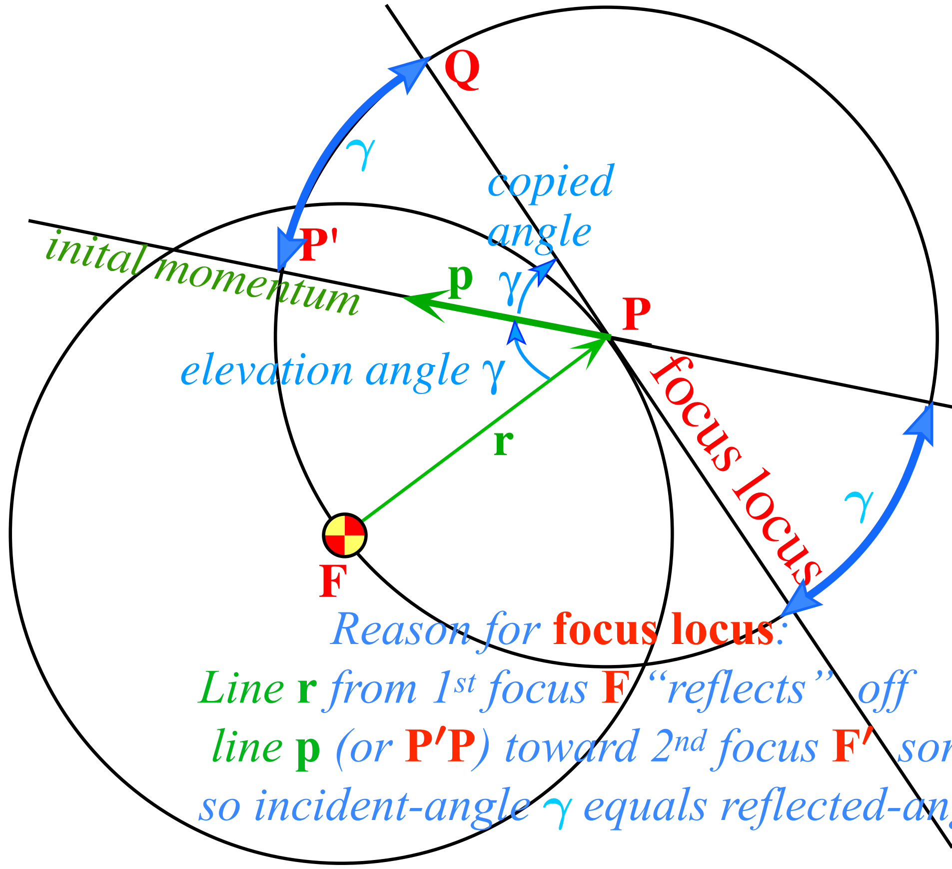
(momentum initial **p** *direction)*



Copy F-center circle around launch point P

Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*

Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus**:

Line **r** *from 1st focus* **F** *"reflects" off*

line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*

so incident-angle γ *equals reflected-angle* γ

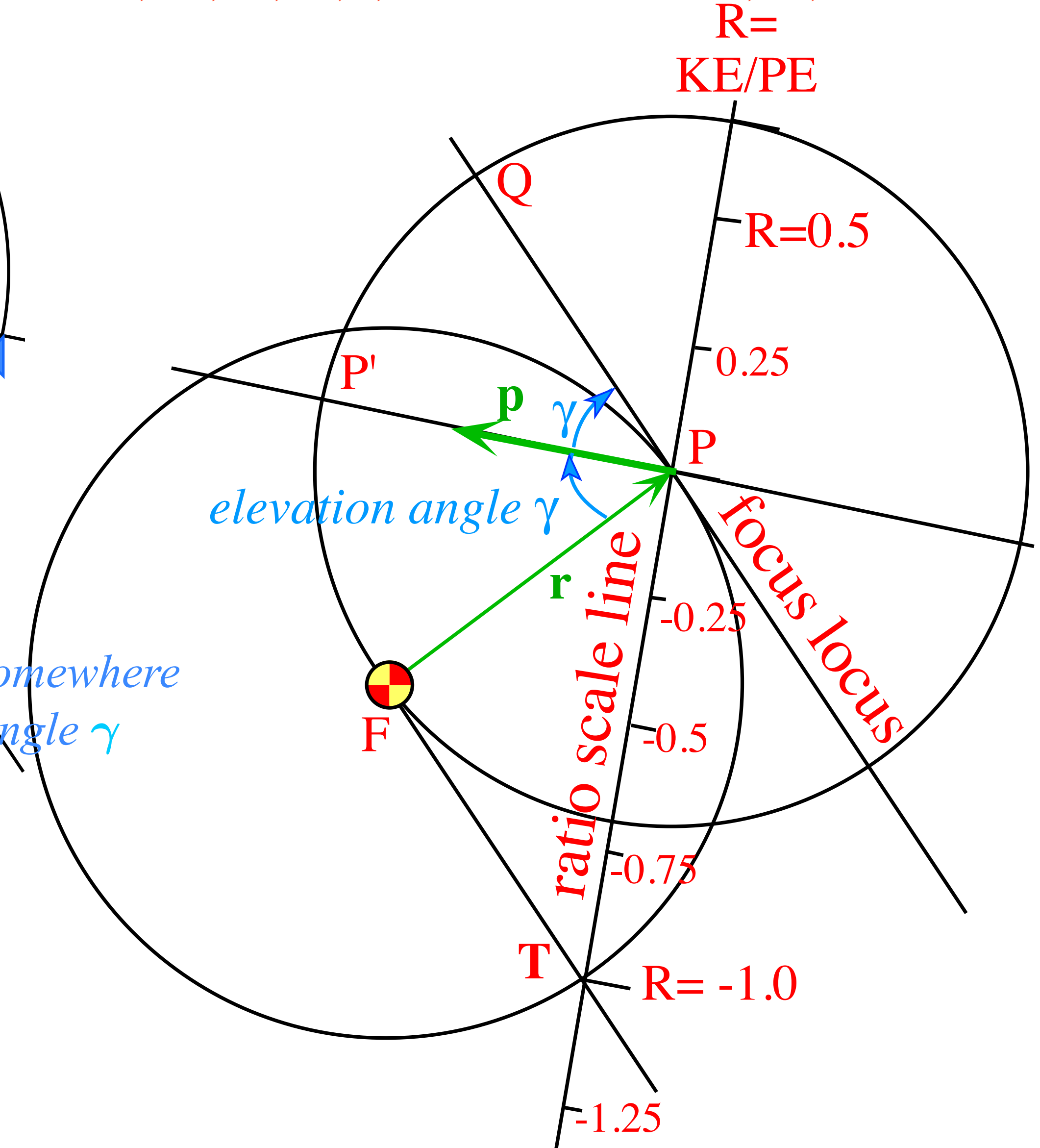
Copy double angle 2γ *(∠FPQ) onto ∠PFT*

Extend ∠PFT chord PT *to make R-***ratio scale line**

Label chord PT *with R=0 at* P *and R=-1.0 at* T.

*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*

R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.



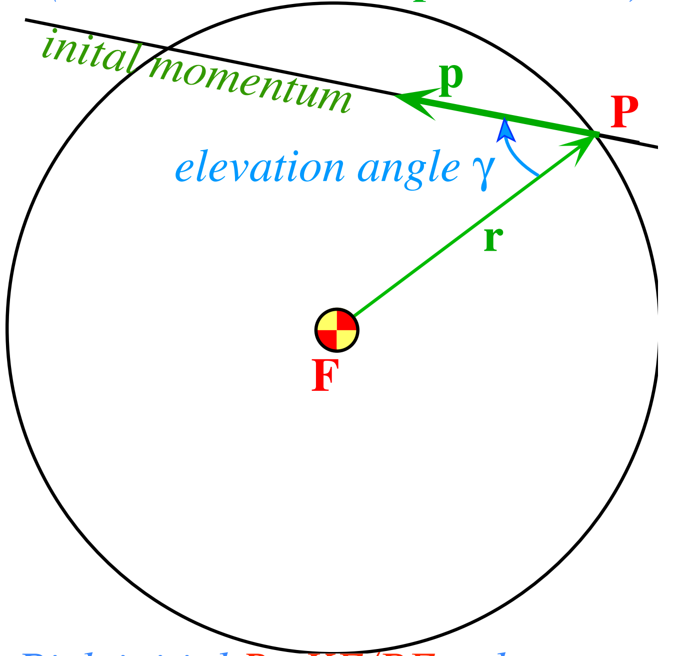
$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

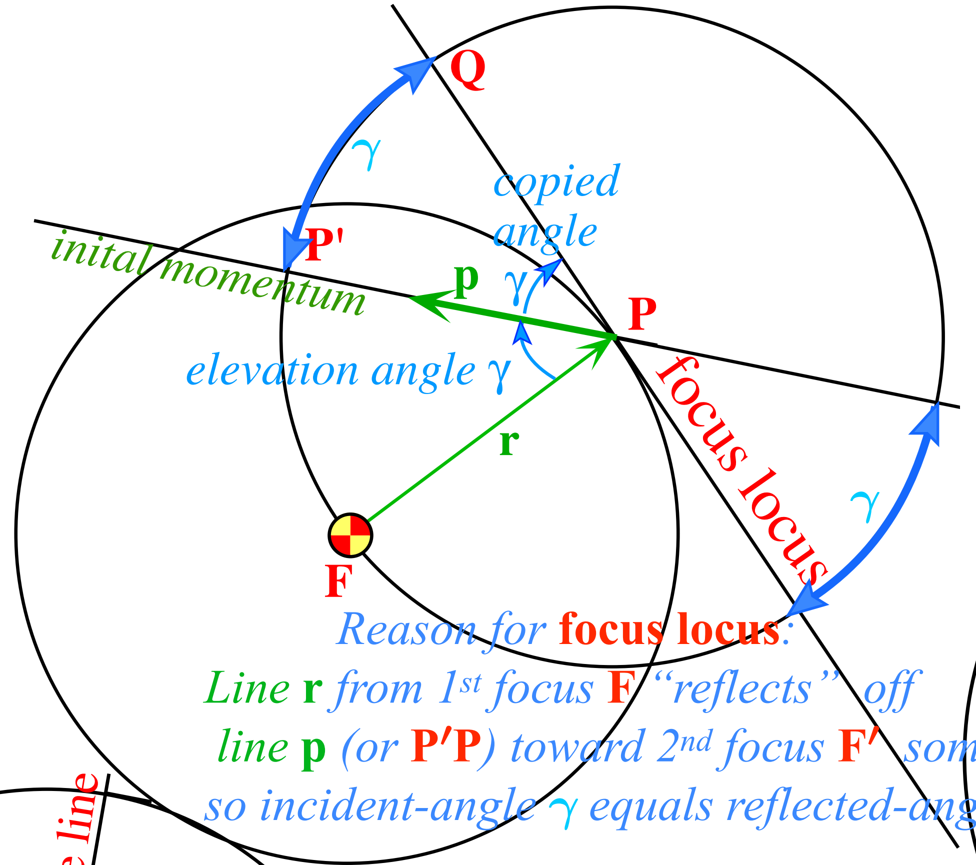
(To be proved on p.74 to p.77)

General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r** *)*
and elevation angle γ from radius
(momentum initial **p** *direction)*

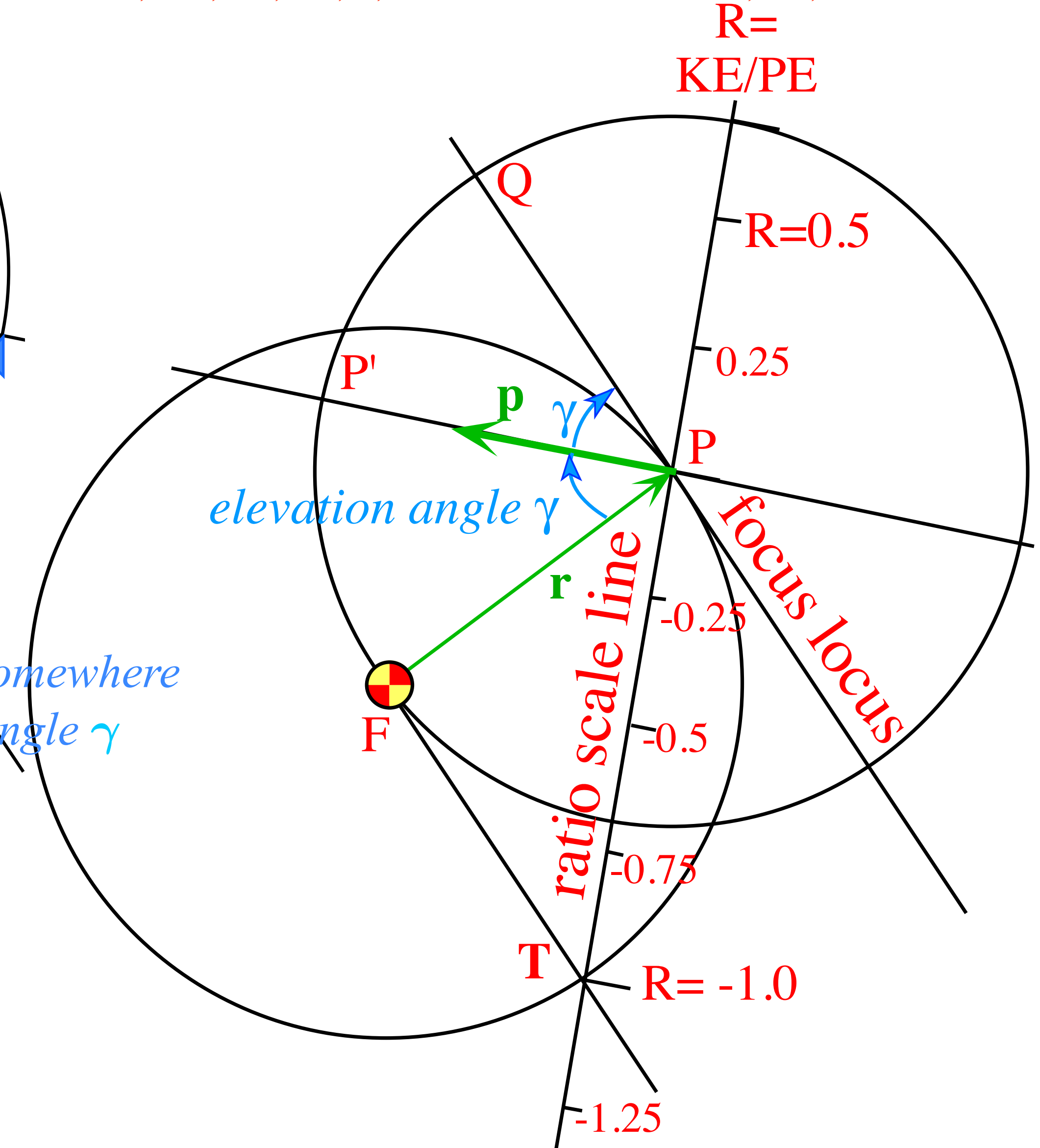


Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ *to make* **focus locus**

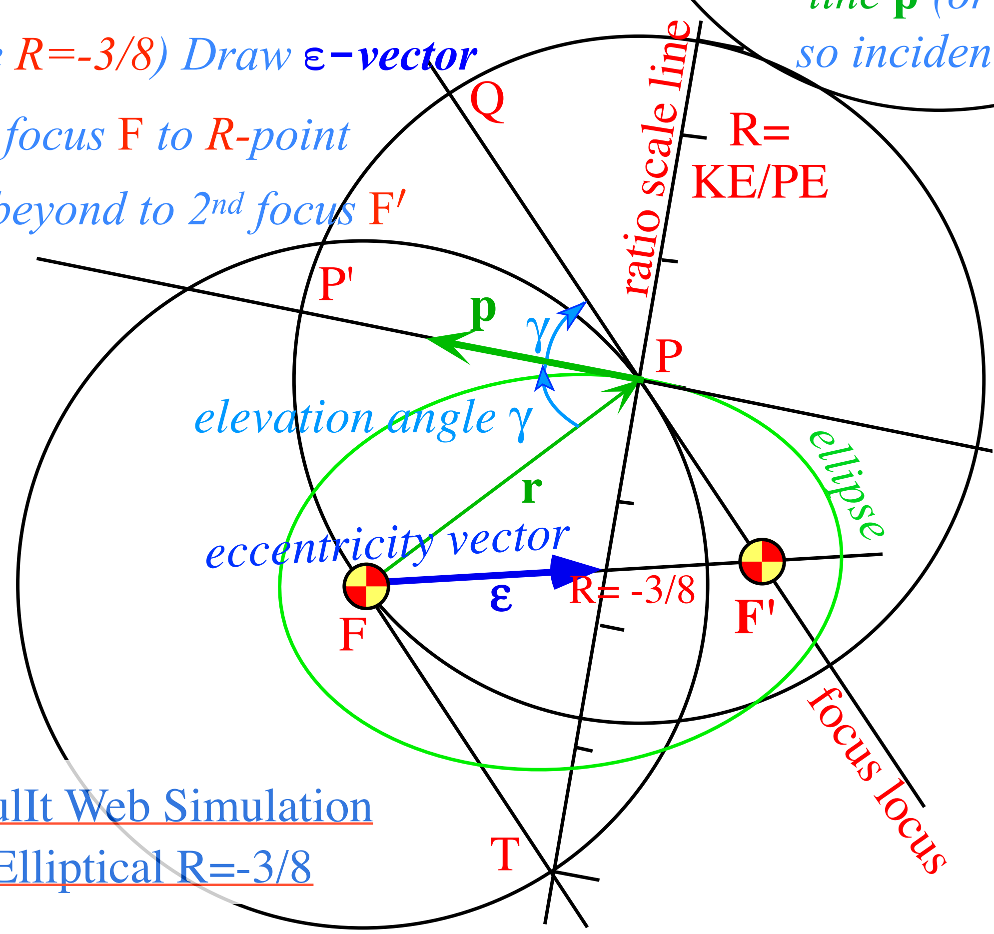


Reason for **focus locus**:
Line **r** *from 1st focus* **F** *"reflects" off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ equals reflected-angle γ

Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.



Pick initial R=KE/PE value
(here R=-3/8) Draw **ε−vector**
from focus F *to R-point*
and beyond to 2nd focus F′



[CoulIt Web Simulation
Elliptical R=-3/8]

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

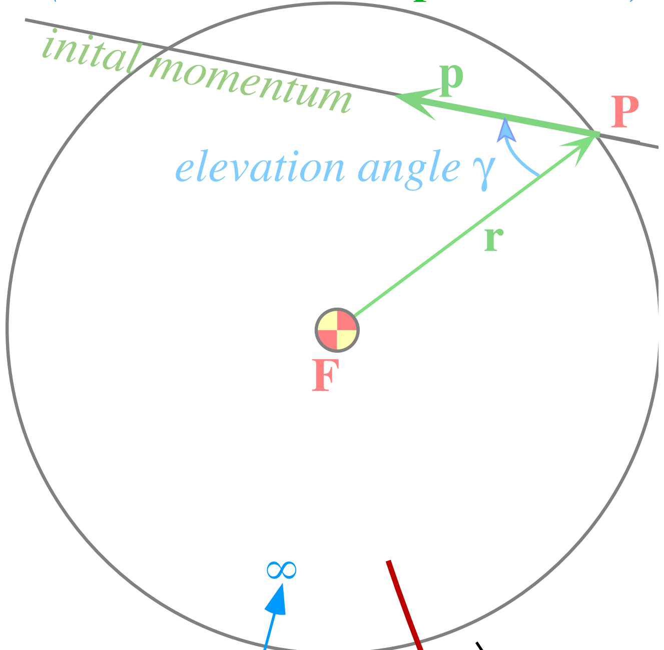
$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

focus F *and 2nd focus* F′ *allow final*
construction of orbital trajectory.
Here it is an R=-3/8 ellipse.

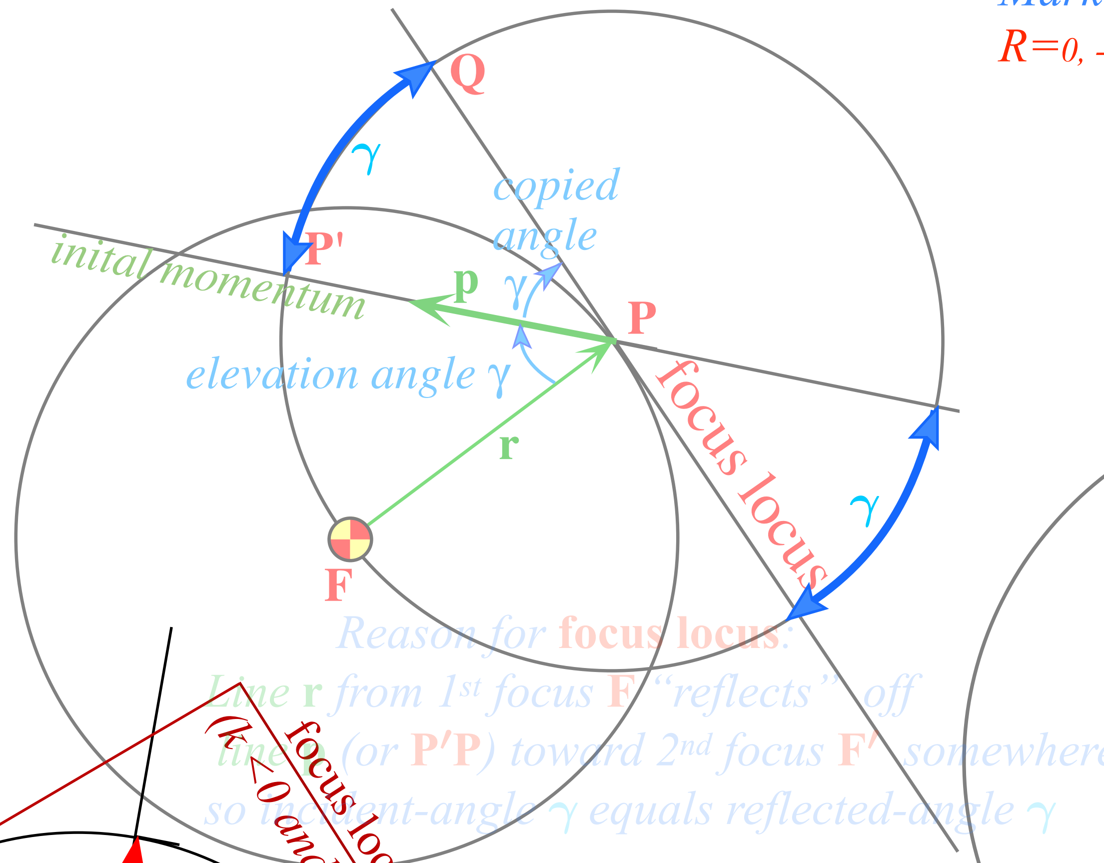
(Detailed geometry of **ε-vector** *follows.)* *(To be proved on p.74 to p.77)*

General geometric orbit construction using **ε**-*vector and* (γ,*R*)-*parameters*

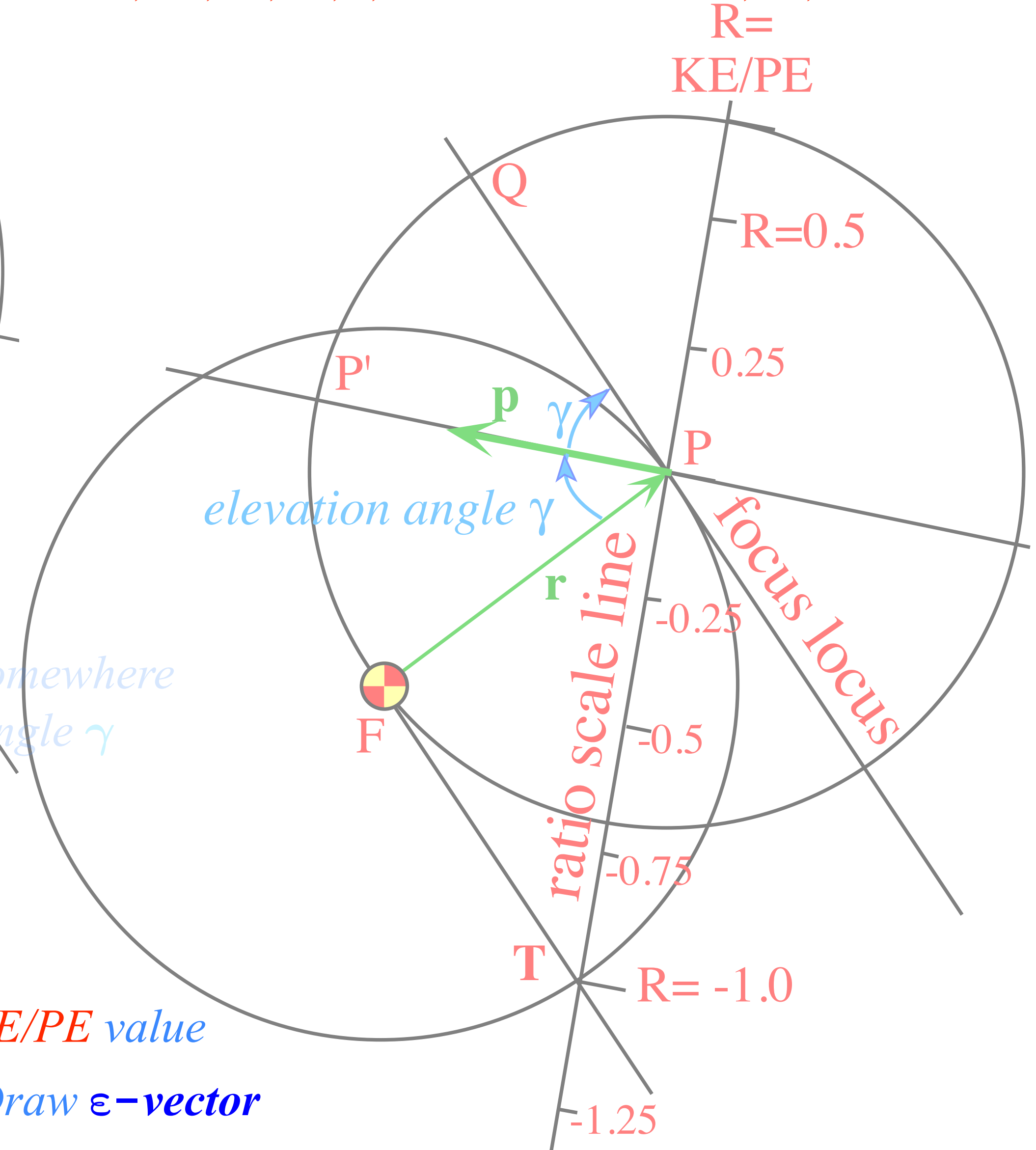
Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

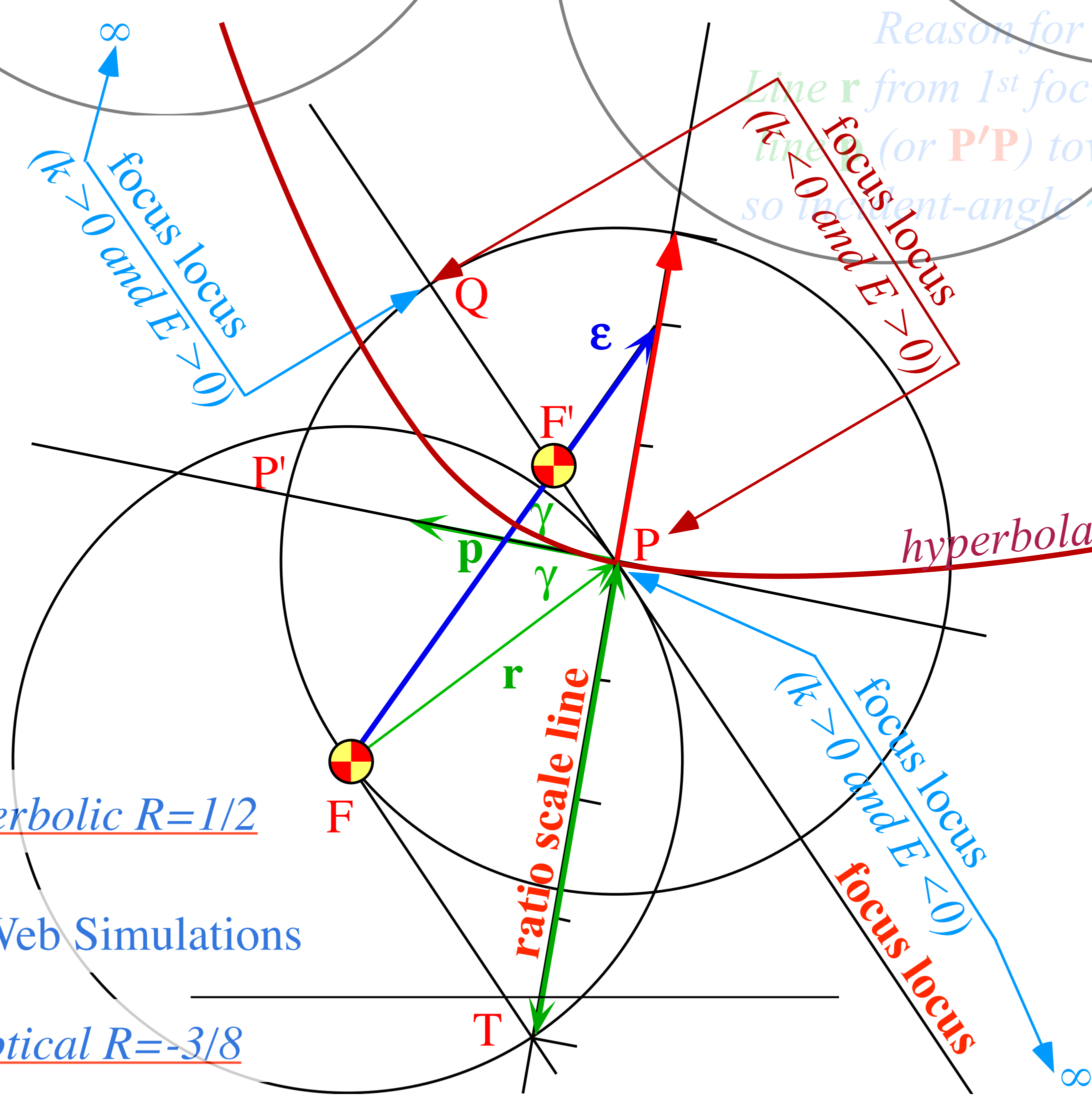


Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**



Copy double angle 2γ *(∠FPQ) onto ∠PFT*
Extend ∠PFT chord PT *to make R*-**ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
Mark R-**line** *fractions R=0, +1/4, +1/2,... above* P *and R=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.





Pick initial R=KE/PE value
(here R=+1/2) Draw **ε−vector**
from focus F *to R-point*
(Here it intersects 2nd focus F′

> *focus* F *and 2nd focus* F′ *allow final construction of orbital trajectory.*
> *Here it is an R=+1/2 hyperbola.*

(Detailed geometry of **ε-vector** *follows.)*

$$R=\frac{\textit{Initial KE}}{\textit{Initial PE}}=\frac{m\,v^2(0)/2}{-k\,/\,r(0)}$$

$$=\pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2=\pm\frac{v^2(0)}{v^2(\infty)}$$

(To be proved on p.74 *to* p.77 *)*

Hyperbolic R=1/2

CoulIt Web Simulations

Elliptical R=-3/8

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

➡ *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Derivation of $\boldsymbol{\varepsilon}$*-construction on* p.69 *by analytic geometry*

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

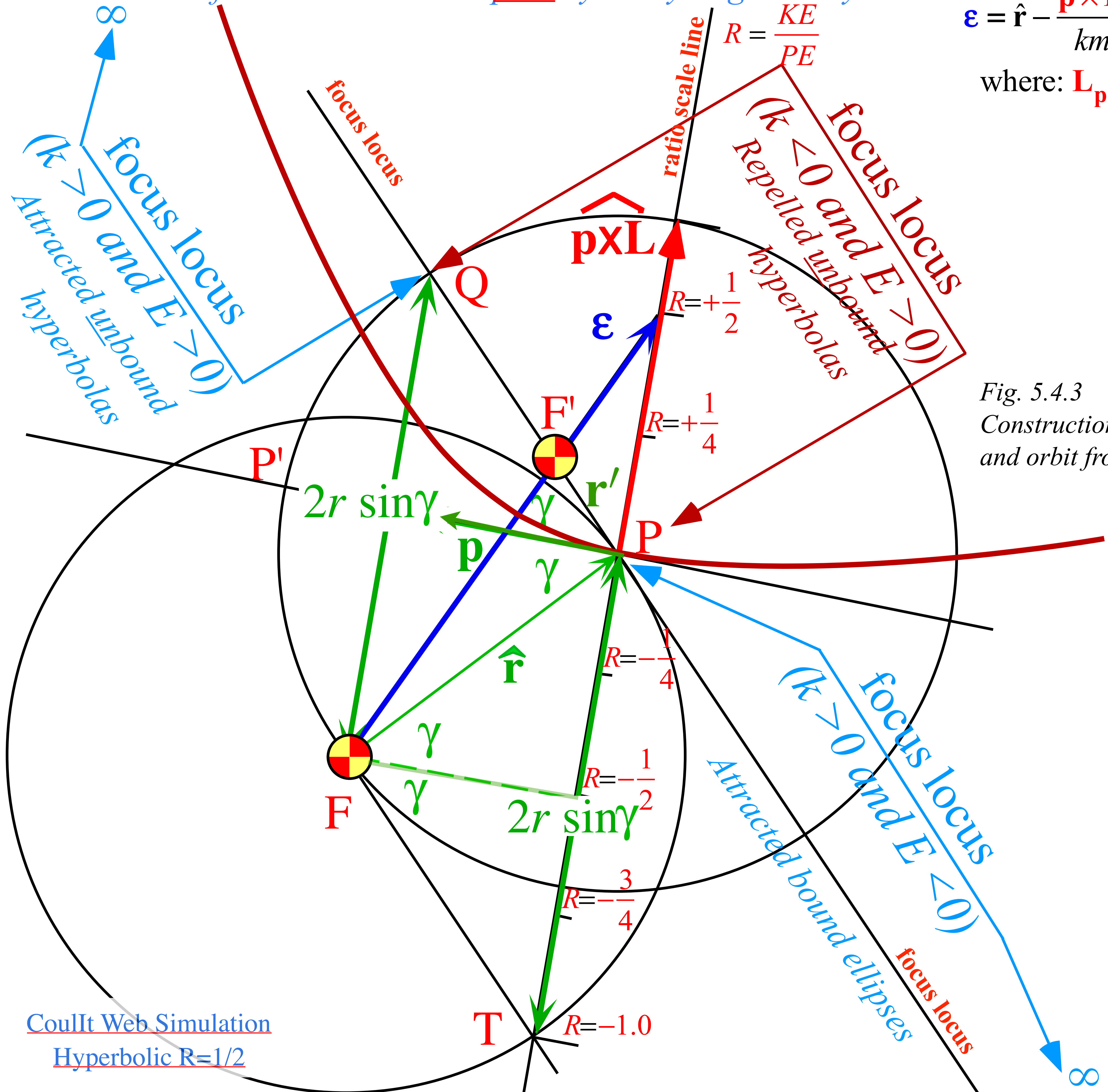


Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

CoulIt Web Simulation
Hyperbolic R=1/2

Derivation of $\boldsymbol{\varepsilon}$*-construction on* *p.69* *by analytic geometry*

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

(Go back to p.67)

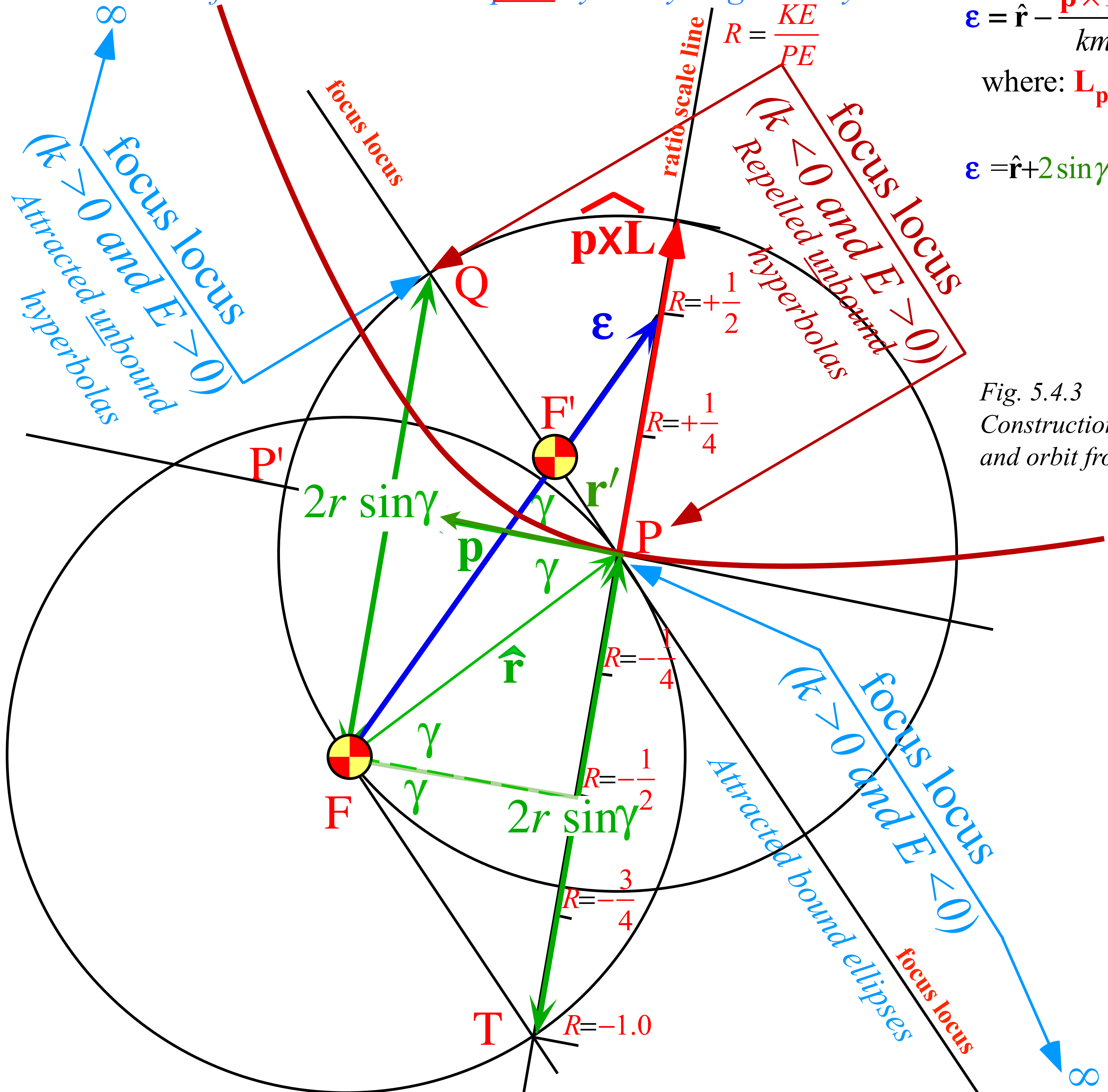
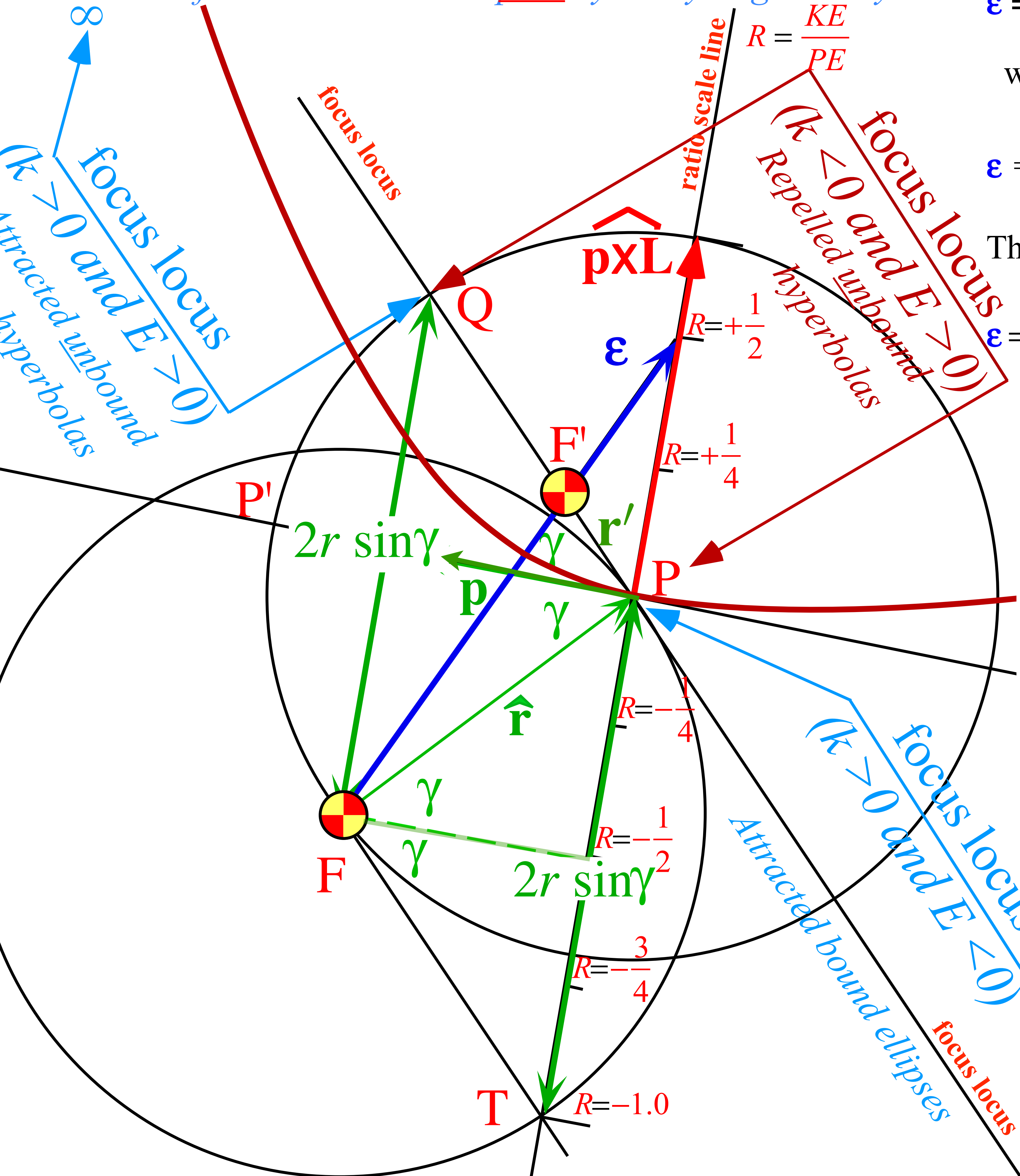


Fig. 5.4.3
Construction of eccentricity vector ε *and orbit from initial* **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of $\boldsymbol{\varepsilon}$-construction on p.69 by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma \frac{m{v_0}^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma \frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

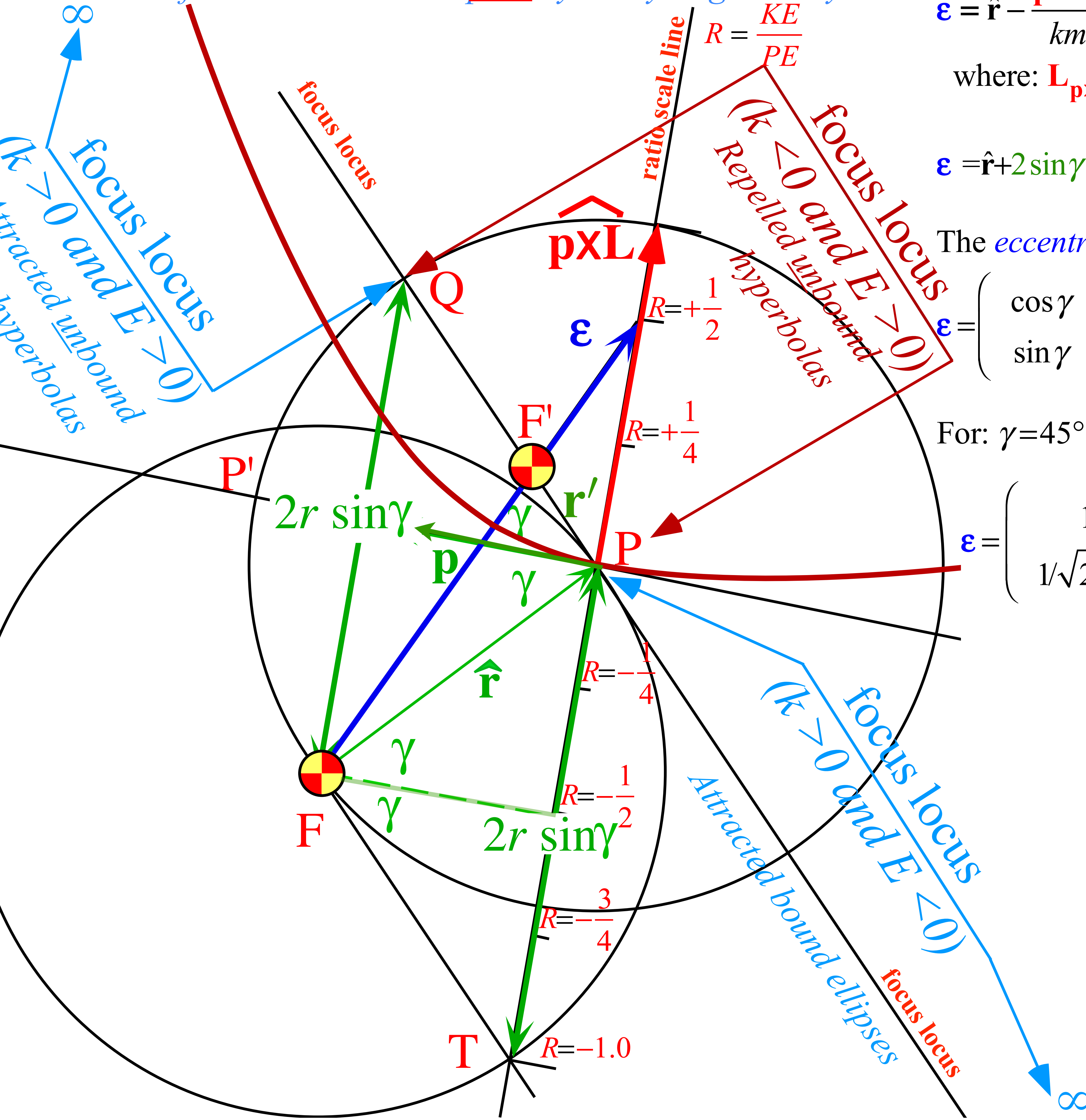
The *eccentricty* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} + 2\sin\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos\gamma \\ (2R+1)\sin\gamma \end{pmatrix}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of $\boldsymbol{\varepsilon}$*-construction on p.69 by analytic geometry*



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\frac{mv_0^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricty* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

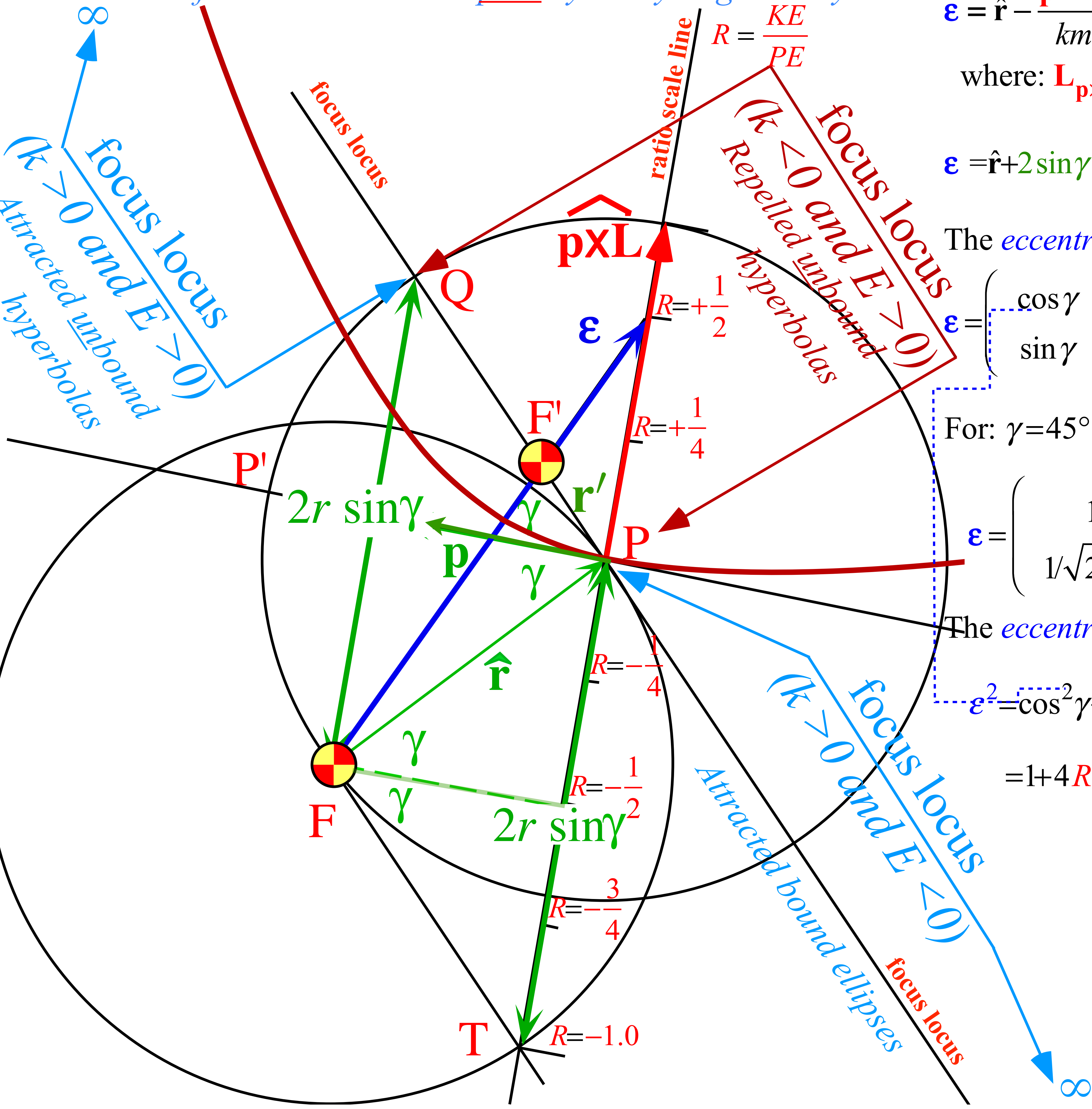
For: $\gamma=45°$ and: $R=+\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε*-construction on* *p.69* *by analytic geometry*



$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\hat{\mathbf{r}}-\frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times}\equiv\mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}+2\sin\gamma\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times}=\hat{\mathbf{r}}+2\sin\gamma\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon}=\begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix}+2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R=\begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

For: $\gamma=45°$ and: $R=+\frac{1}{2}$

$$\boldsymbol{\varepsilon}=\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

The *eccentricity* parameter defined by:

$$\varepsilon^2=\cos^2\gamma+(2R+1)^2\sin^2\gamma=1\pm\frac{a^2}{b^2}$$

$$=1+4R(R+1)\sin^2\gamma=\frac{5}{2}\quad\text{where: }\sin^2\gamma=\frac{1}{2}$$

$$R=\frac{\textit{Initial KE}}{\textit{Initial PE}}=\frac{m\,v^2(0)/2}{-k/r(0)}$$

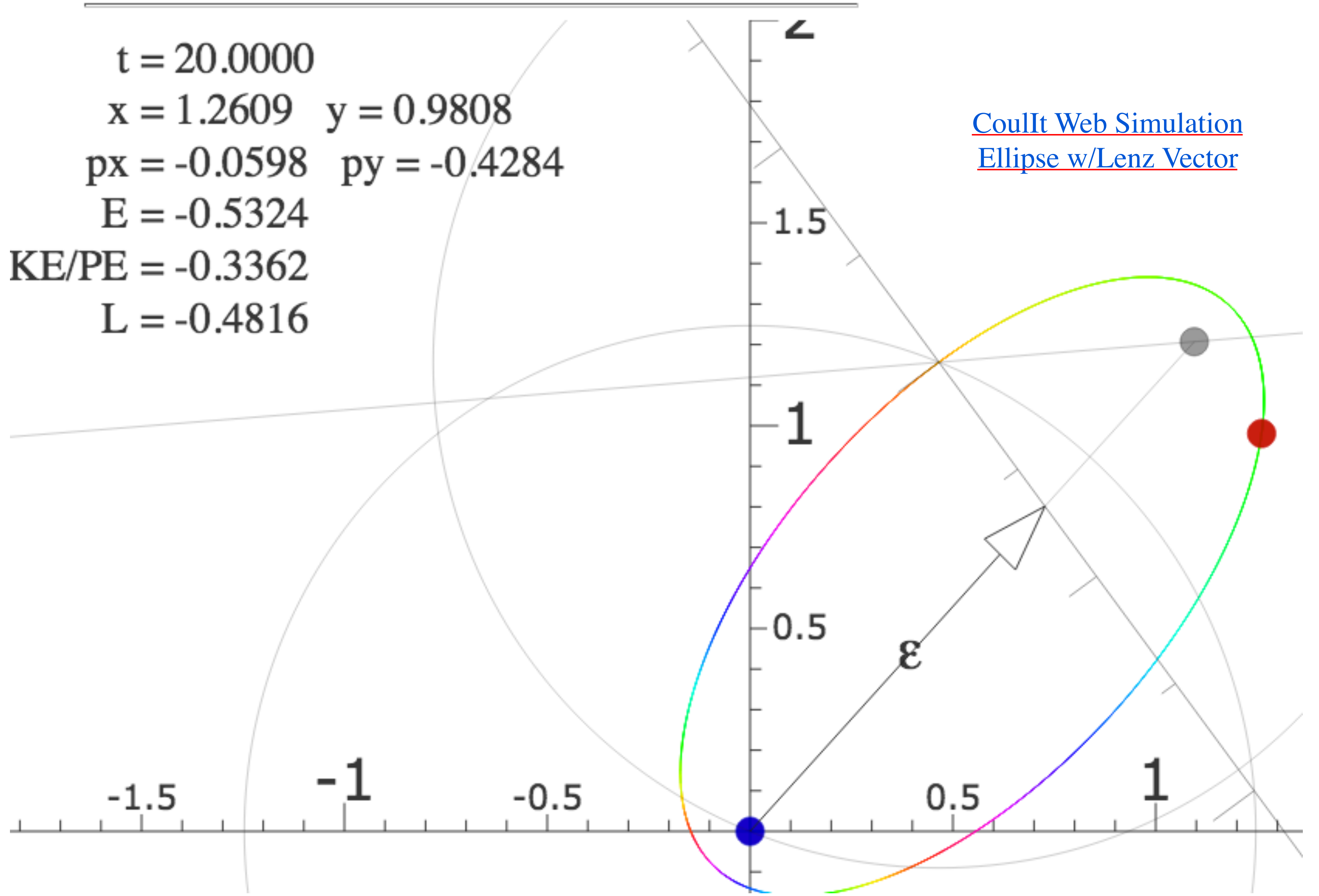
$$=\pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2=\pm\frac{v^2(0)}{v^2(\infty)}$$

Initial position x(0) = 0.465648
Initial position y(0) = 1.156488
Initial momentum px(0) = 0.591603
Initial momentum py(0) = 0.435114

Terminal time t(off) = 20
Maximum step size dt = 0.01

Charge of Nucleus 1 = -1
x-Position of Nucleus 1 = 0
y-Position of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = -1
Core thickness r = 0.000001
x-Stark field Ex = 0
y-Stark field Ey = 0
Zeeman field Bz = 0
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 64
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 9

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☑ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by φ ☐ Elastic ☑ 2 Free ☐
Save to GIF ☐



Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Chapter 4 Molecular Ion Orbits

Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

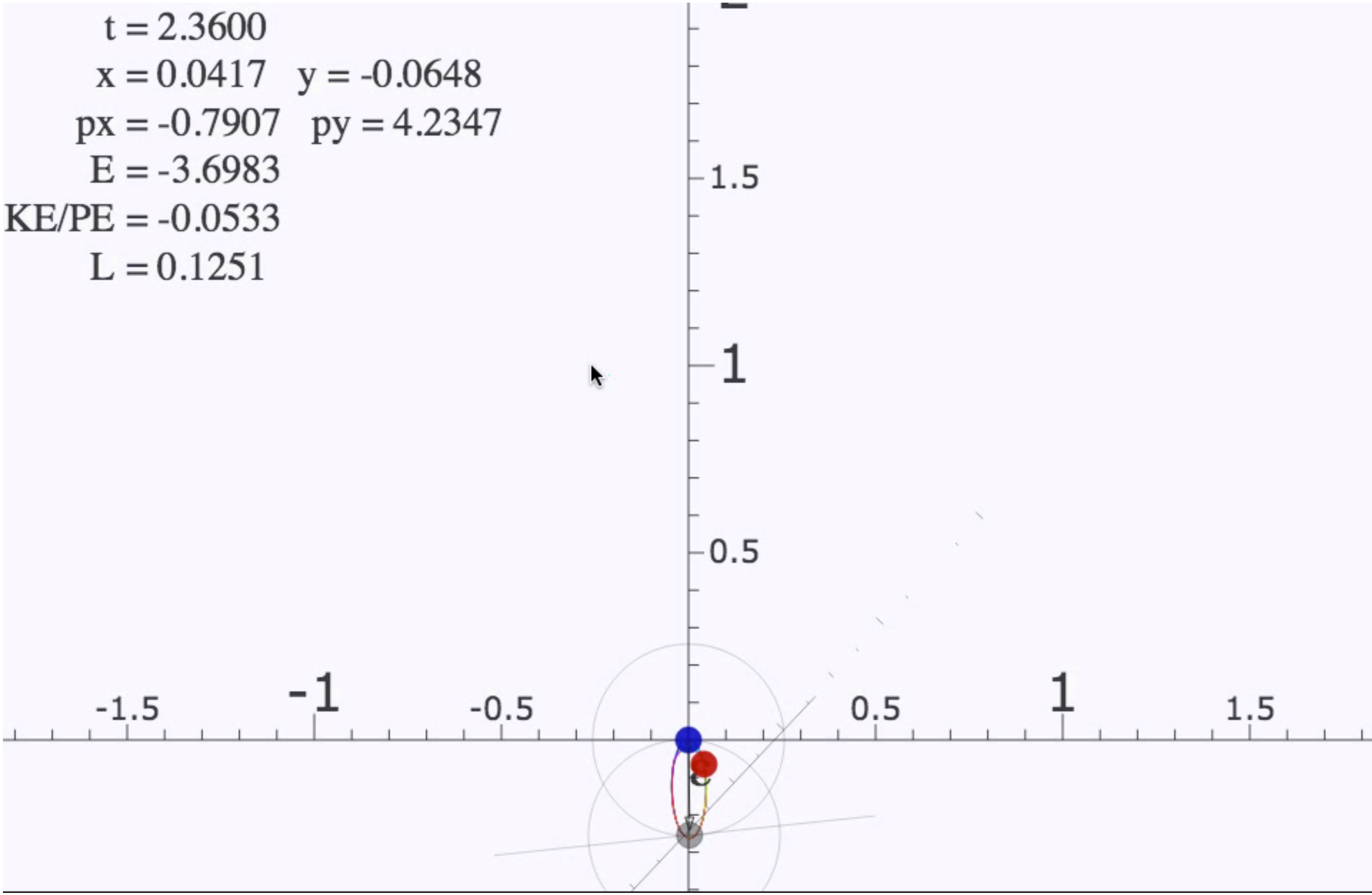
Volcanoes of Io (Paths=180, No color quant.) | Parabolic Fountain (Uniform) | Space Bomb (Coulomb) | Exploding Starlet (IHO) | Synchrotron Motion (Crossed E & B fields)

Rutherford scattering | 2-Electron Orbits

Atomic Orbits

Molecular Ion Orbits

Oscillator Scattering | 2-Particle Orbits | 2-Particle Collision



Play embedded movie with controls above.

[or follow this link to try your hand at ε-construction using the CoulIt Web App]

Just click and drag in main window to set new initial conditions. The Lenz vector will display as part of an overlay.

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

➤ *Coulomb orbit algebra of* $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

NOTE:Standard notation $\boldsymbol{L}$ *for angular momentum* $\boldsymbol{\mu}$

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r: (p. 48 to p.53)

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

Coulomb orbit algebra of **ε**-*vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

NOTE: Standard notation **L** *for angular momentum* **μ**

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity **v** *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}$$

Coulomb orbit algebra of **ε-vector** *and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

NOTE:Standard notation ***L*** *for angular momentum* $\boldsymbol{\mu}$

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

NOTE: Standard notation $\mathbf{L}$ for angular momentum $\boldsymbol{\mu}$

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity $\mathbf{v}$ or momentum $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

NOTE: Standard notation $\mathbf{L}$ *for angular momentum* $\boldsymbol{\mu}$

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}$$

Coulomb orbit algebra of **ε-vector** *and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

*NOTE: Standard notation **L** for angular momentum **μ***

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r}=\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)$

Coulomb orbit algebra of ***ε-vector*** *and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

NOTE:Standard notation ***L*** *for angular momentum* $\boldsymbol{\mu}$

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* $\mathbf{p}=m\mathbf{v}$

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$$\dot r=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot\phi}{(1-\varepsilon\cos\phi)^2}$$

$$\dot r=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2r^2\dot\phi\,\varepsilon\sin\phi$$

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$$\dot r=-\frac{k}{L^2}mr^2\dot\phi\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

again using: $L=mr^2\dot\phi$

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$$\dot r=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot\phi\,\varepsilon\sin\phi$$

$$\boxed{-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 =-\frac{k}{L^2}m}$$

$$\dot r=-\frac{k}{L^2}mr^2\dot\phi\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

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again using: $L=mr^2\dot\phi$

Cartesian $x=r\cos\phi$*:*

$$\dot x=\frac{dx}{dt}=\quad\dot r\cos\phi-\sin\phi\; r\dot\phi$$

Cartesian $y=r\sin\phi$*:*

$$\dot y=\frac{dy}{dt}=\quad\dot r\sin\phi+\cos\phi\; r\dot\phi$$

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$$\dot r=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot\phi\,\varepsilon\sin\phi \qquad \textit{using: } \frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$$

$$-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2=-\frac{k}{L^2}m$$

$$\dot r=-\frac{k}{L^2}mr^2\dot\phi\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi \qquad \textit{again using: } L=mr^2\dot\phi$$

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Cartesian x = r cos φ:

$$\dot x=\frac{dx}{dt}=\dot r\cos\phi-\sin\phi\, r\dot\phi$$
$$=-\frac{k}{L}\varepsilon\sin\phi\cos\phi-\sin\phi\frac{k}{L}(1-\varepsilon\cos\phi)$$

Cartesian y = r sin φ:

$$\dot y=\frac{dy}{dt}=\dot r\sin\phi+\cos\phi\, r\dot\phi$$
$$=-\frac{k}{L}\varepsilon\sin\phi\sin\phi+\cos\phi\frac{k}{L}(1-\varepsilon\cos\phi)$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=m**v**

*NOTE:Standard notation **L** for angular momentum **μ***

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Cartesian x =r cos φ:

$$\dot{x}=\frac{dx}{dt}=\boxed{\dot{r}}\cos\phi-\sin\phi\,\boxed{r\dot{\phi}}$$

$$=\boxed{-\frac{k}{L}\varepsilon\cancel{\sin\phi\cos\phi}}-\sin\phi\,\boxed{\frac{k}{L}(1-\cancel{\varepsilon\cos\phi})}$$

$$=-\frac{k}{L}\sin\phi$$

Cartesian y =r sin φ:

$$\dot{y}=\frac{dy}{dt}=\boxed{\dot{r}}\sin\phi+\cos\phi\,\boxed{r\dot{\phi}}$$

$$=\boxed{-\frac{k}{L}\varepsilon\sin\phi}\sin\phi+\cos\phi\,\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

$$-\varepsilon\sin\phi\sin\phi-\varepsilon\cos\phi\cos\phi=-\varepsilon$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

NOTE: Standard notation $\mathbf{L}$ for angular momentum $\boldsymbol{\mu}$

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity $\mathbf{v}$ or momentum $\mathbf{p}=m\mathbf{v}$

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Cartesian $y=r\sin\phi$:

$$\dot{y}=\frac{dy}{dt}=\boxed{\dot{r}}\sin\phi+\cos\phi\, r\dot{\phi}$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

Velocity:

Momentum:

$$\boxed{p_x=m\dot{x}=-\frac{mk}{L}\sin\phi}$$

$$\boxed{p_y=m\dot{y}=\frac{mk}{L}(\cos\phi-\varepsilon)}$$

p *traces an off-center circle!*

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

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Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

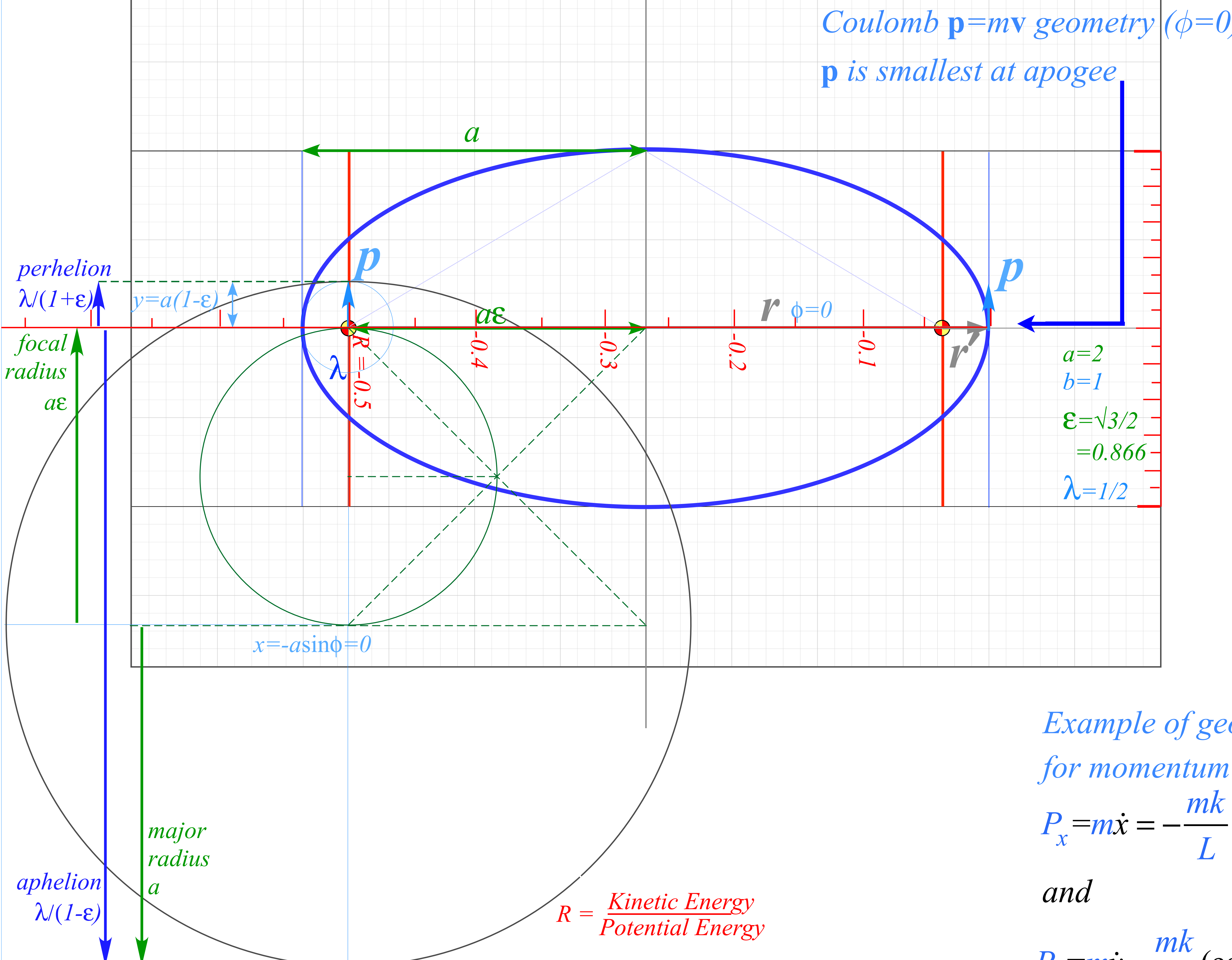
General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

➡ *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)



CoulIt Web Simulation

Elliptical with Momentum

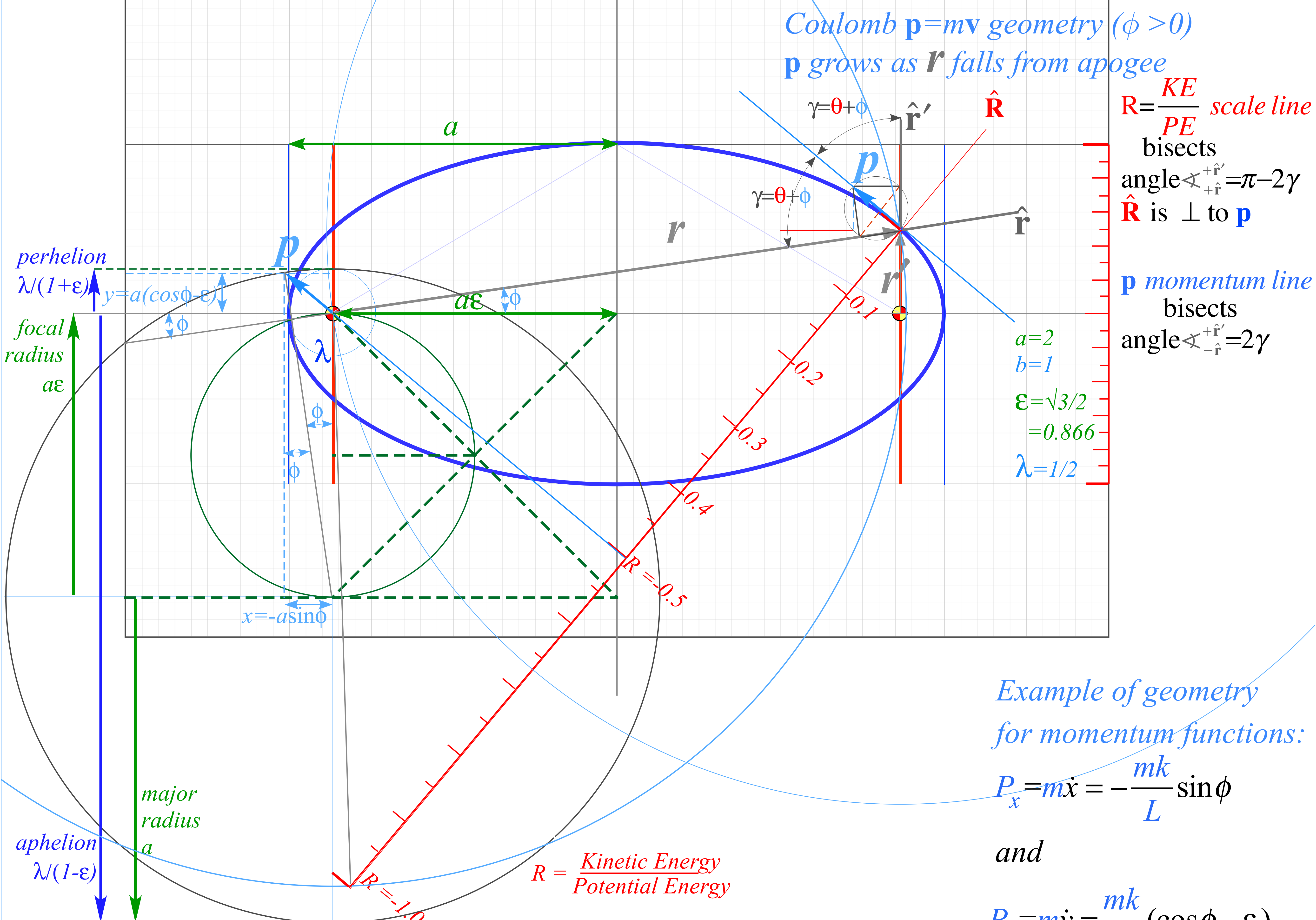
Example of geometry

for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$



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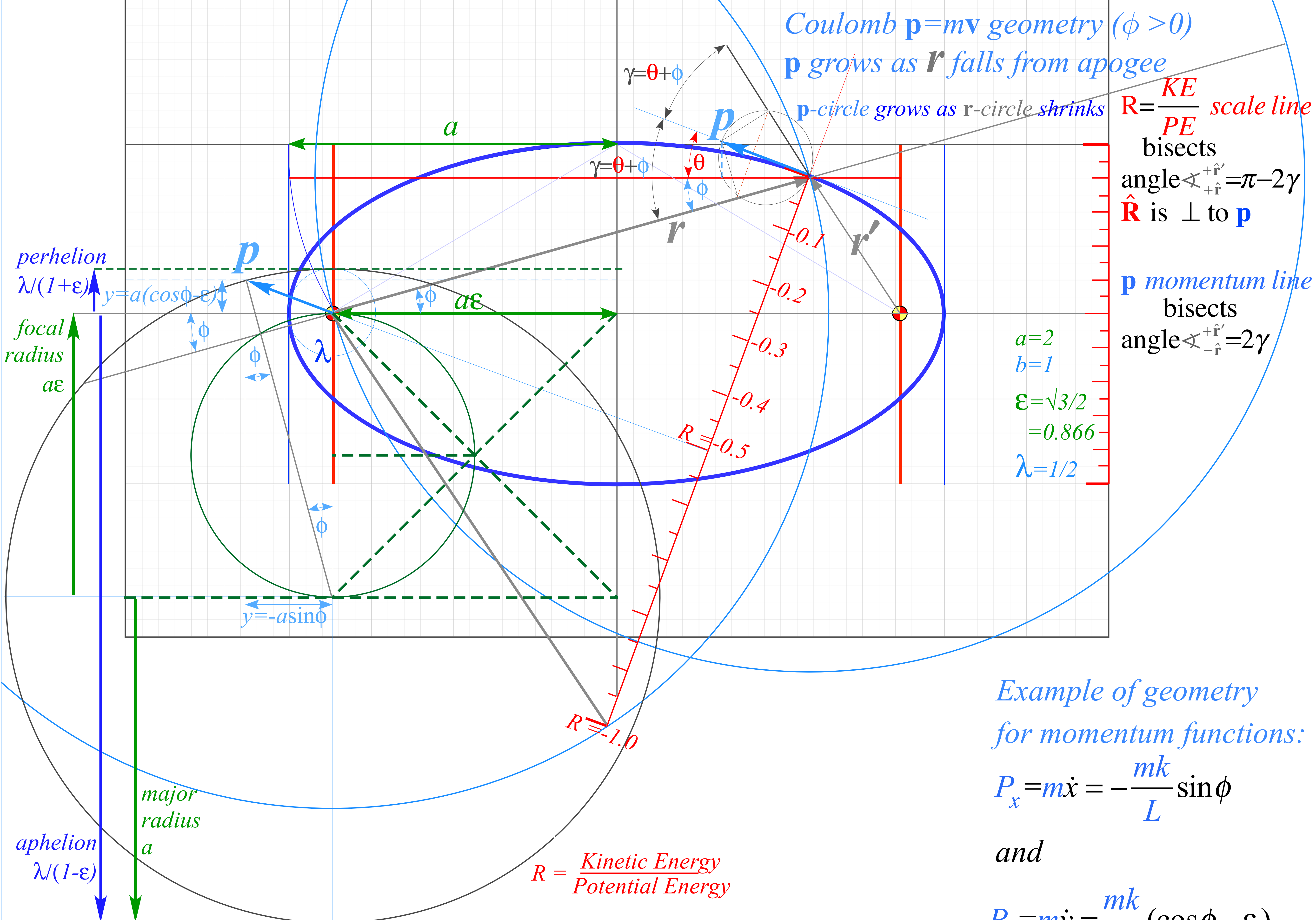
0>R=KE/PE>-1 *scale subtends angle* 2γ *with length* $2r\sin\gamma$ *as is derived before on* *p. 67-71.*

*Note similarity of (***R***,***r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*



Example of geometry for momentum functions:

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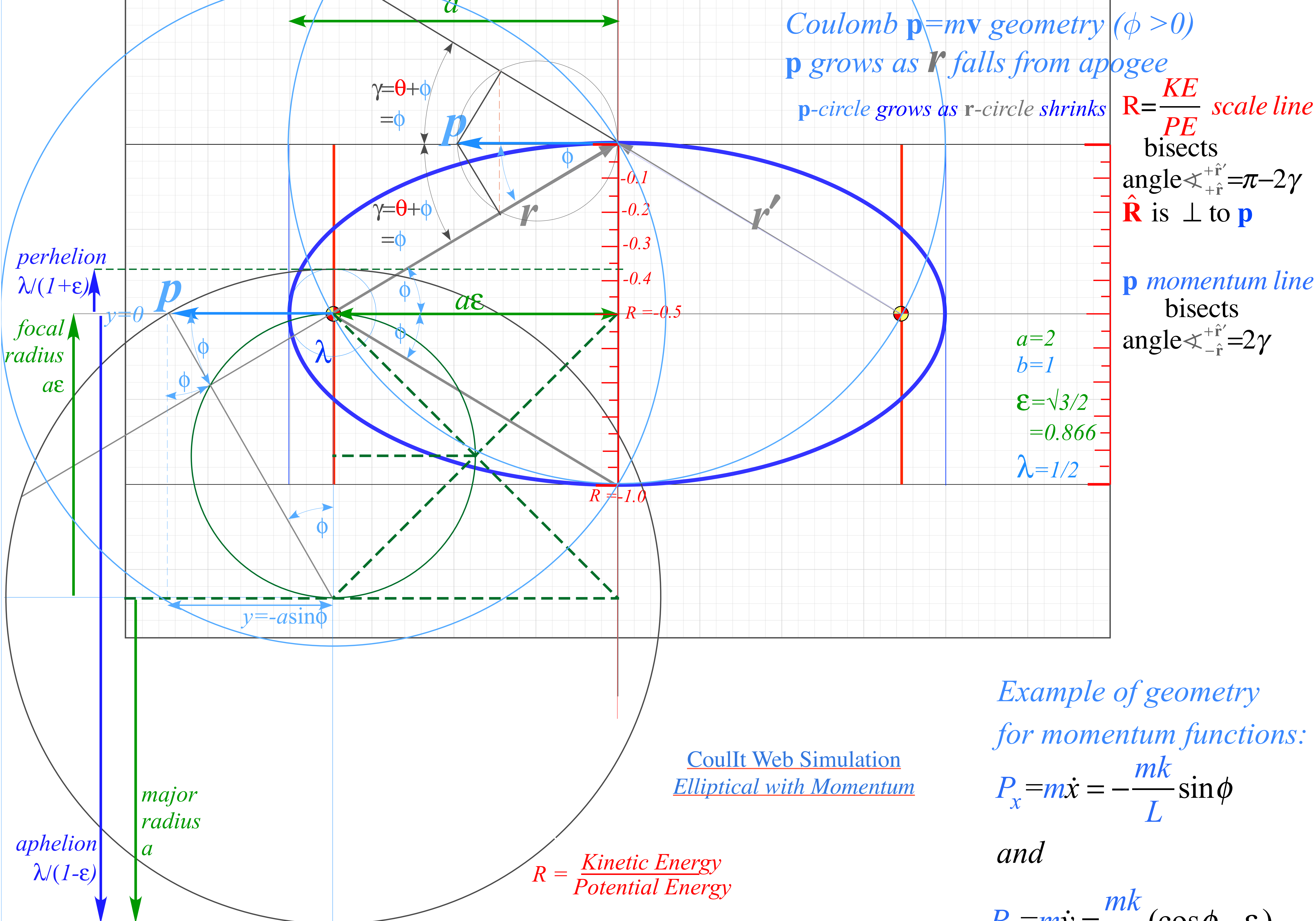
Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.



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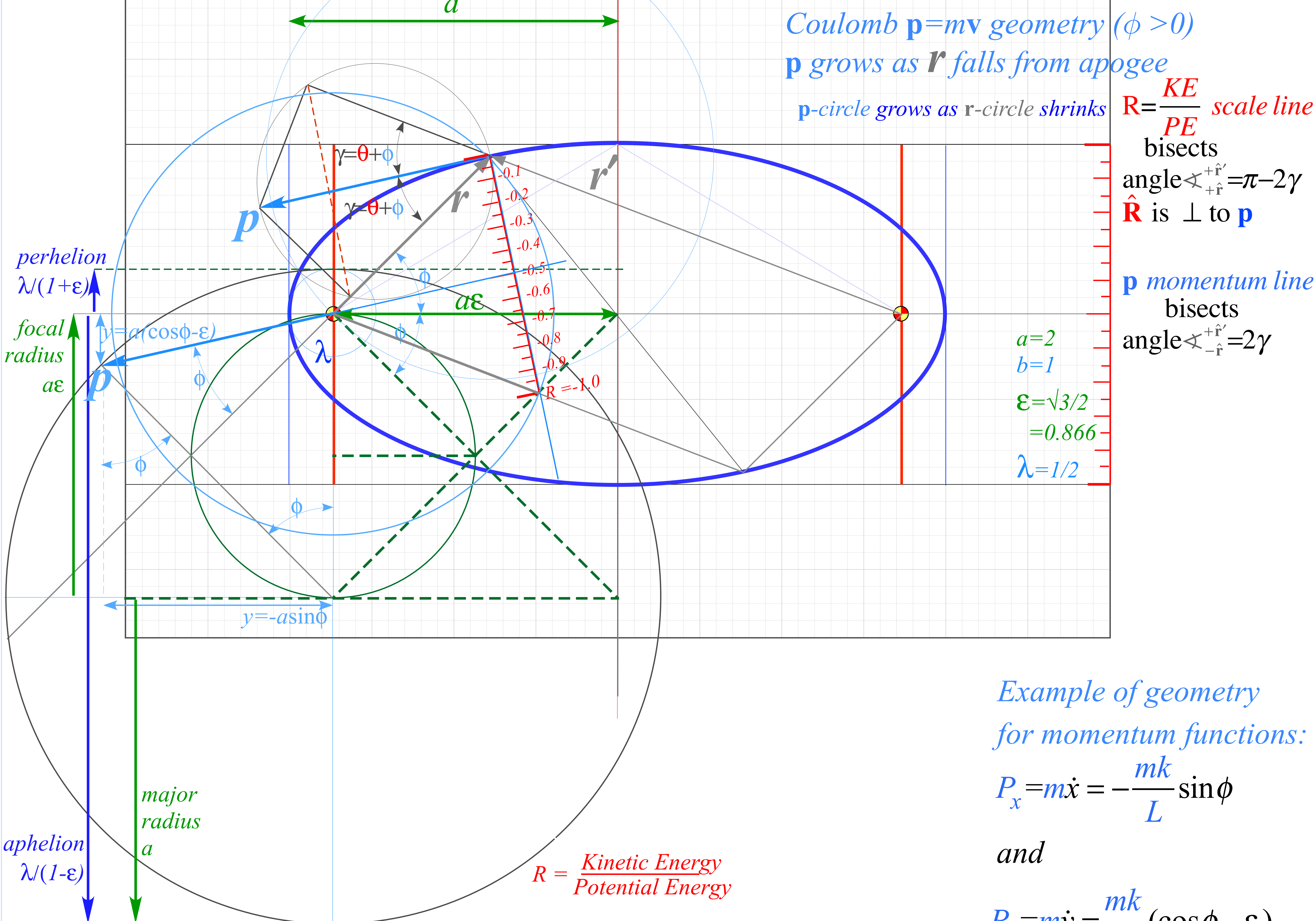
$0 > R = KE/PE > -1$ scale subtends angle 2γ with length $2r\sin\gamma$ as is derived before on p. 66-70.

Note similarity of ($\mathbf{R}$,$\mathbf{r}$)-triangle in $\mathbf{r}$-circle of radius r to that in $\mathbf{p}$-circle of diameter p above.



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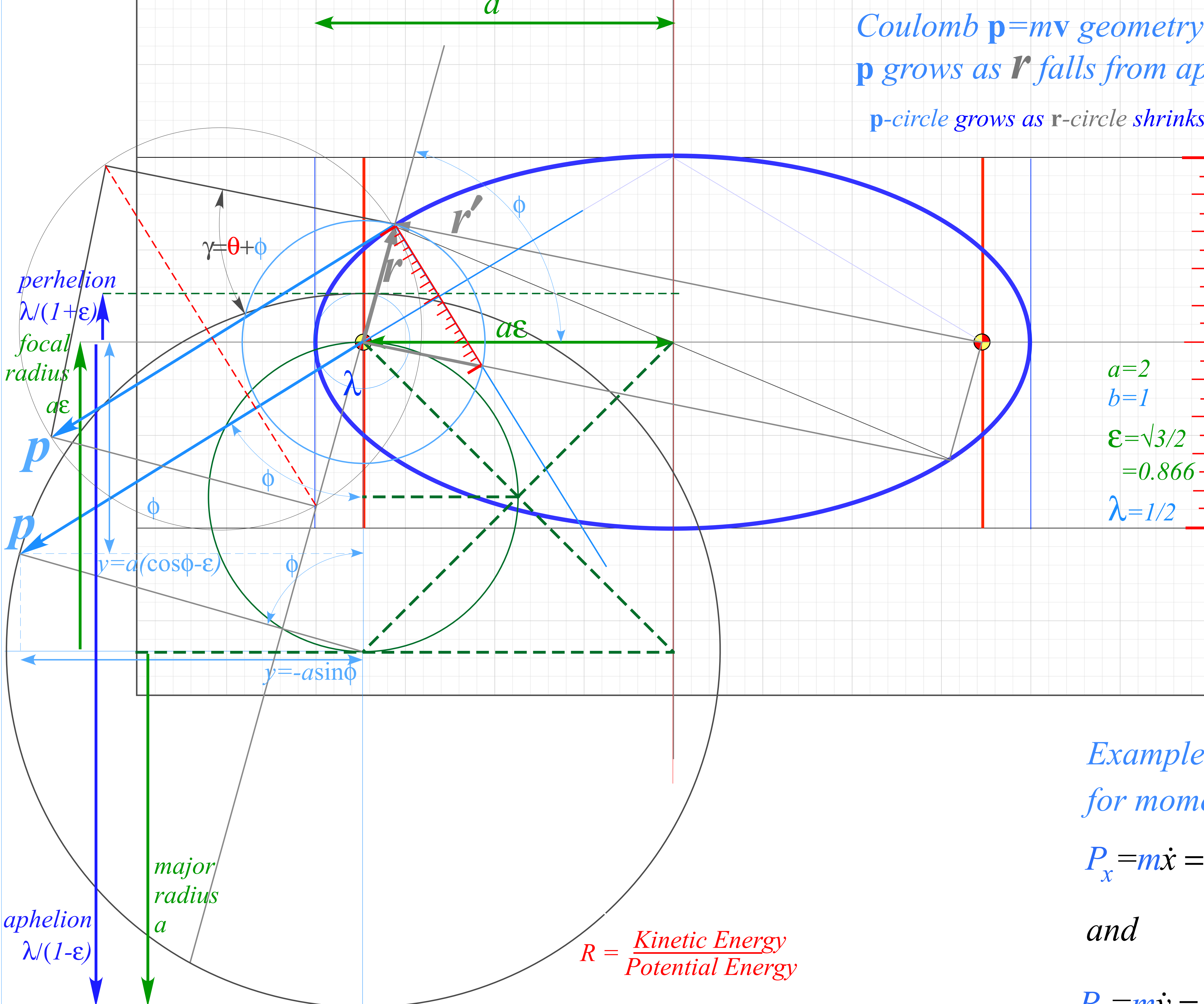

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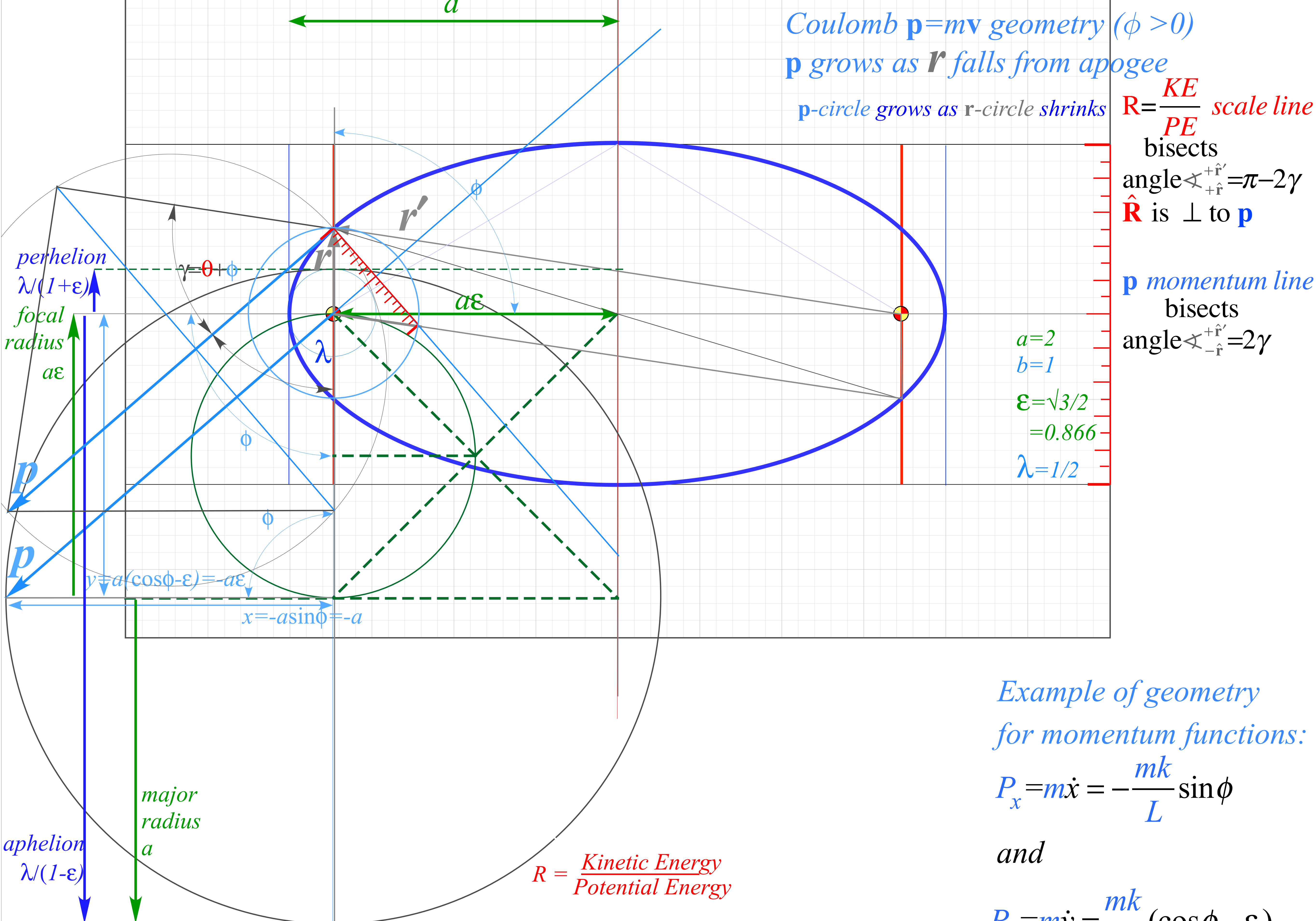

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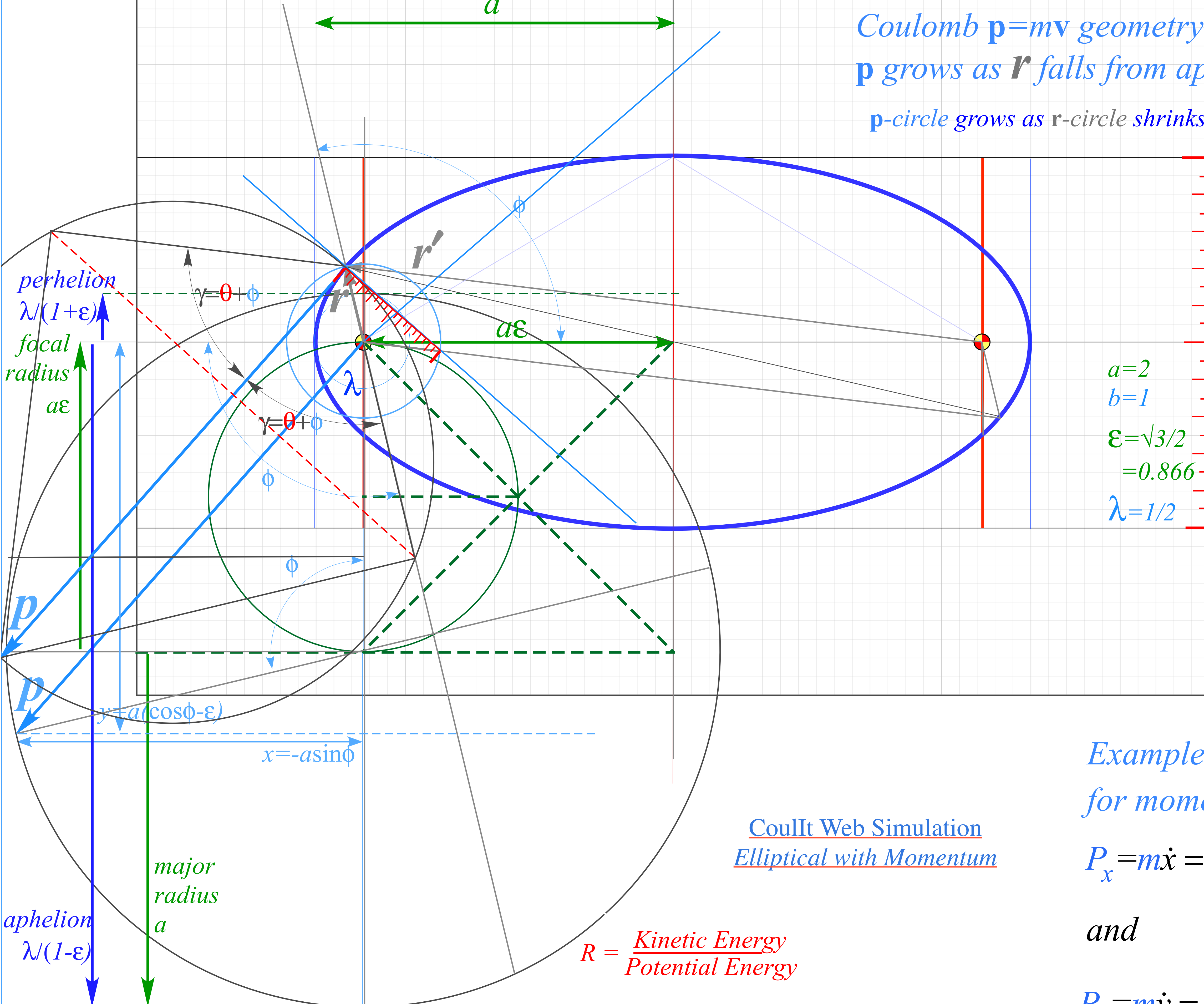
*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*

Coulomb **p**=*m***v** *geometry (ϕ >0)*

p *grows as* **r** *falls from apogee*



$\mathrm{R}=\dfrac{KE}{PE}$ *scale line* bisects angle$\sphericalangle^{+\hat{r}'}_{+\hat{r}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is $\perp$ to **p**

p *momentum line* bisects angle$\sphericalangle^{+\hat{r}'}_{-\hat{r}}=2\gamma$

Example of geometry for momentum functions:

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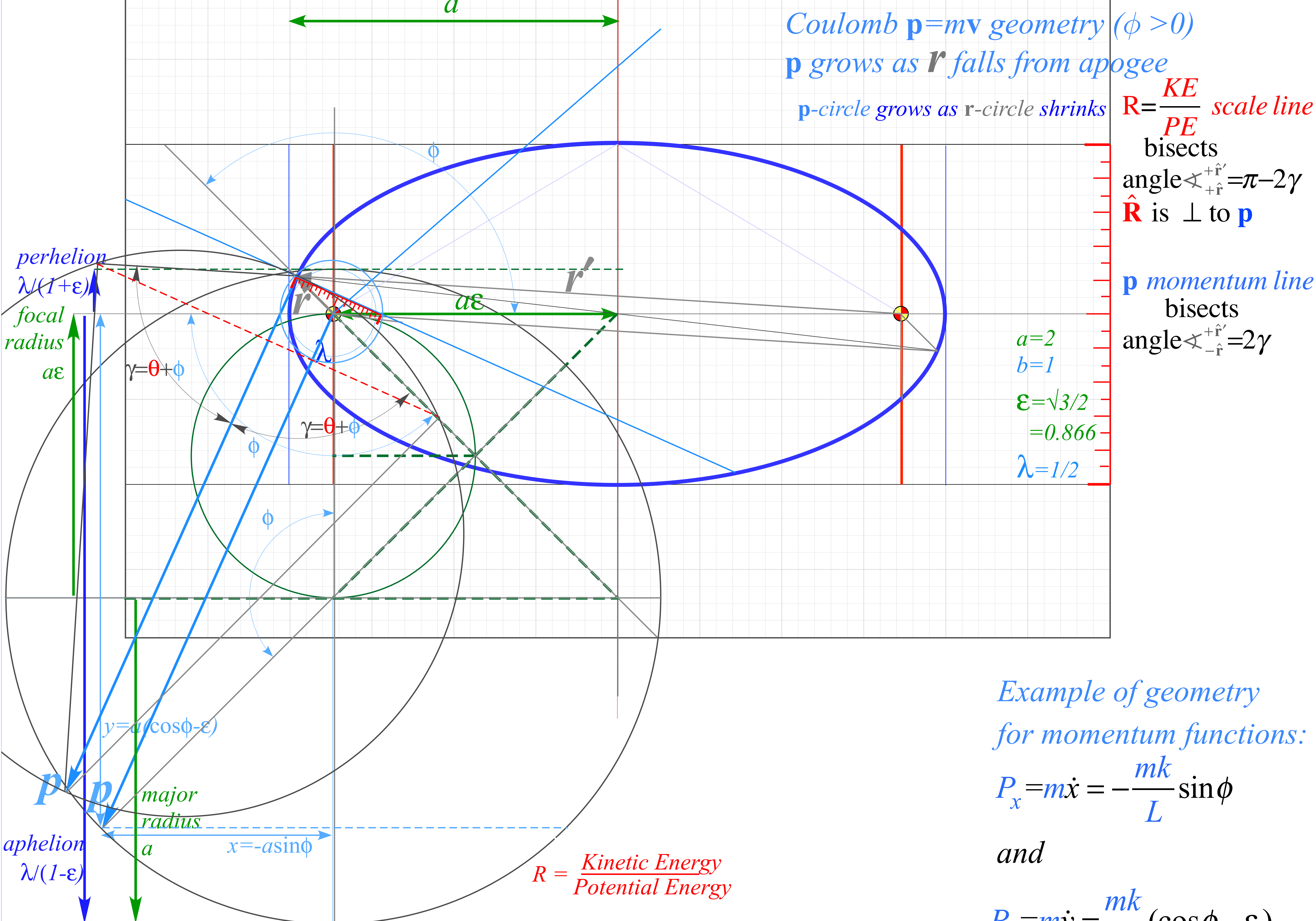

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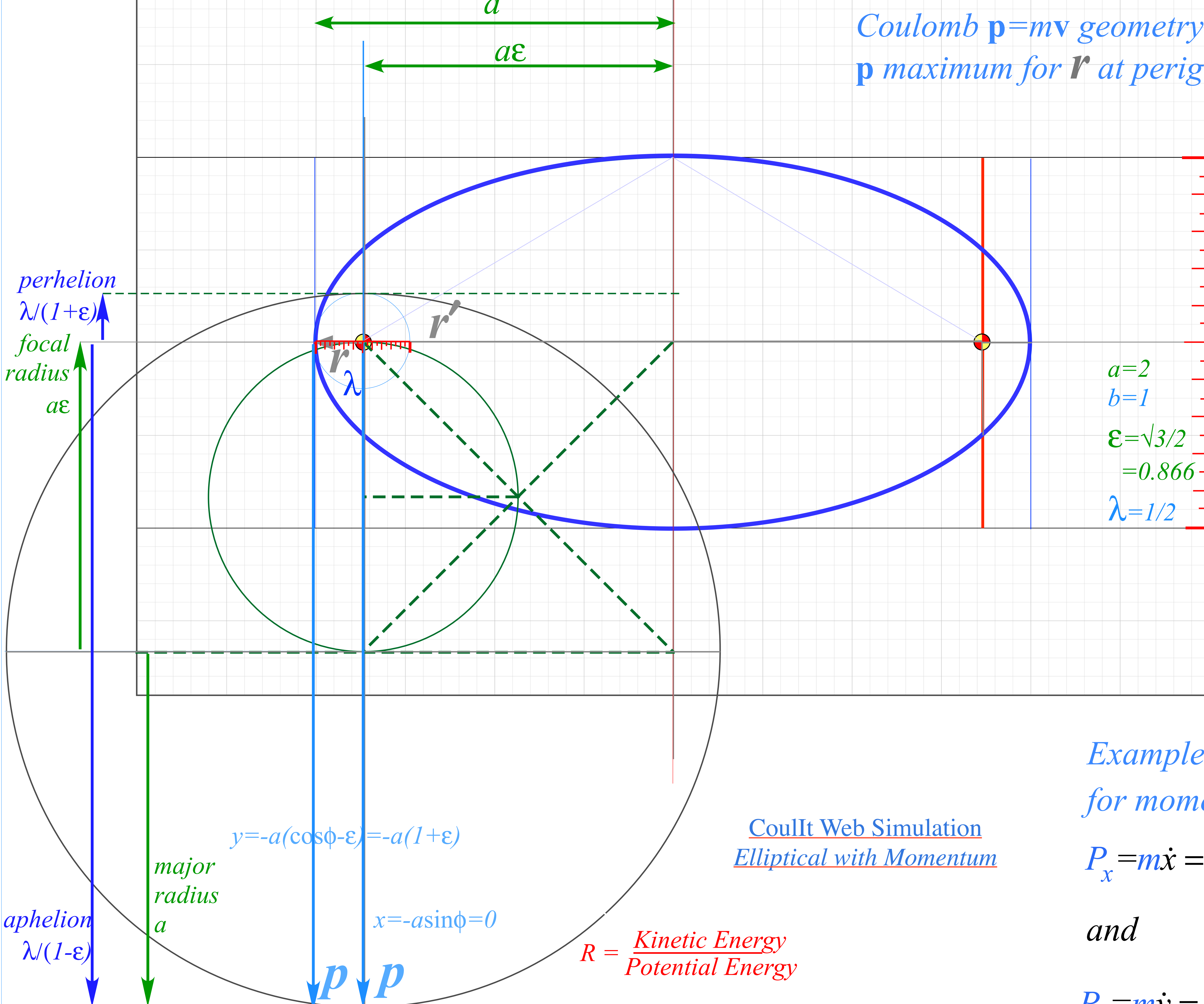
Coulomb **p**=m**v** geometry (ϕ >0)
p maximum for **r** at perigee



CoulIt Web Simulation
Elliptical with Momentum

$R = \dfrac{\text{Kinetic Energy}}{\text{Potential Energy}}$

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Derivation of **ε***-construction by analytic geometry*

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Example of complete (**r**,**p**)*-geometry of elliptical orbit*

➔ *Connection formulas for* (γ,*R*)*-parameters with (a,b) and* (ε,λ)

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:

1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)

Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1)$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \ (\varepsilon > 1)$$

Algebra of ε-construction geometry

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$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
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$$\lambda=\frac{b^2}{a}=\mp 2rR\sin^2\gamma$$

Algebra of $\boldsymbol{\varepsilon}$*-construction geometry*

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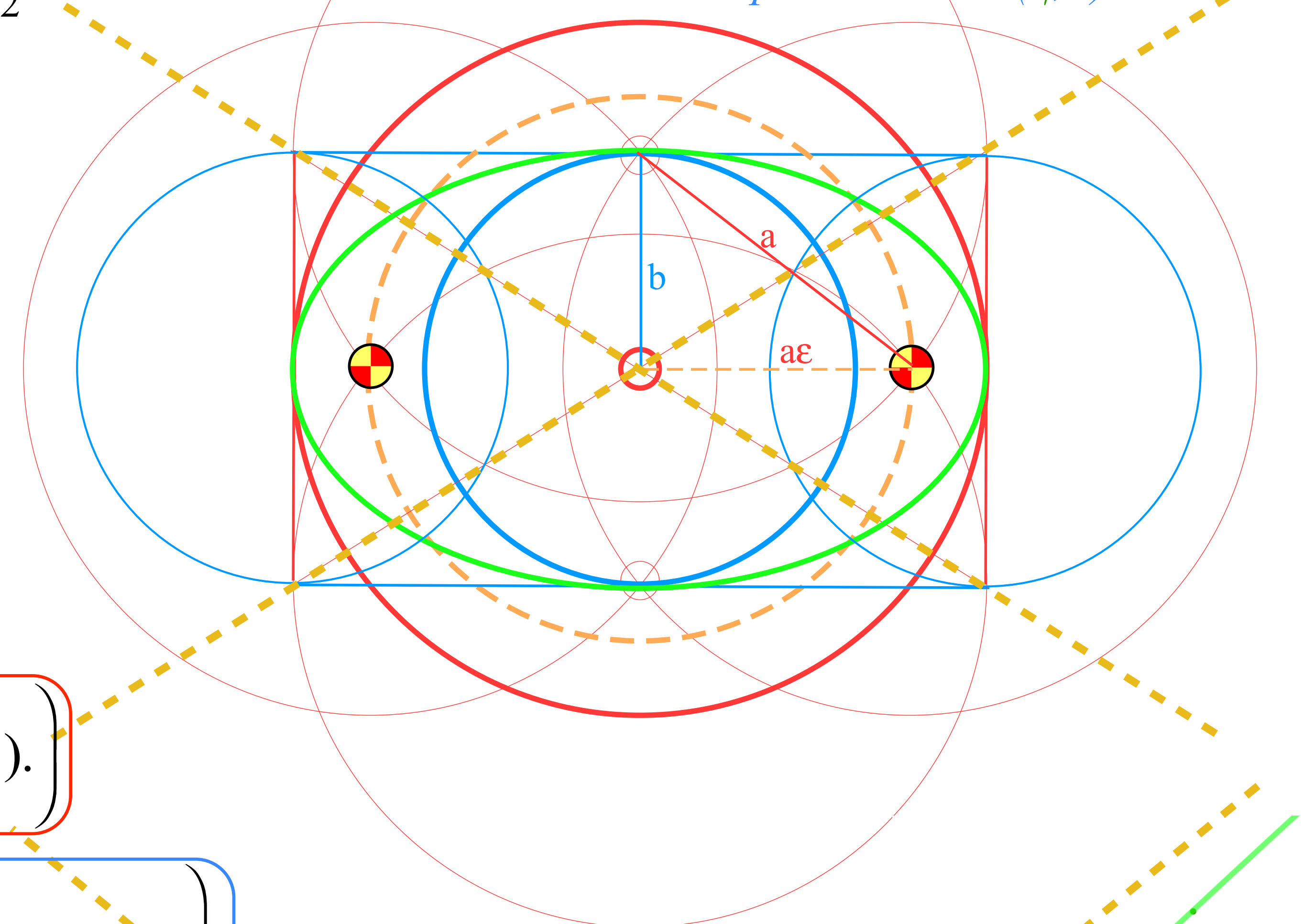
$$=1+\frac{b^2}{a^2}\ \text{hyperbola}\ (\varepsilon>1) \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}$$



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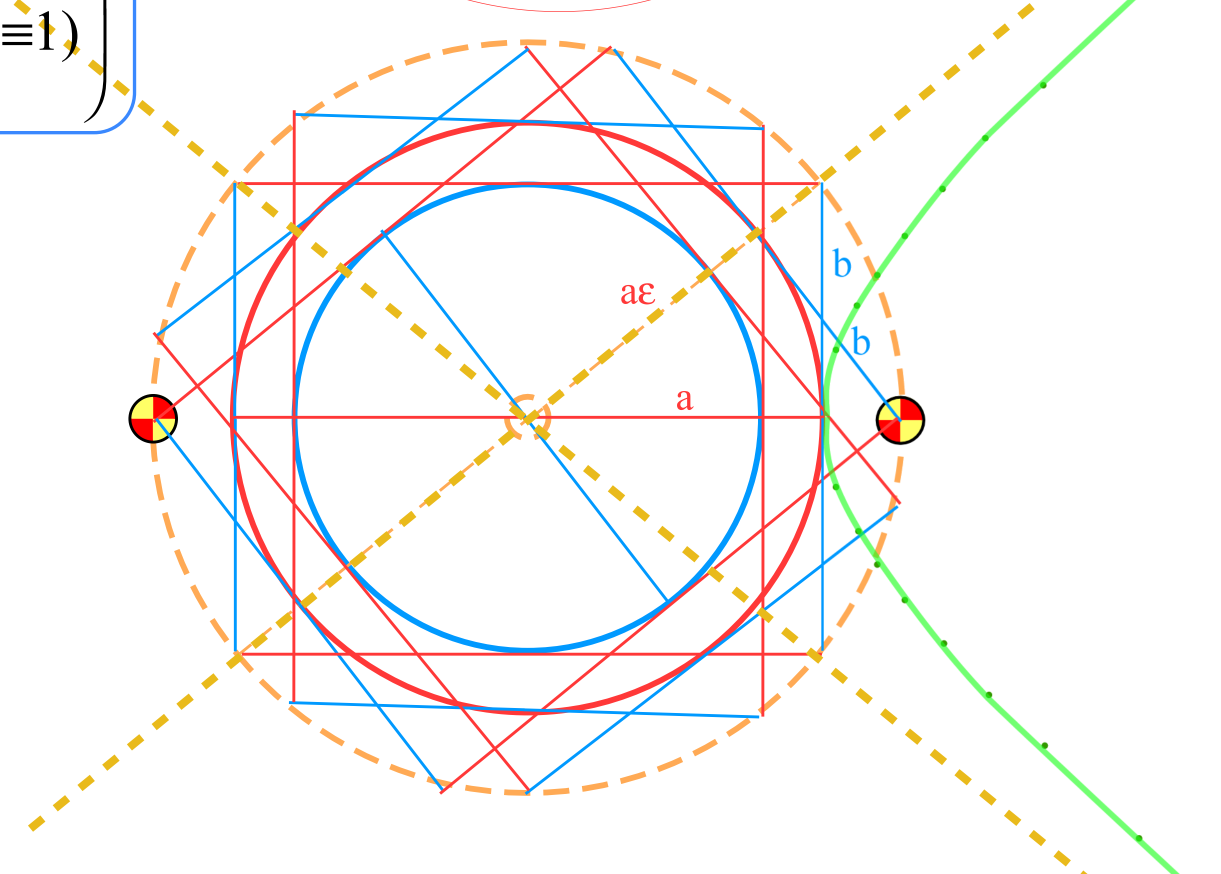
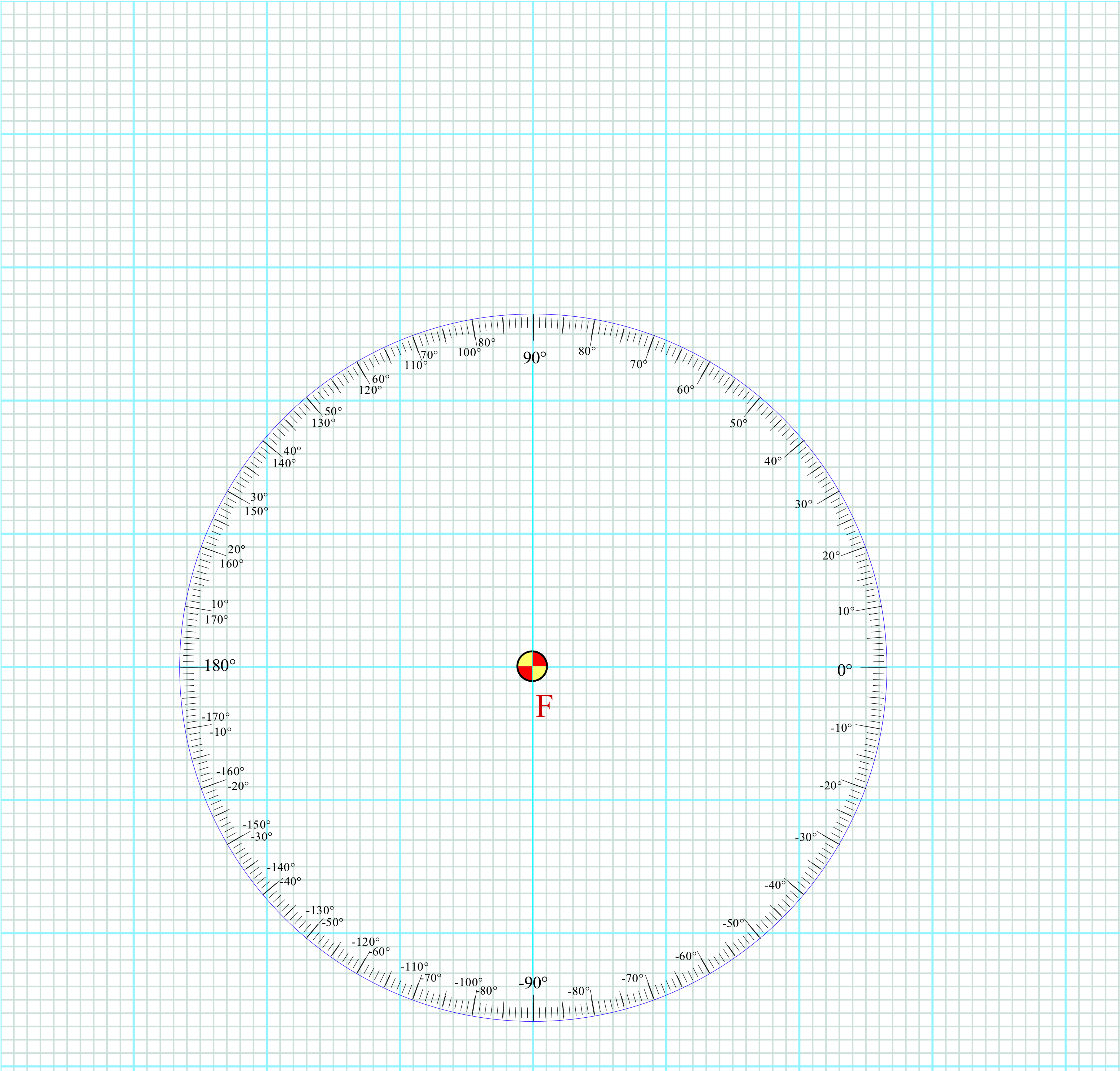
From ε^2 result (at top):

$$\frac{b}{a}=2\sqrt{\mp R(R+1)}\sin\gamma=\sqrt{\pm(1-\varepsilon^2)}$$

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Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) and (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

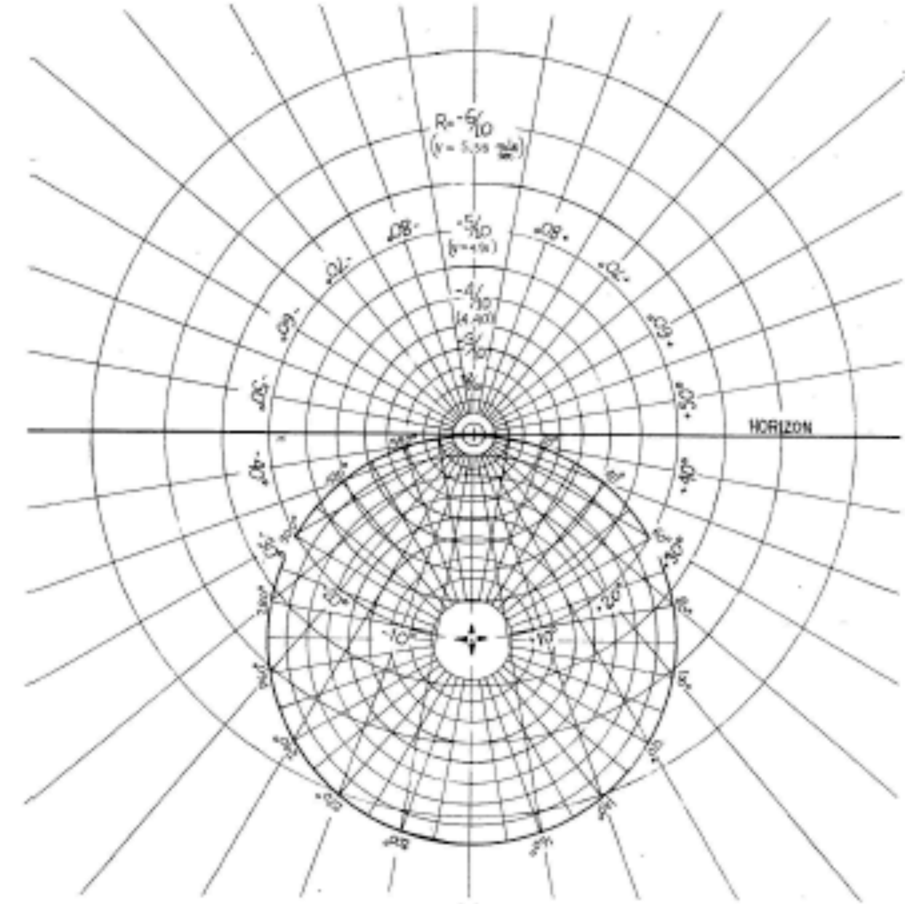
Properties of Coulomb trajectory families and envelopes

Graphical $\boldsymbol{\varepsilon}$*-development of orbits*

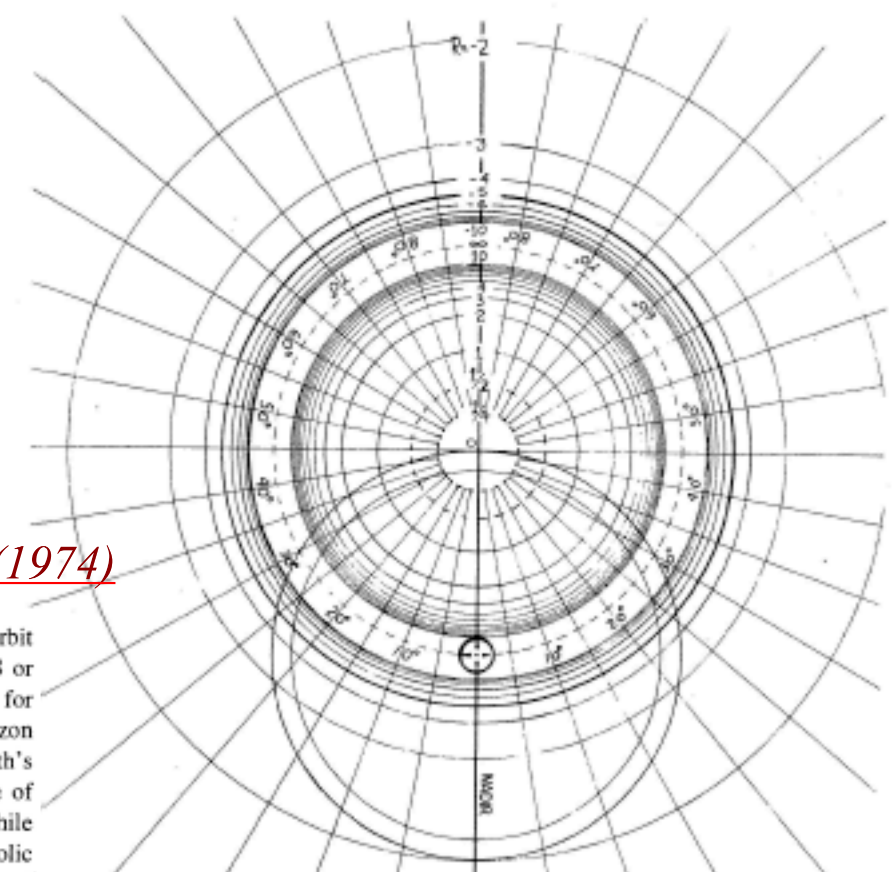
Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

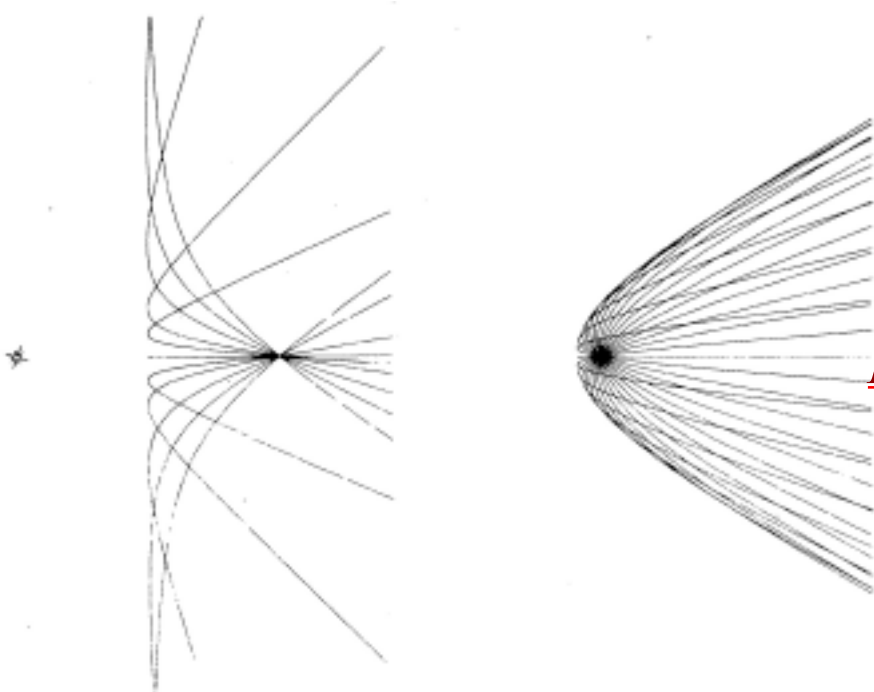
Launch optimization and orbit family envelopes



(a)



(b)



(a) (b)

Fig. 11. Sample computer trajectories. (a) Family of hyperbolic orbits with $R = 1$. (b) Family of hyperbolic orbits with $R < 1$.

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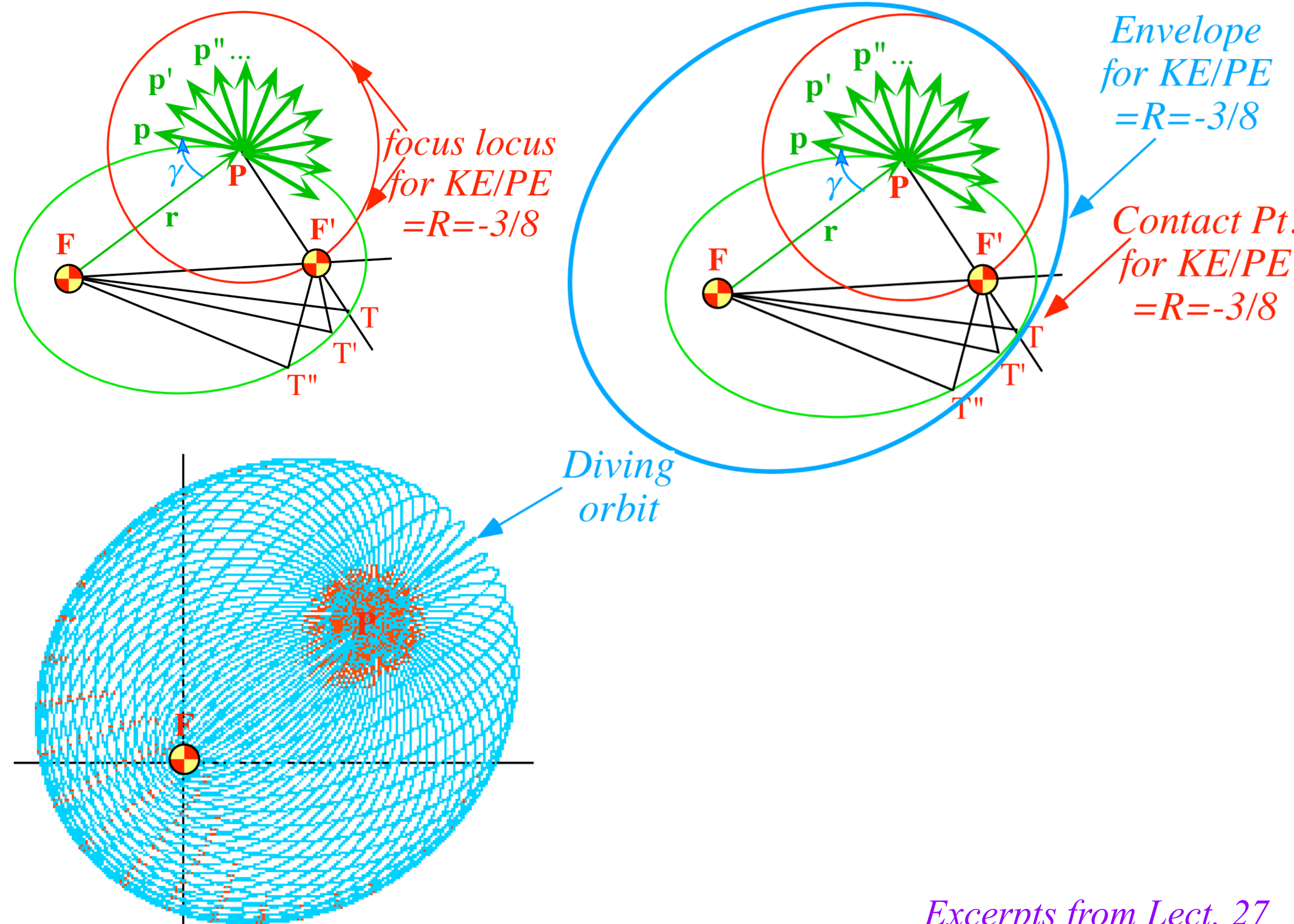
Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

Excerpts from Lect. 27



Excerpts from Lect. 27

Coulomb envelope geometry

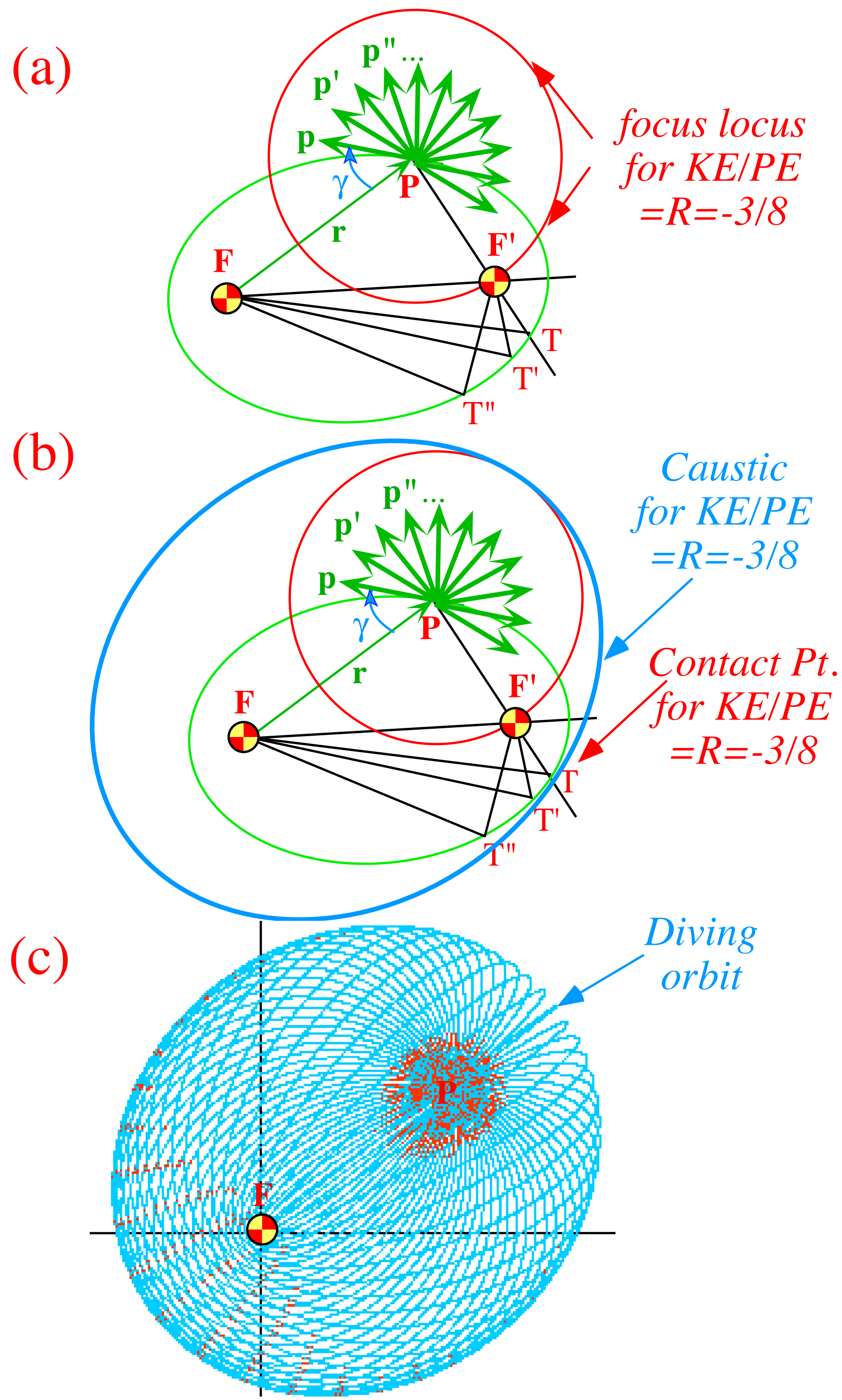
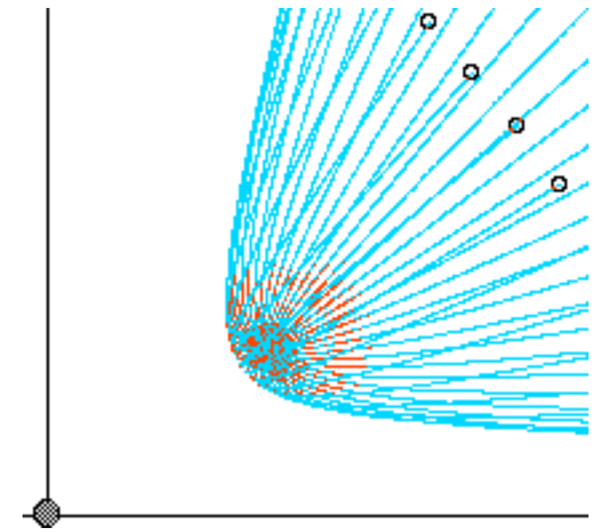


Fig. 5.4.4 in Unit 5 of CMwBANG!



Ideal comet "heads" or "tails" in solar wind

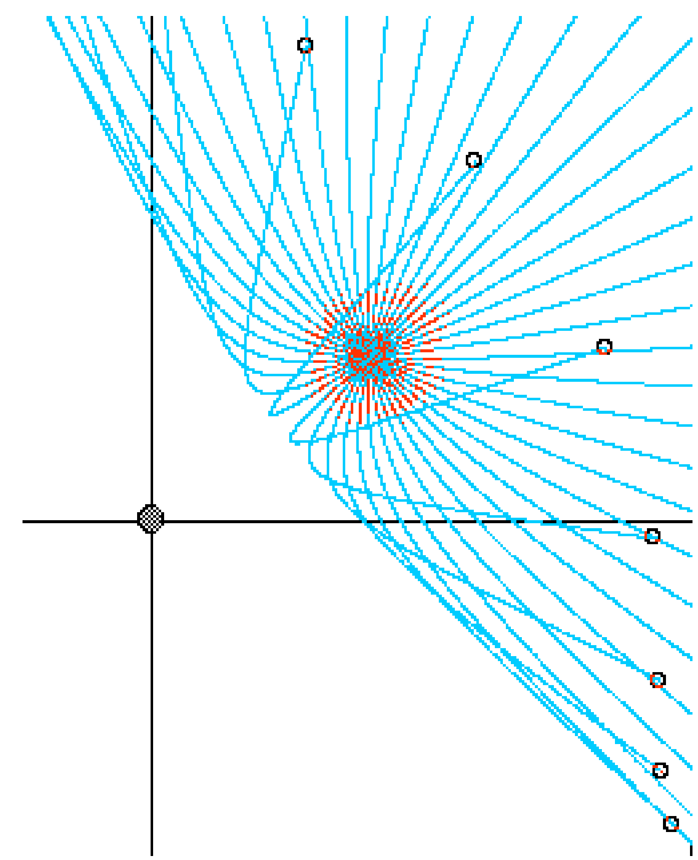
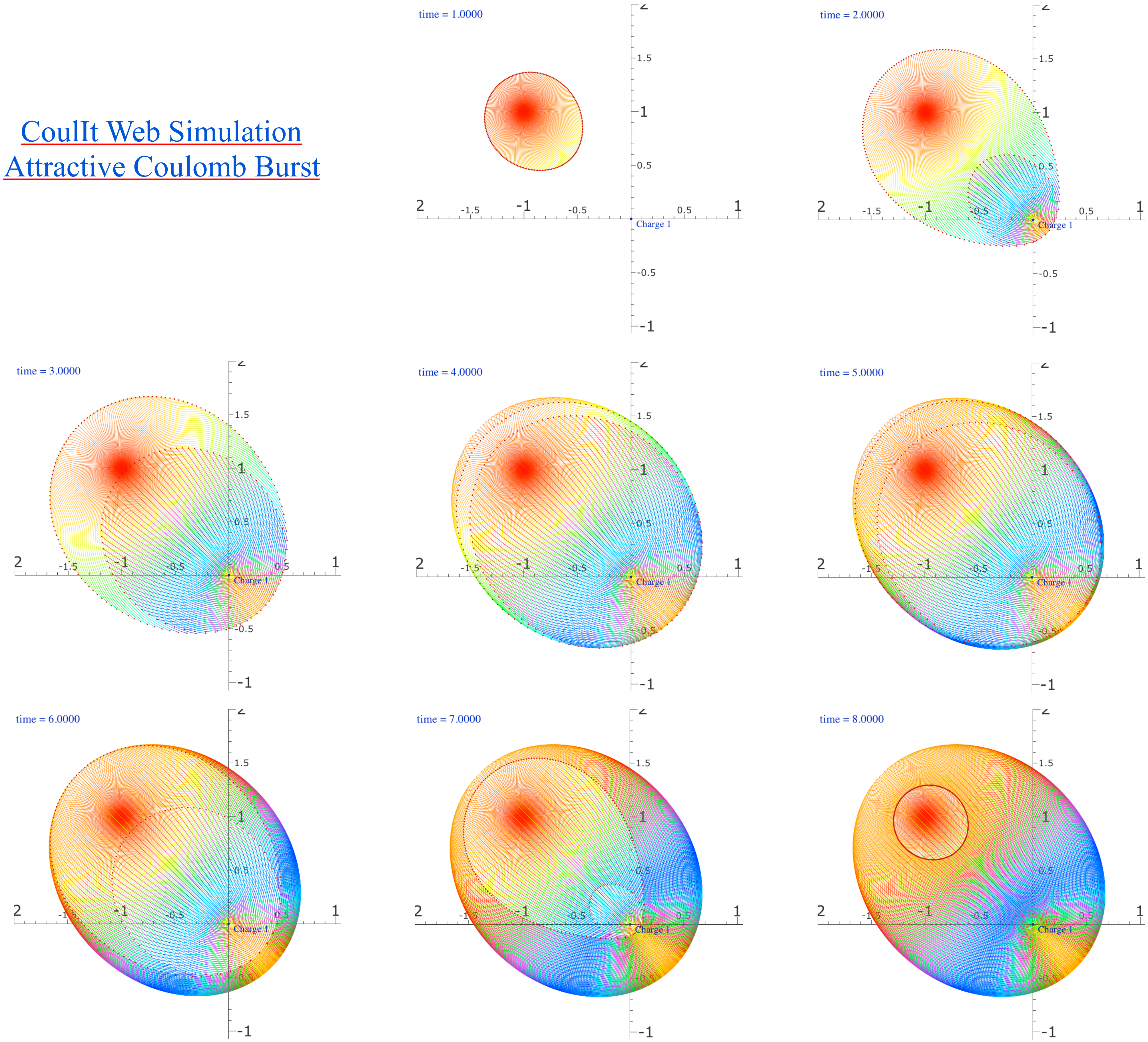


Fig. 5.4.5 in Unit 5 of CMwBANG!

Excerpts from Lect. 27

[CoulIt Web Simulation
Attractive Coulomb Burst](#)



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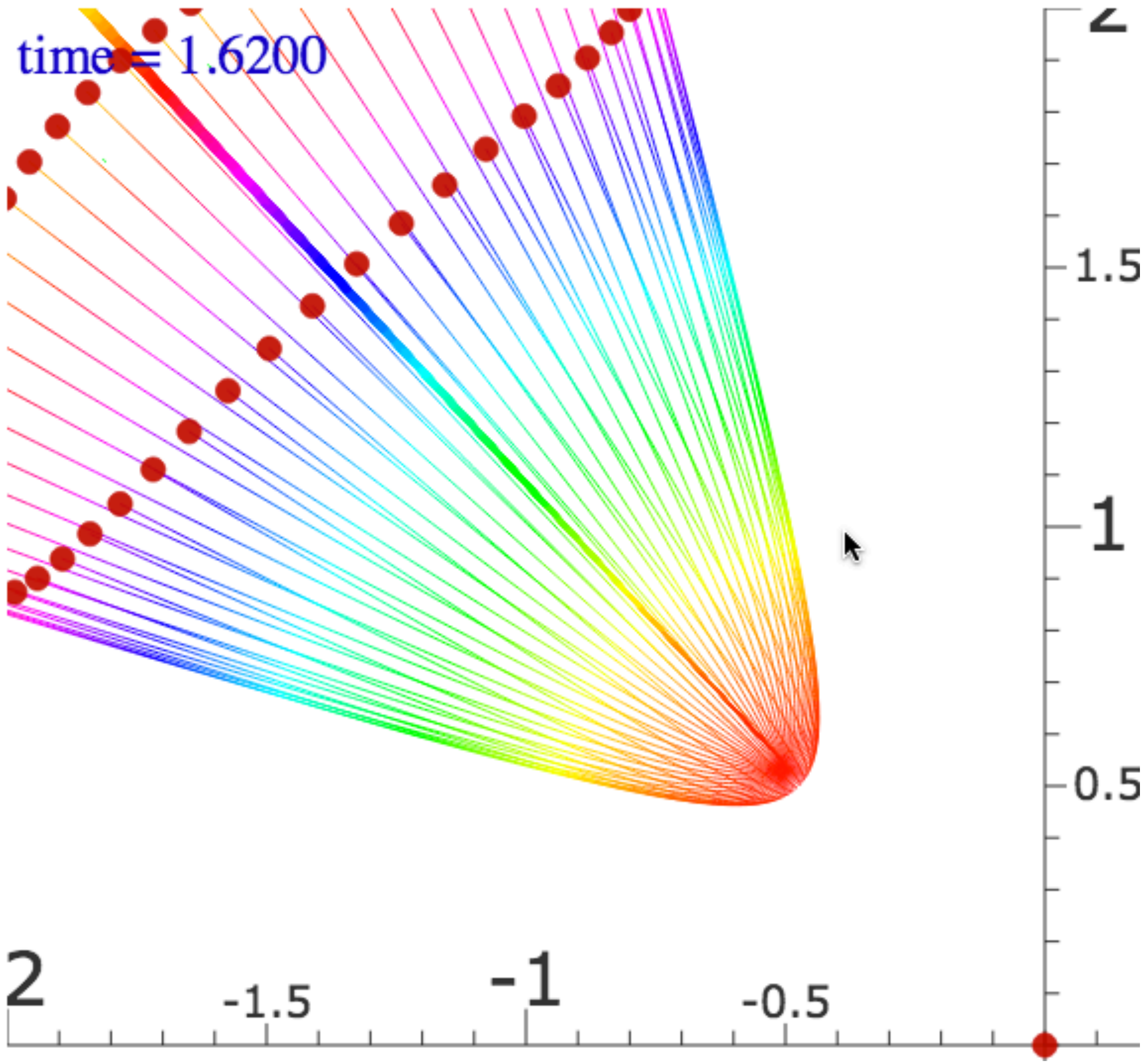
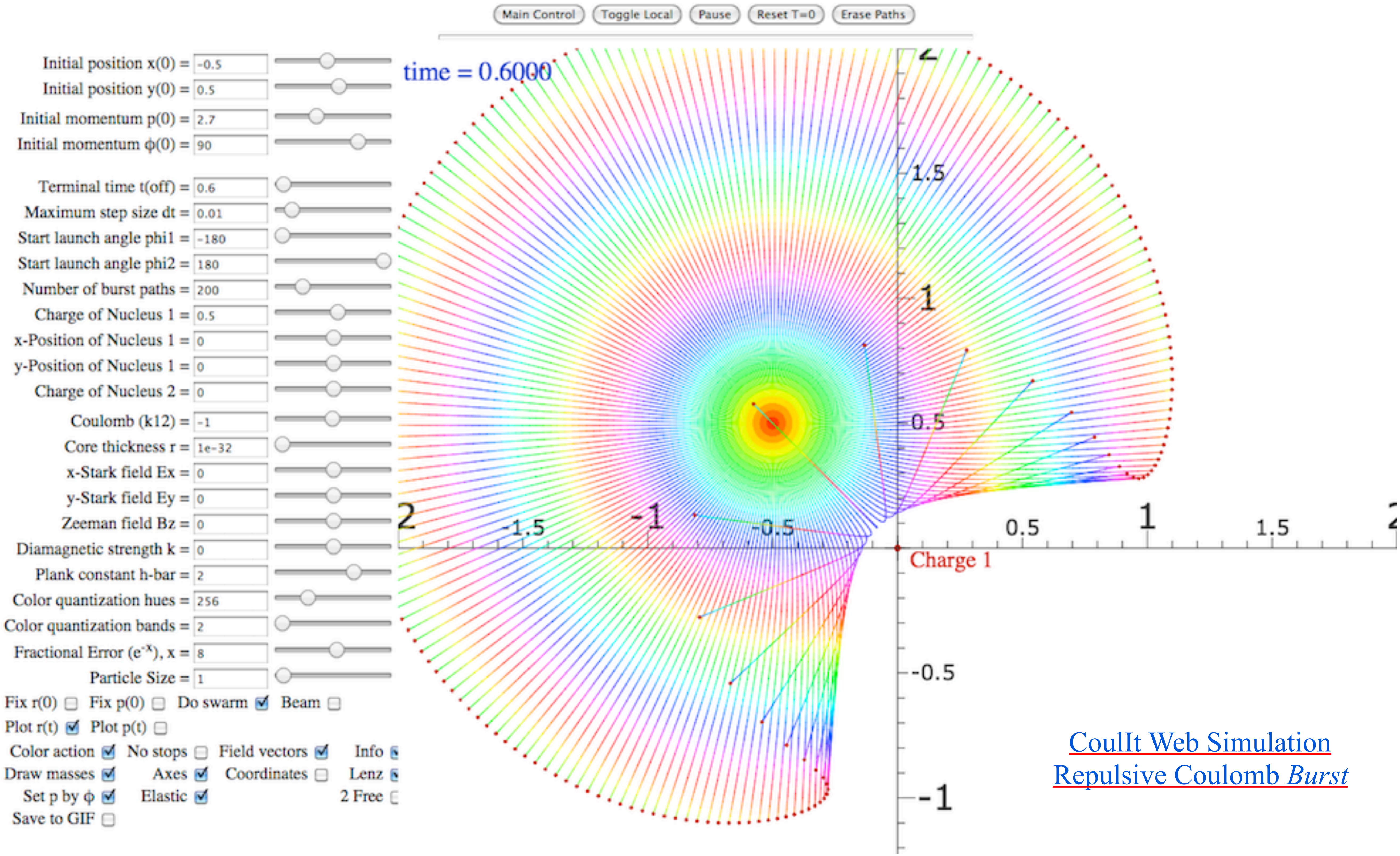


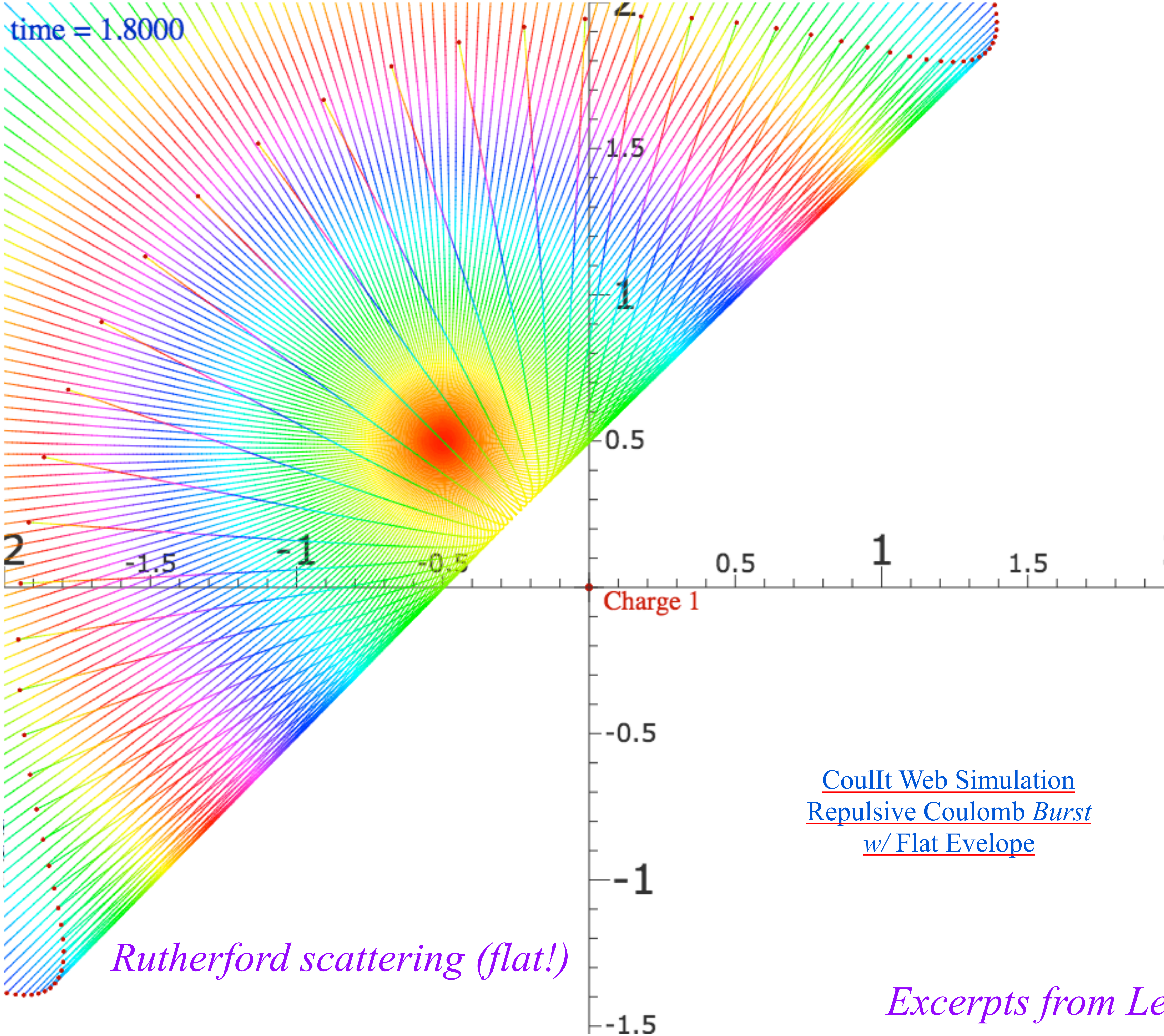
CoulIt Web Simulation Repulsive
Coulomb Burst - Tight



[CoulIt Web Simulation](#)
[Repulsive Coulomb *Burst*](#)

Rutherford scattering (roughly!)

Excerpts from Lect. 27



Rutherford scattering (flat!)

Excerpts from Lect. 27

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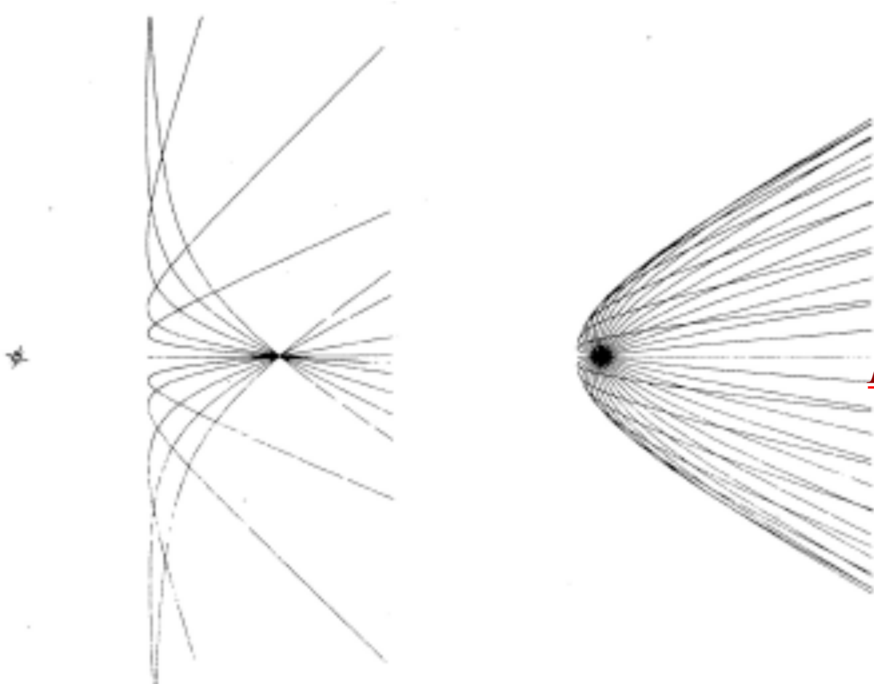

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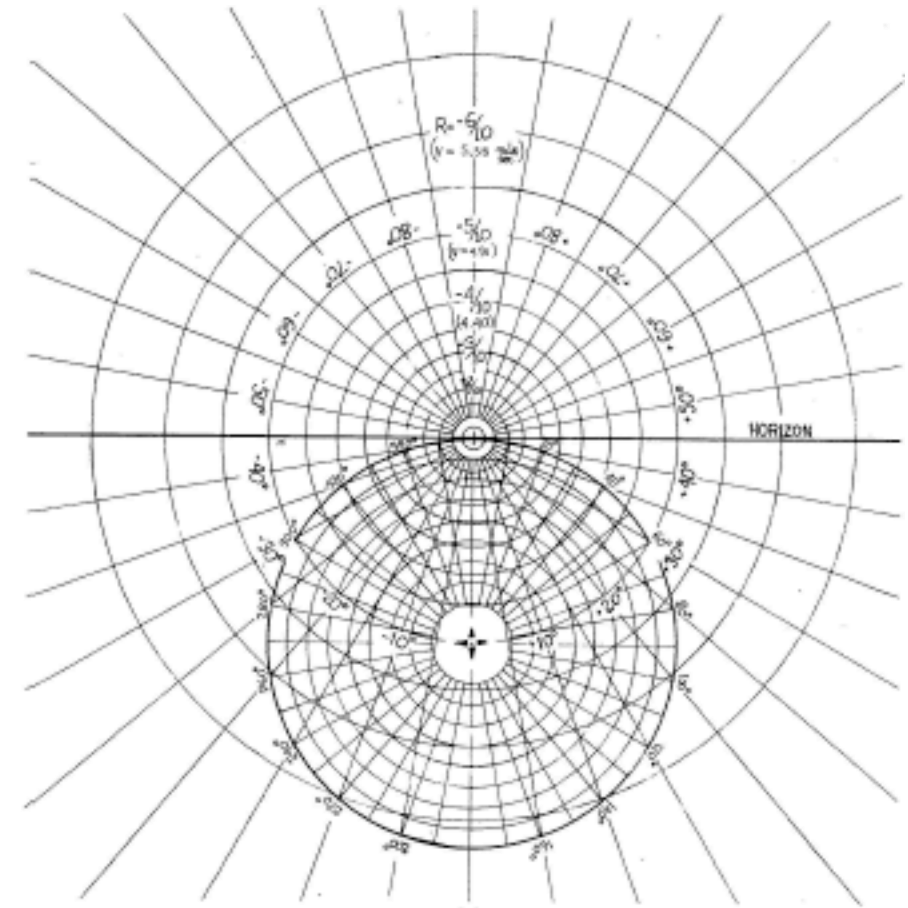
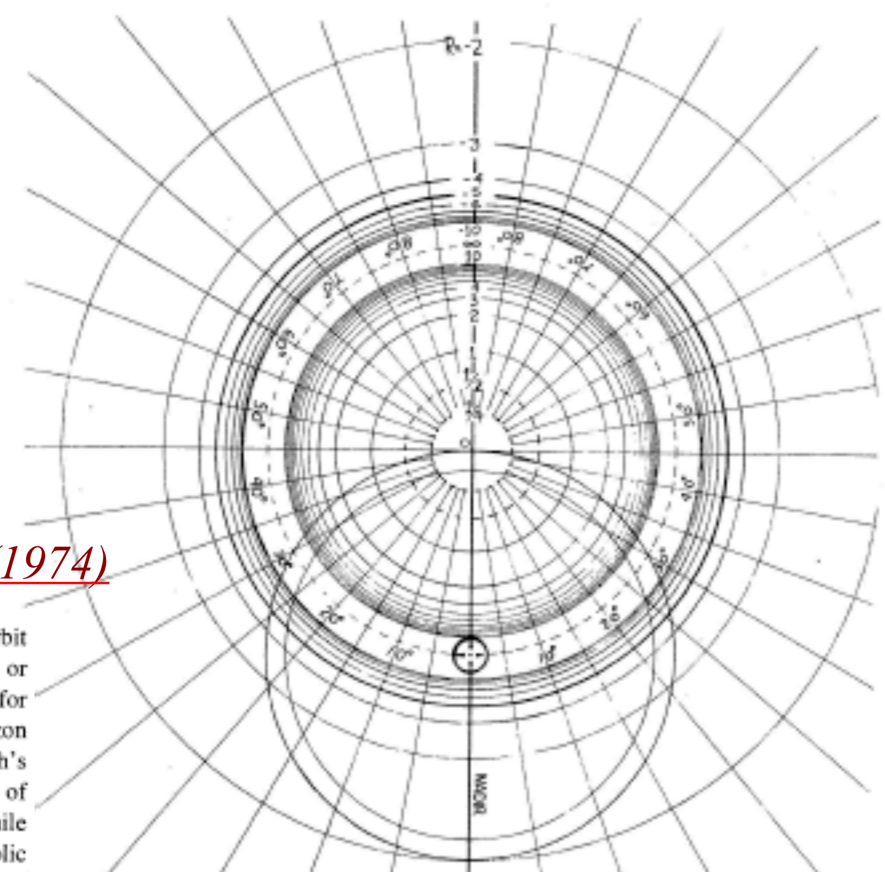


(a)



(b)

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The Lenz vector and orbital analog computers*

W. G. Harter
Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas, Campinas, Sao Paulo, Brasil
(Received 31 March 1975; revised 10 June 1975)

A single geometrical diagram based on the Lenz vector shows the qualitative and quantitative features of all three types of Coulomb orbits. A simple analog computer can be made for an overhead projector by using this theory, and a number of interesting effects can be efficiently demonstrated.

I. INTRODUCTION: THE ECCENTRICITY VECTOR

Occasionally, a geometrical construction and accompanying picture is worth a great deal more to the physicist or the physics student than pages of equations and solutions, especially now that computer graphics are so available. Since Newton's time the geometrical approach has come to be regarded as more clumsy than other methods of thought, and some very pretty pictures and proofs of physical phenomena have undoubtedly been lost. An example of such a construction involving Rutherford scattering was discussed in a recent article[1] by my students and myself, and the following is an improvement of this which describes general Coulombic orbit mechanics.

The generalization we shall describe below is based partly on a more recently discovered quantity called the Lenz–Runge vector[2,3] or the "eccentricity" vector ϵ defined by Eq. (1). There $\mathbf{r}$ is the position vector of the orbiting particle, $\mathbf{L}$ is its angular momentum, $\mathbf{p}$ is its linear momentum, m is its mass, and k is the gravitational (or electrostatic) coefficient:

$$\epsilon = \mathbf{r}/r - \mathbf{L}\times\mathbf{p}/km. \tag{1}$$

Lately this quantity has received a flurry of attention in group theoretical studies of the hydrogen atom[4]; however, we shall use only its geometrical and classical properties.

In particular, the main property of ϵ is that it is a constant vector for any particle moving according to a Coulomb field. Vector ϵ points along the major axis of ellipse, parabola, or hyperbola, whichever is the appropriate orbit of the particle. Furthermore, the magnitude ϵ of this vector is the eccentricity of the orbit.

To show that this is consistent with the usual formulation, we take the dot product of this vector ϵ with the position $\mathbf{r}$ as in Eq. (2). This then reduces to the following equation (3) of a conic section in polar coordinates, which is the general orbit equation[5]:

$$\begin{aligned} \epsilon r\cos\theta &= \epsilon\cdot\mathbf{r} = r - \mathbf{L}\times\mathbf{p}\cdot\mathbf{r}/km \qquad (2)\\ &= r + \mathbf{L}\cdot\mathbf{L}/km,\\ r &= -(L^2/km)(1-\epsilon\cos\theta)^{-1}. \qquad (3)\end{aligned}$$

In Sec. II a simple geometric construction using these properties is shown to describe qualitatively and quantitatively the Coulomb orbits for all three cases: namely, the attractive case ($k < 0$) with positive energy, with negative energy, and the repulsive case ($k > 0$).

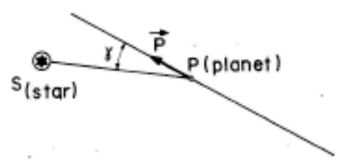

Fig. 1. Initial position and momentum must be given before construction of the resulting orbit is possible.

Finally, it is shown how this construction leads to an analog computer of orbits that can be made for a few dollars to fit onto an overhead projector. This device can be appreciated by elementary classes (even *large* elementary classes if you use the right projector) when they know only a little about conic sections, since the more tedious mathematics is built into the device.

II. COMPUTING ORBITS BY RULER AND COMPASS

We start by simply listing three steps of an orbit construction while demonstrating their application to a particular case of a satellite orbiting a star. Then a general proof of the steps will be provided along with further discussion and applications.

Suppose you are given the initial position and velocity of a satellite relative to some very massive star. If these quantities are given in a pictorial form which shows the angle γ between momentum vector vector $\mathbf{p} = m\mathbf{v}$ and the radius line PS in Fig. 1, and if the magnitude of $\mathbf{v}$ is given by the ratio $R = T/V$ of the kinetic energy ($T = mv^2/2$) to the potential energy ($V = k/r$), then the construction below proceeds immediately. Otherwise, these quantities must be calculated before proceeding. (In the potential energy of the star's gravity we have $k = -GMm$, where M is the star mass and G is the universal constant of gravitation.) Note that R is minus the squared ratio of initial velocity to the escape velocity in

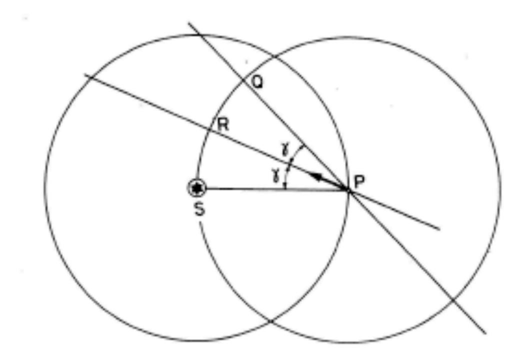

Fig. 2. Doubling the angle between the momentum and the position vectors gives a line QP which must contain the orbit focus.

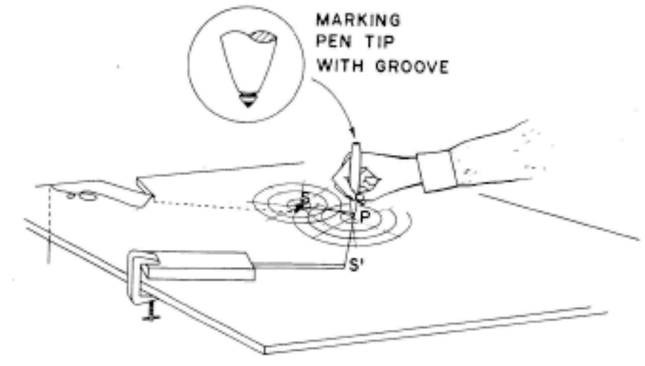

Fig. 8. Orbital computer design: cheaper model that computes elliptical orbits only using the scale of Fig. 7(a). A Plexiglas sheet that is about ⅛ in. thick has a string hole at the orbit's center. A transparency (Xerox, 3-M, etc.) of Fig. 7(a) is taped in position on the underside. (*Caution:* One should avoid marking pens that permanently mark plastic.)

8 go with Fig. 7(a) and can be assembled in a few minutes by using odds and ends. A more sophisticated apparatus is shown in Fig. 9. The simple apparatus of Fig. 8 produces the well-known elliptical orbits and trajectories of planets and satellites, but not the hyperbolic trajectories characteristic of higher-than-escape velocity meteors or of the repulsive Coulomb force problems. The second apparatus (Fig. 9) is designed to handle all cases, provided that the appropriate focal point scale is inserted.

The operation of either plotter begins with the positioning of second focal point S′ according to the scale on the plotting board. Then the marking pen is poked into a small identation at P and held while the strings to S and S′ are tightened. Finally, you slide the pen out along the board in such a way that the strings stay tight and the desired trajectory is drawn.

The apparatus in Fig. 8 will thus make an ellipse since the sum of distances SP and S′P is constant. The apparatus in Fig. 9 does the same when the spool brakes are

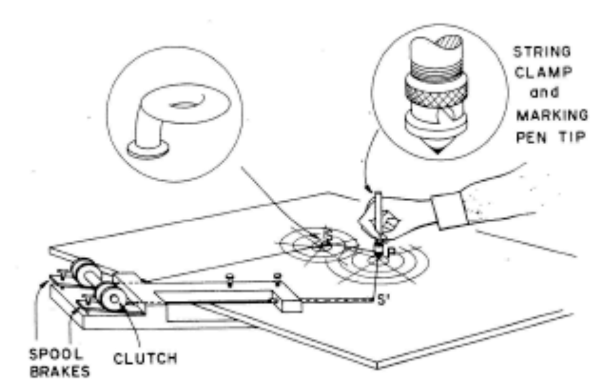

Fig. 9. Orbital computer design: a more elaborate model that computes general Coulomb orbits. The ellipse drawing mode is obtained when, first, the clamp is opened to allow the string to slide and then the spool brakes are tightened after the initial adjustment has been made with the use of the scale in Fig. 7(a). The hyperbola drawing mode is obtained when the string clamp and spool clutch are tightened but the brakes are released. The spools must turn together after the initial adjustment has been made with the use of the scale in Fig. 7(b). One hand can maintain the string tension while drawing the orbit, and the other hand can control the paying out of string. (Alternately, springs on the spool axis accomplish the same thing.)

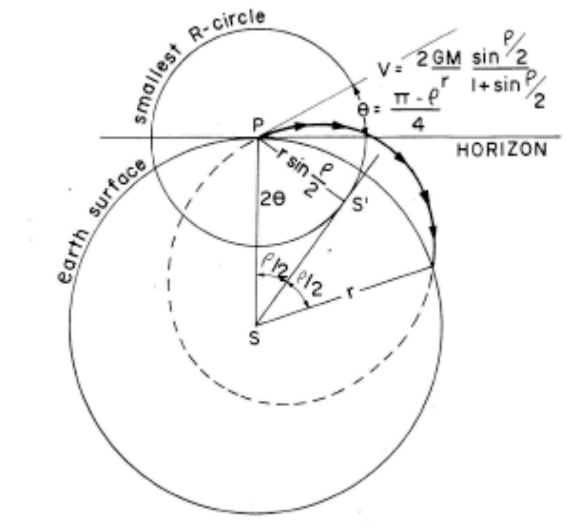

Fig. 10. Sample computer trajectory problem. One finds the minimum-energy trajectory for a given range ρ. Initial velocity v and θ follow easily from the geometry of the computer scale in Fig. 7(a).

tightened and the string clamp on the marking pen is loosened.

The apparatus in Fig. 9 will produce a hyperbola if the *difference* between distances SP and S′P is constant. This is accomplished by tightening the string clamp and the clutch so that two string spools reel equal amounts of string in or out and the constant difference in length is maintained.

IV. SOME USES FOR COMPUTERS

For the computer to be set up to draw elliptical orbits, there is one important question that can be answered immediately: To throw with *minimum* initial velocity a free falling spacecraft between two fixed points near the earth and a distance ρ apart, what initial angle θ and speed v are needed? (We imagine a fixed coordinate system here, not a rotating one.)

Measuring this range by a great circle angle ρ, we see that the focus of the orbit must be on a line through the Earth's center, making an angle $\rho/2$ with the launch point P. The smallest R circle [recall Fig. 7(a)] intersecting this line is the one tangent to it and represents the solution to the problem. Indeed, the algebraic solution to this problem follows from the diagram in Fig. 10 and is given there. Note that the angle θ approaches 45° as the range becomes small compared to the radius of the Earth.

Note that one may change the radius of the starting point P by simply reinterpreting the scale of the computer. For example, if the starting point is located at a height of, say, four times the Earth's radius, then the velocities marked on this scale are all divided by the square root of this factor, in this case, by 2.

With the computer set up to draw hyperbolic orbits and the appropriate scales available, there are a number of interesting problems to examine. For example, the attractive-field positive-energy scale allows one to exhibit the paths of meteorites. Given the impact direction and speed, one can extrapolate to find its origin.

The hyperbolic computer setup can be used to demonstrate Rutherford scattering for either the repulsive field (see Ref. 1) or the attractive field. At the same time