Lecture 24
Mon. 11.18.2019

Parametric Resonance and Multi-particle Wave Modes

(Ch. 7-8 of Unit 4   11.24.17)

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)
- *Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)*
- *Schrodinger wave equation related to Parametric resonance dynamics*
  - *Electronic band theory and analogous mechanics*

Wave resonance in cyclic symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ...)*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles

Relating C_N symmetric H and K matrices to differential wave operators

This Lecture's Reference Link Listing

Web Resources - front page | Quantum Theory for the Computer Age | 2017 Group Theory for QM

UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2018 Adv CM

Classical Mechanics with a Bang! | 2018 AMOP

Modern Physics and its Classical Foundations | **2019 Advanced Mechanics**

Lecture #22-24

In reverse order

JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap(1)
MolVibes Web App: C3vN3
WaveIt Web App:
 Dim = 3 w/Wave Components;
 Static Char Table: 6, 12, 12(b), 16, 36, 256
 Quantum Carpet with N=20: Gaussian, Boxcar
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015
QTCA Unit_5 Ch14 2013
Lester. R. Ford, Am. Math. Monthly 45,586(1938)
John Farey, Phil. Mag.(1816) Wolfram
Harter, J. Mol. Spec. 210, 166-182 (2001)
Harter, Li IMSS (2013)
Li, Harter, Chem.Phys.Letters (2015)

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
 Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
 Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
 Pure C-Type A,D=4.055, B=0, C=0.1
 Mixed AB-Type w/Cosine
 Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
 Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
 Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
 Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Select, exciting, and/or related Research

***This Indestructible NASA Camera Revealed Hidden Patterns on Jupiter** - seeker-yt-2019*
What did NASA's New Horizons discover around Pluto? - Astrum-yt-2018
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019

Classical Mechanics with a Bang! 2018
 Lectures 8, 9, 23 page 93
 Text Unit 6, page=27
ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
 and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Spectral Decomposition with Repeated Eigenvalues - 2017 GTQM - Lecture 5

Web based 3D & XR (x∈{A,M,V}, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

Recent In-House draft Articles:

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 1 - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 2 - Reimer-www-2019

Quantum Computing (*QC*) and Geometric Algebra (*GA*) references:

Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Quantum Computing for Computer Scientists - Helwer-mr-yt-2018, Slides
Quantum Computing and Workforce, Curriculum, and App Devel - Roetteler-MS-2019

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Excerpts (Page 44-47 in *Preliminary Draft*) for a GA take on the Complex Numbers
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
GA & QC references (Page 11-16 in Preliminary Draft)

In development, but close to role out.
More Advanced QM and classical references will *soon* be available through our: ***References Page***
Would be great to have our Apache SOLR *Search & Index system up for a bigger* Bang!*)*

This Lecture's Reference Link Listing

Web Resources - front page | Quantum Theory for the Computer Age | 2017 Group Theory for QM
UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2018 Adv CM
Classical Mechanics with a Bang! | 2018 AMOP
Modern Physics and its Classical Foundations | **2019 Advanced Mechanics**

Lectures #12 through #21

In reverse order

Wiki on Pafnuty Chebyshev
Nobelprize.org
2005 Physics Award

BoxIt **Web Simulations:**
A-Type w/Cosine, A-Type w/Freq ratios,
AB-Type w/Cosine, AB-Type 2:1 Freq ratio

OscillIt **Web Simulations:**
Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
Way above resonance,Underdamped
Complex Response Plot

CoulIt **Web Simulations:**
Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

Trebuchet **Web Simulations:**
Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)
Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:
A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for …
Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*
Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research
Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

An assist from *Physics Girl* (YouTube Channel):
How to Make VORTEX RINGS in a Pool
Crazy pool vortex - pg-yt-2014
Fun with Vortex Rings in the Pool - pg-yt-2014

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74, Page=75, Page=79*
Pendulum Web Sim
Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted, FVPlot*

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io
Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_* realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_, Nuclear_Collisions - Mishustin-PhysRevC-2007, APS Link & Abstract
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
$m_1:m_2 = 3:1$
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$
$m_1:m_2 = 4:1$
v2 vs v1, *y2 vs y1*
$m_1:m_2 = 100:1$, $(v_1, v_2)=(1, 0)$: *V2 vs V1 Estrangian plot*, *y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<Under Construction>>

Two Kinds of Resonance

Linear or *additive resonance*.
Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos(\omega_s t)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

Two Kinds of Resonance

Linear or *additive resonance*.
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$$\ddot{x} + \omega_0^2 x = E_s \cos(\omega_s t)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

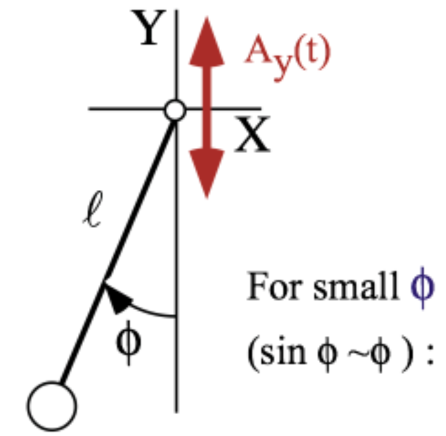
Nonlinear or *multiplicative resonance.*
Example: oscillating magnetic **B**-field is applied to a cyclotron orbit.

$$\ddot{x} + \left(\omega_0^2 + B\cos(\omega_s t)\right)x = 0$$

Chapter 4.7

Also called *parametric resonance.*
Frequency parameter or spring constant $k = m\omega^2$ is being stimulated.
...Or pendulum accelerated up and down *(model to be used here)*

Y-stimulated pendulum:
(Non-Linear Resonance)



Parametric Resonance

Two Kinds of Resonance

Linear or *additive resonance*.
Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos(\omega_s t)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

Nonlinear or *multiplicative resonance.*
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Chapter 4.7

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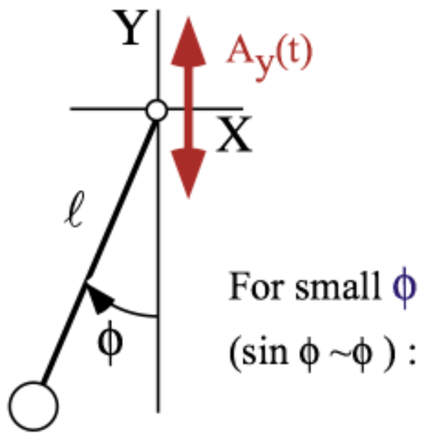


Parametric Resonance

It's quite like the Schrodinger $\psi(x)$-wave equation (except t replaces x)

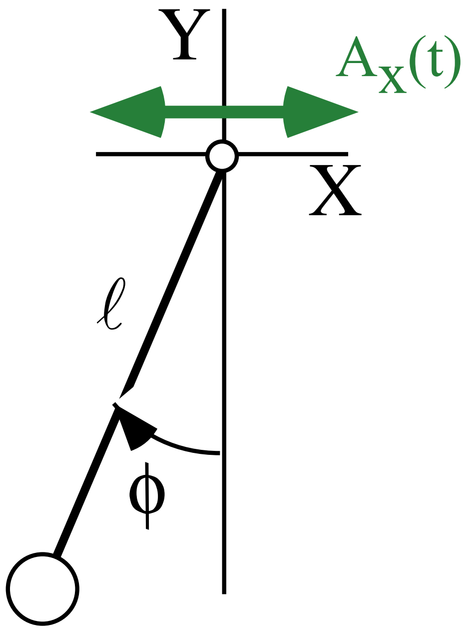
$$\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + [E - V(x)]\psi(x) = 0$$

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)
Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)
Schrodinger wave equation related to Parametric resonance dynamics
Electronic band theory and analogous mechanics

Coupled Rotation and Translation (Throwing and Trebucheting)

Early non-human (or in-human) machines: trebuchets, whips.. (3000 BCE-1542 CE)

X-stimulated pendulum:
(Quasi-Linear Resonance)



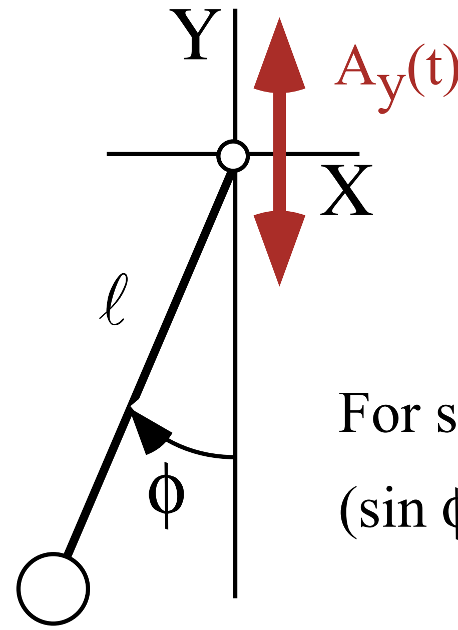
For small ϕ
($\cos\phi \sim 1$) :

Forced Harmonic Resonance

$$\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\phi = \frac{A_x(t)}{\ell}$$

A Newtonian F=Ma equation
Lorentz equation (with Γ=0)

Y-stimulated pendulum:
(Non-Linear Resonance)

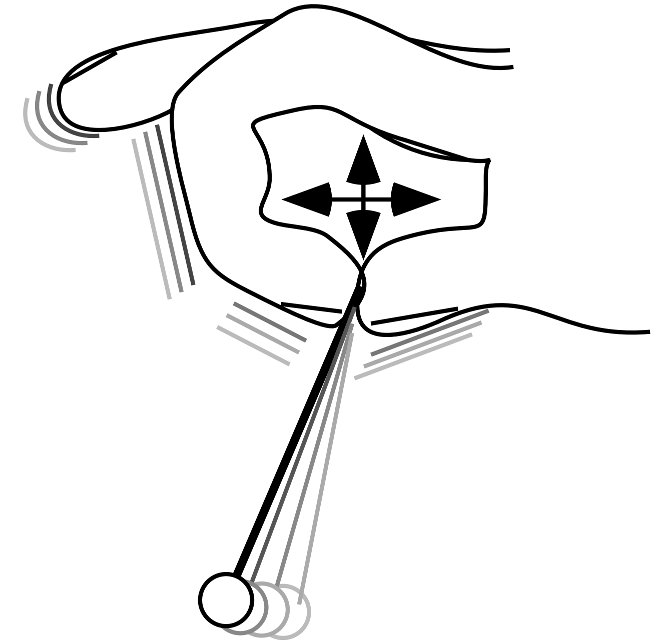


For small ϕ
($\sin\phi \sim \phi$) :

Parametric Resonance

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

A Schrodinger-like equation
(Time *t* replaces coord. *x*)



General ϕ :

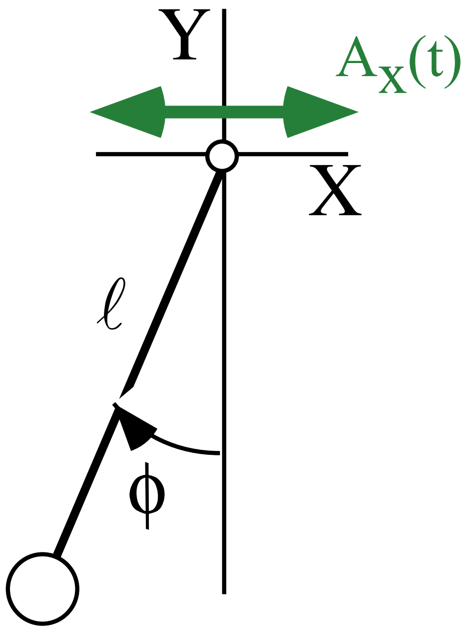
(1542-2019 CE)

General case: A Nasty equation! $\frac{d^2\phi}{dt^2} + \frac{g+A_y(t)}{\ell}\sin\phi + \frac{A_x(t)}{\ell}\cos\phi = 0$

Coupled Rotation and Translation (Throwing and Trebucheting)

Early non-human (or in-human) machines: trebuchets, whips.. (3000 BCE-1542 CE)

X-stimulated pendulum:
(Quasi-Linear Resonance)



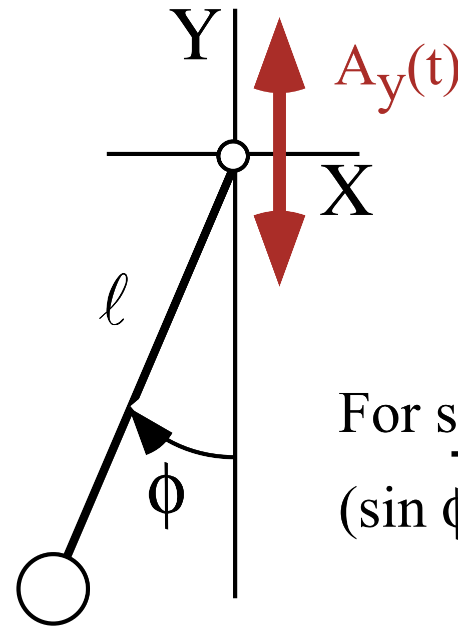
For small ϕ
($\cos\phi \sim 1$) :

Forced Harmonic Resonance

$$\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\phi = \frac{A_x(t)}{\ell}$$

A Newtonian F=Ma equation
Lorentz equation (with Γ=0)

Y-stimulated pendulum:
(Non-Linear Resonance)

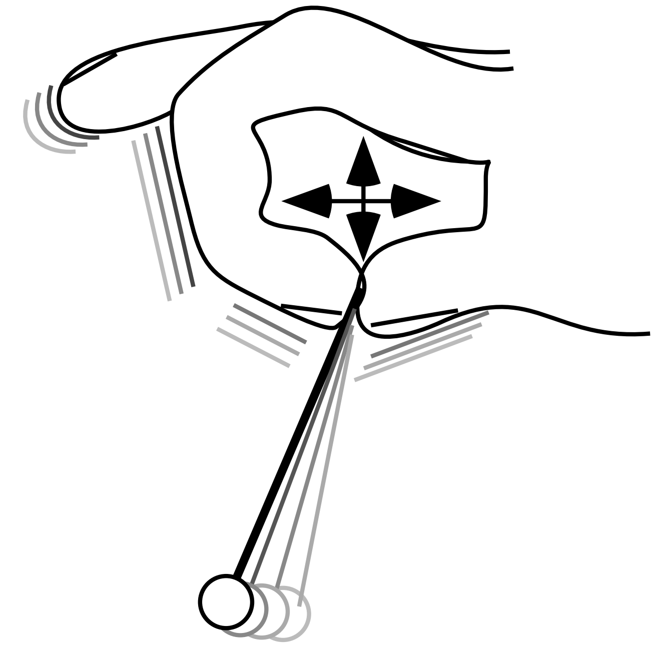


For small ϕ
($\sin\phi \sim \phi$) :

Parametric Resonance

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

A Schrodinger-like equation
(Time *t* replaces coord. *x*)



General ϕ:

(1542-2019 CE)

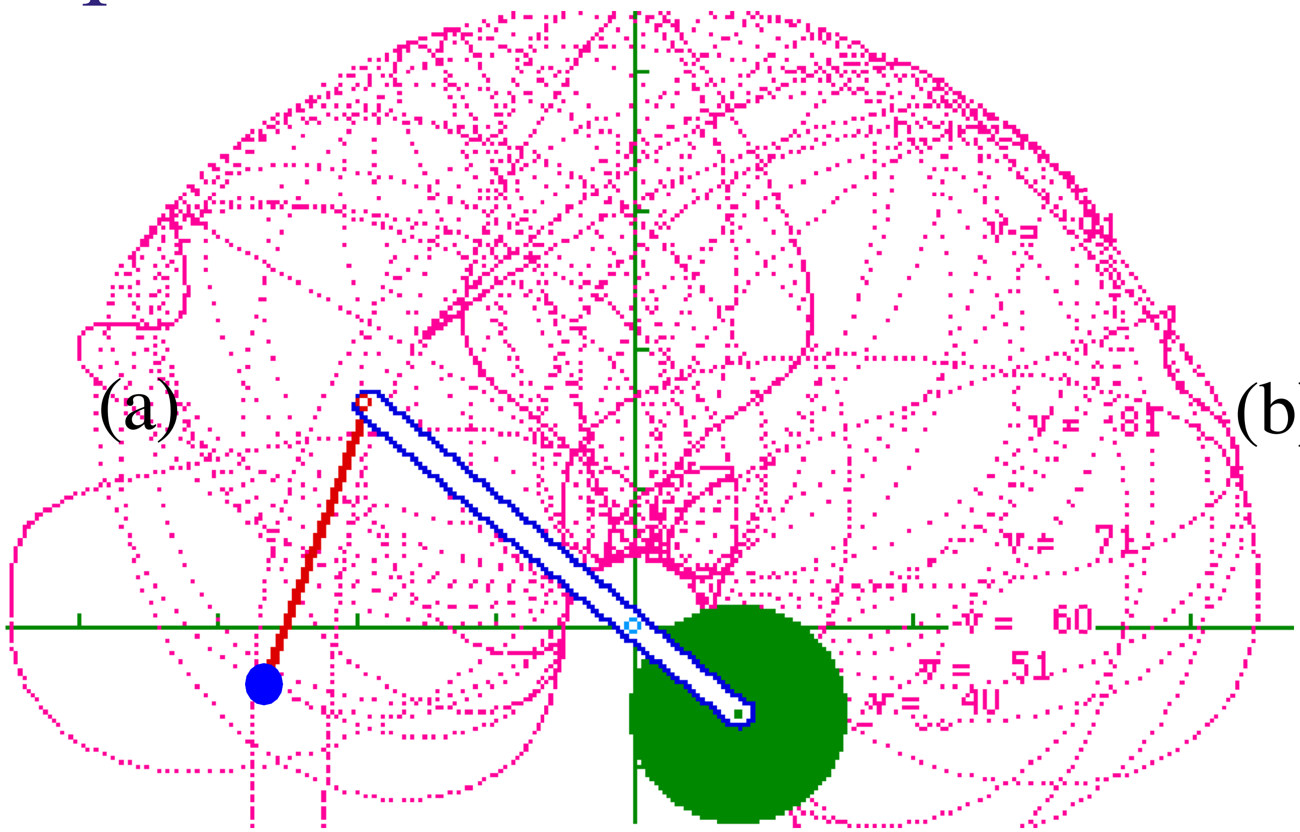
Need small angle approximation to avoid this:

General case: A Nasty equation! $$\frac{d^2\phi}{dt^2} + \frac{g+A_y(t)}{\ell}\sin\phi + \frac{A_x(t)}{\ell}\cos\phi = 0$$

…but ϕ can be near π…

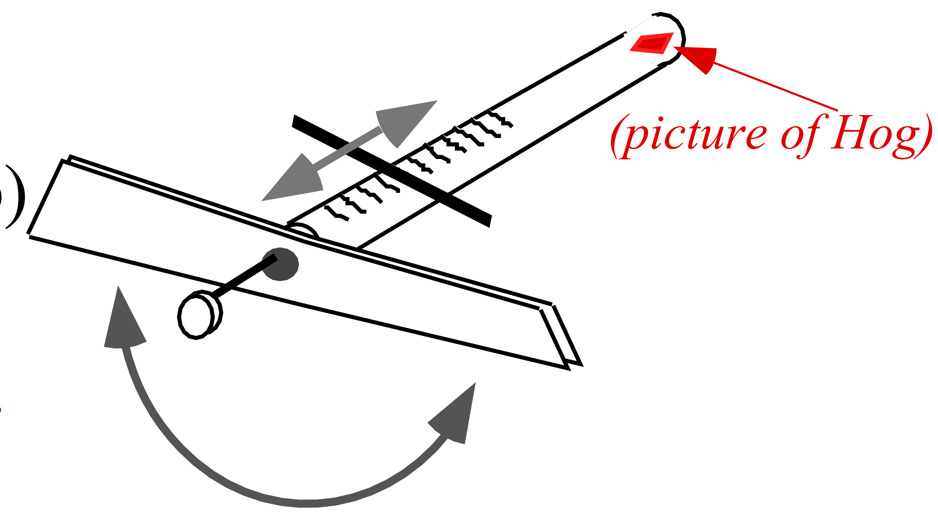
Coupled Rotation and Translation (Throwing)

(a)



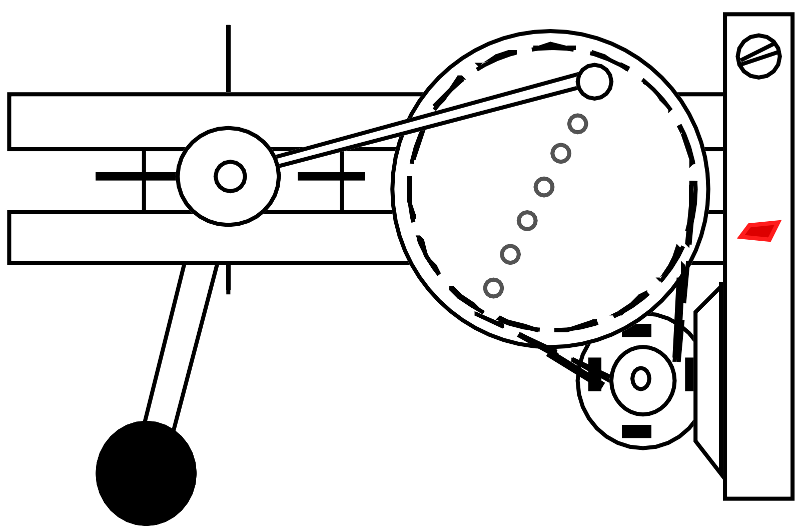
The “Arkansas Whirler”

(b)

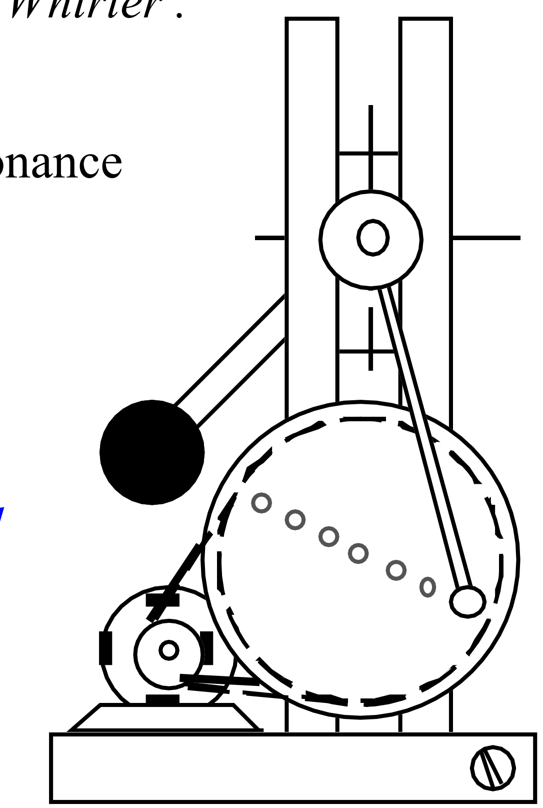


Chaotic motion from both linear and non-linear resonance (a) Trebuchet, (b) Whirler .

Positioned for linear resonance



Positioned for nonlinear resonance



device we hope to build (...someday)

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)

Schrodinger wave equation related to Parametric resonance dynamics

Electronic band theory and analogous mechanics

Schrodinger Equation Parametric Resonance ⟺ Related to ⟺ Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation (With m=1 and ħ=1)

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

main difference: independent variable

space=x ← *becomes* → *time=t*

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos \omega_y t$

$$A_y(t) = \omega_y{}^2 A_y \cos(\omega_y t)$$

(it's periodic acceleration)

Erwin Rudolf Josef Alexander Schrödinger. (1887-1961)

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Émile Mathieu (1835-1890)

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Connection Relations

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$$\frac{N}{\omega_y}dx=dt \longrightarrow \frac{N^2}{\omega_y{}^2}dx^2=dt^2$$

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We let N=2 to get Band-edge modes

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$$\boxed{E=\frac{N^2}{\omega_y{}^2}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

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$$\boxed{E=\frac{N^2}{\omega_y{}^2}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

$$\boxed{V_0=\frac{N^2A_y}{\ell}}$$

QM Potential V_0-A_y Amplitude Connection

Applied on page 32

Applied on page 33

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We let N=2 to get Band-edge modes

$$\boxed{E=\frac{4}{\omega_y^{2}}g}$$

QM Energy E-to-ω_y Jerk frequency Connection

For N=2 and ℓ=1

$$\boxed{V_0=4A_y}$$

QM Potential V_0-A_y Amplitude Connection

Applied
on page 32, 33

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)

Schrodinger wave equation related to Parametric resonance dynamics

➔ *Electronic band theory and analogous mechanics*

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
← *space=x*
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time=t $\rightarrow$

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Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E{=}k^2$$

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Bohr has *periodic boundary conditions* x between 0 and L

$$\phi(0)=\phi(L) \Rightarrow e^{ikL}=1, \text{ or: } kL=2\pi m, \text{ or: } k=\frac{2\pi m}{L}$$

Pendulum repeats perfectly after a time T.

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Limit $L=2\pi=T$ for both analogies. Then the allowed energies and frequencies follow

$$E = k^2 = 0,1,4,9,16...$$

$$\omega_0 = m = 0, \pm 1, \pm 2, \pm 3, \pm 4, ...$$

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Schrodinger equation with non-zero V solved in Fourier basis

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Fourier representation: $\langle j|\mathbf{D}|k\rangle = j^2\delta_j^k$

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Matrix eigenvalue equation

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$$= \int_0^{2\pi} \frac{V_0}{2} dx \frac{e^{-i(j-k+N)x} + e^{-i(j-k-N)x}}{2\pi}$$

$$\sum \langle j|(\mathbf{D}+\mathbf{V})|k\rangle\langle k|\phi\rangle = E\langle j|\phi\rangle$$

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Matrix eigenvalue equation

(Move Fourier reps. to top)

Electronic band theory and analogous mechanics

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Matrix eigenvalue equation

$\langle j|(\mathbf{D}+\mathbf{V})|k\rangle =$ (for j and k *even*)

$$\cdots\ |-6\rangle,|-4\rangle,|-2\rangle,|0\rangle,|2\rangle,\ |4\rangle,\ |6\rangle,\cdots$$

$$\begin{array}{c} \vdots \\ \langle -6| \\ \langle -4| \\ \langle -2| \\ \langle -0| \\ \langle +2| \\ \langle +4| \\ \langle +6| \\ \vdots \end{array}
\begin{pmatrix}
\ddots & & & & & & & \\
& 6^2 & v & & & & & \\
& v & 4^2 & v & & & & \\
& & v & 2^2 & v & & & \\
& & & v & 0 & v & & \\
& & & & v & 2^2 & v & \\
& & & & & v & 4^2 & v \\
& & & & & & v & 6^2 \\
& & & & & & &
\end{pmatrix},$$

$\langle j|(\mathbf{D}+\mathbf{V})|k\rangle =$ (for j and k *odd*)

$$\cdots\ |-7\rangle,|-5\rangle,|-3\rangle,|-1\rangle,|1\rangle,|3\rangle,\ |5\rangle,\cdots$$

$$\begin{array}{c} \vdots \\ \langle -7| \\ \langle -5| \\ \langle -3| \\ \langle -1| \\ \langle +1| \\ \langle +3| \\ \langle +5| \\ \\ \end{array}
\begin{pmatrix}
\ddots & & & & & & & & \\
& 7^2 & v & & & & & & \\
& v & 5^2 & v & & & & & \\
& & v & 3^2 & v & & & & \\
& & & v & 1^2 & v & & & \\
& & & & v & 1^2 & v & & \\
& & & & & v & 3^2 & v & \\
& & & & & & v & 5^2 & \\
& & & & & & & & \ddots
\end{pmatrix}$$

Connection relations from p. 20-21

Here: $v=\frac{V_0}{2}=\frac{4A_y}{2\ell}=\frac{2A_y}{\ell}=2A_y$

For N=2 and ℓ=1

E_m-values vary with amplitude V_0 or wiggle amplitude $A_y = V_0\ell/N^2 = 2v/N^2 = v/2$. *($N=2$ and $\ell=1$ here)*

Eigenvalues for $V_0=0.2$ or $v=0.1$ and $V_0=2.0$ or $v=1.0$.

$E_0 =$	-0.0050
$E_{1-} =$	0.8988
$E_{1+} =$	1.0987
$E_{2-} =$	3.9992
$E_{2+} =$	4.0042
$E_{3-} =$	9.0006
$E_{3+} =$	9.0006

minus means inverted pendulum

$E_0 =$	-0.4551
$E_{1-} =$	-0.1102
$E_{1+} =$	1.8591
$E_{2-} =$	3.9170
$E_{2+} =$	4.3713
$E_{3-} =$	9.0477
$E_{3+} =$	9.0784

minus means inverted pendulum

Connection relations from p. 20 21

When pendulum is "normal" and near its lowest point ($\phi \sim 0$) then $\cos\phi \sim 1$ and $\sin\phi \sim \phi$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\omega_y^2}\left(\frac{g}{\ell} - \frac{\omega_y^2 A_y}{\ell}\cos(Nx)\right)\phi = 0 = \frac{d^2\phi}{dx^2} + \left(\frac{N^2}{\omega_y^2}\frac{g}{\ell} - \frac{N^2 A_y}{\ell}\cos(Nx)\right)\phi, \quad \text{(where: } \phi \sim 0\text{)}$$

When pendulum is "inverted" near highest point ($\phi \sim \pi$) then $\cos\phi \sim -1$ and $\sin\phi \sim \pi - \phi$.

$$\frac{d^2\phi}{dt^2} - \left(\frac{g}{\ell} - \frac{\omega_y^2 A_y}{\ell}\cos(\omega_y t)\right)(\phi - \pi) = 0 , \qquad \text{(where: } \phi \sim \pi\text{)}$$

E_m-eigenvalue determines pendulum Y-wiggle frequency $\omega_{y(m)}$.

$$E_m = \frac{N^2}{\omega_{y(m)}^2}\frac{g}{\ell} \qquad \text{implies:} \qquad \omega_{y(m)} = \frac{N}{\sqrt{E_m}}\sqrt{\frac{g}{\ell}} = \frac{2}{\sqrt{E_m}} \qquad (g=1, \text{ too})$$

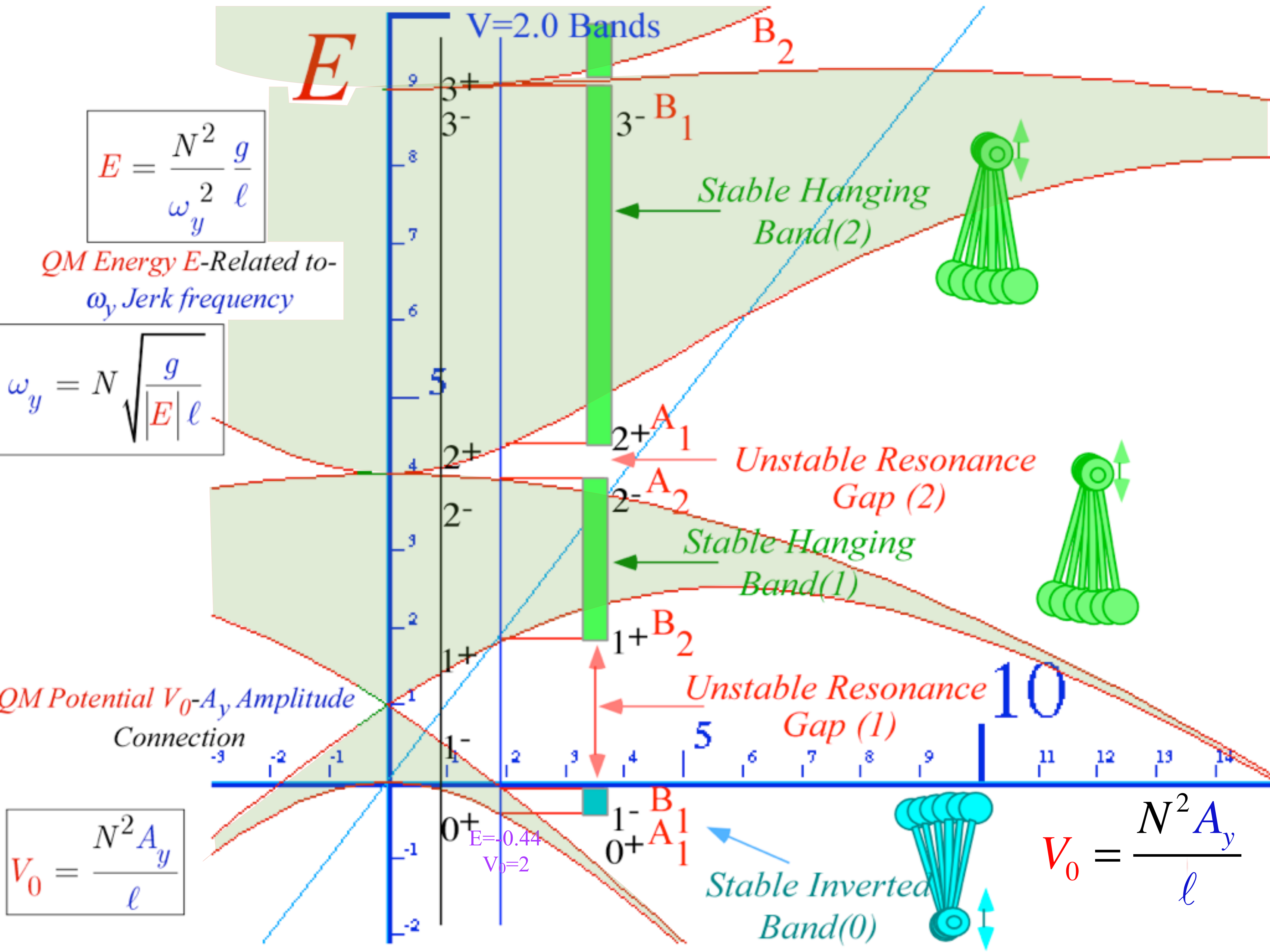
Pendulum Y-wiggle frequency $\omega_{y(m)}$ for $V_0=0.2$ and for $V_0=2.0$.

$\omega_{y(0)} = 2/\sqrt{.0050}$	$=28.2843$
$\omega_{y(1^-)} = 2/\sqrt{.8988}$	$=2.10959$
$\omega_{y(1^+)} = 2/\sqrt{1.0987}$	$=1.90805$
$\omega_{y(2^-)} = 2/\sqrt{3.9992}$	$=1.00010$
$\omega_{y(2^+)} = 2/\sqrt{4.0042}$	$=0.99948$

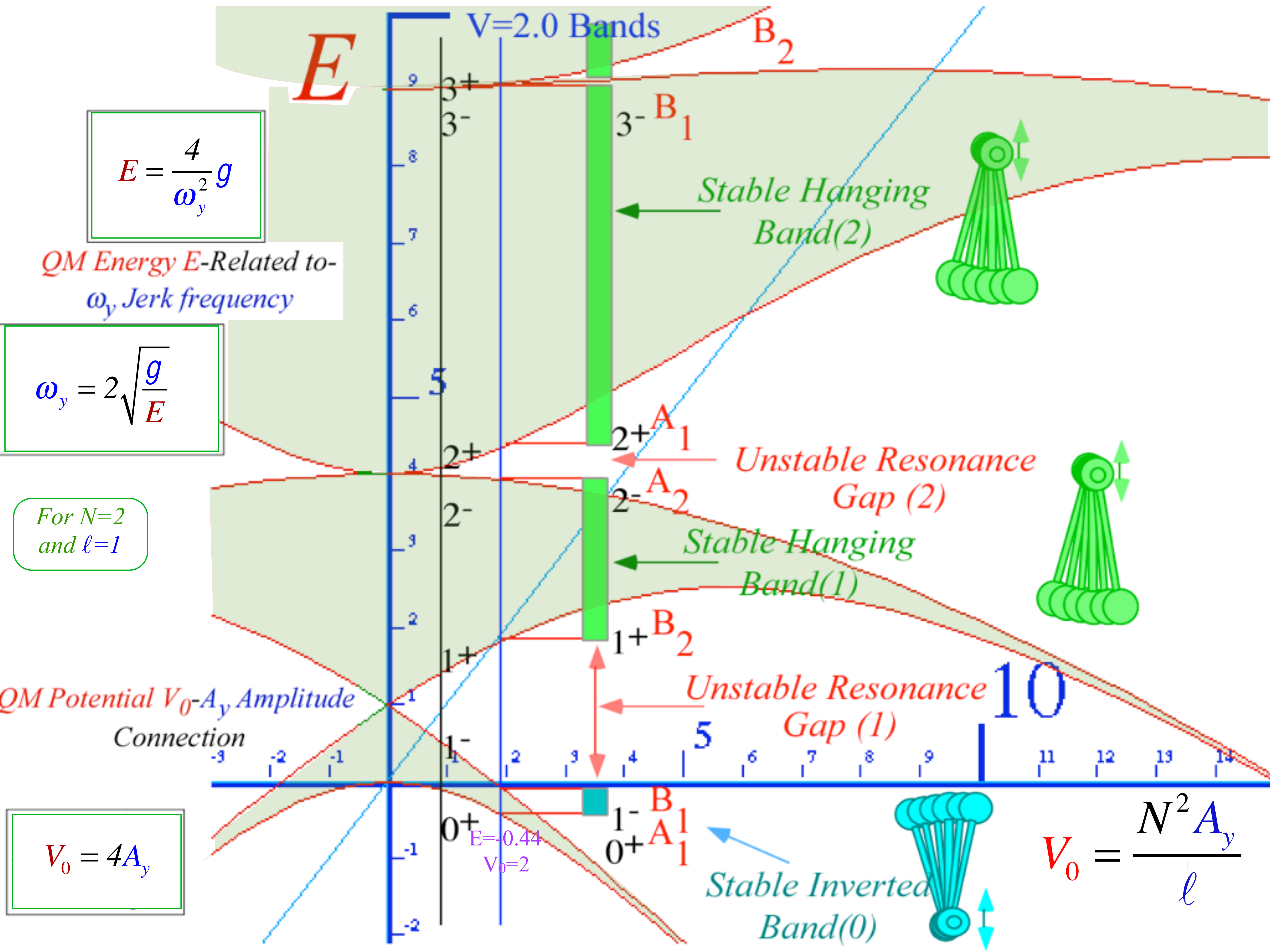
minus on E_0 means inverted pendulum

$\omega_{y(0)} = 2/\sqrt{.4551}$	$=2.9646$
$\omega_{y(1^-)} = 2/\sqrt{.1102}$	$=6.02475$
$\omega_{y(1^+)} = 2/\sqrt{1.8591}$	$=1.4668$
$\omega_{y(2^-)} = 2/\sqrt{3.9170}$	$=1.0105$
$\omega_{y(2^+)} = 2/\sqrt{4.3713}$	$=0.9566$

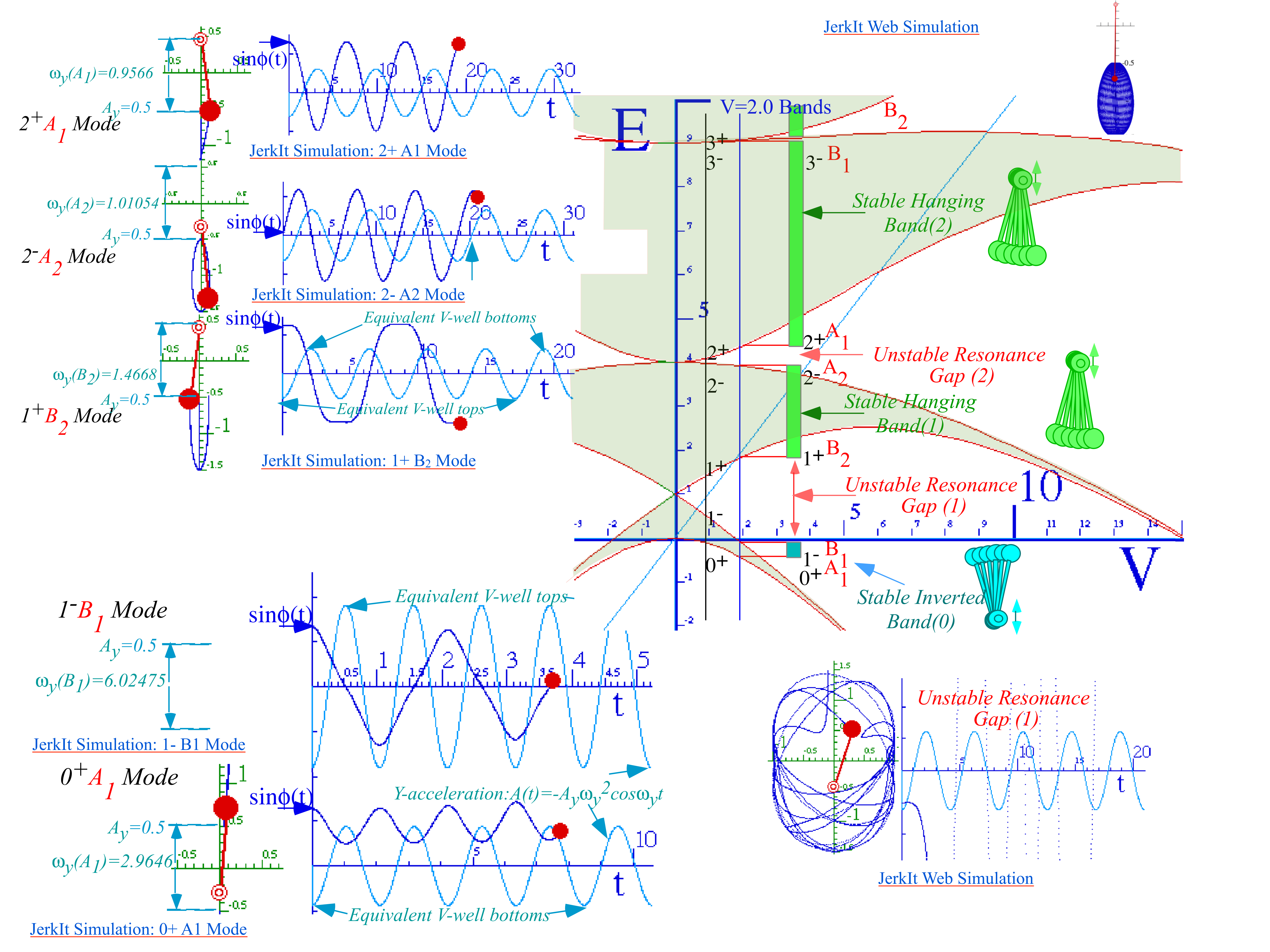
minus on $E_{0,or\ 1}$ means inverted pendulum



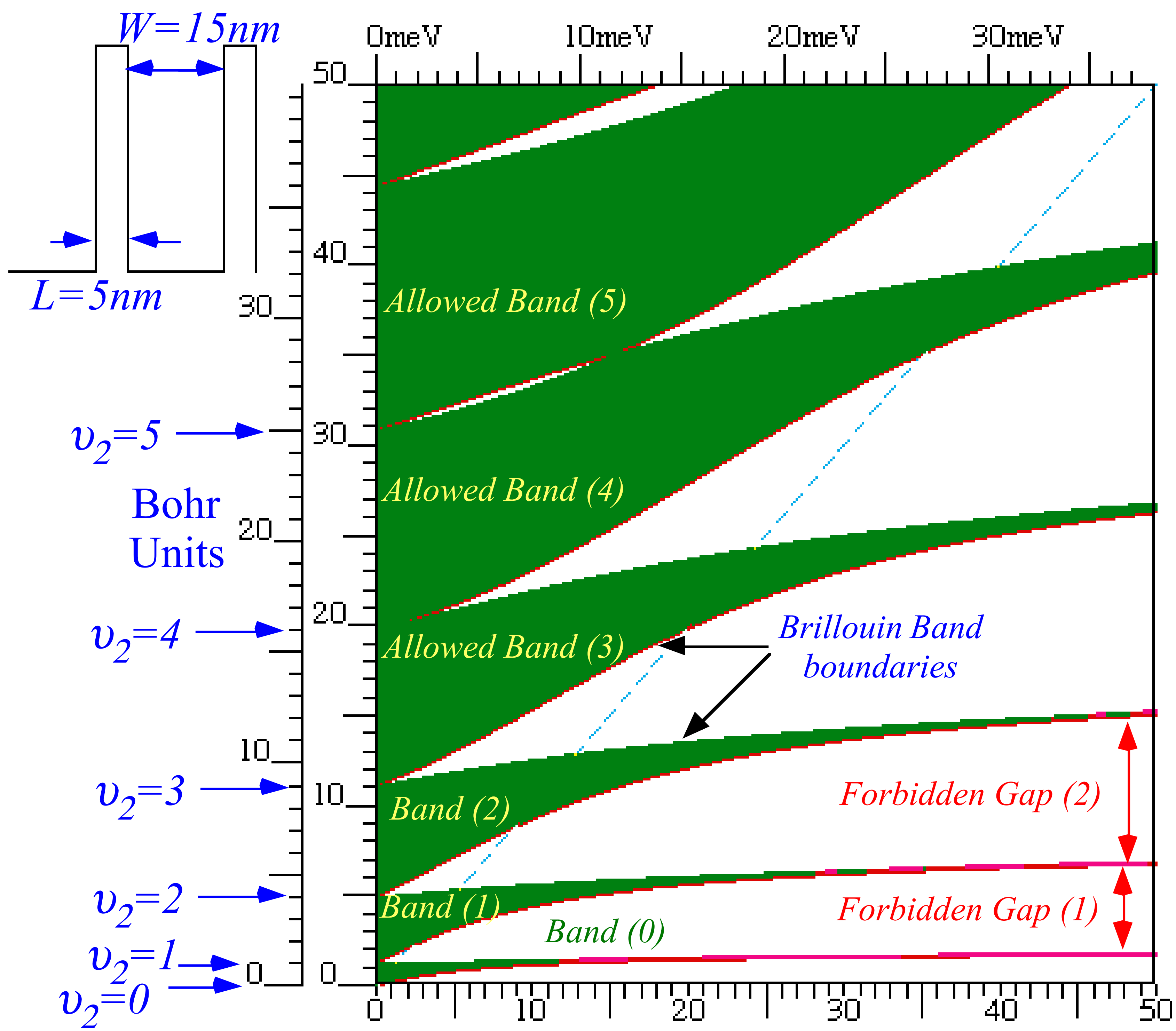
V=2.0 Bands
E
$E = \frac{N^2}{\omega_y^2}\frac{g}{\ell}$
QM Energy E-Related to-
ω_y Jerk frequency
$\omega_y = N\sqrt{\frac{g}{|E|\ell}}$
QM Potential V_0-A_y Amplitude
Connection
$V_0 = \frac{N^2 A_y}{\ell}$
B_2
3^+
3^-
3^- B_1
Stable Hanging
Band(2)
2^+ A_1
2^+
Unstable Resonance
Gap (2)
2^- A_2
2^-
Stable Hanging
Band(1)
1^+ B_2
1^+
Unstable Resonance
Gap (1)
1^-
1^- B_1
0^+ A_1
0^+
E=-0.44
V_0=2
Stable Inverted
Band(0)
$V_0 = \frac{N^2 A_y}{\ell}$



V=2.0 Bands
E
B_2
B_1
3+
3-
$E = \frac{4}{\omega_y^2} g$
QM Energy E-Related to-
ω_y Jerk frequency
Stable Hanging
Band(2)
$\omega_y = 2\sqrt{\frac{g}{E}}$
2+
A_1
Unstable Resonance
Gap (2)
A_2
2-
For N=2
and ℓ=1
Stable Hanging
Band(1)
1+
B_2
1-
QM Potential V_0-A_y Amplitude
Connection
Unstable Resonance
Gap (1)
10
5
0+
E=-0.44
V_0=2
B_1
A_1
$V_0 = 4A_y$
Stable Inverted
Band(0)
$V_0 = \frac{N^2 A_y}{\ell}$


JerkIt Web Simulation
$2^{+}A_{1}$ Mode
$\omega_y(A_1)=0.9566$
$A_y=0.5$
sinϕ(t)
t
JerkIt Simulation: 2+ A1 Mode
$2^{-}A_{2}$ Mode
$\omega_y(A_2)=1.01054$
$A_y=0.5$
JerkIt Simulation: 2- A2 Mode
$1^{+}B_{2}$ Mode
$\omega_y(B_2)=1.4668$
$A_y=0.5$
Equivalent V-well bottoms
Equivalent V-well tops
JerkIt Simulation: 1+ B2 Mode
E
V=2.0 Bands
B_2
B_1
3^+
3^-
Stable Hanging Band(2)
2^+A_1
Unstable Resonance Gap (2)
2^-A_2
Stable Hanging Band(1)
1^+B_2
Unstable Resonance Gap (1)
1^-B_1
0^+A_1
Stable Inverted Band(0)
V
$1^{-}B_{1}$ Mode
$A_y=0.5$
$\omega_y(B_1)=6.02475$
JerkIt Simulation: 1- B1 Mode
$0^{+}A_{1}$ Mode
$A_y=0.5$
$\omega_y(A_1)=2.9646$
JerkIt Simulation: 0+ A1 Mode
Y-acceleration: $A(t)=-A_y\omega_y^2\cos\omega_y t$
Unstable Resonance Gap (1)
JerkIt Web Simulation

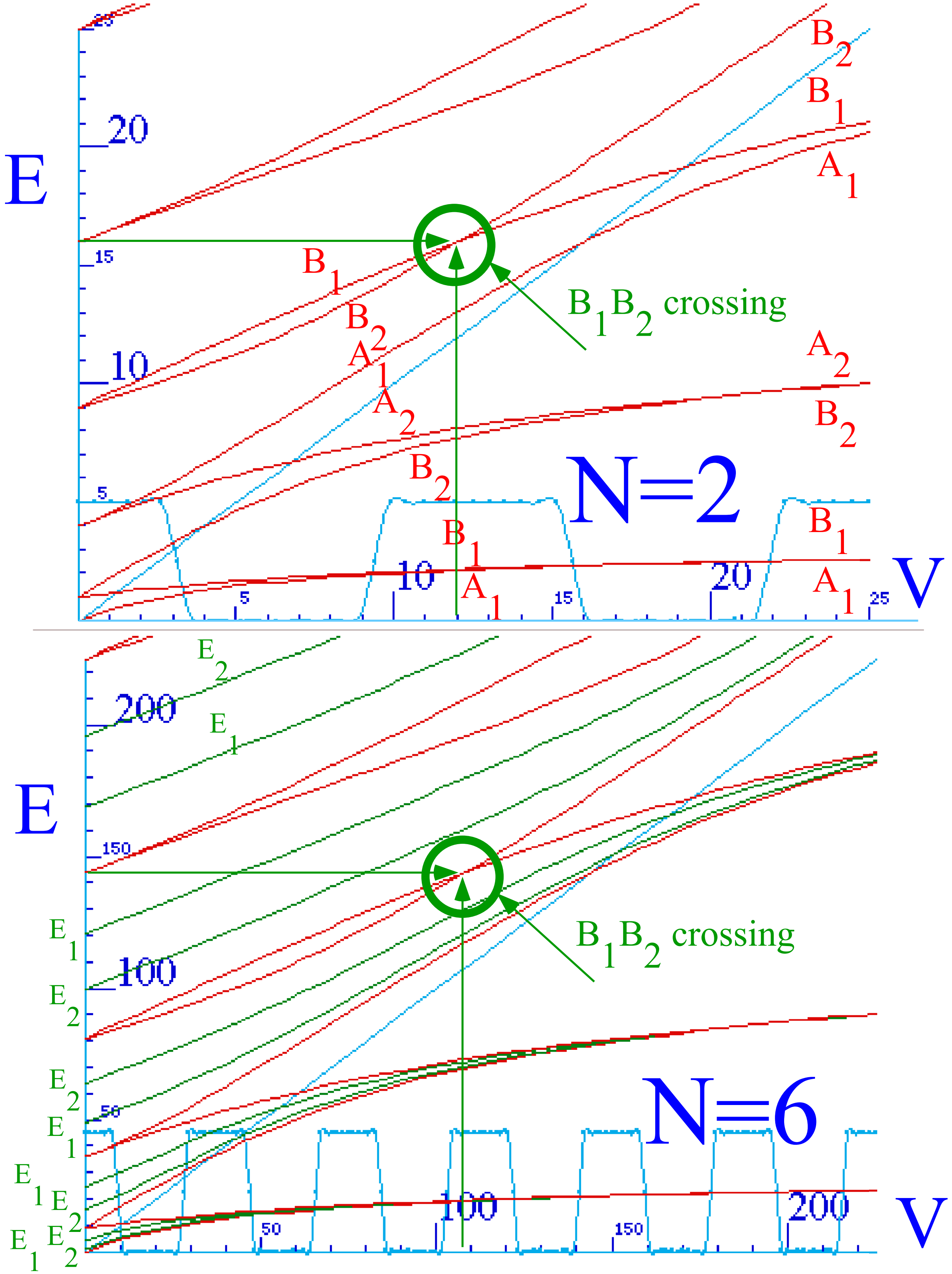
A quick look at band splitting for a square periodic potential (Kronig-Penney Model)



(From Ch. 14 Unit 5 Quantum Theory for the Computer Age (QTftCA))

Fig. 14.2.7 Bands vs. V.(W=15nm well ,L=5nm barrier) showing Bohr splitting for (N=2)-ring.

A quick look at band splitting for a square periodic potential (Kronig-Penney Model)



(From Ch. 14 Unit 5 Quantum Theory for the Computer Age (QTftCA))

Fig. 14.2.13 (B_1, B_2) crossing for:(N=2) at V=12 and E=16, and (N=6) at V=144 and E=108.

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, .*
- *C_N symmetric mode models: Made-to order dispersion functions*
    - *Quadratic dispersion models: Super-beats and fractional revivals*
    - *Phase arithmetic*

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} \qquad + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} + B\cdot\boldsymbol{\sigma}_B$$

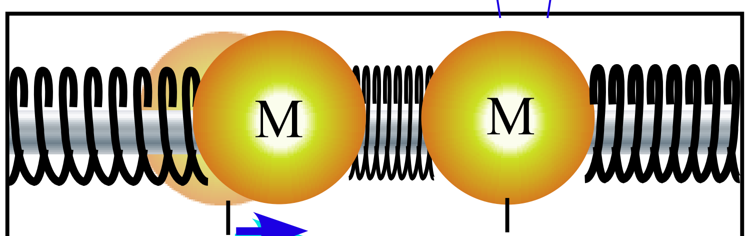
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

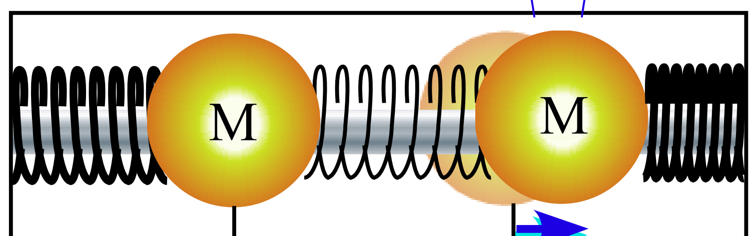
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

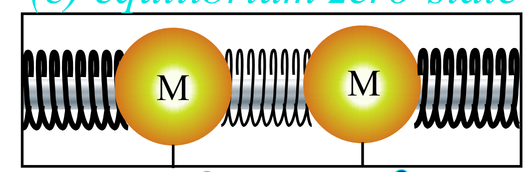


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$=A\cdot\mathbf{1} \qquad +B\cdot\boldsymbol{\sigma}_B$$

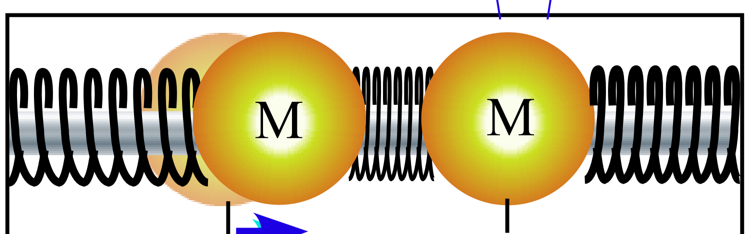
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

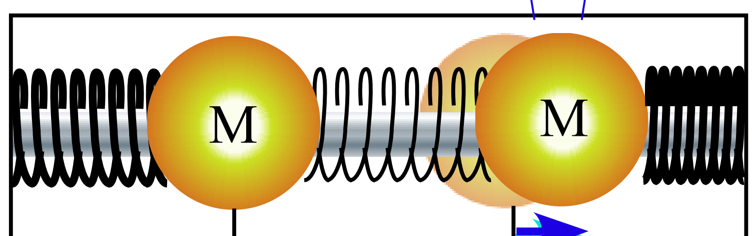
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $\quad x_0=0$

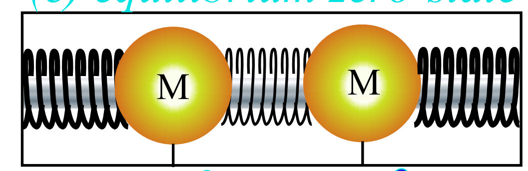
(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $\quad x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $\quad x_1=0$

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2$**-1=0** gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\,\mathbf{p}^{(-1)}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

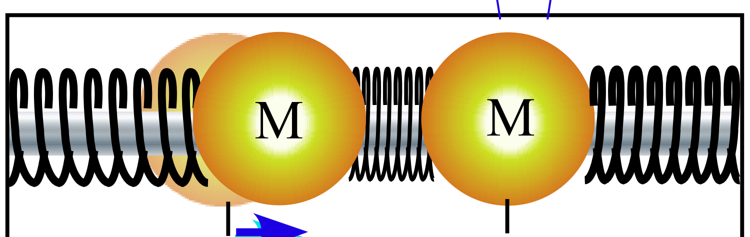
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1} \qquad + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

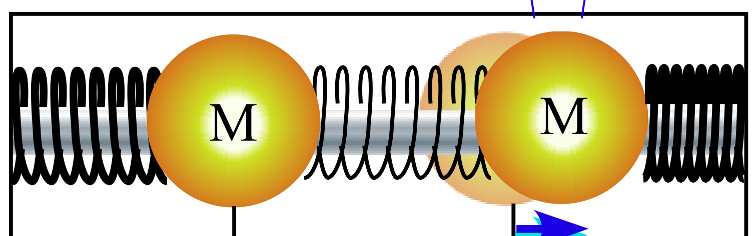
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

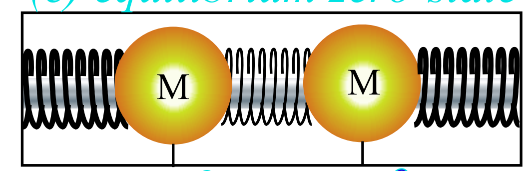


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|{-1}\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\mathbf{1}+\sigma_B)/2$ and $\mathbf{P}^{(-)}=(\mathbf{1}-\sigma_B)/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$=A\cdot\mathbf{1} \qquad +B\cdot\boldsymbol{\sigma}_B$$

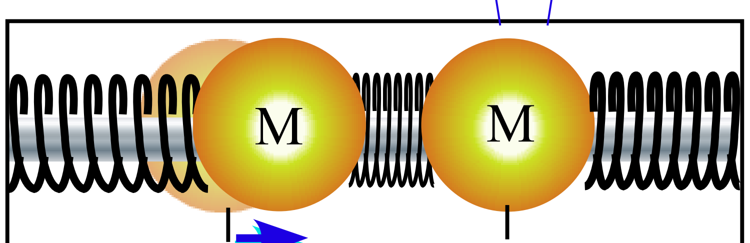
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2$=**1** in C_2 group product table.

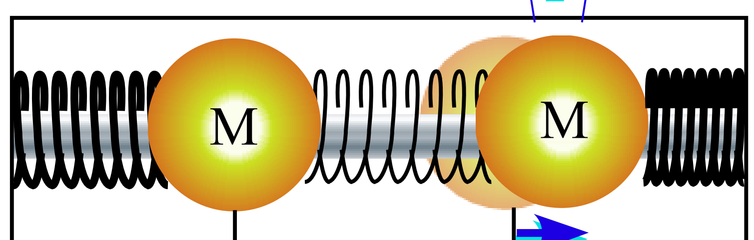
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

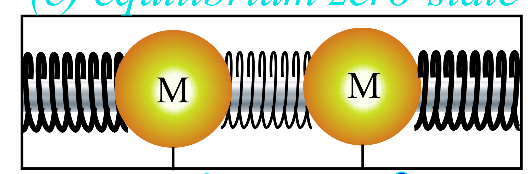


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2$=**1** or: $(\sigma_B)^2$**-1=0** gives projectors:

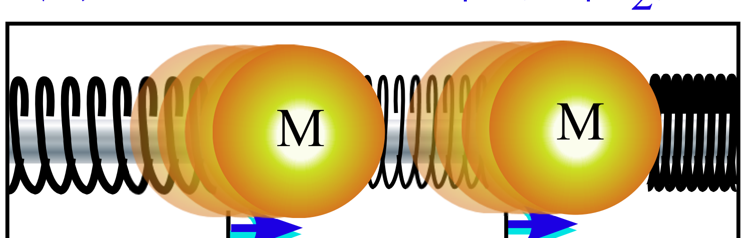
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\mathbf{1}+\sigma_B)/2$ and $\mathbf{P}^{(-)}=(\mathbf{1}-\sigma_B)/2$

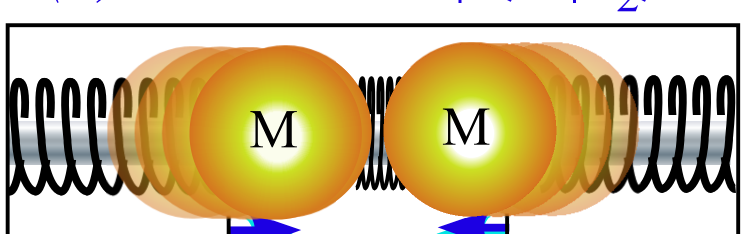
(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H²** with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=A\cdot\mathbf{1}+B\cdot\boldsymbol{\sigma}_B$$

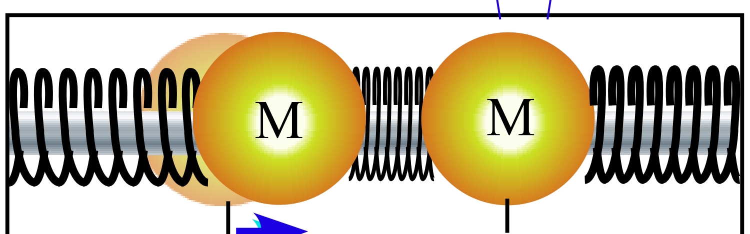
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}=(A^2+B^2)\cdot\mathbf{1}+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

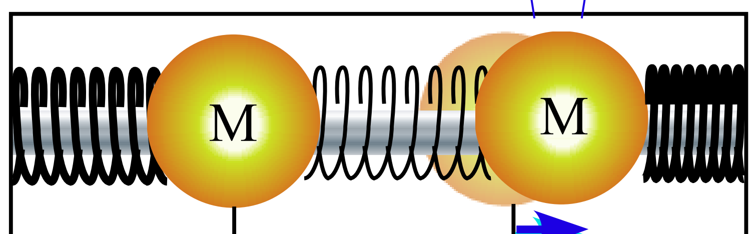
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $x_0=0$

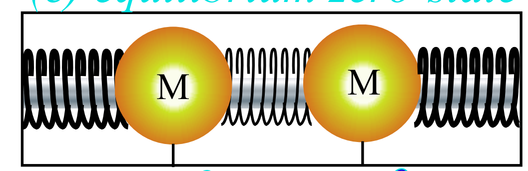
(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $x_1=0$

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

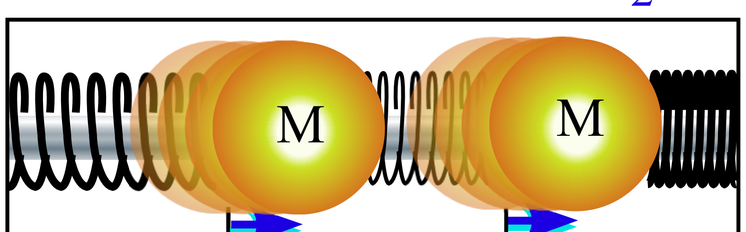
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\mathbf{1}+\sigma_B)/2$ and $\mathbf{P}^{(-)}=(\mathbf{1}-\sigma_B)/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

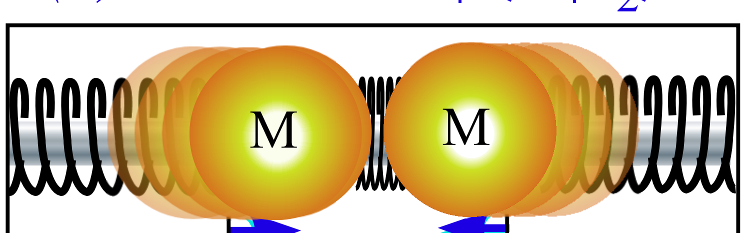
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $x_1=1/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $x_1=-1/\sqrt{2}$

Mode state projection:

$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$$
$$=(|0\rangle+|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$$
$$=(|0\rangle-|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H²** with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

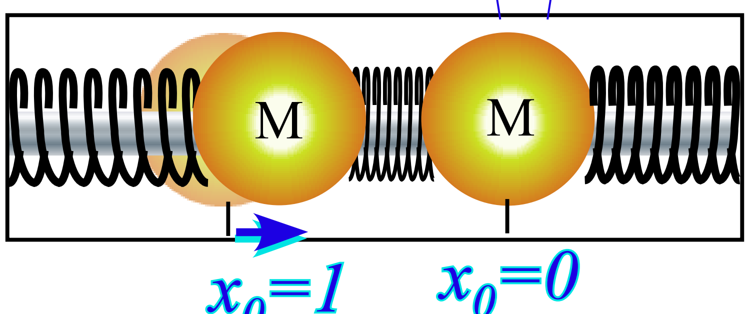
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} \qquad + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

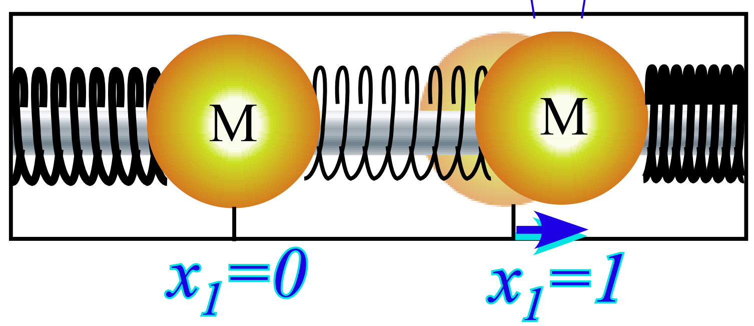
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

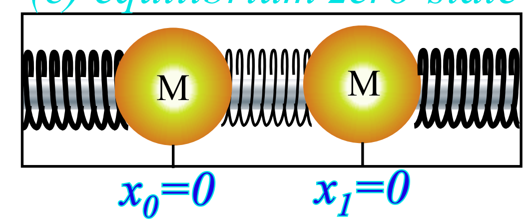


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\mathbf{1}+\sigma_B)/2$ and $\mathbf{P}^{(-)}=(\mathbf{1}-\sigma_B)/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

C_2 symmetry (B-type) modes

Mode state projection:

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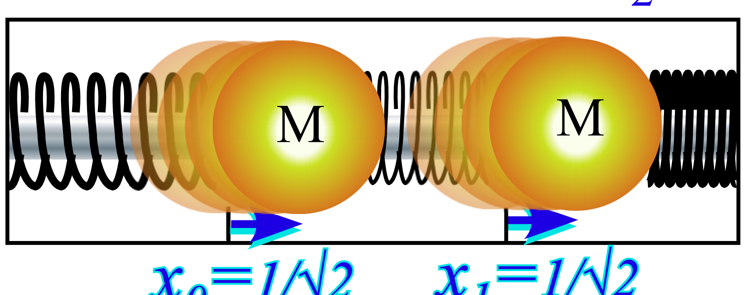
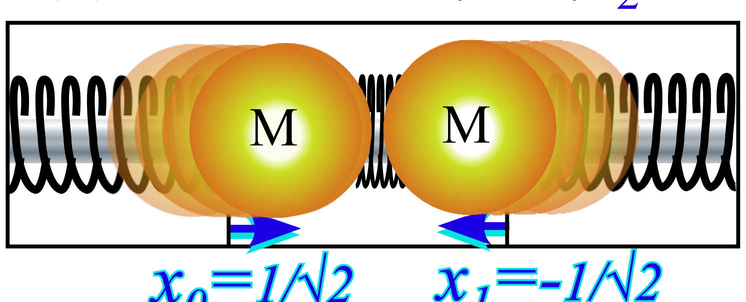


$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$$
$$=(|0\rangle+|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

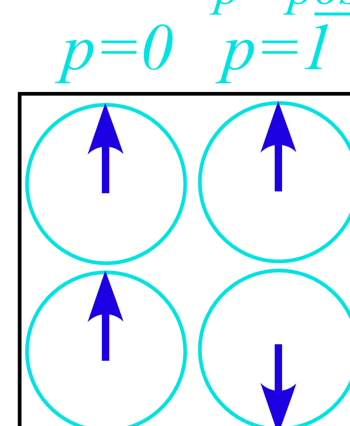
(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$$
$$=(|0\rangle-|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

C_2 mode phase & character tables

p= position point (modulo-2)



	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1

State norm: $1/\sqrt{2}$

m=wave-number or "momentum" (modulo-2)

Operator norm: 1/2

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

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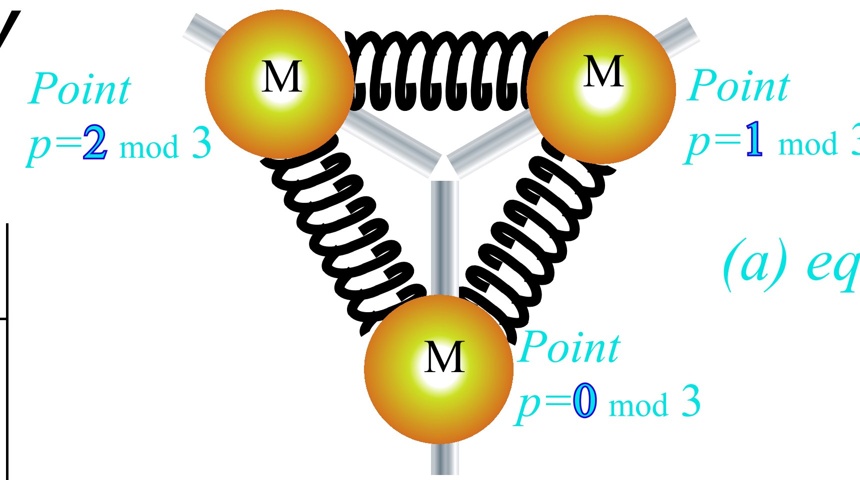
Phase arithmetic

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$x_0=x_1=x_2=0$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} \quad = \quad r_0\cdot\mathbf{1} \quad + r_1\cdot\mathbf{r}^1 \quad + r_2\cdot\mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

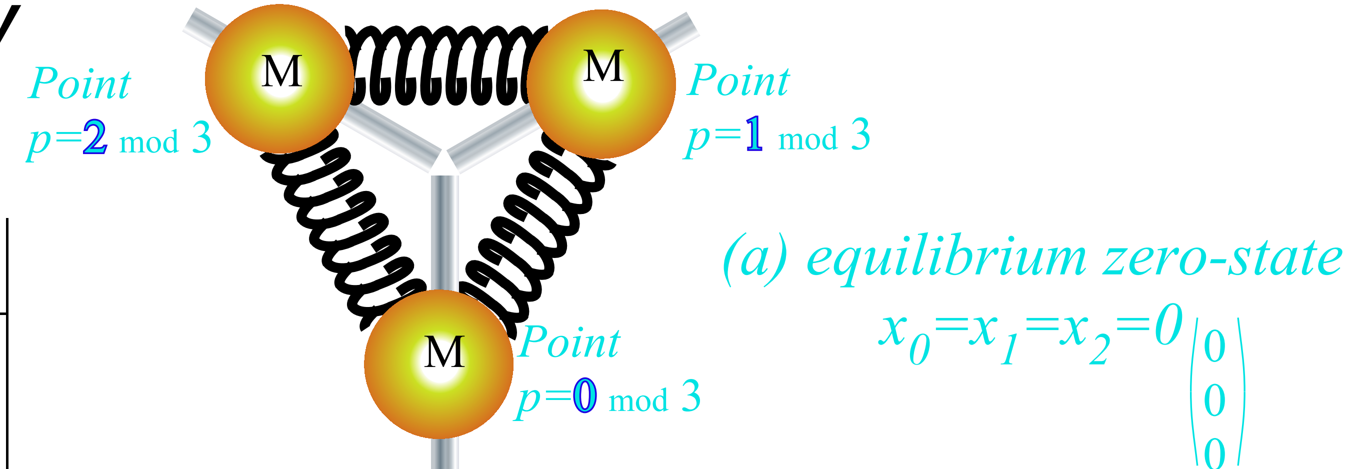
Fig. 4.8.1
Unit 4
CMwBang

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

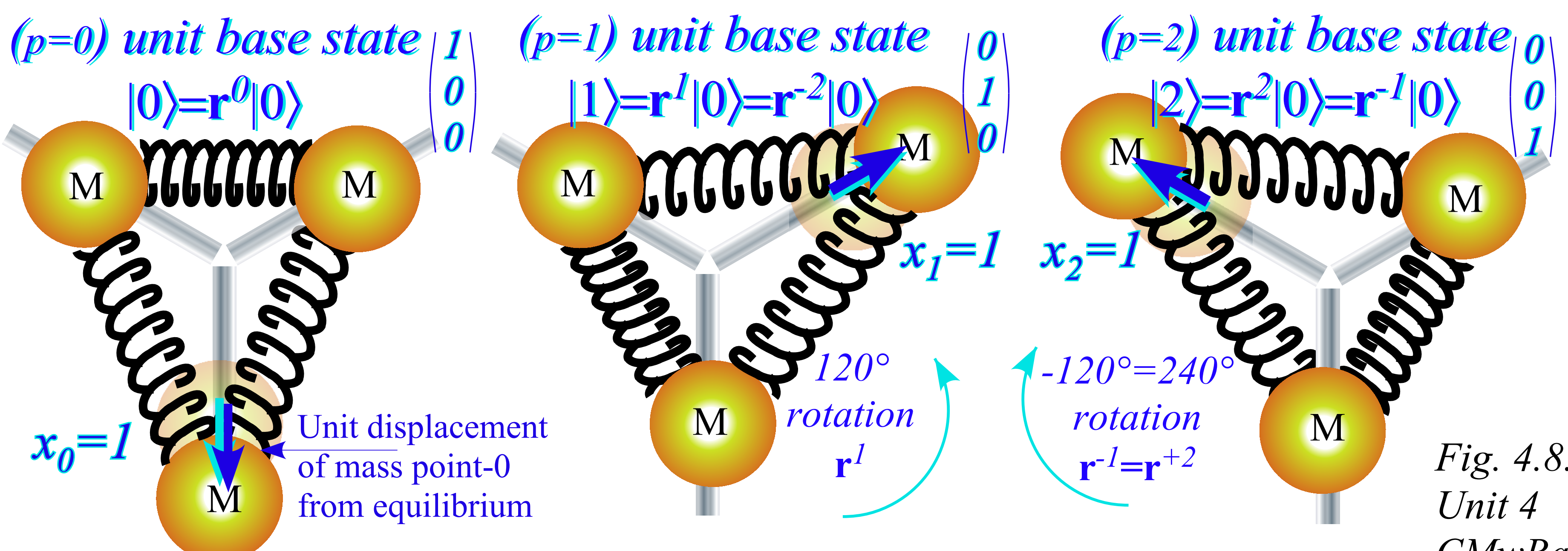
H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = r_0\cdot\mathbf{1} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

C_3 unit base states



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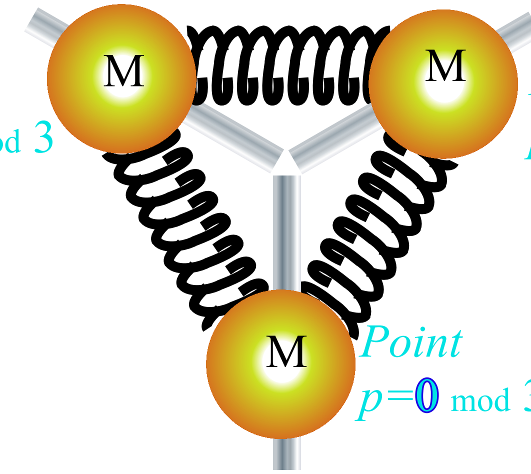
C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state
$x_0=x_1=x_2=0$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$

g$=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

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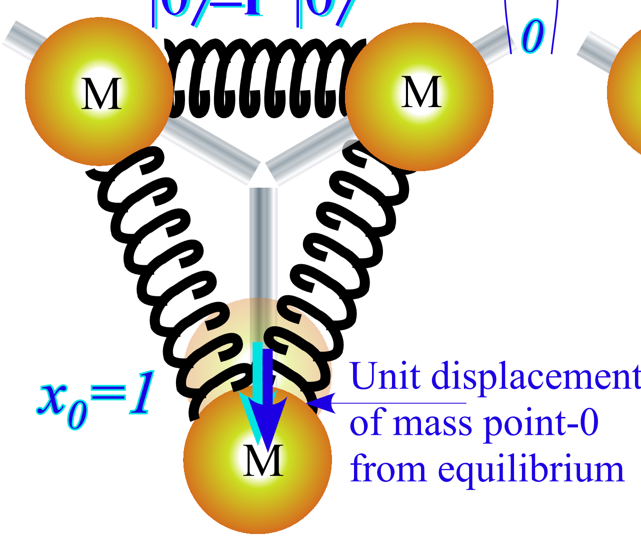
$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix} + r_1\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0\end{pmatrix} + r_2\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\mathbf{H} = r_0\cdot\mathbf{1} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

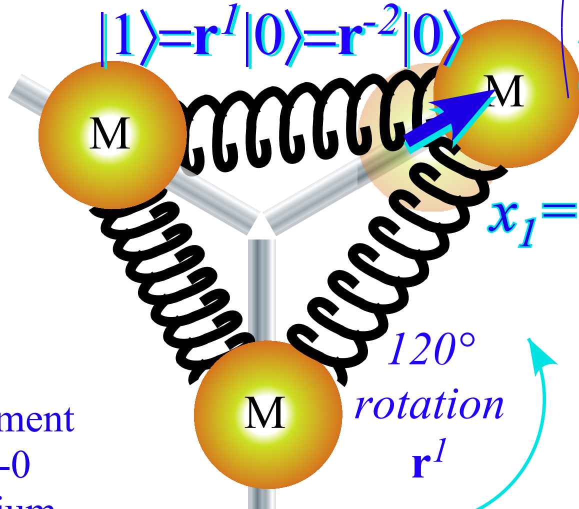
$\mathbf{r}^0=\mathbf{1}$

C_3 unit base states

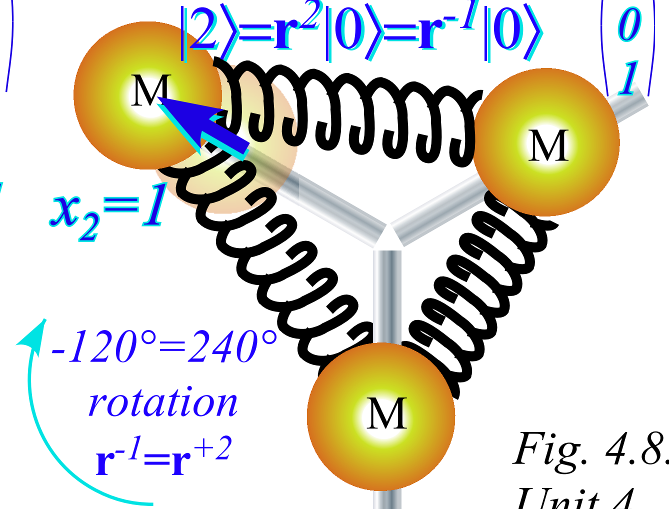
(p=0) unit base state $\begin{pmatrix}1\\0\\0\end{pmatrix}$
$|0\rangle=\mathbf{r}^0|0\rangle$

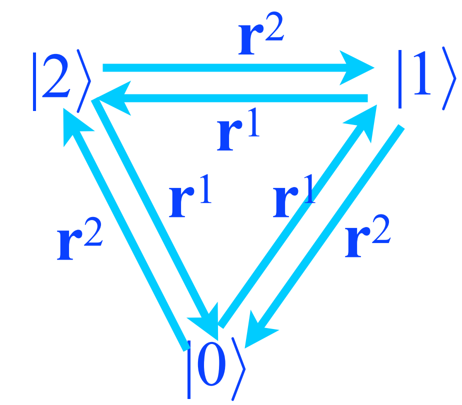


(p=1) unit base state $\begin{pmatrix}0\\1\\0\end{pmatrix}$
$|1\rangle=\mathbf{r}^1|0\rangle=\mathbf{r}^{-2}|0\rangle$



(p=2) unit base state $\begin{pmatrix}0\\0\\1\end{pmatrix}$
$|2\rangle=\mathbf{r}^2|0\rangle=\mathbf{r}^{-1}|0\rangle$





$\mathbf{r}^0=\mathbf{1}$
$=\mathbf{r}^3$

Fig. 4.8.1
Unit 4
CMwBang

Each **H**-matrix coupling constant $r_p=\{r_0, r_1, r_2\}$ is amplitude of its operator power $\mathbf{r}^p=\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$

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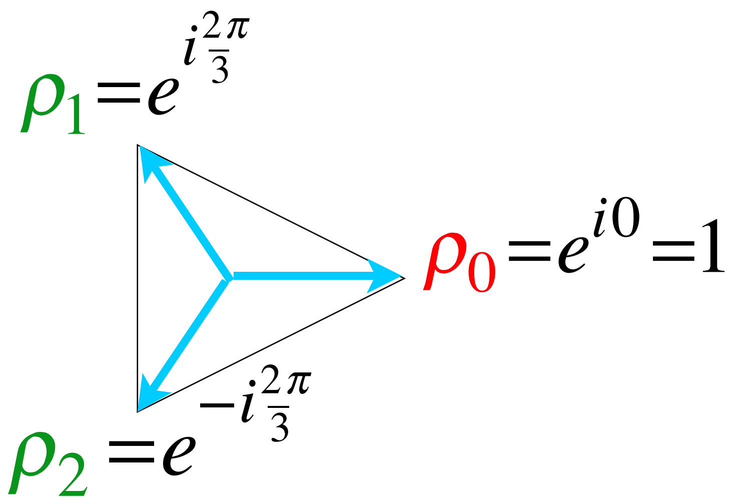
C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

C_3 Spectral resolution: 3^{rd} roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.



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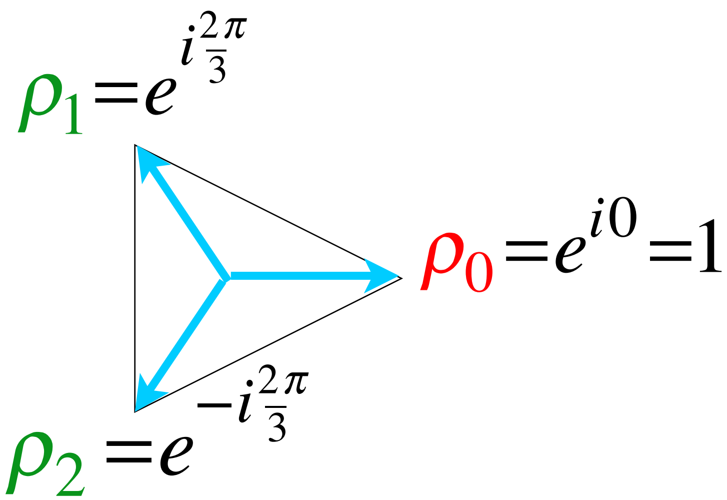
$$\mathbf{1}=\mathbf{r}^3 \ \ implies: \ \ \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \ \ where: \ \ \rho_m = e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.



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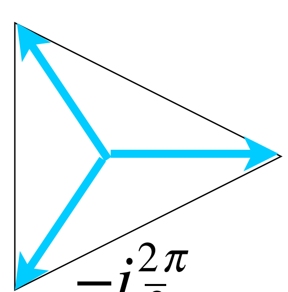
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).



$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

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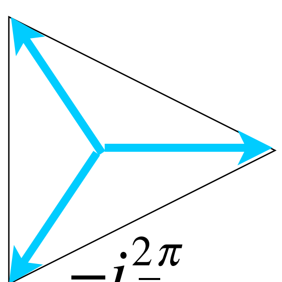

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\,\mathbf{P}^{(0)} + \rho_1\,\mathbf{P}^{(1)} + \rho_2\,\mathbf{P}^{(2)}$$

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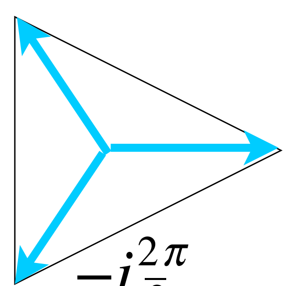
$\rho_1=e^{i\frac{2\pi}{3}}$



$\rho_0=e^{i0}=1$

$\rho_2=e^{-i\frac{2\pi}{3}}$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\,\mathbf{P}^{(0)} + \rho_1\,\mathbf{P}^{(1)} + \rho_2\,\mathbf{P}^{(2)}$$

$$\mathbf{r}^2 = (\rho_0)^2\mathbf{P}^{(0)} + (\rho_1)^2\mathbf{P}^{(1)} + (\rho_2)^2\mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3^{rd} roots of unity

We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

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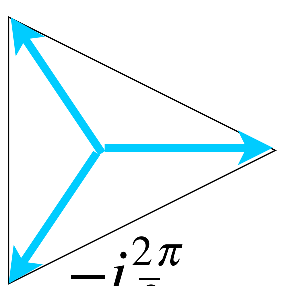
$\rho_1=e^{i\frac{2\pi}{3}}$



$\rho_0=e^{i0}=1$

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$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\mathbf{P}^{(0)} + \rho_1\mathbf{P}^{(1)} + \rho_2\mathbf{P}^{(2)}$$

$$\mathbf{r}^2 = (\rho_0)^2\mathbf{P}^{(0)} + (\rho_1)^2\mathbf{P}^{(1)} + (\rho_2)^2\mathbf{P}^{(2)}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ (because they're UNITARY operators $\mathbf{r}^\dagger=\mathbf{r}^{-1}$)
(and $\mathbf{P}$ is HERMITIAN $\mathbf{P}^\dagger=\mathbf{P}$)

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C_3 Spectral resolution: 3rd roots of unity

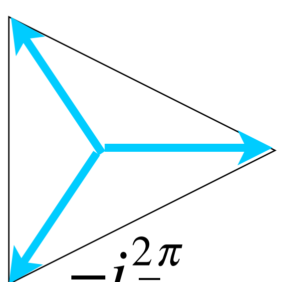
We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p \mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \;\; implies: \;\; \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \;\; where: \;\; \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).

$\rho_1=e^{i\frac{2\pi}{3}}$



$\rho_0=e^{i0}=1$

$\rho_2=e^{-i\frac{2\pi}{3}}$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

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Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - ➤ *Resolving C_3 projectors and moving wave modes*
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- *C_6 symmetric mode model:Distant neighbor coupling*
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    - *Phase arithmetic*

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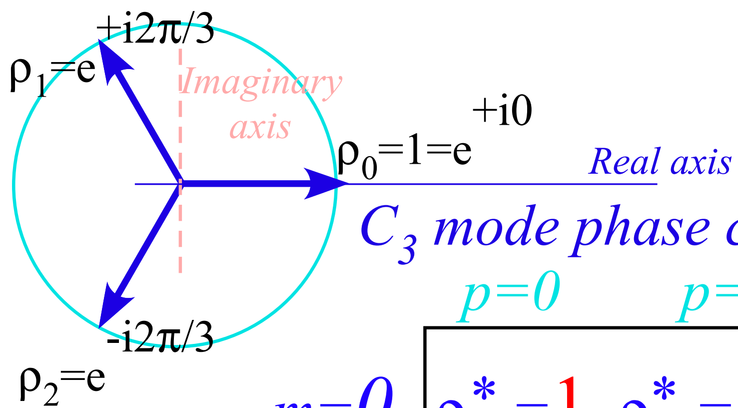

C_3 mode phase character tables

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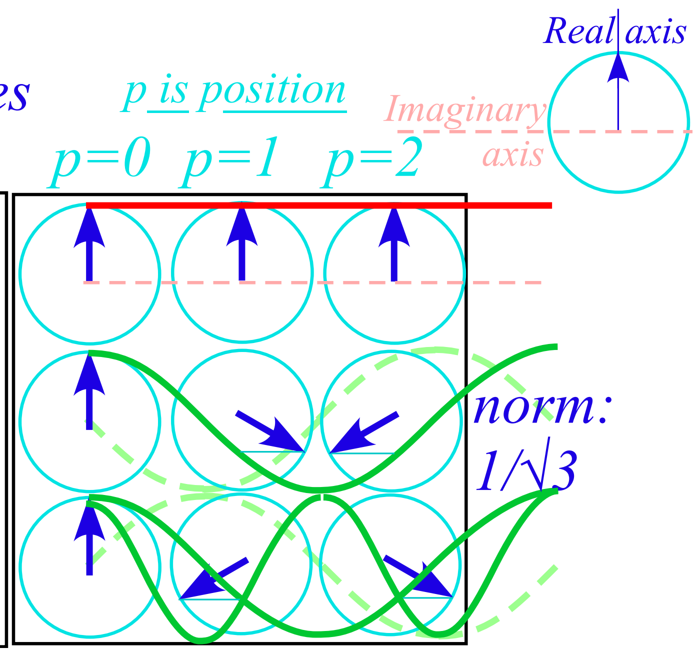

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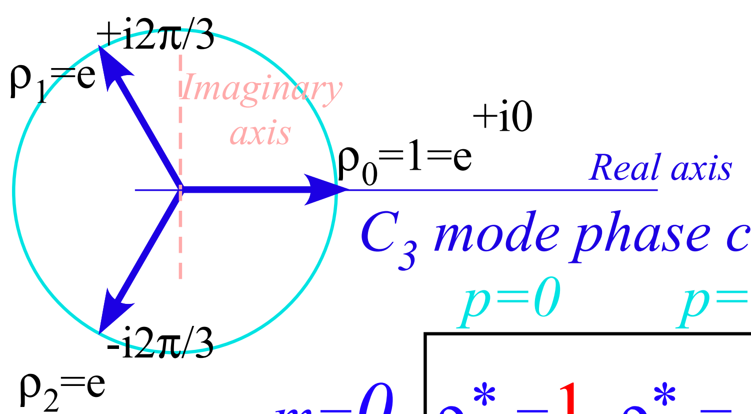

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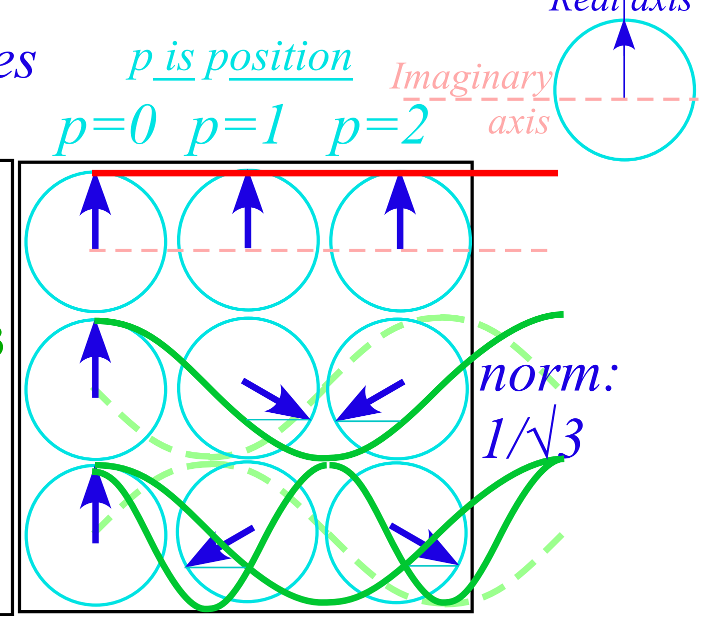

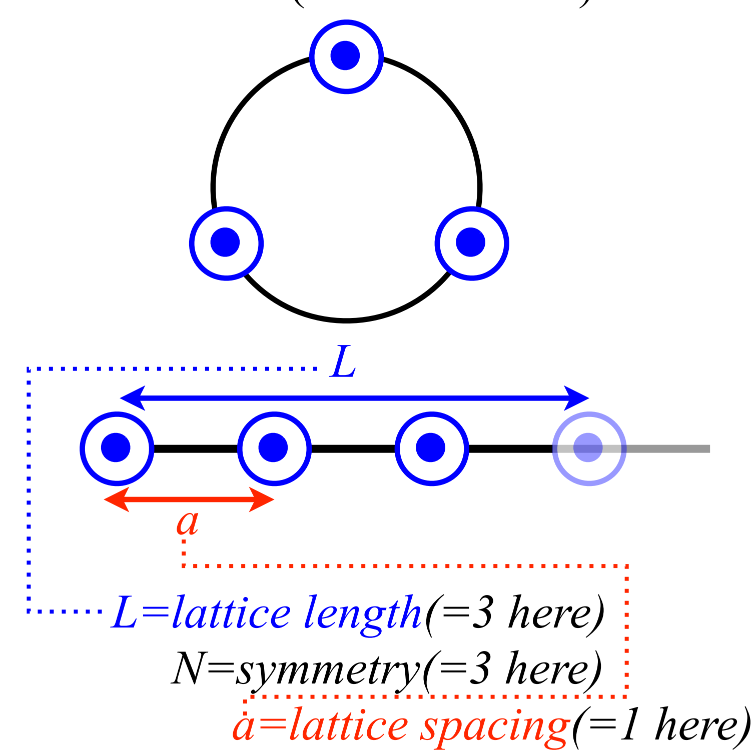

WaveIt App - N3 Wave

MolVibes App C3v N3

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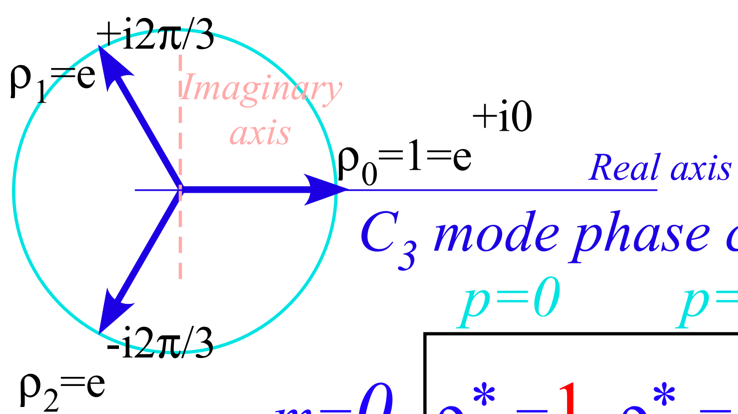

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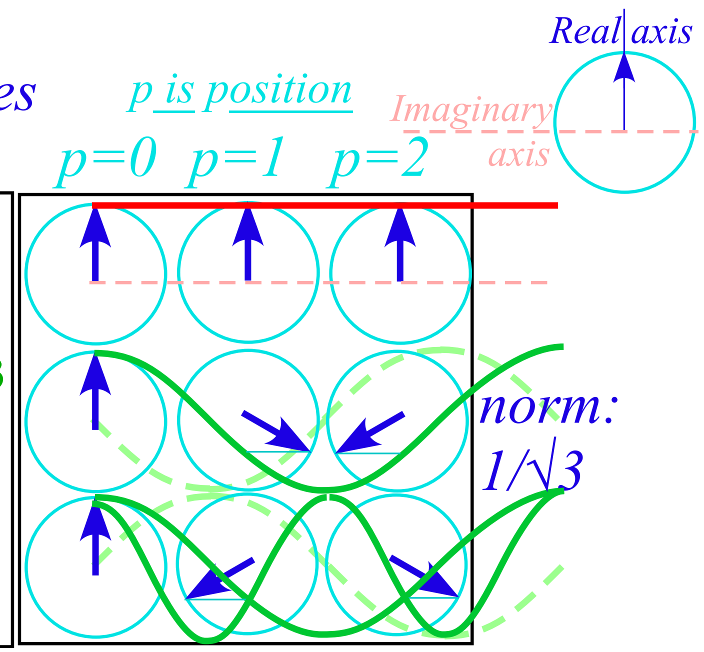

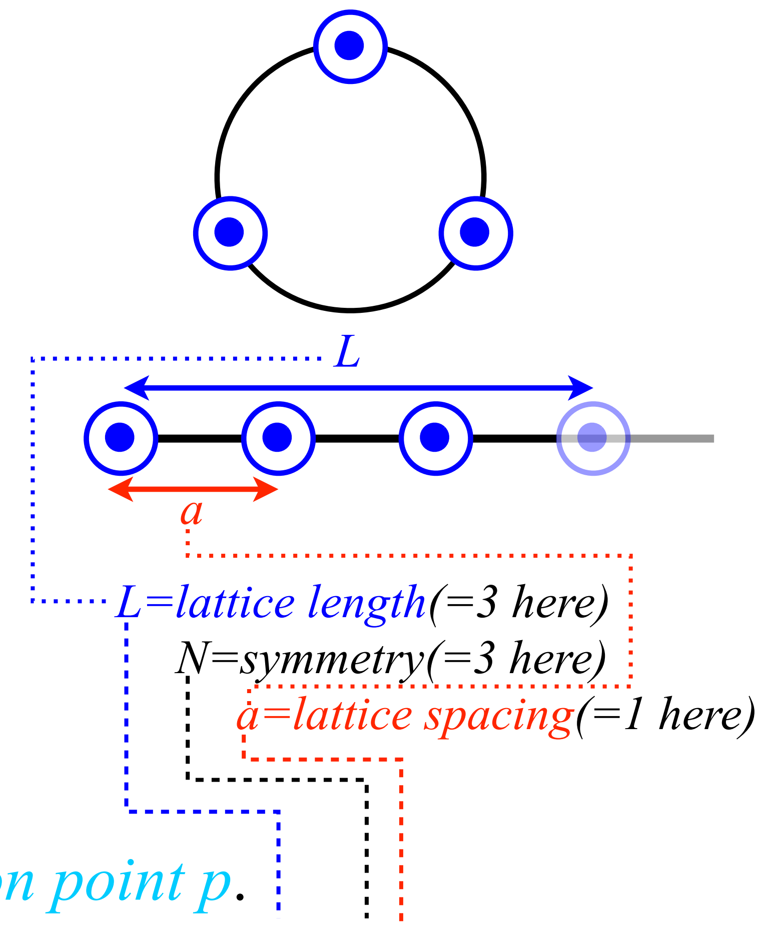

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WaveIt App - N3 Wave

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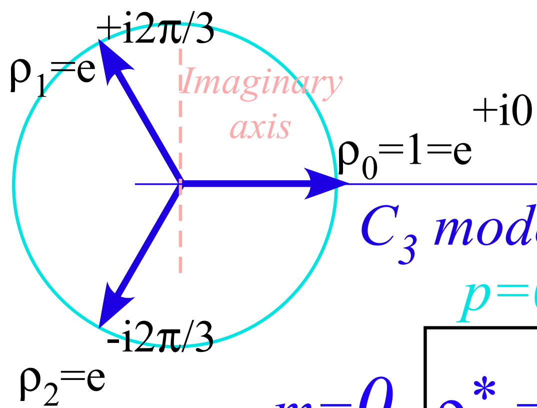

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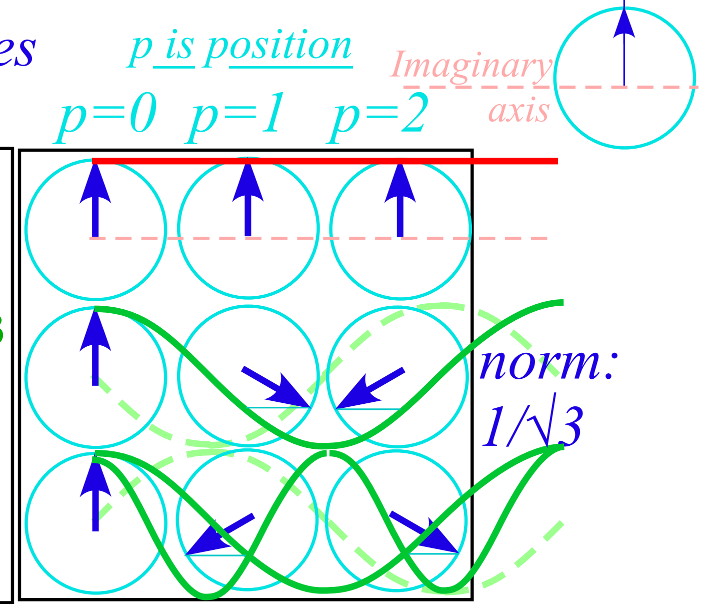

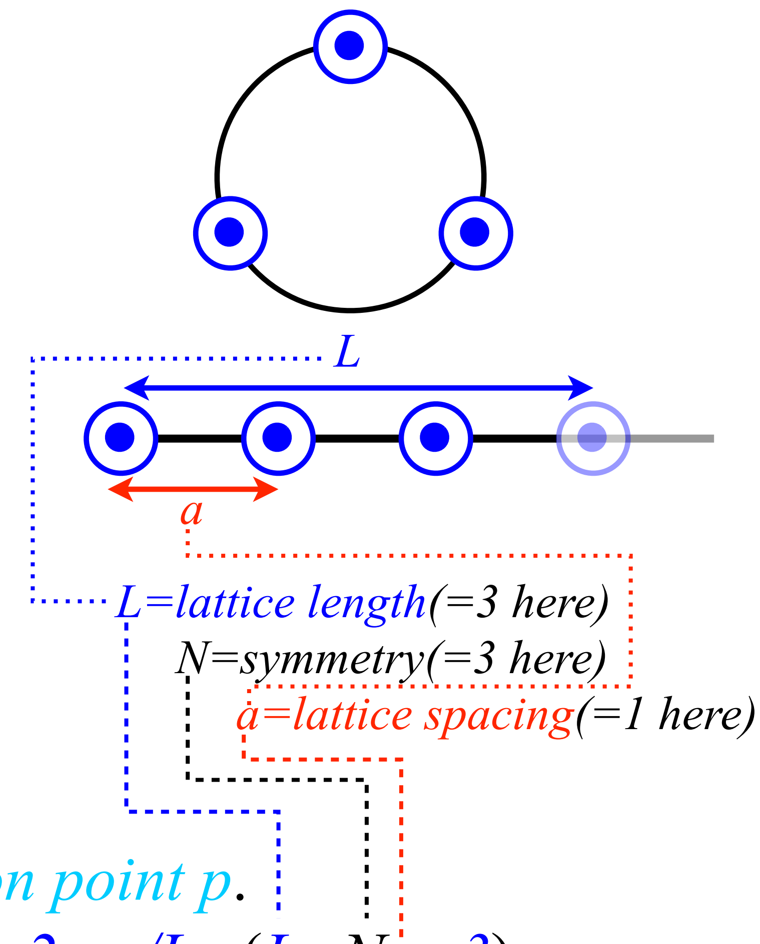

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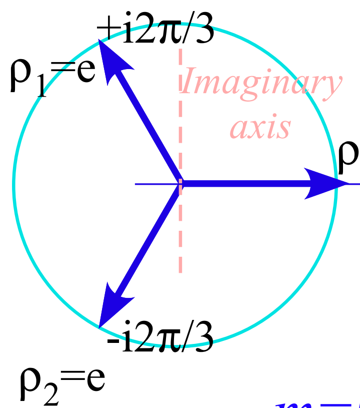

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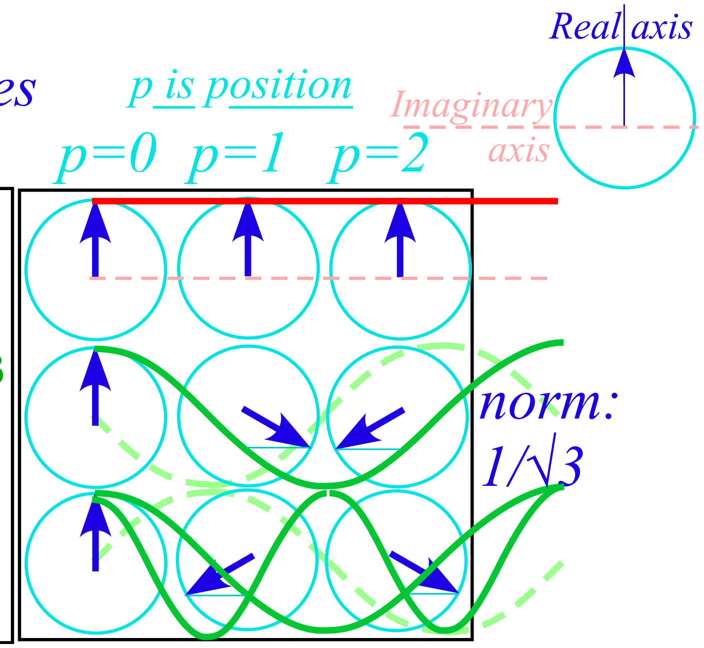

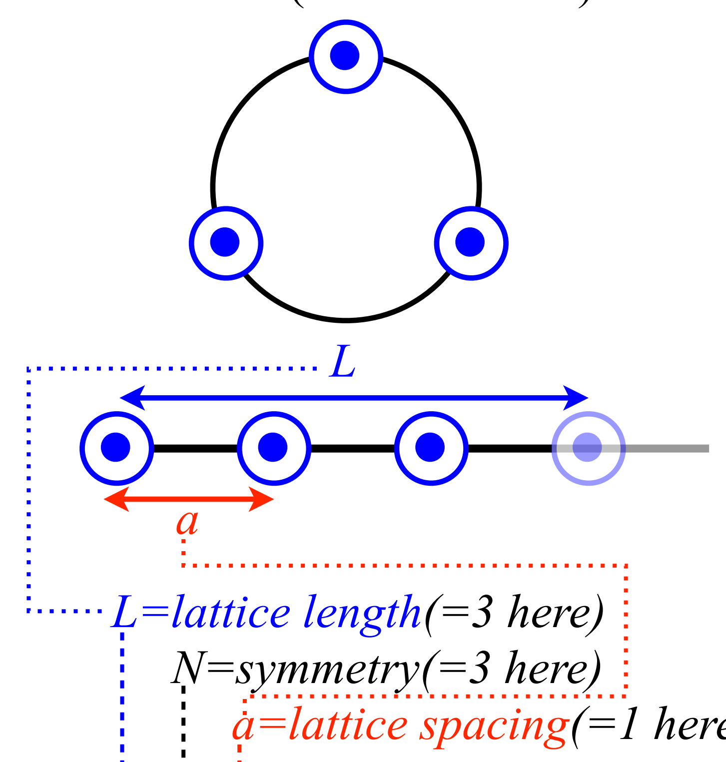

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For example, for $m=2$ and $p=2$ the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}=e^{i4\cdot 2\pi/3}=e^{i1\cdot 2\pi/3}\,e^{i2\pi}=e^{i2\pi/3}=\rho_1$.

That is, (*2-times-2*) mod *3* is not *4* but *1* (*4* mod *3=1*, the remainder of *4* divided by *3*.)

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m^{th} Eigenvalue of $\mathbf{r}^p$

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$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)\right) \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\, m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\, m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\, m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th} \text{ Eigenvalue of } \mathbf{r}^p \\ \langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\, m\cdot p\; 2\pi/3} \end{array}}$$

$$= r_0 e^{i\, m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} Eigenvalue of $\mathbf{r}^p$

$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r \ \ (\text{for } m = 0) \\ r_0 - r \ \ (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th}\ \textit{Eigenvalue of}\ \mathbf{r}^p \\ \langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3} \end{array}}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

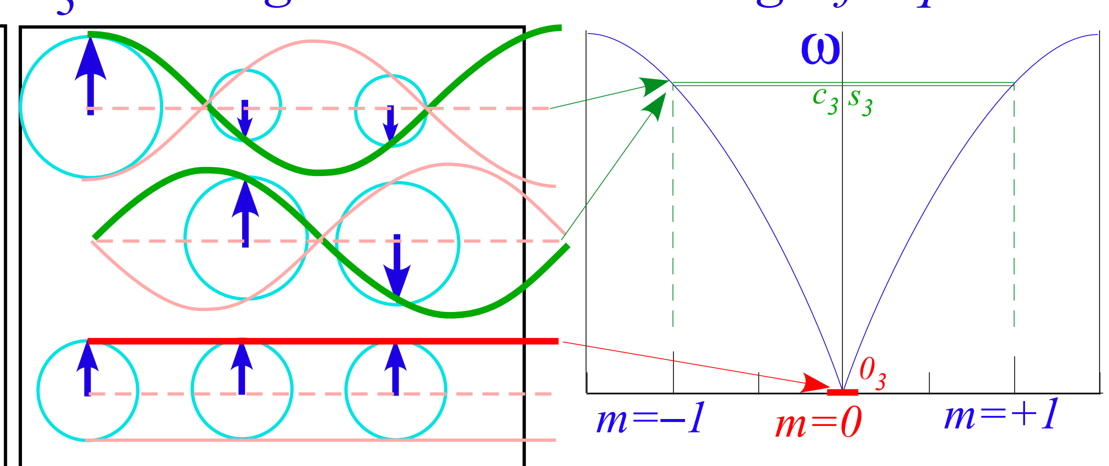
K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\vert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\vert c_3\rangle = \frac{\vert(+1)_3\rangle + \vert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\vert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\vert s_3\rangle = \frac{\vert(+1)_3\rangle - \vert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\vert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

	$p=0$	$p=1$	$p=2$
c_3	$2/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
s_3	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
$m=0_3$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

C_3 standing wave modes and eigenfrequencies of **K**



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

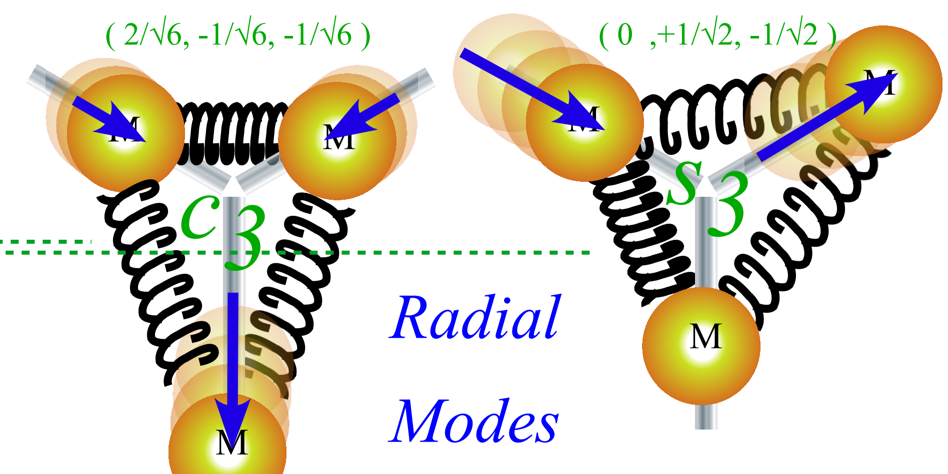
$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

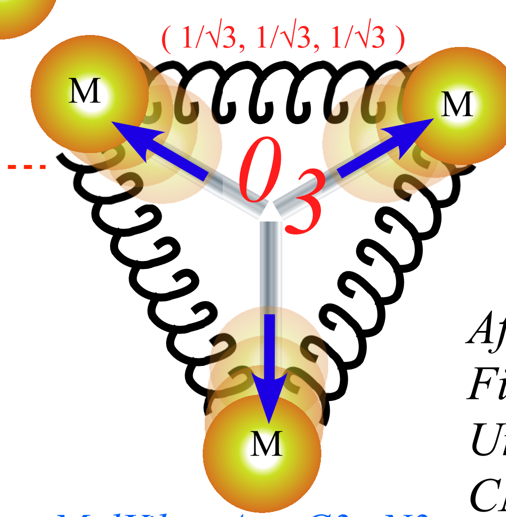
K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

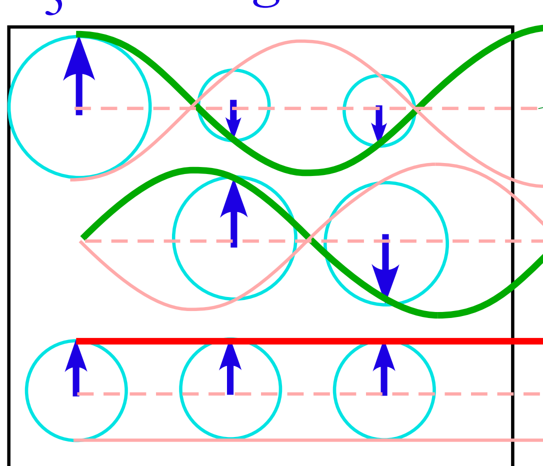
Transverse (to k) Waves

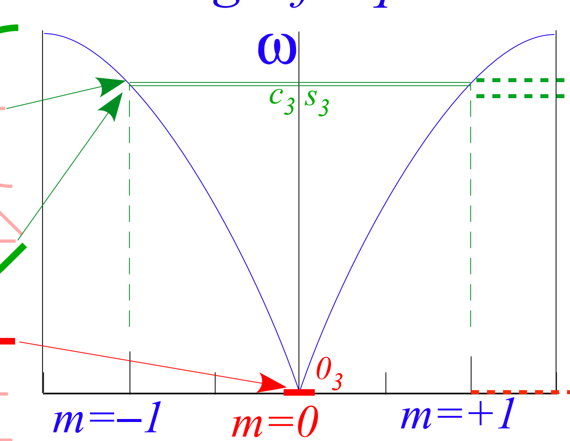




C₃ standing wave modes and eigenfrequencies of **K**

	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3





WaveIt App - N3 Wave

MolVibes App C3v N3

After:
Fig. 4.8.3
Unit 4
CMwBang

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} Eigenvalue of $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

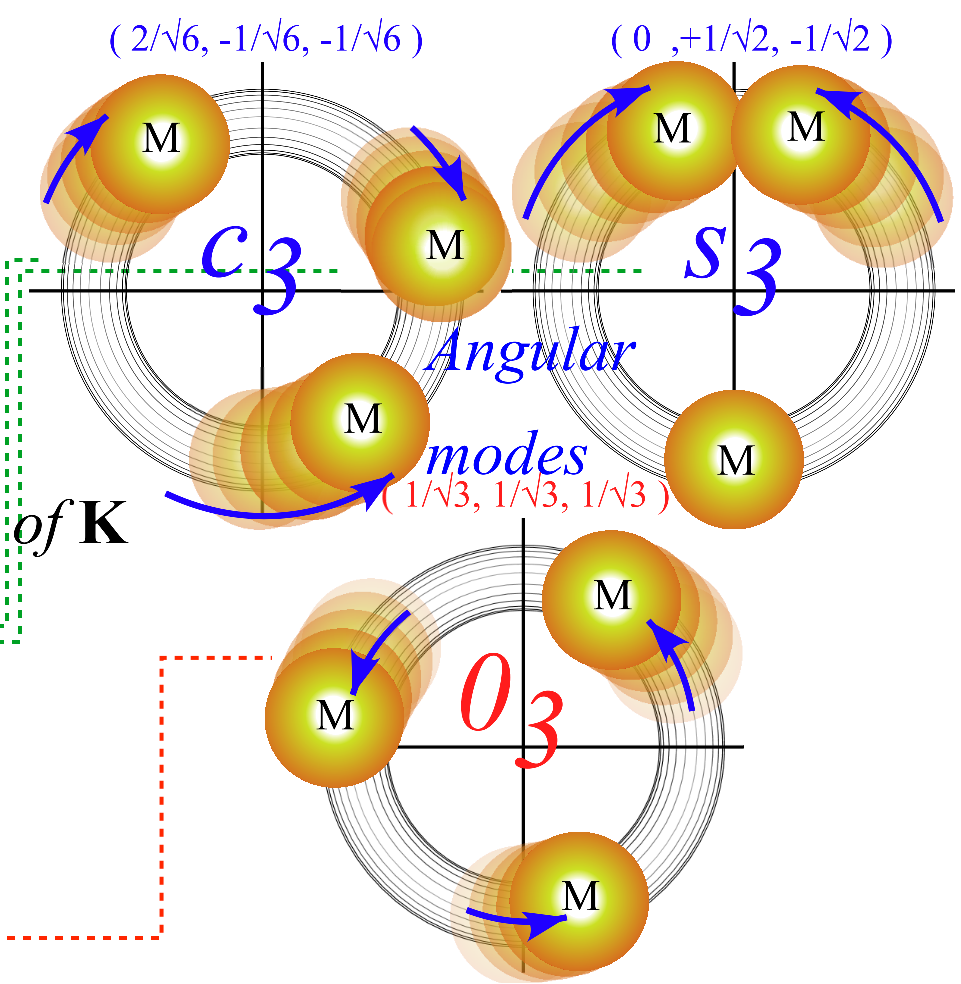
$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}$ – *eigenfrequencies*	$\mathbf{K}$ – *eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

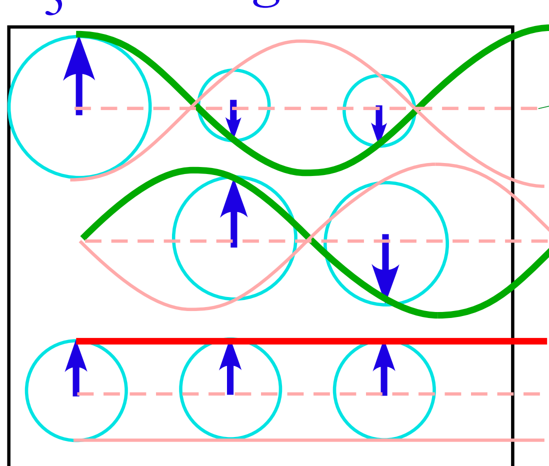
Longitudinal (to k) Waves

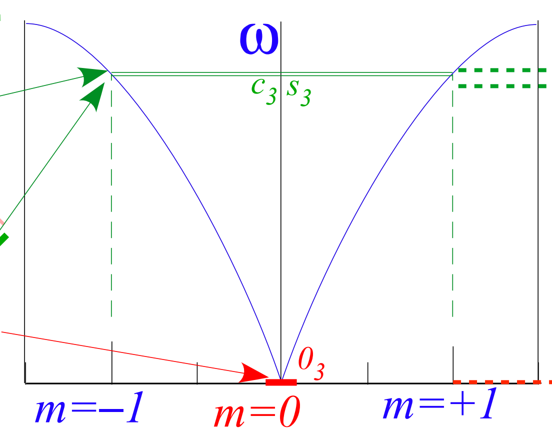


MolVibes App C3v N3

C_3 standing wave modes and eigenfrequencies of **K**

	$p=0$	$p=1$	$p=2$
c_3	$2/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
s_3	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
$m=0_3$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$





WaveIt App - N3 Wave

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

➡ *C_6 symmetric mode model:Distant neighbor coupling*

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N2
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N3
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N4
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N5
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N6(Snap below)

https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N2
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N3
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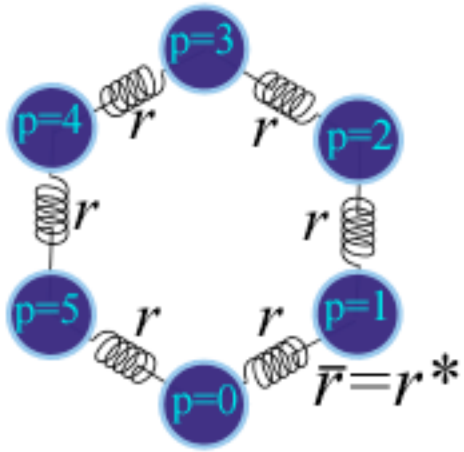
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N6

C_6 Symmetric Mode Model: Distant neighbor coupling

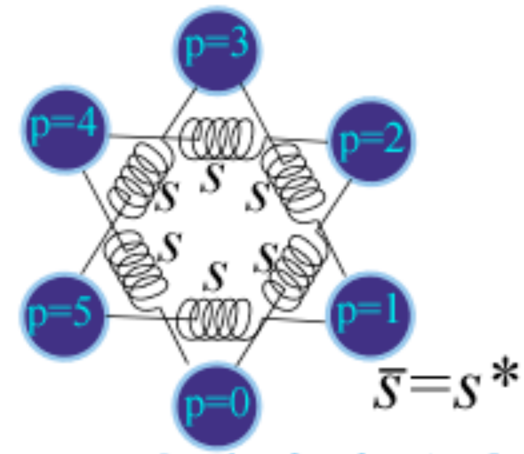
(a) 1^{st} Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{c} \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

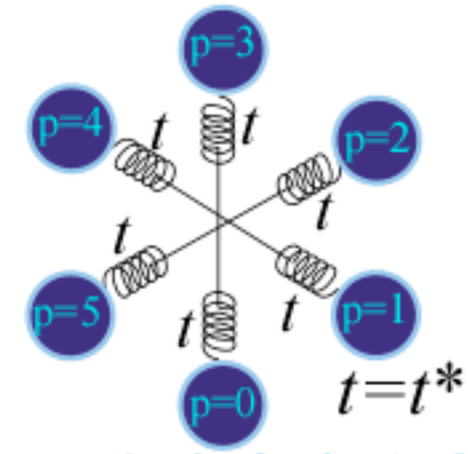
(b) 2^{nd} Neighbor C_6



$$\mathbf{H}^{B2(6)}=\begin{array}{c} \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot \\ \cdot & H_2 & \cdot & -s & \cdot & -\bar{s} \\ -\bar{s} & \cdot & H_2 & \cdot & -s & \cdot \\ \cdot & -\bar{s} & \cdot & H_2 & \cdot & -s \\ -s & \cdot & -\bar{s} & \cdot & H_2 & \cdot \\ \cdot & -s & \cdot & -\bar{s} & \cdot & H_2 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

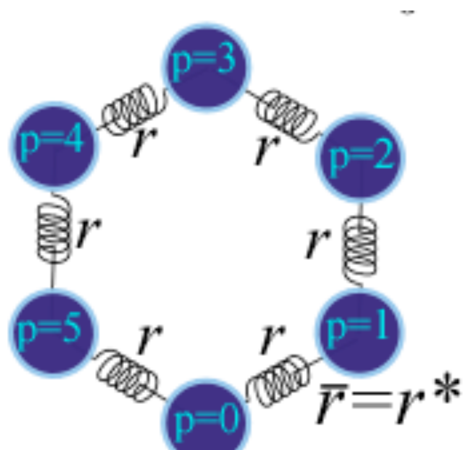
$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$

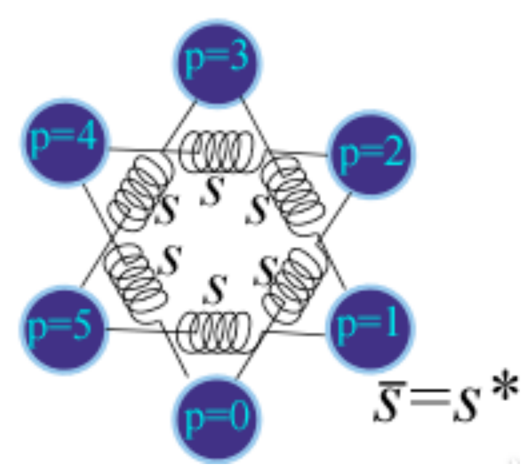
(c) 3^{rd} Neighbor C_6



$$\mathbf{H}^{B3(6)}=\begin{array}{c} \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} H_3 & \cdot & \cdot & -t & \cdot & \cdot \\ \cdot & H_3 & \cdot & \cdot & -t & \cdot \\ \cdot & \cdot & H_3 & \cdot & \cdot & -t \\ -t & \cdot & \cdot & H_3 & \cdot & \cdot \\ \cdot & -t & \cdot & \cdot & H_3 & \cdot \\ \cdot & \cdot & -t & \cdot & \cdot & H_3 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$= H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$$





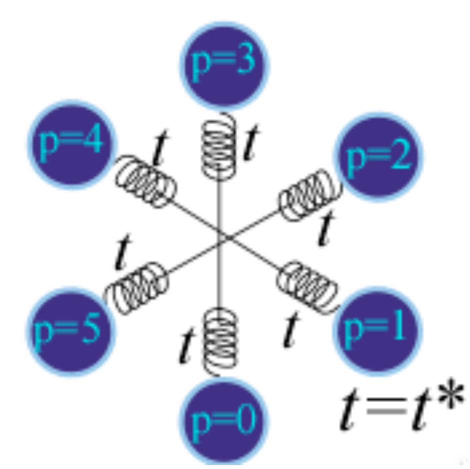
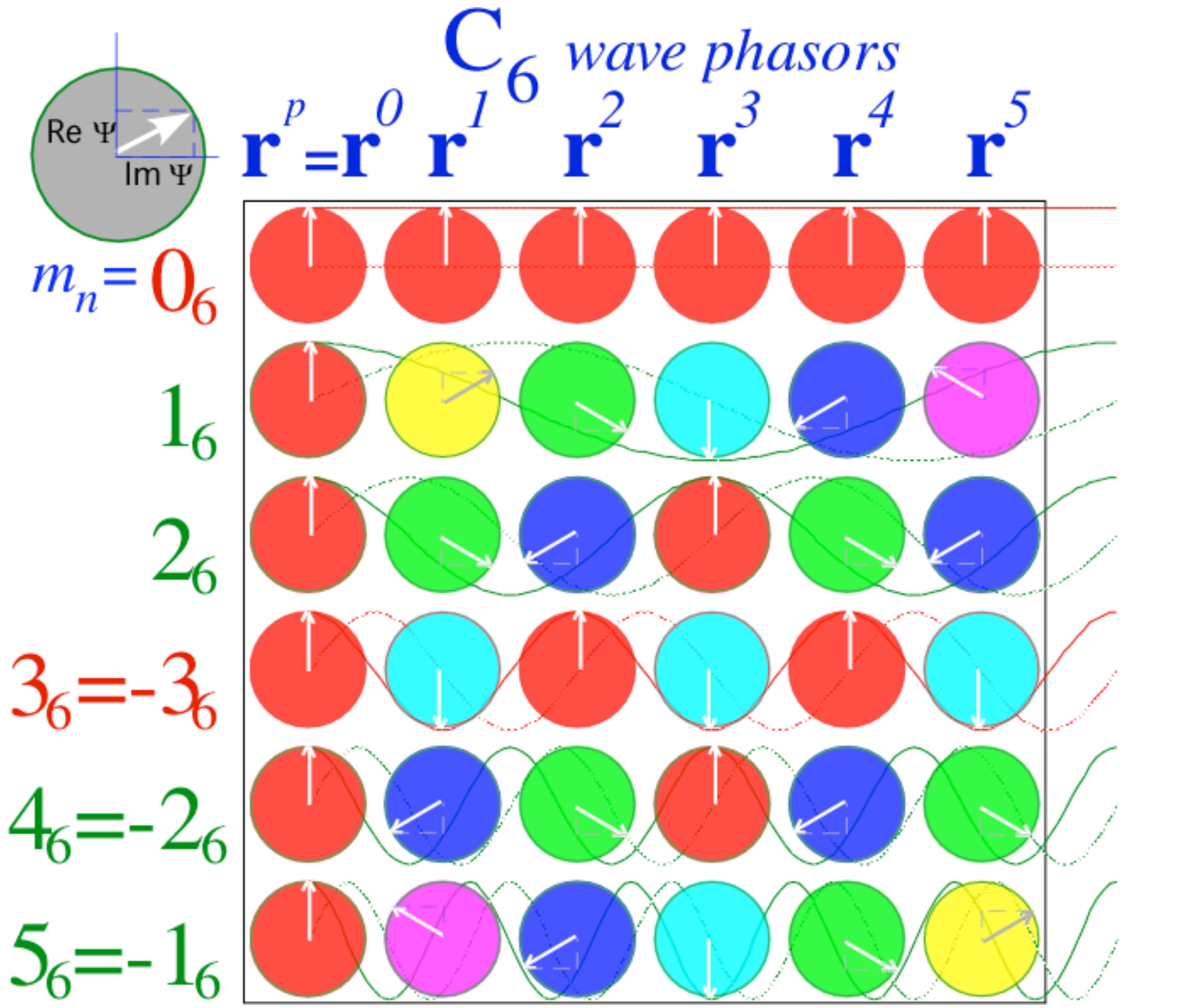


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Spectral resolution: 6th roots of unity

$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

$$\textit{Wavefunction}: \ \Psi^m(x_p) = \chi_p^{m*} = D^{m*}(\mathbf{r}^p)$$



$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{6}}$$

Fig. 13 International Journal of Molecular Science 14, 752 (2013) WaveIt C6 Character Phasors Simulation

C_6 Spectral resolution of n^{th} Neighbor H: Same modes but different dispersion

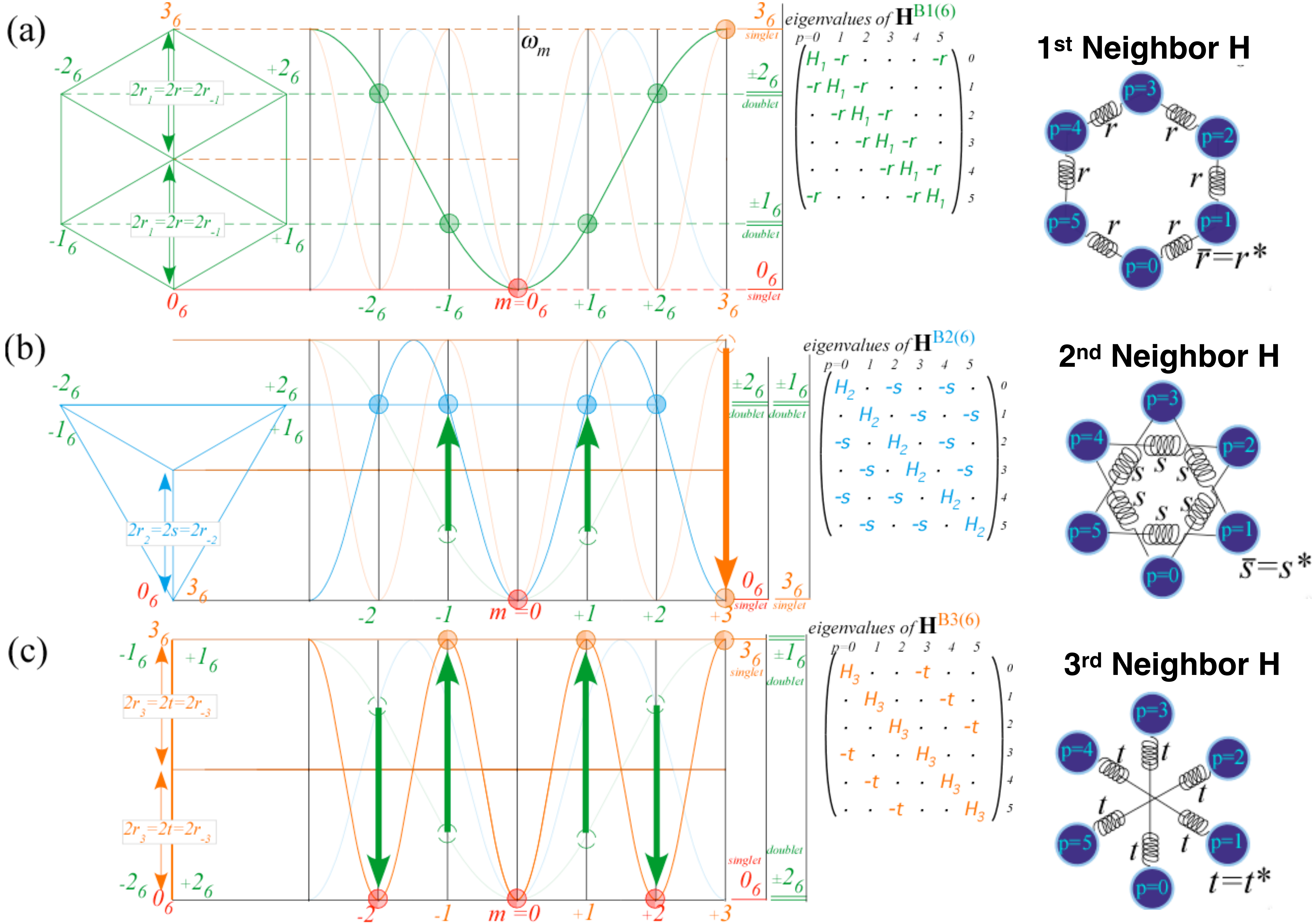


Fig. 14 International Journal of Molecular Science 14, 754 (2013)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

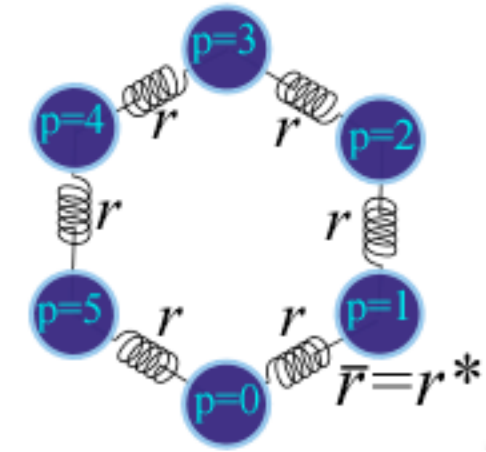
C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

C_6 Spectra of 1st neighbor gauge splitting by C-type (Chiral, Coriolis,…,

1st Neighbor H



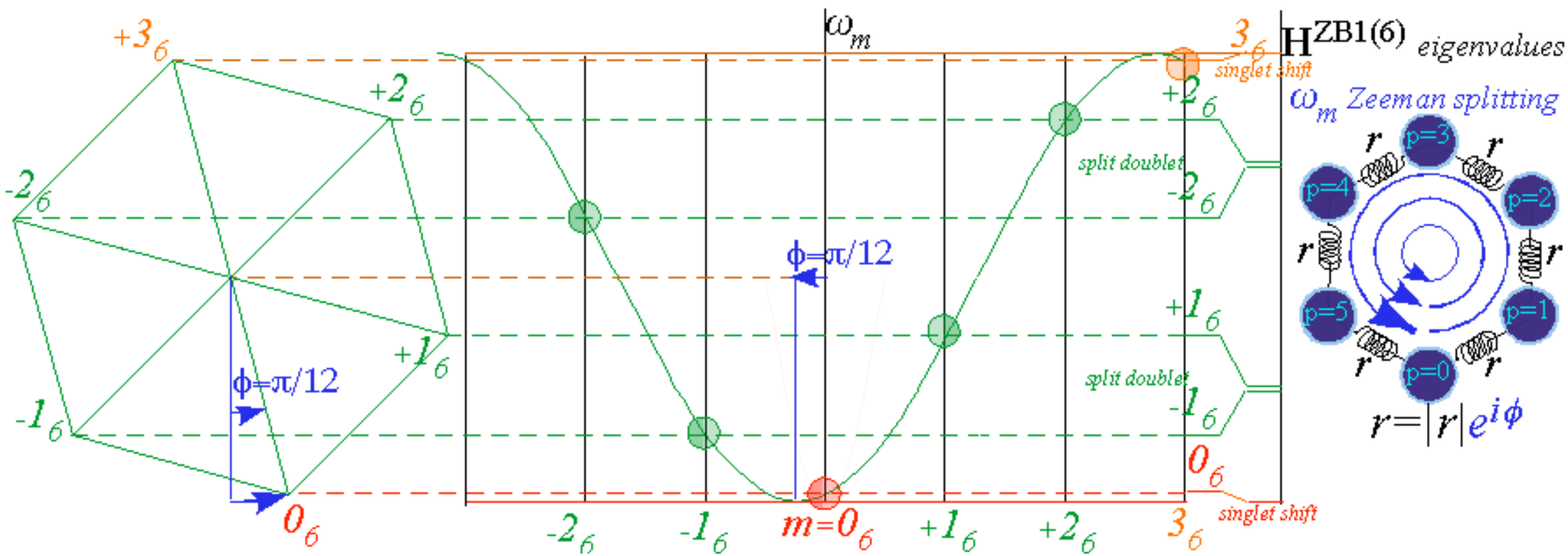


Fig. 15 International Journal of Molecular Science 14, 755 (2013)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

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C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

C_N Symmetric Mode Models:

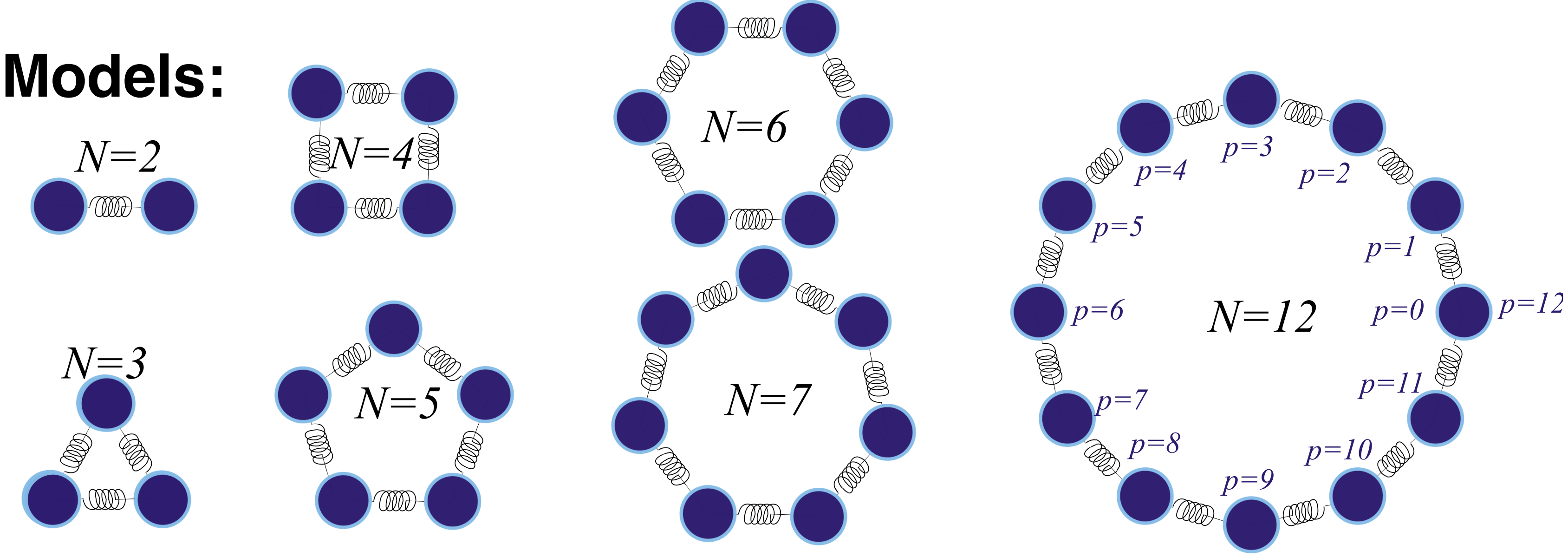


Fig. 4.8.4
Unit 4
CMwBang

C_N Symmetric Mode Models:

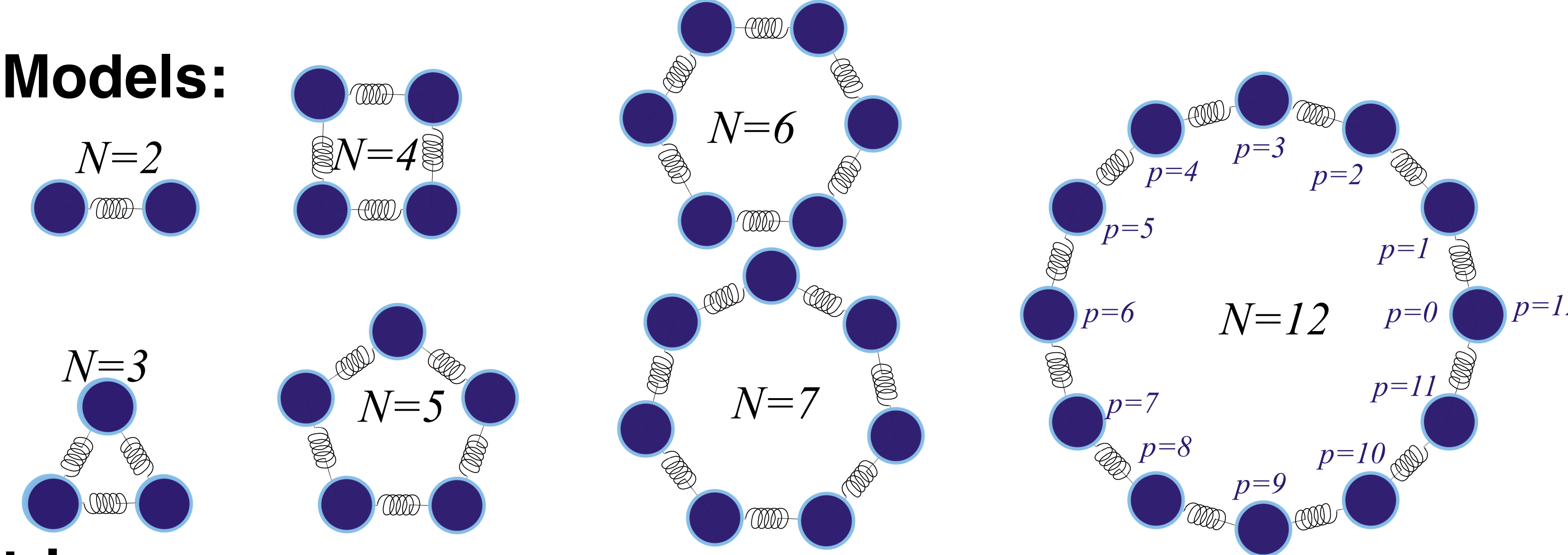


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix} \quad \text{where:} \quad \begin{gathered} K = k + 2k_{12} \\ k = \frac{Mg}{\ell} \\ (\cdot) = 0 \end{gathered}$$

C_N Symmetric Mode Models:

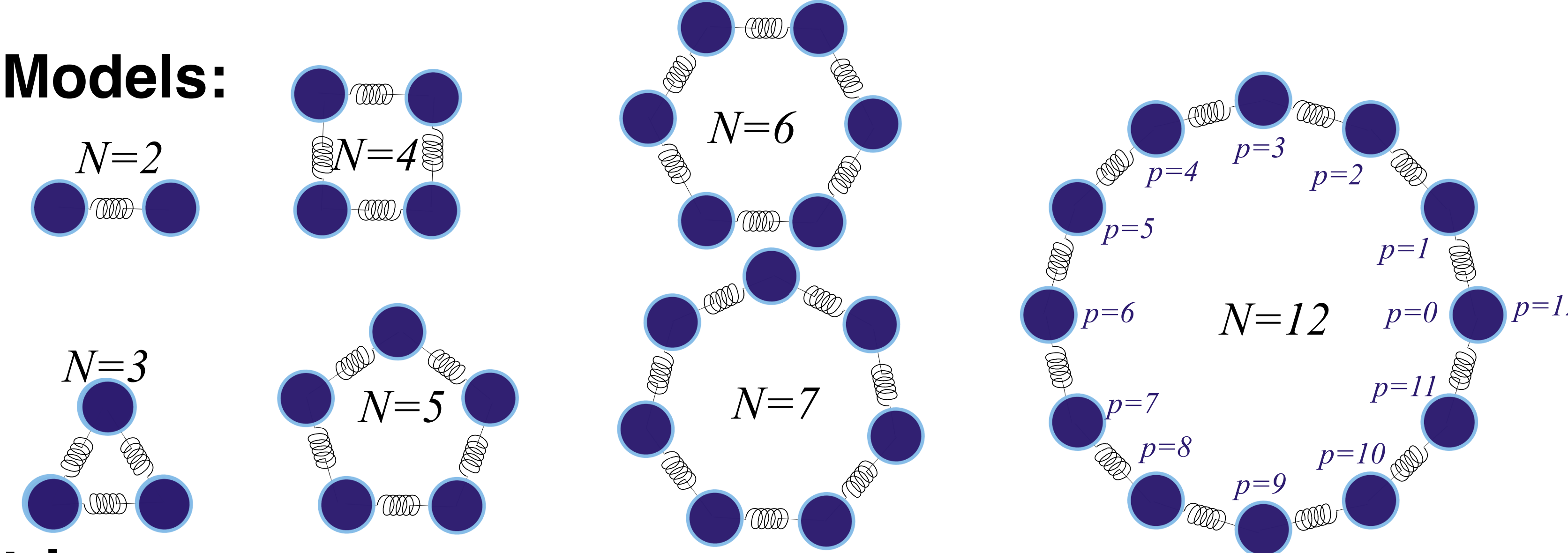


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

where: $K = k + 2k_{12}$

$k = \dfrac{Mg}{\ell}$

$(\cdot) = 0$

N^{th} roots of 1 $e^{\,i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

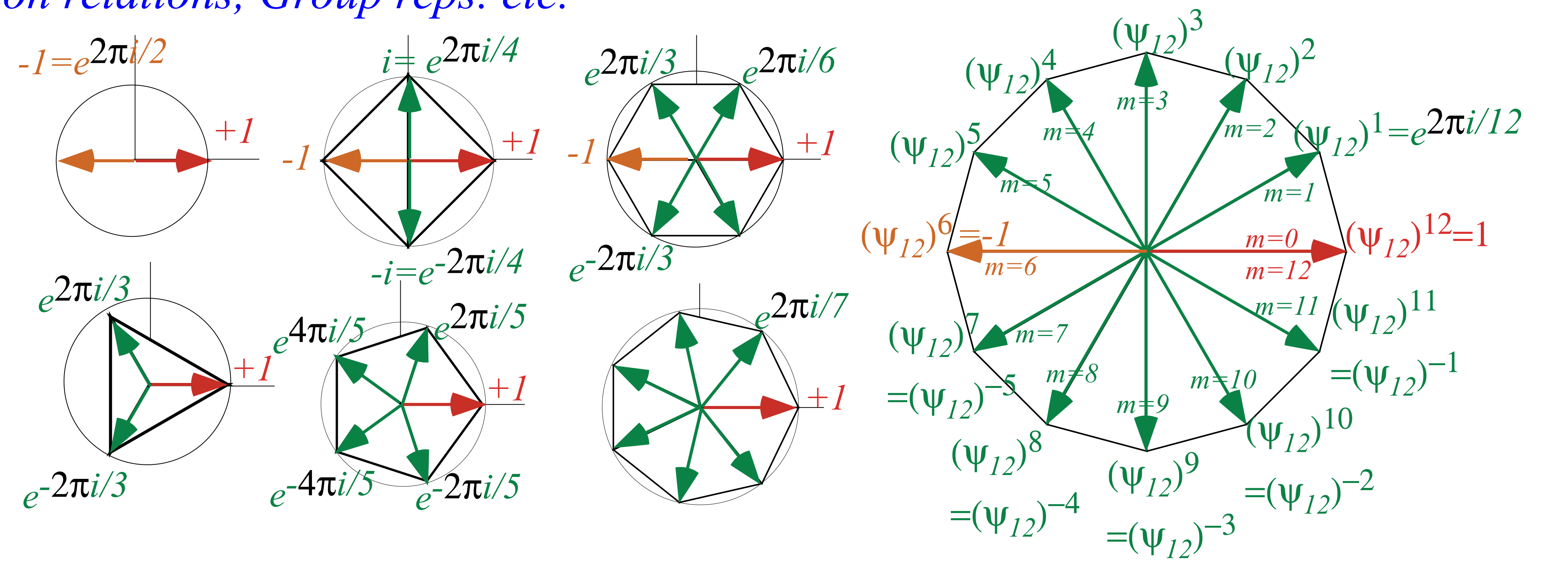
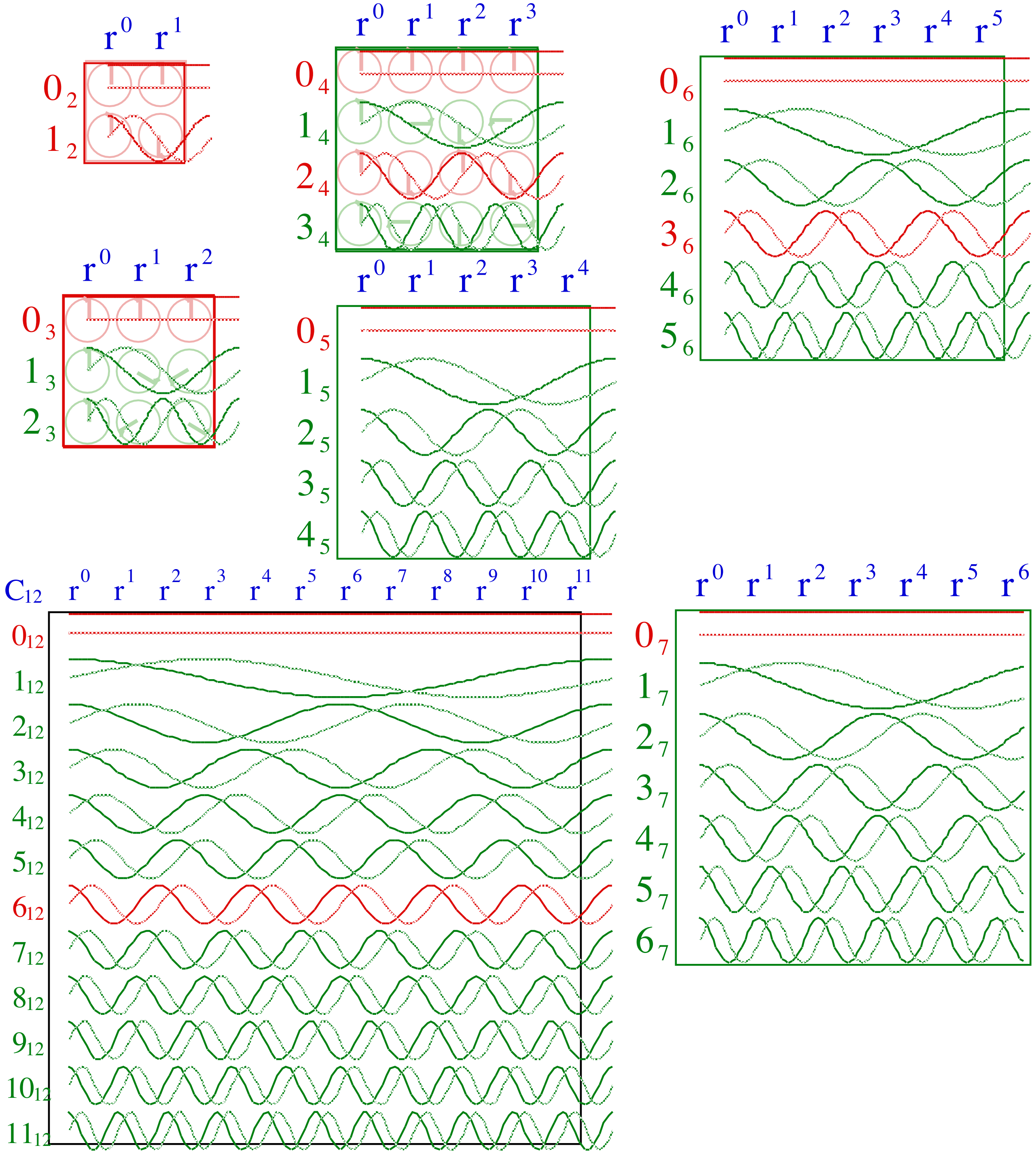


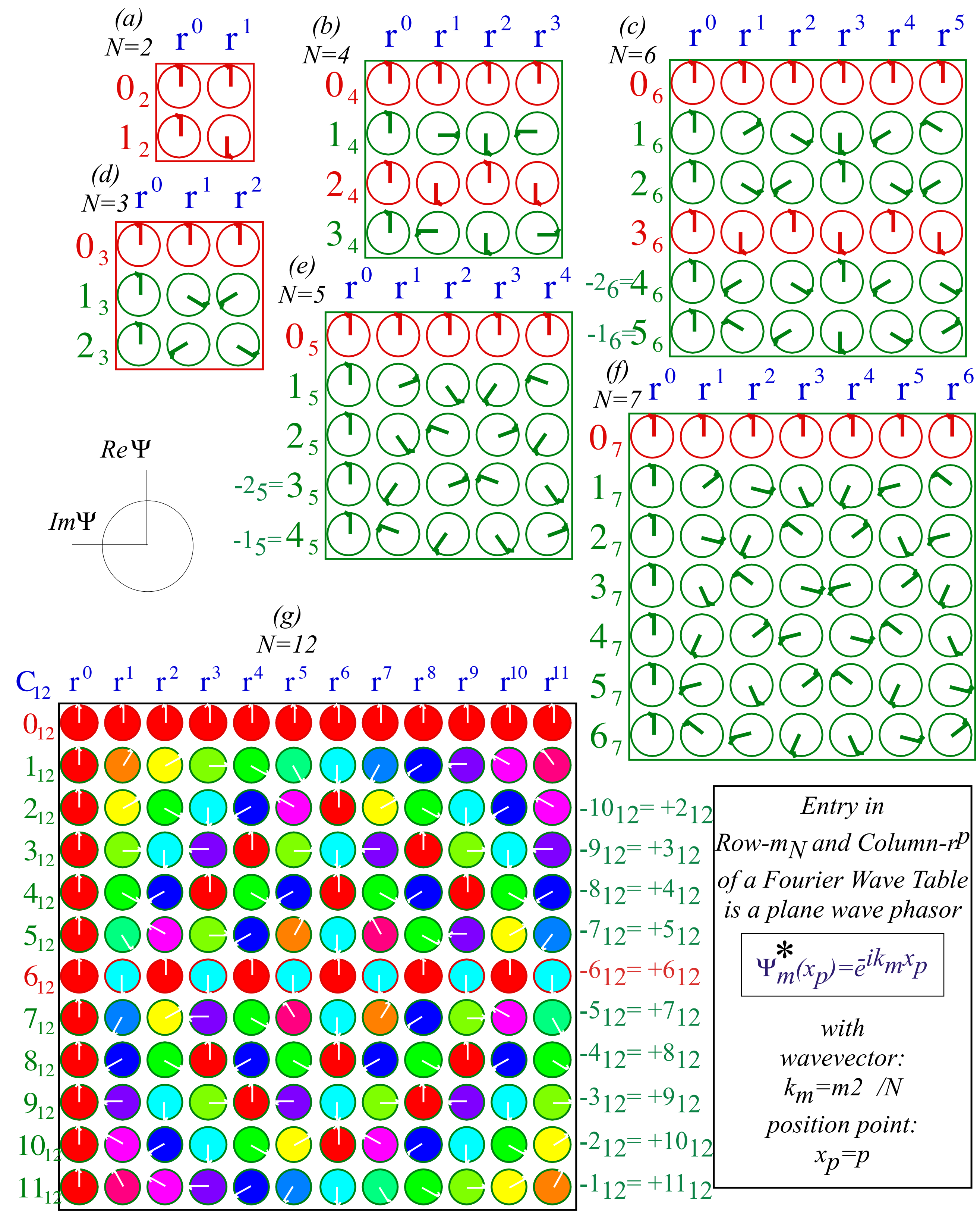
Fig. 4.8.5
Unit 4
CMwBang

C_N Symmetric Mode Models:

N^{th} roots of 1 *$e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*



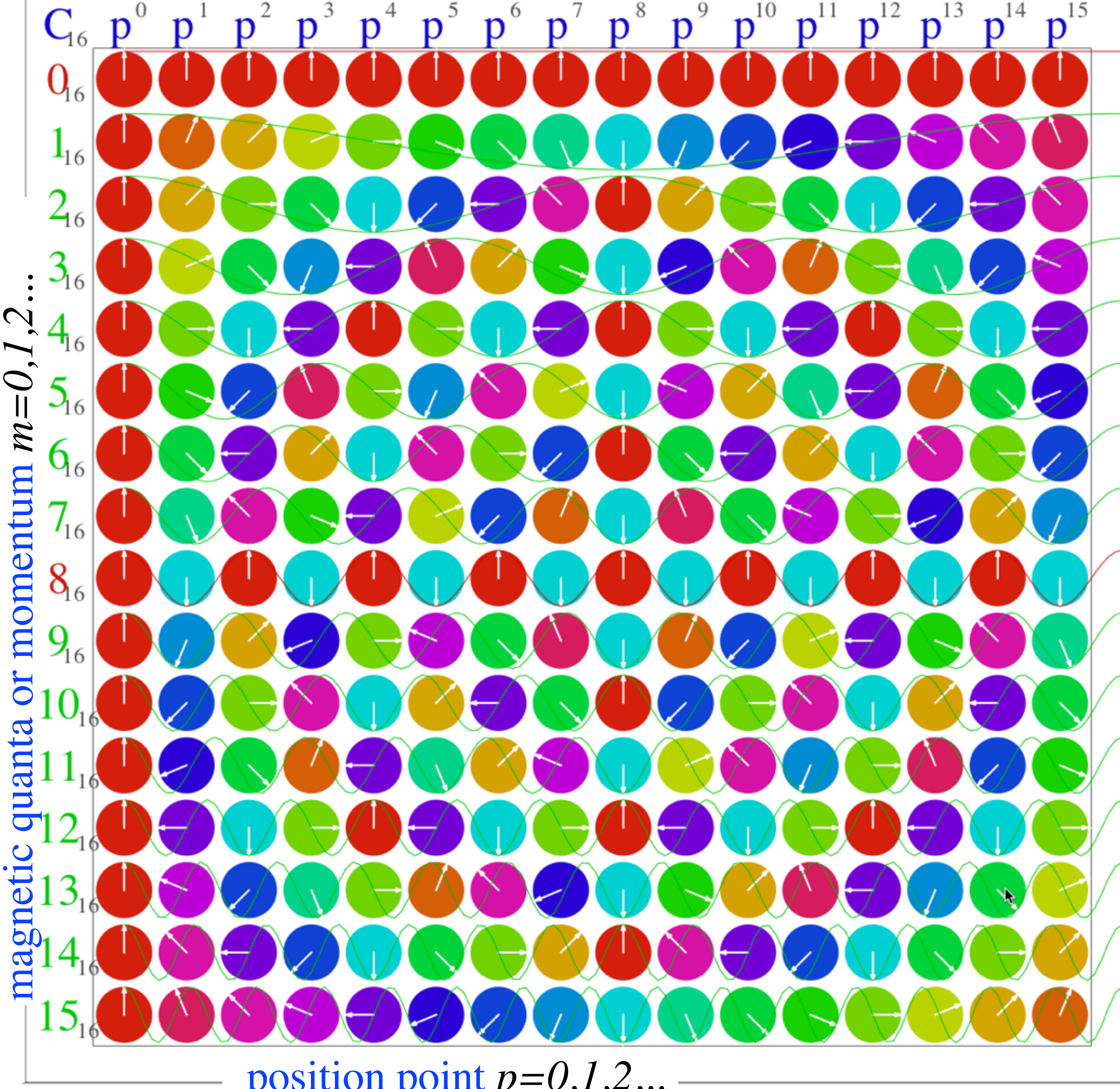
[WaveIt C12 Web Simulation]



[WaveIt C12 Character Phasors Web Simulation]

Fig. 4.8.6-7
Unit 4
CMwBang

Fourier transformation matrices

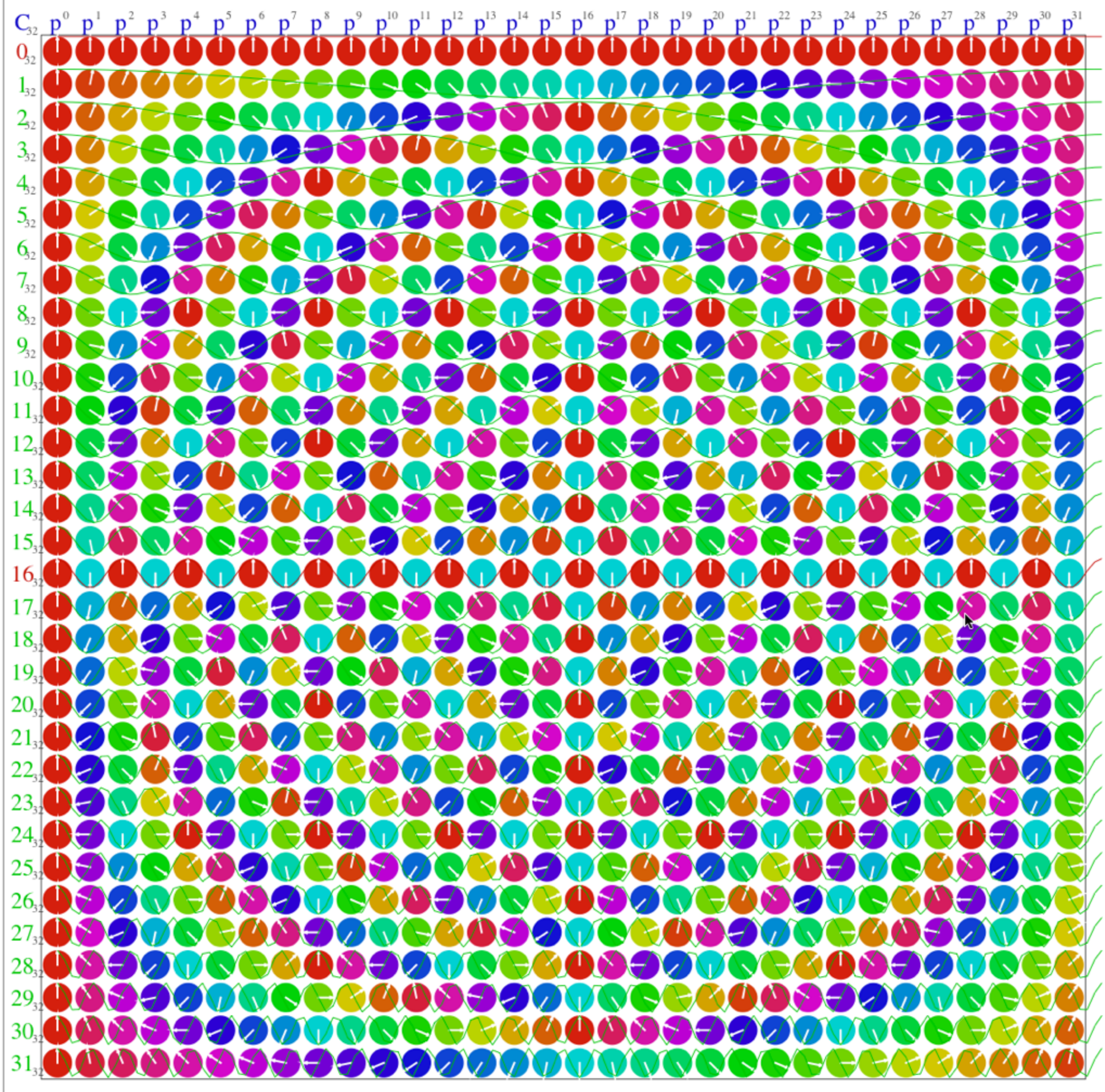


C_{16}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{16}}$$

WaveIt C16 Character Phasors
Web Simulation



magnetic quanta or momentum $m=0,1,2\ldots$

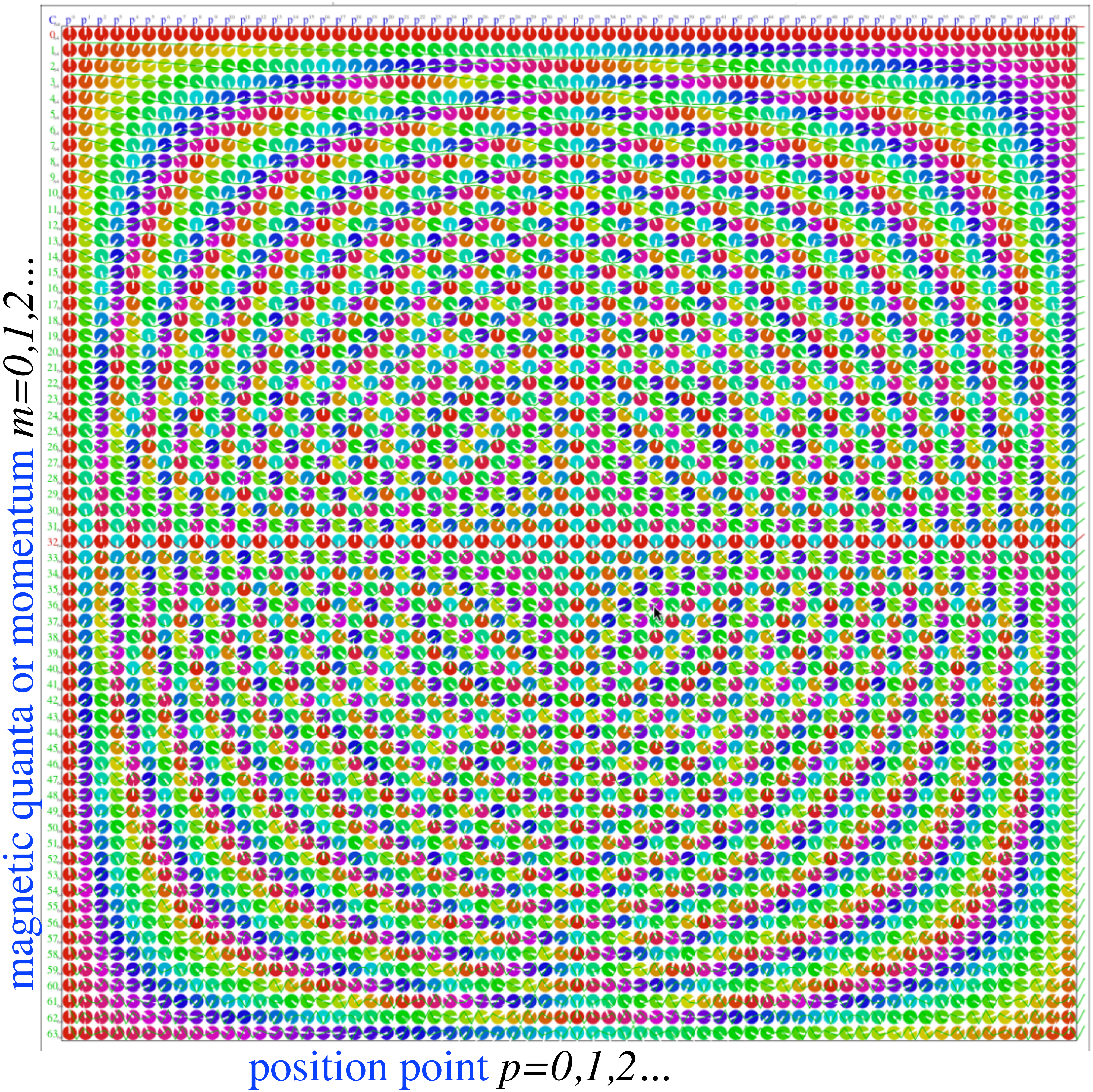
position point $p=0,1,2\ldots$

C_{32}

phasor character table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{32}}$$

[WaveIt C32 Character Phasors Web Simulation]



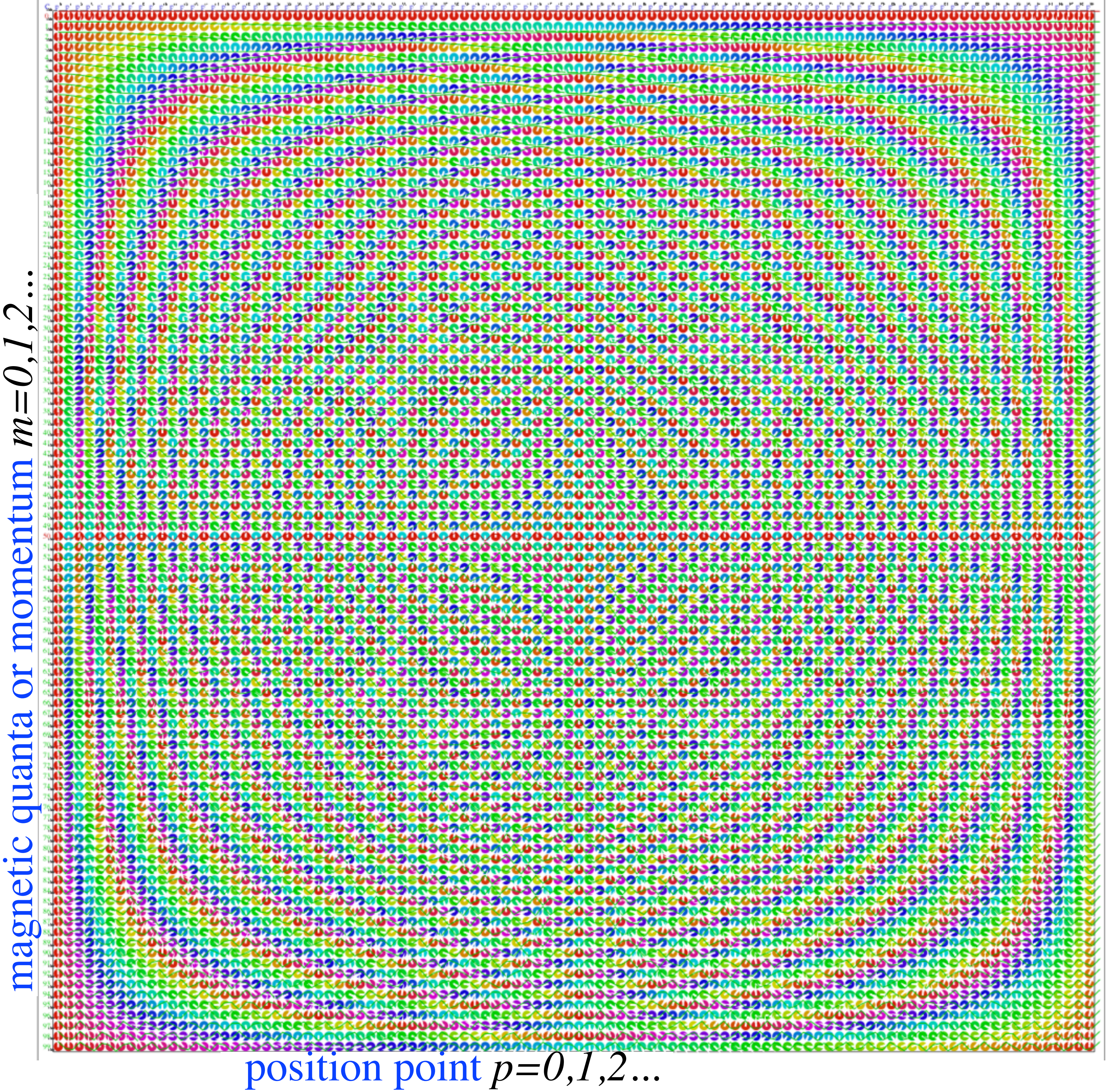
C_{64}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{64}}$$

Invariant phase "Uncertainty" hyperbolas: $m{\cdot}p=const.$



C_{100}

phasor
character
table

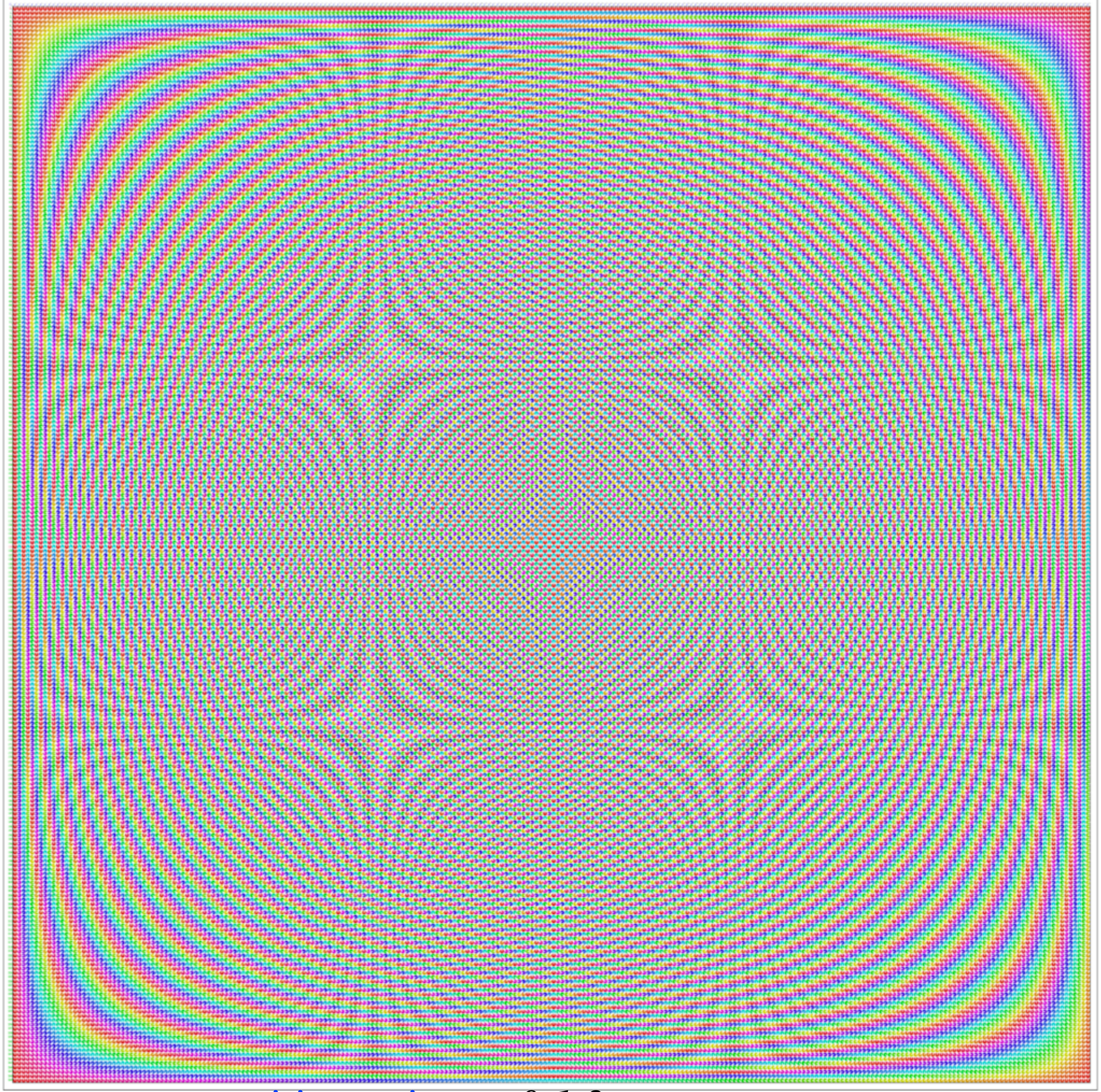
$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{100}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$

C_{256}

phasor character table



magnetic quanta or momentum $m=0,1,2...$

position point $p=0,1,2...$

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{256}}$$

Invariant phase "Uncertainty" hyperbolas:

$m \cdot p = const.$

[WaveIt C256 Character Phasors Web Simulation]

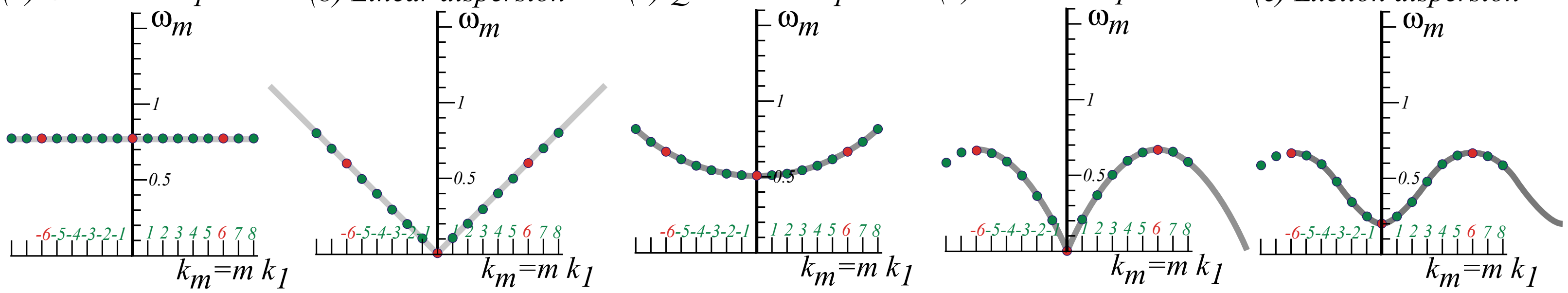
Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
- *C_N symmetric mode models: Made-to order dispersion functions*
    - ➔ *Quadratic dispersion models: Super-beats and fractional revivals*
    - *Phase arithmetic*

C_N Symmetric Mode Models: Made-to-Order Dispersion *(and wave dynamics)*

(Making pure linear $\omega=ck$, quadratic $\omega=ck^2$, etc. ?)

Archetypical Examples of Dispersion Functions



Applications:

(a)	(b)	(c)	(d)	(e)
Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis Xmas lights	*Light in vacuum (Exactly) Sound (Approximately)*	*Light in fiber (Approx) Non-relativistic Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids Relativistic matter (If exact hyperbola)*

$$a = k_a \cdot x - \omega_a \cdot t$$

$$b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right)$$

$$= e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

Reading Wave Velocity From Dispersion Function by (k,ω) Vectors

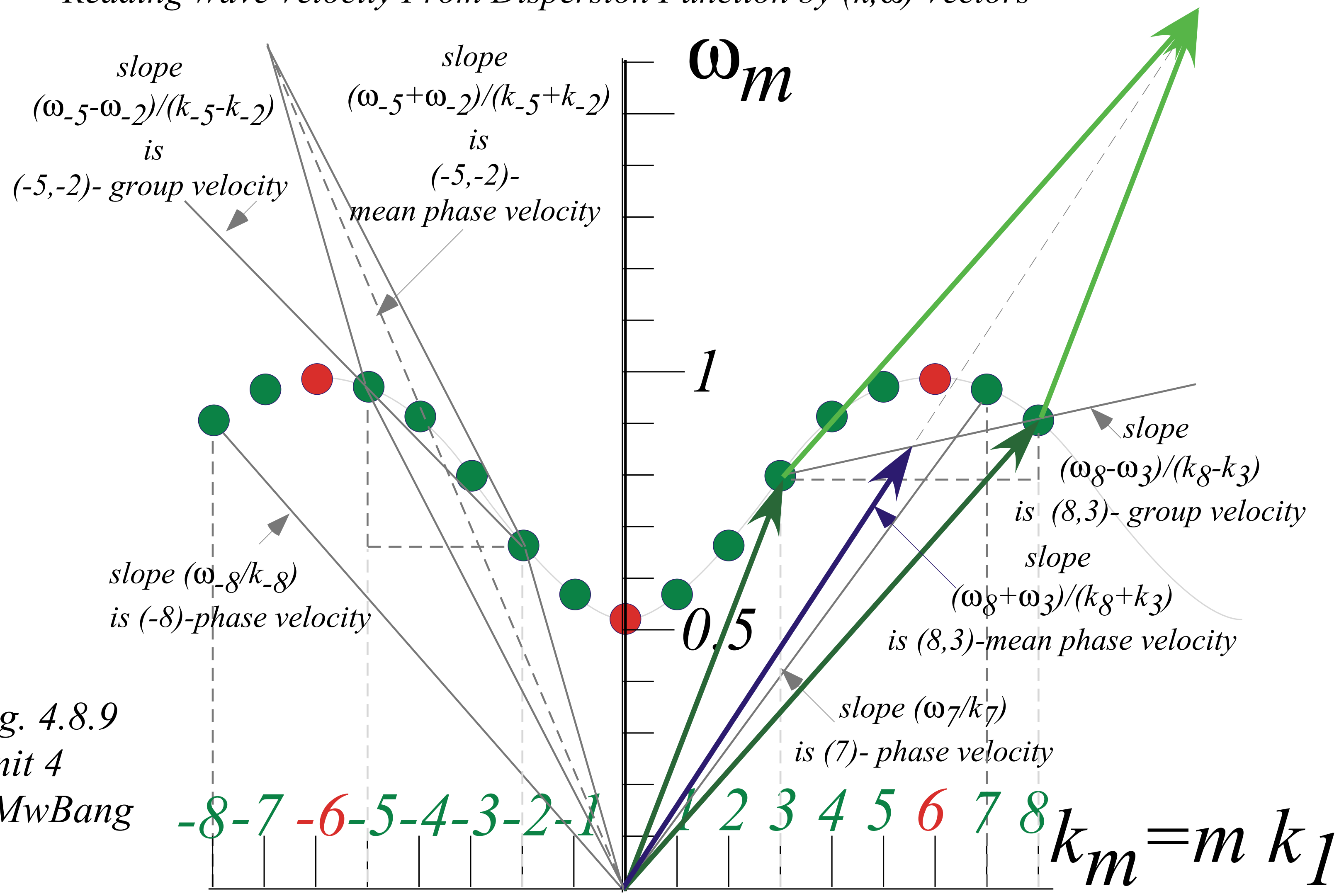


Fig. 4.8.9
Unit 4
CMwBang

Things determined by

Dispersion $\omega = \omega(k)$

Individual phase velocity:

$$V_{phase-1} = \frac{\omega(k)}{k}$$

Pairwise phase velocity:

$$V_{phase-2} = \frac{\omega(k_a)+\omega(k_b)}{k_a+k_b}$$

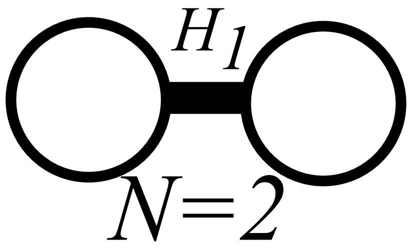
Pairwise group velocity:

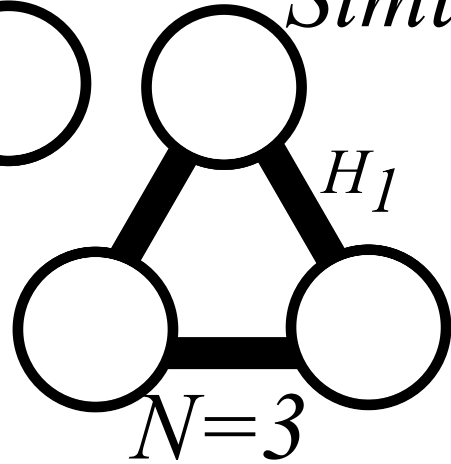
$$V_{group-2} = \frac{\omega(k_a)-\omega(k_b)}{k_a-k_b}$$

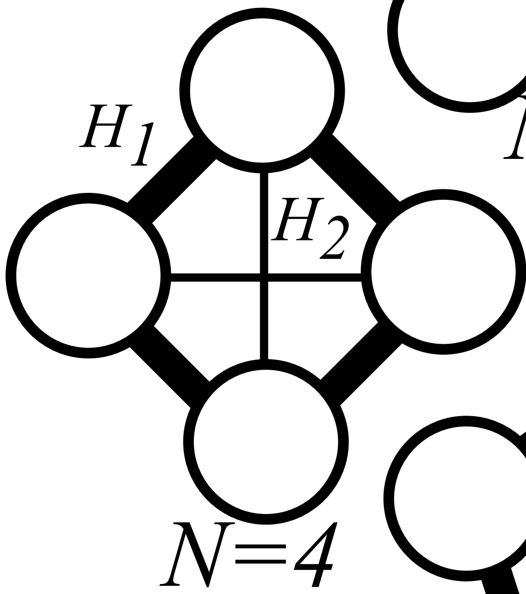
Simulating Complex Systems With Simpler Ones

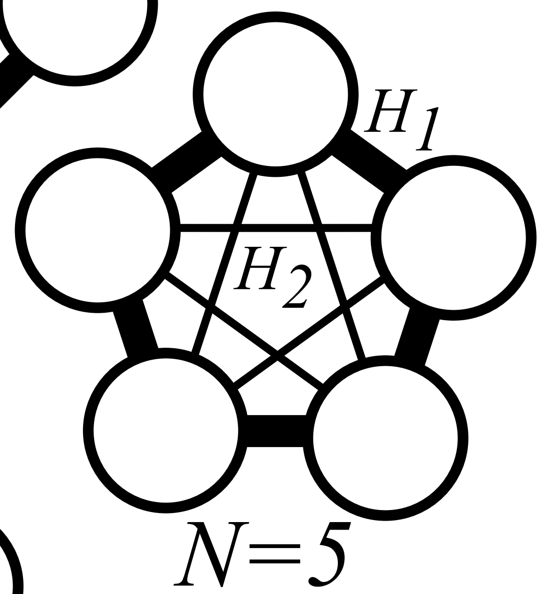
Made of Quantum Dots

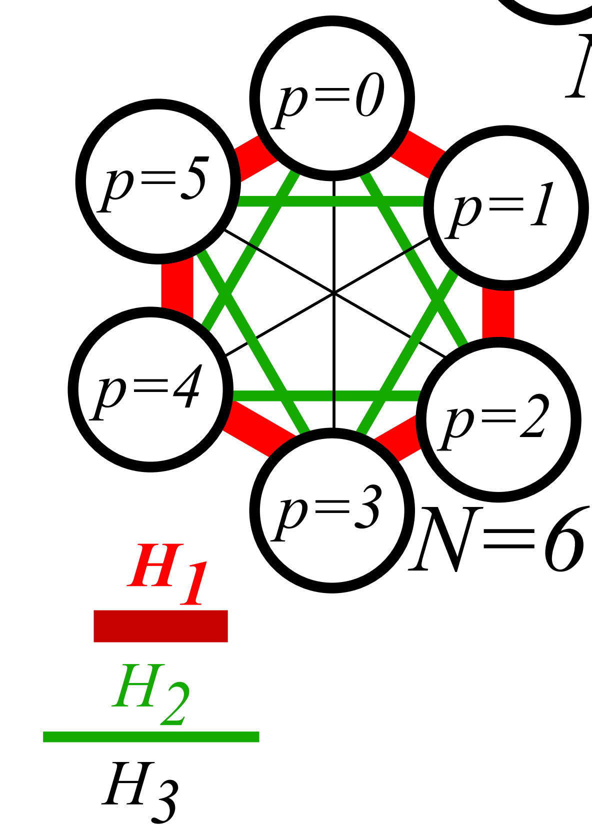
[Harter, J. Mol. Spec. 210, 166-182 (2001)]











Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H_1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

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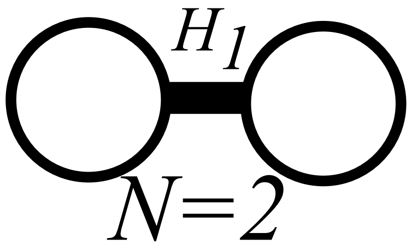
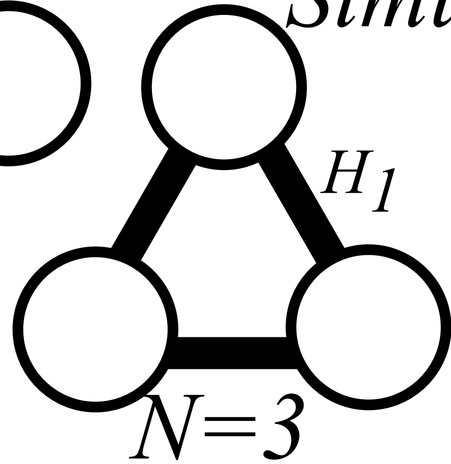
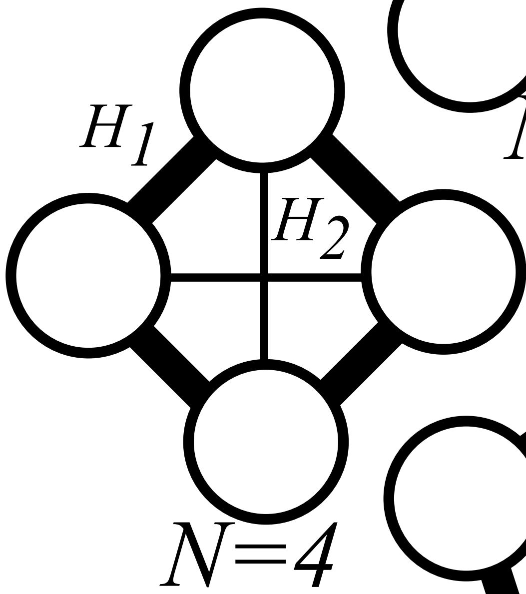
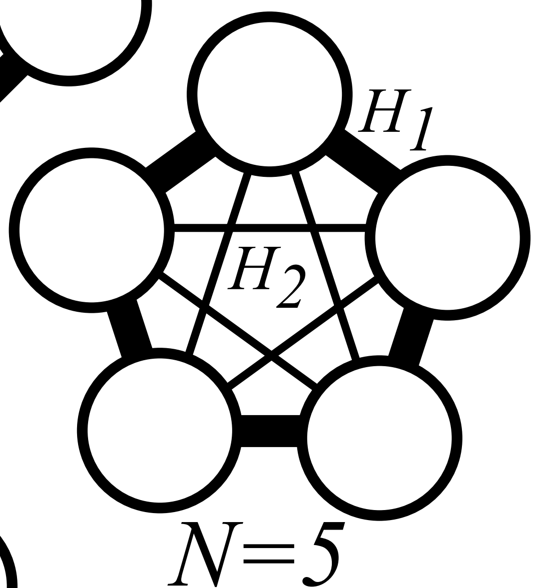
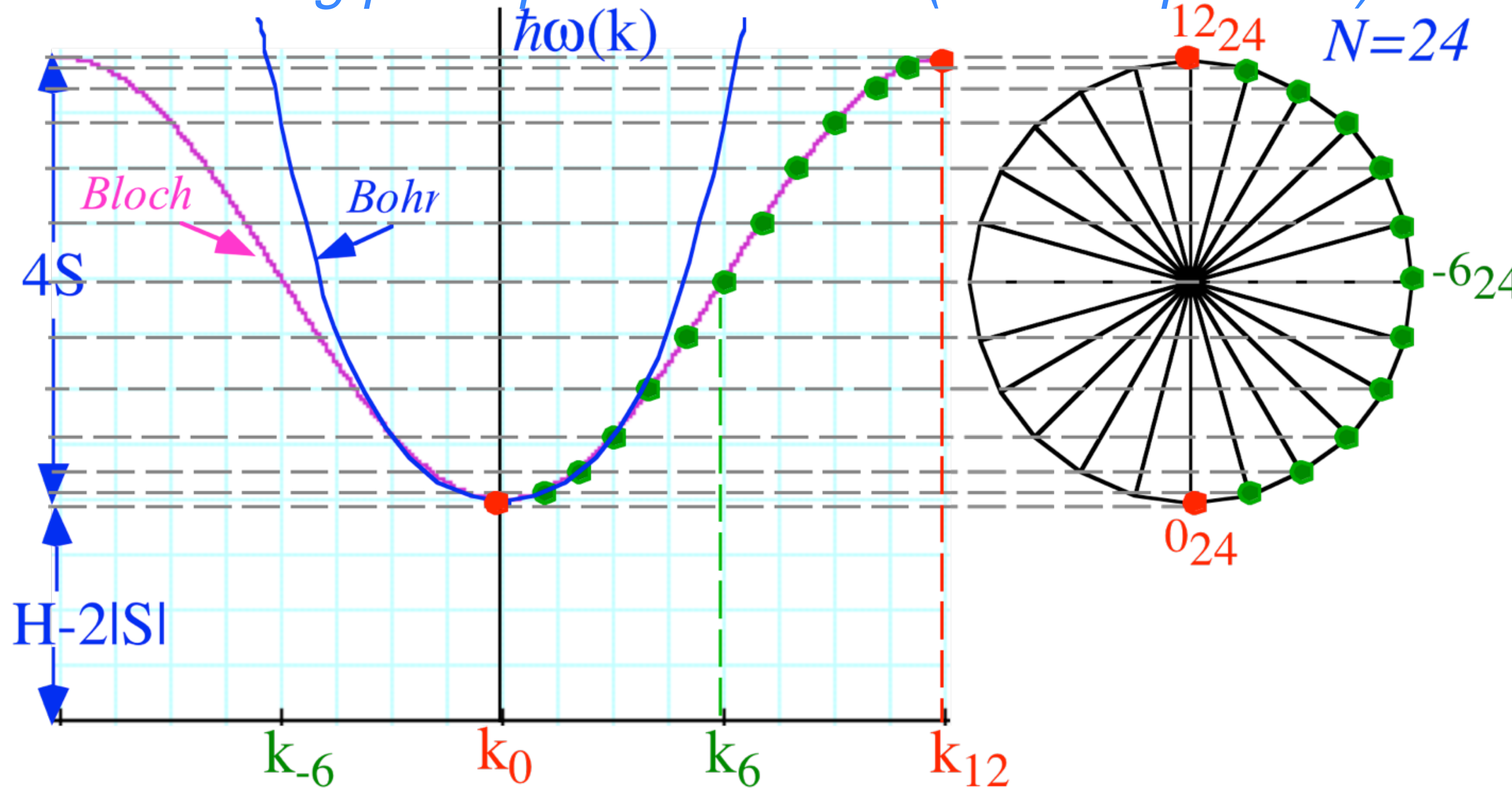


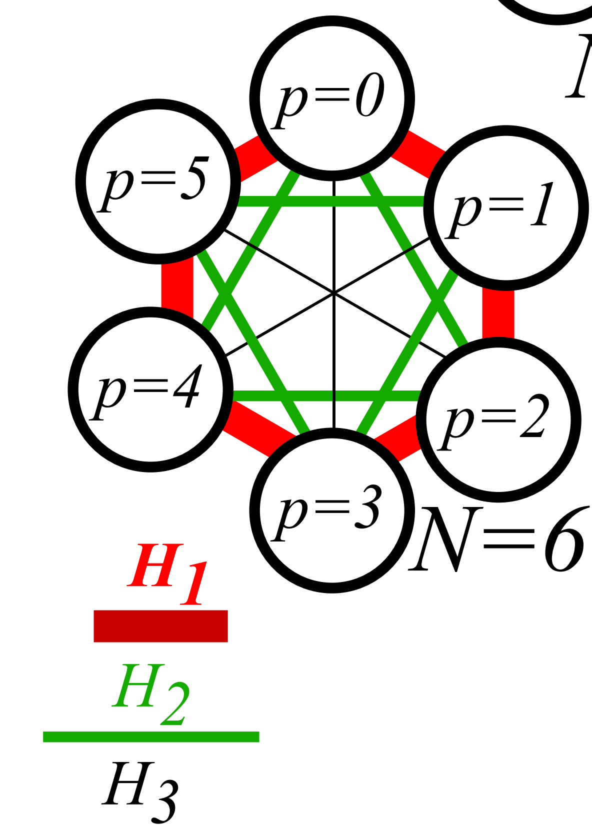






Making pure quadratic $\omega = ck^2$ (Bohr dispersion)





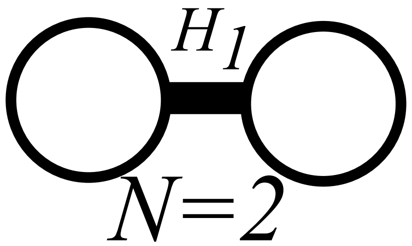
Hexagonal 2D Rotor

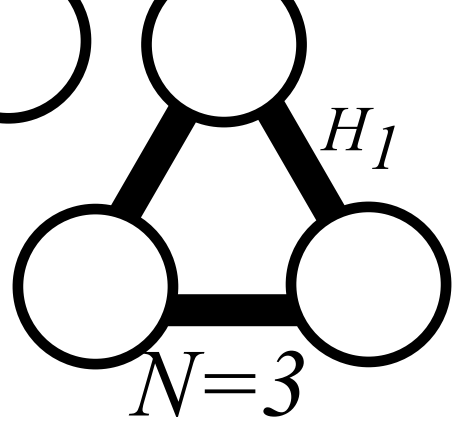
$$\begin{pmatrix} H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} \\ \mathbf{H1} & H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 \\ H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 & H_3 \\ H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 \\ H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} \\ \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 \end{pmatrix}$$

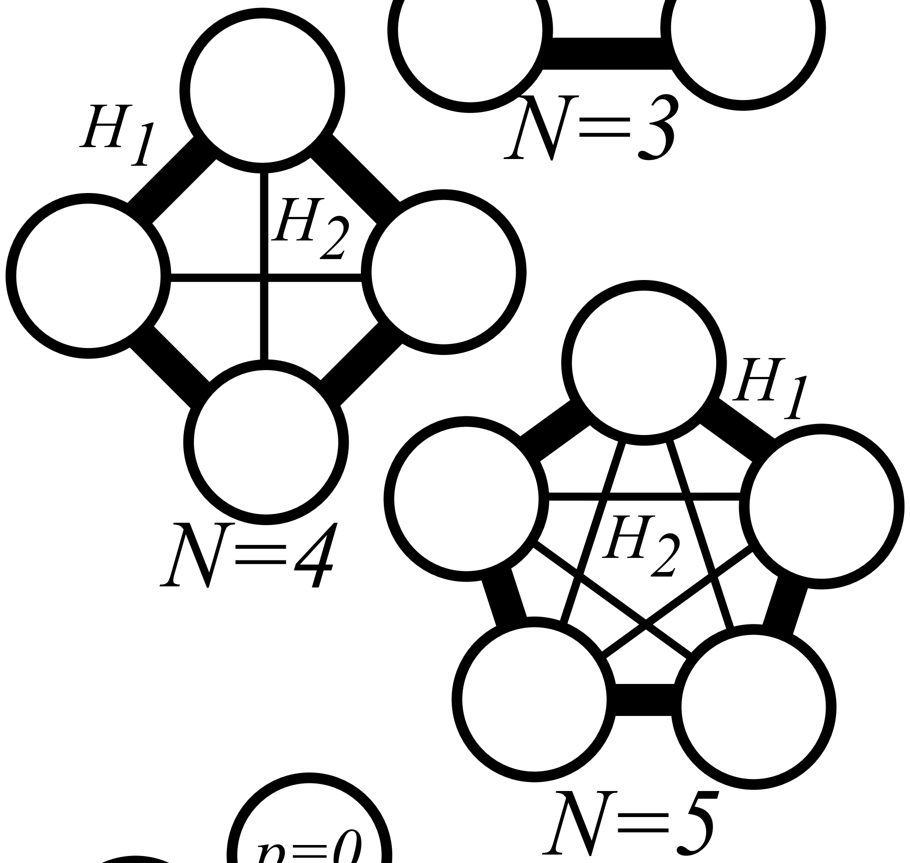
Simulating Complex Systems With Simpler Ones

Made of Quantum Dots

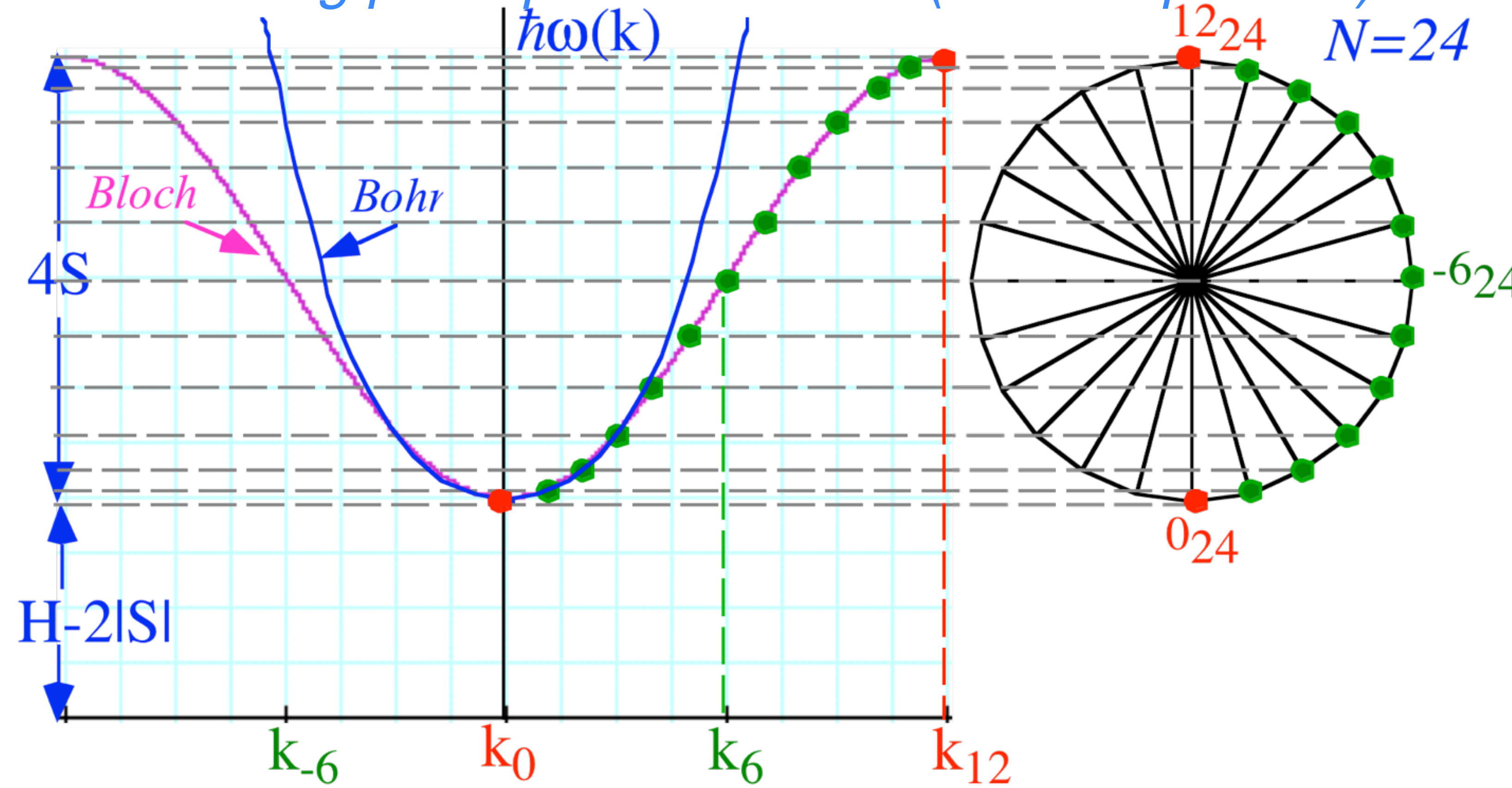
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

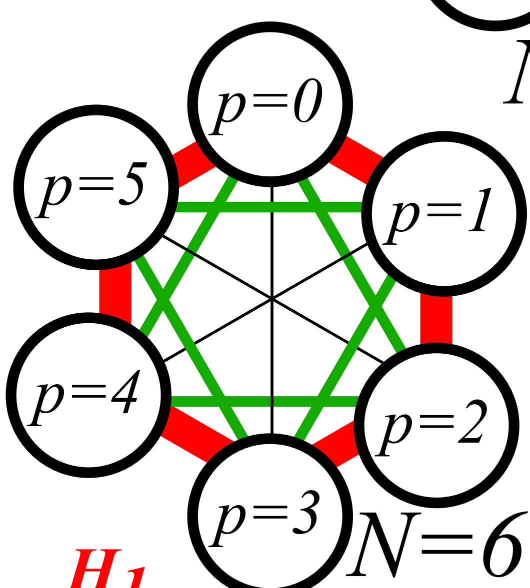


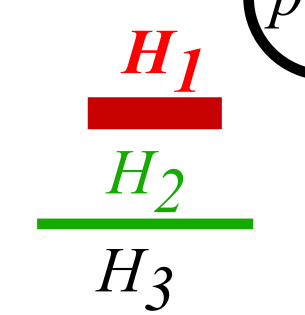




Making pure quadratic $\omega = ck^2$ (Bohr dispersion)







Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} \\ \mathbf{H1} & H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 \\ H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 & H_3 \\ H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 \\ H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} \\ \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 \end{pmatrix}$$

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$N=2$	1/2	-1/2							
$N=3$	2/3	-1/3							
$N=4$	3/2	-1	1/2						
$N=5$	2	-1.1708	0.1708						
$N=6$	19/6	-2	2/3	-1/2					
$N=7$	4	-2.393	0.51	-0.1171					
$N=8$	11/2	-3.4142	1	-0.5858	1/2				
$N=9$	20/3	-4.0165	0.9270	-1/3	0.0895				
$N=10$	17/2	-5.2361	1.4472	-0.7639	0.5528	-1/2			
$N=11$	10	-6.0442	1.4391	-0.5733	0.2510	-0.0726			
$N=12$	73/6	-7.4641	2	-1	2/3	-0.5359	1/2		
$N=13$	14	-8.4766	2.0500	-0.8511	0.4194	-0.2028	0.06116		
$N=14$	33/2	-10.098	2.6560	-1.2862	0.8180	-0.6160	0.5260	-1/2	
$N=15$	57/3	-11.314	2.7611	-1.1708	0.6058	-1/3	0.1708	-0.0528	
$N=16$	43/2	-13.137	3.4142	-1.6199	1	-0.7232	0.5858	-0.5198	1/2
$N=17$	24	-14.557	3.5728	-1.5340	0.81413	-0.4732	0.2781	-0.1479	0.0465

Wave resonance in cyclic symmetry

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C_2 symmetric (B-type) modes

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C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

➡ *Phase arithmetic*

https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N2
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N3
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N4
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N5
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N6(Snap below)

https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N2
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N3
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N4
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N5
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N6

2-level-system and C_2 symmetry phase dynamics

C_2 Character Table describes eigenstates

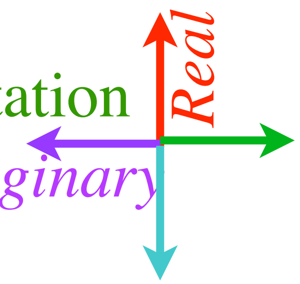
symmetric A_1

vs.

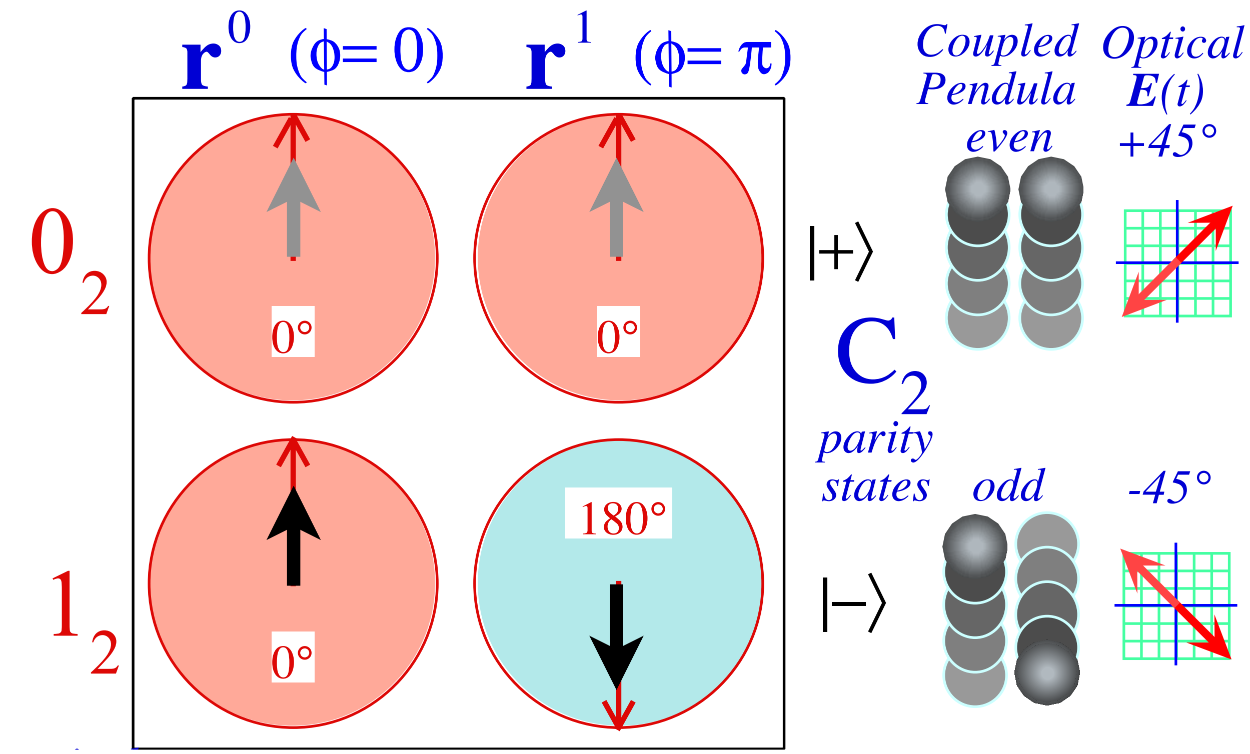
antisymmetric A_2

	$1 = r^0$	$r = r^1$
$0 \bmod 2$	1	1
$\pm 1 \bmod 2$	1	-1

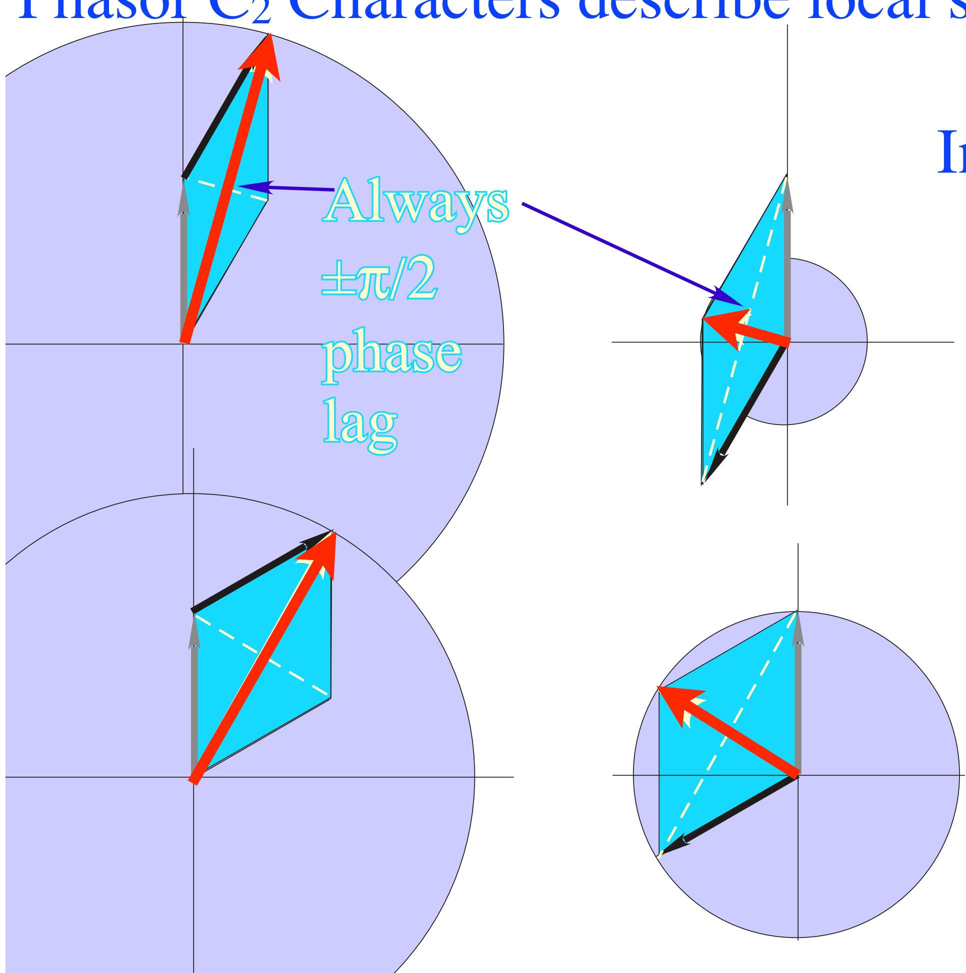
Phasor notation



C_2 *Phasor*-Character Table



Phasor C_2 Characters describe local state beats



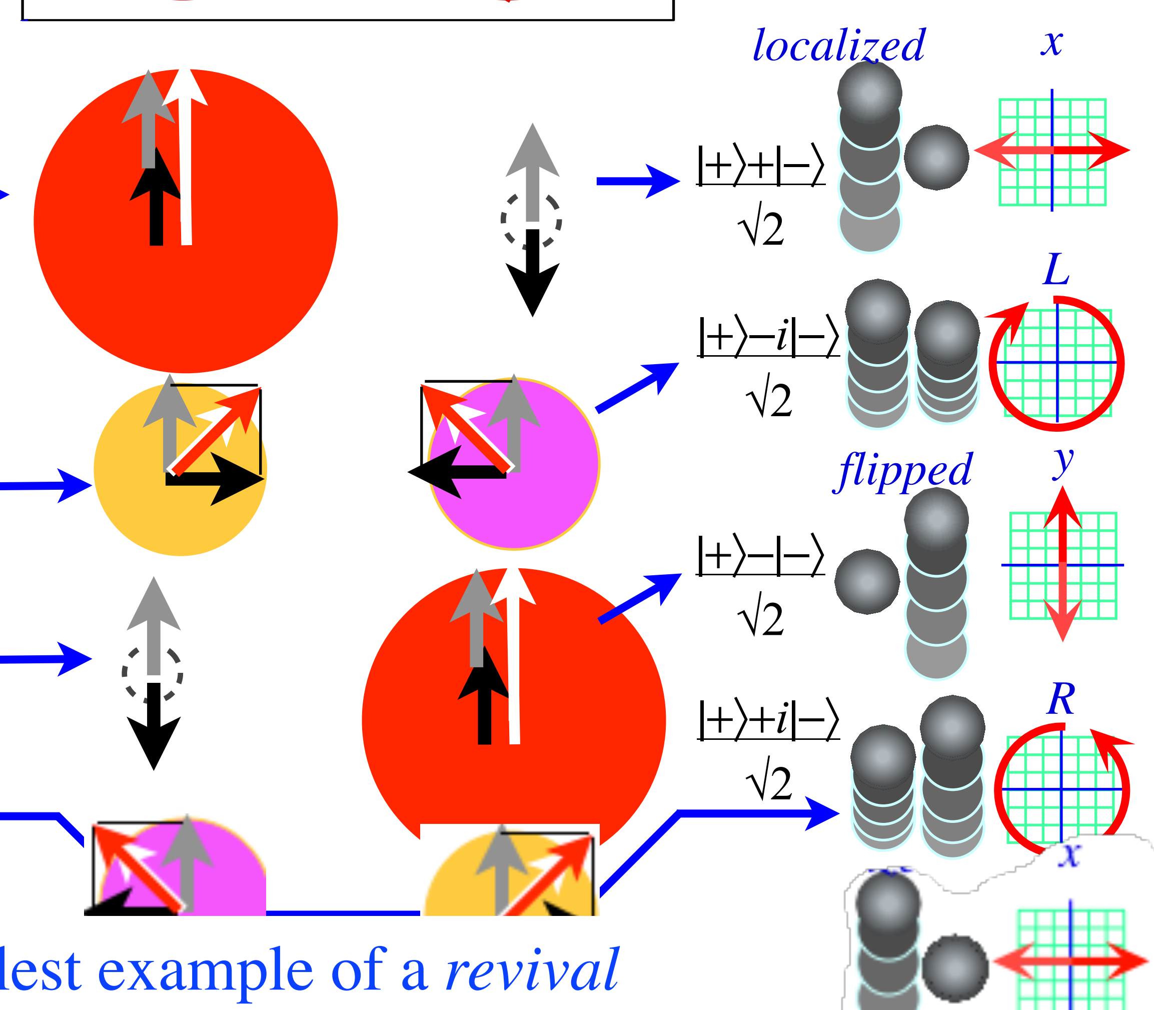
Initial sum

1/4-beat

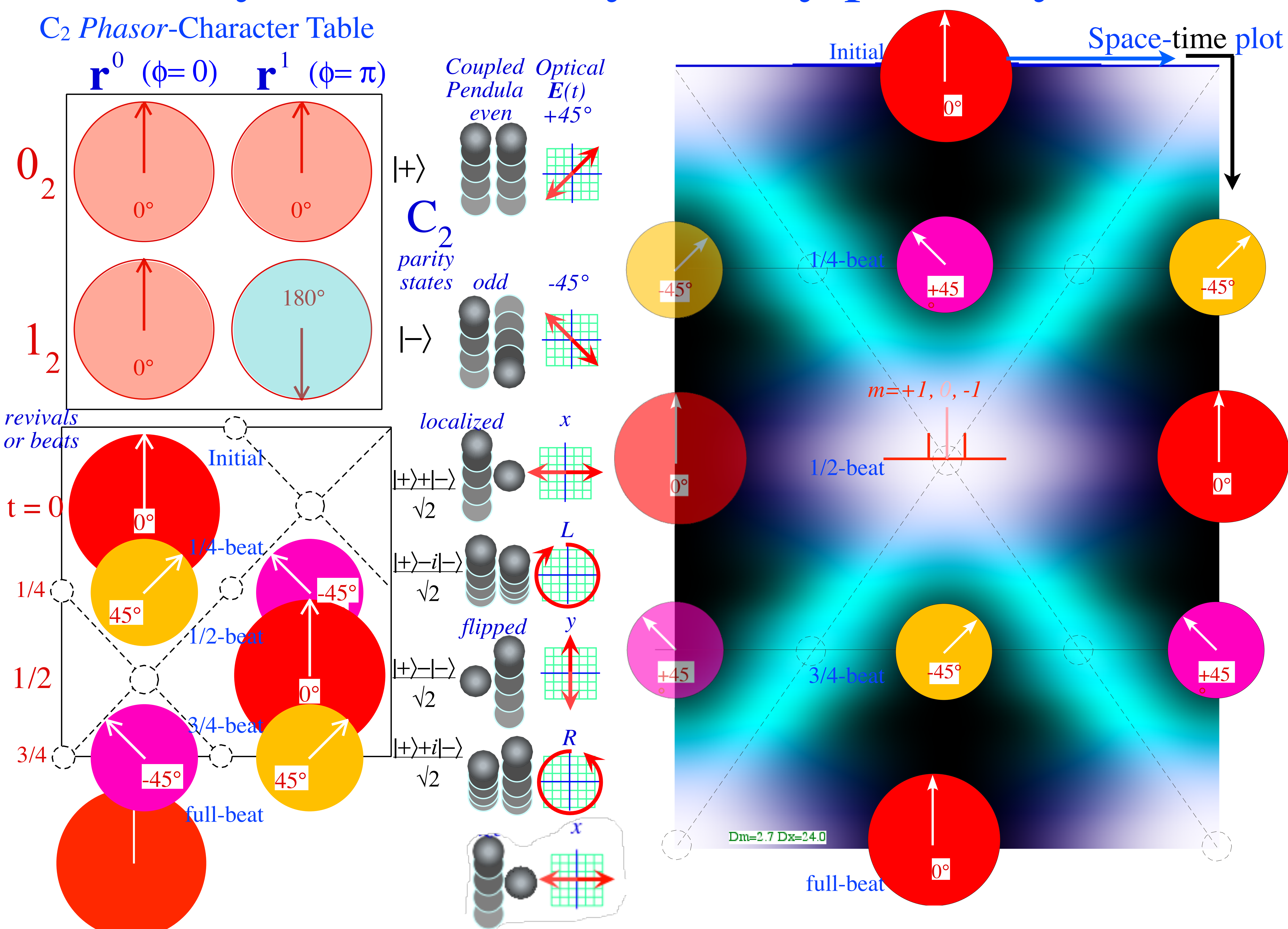
1/2-beat

3/4-beat

full-beat is simplest example of a *revival*

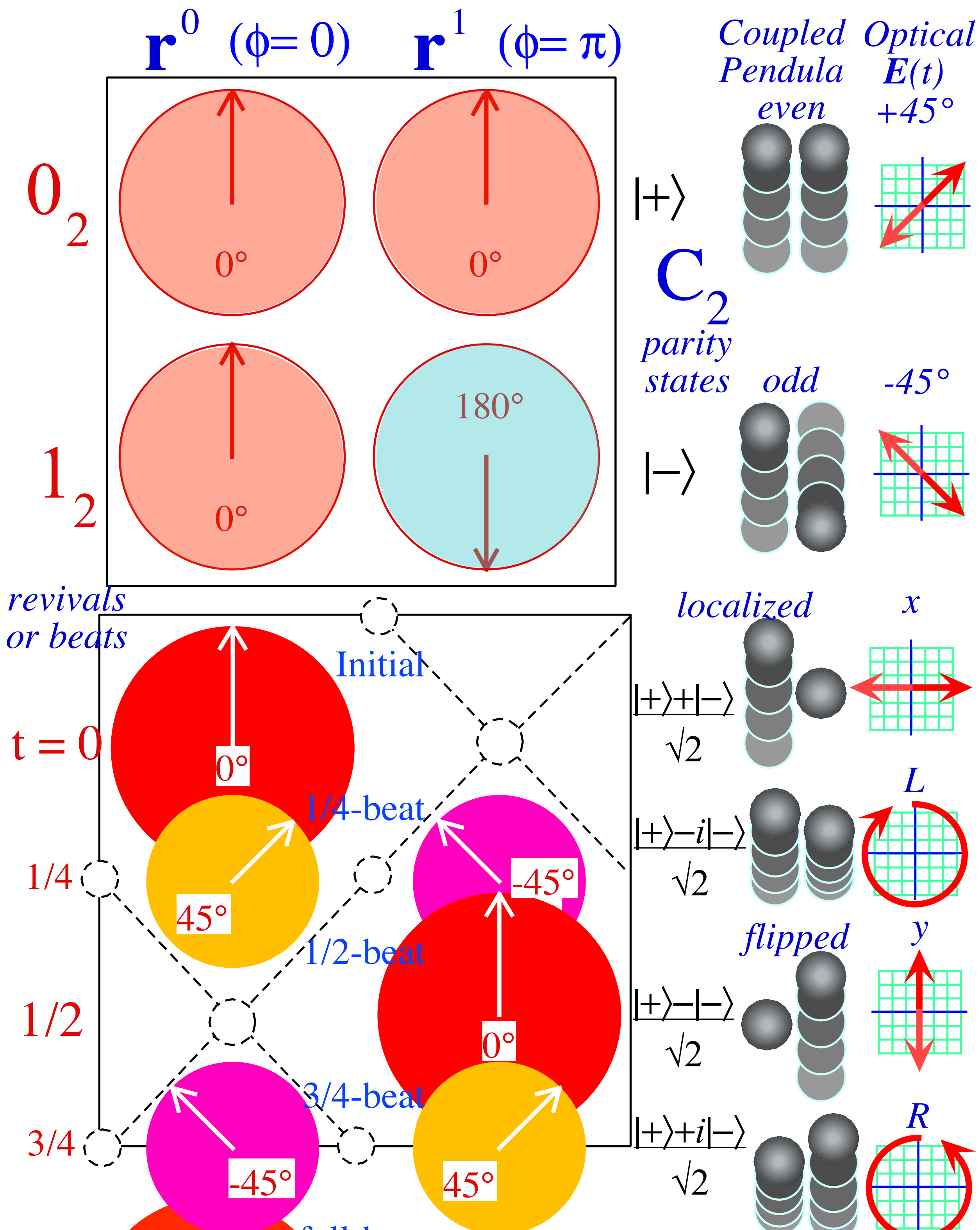


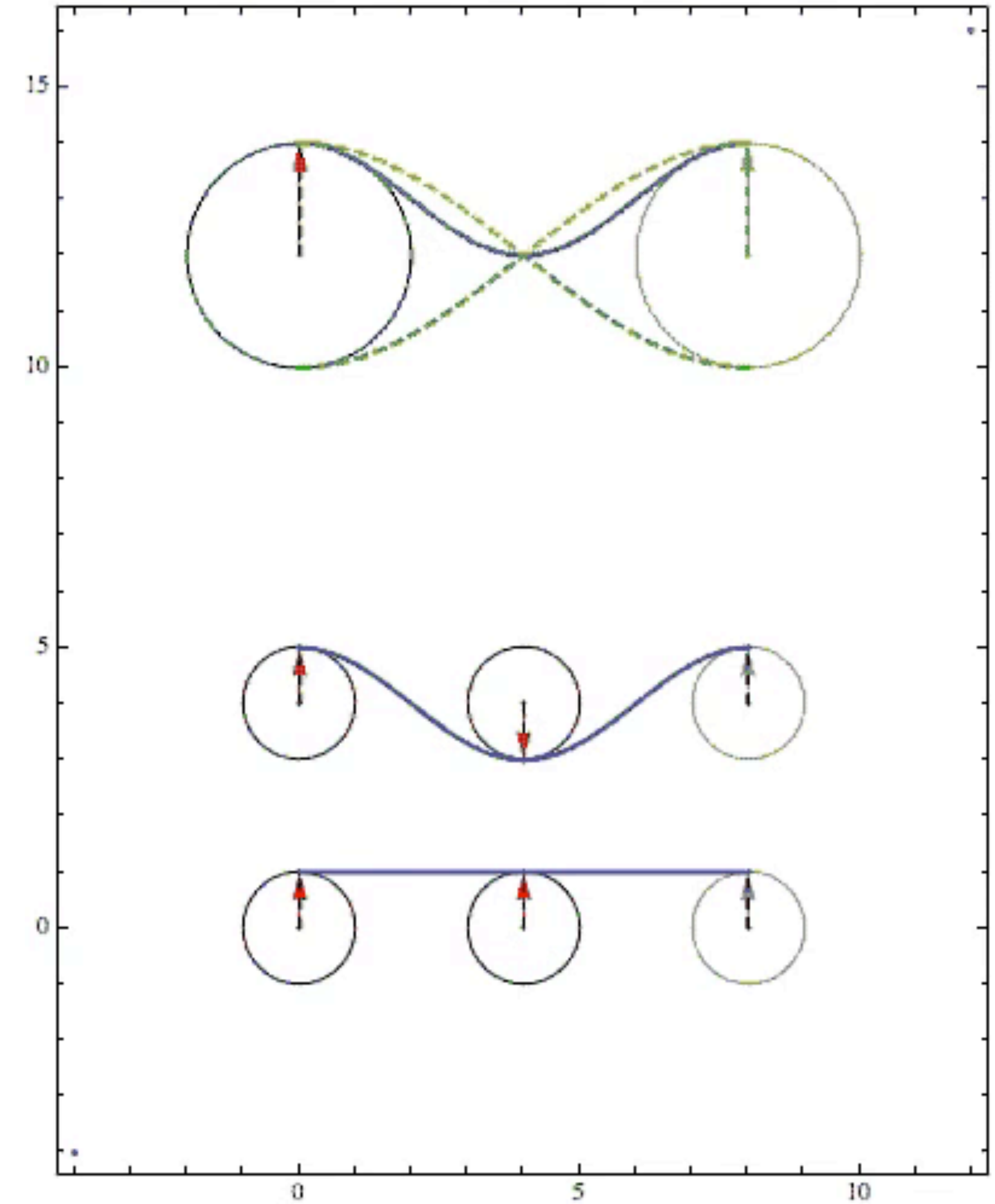
2-level-system and C_2 symmetry phase dynamics

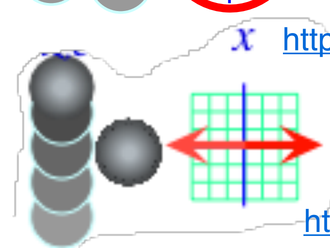


2-level-system and C_2 symmetry phase dynamics

C_2 *Phasor*-Character Table







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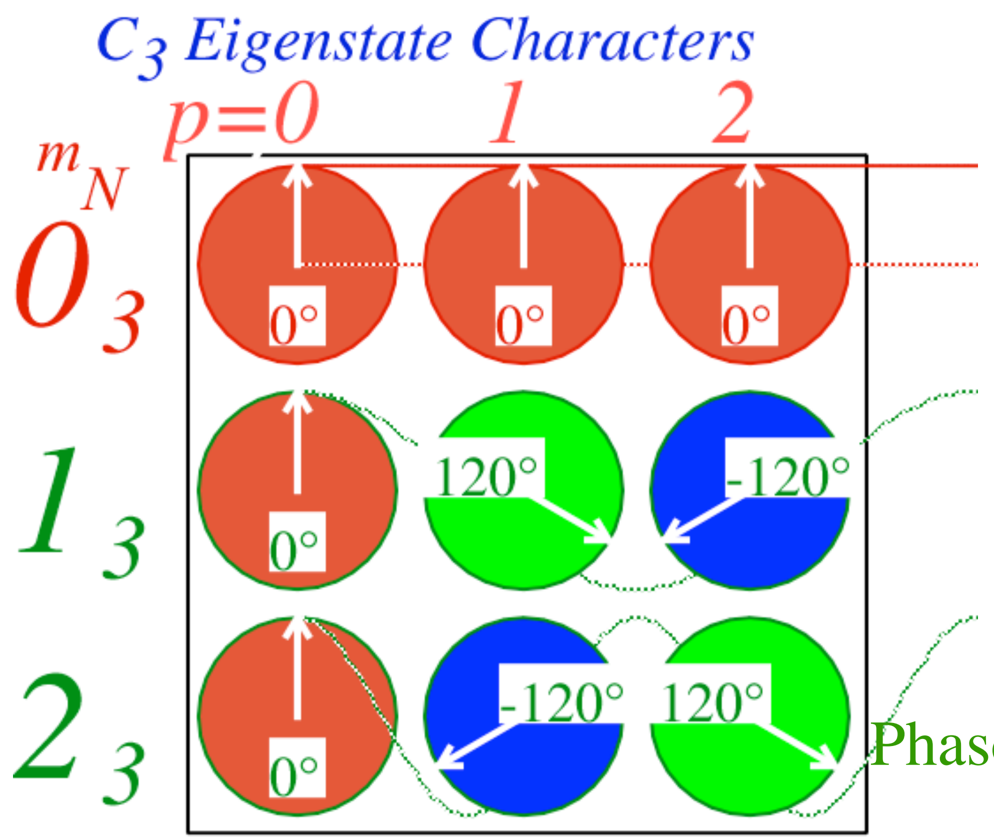
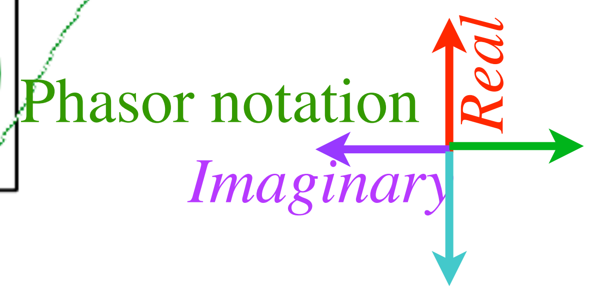
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N2

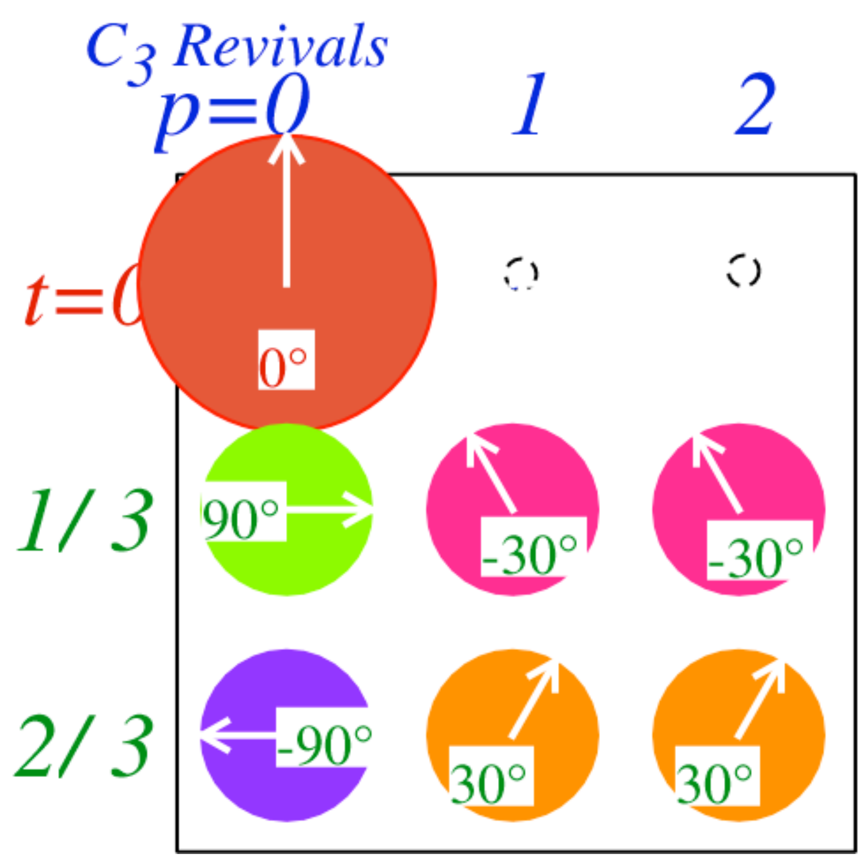
C_3 symmetry phase in *1*, *2*, *or 3*-level-systems

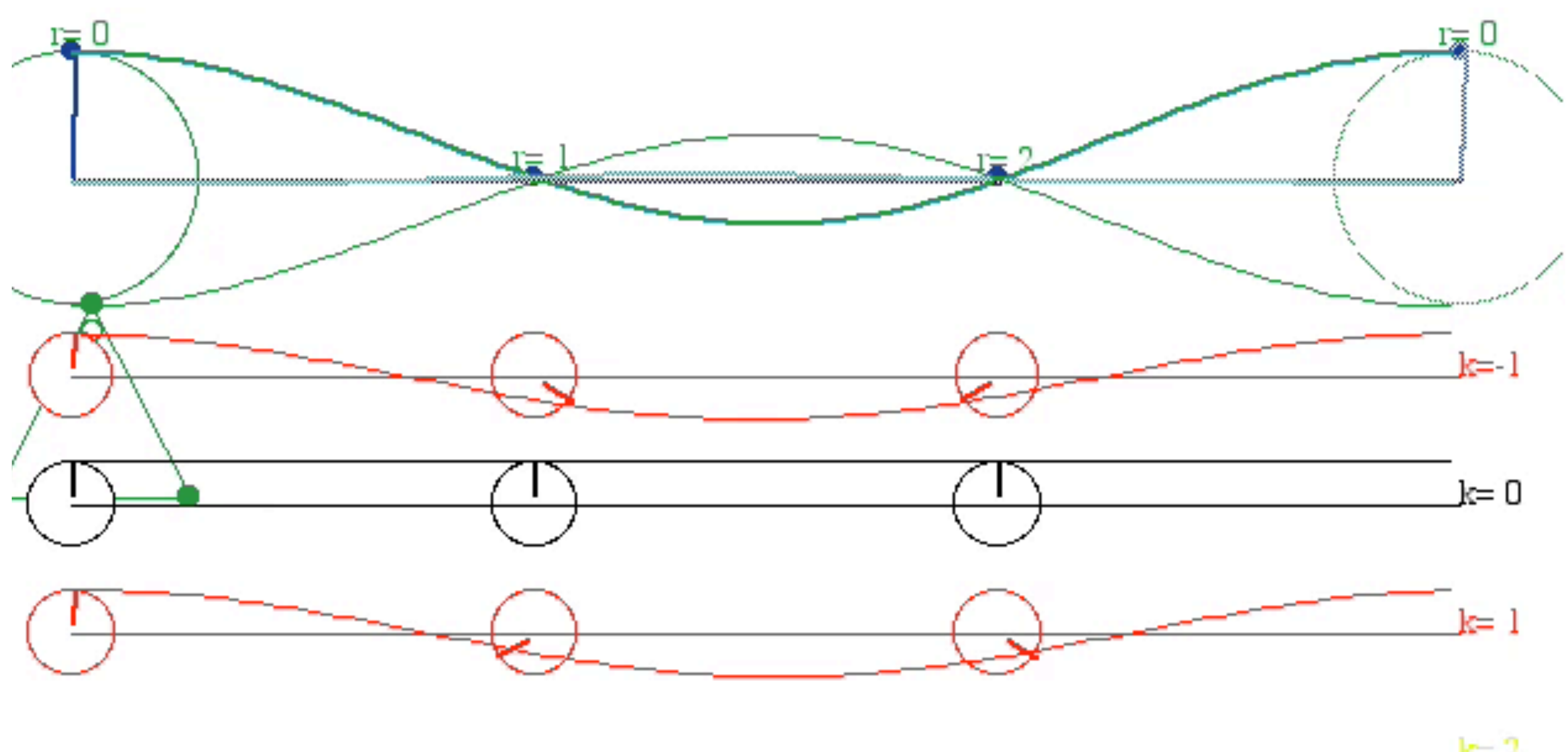


Non - chiral
C_{3v} system

Chiral
"quantum-Hall-like"
systems
deserve special treatment



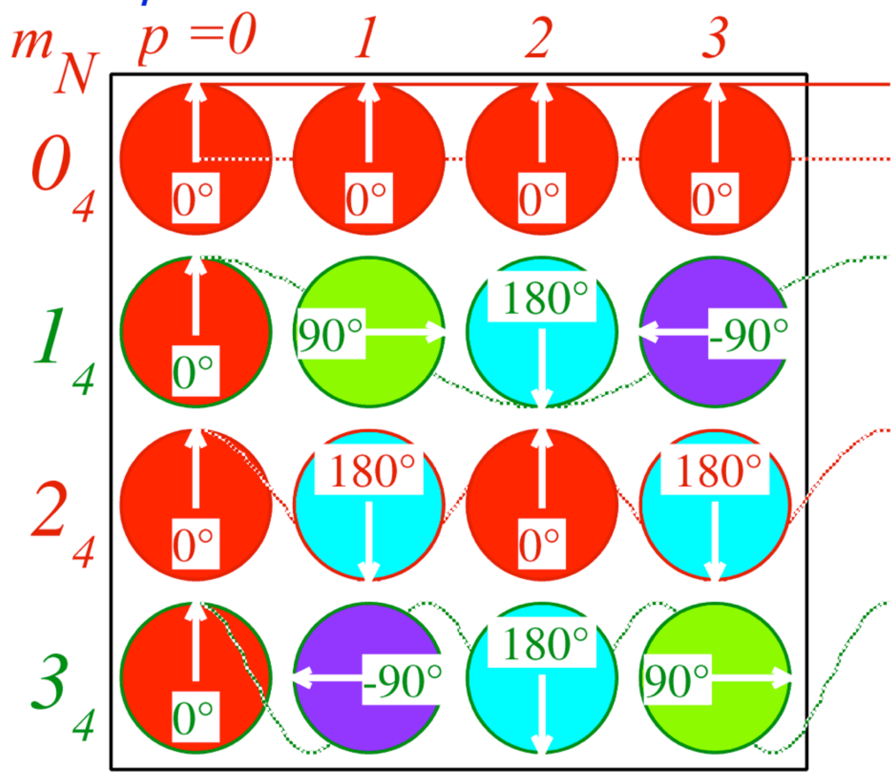




https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N3
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N3

C_4 symmetry phase in *1*, *2*, *3* , *or 4* level-systems

C_4 Eigenstate Characters

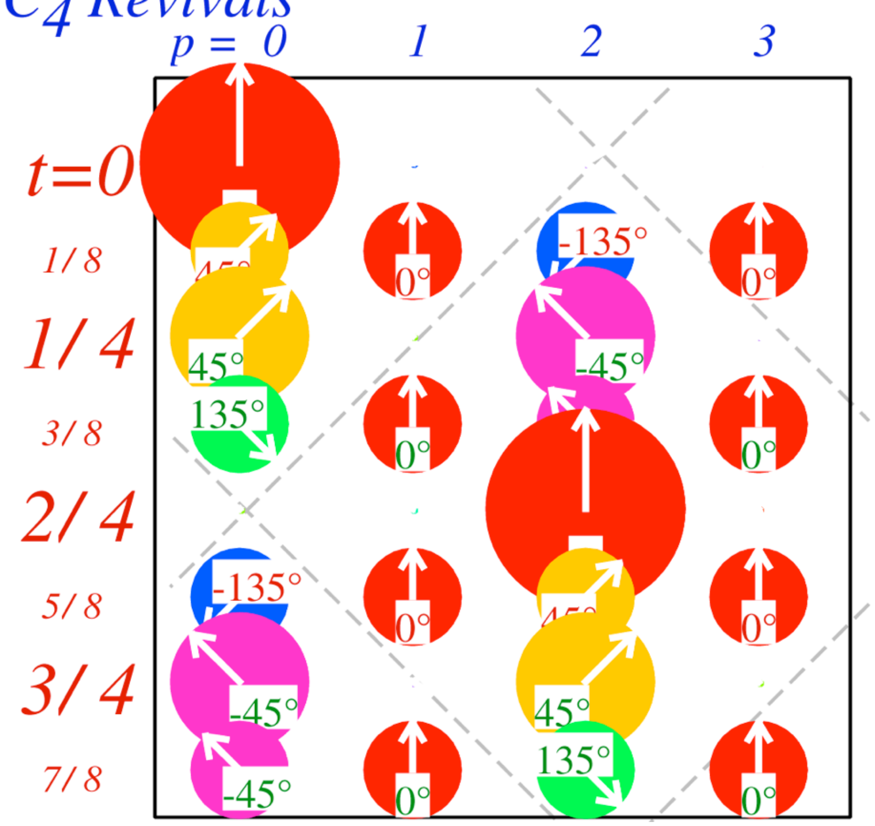


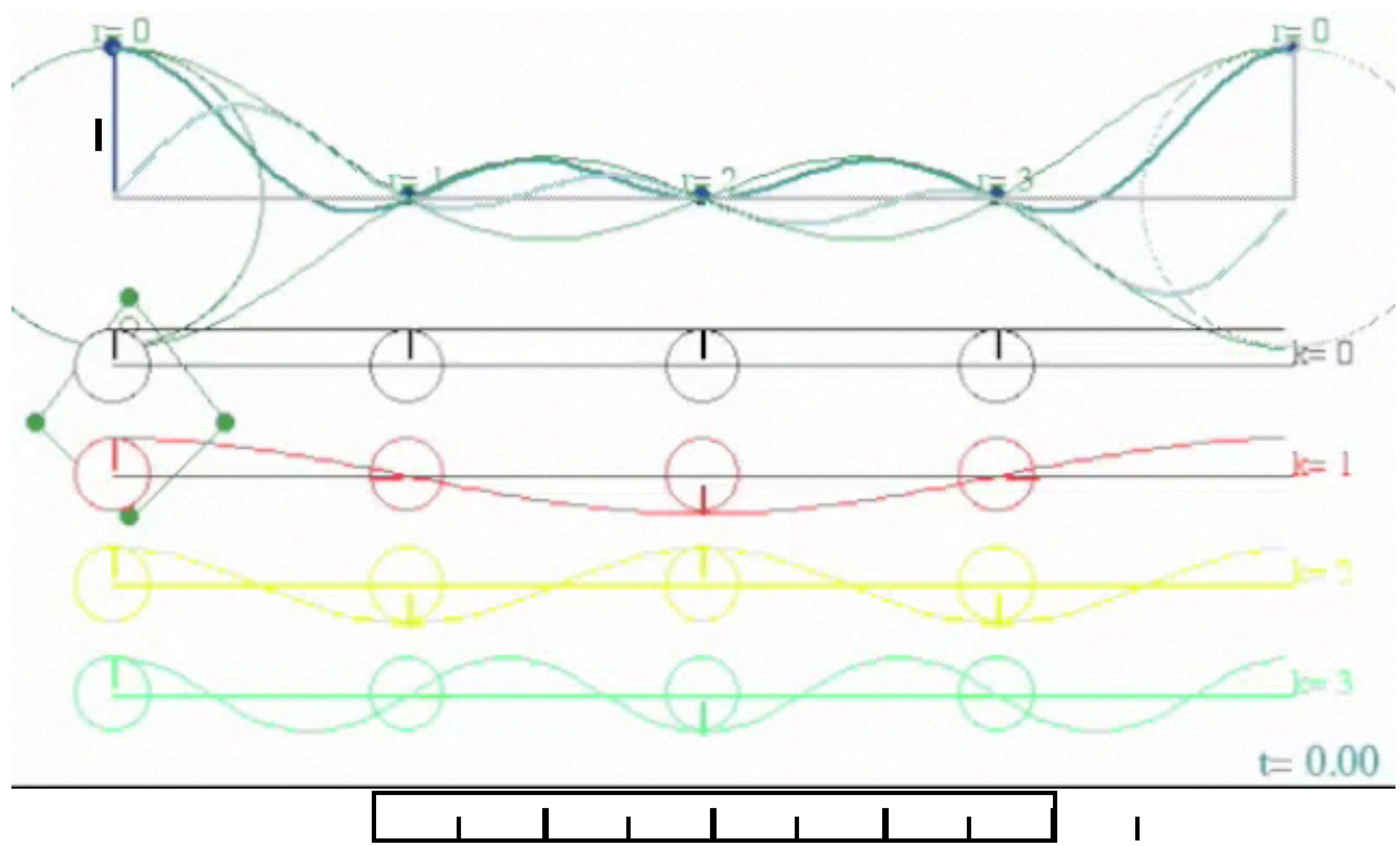
Non - chiral
C_{4v} system

https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N4

https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N4

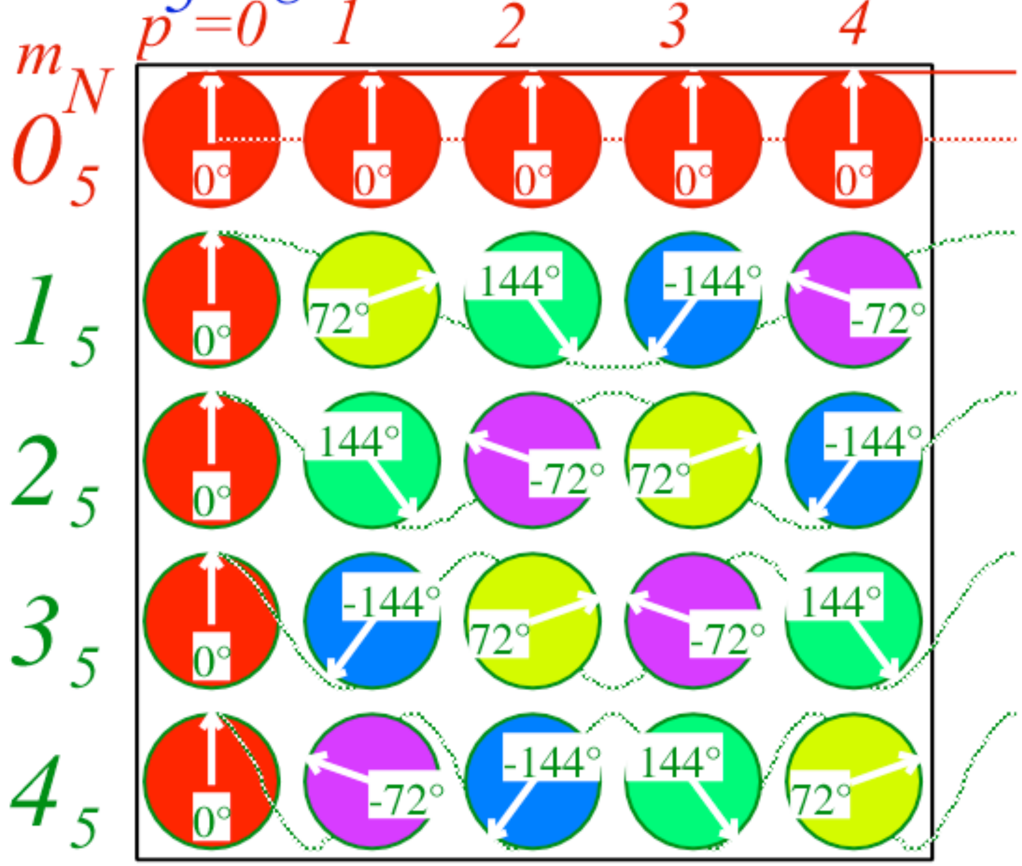
C_4 Revivals

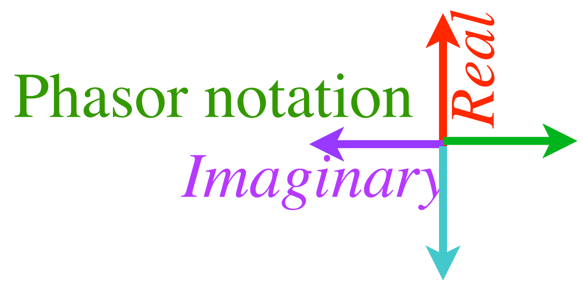




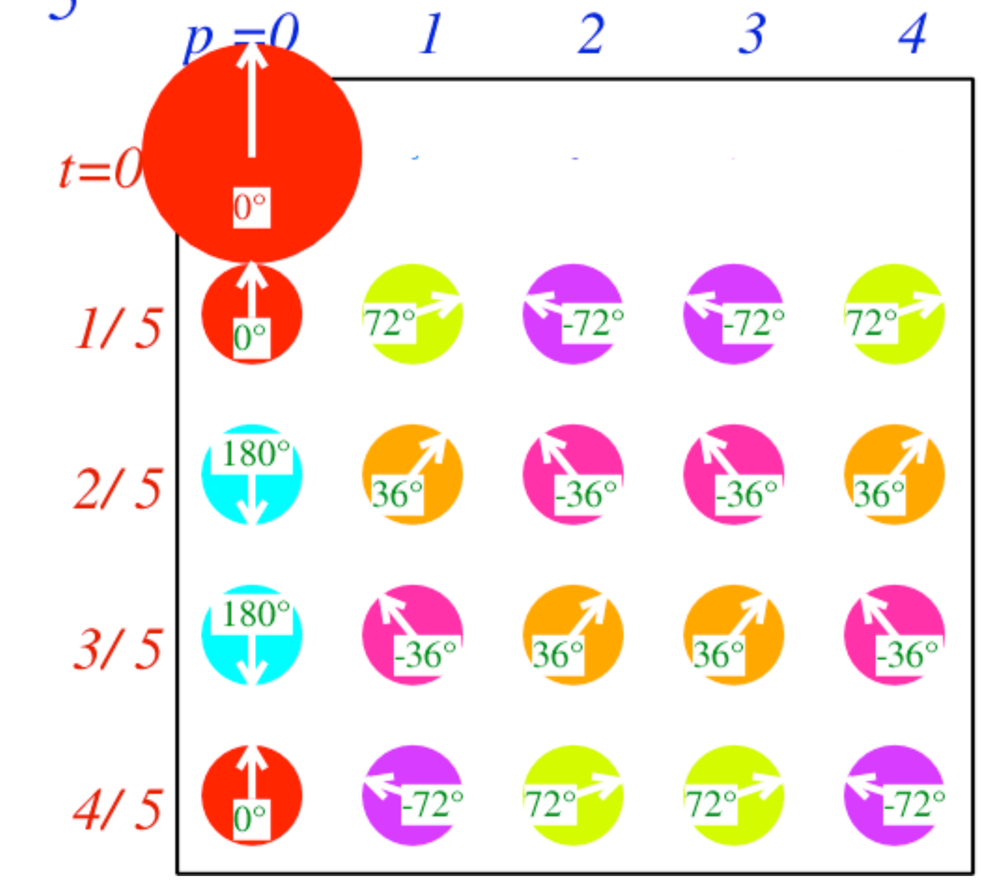
C_5 symmetry phase in *1*, *2*,…*5* level-systems

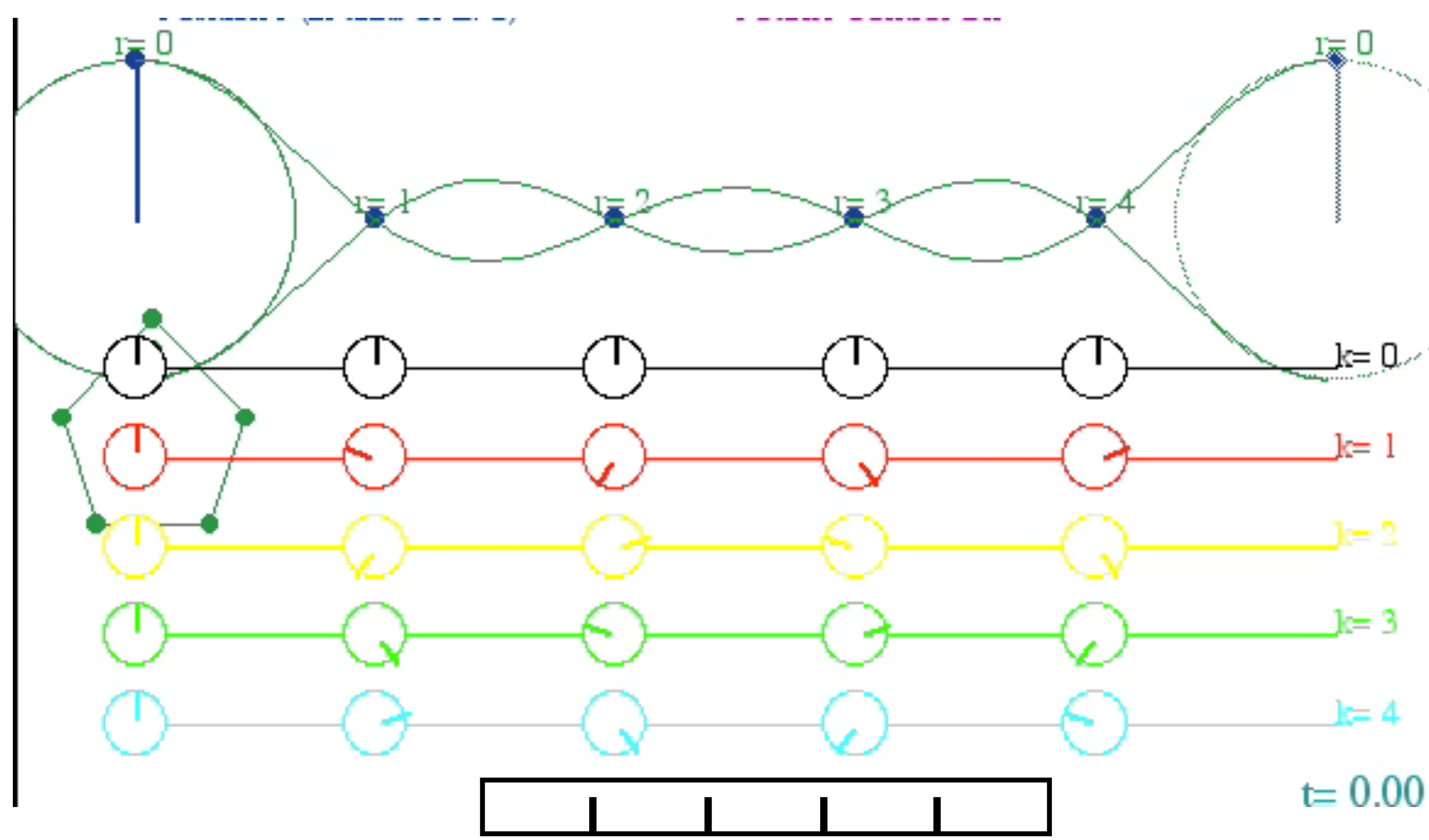
C_5 *Eigenstate Characters*





C_5 *Revivals*

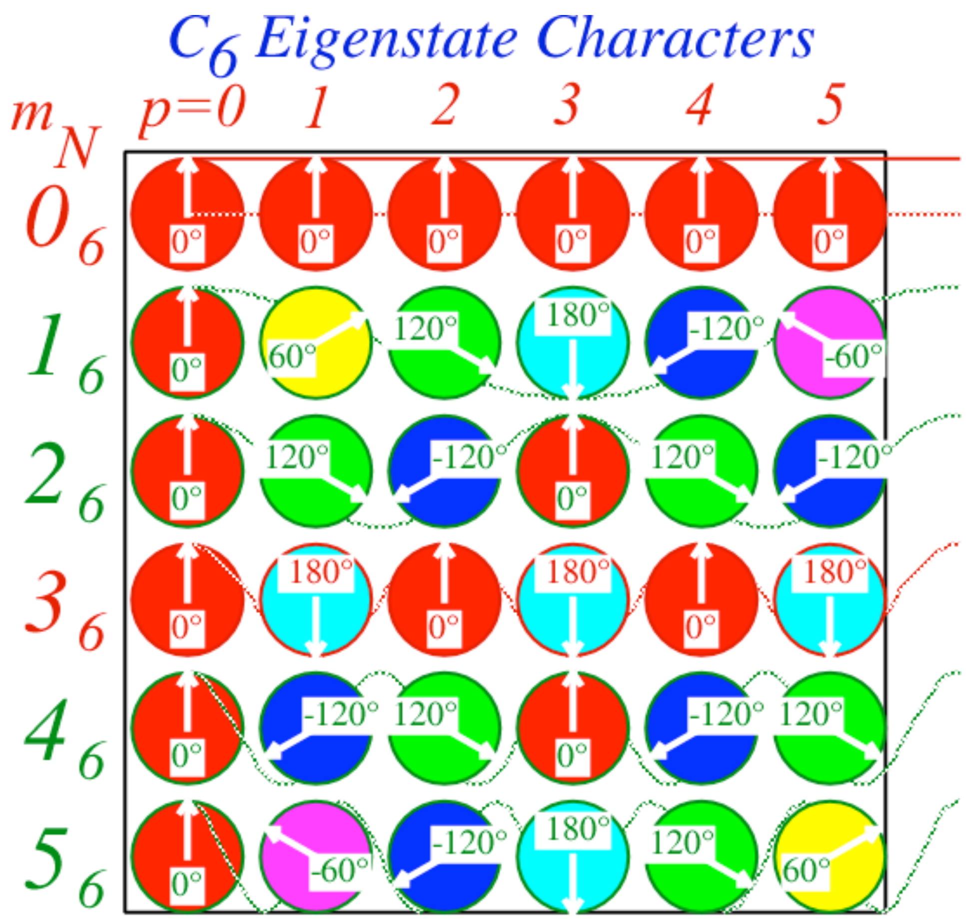


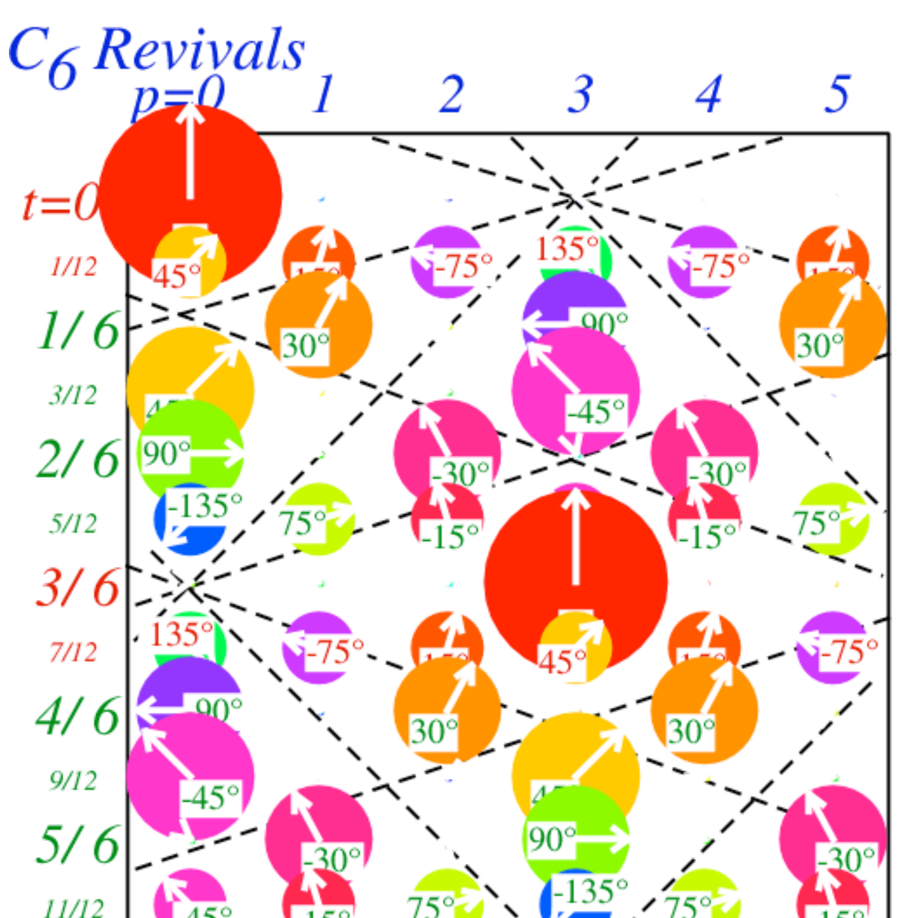


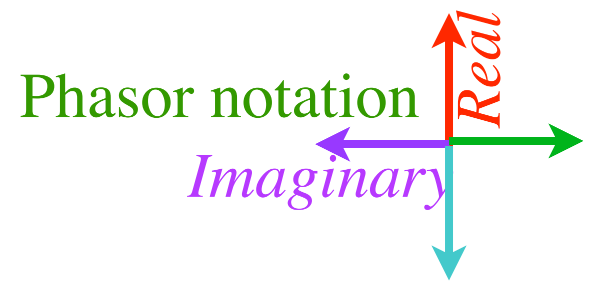
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N5

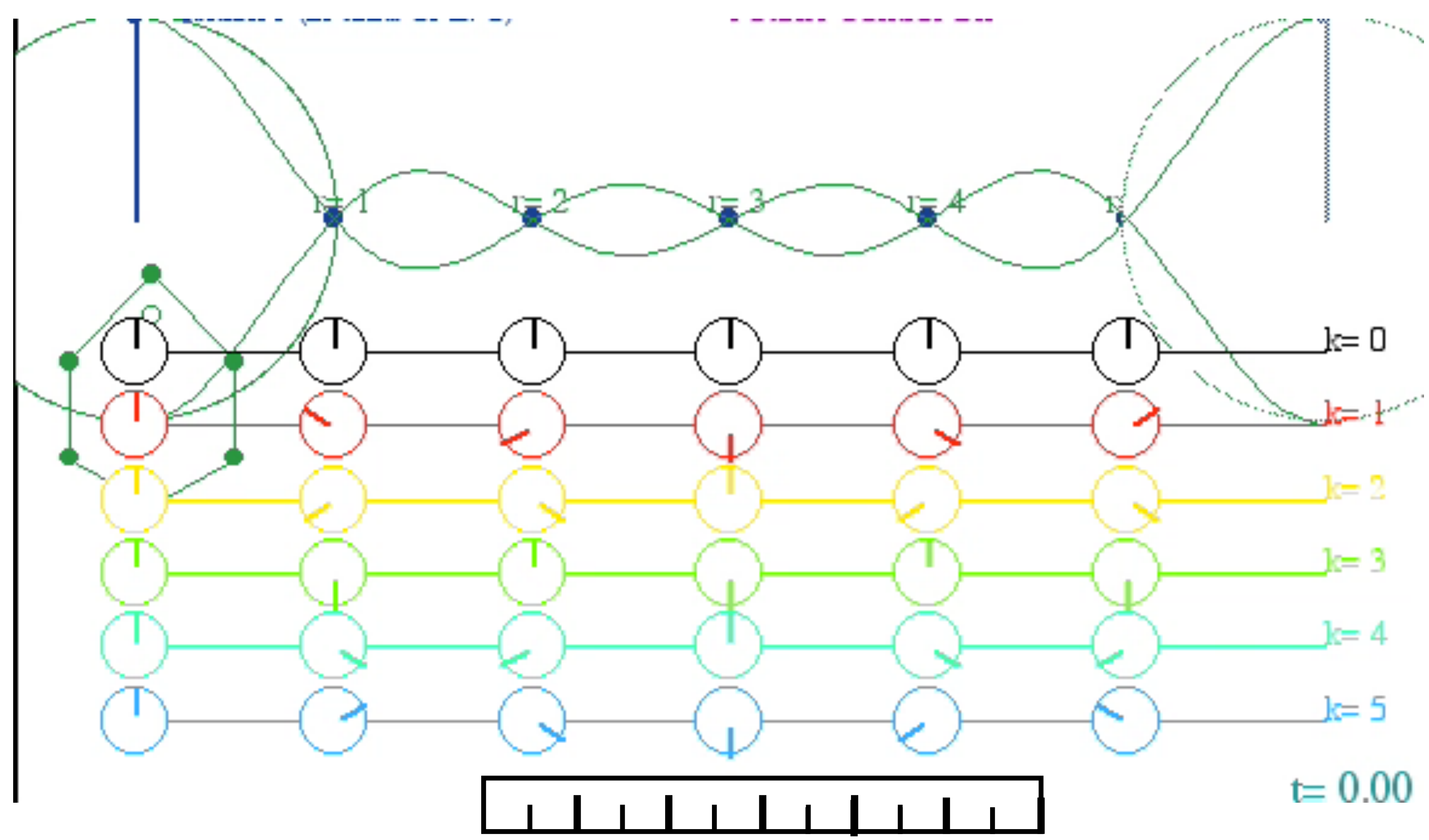
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N5

C_6 symmetry phase in *1*, …*6* level-systems







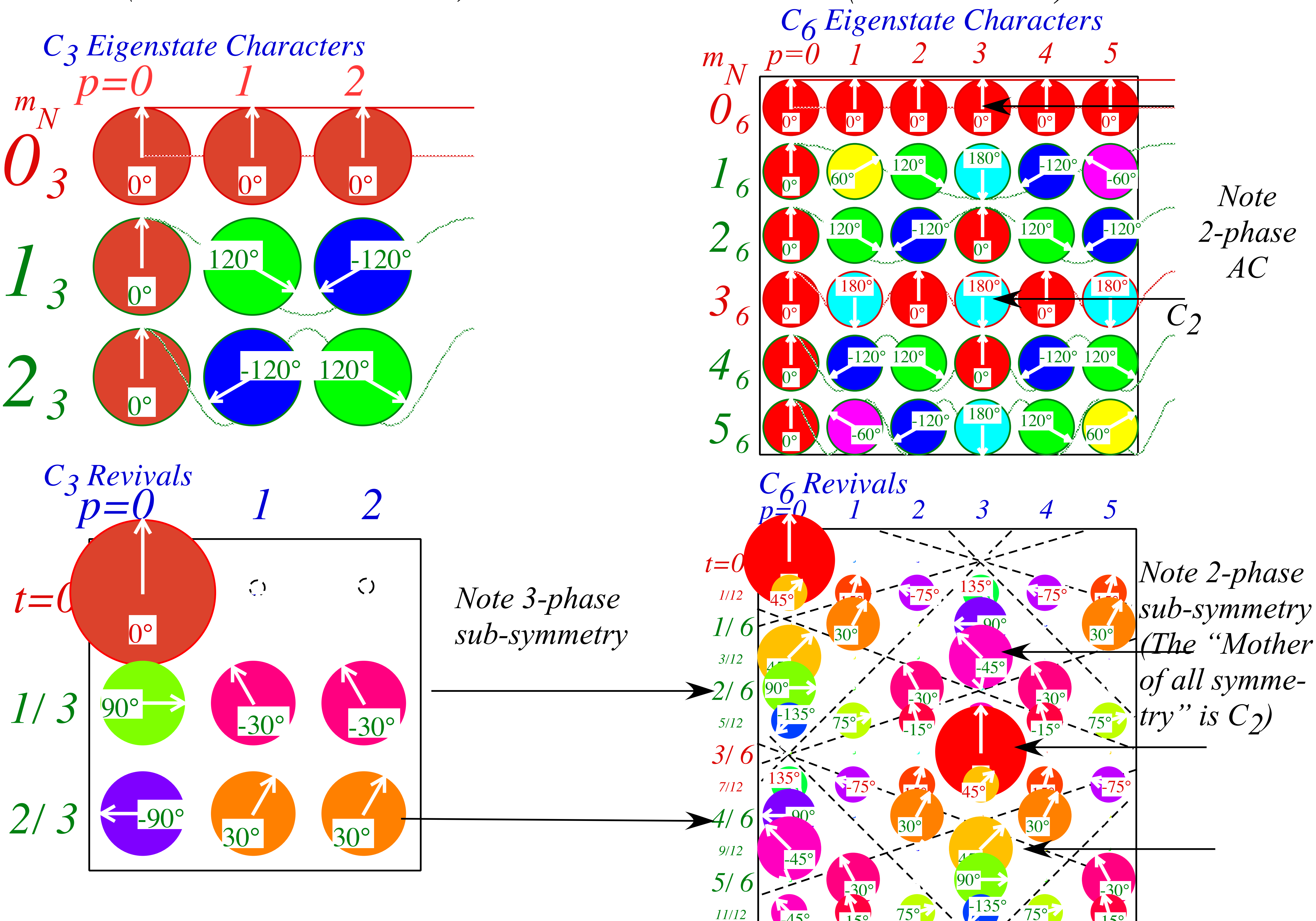


https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=1PW_R_Stacked_2018CM_N6
https://modphys.hosted.uark.edu/markup/WaveItWeb.html?scenario=2PW_Stacked_2018CM_N6

C_m algebra of revival-phase dynamics

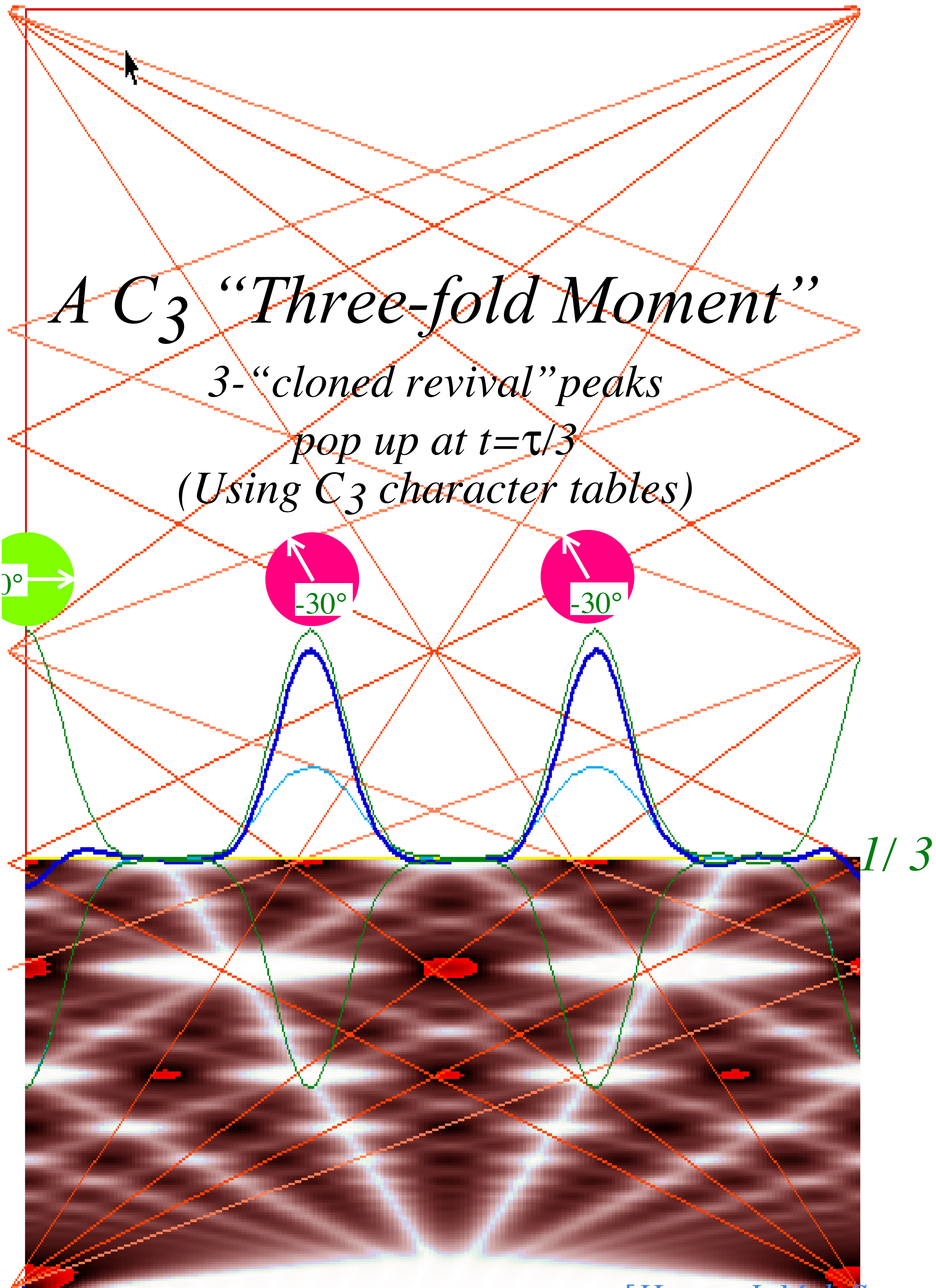
Discrete 3-State or Trigonal System (Tesla's 3-Phase AC)

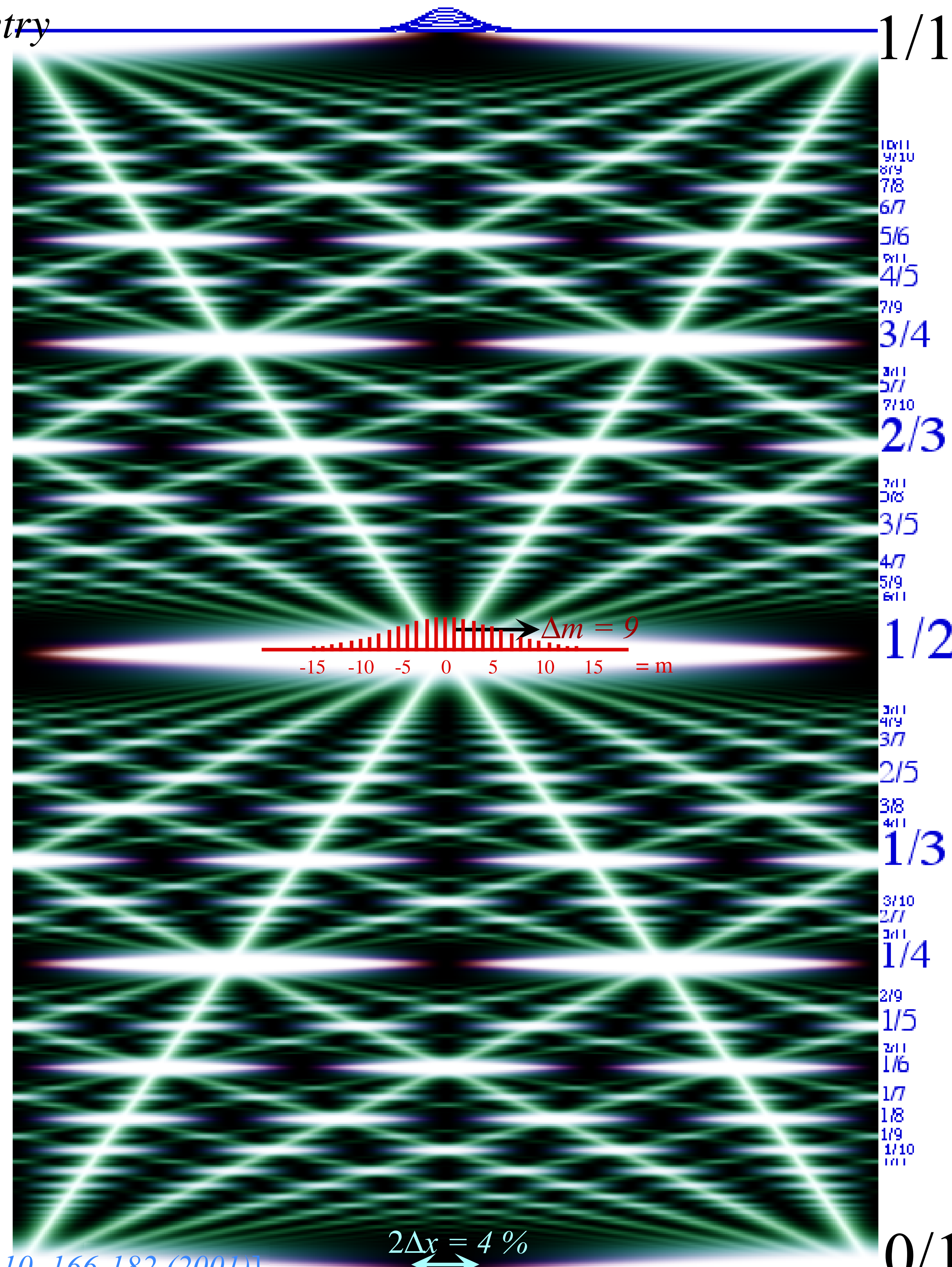
Discrete 6-State or Hexagonal System (6-Phase AC)



C_m algebra of revival-phase dynamics

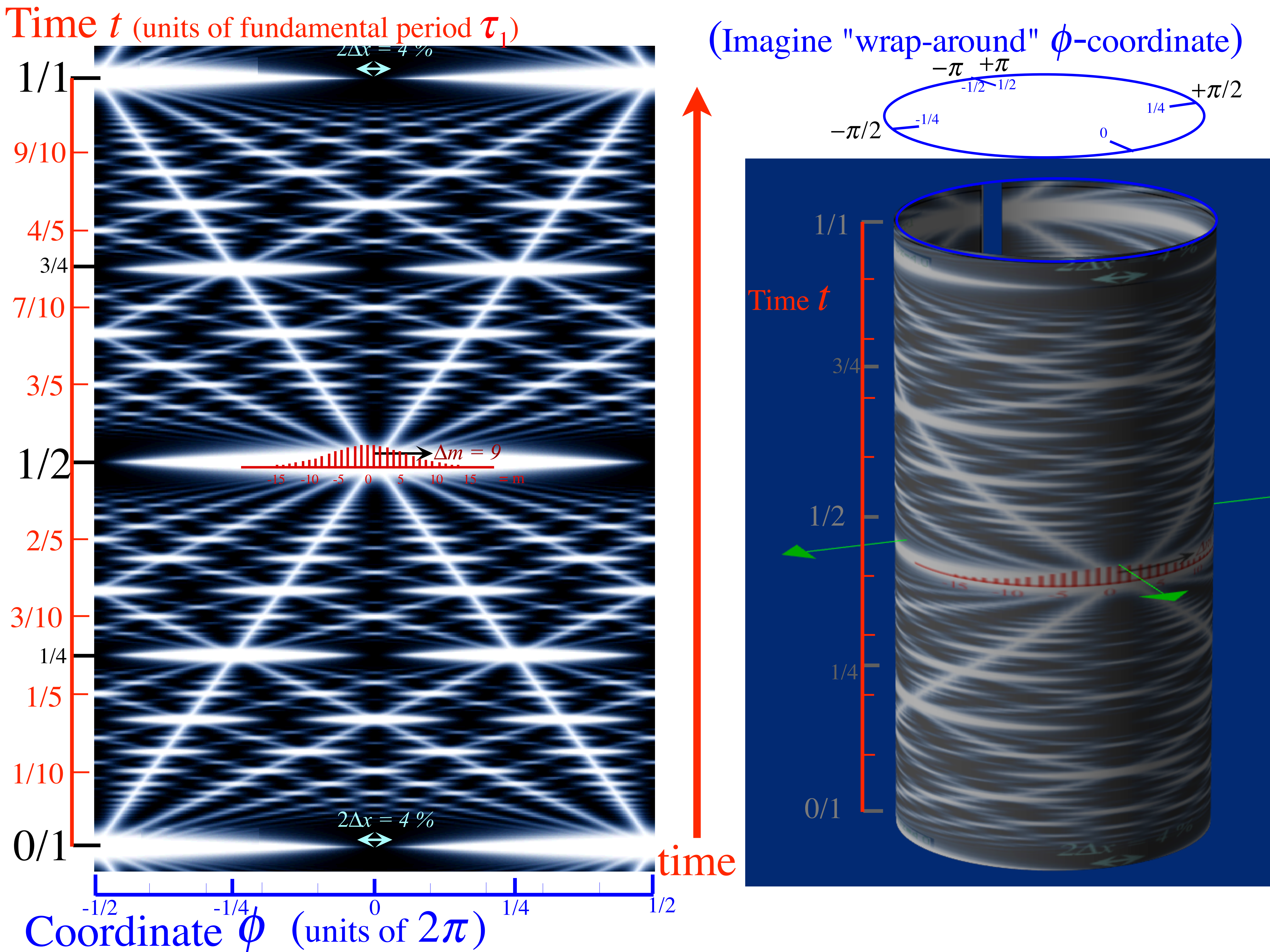
Quantum rotor fractional take turns at Cn symmetry





[Harter, J. Mol. Spec. 210, 166-182 (2001)]

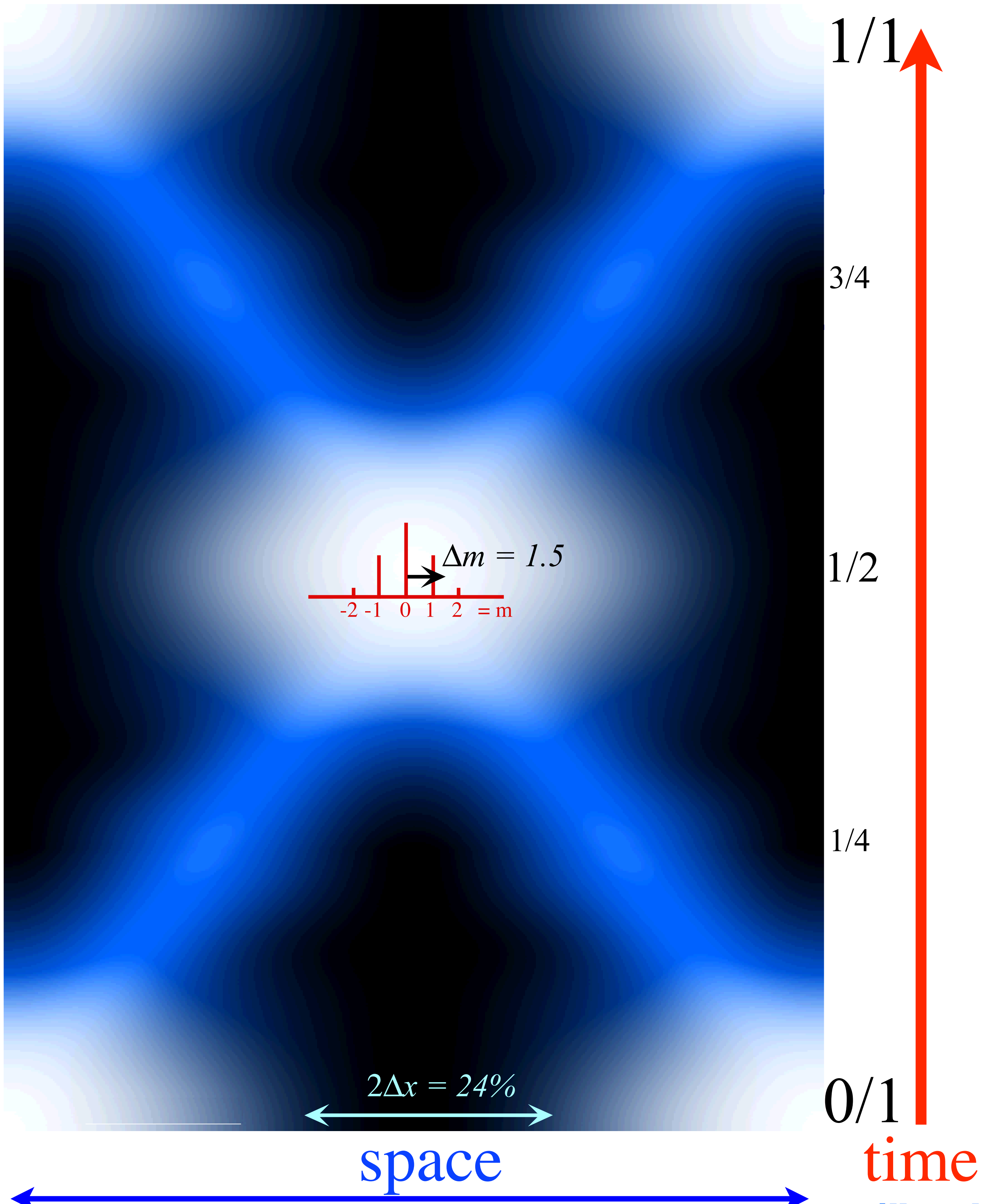
Algebra and geometry of resonant revivals: Farey Sums and Ford Circles



Time t (units of fundamental period τ_1)
1/1
9/10
4/5
3/4
7/10
3/5
1/2
2/5
3/10
1/4
1/5
1/10
0/1
$2\Delta x = 4$ %
$\Delta m = 9$
-15 -10 -5 0 5 10 15 = m
time
-1/2 -1/4 0 1/4 1/2
Coordinate ϕ (units of 2π)
(Imagine "wrap-around" ϕ-coordinate)
$-\pi$ $+\pi$
-1/2 1/2
$+\pi/2$
1/4
$-\pi/2$
-1/4
0
Time t
0/1

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



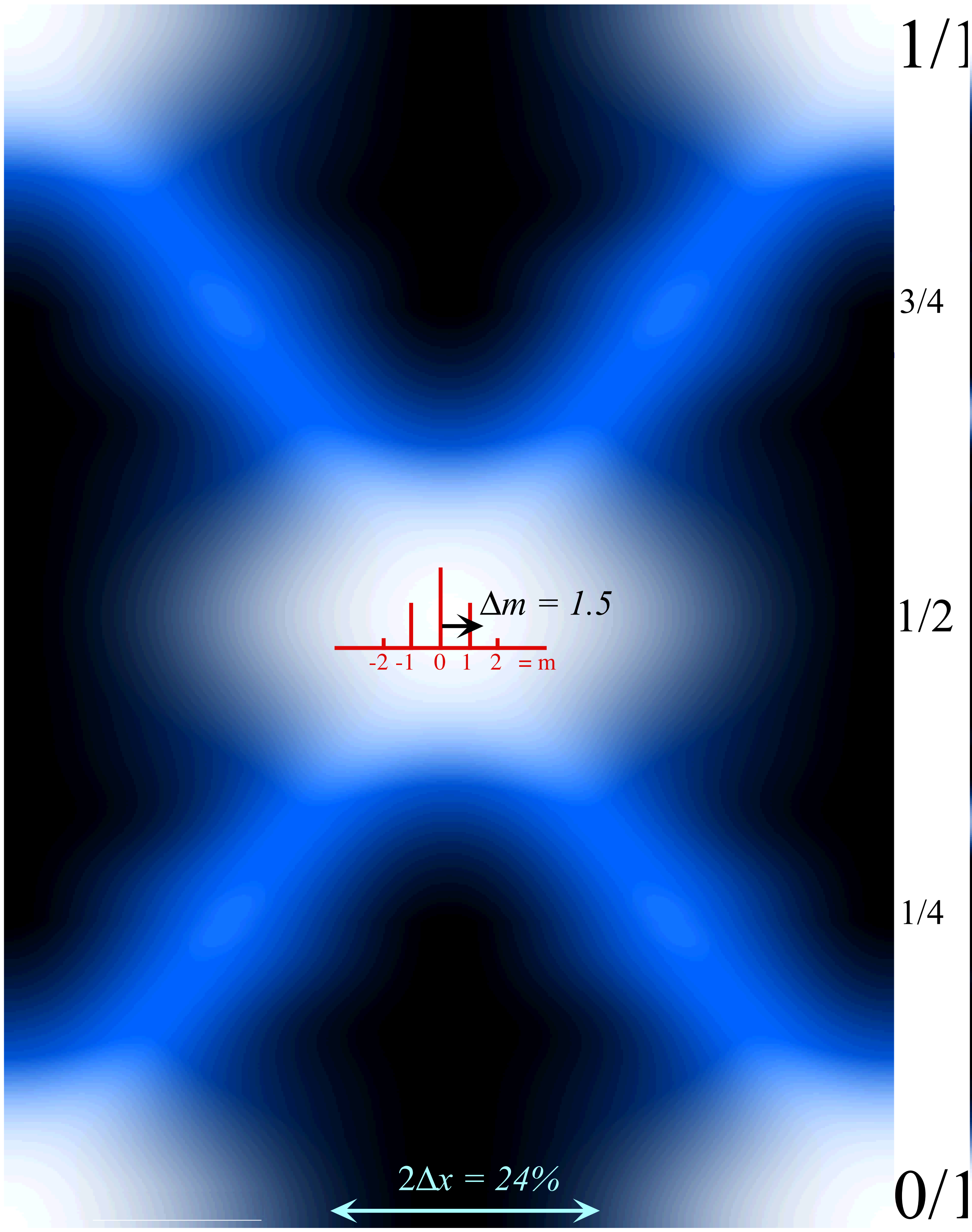
|Ψ(x,t)| in space-time

Simplest quantum revival:
Exciting first two levels
(ℓ=0 and ℓ=±1)
is like a
2-level system quantum beat
in space-time

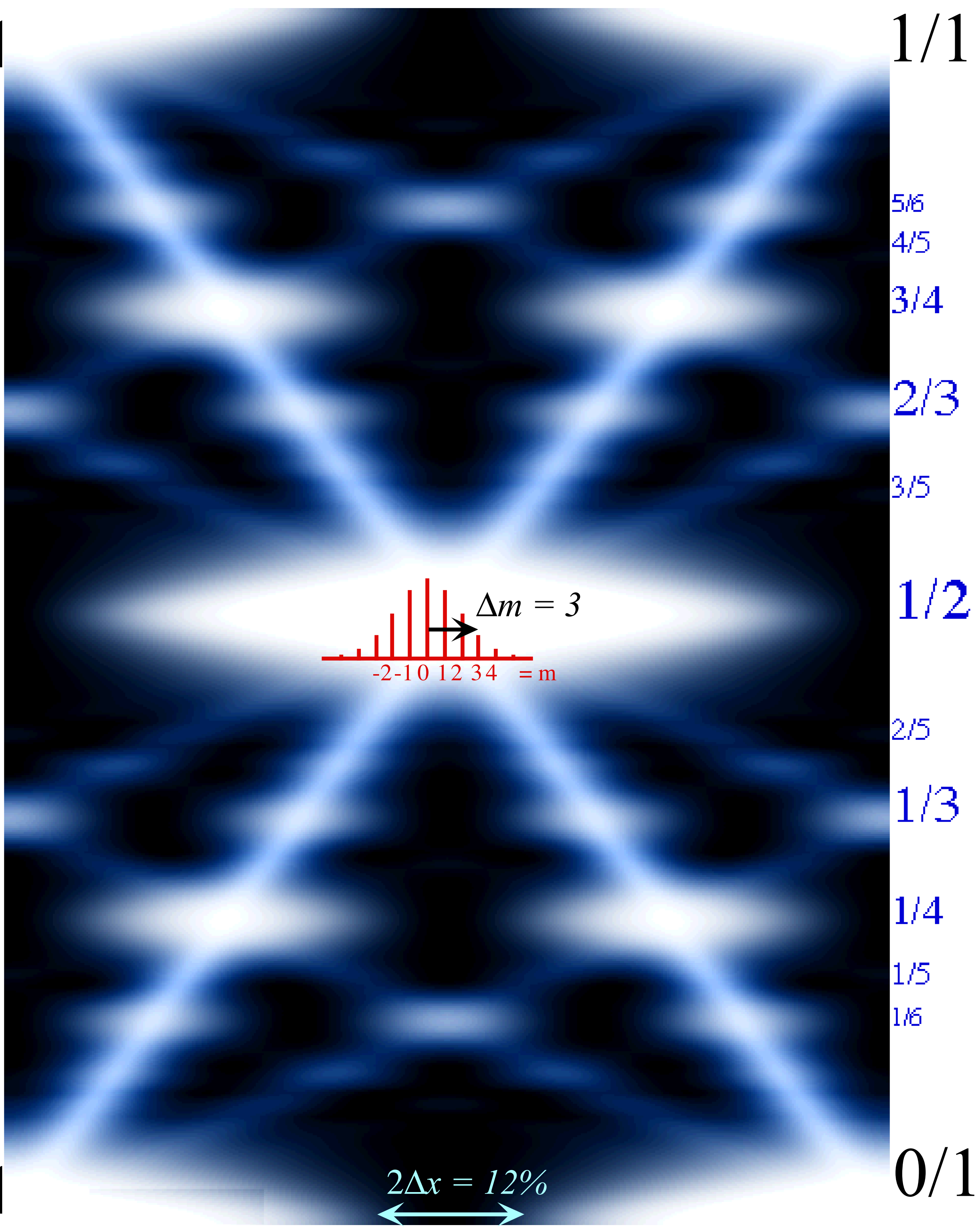
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



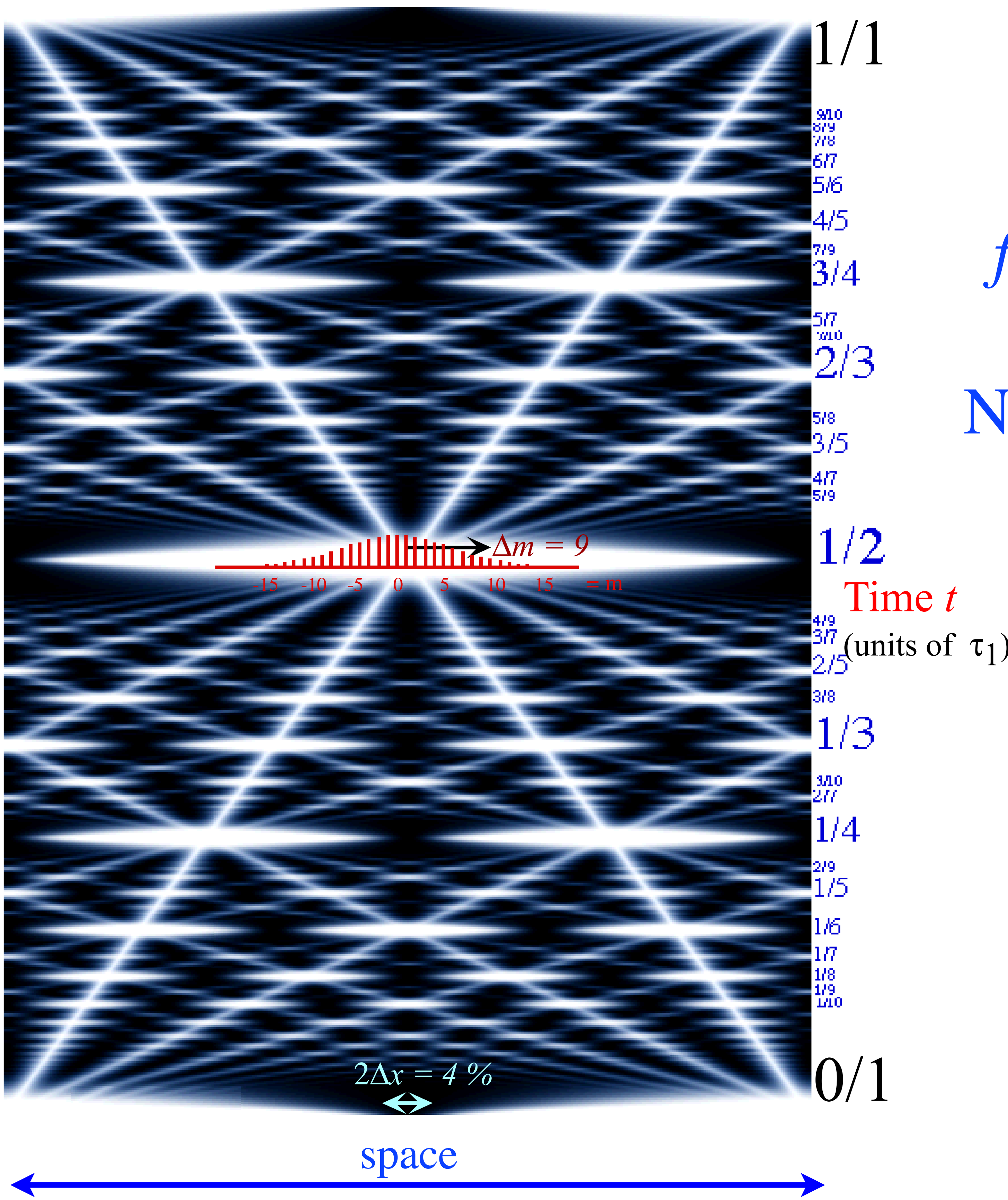
(4-levels (0, ±1,±2,±3) (and some ±4) excited)



Simplest *fractional* quantum revivals: 3,4,5-level systems

N-level-rotor system revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)



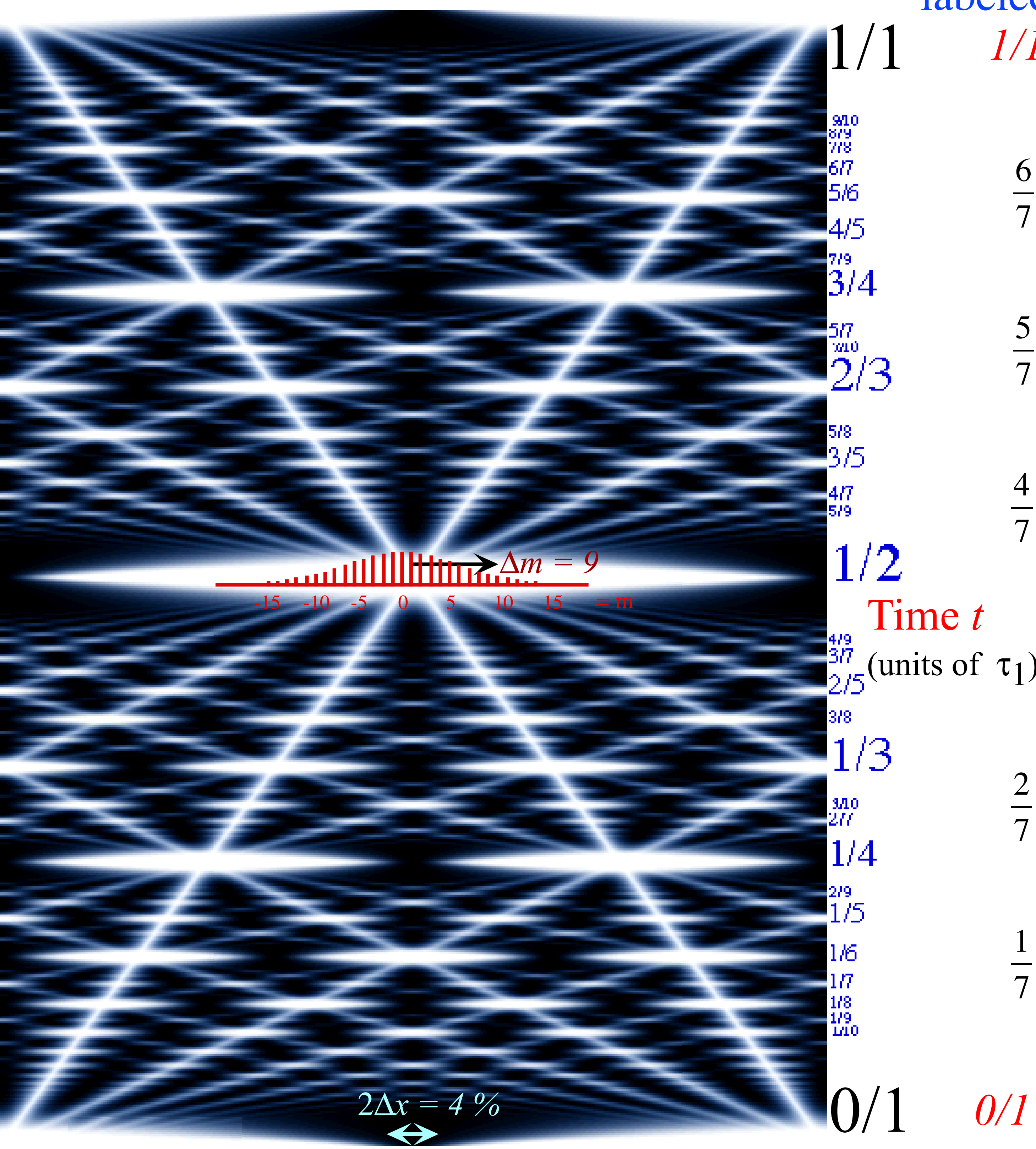
fractional quantum revivals:
in 3,4,..., N-level systems
Number increases rapidly with
number of levels
and/or bandwidth
of excitation

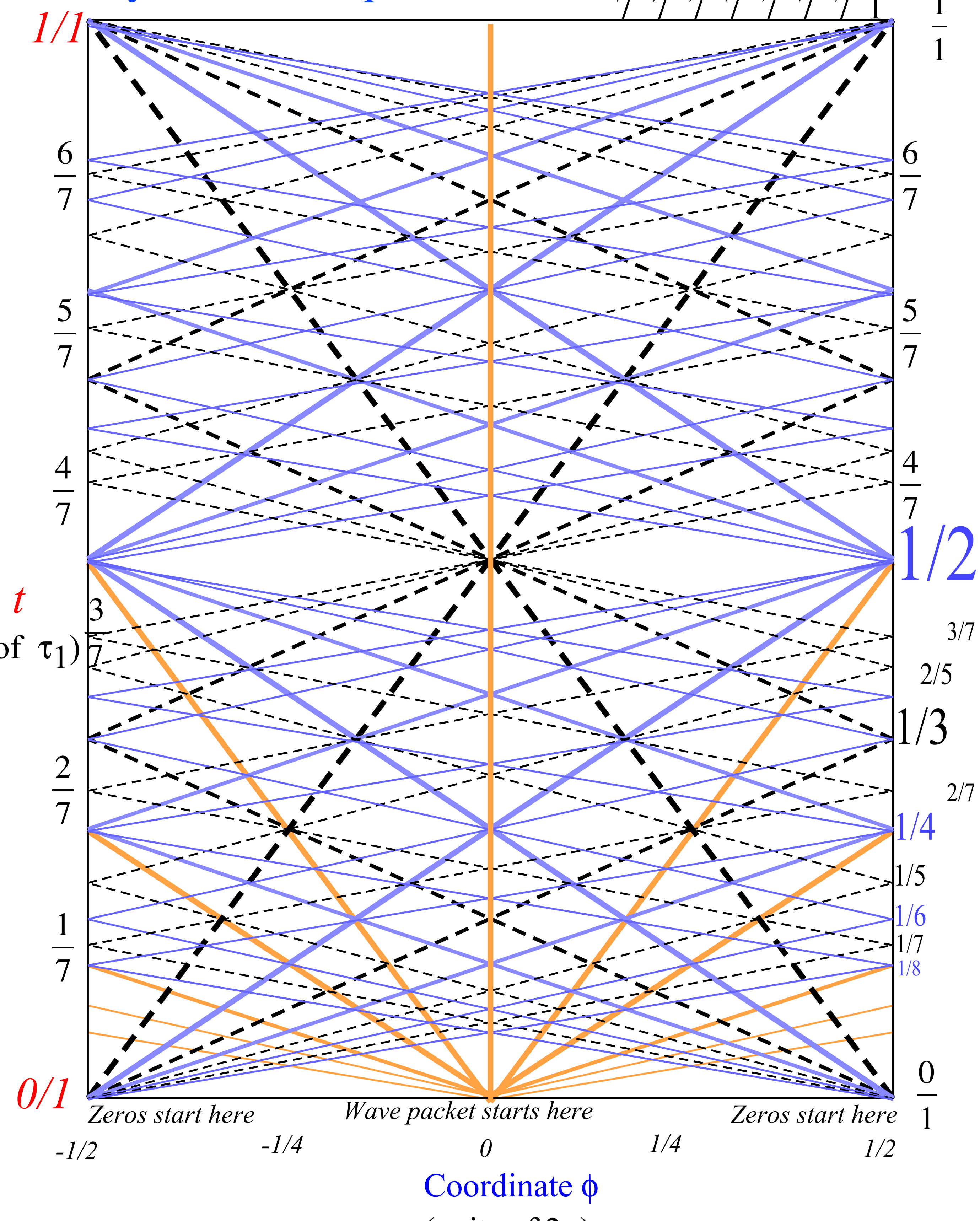
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-rotor system revival-beat wave dynamics

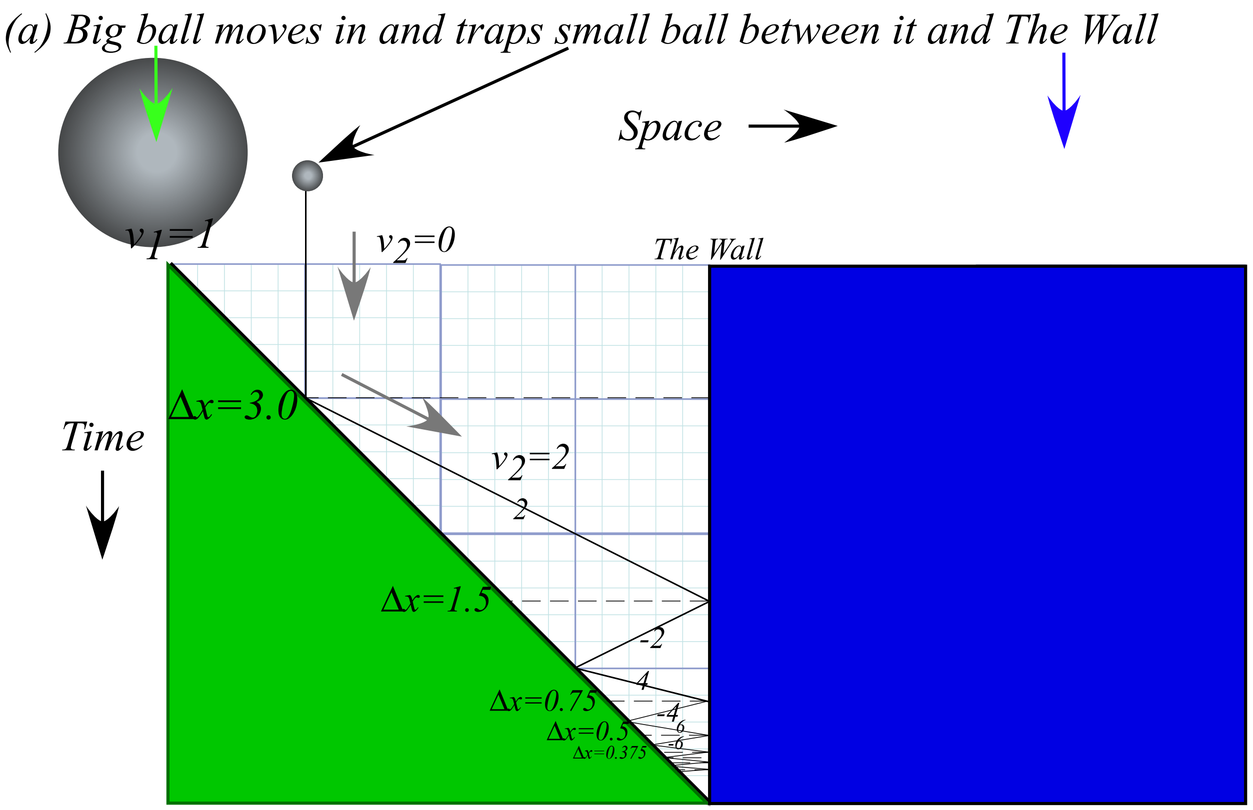
(9 or10-levels $(0, \pm1, \pm2, \pm3, \pm4, \ldots, \pm9, \pm10, \pm11\ldots)$ excited)

Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$





[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Lect. 5 (9.11.14)

The Classical "Monster Mash"

Classical introduction to

Heisenberg "Uncertainty" Relations

$$v_2 = \frac{const.}{Y} \quad \text{or:} \quad Y \cdot v_2 = const.$$

is analogous to: $\Delta x \cdot \Delta p = N \cdot \hbar$

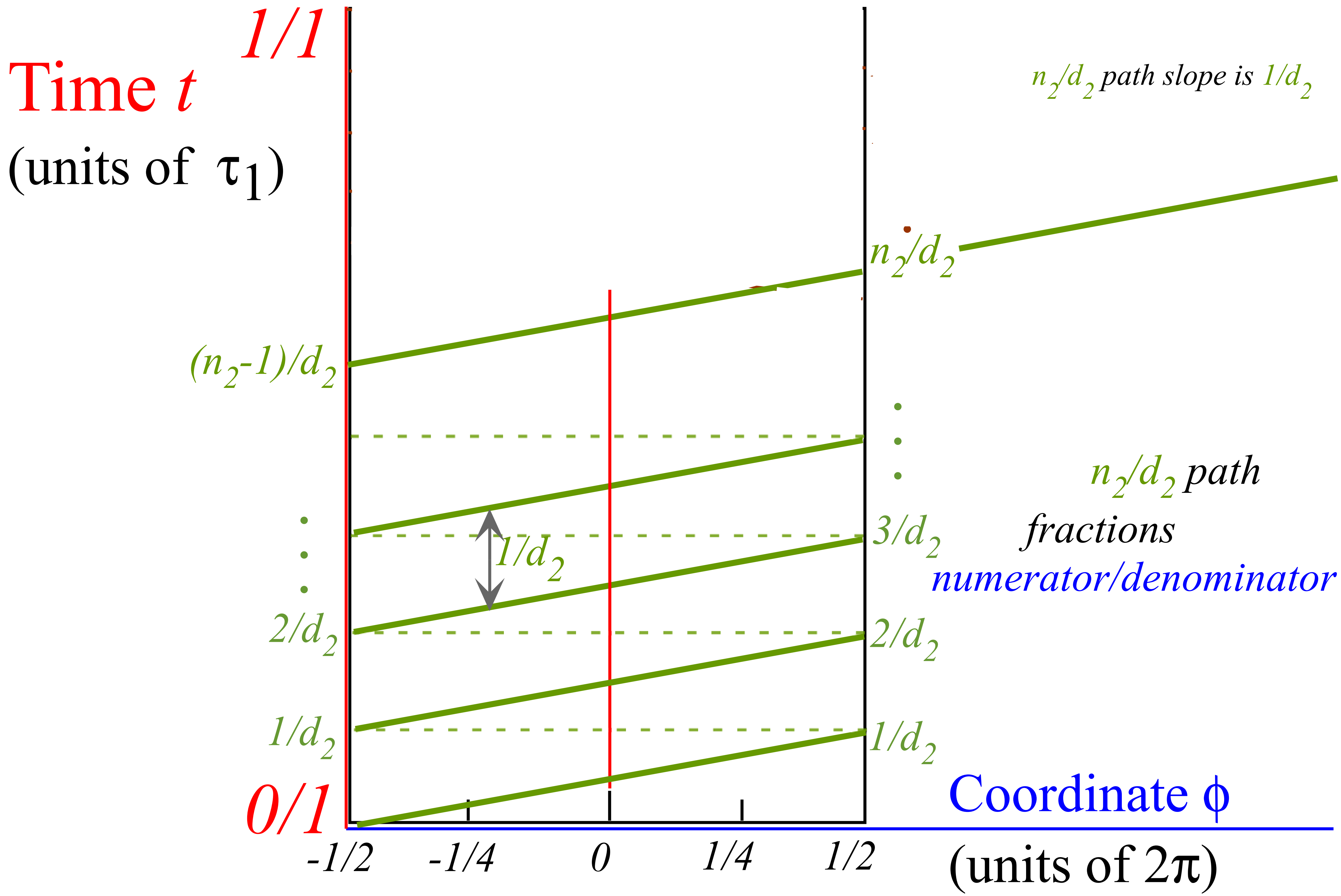
Recall classical "Monster Mash" in Lecture 5

with small-ball trajectory paths having same geometry
as revival beat wave-zero paths

Farey-Sum arithmetic of revival wave-zero paths
(How *Rational Fractions N/D* occupy real space-time)

Farey Sum algebra of revival-beat wave dynamics

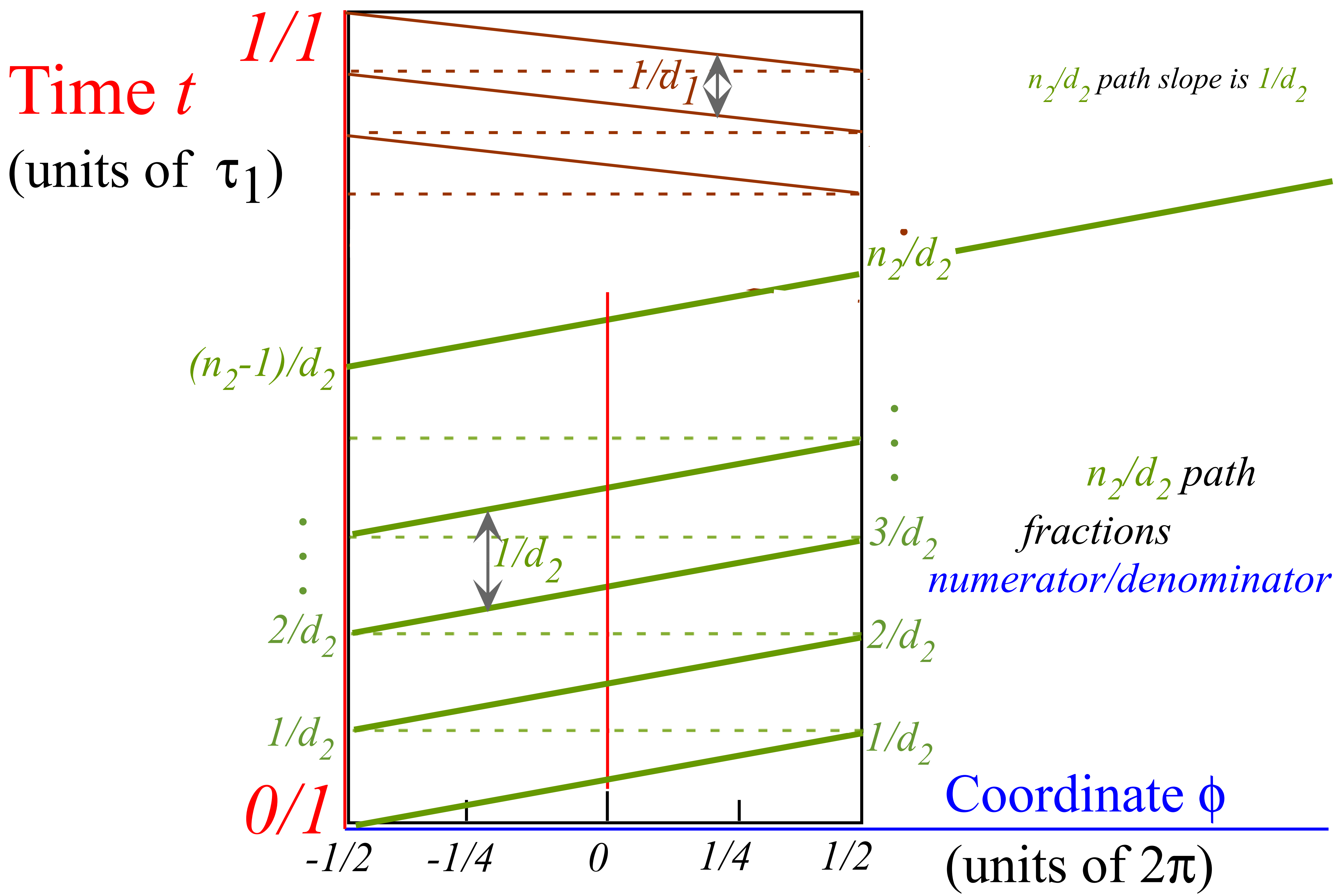
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

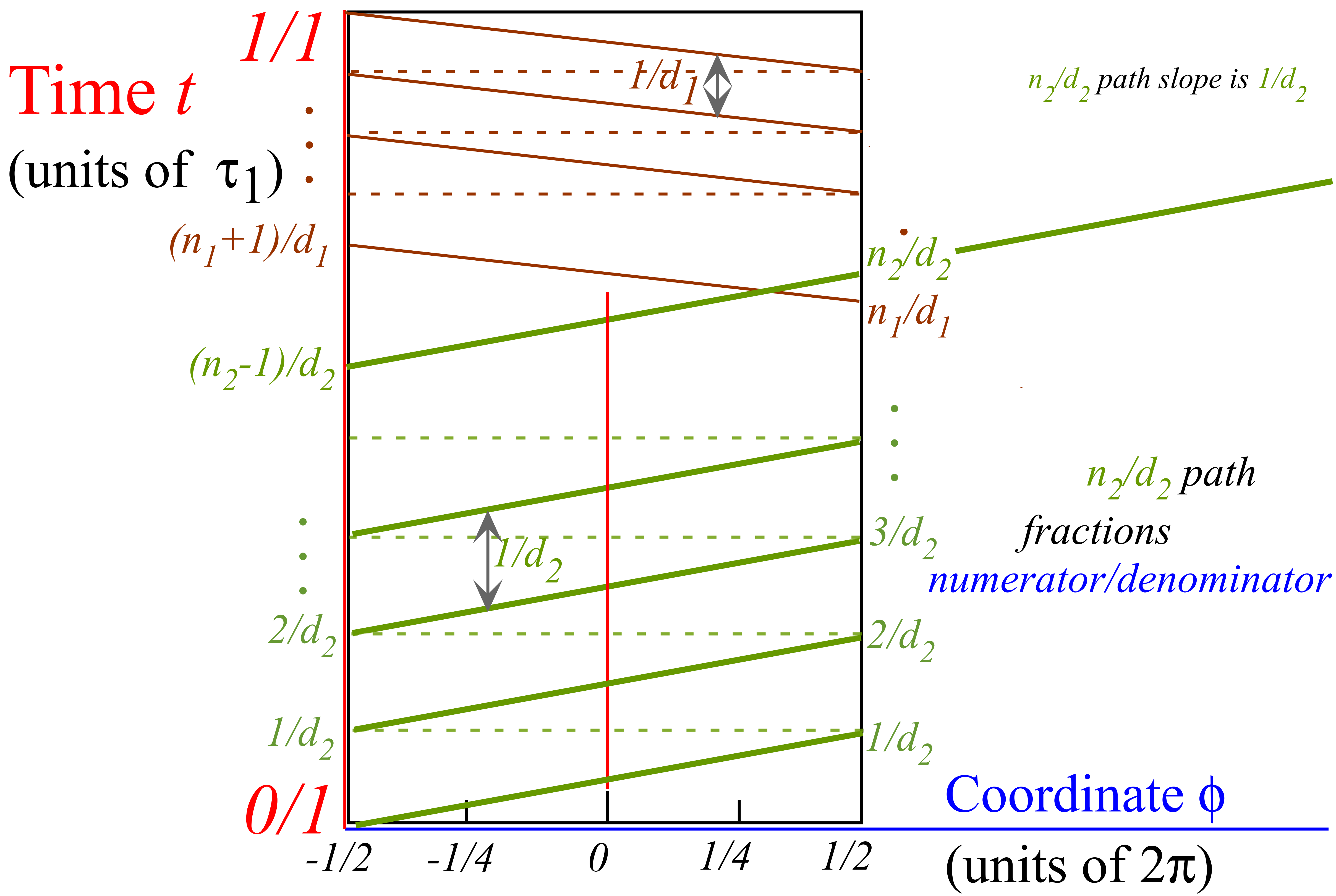
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

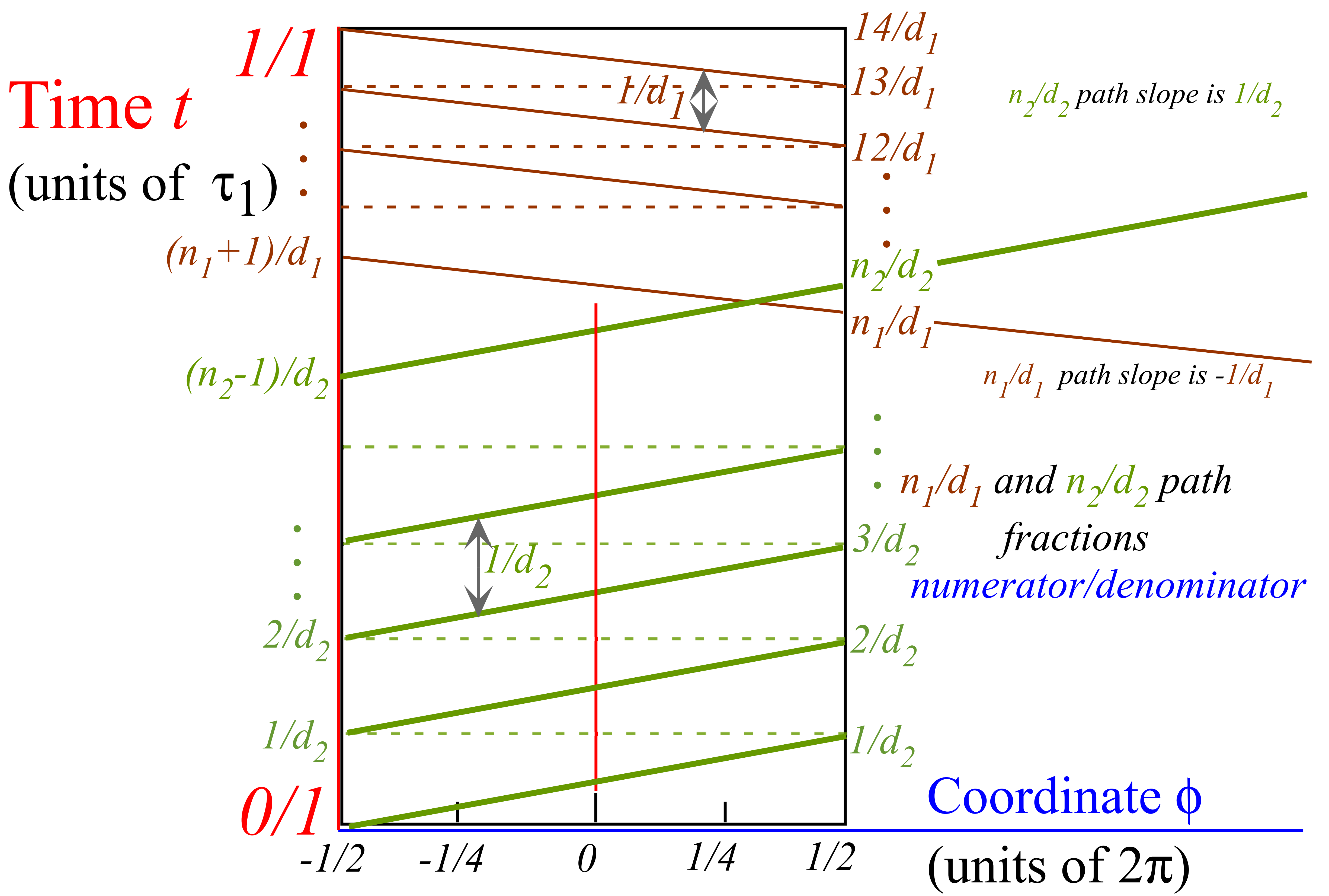
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

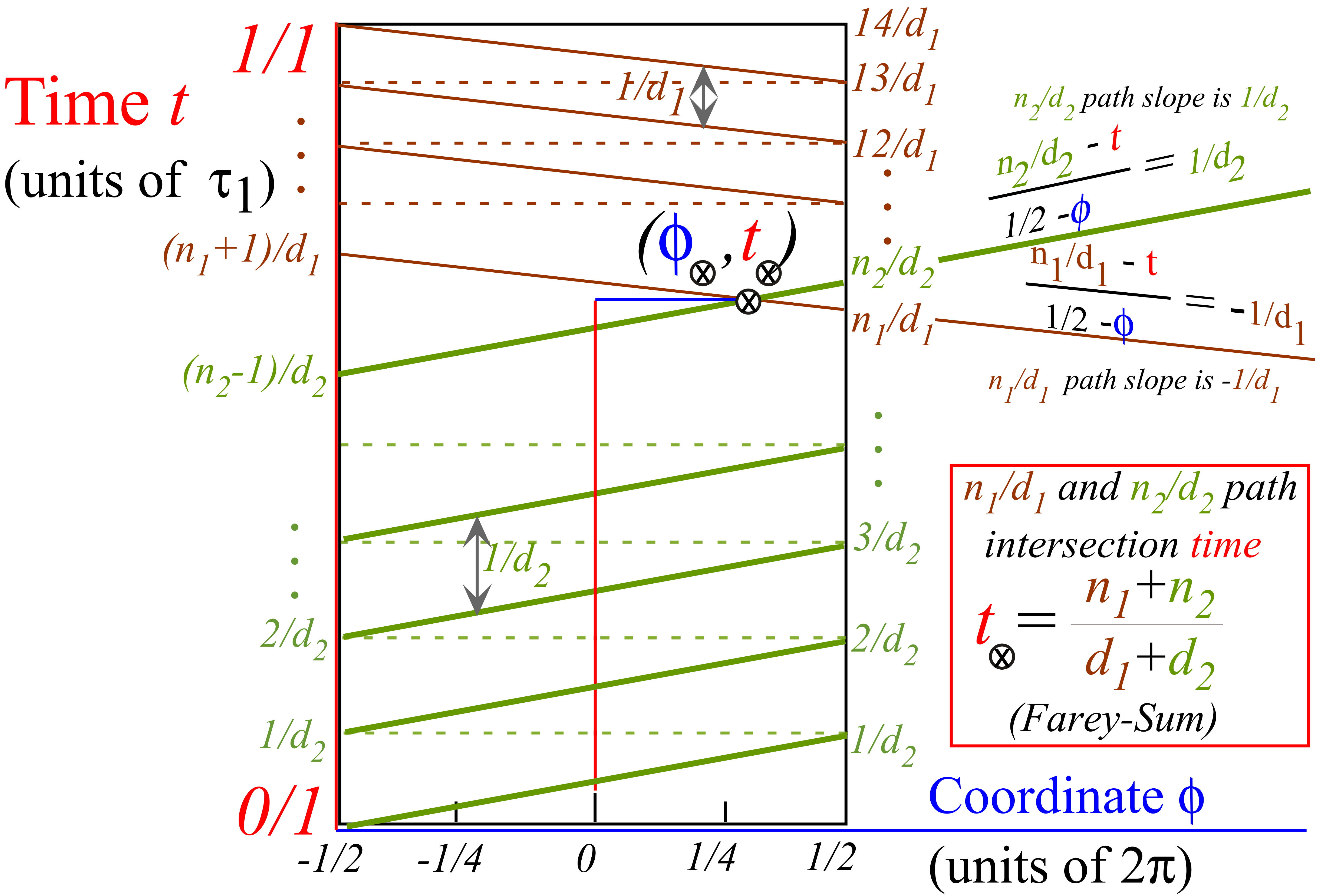
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*

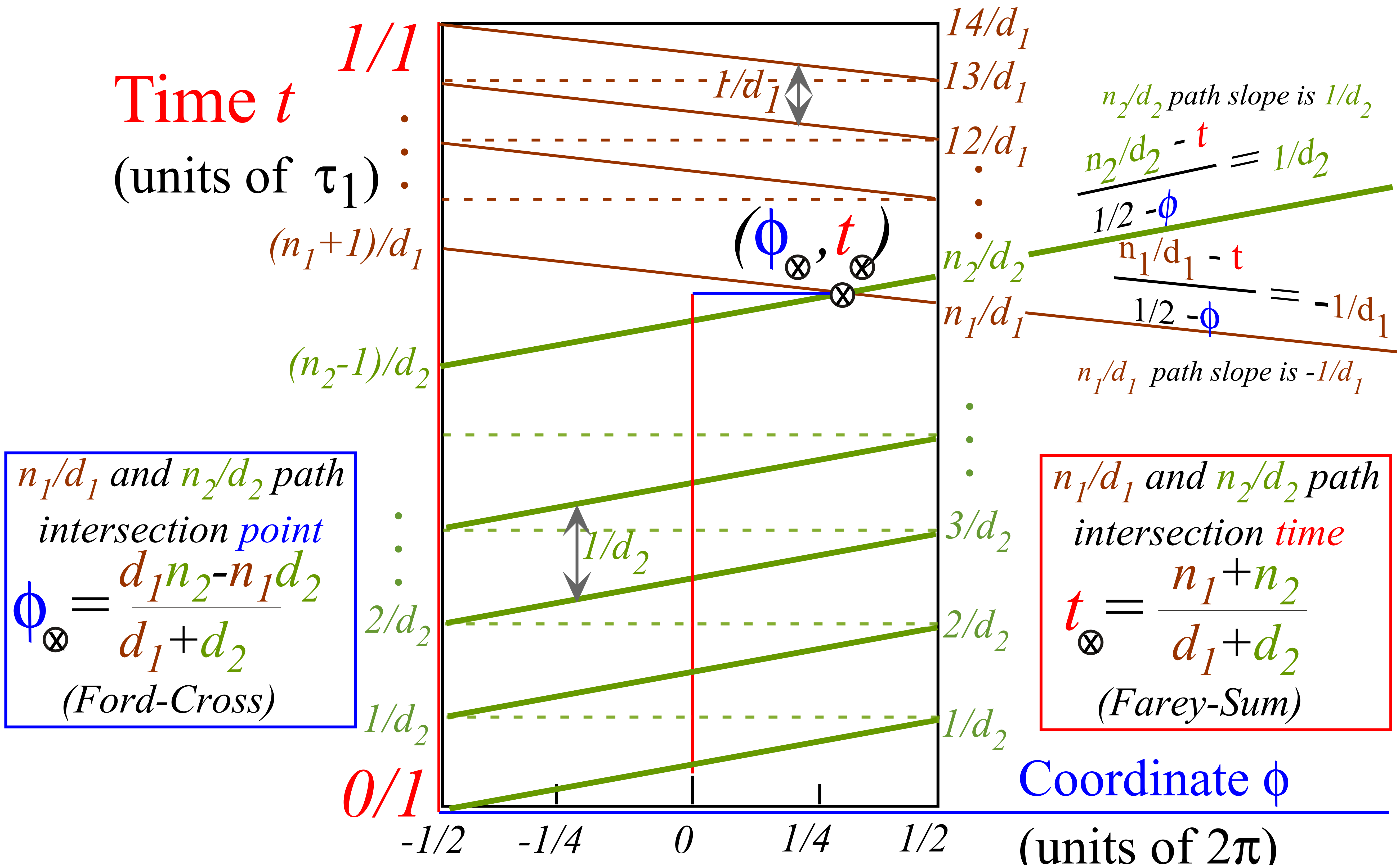


[John Farey, Phil. Mag.(1816)]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*

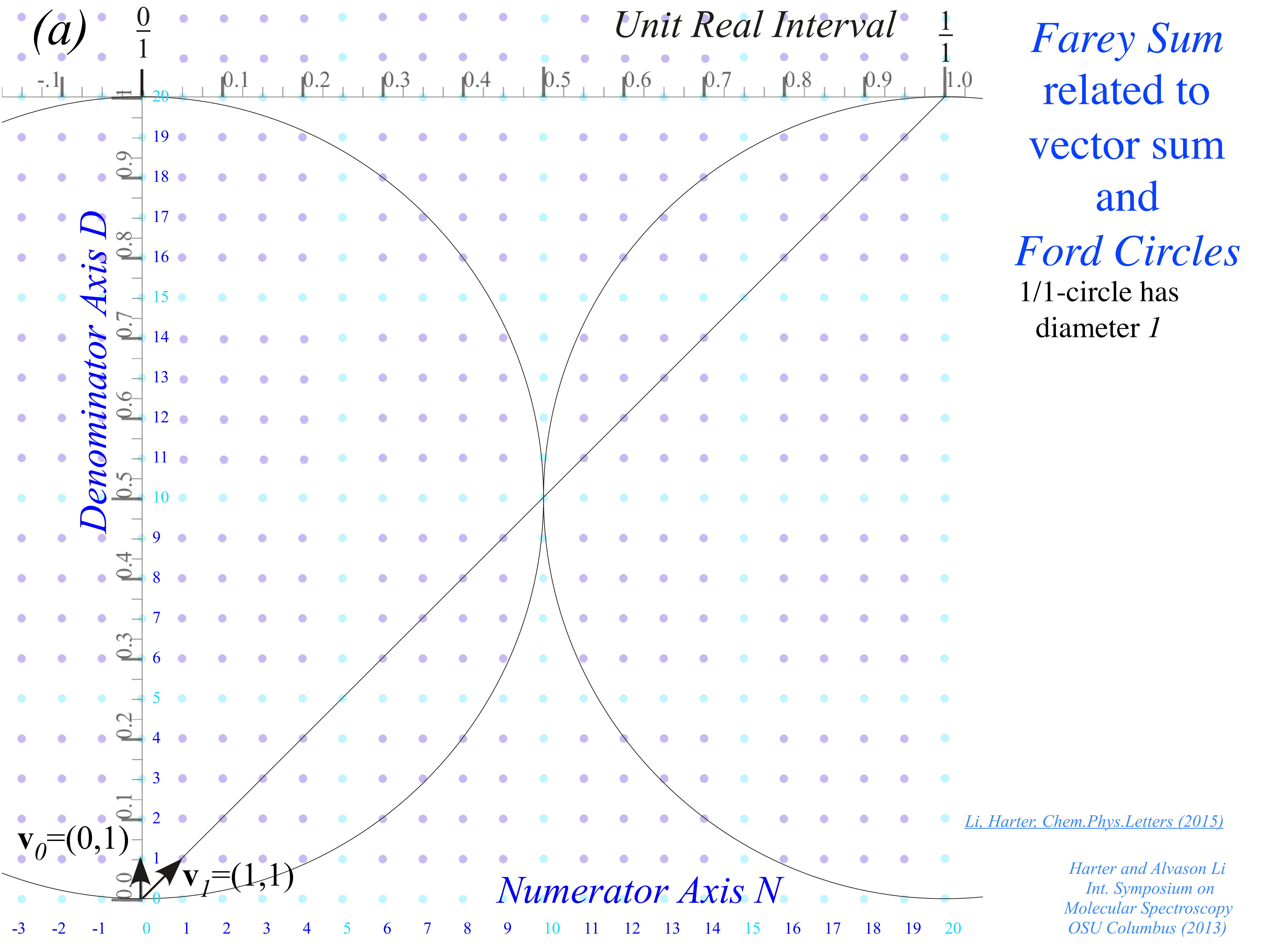


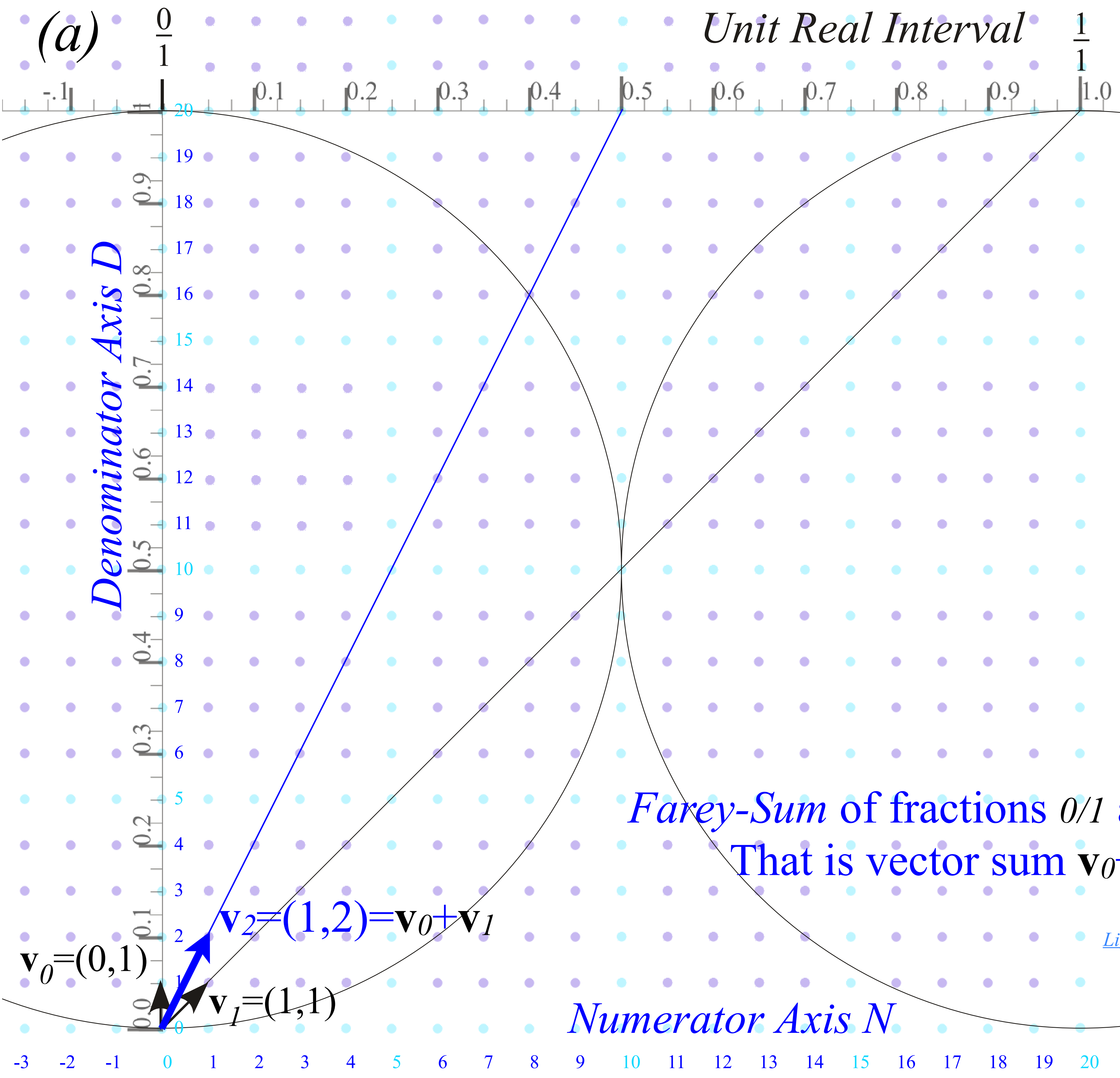
[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

[John Farey, Phil. Mag.(1816)]

Ford-Circle geometry of revival paths
(How *Rational Fractions N/D* occupy real space-time)


(a)
Unit Real Interval
0/1
1/1
Denominator Axis D
Numerator Axis N
$\mathbf{v}_0$=(0,1)
$\mathbf{v}_1$=(1,1)
Farey Sum
related to
vector sum
and
Ford Circles
1/1-circle has
diameter *1*
Li, Harter, Chem.Phys.Letters (2015)
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

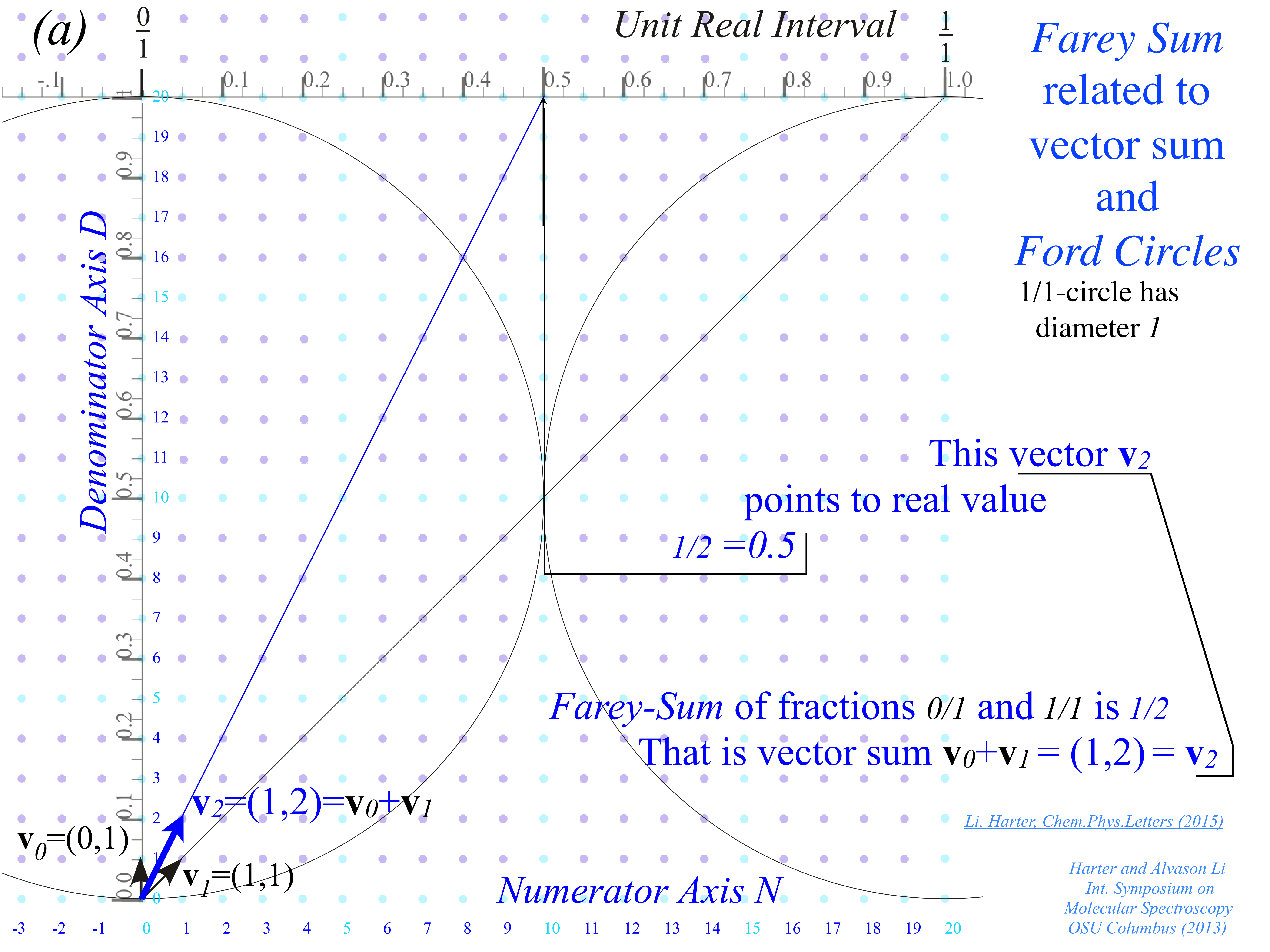
1/1-circle has diameter *1*

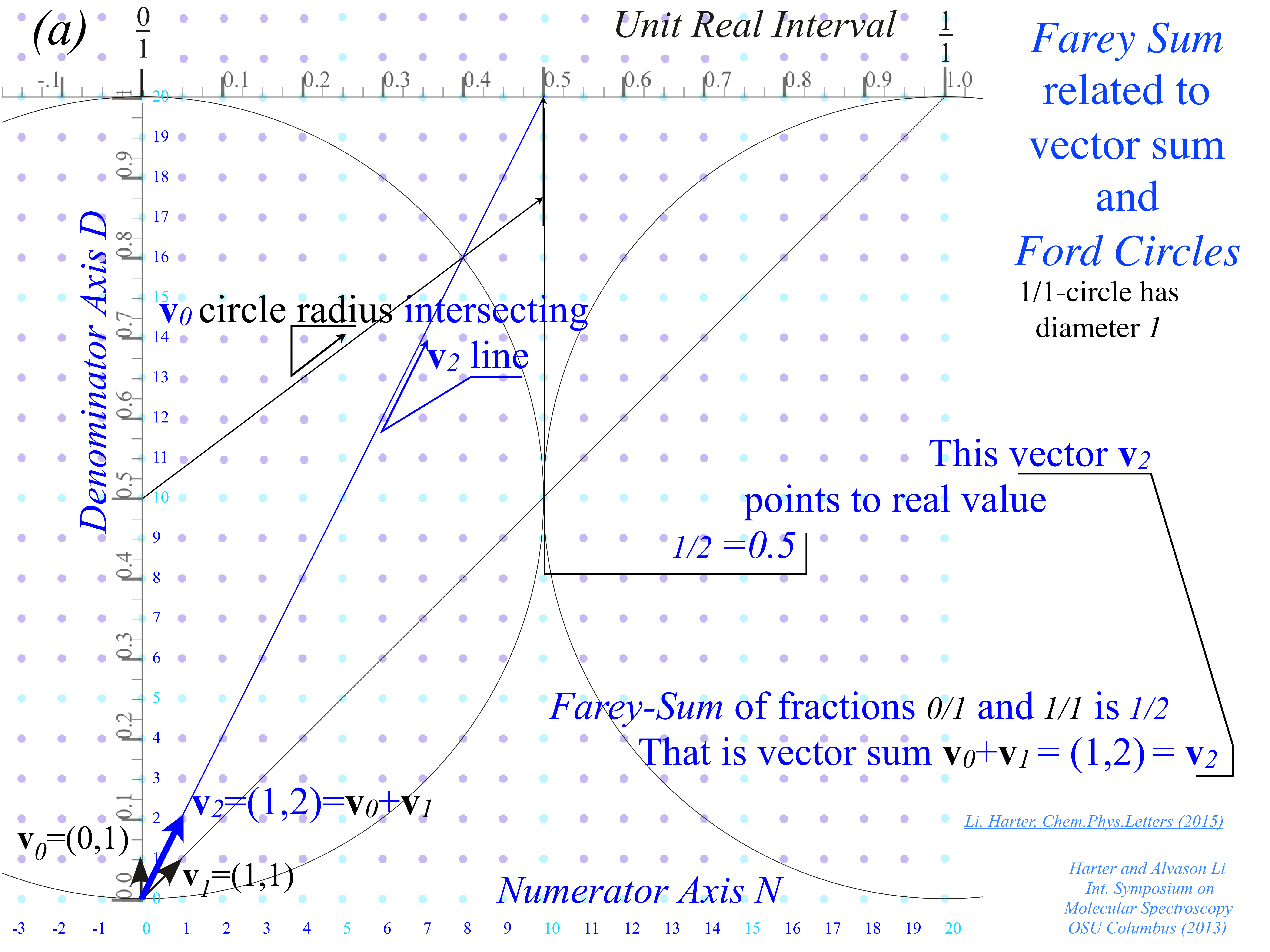
Farey-Sum of fractions *0/1* and *1/1* is *1/2*

That is vector sum $\mathbf{v}_0+\mathbf{v}_1=(1,2)=\mathbf{v}_2$

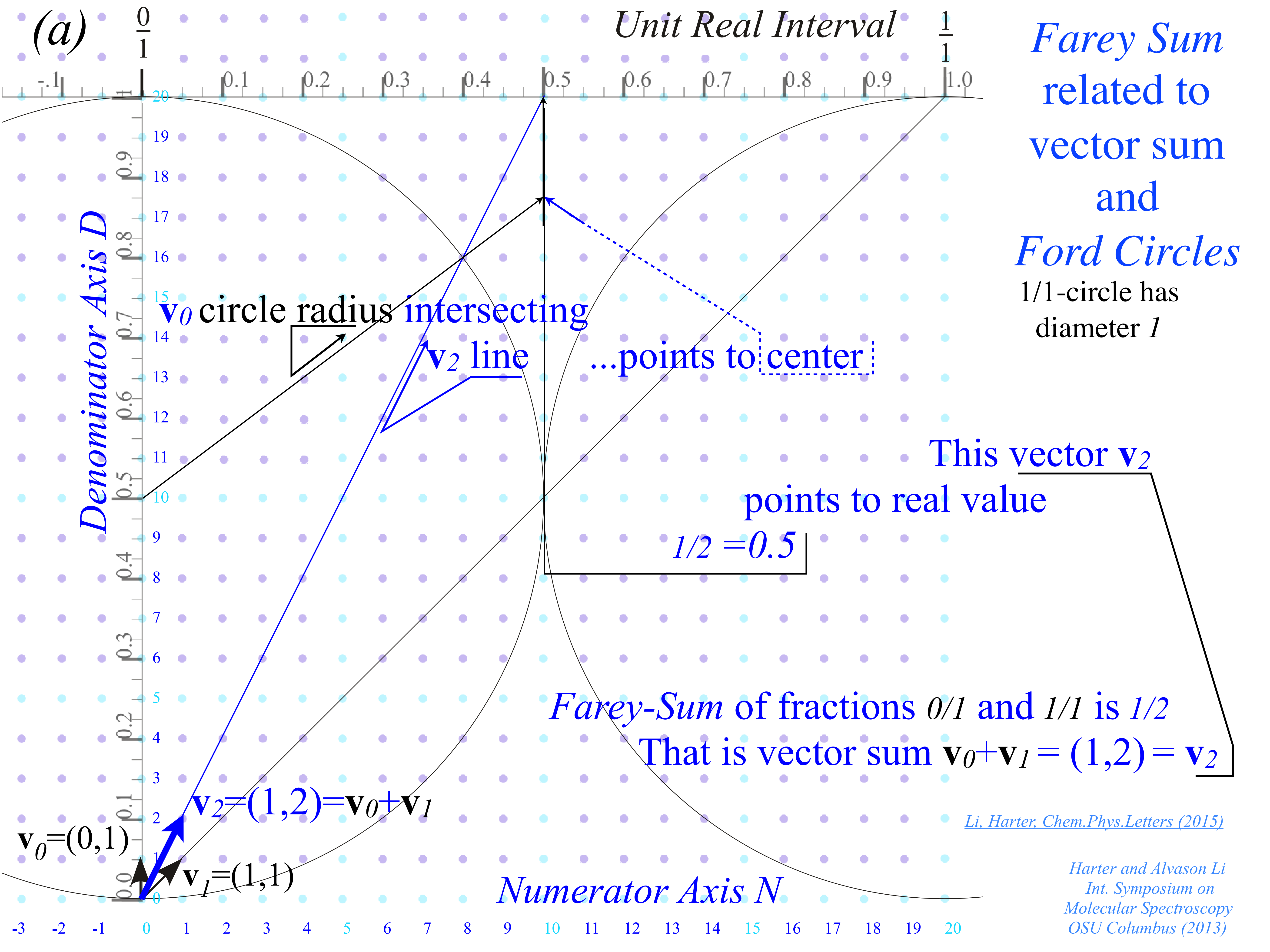
Li, Harter, Chem.Phys.Letters (2015)

Harter and Alvason Li
Int. Symposium on Molecular Spectroscopy
OSU Columbus (2013)


(a)
Unit Real Interval
0/1
1/1
-.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Farey Sum
related to
vector sum
and
Ford Circles
1/1-circle has
diameter 1
Denominator Axis D
This vector $\mathbf{v}_2$
points to real value
1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$
$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$
$\mathbf{v}_0=(0,1)$
$\mathbf{v}_1=(1,1)$
Numerator Axis N
Li, Harter, Chem.Phys.Letters (2015)
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
v0 circle radius intersecting
v2 line
This vector v2 points to real value
1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Numerator Axis N
Li, Harter, Chem.Phys.Letters (2015)
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
Numerator Axis N
v0 circle radius intersecting v2 line
...points to center
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Li, Harter, Chem.Phys.Letters (2015)
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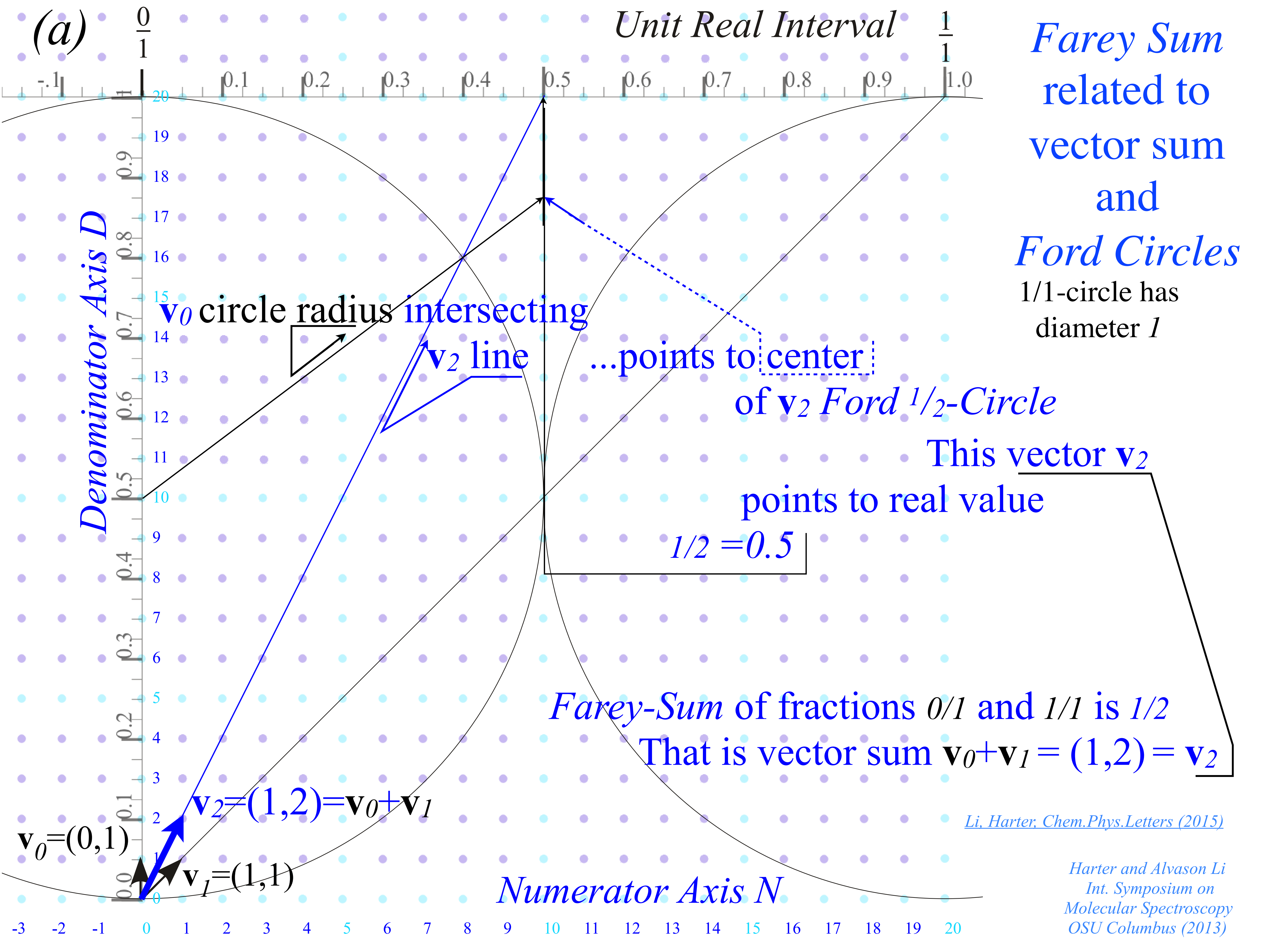
(a)
0/1
Unit Real Interval
1/1
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
Numerator Axis N
v0 circle radius intersecting
v2 line
...points to center
of v2 Ford 1/2-Circle
This vector v2
points to real value
1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
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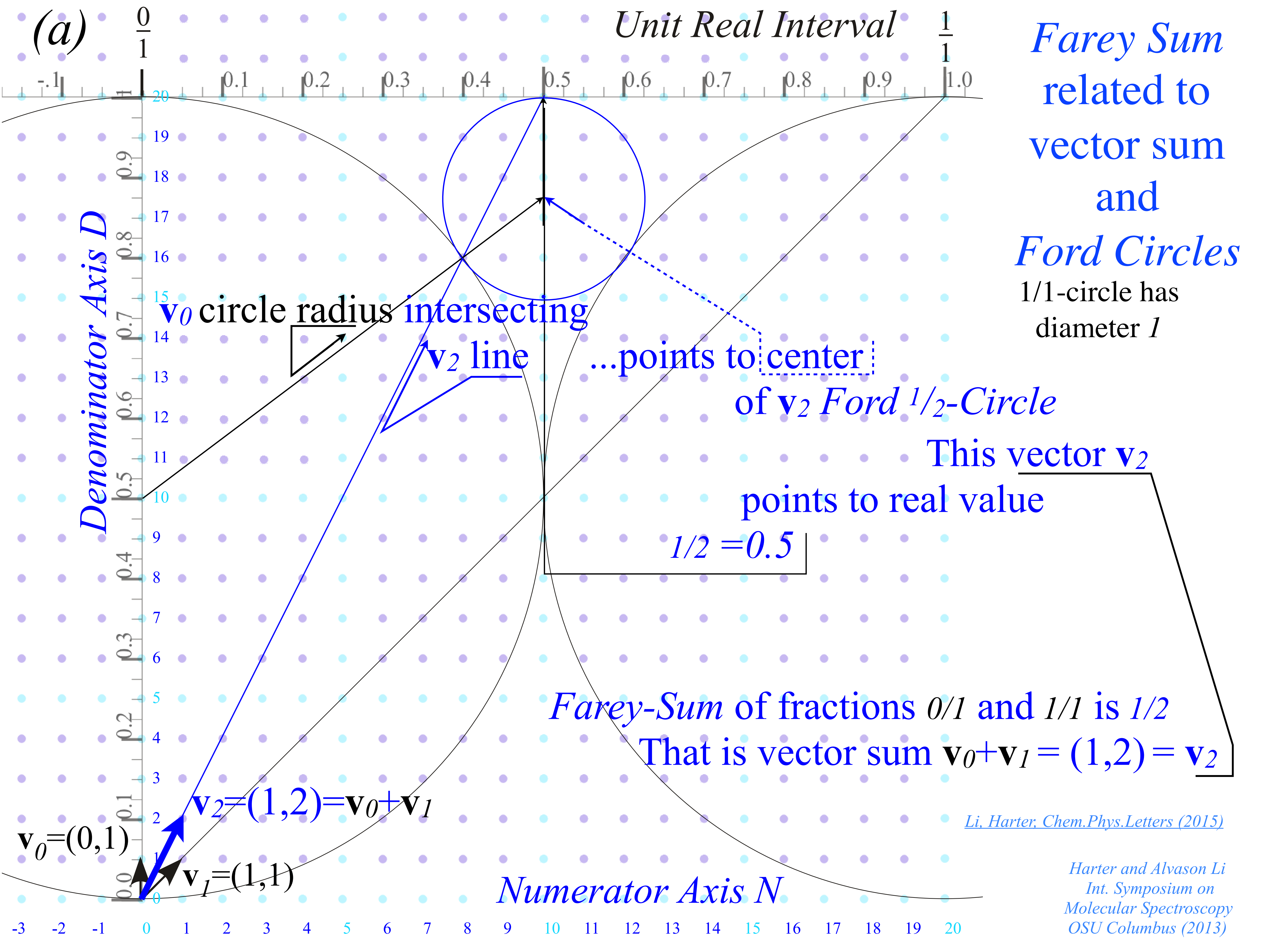

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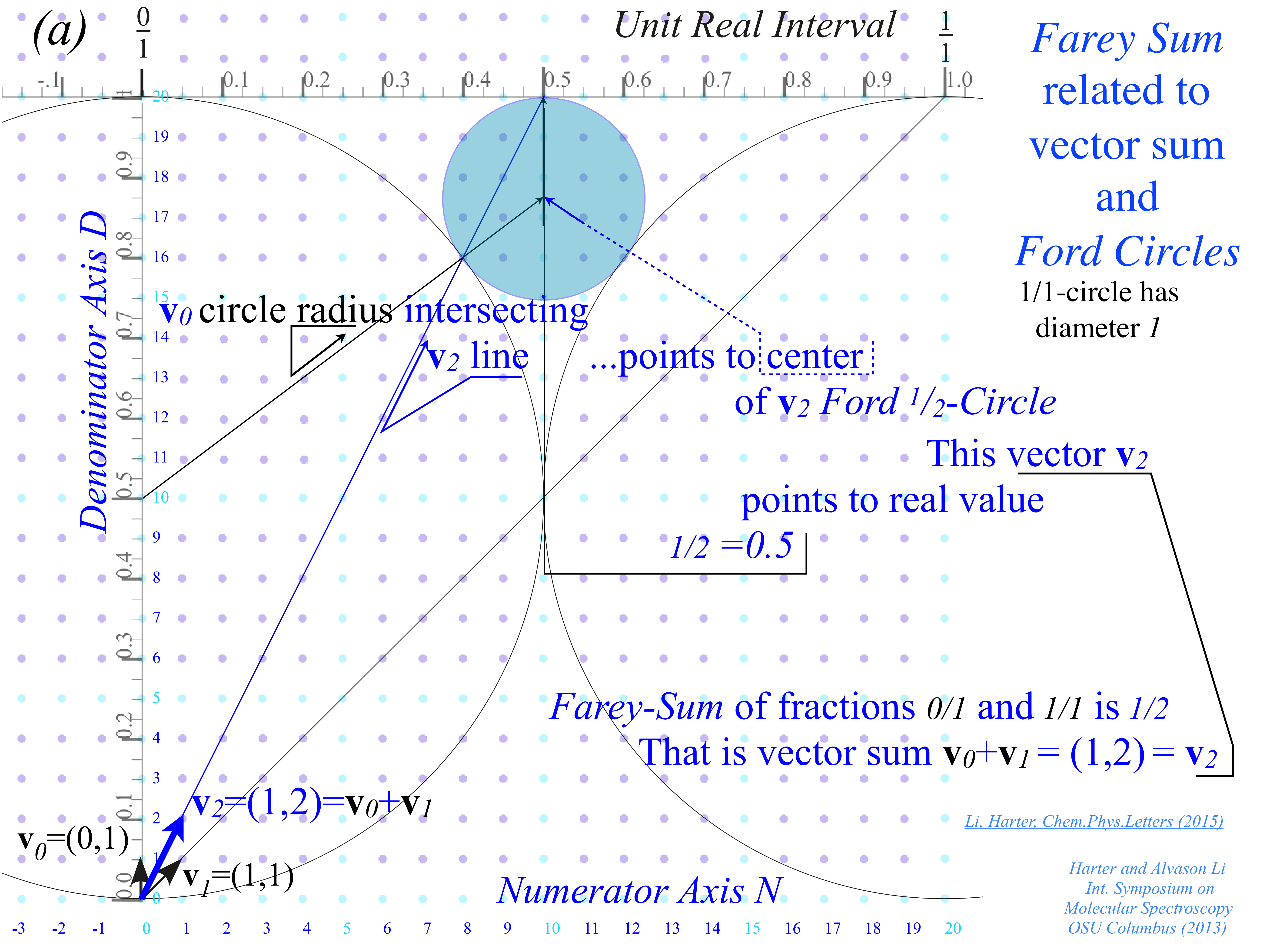
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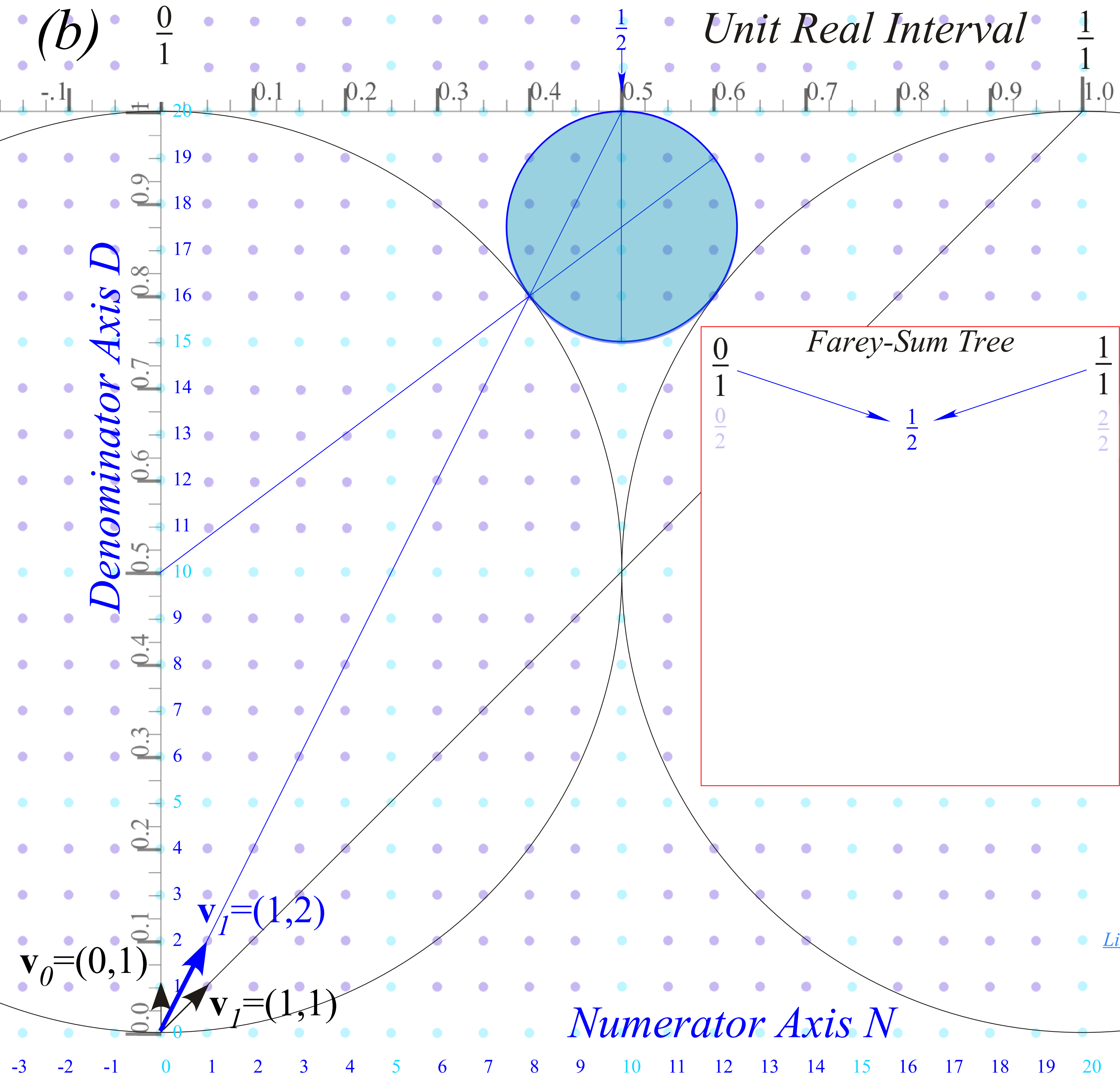

Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

1/2-circle has diameter *1/2²=1/4*

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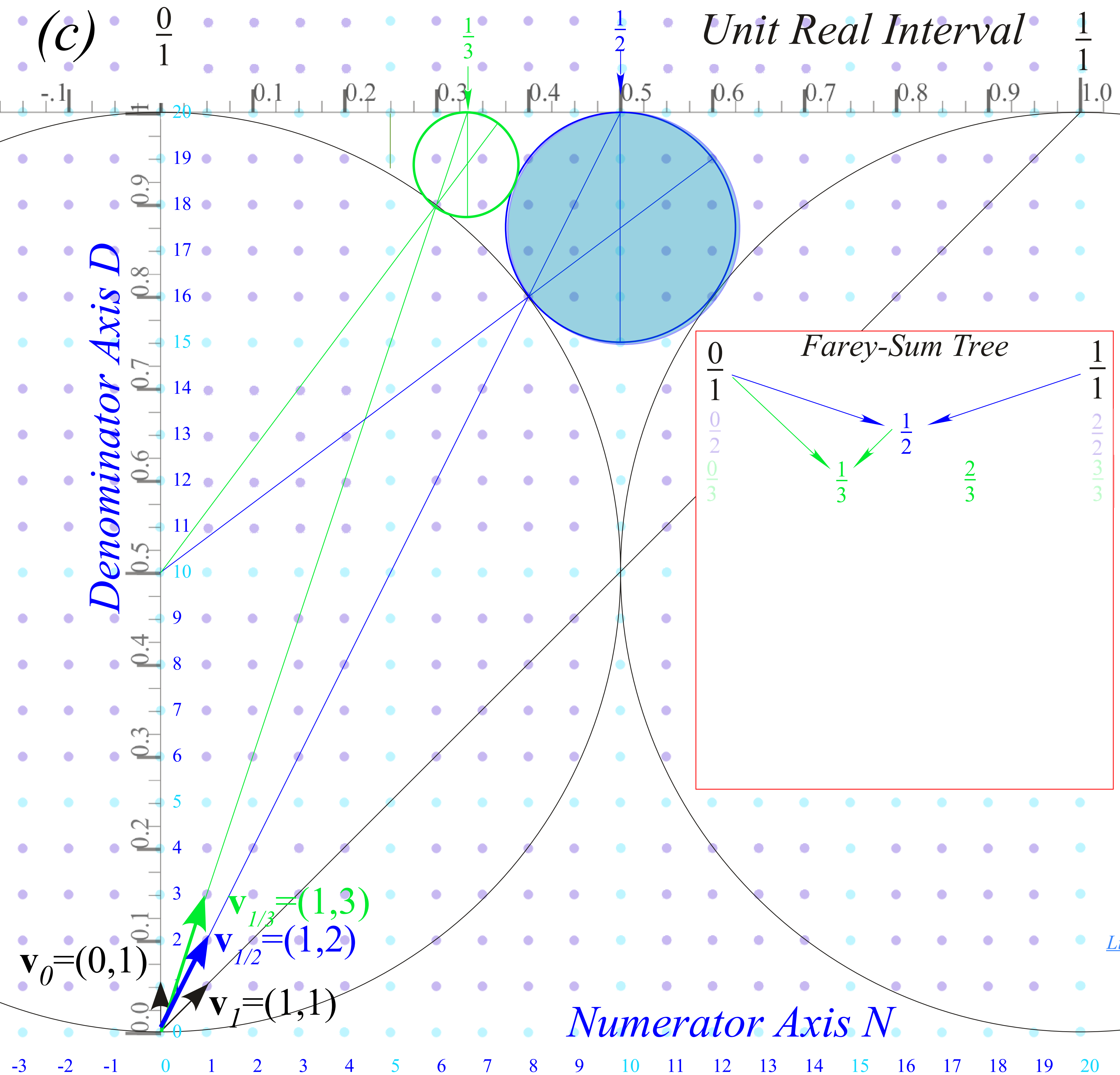

Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

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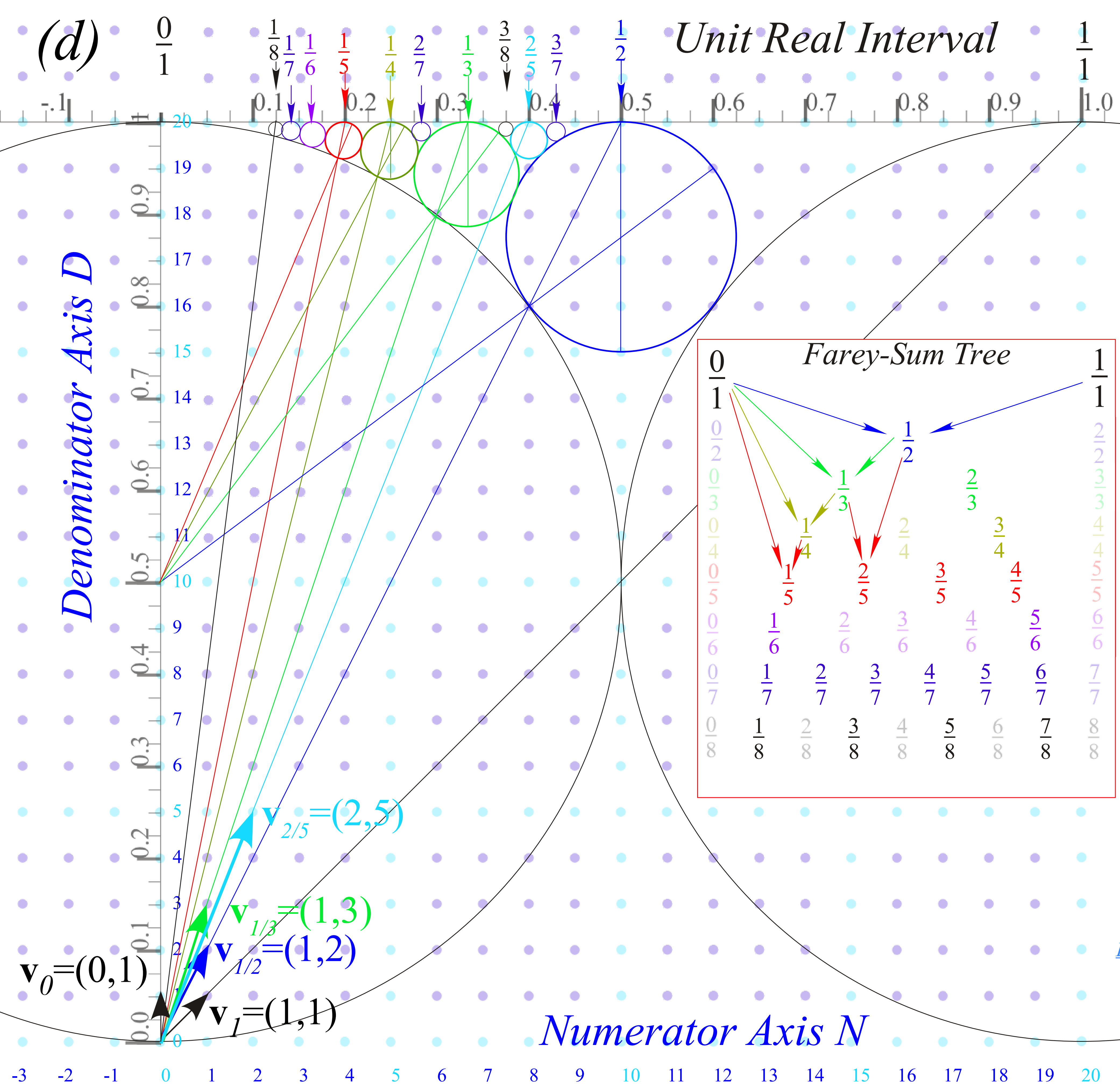

Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

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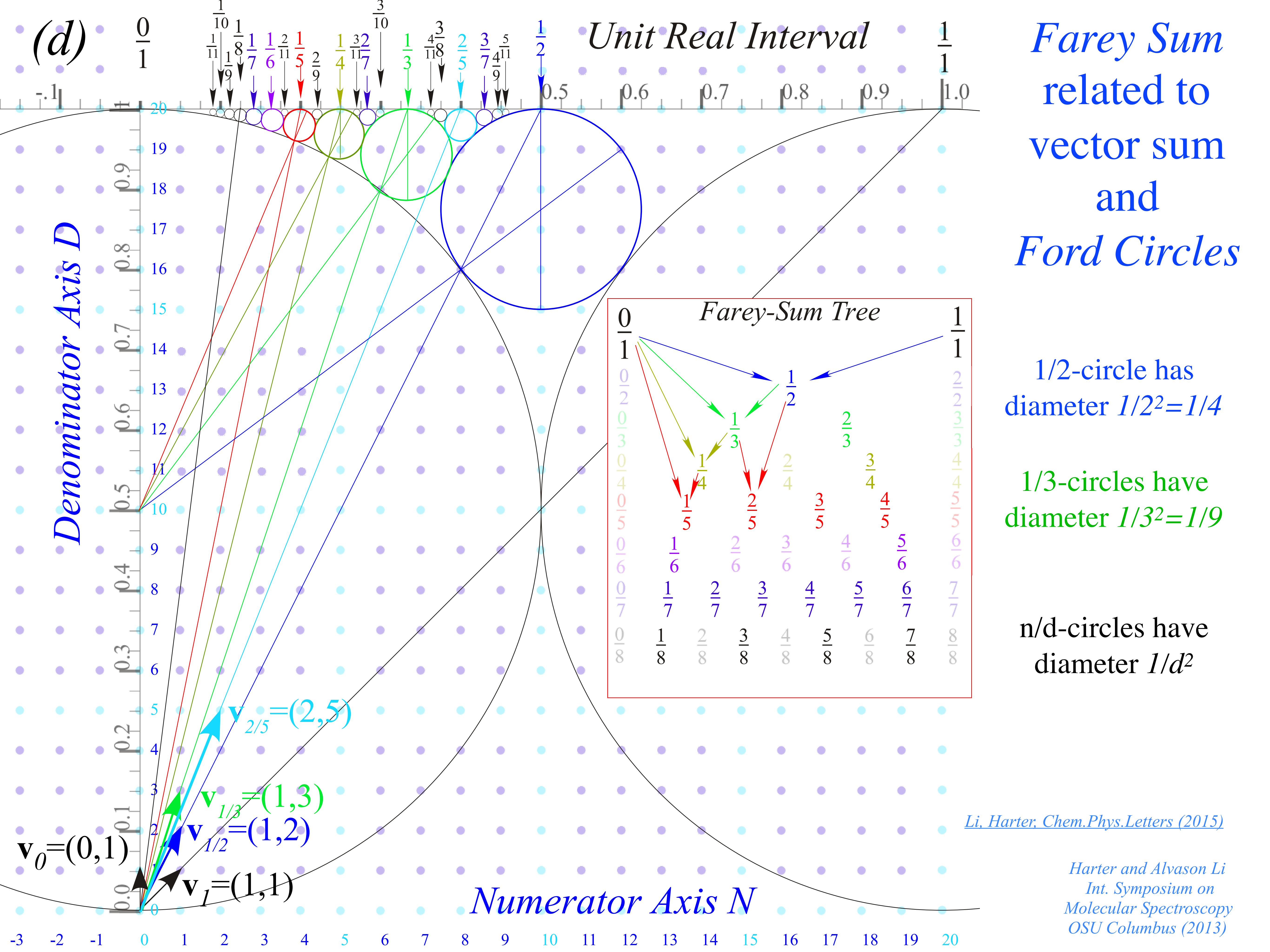
(d)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
Farey-Sum Tree
1/2-circle has diameter $1/2^2=1/4$
1/3-circles have diameter $1/3^2=1/9$
n/d-circles have diameter $1/d^2$
Denominator Axis D
Numerator Axis N
$\mathbf{v}_{2/5}=(2,5)$
$\mathbf{v}_{1/3}=(1,3)$
$\mathbf{v}_{1/2}=(1,2)$
$\mathbf{v}_0=(0,1)$
$\mathbf{v}_1=(1,1)$
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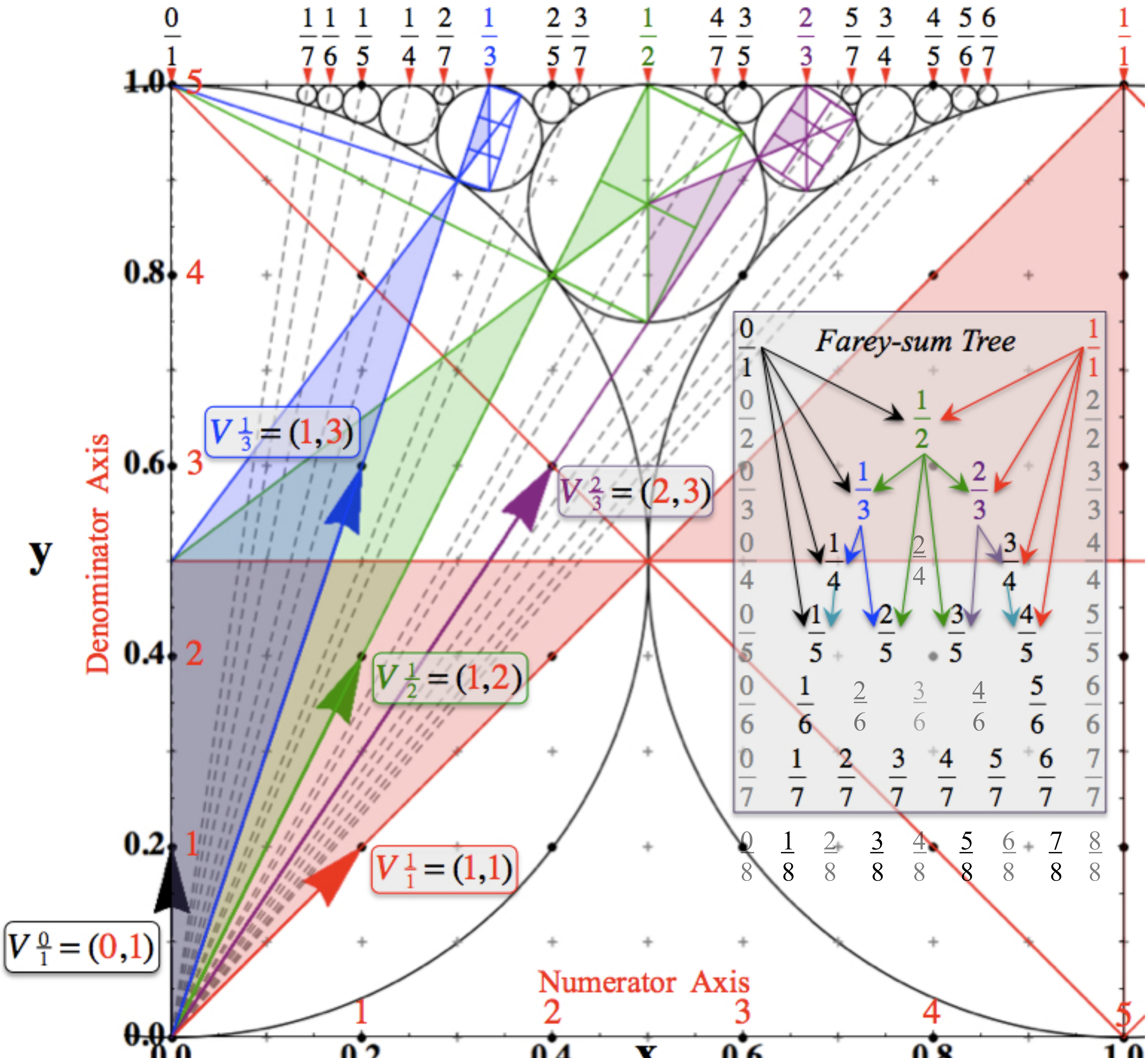

Thales Rectangles

provide
analytic geometry
of
fractal structure

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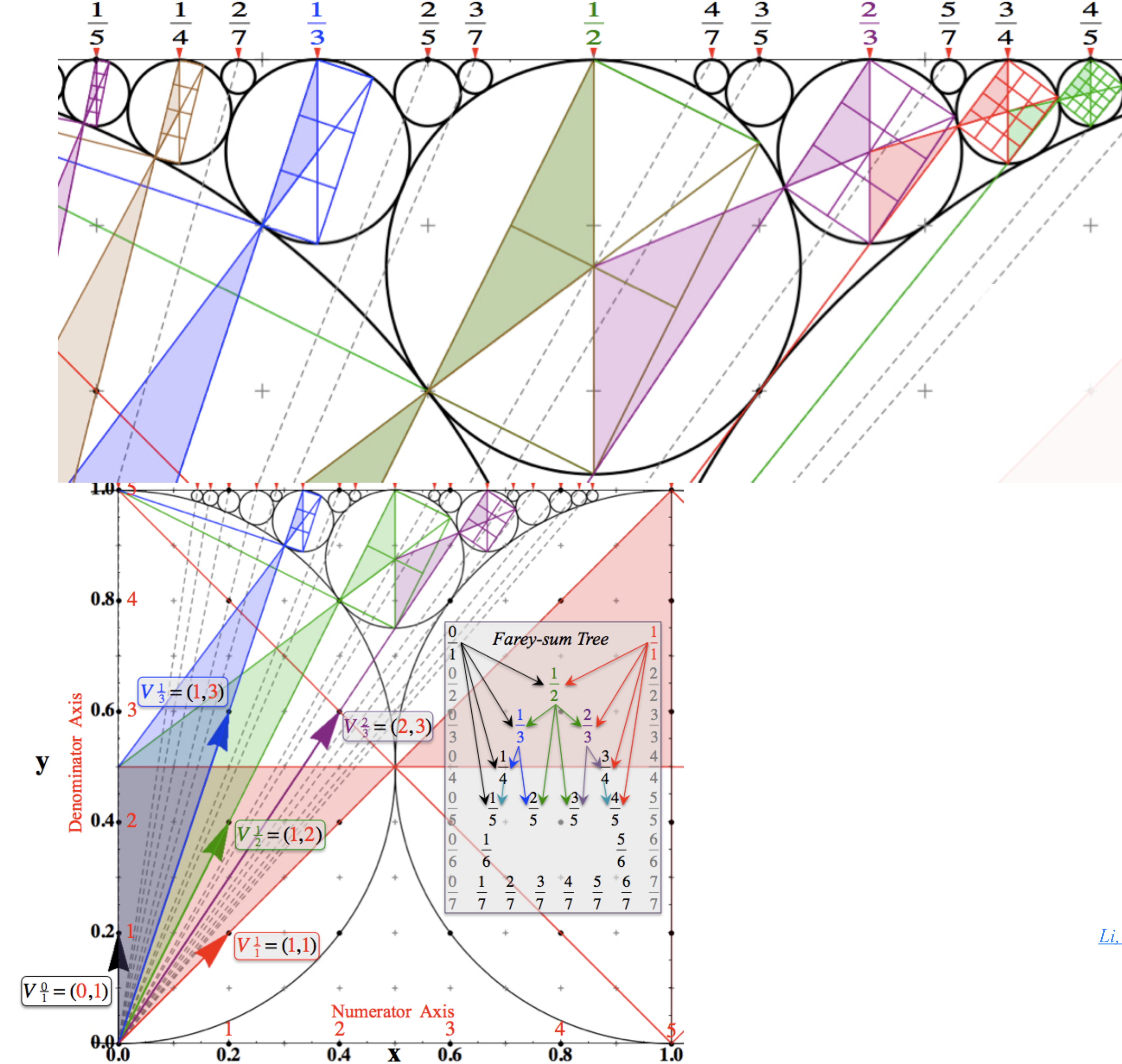

"Quantized" Thales Rectangles

provide
analytic geometry
of
fractal structure

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Relating C_N symmetric H and K matrices to differential wave operators

Relating C_N symmetric H and K matrices to wave differential operators

The 1st neighbor **K** matrix relates to a 2nd *finite-difference* matrix of 2nd x-derivative for high C_N.

$$\mathbf{K} = k(2\mathbf{1} - \mathbf{r} - \mathbf{r}^{-1}) \text{ analogous to:} -k\frac{\partial^2}{\partial x^2}$$

1st derivative momentum: $p = \frac{\hbar}{i}\frac{\partial y}{\partial x} \approx \frac{\hbar}{i}\frac{y(x+\Delta x) - y(x)}{(\Delta x)}$

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \frac{\hbar}{i}\begin{pmatrix} \cdot \\ y_1 - y_0 \\ y_2 - y_1 \\ y_3 - y_2 \\ y_4 - y_3 \\ \cdot \end{pmatrix}$$

2nd derivative KE: $2mE = -\hbar^2\frac{\partial^2 y}{\partial x^2} \approx \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2}$

$$-\hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \end{pmatrix}$$

H and **K** matrix equations are finite-difference versions of quantum and classical wave equations.

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \mathbf{H}\left|\psi\right\rangle \quad \textit{(}\mathbf{H}\textit{-matrix equation)}$$

$$-\frac{\partial^2}{\partial t^2}\left|y\right\rangle = \mathbf{K}\left|y\right\rangle \quad \textit{(}\mathbf{K}\textit{-matrix equation)}$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\left|\psi\right\rangle \quad \textit{(Scrodinger equation)}$$

$$-\frac{\partial^2}{\partial t^2}\left|y\right\rangle = -k\frac{\partial^2}{\partial x^2}\left|y\right\rangle \quad \textit{(Classical wave equation)}$$

Square p^2 gives 1st neighbor **K** matrix. Higher order p^3, p^4,.. involve 2nd, 3rd, 4th..neighbor **H**

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$p^4 \cong \begin{pmatrix} \ddots & \vdots & 1 & \cdot & \cdot & \cdot \\ \cdots & 6 & -4 & 1 & \cdot & \cdot \\ 1 & -4 & 6 & -4 & 1 & \cdot \\ \cdot & 1 & -4 & 6 & -4 & 1 \\ \cdot & \cdot & 1 & -4 & 6 & -4 \\ \cdot & \cdot & \cdot & 1 & -4 & 6 \end{pmatrix}$$

Symmetrized finite-difference operators

$$\bar{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \quad \bar{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\bar{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \quad \bar{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$