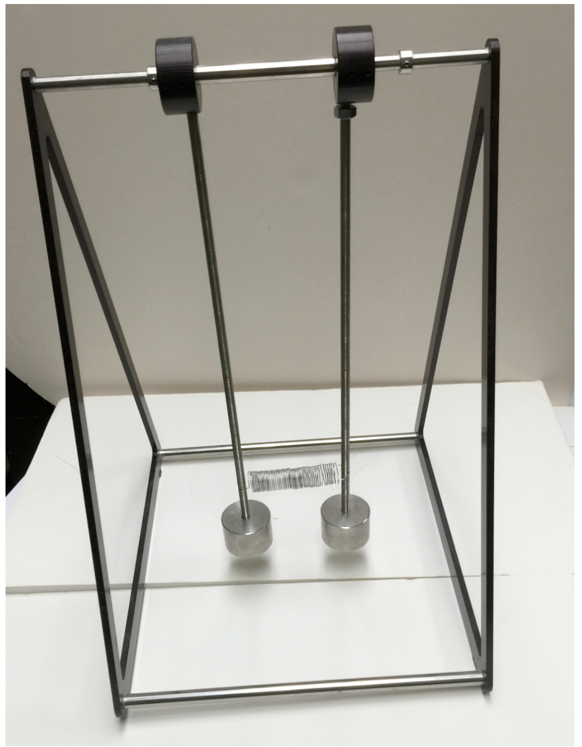
Lecture 21
Wed. 11.06.2019

Introduction to coupled oscillation and eigenmodes

(Ch. 2-4 of Unit 4)



2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*

Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #21

In reverse order

Wiki on Pafnuty Chebyshev
Nobelprize.org
2005 Physics Award

BoxIt Web Simulations:
A-Type w/Cosine, A-Type w/Freq ratios,
AB-Type w/Cosine, AB-Type 2:1 Freq ratio

OscillIt Web Simulations:
Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
Way above resonance,Underdamped
Complex Response Plot

CoulIt Web Simulations:
Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

Trebuchet Web Simulations:
Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)
Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:
Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
This Indestructible NASA Camera Revealed Hidden Patterns on Jupiter - seeker-yt-2019
What did NASA's New Horizons discover around Pluto? - Astrum-yt-2018
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019
Quantum Computing for Computer Scientists - Helwer-mr-yt-2018, Slides
Quantum Computing and Workforce, Curriculum, and App Devel - Roetteler-MS-2019
A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for ...
Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Quantum_Chaos_-_An_Introduction_-_Stockmann-cup-2006, Review by E. Heller
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*
Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research
Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018
Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

An assist from *Physics Girl* (YouTube Channel):
How to Make VORTEX RINGS in a Pool
Crazy pool vortex - pg-yt-2014
Fun with Vortex Rings in the Pool - pg-yt-2014

Excerpts (Page 44-47 in *Preliminary Draft*) from the
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74*, *Page=75*, *Page=79*
Pendulum Web Sim
Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
*Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_*,
Nuclear_Collisions - Mishustin-PhysRevC-2007, *APS Link & Abstract*
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0)
y_2 vs y_1 plots: (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0), (v_1, v_2)=(1, -1)
Estrangian plot *V_2 vs V_1: (v_1, v_2)=(0, 1), (v_1, v_2)=(1, -1)*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *(v_1, v_2)=(1, 0): V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

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Lagrange functional interpolation formula

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Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

2D harmonic oscillators

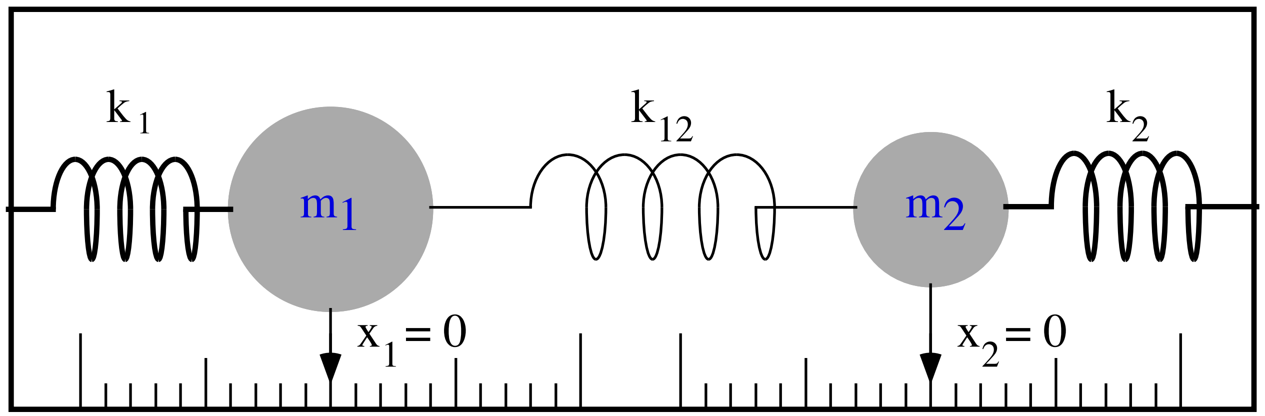
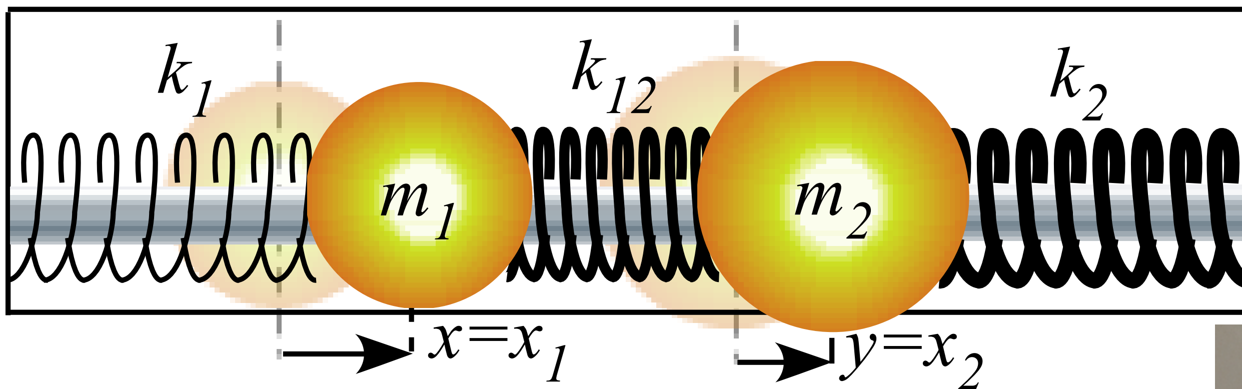


Fig. 3.3.1 Two 1-dimensional coupled oscillators



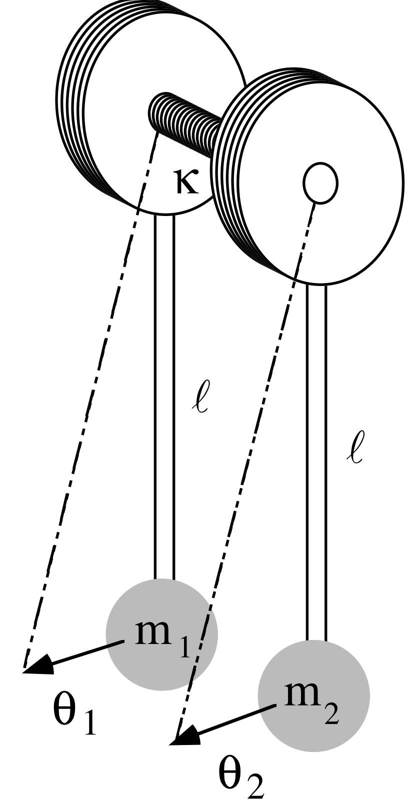


Fig. 3.3.2 Coupled pendulums

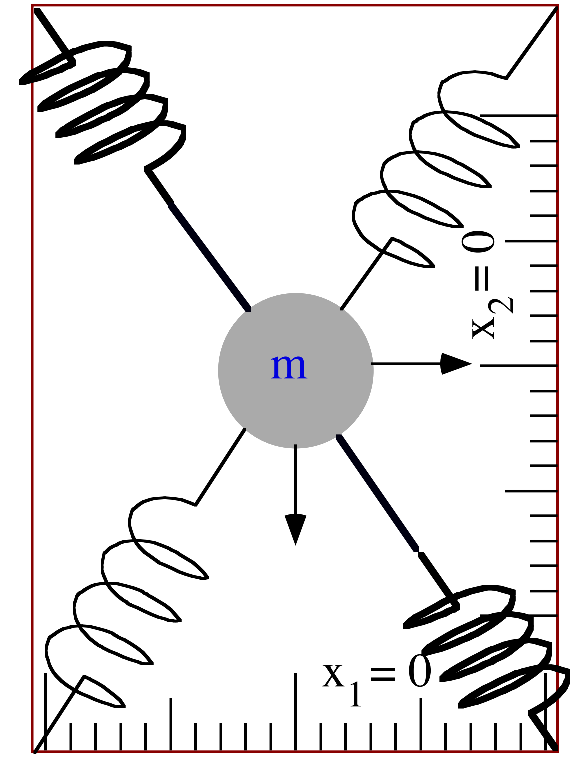
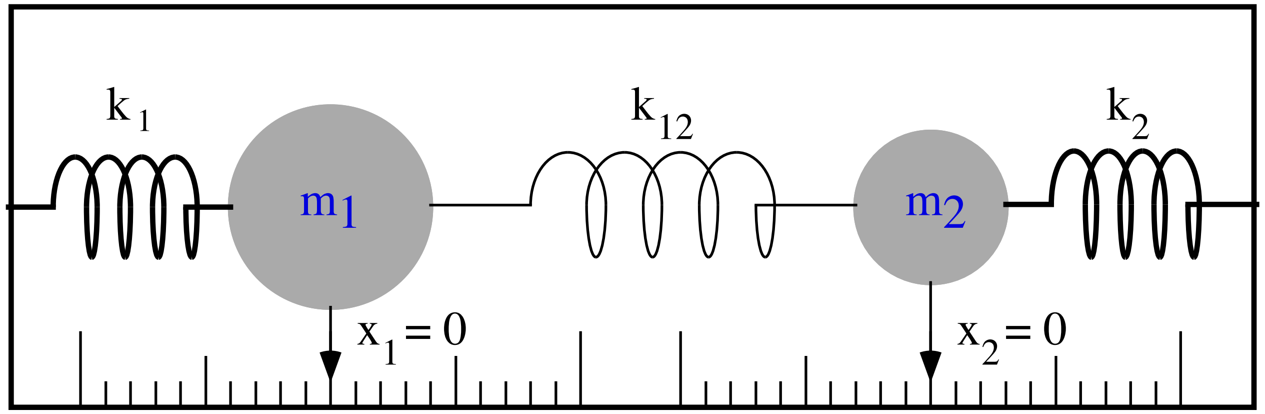


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D harmonic oscillator energy



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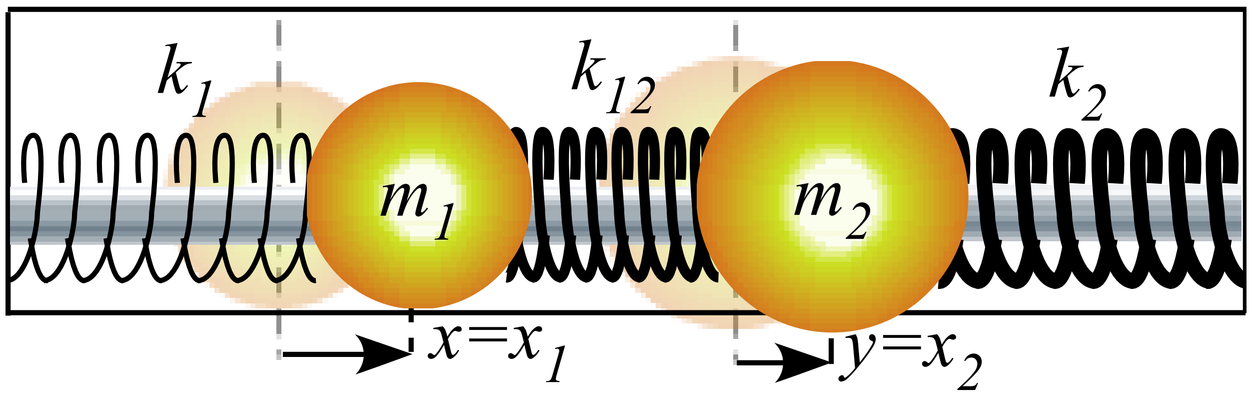


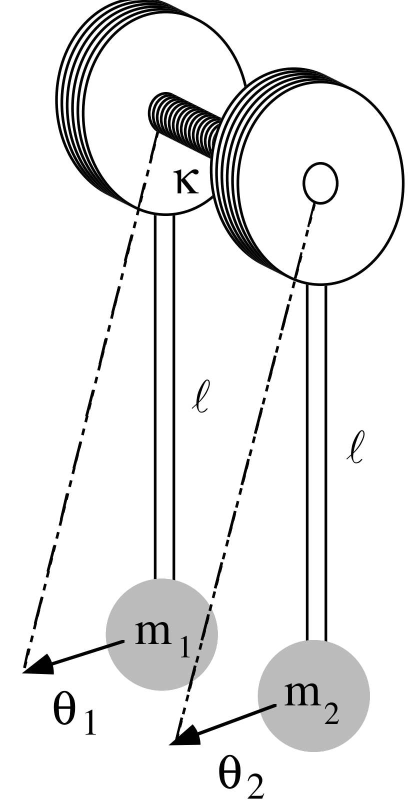


Fig. 3.3.2 Coupled pendulums

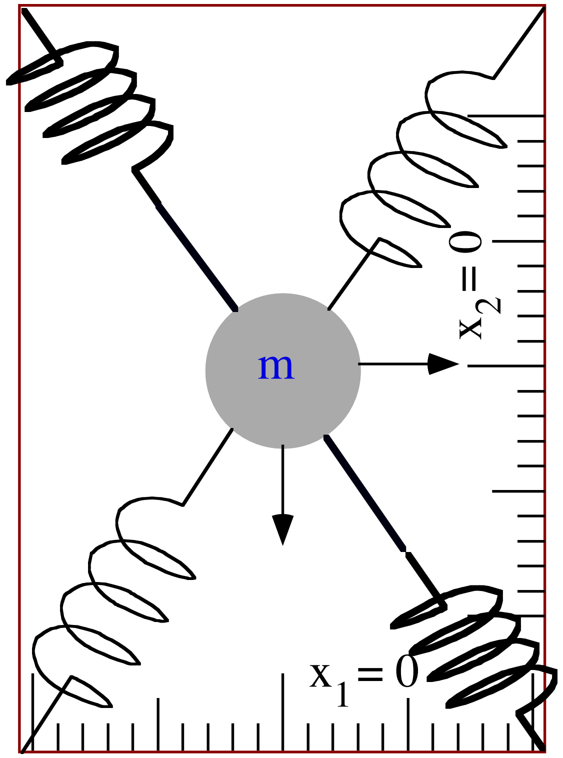


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D harmonic oscillator energy

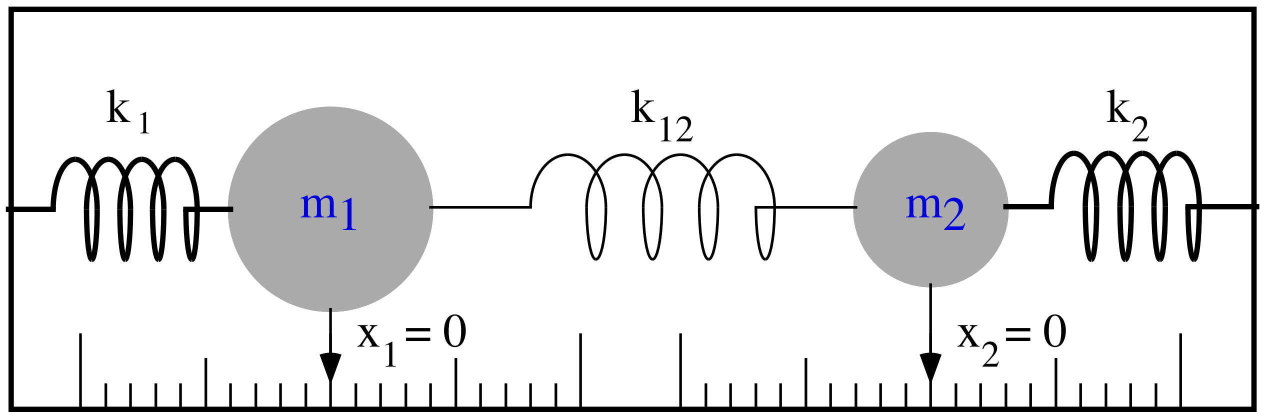
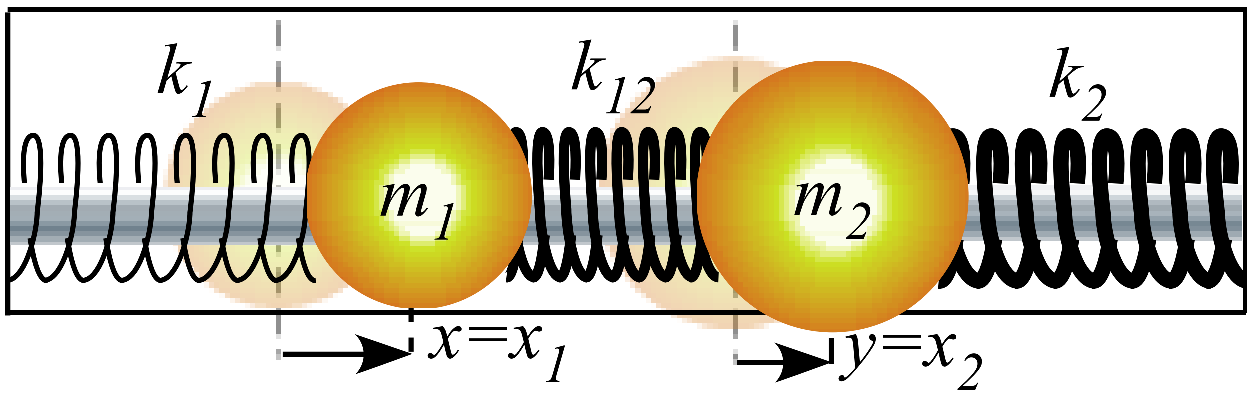


Fig. 3.3.1 Two 1-dimensional coupled oscillators



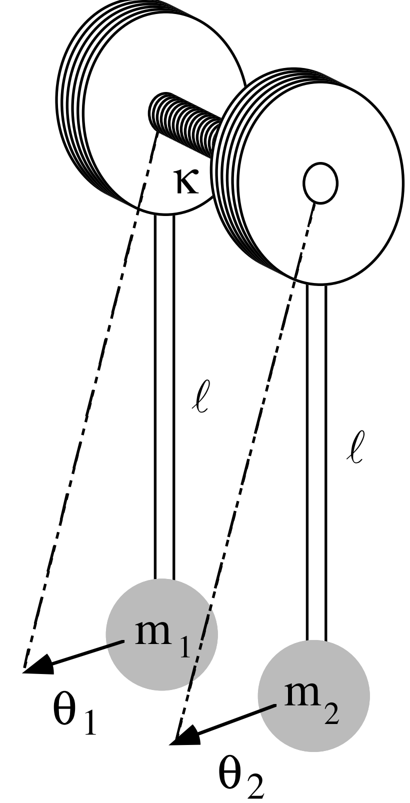


Fig. 3.3.2 Coupled pendulums

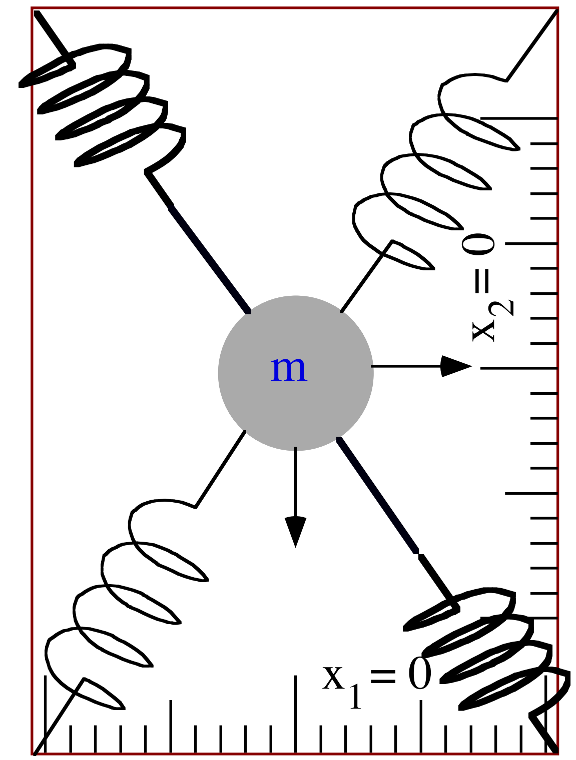


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_{12}\left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2$$

Lagrangian L=T-V

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

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Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

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Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

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2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

2D harmonic oscillator equations

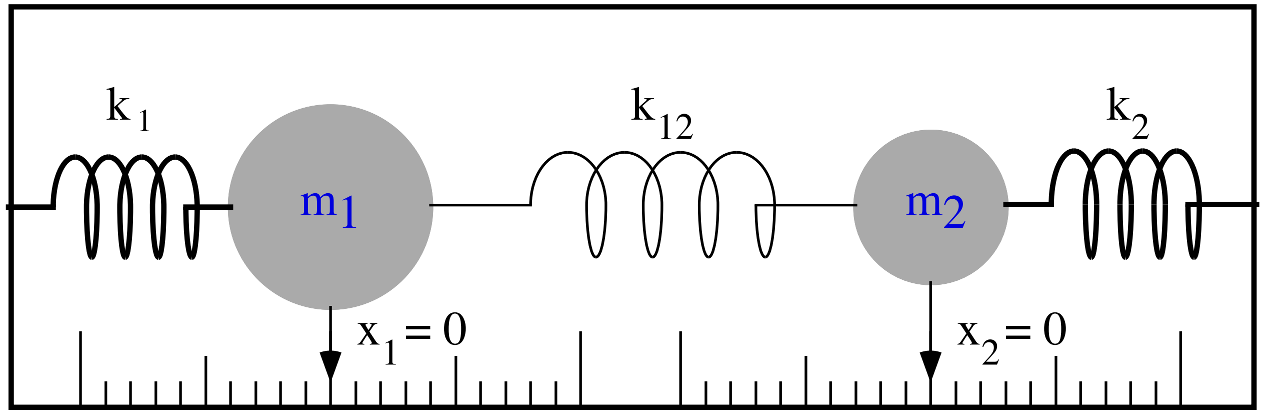
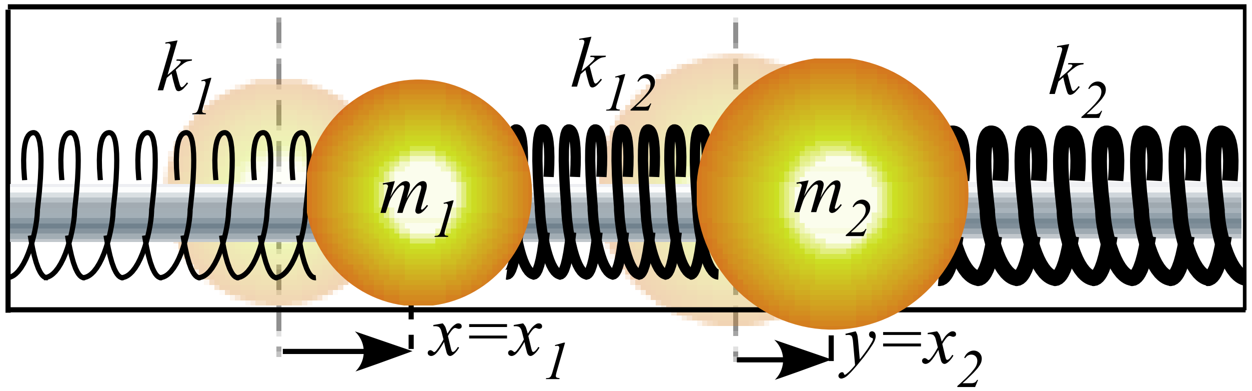


Fig. 3.3.1 Two 1-dimensional coupled oscillators



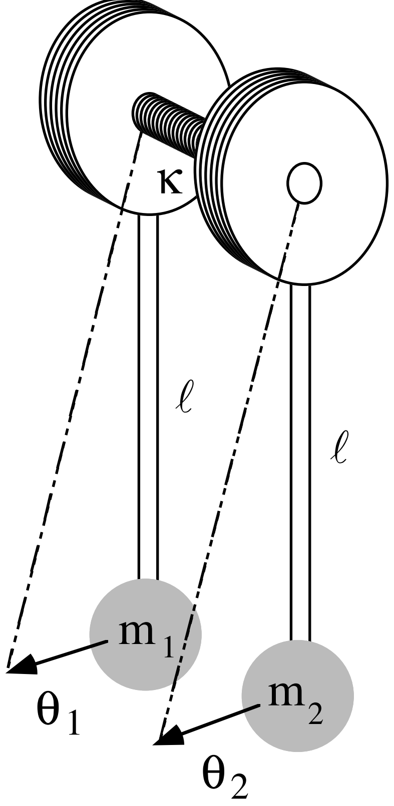


Fig. 3.3.2 Coupled pendulums

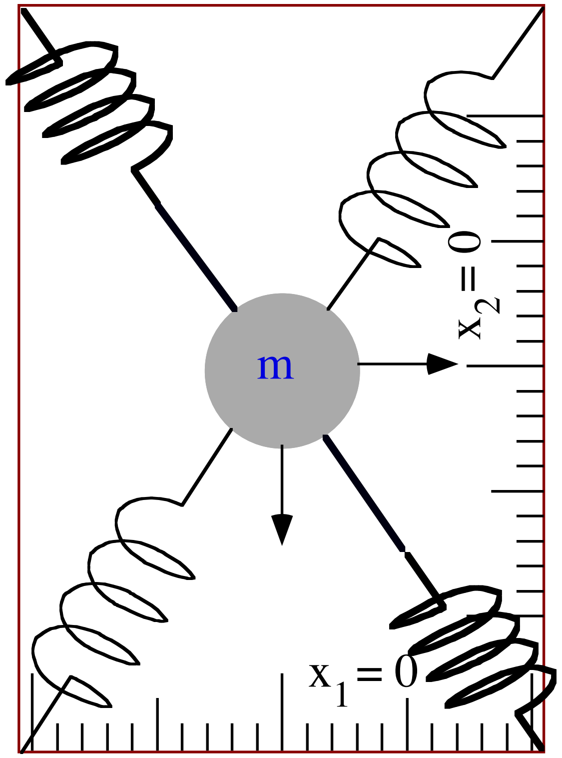


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} \left(x_1 - x_2 \right)^2$$

$$= \frac{1}{2} \left(k_1 + k_{12} \right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} \left(k_2 + k_{12} \right) x_2^2$$

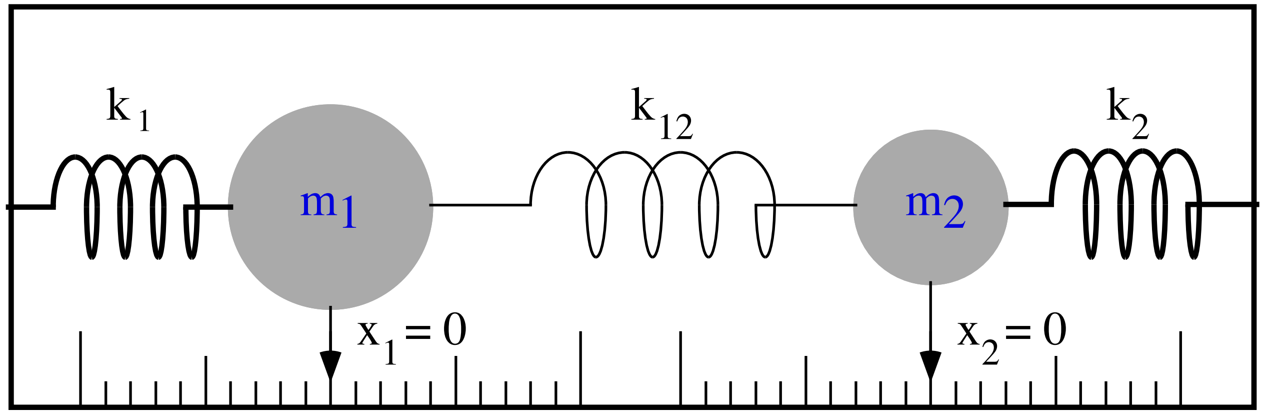
Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12} \right) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12} \right) x_2$$

2D harmonic oscillator equations



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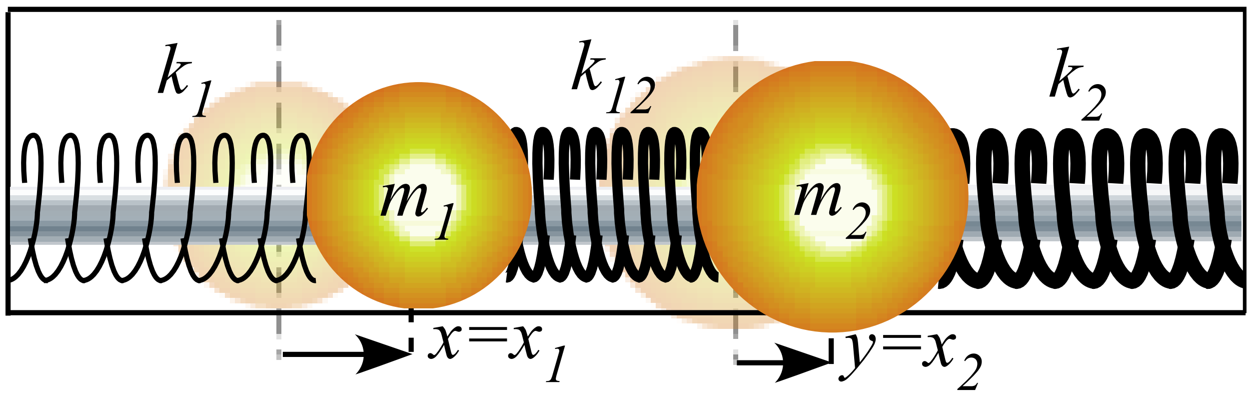


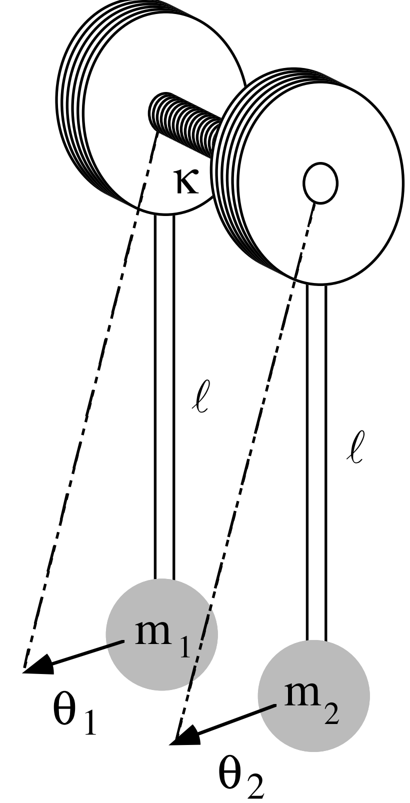
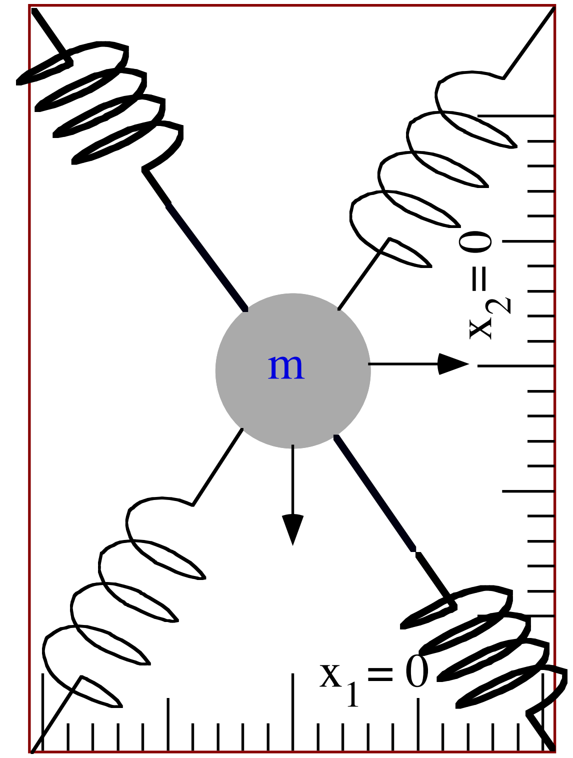


Fig. 3.3.2 Coupled pendulums



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$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_{12}\left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12}\right)x_1 + k_{12}x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - \left(k_2 + k_{12}\right)x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations

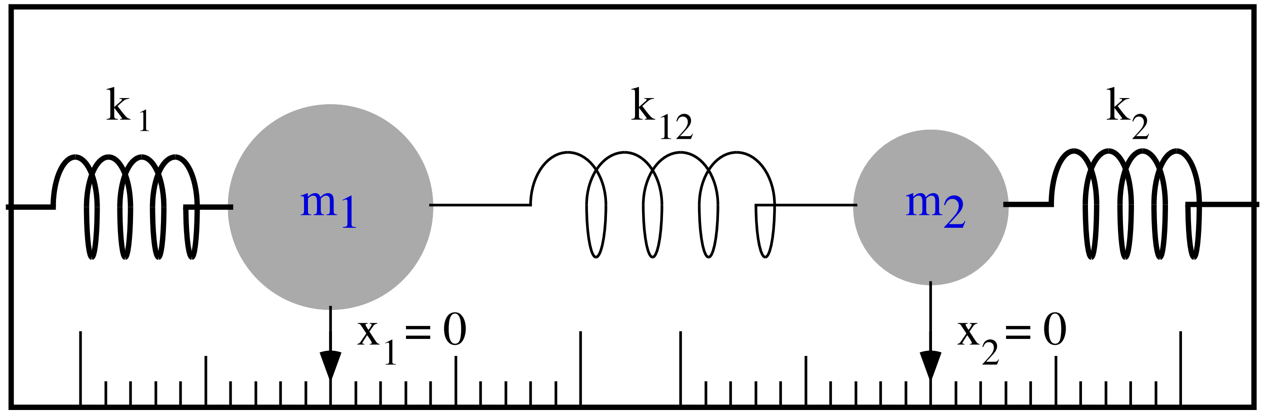
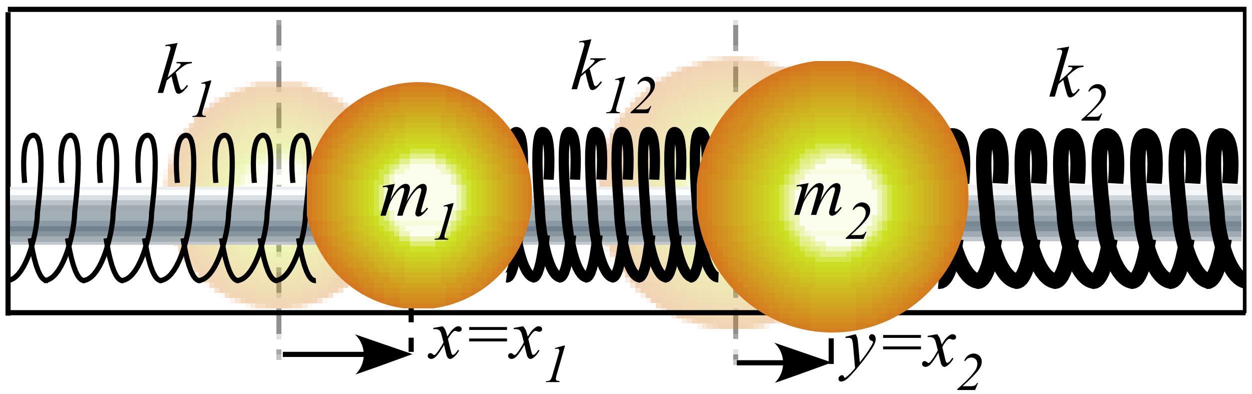


Fig. 3.3.1 Two 1-dimensional coupled oscillators



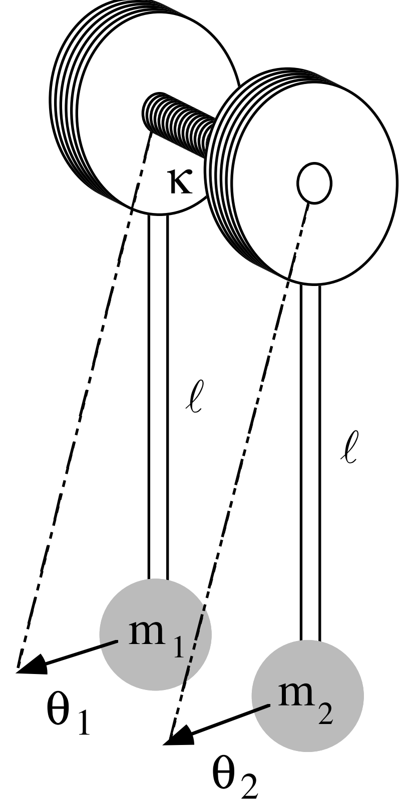


Fig. 3.3.2 Coupled pendulums

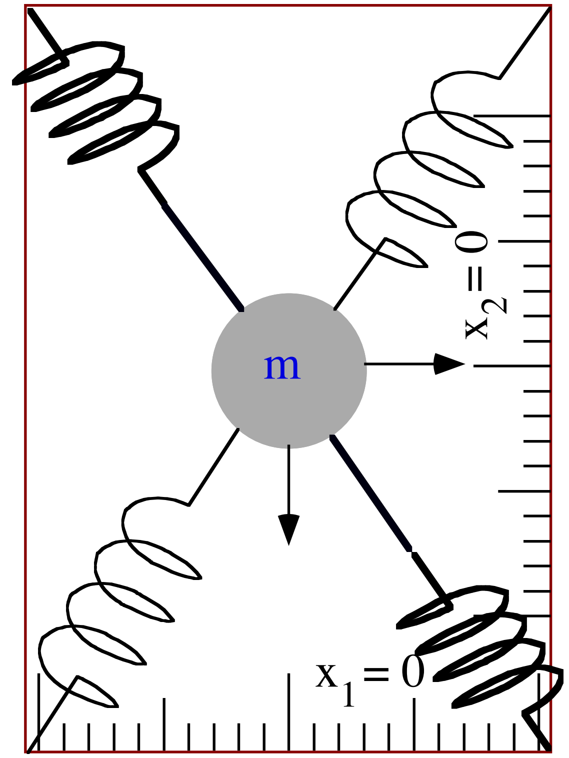


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} \left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}\left(k_2 + k_{12}\right) x_2^2$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12}\right) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12}\right) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \bullet \left|\ddot{\mathbf{x}}\right\rangle = - \mathbf{K} \bullet \left|\mathbf{x}\right\rangle$$

2D harmonic oscillator equations

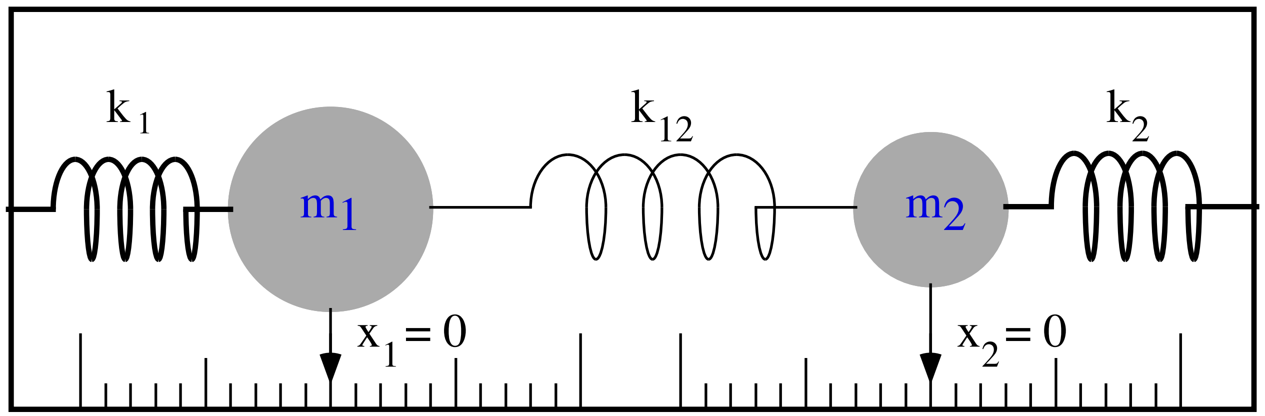
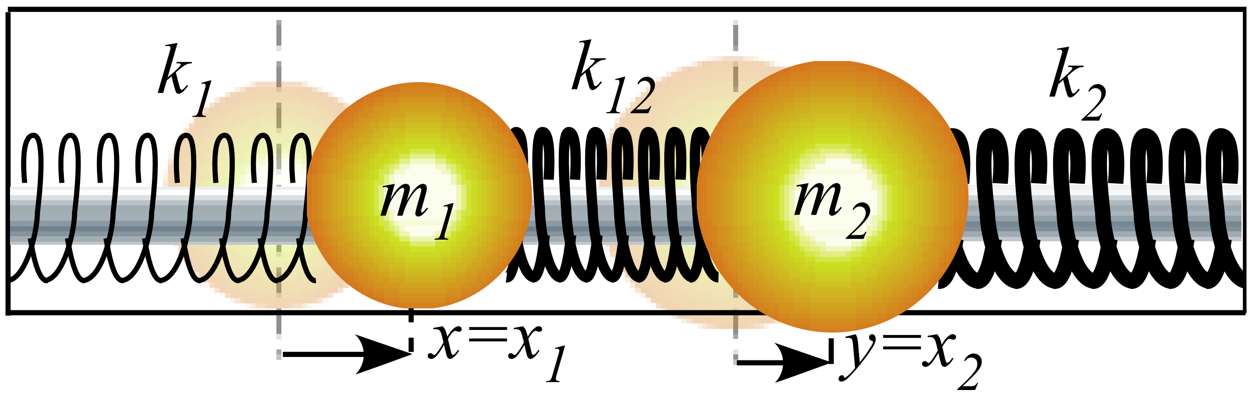


Fig. 3.3.1 Two 1-dimensional coupled oscillators



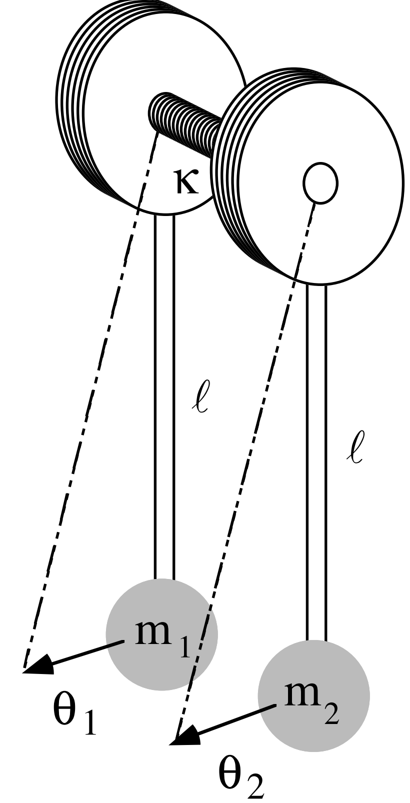


Fig. 3.3.2 Coupled pendulums

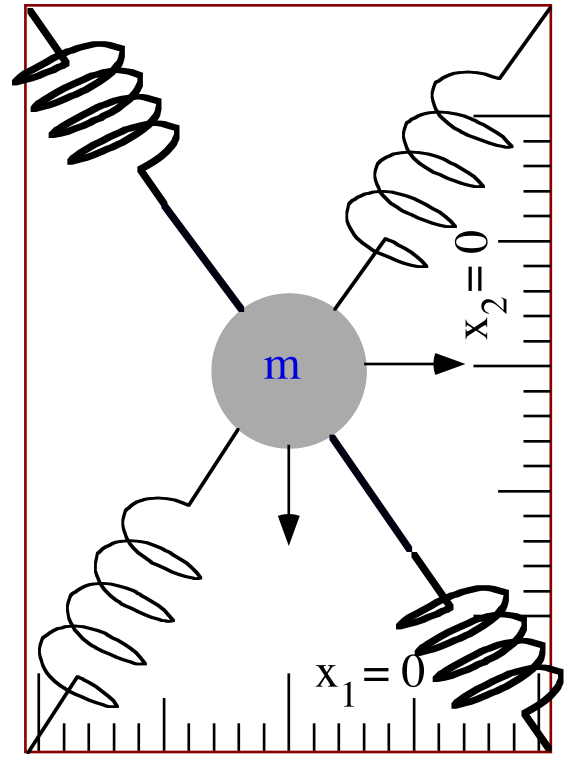


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}(k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \bullet | \ddot{\mathbf{x}} \rangle \quad = \quad - \quad \mathbf{K} \bullet | \mathbf{x} \rangle$$

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Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

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Matrix equations and reciprocity symmetry

General form of 2D-HO equation of motion has force matrix components: $\kappa_{11} = k_1 + k_{11}, \quad \kappa_{22} = k_2 + k_{22}$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Off-diagonal force constants satisfy *Reciprocity Relations:* $-\kappa_{12} = k_{12} = \frac{\partial F_1}{\partial x_2} = -\frac{\partial^2 V}{\partial x_2 \partial x_1} = -\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial F_2}{\partial x_1} = k_{21} = -\kappa_{21}$

Rescaling and symmetrization

Each coordinate (x_1, x_2) is rescaled $(q_1 = s_1 x_1, q_2 = s_2 x_2)$ to symmetrize mass factors on $\ddot{q}_j$-terms.

$$-\frac{m_1}{s_1}\ddot{q}_1 = \kappa_{11}\frac{q_1}{s_1} + \kappa_{12}\frac{q_2}{s_2} \qquad -\ddot{q}_1 = \frac{\kappa_{11}}{m_1}q_1 + \frac{\kappa_{12}s_1}{m_1 s_2}q_2 \equiv \mathrm{K}_{11}q_1 + \mathrm{K}_{12}q_2$$

$$-\frac{m_2}{s_2}\ddot{q}_2 = \kappa_{12}\frac{q_1}{s_1} + \kappa_{22}\frac{q_2}{s_2} \qquad -\ddot{q}_2 = \frac{\kappa_{12}s_2}{m_2 s_1}q_1 + \frac{\kappa_{22}}{m_2}q_2 \equiv \mathrm{K}_{21}q_1 + \mathrm{K}_{22}q_2$$

New constants K_{ij} have pseudo-reciprocity symmetry for a special scale factor ratio: $\frac{s_2}{s_1} = \sqrt{\frac{m_2}{m_1}}$

$$\mathrm{K}_{21} = \frac{\kappa_{12}s_2}{m_2 s_1} = \mathrm{K}_{12} = \frac{\kappa_{12}s_1}{m_1 s_2} = \frac{-k_{12}}{\sqrt{m_1 m_2}}$$

Diagonal constants K_{jj} are not affected by scaling. To be equal requires: $\frac{\kappa_{11}}{m_1} = \frac{\kappa_{22}}{m_2}$ or: $\frac{\kappa_{11}}{\kappa_{22}} = \frac{m_1}{m_2}$

$$\mathrm{K}_{11} = \frac{\kappa_{11}}{m_1} = \frac{k_1 + k_{12}}{m_1} \qquad \mathrm{K}_{22} = \frac{\kappa_{22}}{m_2} = \frac{k_2 + k_{12}}{m_2}$$

Caution is advised since such forced symmetry may give modes with imaginary frequency.

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2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \dfrac{k_1 + k_{12}}{m_1} & \dfrac{-k_{12}}{m_1} \\ \dfrac{-k_{12}}{m_2} & \dfrac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

To introduce eigensolutions we take a simple case of unit masses *($m_1=1=m_2$)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

$\boxed{\text{Eigenvectors } |\mathbf{x}\rangle = |\mathbf{e}_n\rangle \text{ are in special directions where } |\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle \text{ is in same direction as } |\mathbf{x}\rangle}$

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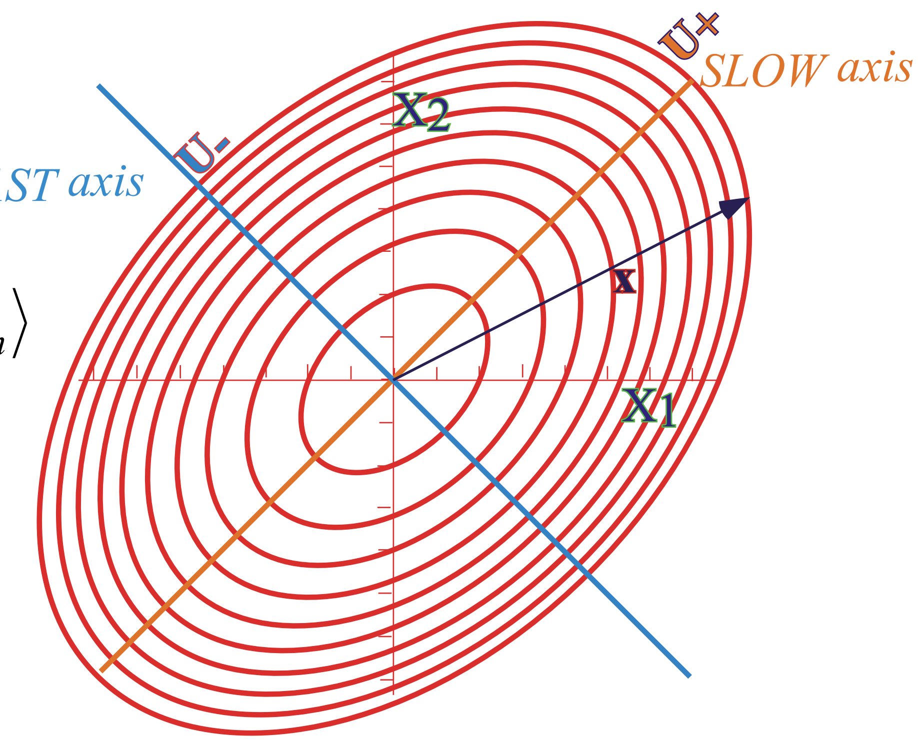
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(if any)

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??



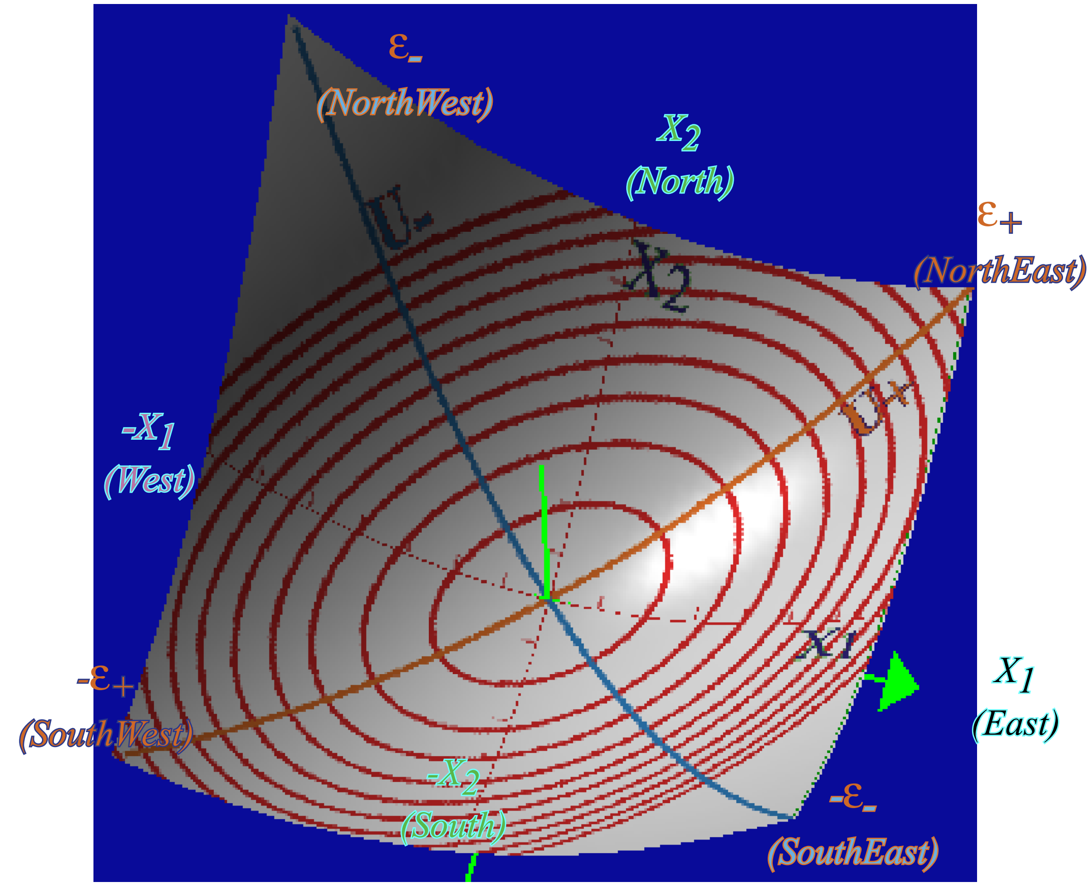
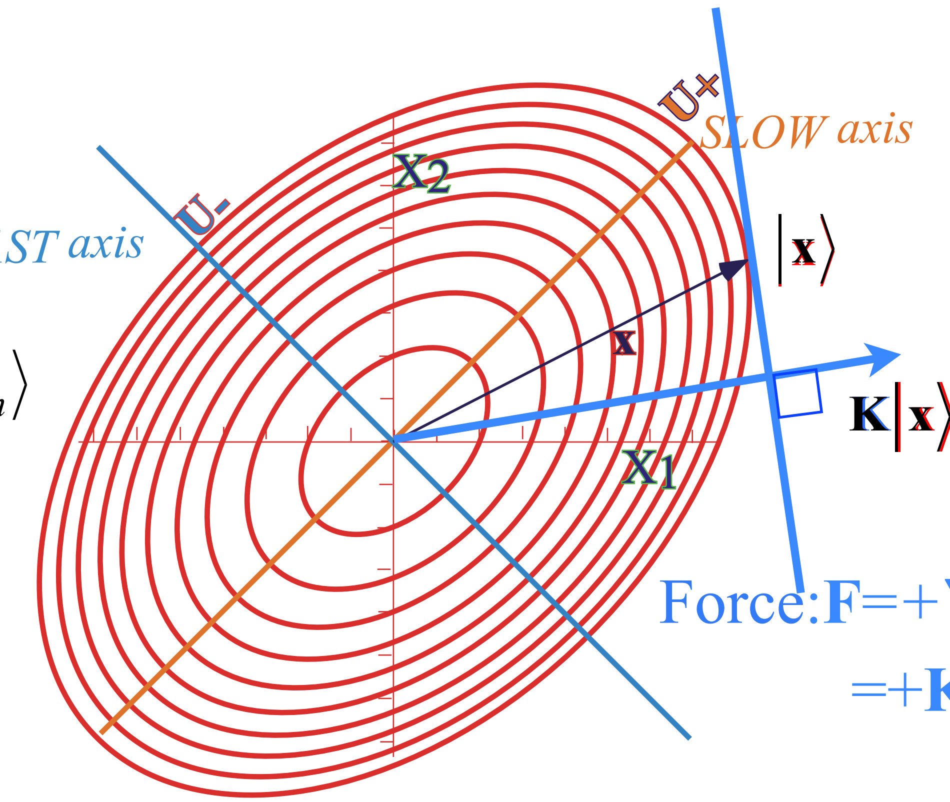


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours



What *(if any)* direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

Not <u>most</u> directions!

Force: $\mathbf{F} = +\nabla V$

$= +\mathbf{K}\bullet\mathbf{x}$

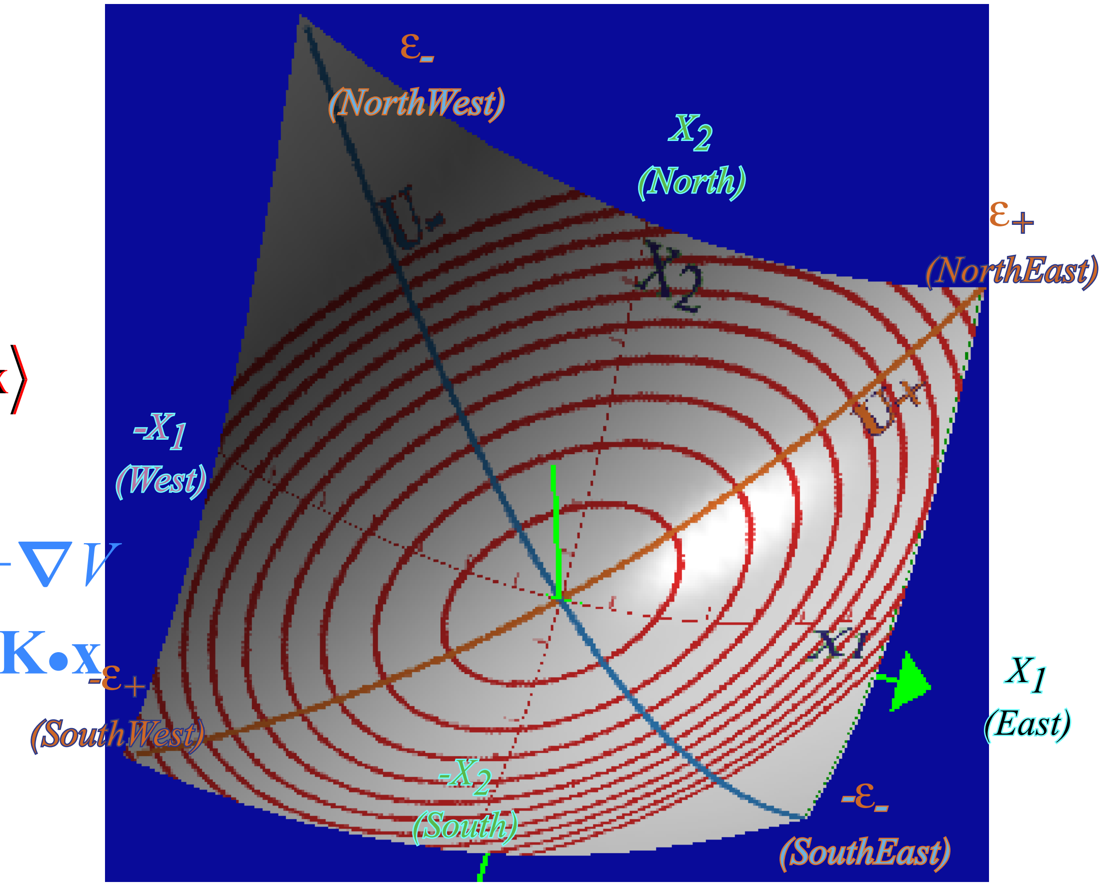


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

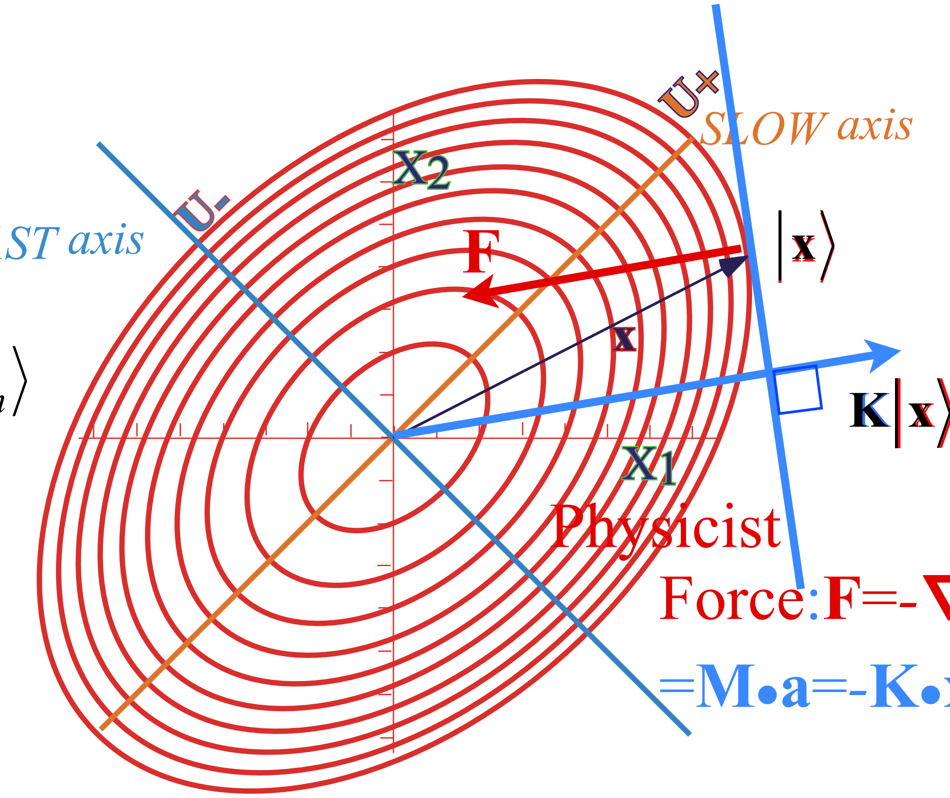
$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours

(if any)

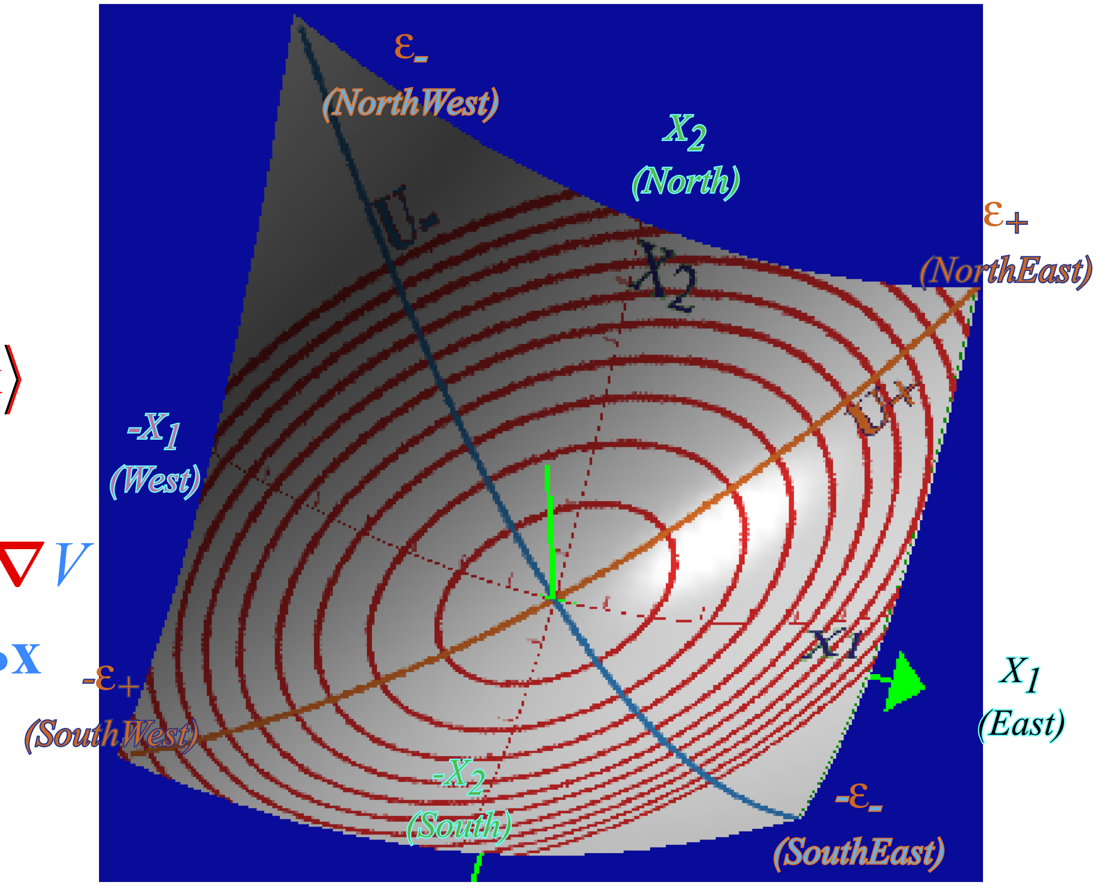
What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

Not <u>most</u> directions!



Force: $\mathbf{F} = -\nabla V$

$= \mathbf{M}\bullet\mathbf{a} = -\mathbf{K}\bullet\mathbf{x}$



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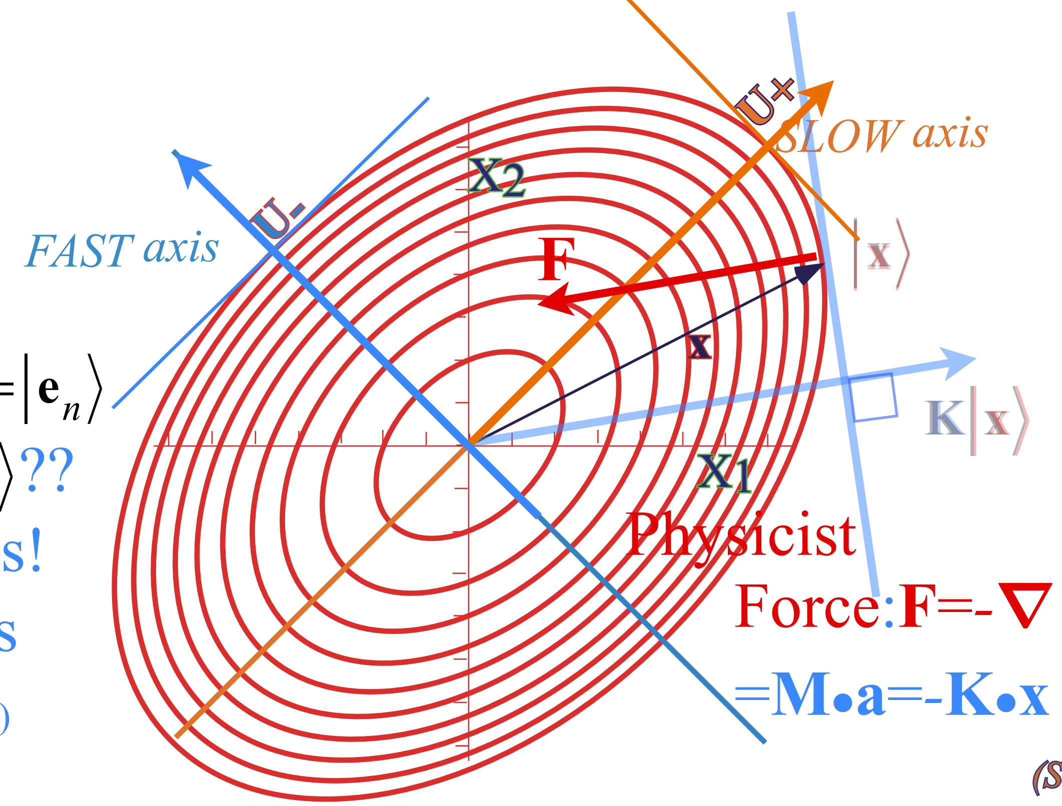
$$V=\frac{1}{2}(k_1+k_{12})x_1^2-k_{12}x_1x_2+\frac{1}{2}(k_2+k_{12})x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours

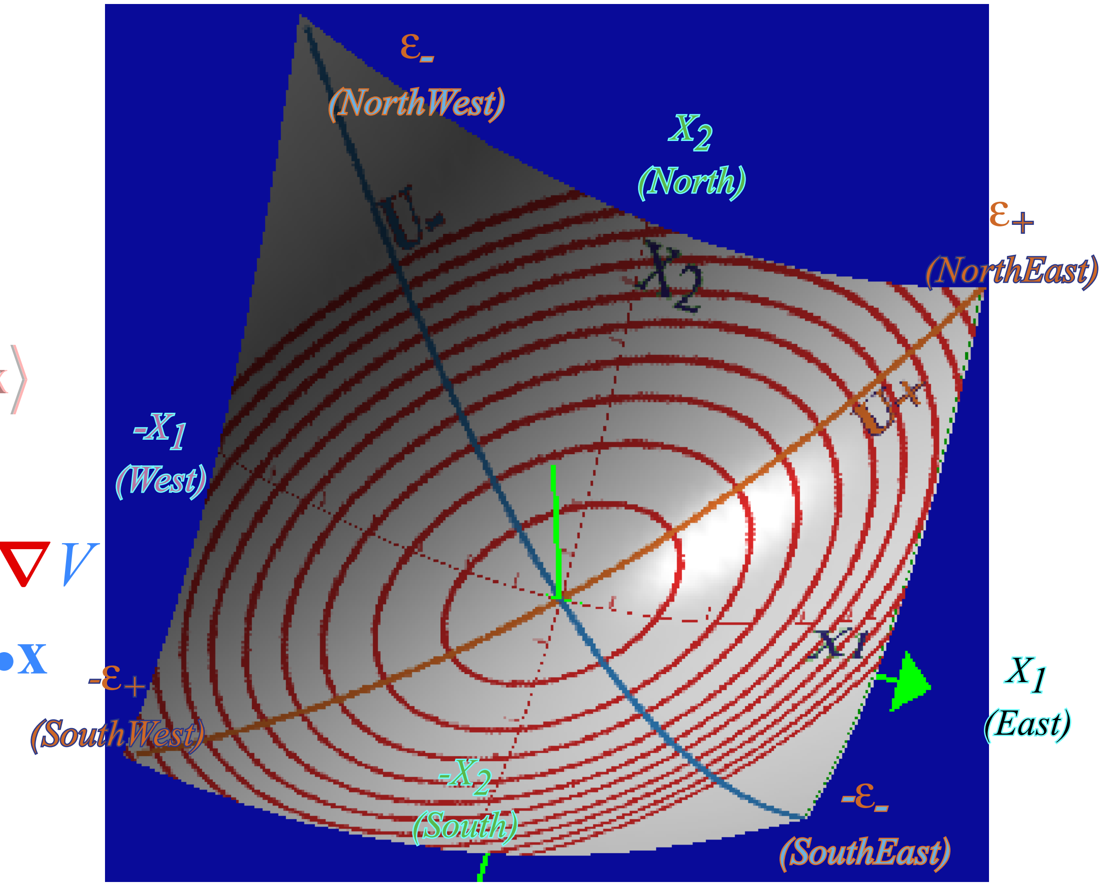


(if any)

What direction $|\mathbf{x}\rangle=|\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

Not most directions!

Only extremal axes work. (major or minor axes)



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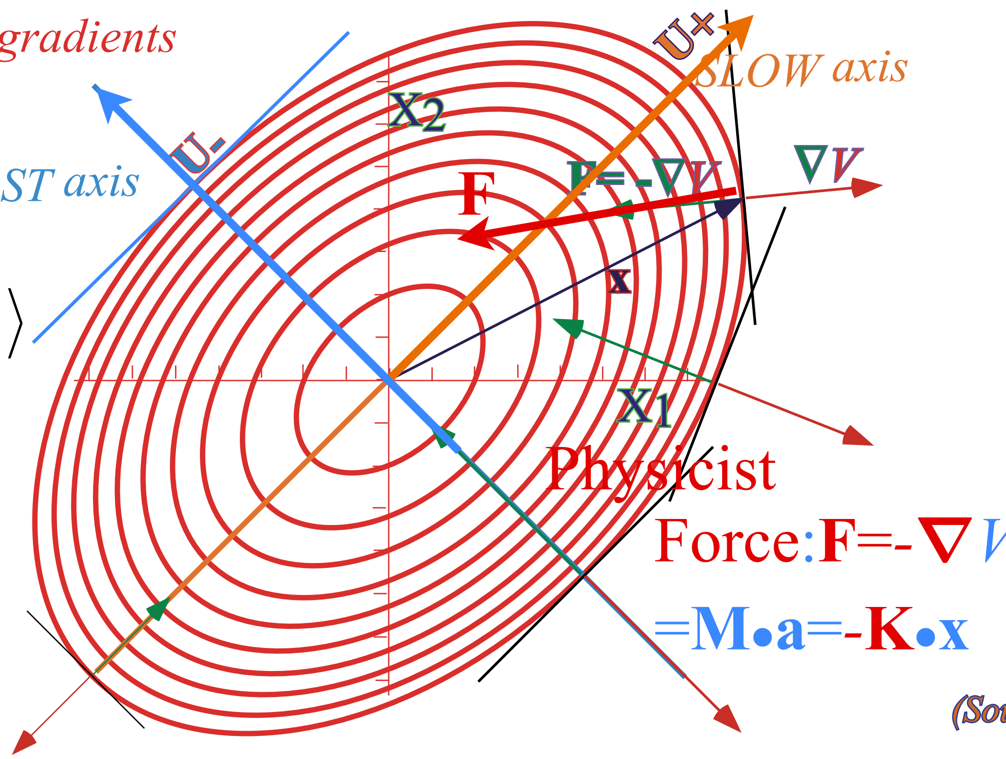
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k+k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ *(if any)*
is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not <u>most</u> directions!
Only extremal axes
work. (major or minor axes)

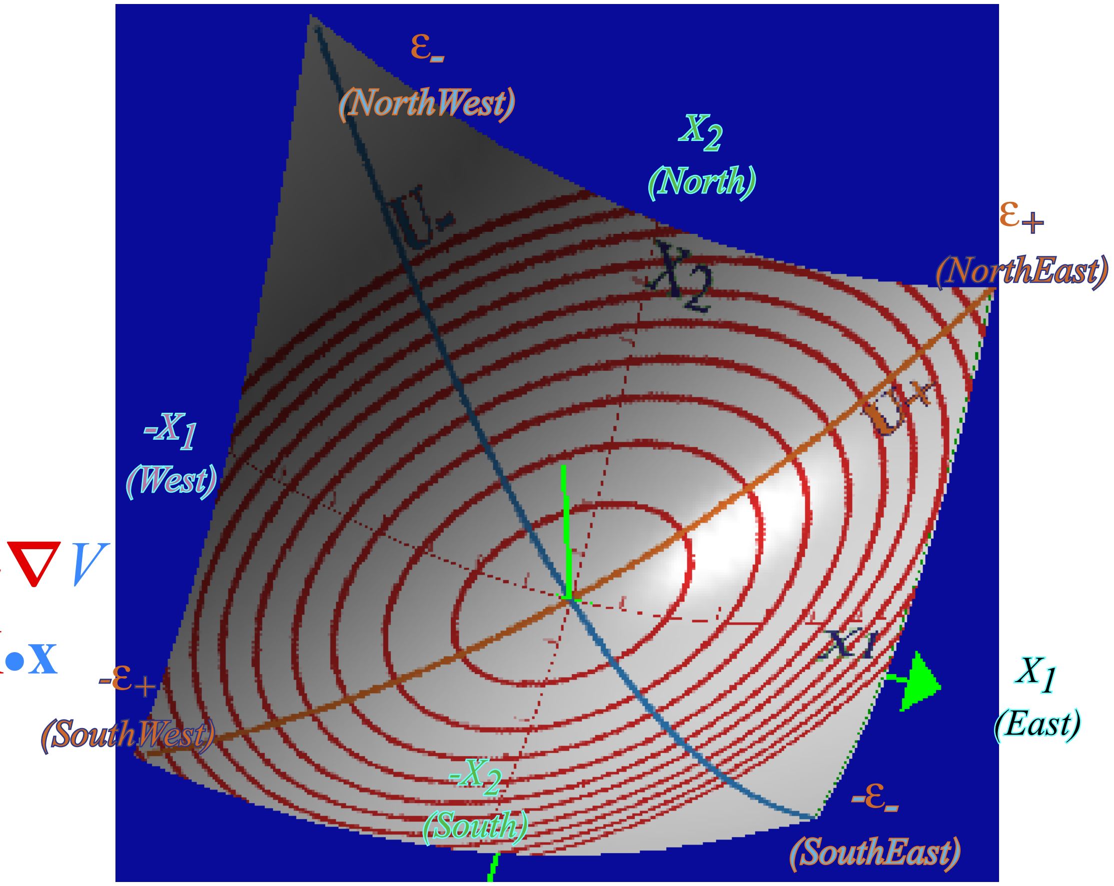
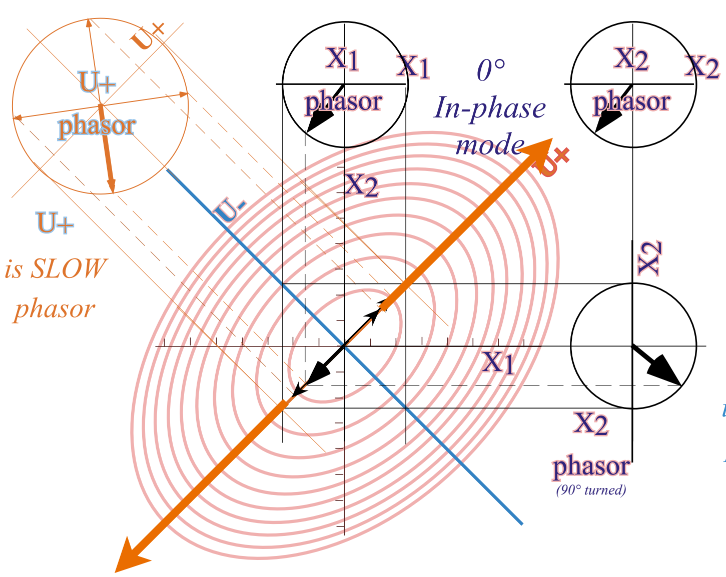
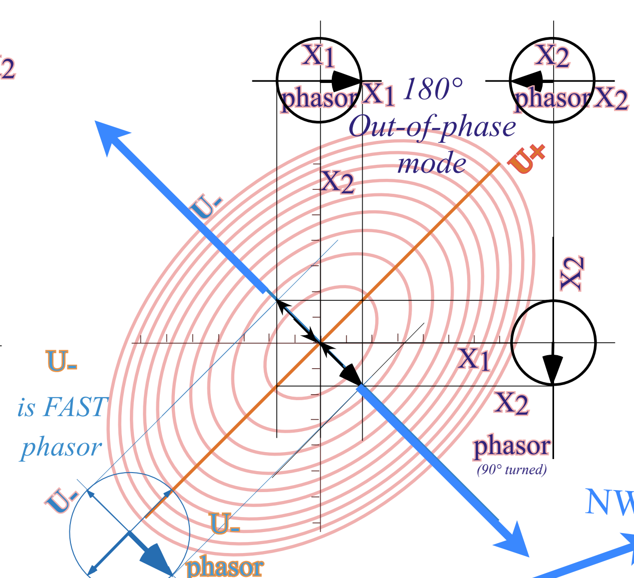


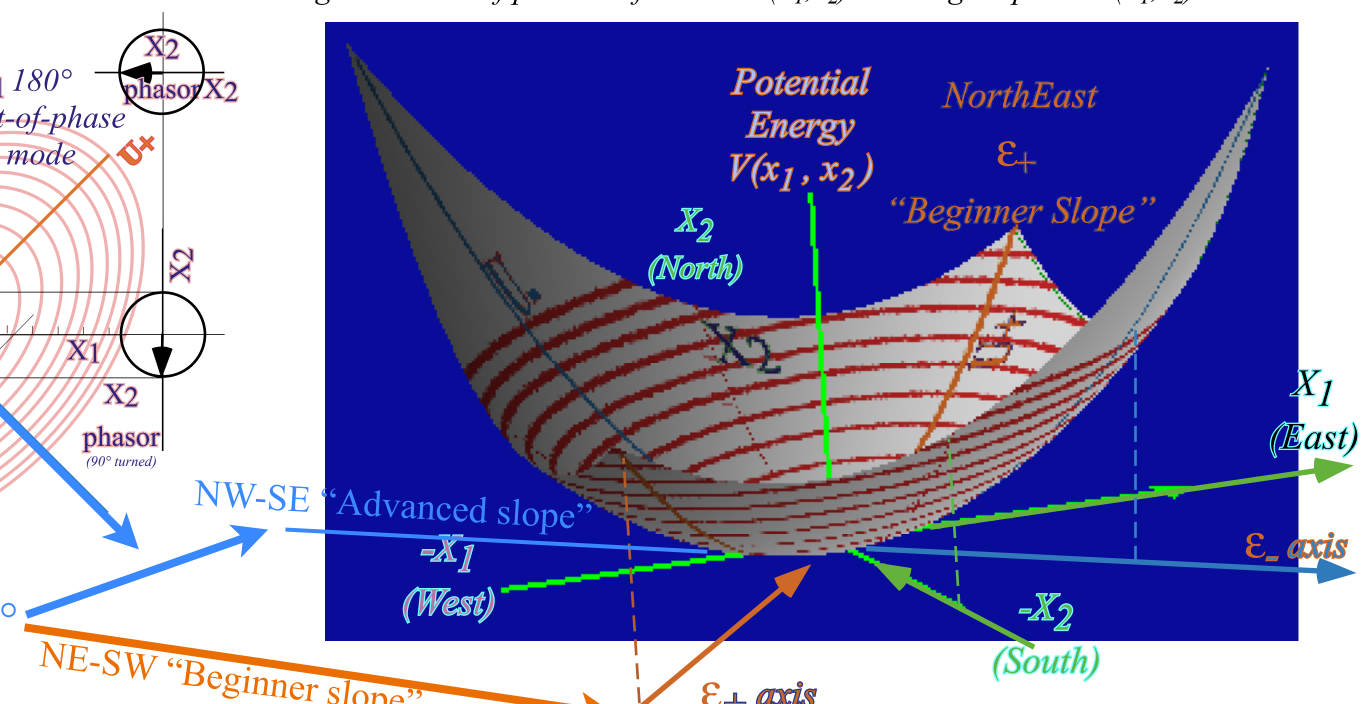
Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate
SLOW Mode



(c) Anti-symmetric U- Coordinate
FAST Mode





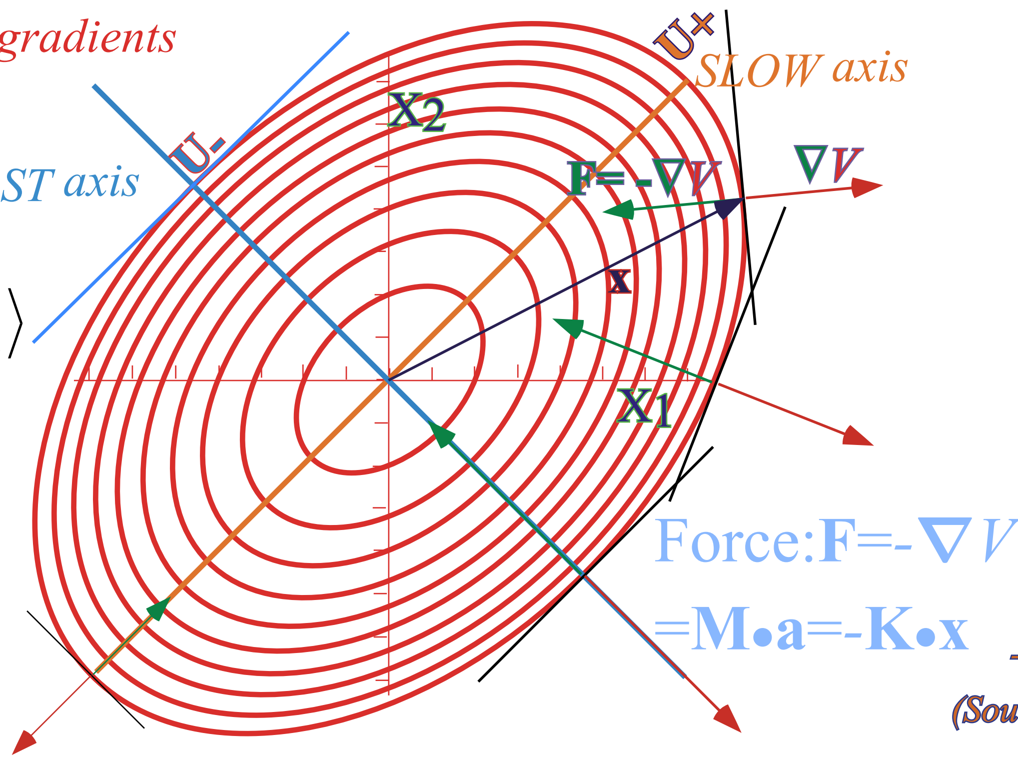
With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V=\frac{1}{2}(k+k_{12})x_1^2-k_{12}x_1x_2+\frac{1}{2}(k+k_{12})x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ *(if any)*
is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!
Only extremal axes work. (major or minor axes)

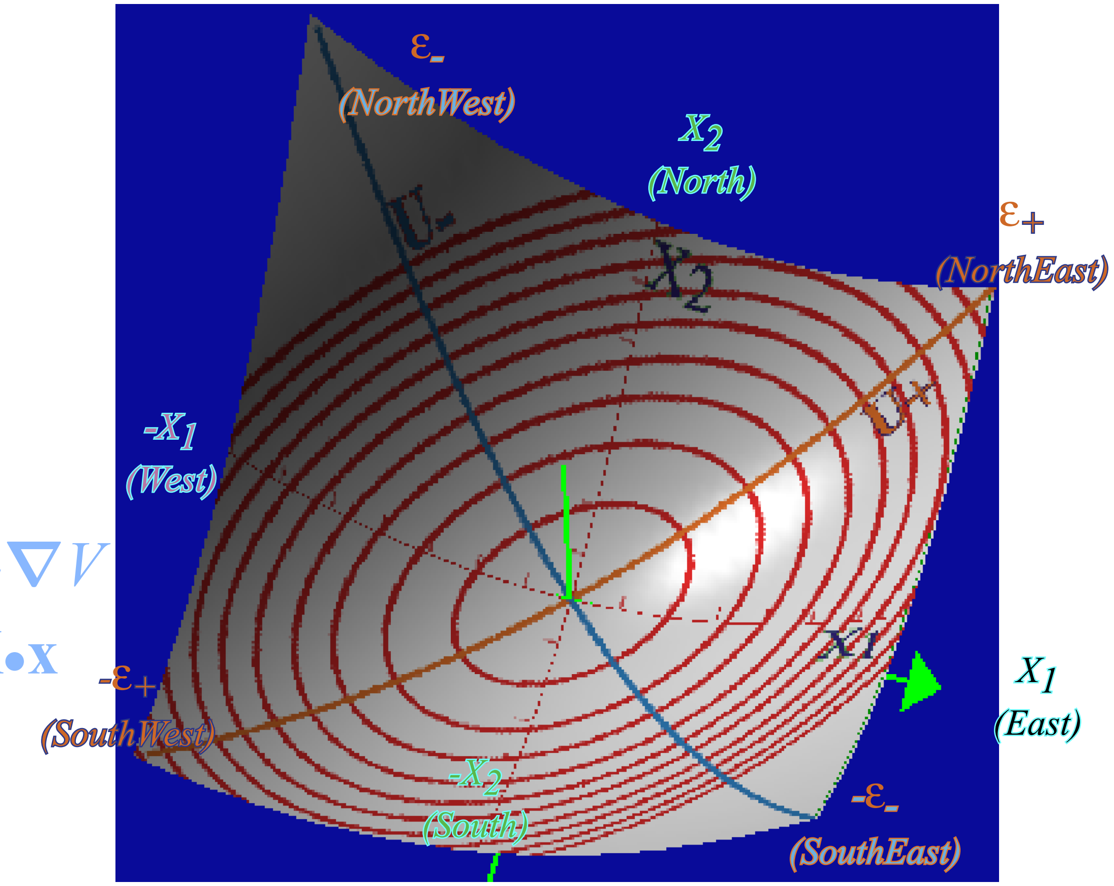
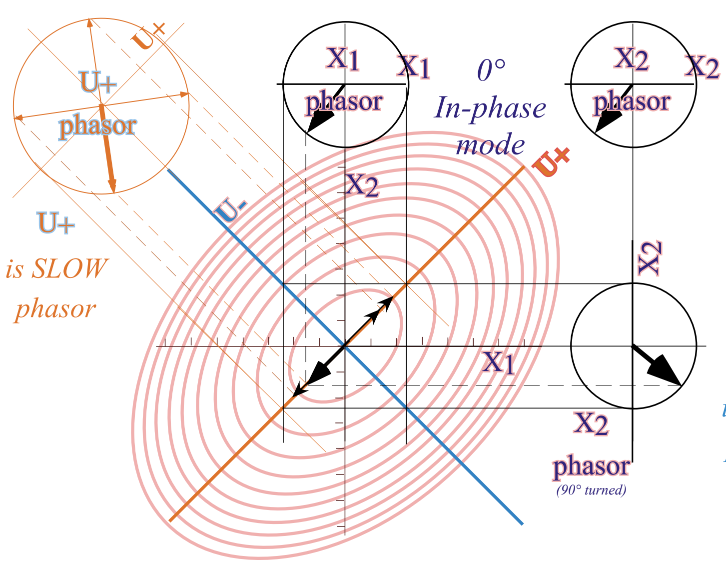
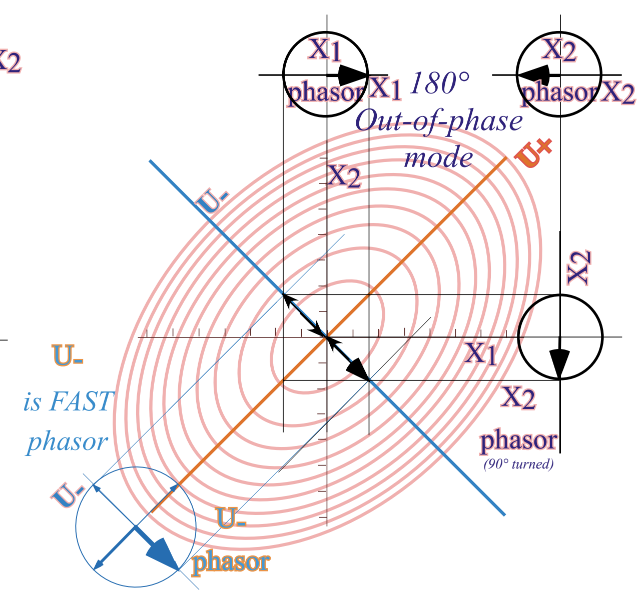


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode



BoxIt (Beating) Web Simulation (A=1, B=-0.1, C=0, D=1) *with Comparison Cosine wave ($T=2\pi$)*

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An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle = \varepsilon_k|\varepsilon_k\rangle, \text{ or: } (\mathbf{M} - \varepsilon_k\mathbf{1})|\varepsilon_k\rangle = \mathbf{0}$$

$$\mathbf{M}|\varepsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle, \cdots |\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

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Trying to solve by Kramer's inversion: (A fool's errand!)

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

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where:

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}-\varepsilon\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon & 1\\ 3 & 2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})$$

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}| = \det\left|\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \varepsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|$$

$$0 = (4-\varepsilon)(2-\varepsilon) - 1\cdot 3 = 8 - 6\varepsilon + \varepsilon^2 - 1\cdot 3 = \varepsilon^2 - 6\varepsilon + 5$$

$$0 = \varepsilon^2 - Trace(\mathbf{M})\varepsilon + \det(\mathbf{M}) = \varepsilon^2 - 6\varepsilon + 5$$

$$0 = (\varepsilon - 1)(\varepsilon - 5) \text{ so let: } \varepsilon_1 = 1 \text{ and: } \varepsilon_2 = 5$$

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Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

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$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

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Obviously true if $\mathbf{M}$ has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by $(\mathbf{1})$ to make *projection operators* .

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

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$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

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A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

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Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

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$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or:}\quad\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

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Matrix-algebraic method for finding eigenvector and eigenvalues   *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

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Notice $\mathbf{p}_k$ *commutes with* $\mathbf{M}$,..

since $\mathbf{M}^1$, $\mathbf{M}^2$,..*commute with* $\mathbf{M}$.

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With example matrix

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(tricky step)

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

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Eigen-bra-ket projectors of matrix:

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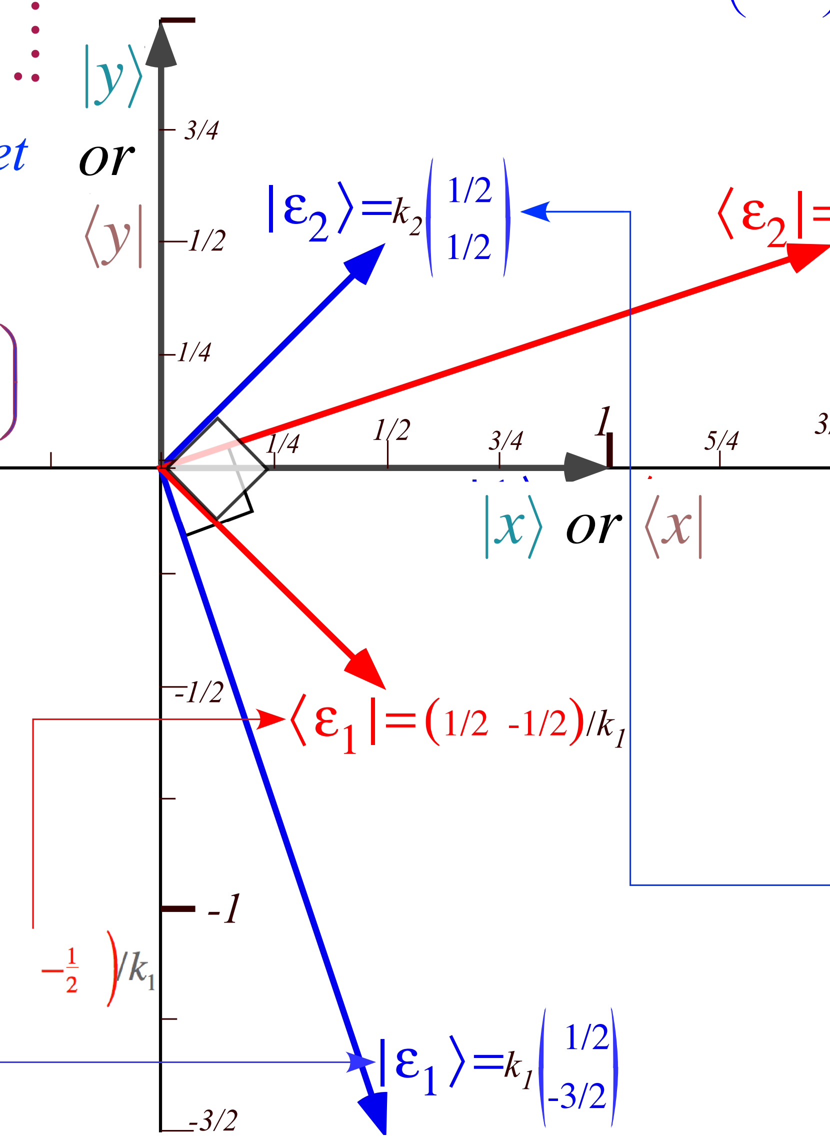

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$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}/k_2=|\varepsilon_2\rangle\langle\varepsilon_2|$$

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$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}\left(\mathbf{M}-\varepsilon_m\mathbf{1}\right) = \prod_{m\neq k}\left(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}\right) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

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With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

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(tricky step)

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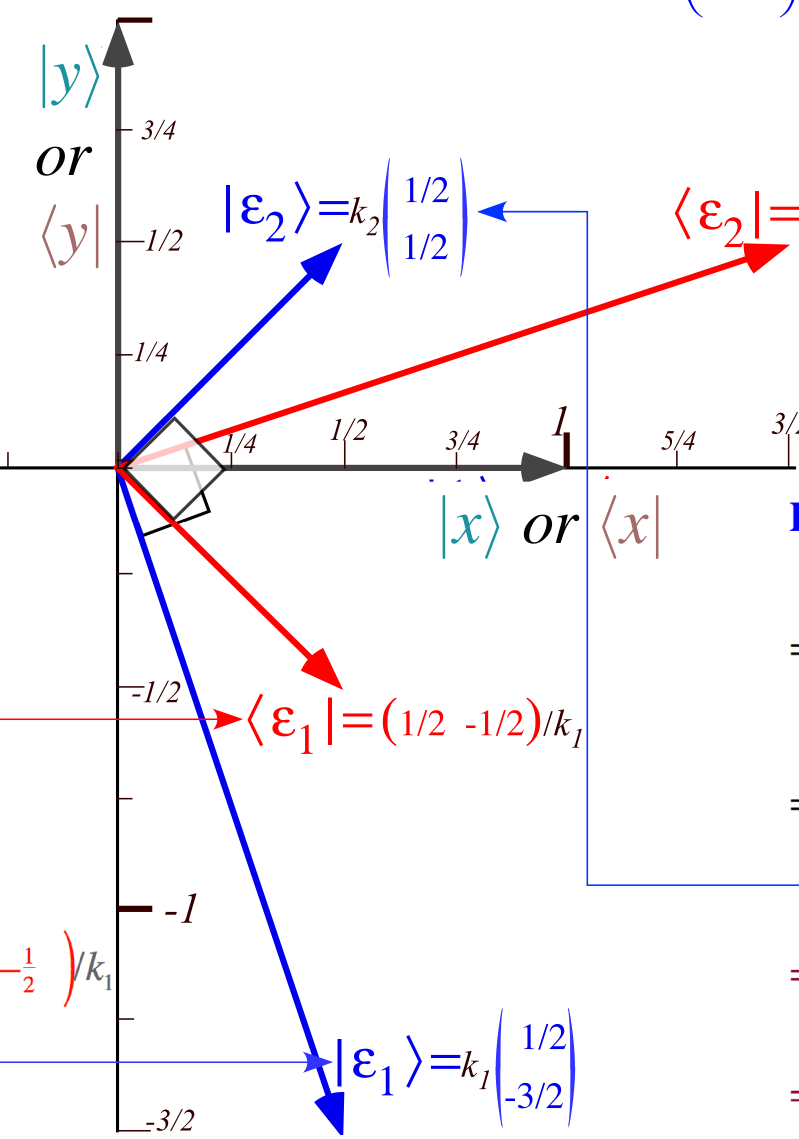

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(tricky step)

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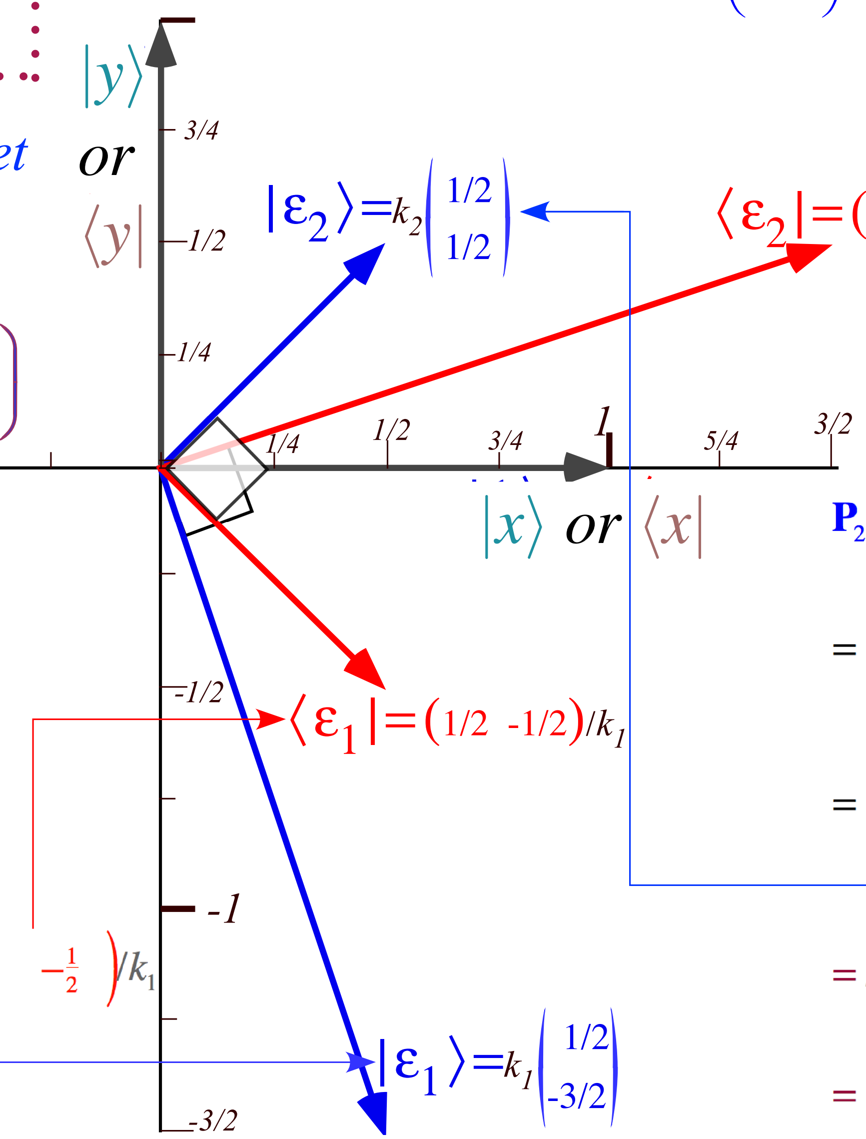

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With example matrix $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

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$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

(*tricky step*)

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}=k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}\otimes\frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

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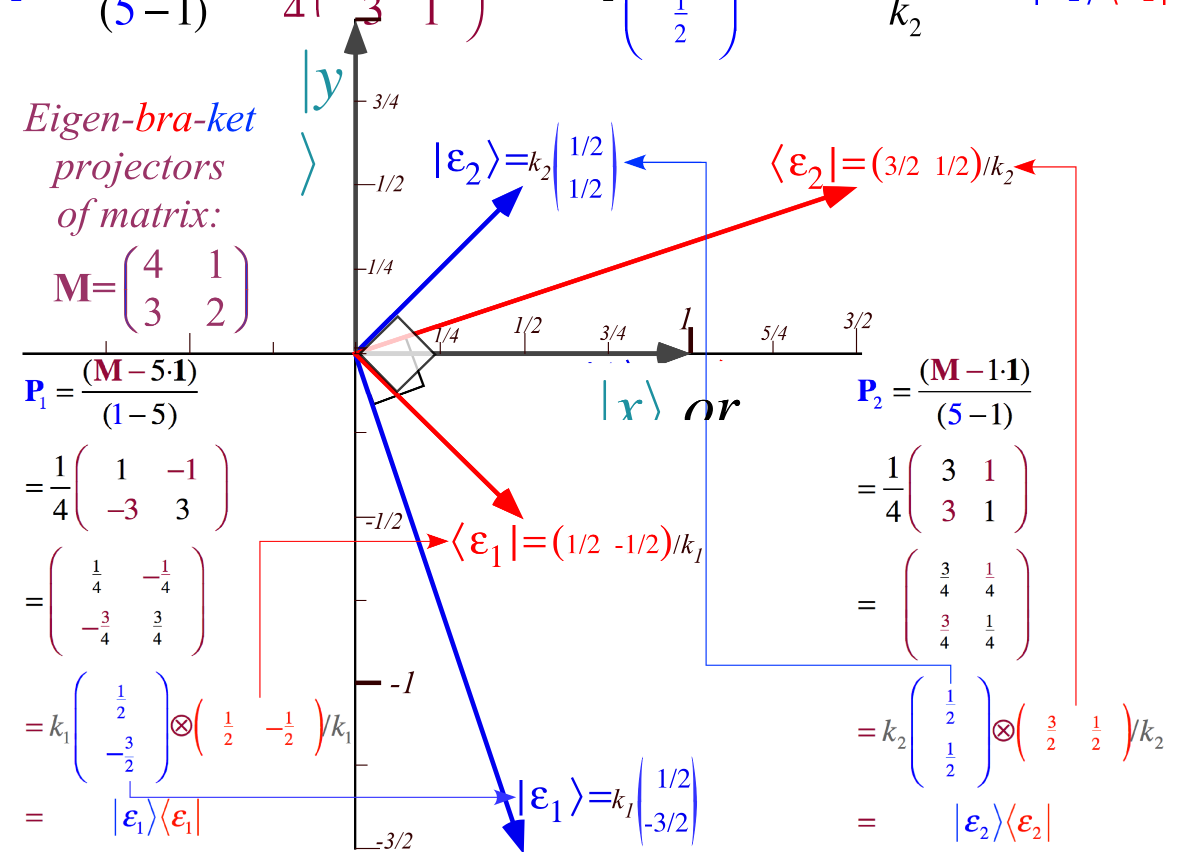

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Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

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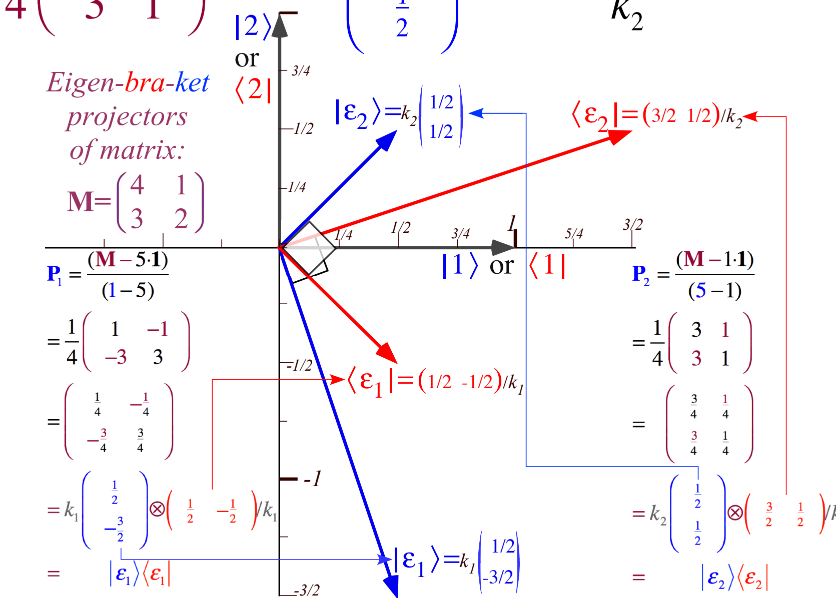

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$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$
$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} + 5\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}\left(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}\right) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}\left(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j\right) = \mathbf{p}_j\prod_{m\neq k}\left(\varepsilon_j-\varepsilon_m\right) = \begin{cases}\mathbf{0} & \text{if}: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & \text{if}: j=k\end{cases}$$

Last step:
make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$
(Idempotent means: **P·P=P)**

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & \text{if}: j\neq k\\ \mathbf{P}_k & \text{if}: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$
implies :
$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

(tricky step)
Factoring bra-kets into “Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=\left|\varepsilon_1\right\rangle\left\langle\varepsilon_1\right|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=\left|\varepsilon_2\right\rangle\left\langle\varepsilon_2\right|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
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$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

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$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle1|+5|2\rangle\langle2|=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Example:

$$\mathbf{M}^{50}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1^{50}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5^{50}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1+3\cdot5^{50}&5^{50}-1\\3\cdot5^{50}-3&5^{50}+3\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

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$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$

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$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

(tricky step)

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} + 5\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

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$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Examples:

$$\mathbf{M}^{50} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = 1^{50}\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} + 5^{50}\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1+3\cdot 5^{50} & 5^{50}-1\\ 3\cdot 5^{50}-3 & 5^{50}+3\end{pmatrix}$$

$$\sqrt{\mathbf{M}} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = \pm\sqrt{1}\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} \pm\sqrt{5}\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

➤ *Orthonormality vs. Completeness vis-a`-vis Operator vs. State* ➤

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

Orthonormality vs. Completeness

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

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(tricky step)

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$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

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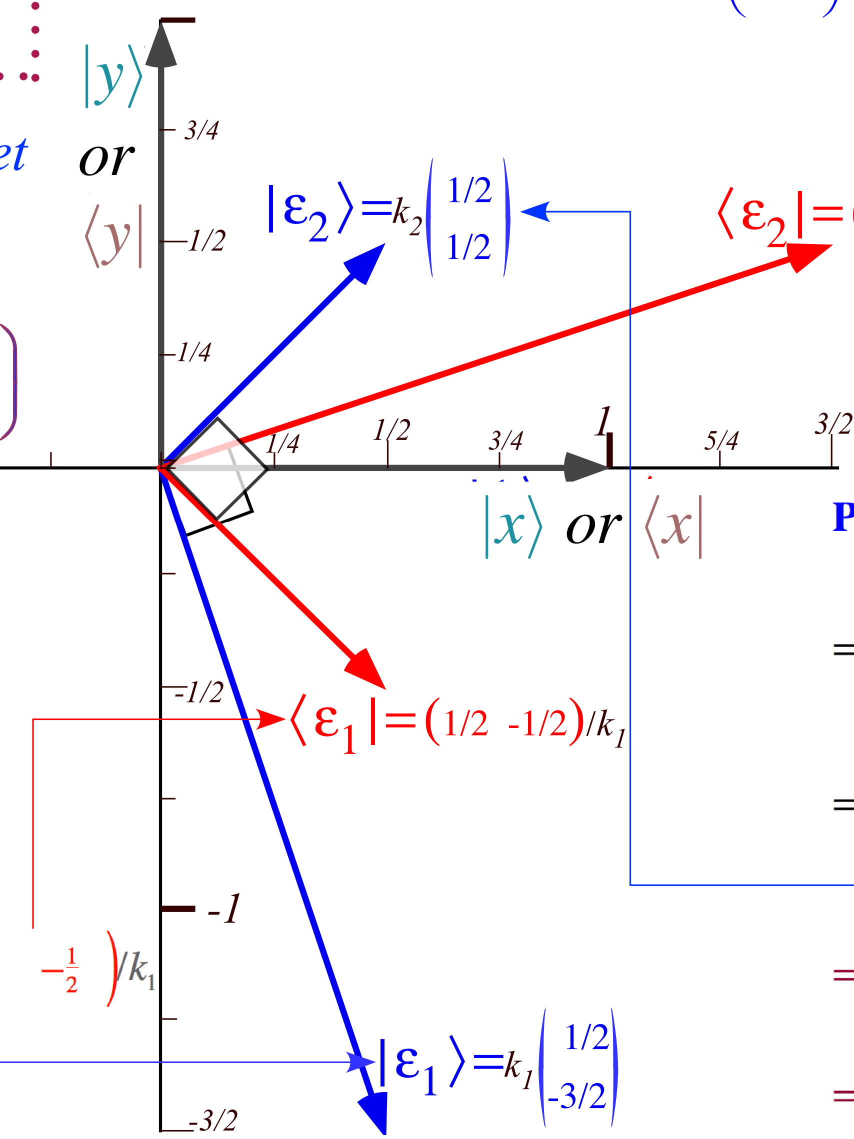
Eigen-bra-ket projectors of matrix:

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$



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$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}/k_1 = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}/k_2 = |\varepsilon_2\rangle\langle\varepsilon_2|$$

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

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$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

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State vector representations of orthonormality are quite ***similar*** *to representations of completeness. Like 2-sides of the same coin.*

$\{|x\rangle, |y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

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$$\langle x|y\rangle = \delta(x,y) = \qquad \psi_1(x)\psi_1^*(y) + \psi_2(x)\psi_2^*(y) + ..$$

Dirac δ-function

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle, |y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

However Schrodinger wavefunction notation $\psi(x) = \langle x|\psi\rangle$ *shows quite a* ***difference****...*

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite ***different*** *from expressions for completeness.*

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$$\langle x|y\rangle = \delta(x,y) = \qquad \psi_1(x)\psi_1^*(y) + \psi_2(x)\psi_2^*(y) + ..$$

Dirac δ-function

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle, |y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \qquad \ldots + \psi_i^*(x)\psi_j(x) + \psi_2(y)\psi_2^*(y) + \ldots \rightarrow \int dx\, \psi_i^*(x)\psi_j(x)$$

However Schrodinger wavefunction notation $\boldsymbol{\psi}(x) = \langle x|\boldsymbol{\psi}\rangle$ *shows quite a* ***difference****...*

...particularly in the orthonormality integral.

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

A Proof of Projector Completeness *(Truer-than-true by Lagrange interpolation)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$$f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

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$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)\cdot P_k(x) \quad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

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$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n = \sum_{\varepsilon_k} \mathbf{P}_k = \sum_{\varepsilon_k} \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} \qquad f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n = \sum_{\varepsilon_k} f(\varepsilon_k)\mathbf{P}_k = \sum_{\varepsilon_k} f(\varepsilon_k) \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)}$$

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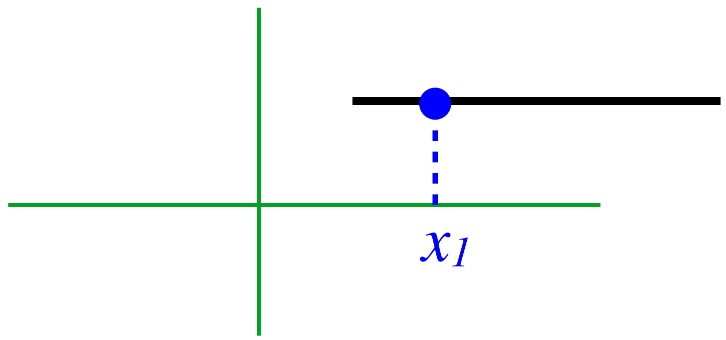

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Compare matrix *completeness relation* and *functional spectral decompositions*

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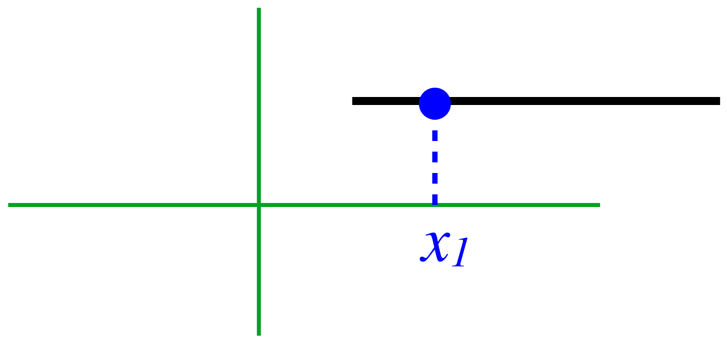

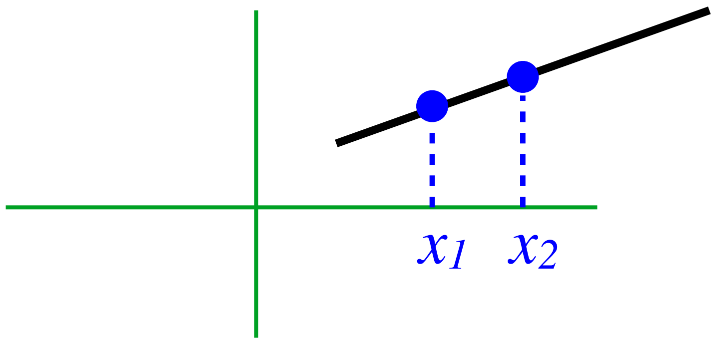

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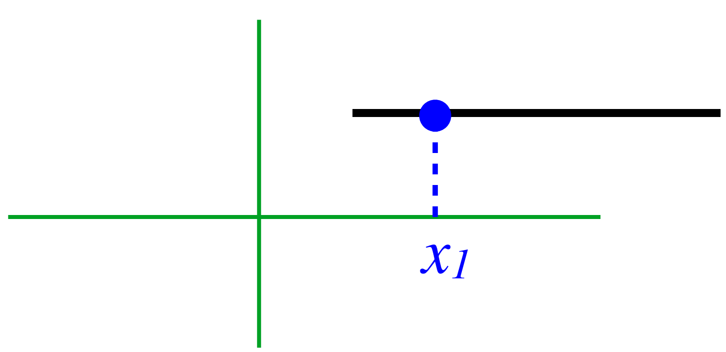

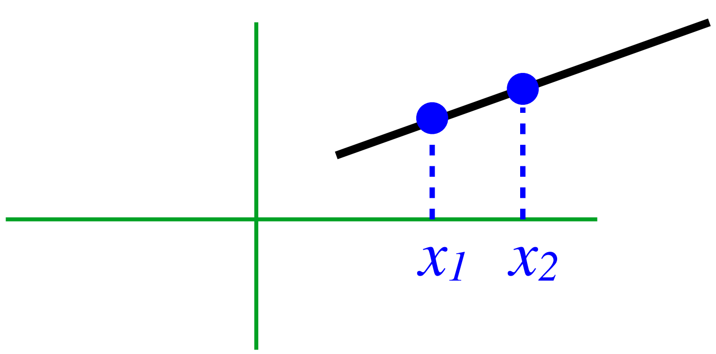

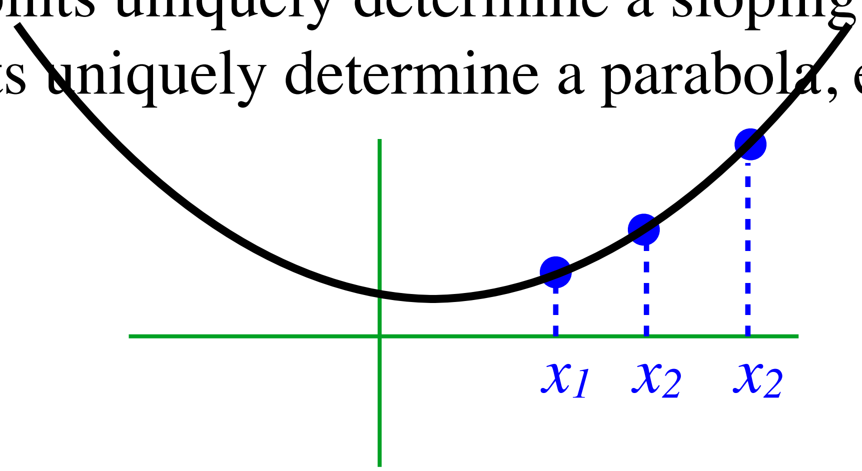

A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

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However, only *select* values ε_k work for eigen-forms $\mathbf{MP}_k=\varepsilon_k\mathbf{P}_k$ or orthonormality $\mathbf{P}_j\mathbf{P}_k=\delta_{jk}\mathbf{P}_k$.

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2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

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Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

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Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix} \langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle \\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle \\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

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Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are "easy"

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2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
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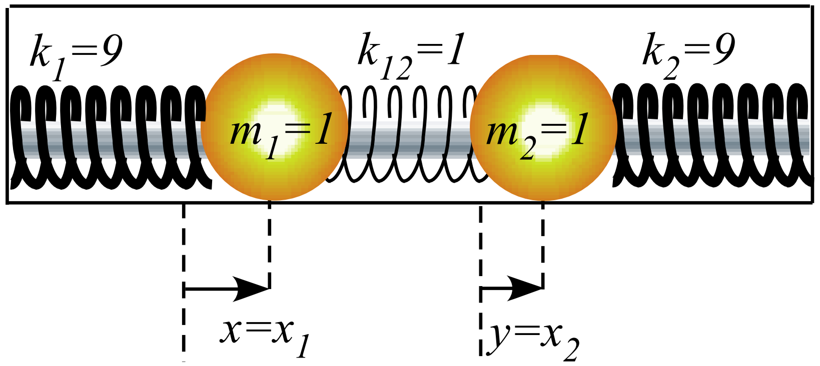


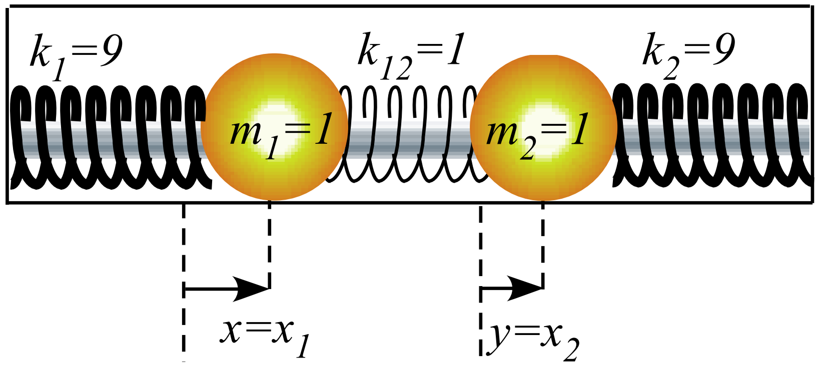
Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$Det(\mathbf{K})=10\cdot 10-1=99$$
$$Trace(\mathbf{K})=10+10=20$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)=(K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=9, \quad K_2=\omega_0^2(\varepsilon_2)=11,$



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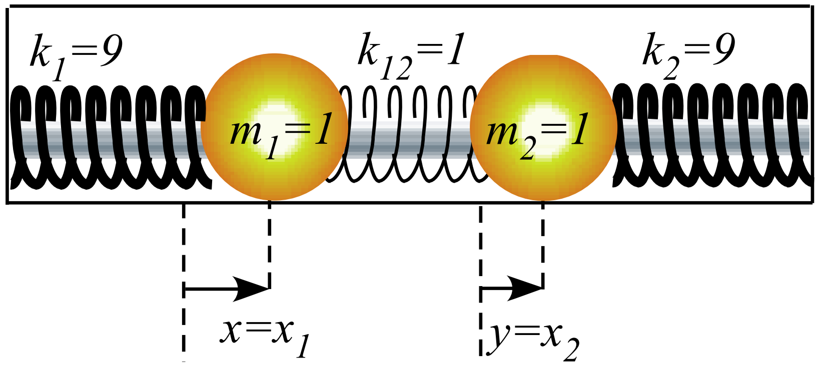

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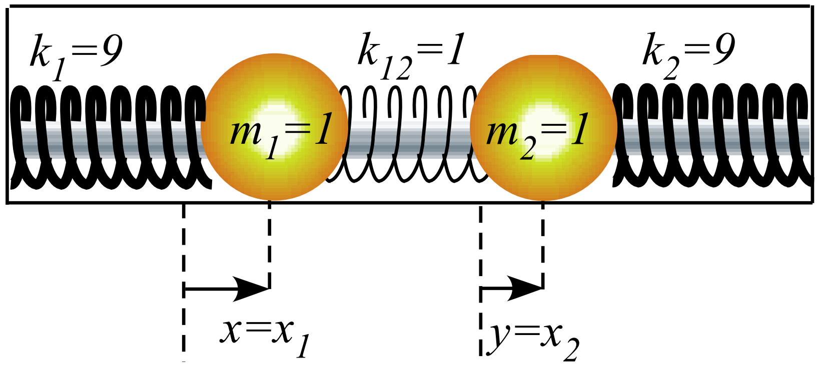
Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$



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$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9\end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}}{2}$$

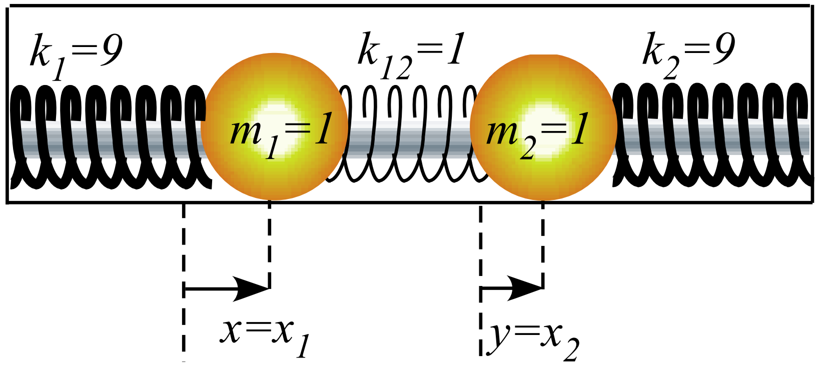
$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$Det(\mathbf{K})=10\cdot 10-1=99$$
$$Trace(\mathbf{K})=10+10=20$$

The $\mathbf{K}$ secular equation $\quad K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)=(K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=9,\quad K_2=\omega_0^2(\varepsilon_2)=11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11}=\frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$
$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$
$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

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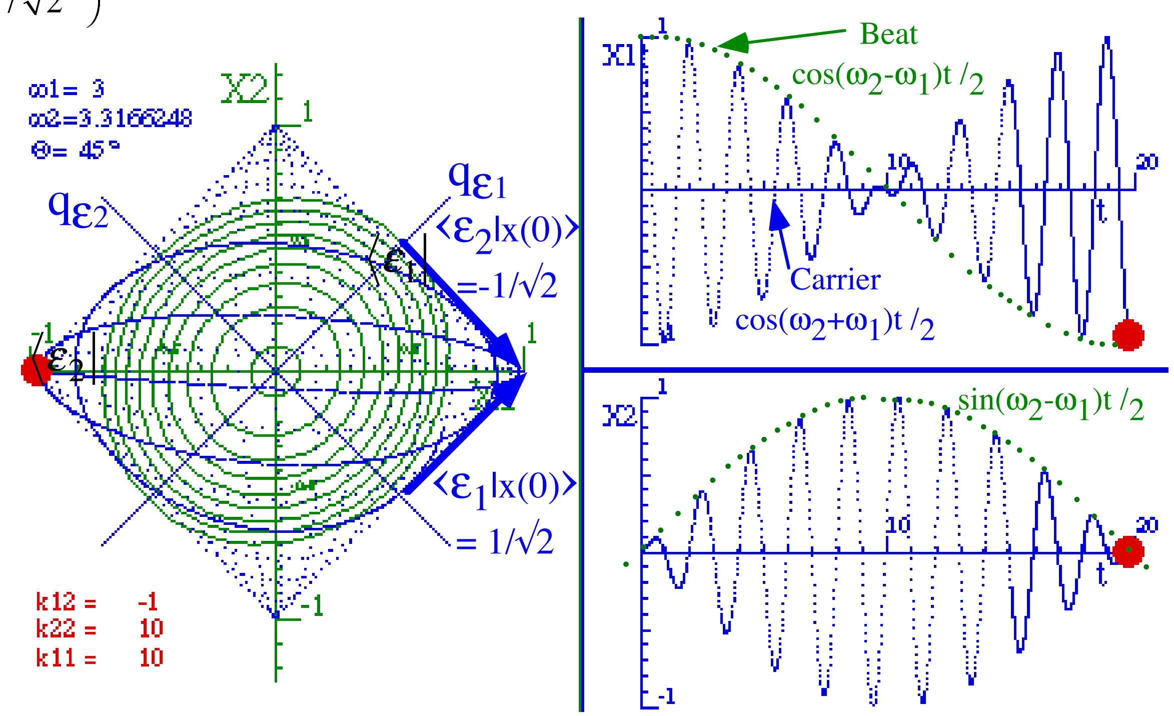
100% modulation (SWR=0) $\quad \frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}$

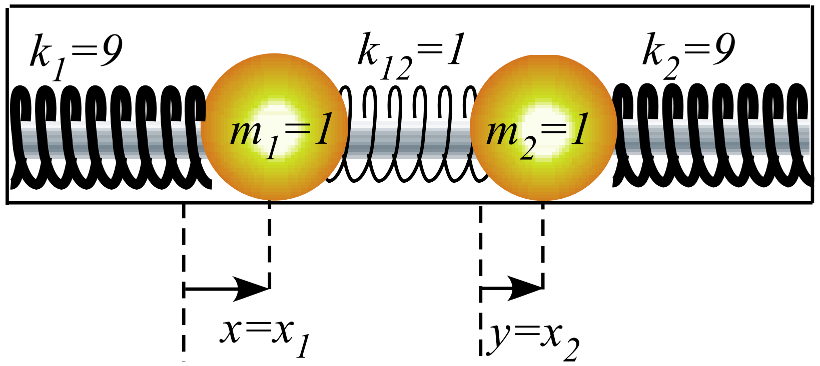
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} \dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2} \end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t} \end{pmatrix}$$



BoxIt (Beating) Web Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10\end{pmatrix}$$

$$Det(\mathbf{K})=10\cdot 10-1=99$$
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$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

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$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

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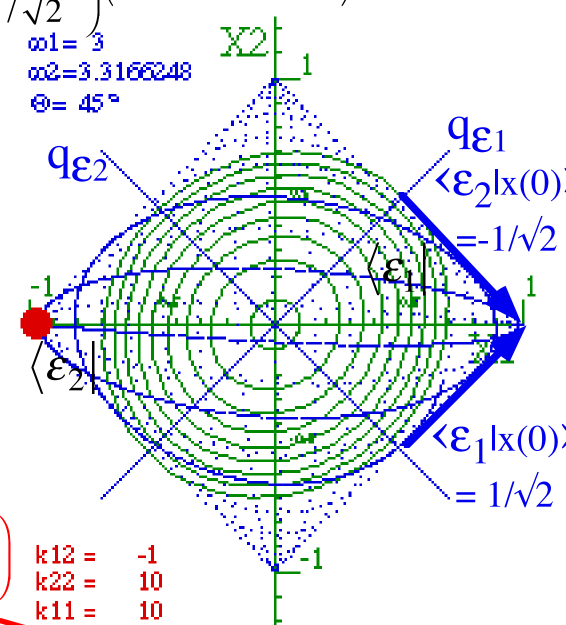
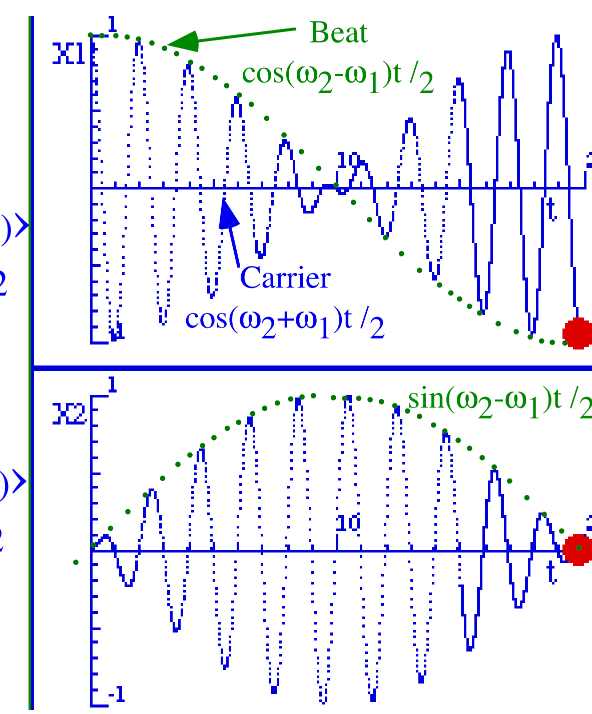
$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$





100% modulation (SWR=0)

$$\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}=e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} \frac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}=e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\frac{(\omega_2-\omega_1)t}{2} \\ i\sin\frac{(\omega_2-\omega_1)t}{2}\end{pmatrix}$$

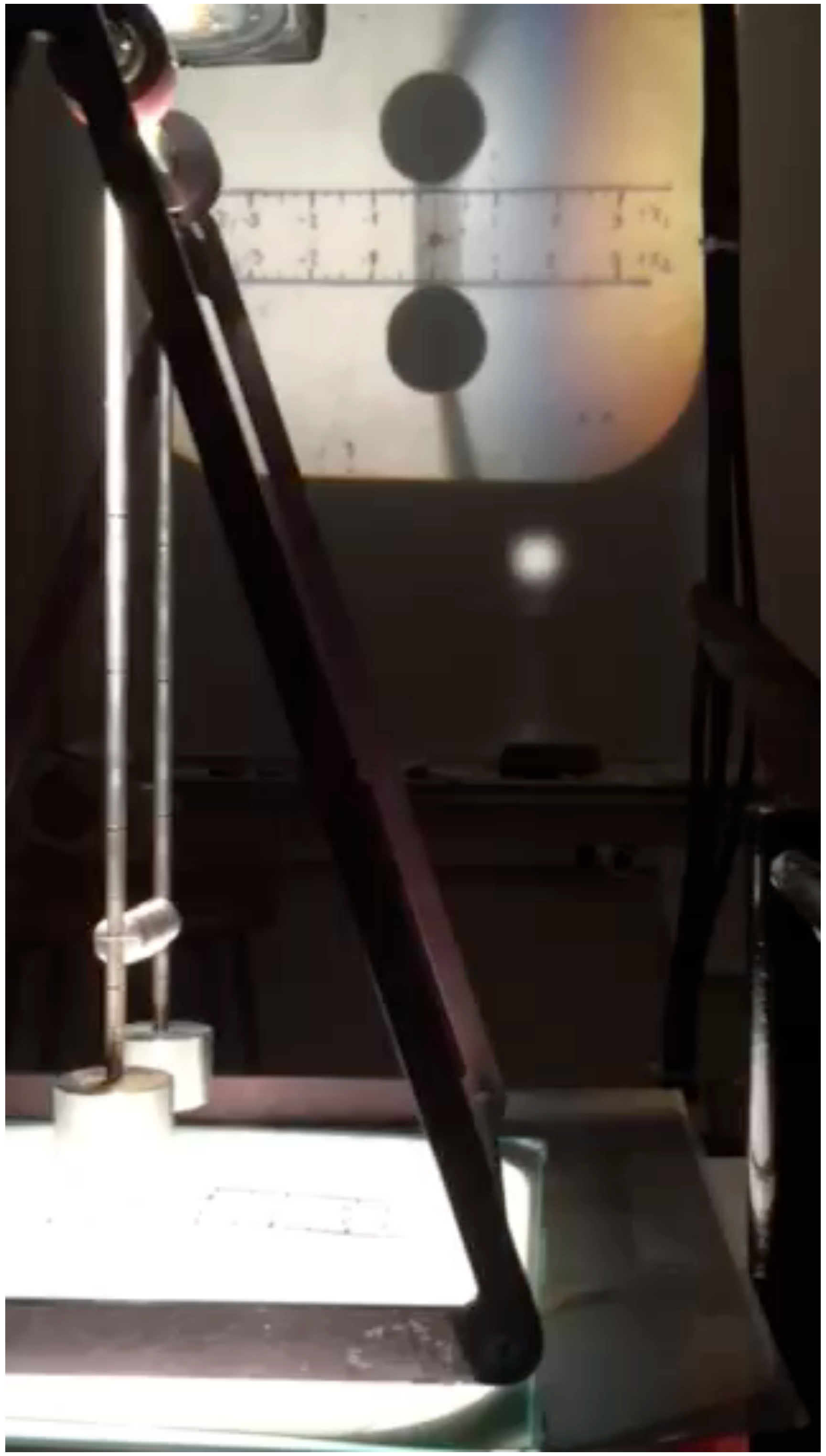
Note the i phase

BoxIt (Beating) Web Simulation

Note: Beat frequency is <u>half-difference</u> of eigenvalues

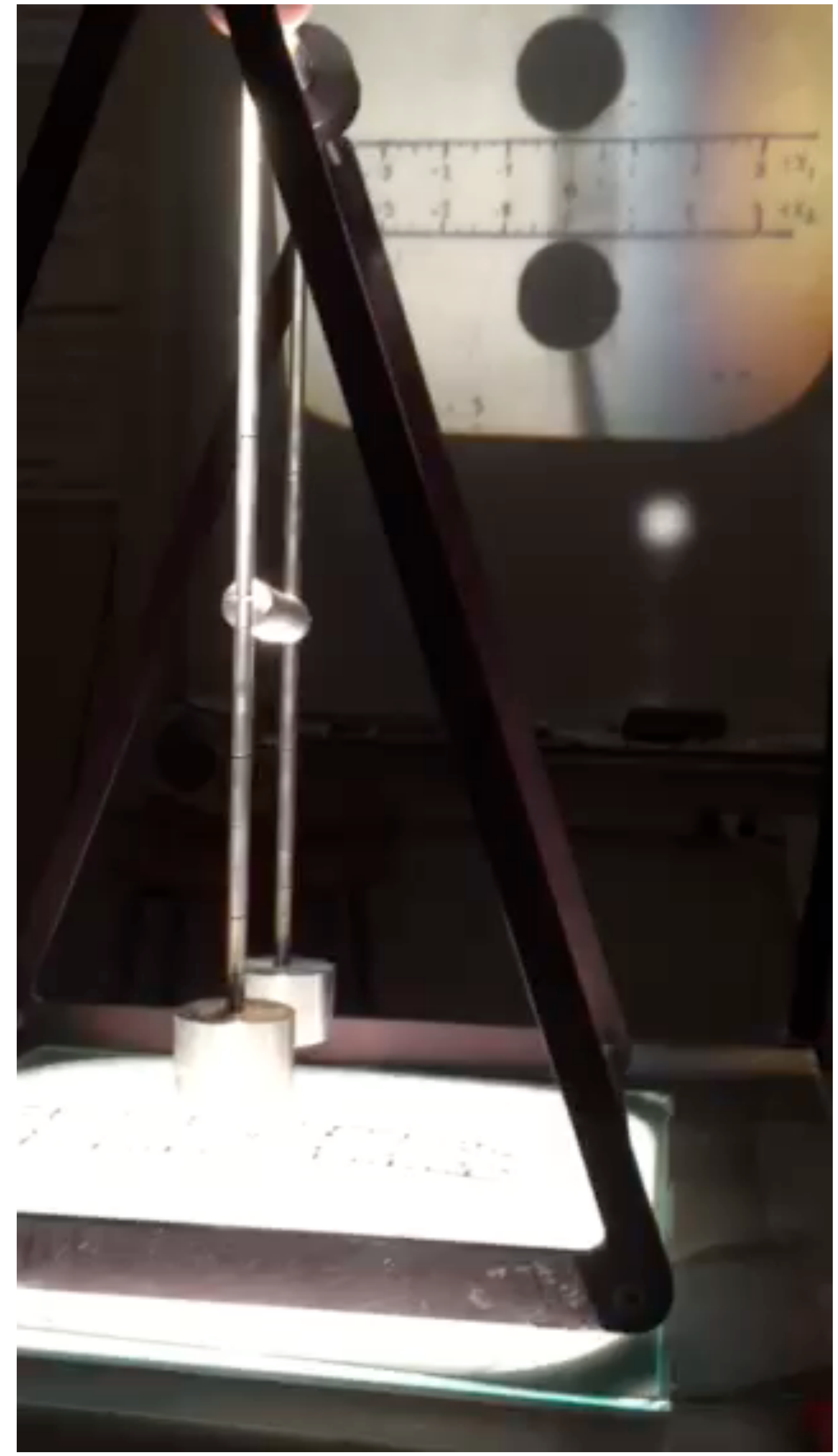
Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

Videos of Coupled Pendula aided by Overhead Projector



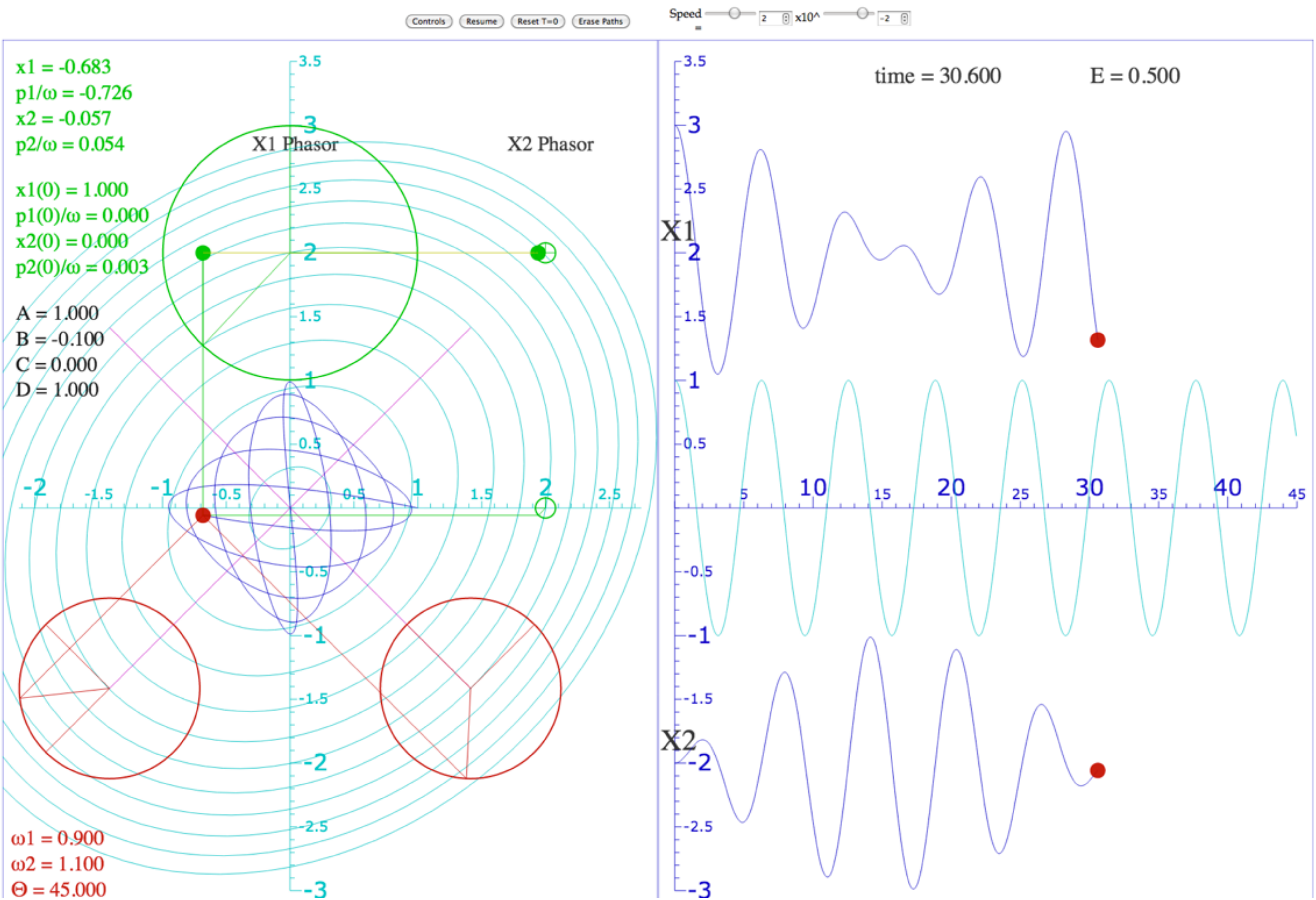
[View on YouTube]

Launch embedded videos using your browser/App
or
⇐ view on YouTube ⇒

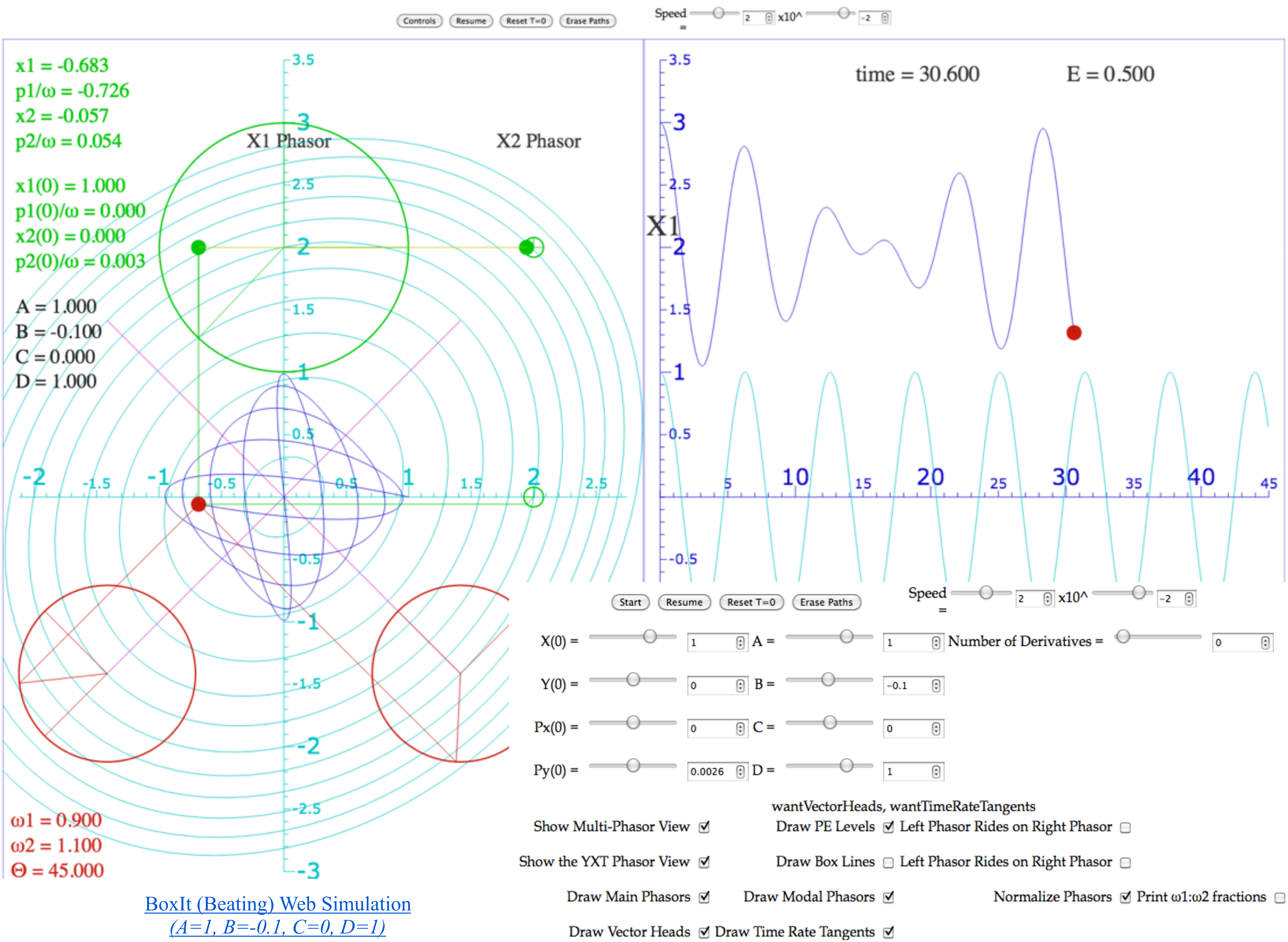


[View on YouTube]

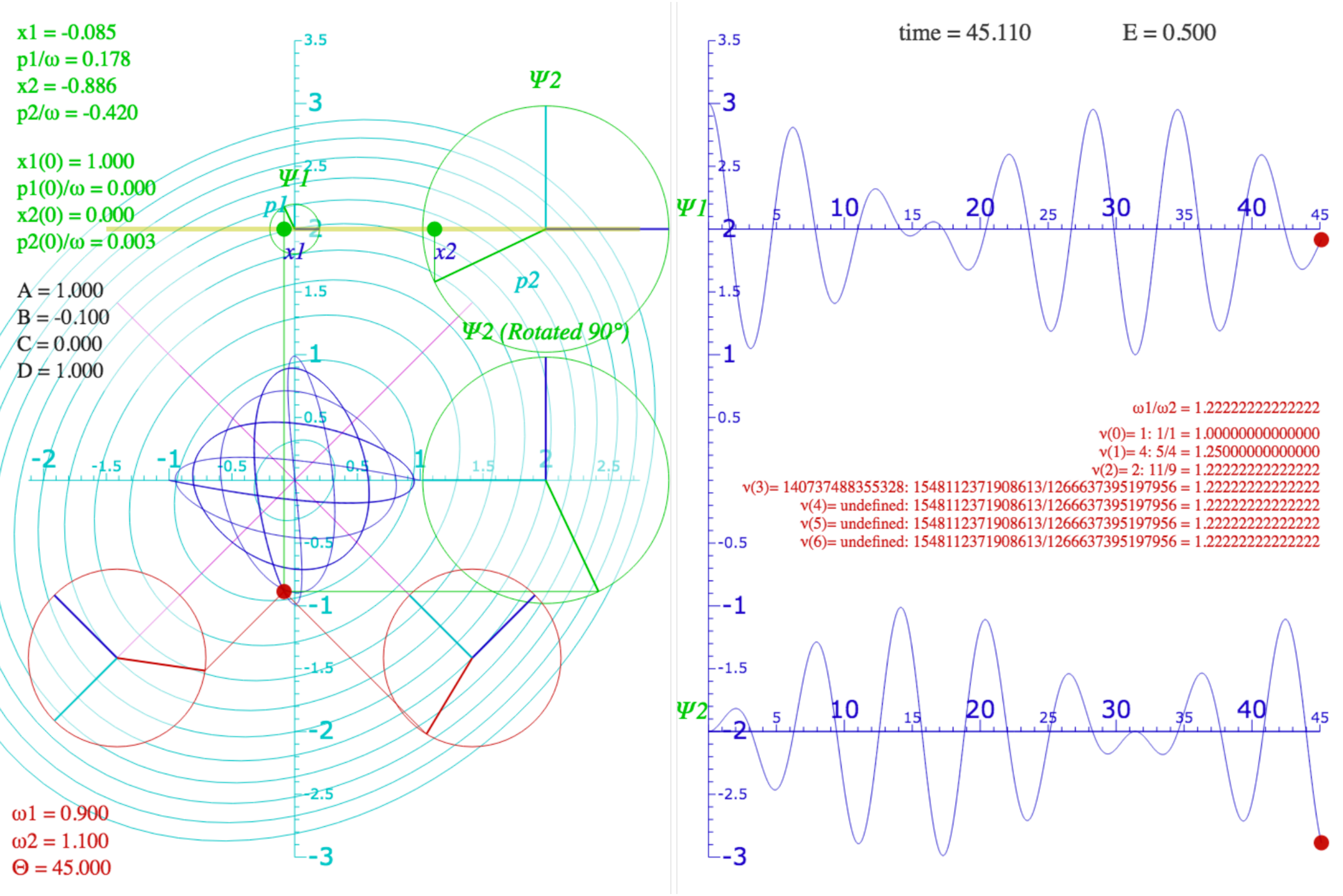
Stronger coupling on the left, illustrated indirectly by a darker looking spring on screen



[BoxIt (Beating) Web Simulation *(A=1, B=-0.1, C=0, D=1)*]



BoxIt (Beating) Web Simulation
(A=1, B=-0.1, C=0, D=1)



BoxIt (Beating) Web Simulation *(A=1, B=-0.1, C=0, D=1) with frequency ratios*

Approximating decimal frequencies ω=α using successive levels of *continued fractions*.

$$\alpha = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \ddots}}}}$$

Pi (π=3.14159265…) converges rather quickly by cf.

$$A_0 = \alpha = 3.14159265... \qquad n_0 = INT(A_0) = 3 \qquad \pi \cong \quad = 3.000..$$

$$A_1 = \frac{1}{A_0 - n_0} = 7.06... \qquad n_1 = INT(A_1) = 7 \qquad \pi \cong 3 + \frac{1}{7} = \frac{22}{7} = 3.1428$$

$$A_2 = \frac{1}{A_1 - n_1} = 15.99... \qquad n_2 = INT(A_2) = 15 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.141509$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.003... \qquad n_3 = INT(A_3) = 1 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15+1}} = \frac{355}{113} = 3.14159292$$

Not so much for the *Golden Mean G*=(1+√5)/2=1.618...

$$A_0 = G = 1.618033989... \qquad n_0 = INT(A_0) = 1 \qquad G \cong \quad = 1.000..$$

$$A_1 = \frac{1}{A_0 - n_0} = 1.6180... \qquad n_1 = INT(A_1) = 1 \qquad G \cong 1 + \frac{1}{1} = \frac{2}{1} = 2.000$$

$$A_2 = \frac{1}{A_1 - n_1} = 1.6180... \qquad n_2 = INT(A_2) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = 1.500$$

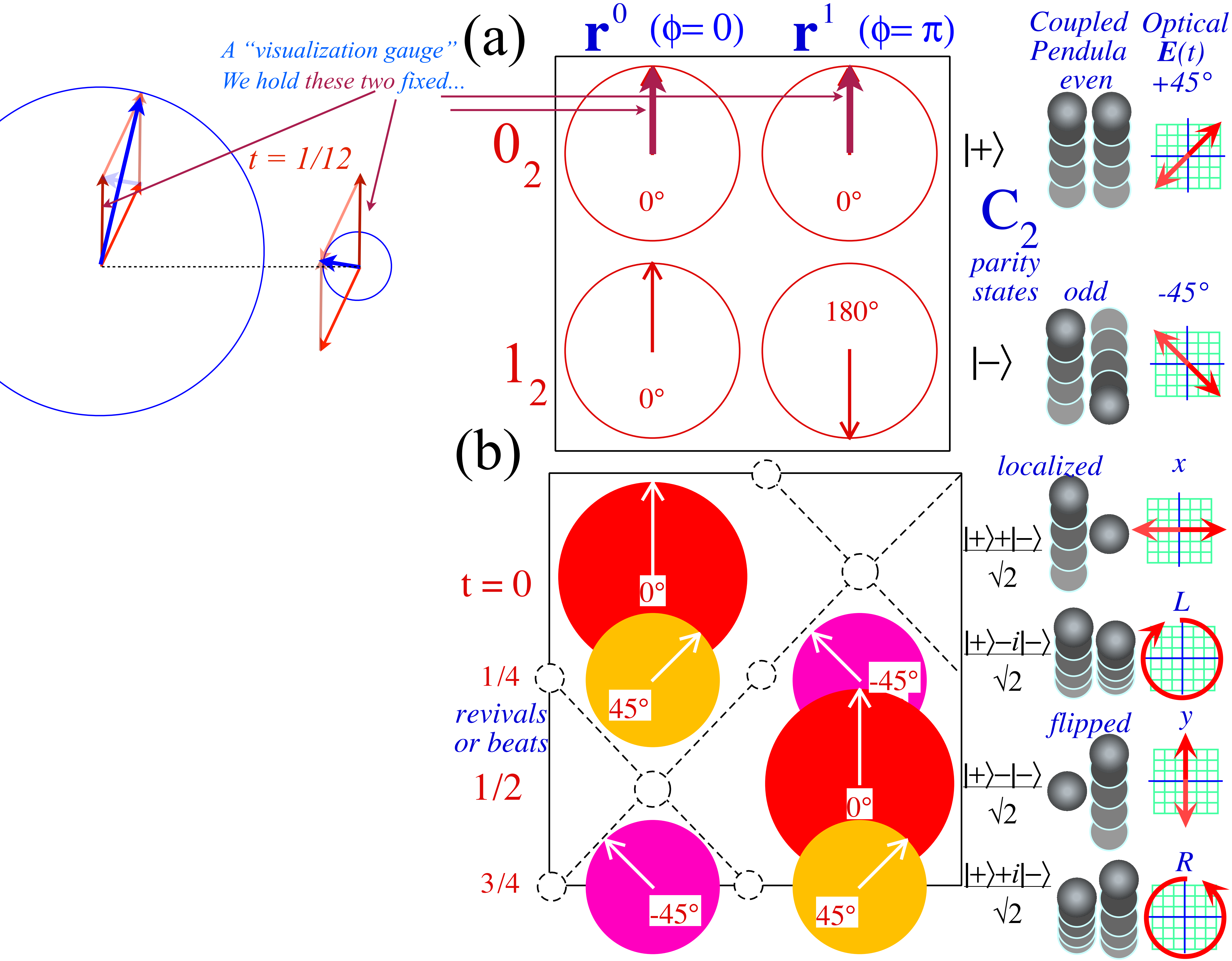
$$A_3 = \frac{1}{A_2 - n_2} = 1.6180... \qquad n_3 = INT(A_3) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1+1}} = \frac{5}{3} = 1.666...$$

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

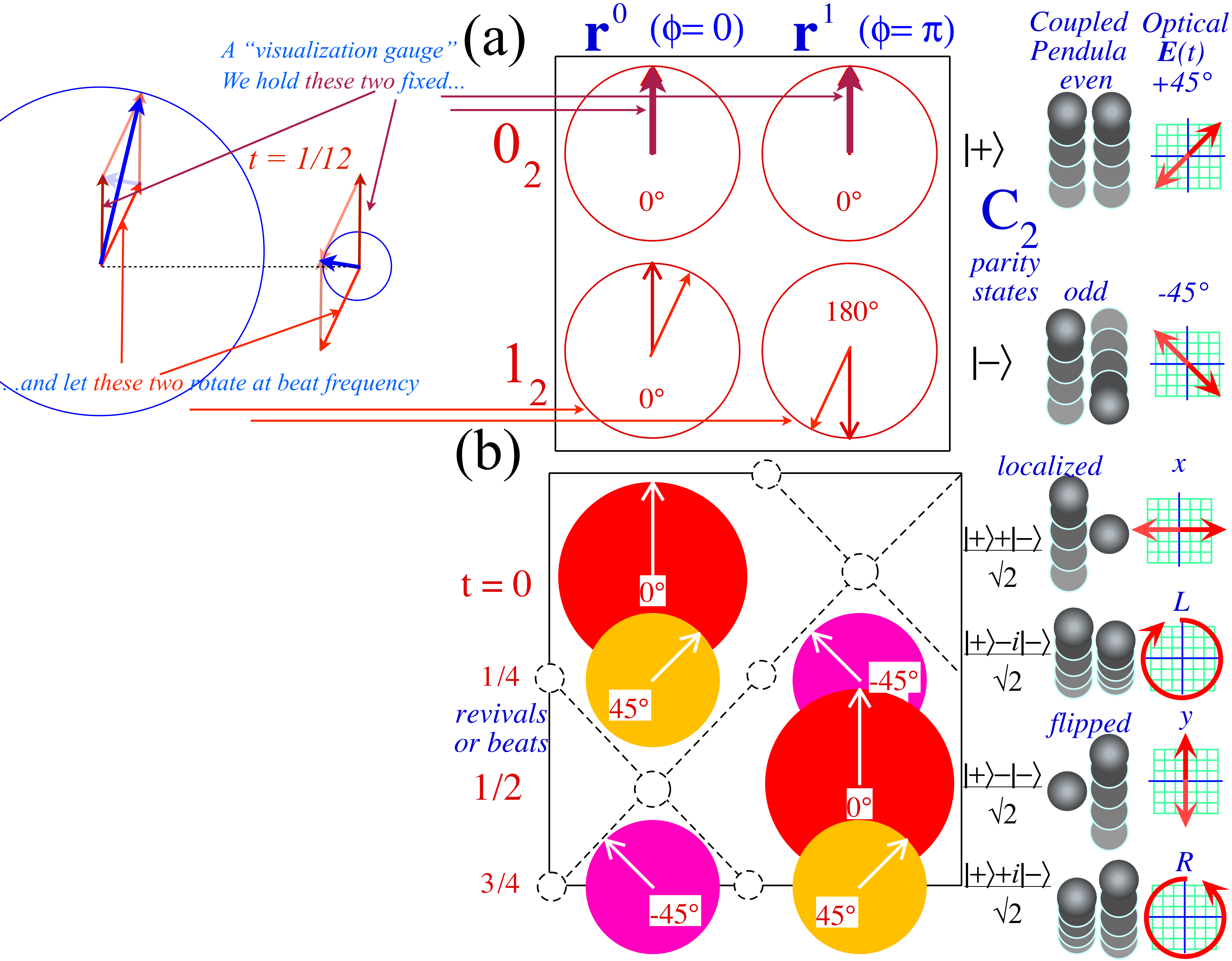
2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P\cdot P=P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase
2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

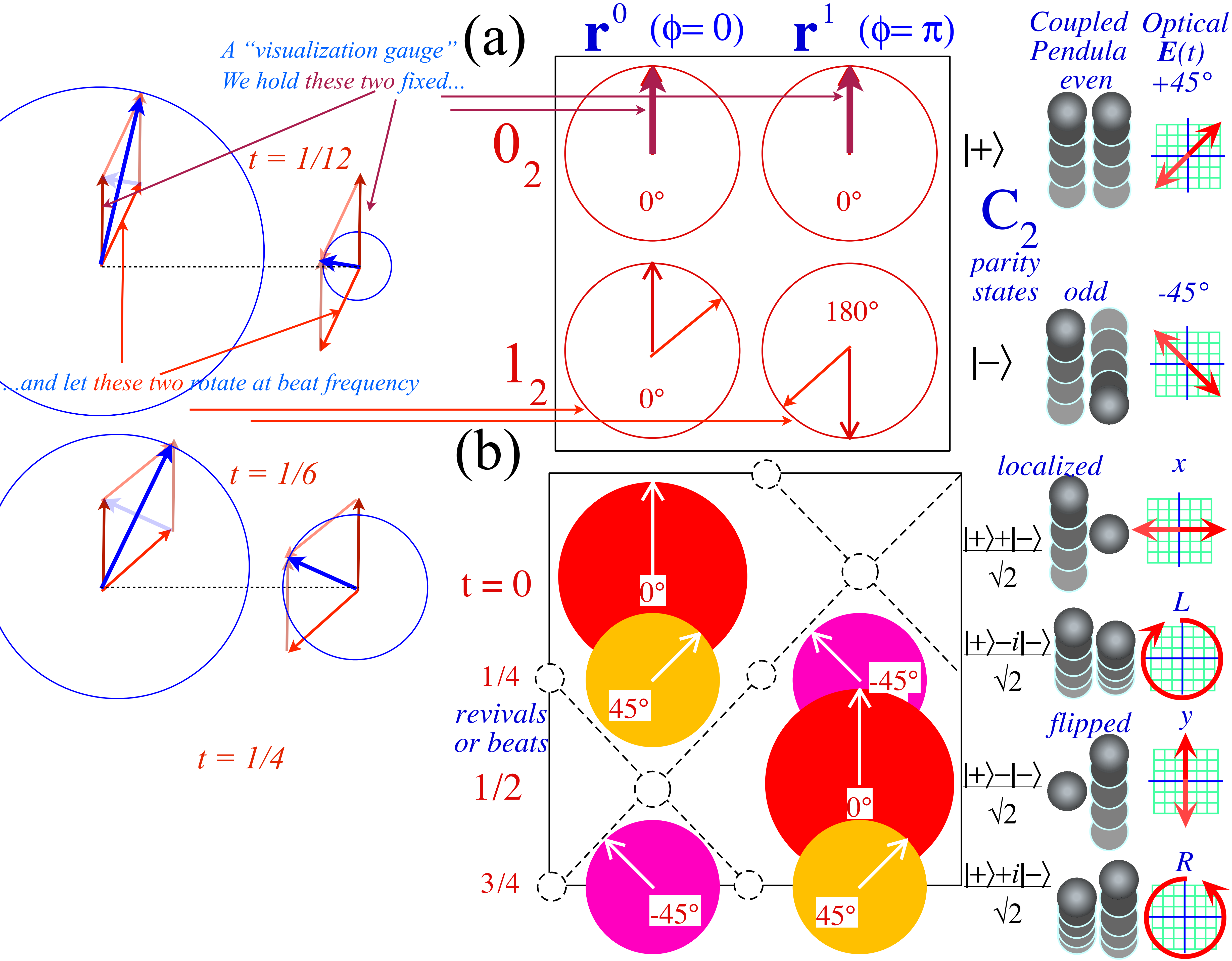
2D-HO beats and mixed mode geometry



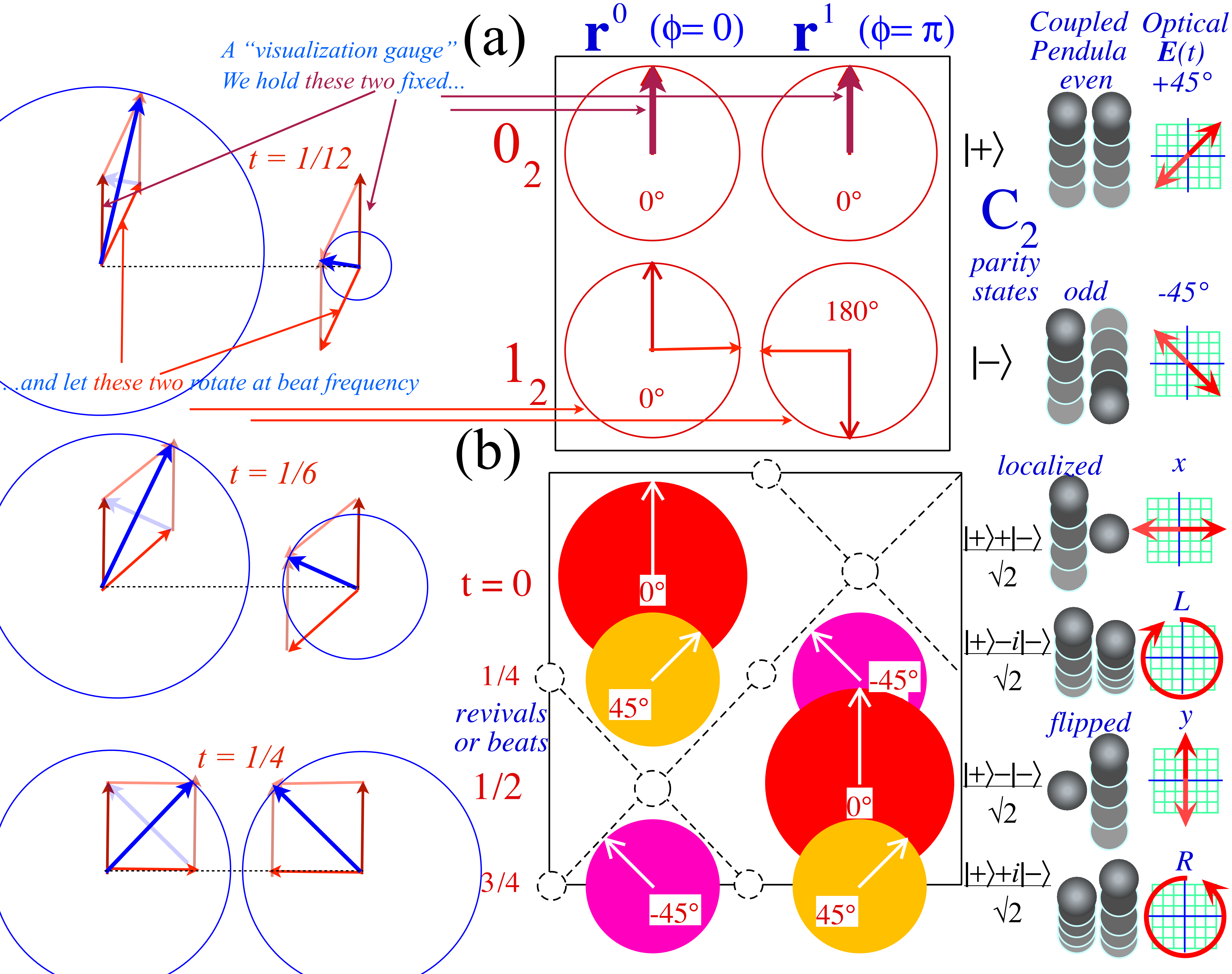
2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



2D harmonic oscillator equations

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Geometric method

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Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

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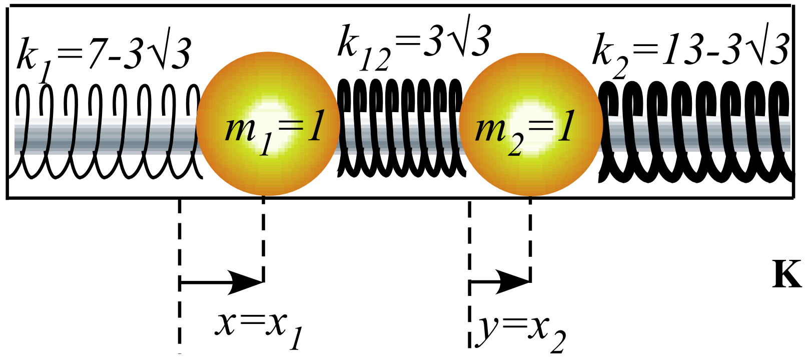
Mixed mode beat dynamics and fixed π/2 phase

➔ *2D-HO eigensolution example with asymmetric (A-Type) symmetry* ⬅

Initial state projection, mixed mode beat dynamics with variable phase

Spectral decomposition of 2D-HO mode dynamics for lower symmetry

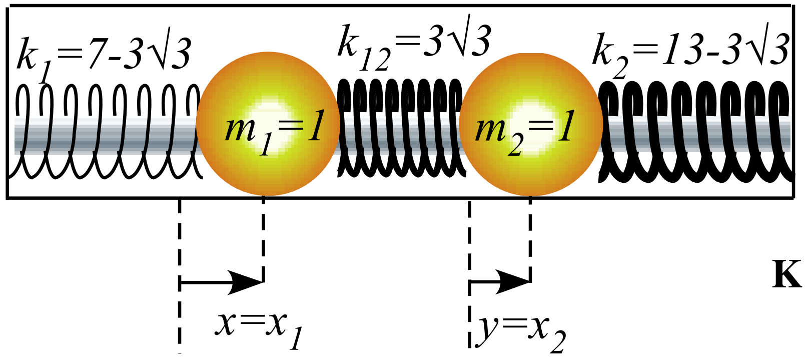


$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$$Det(\mathbf{K})=7\cdot 13-27=91-27=64$$

$$Trace(\mathbf{K})=7+13=20$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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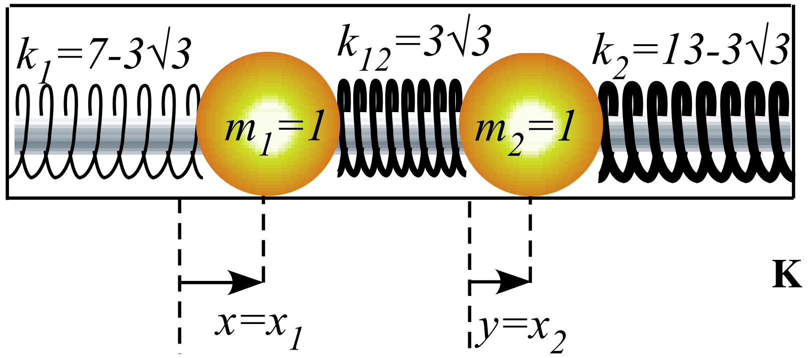
The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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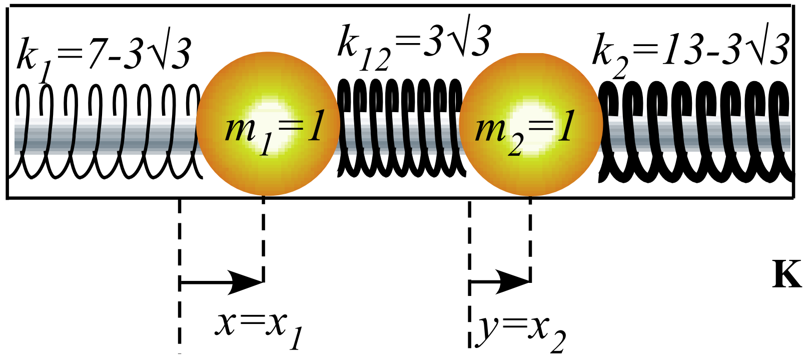
The $\mathbf{K}$ secular equation $\quad K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

$Trace(\mathbf{K})=7+13=20$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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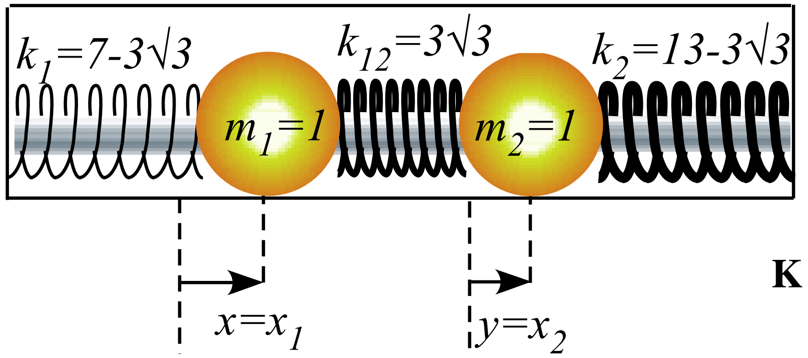
Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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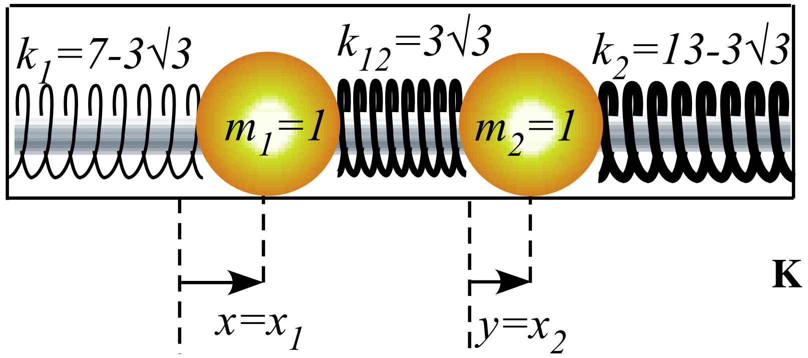
$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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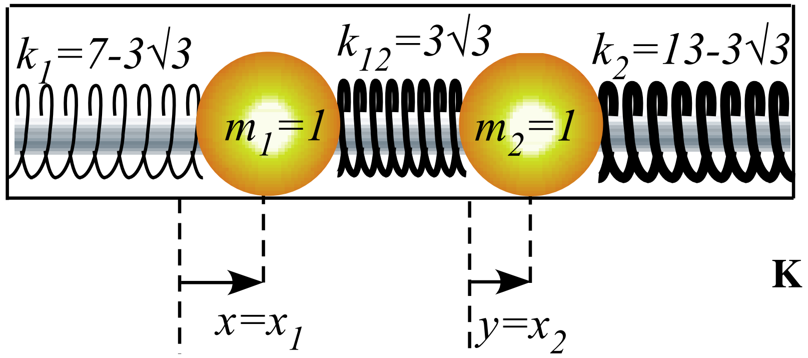
$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

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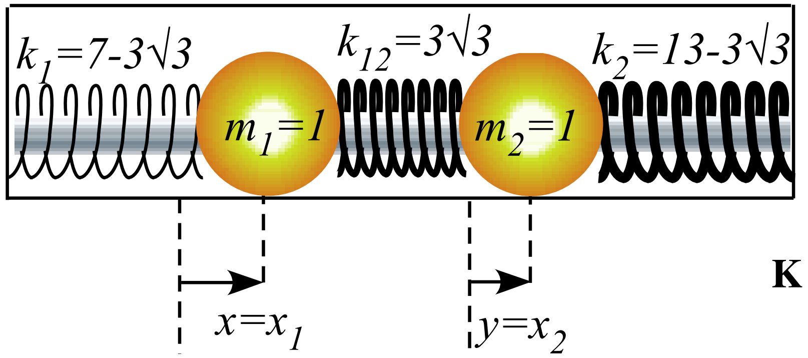
2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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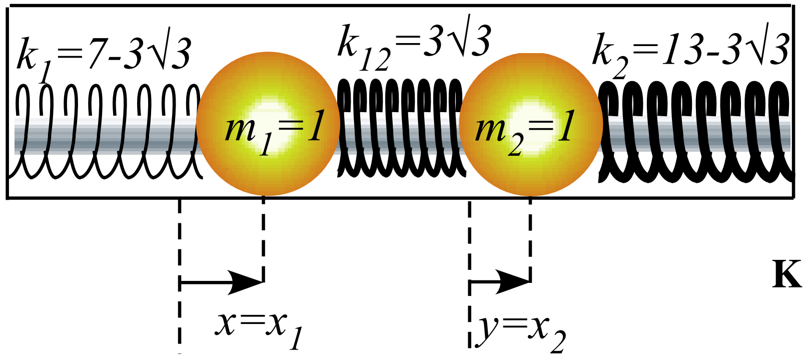
Spectral decomposition of initial state ***x****(0)=(1,0):*

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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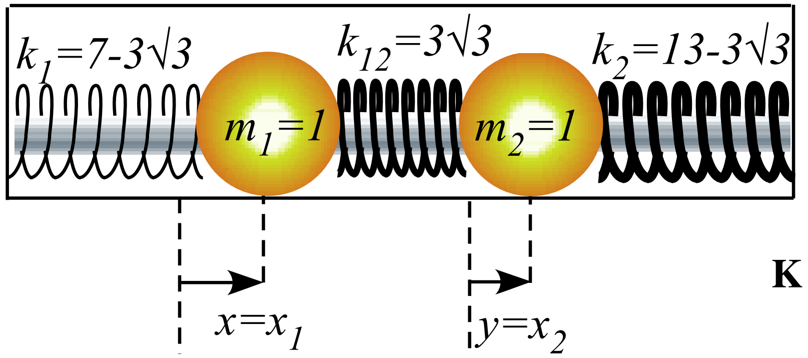
$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\frac{\sqrt{3}}{2}\right)+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\frac{1}{2}\right)$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

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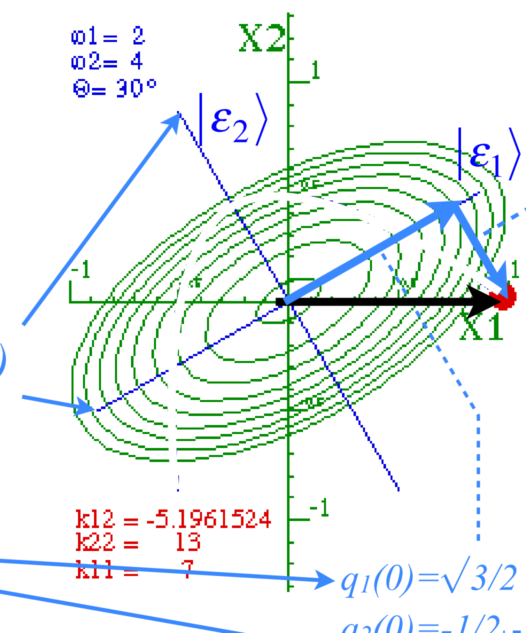
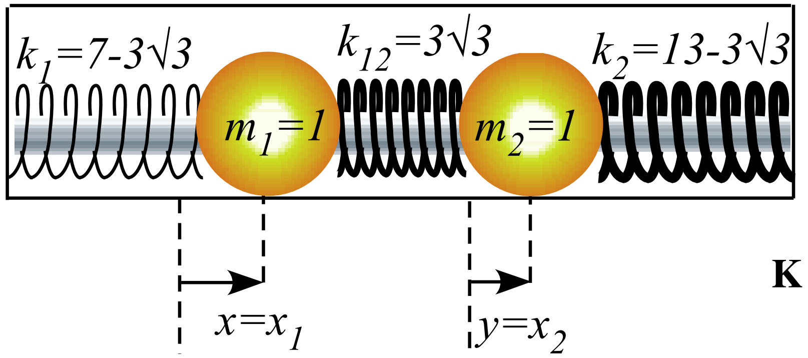


Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

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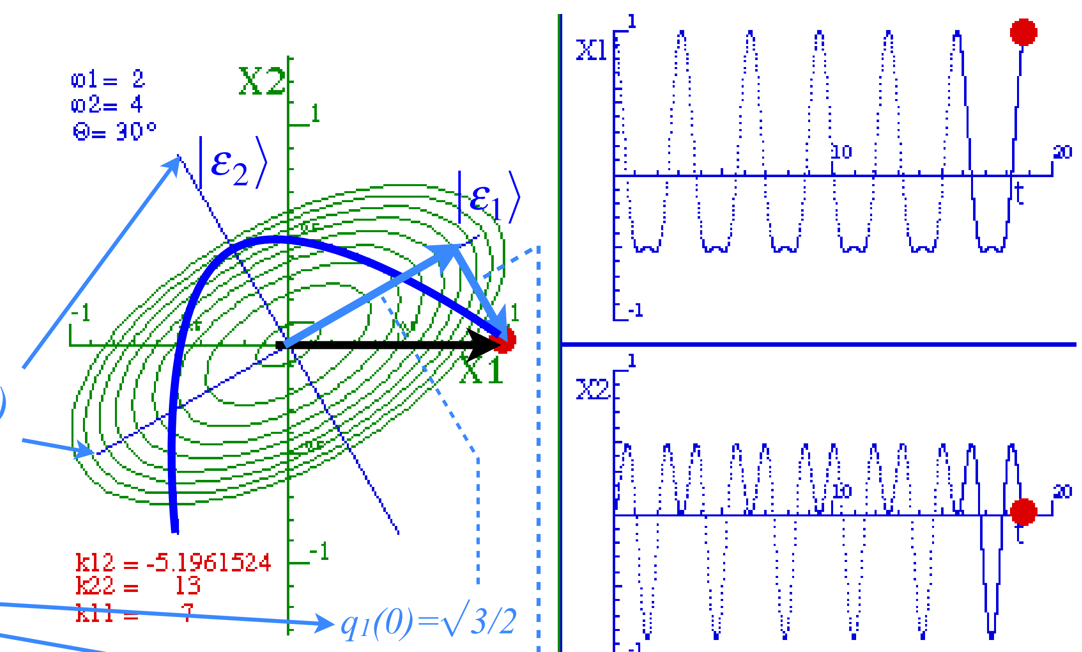
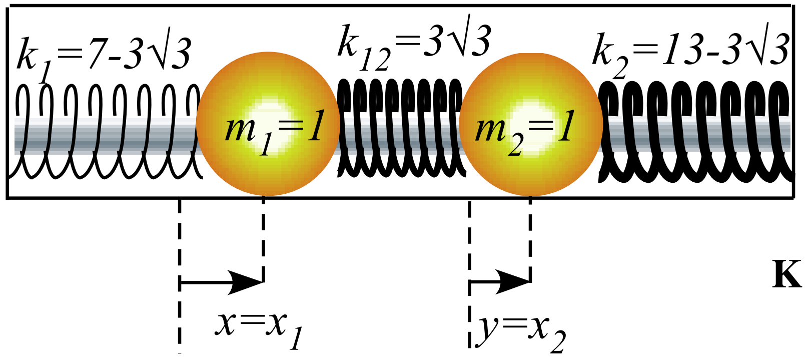


BoxIt Web Simulation

Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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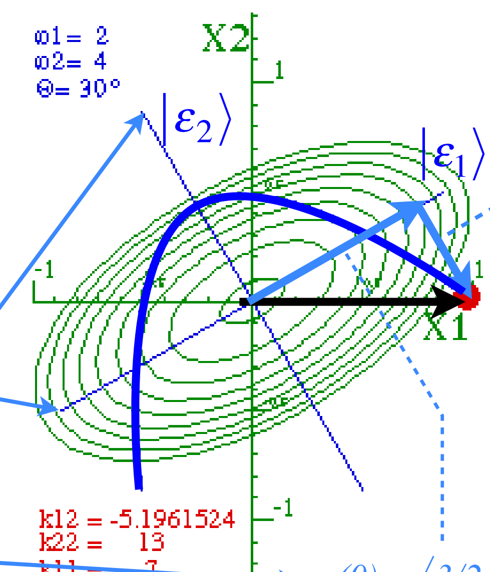

[BoxIt Web Simulation]

$$\mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$

$q_1(0)=\sqrt{3}/2$

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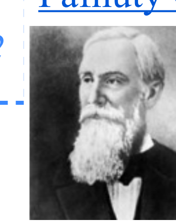
Example of a Tschebycheff Polynomial order 2

[Pafnuty Chebyshev]



Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshov, Tchebychev or Tchebycheff, or Tschebyschev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk

Died: December 8, 1894, Saint Petersburg