Lecture 16
Mon. 10.28.2019

Hamilton Equations for Trebuchet and Related Things (Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)
- Hamiltonian definition from Lagrangian and γ_{mn} tensor
- Hamilton's equations and Poincare invariant relations
- Hamiltonian expression and contravariant γ^{mn} tensor
  - Hamilton p_μ equation log-jam

Hamiltonian energy and momentum conservation and symmetry coordinates
- Coordinate transformation helps reduce symmetric Hamiltonian
- Free-space trebuchet kinematics by symmetry
  - Algebraic approximate approach
  - Direct approximate approach and Superball analogy
- Trebuchet vs Flinger and sports kinematics
  - The multiple approaches to Mechanics (and physics in general)
  - A simplified trebuchet approximation (but unfinished)

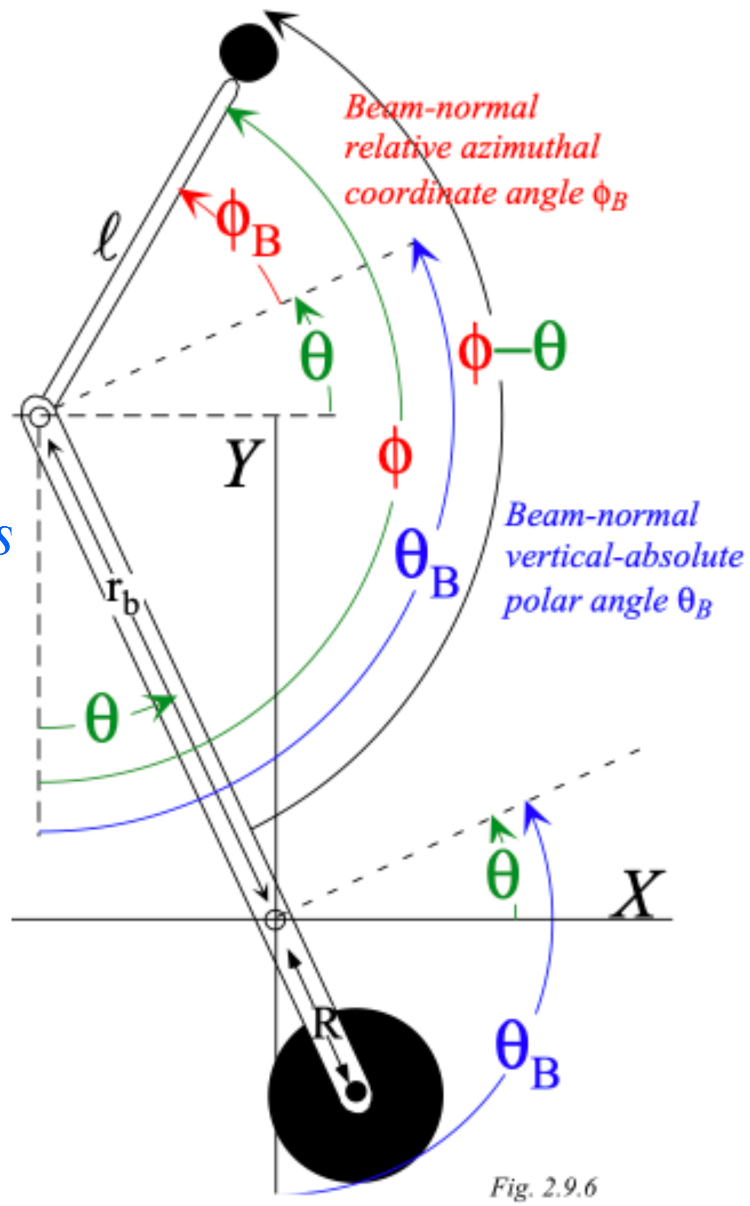


Fig. 2.9.6

This Lecture's Reference Link Listing

Web Resources - front page | Quantum Theory for the Computer Age | 2017 Group Theory for QM
UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2018 Adv CM
Classical Mechanics with a Bang! | 2018 AMOP
Modern Physics and its Classical Foundations | **2019 Advanced Mechanics**

Lectures #12 through #16

In reverse order

Trebuchet Web Animations:

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:

(Many of these may be a bit beyond this course, but are included to lend added insight):

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

An assist from *Physics Girl* (YouTube Channel):

Posted this year:
How to Make VORTEX RINGS in a Pool

Crazy pool vortex (new inclusion with more background)
Crazy pool vortex - pg-yt-2014

Posting with the best visuals:
Fun with Vortex Rings in the Pool - pg-yt-2014

She covers it beautifully!

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.

CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,

RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Excerpts (Page 44-47 in *Preliminary Draft*) from the
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019

Select, exciting, and related Research

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces - Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74*, *Page=75*, *Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20

WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,
Nuclear_Collisions - Mishustin-PhysRevC-2007, *APS Link & Abstract*
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot *V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *$(v_1, v_2)=(1, 0)$: V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

Chapter 1. The Trebuchet: A dream problem for Galileo?

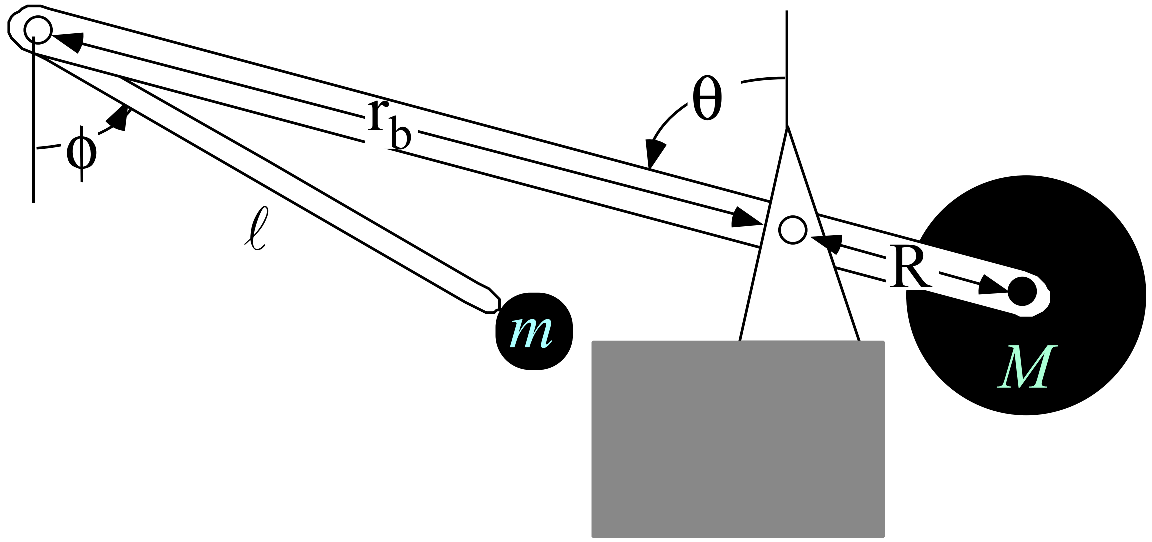
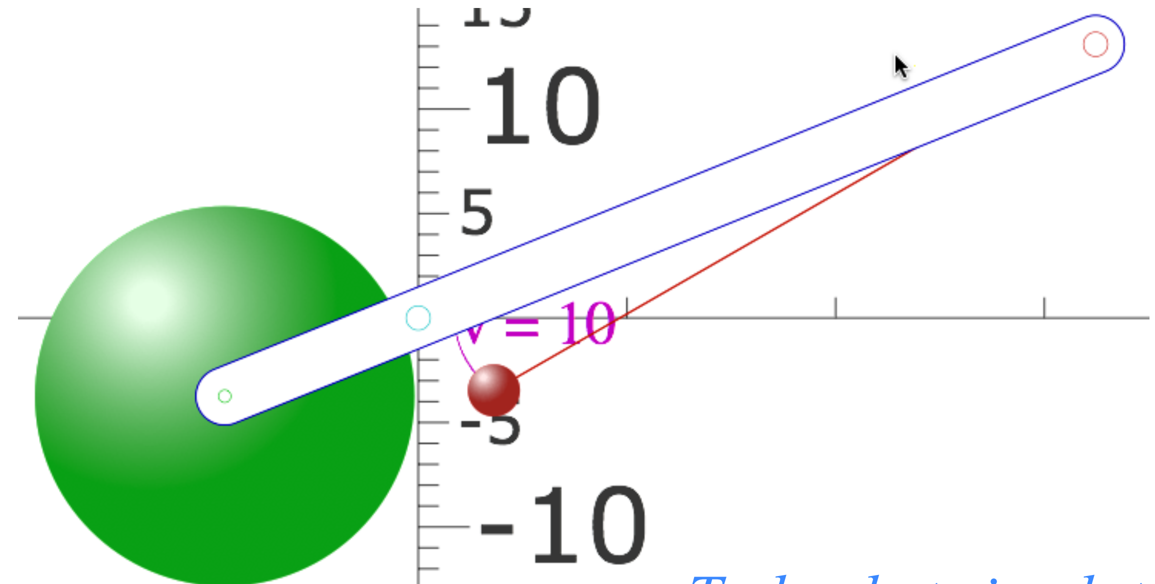
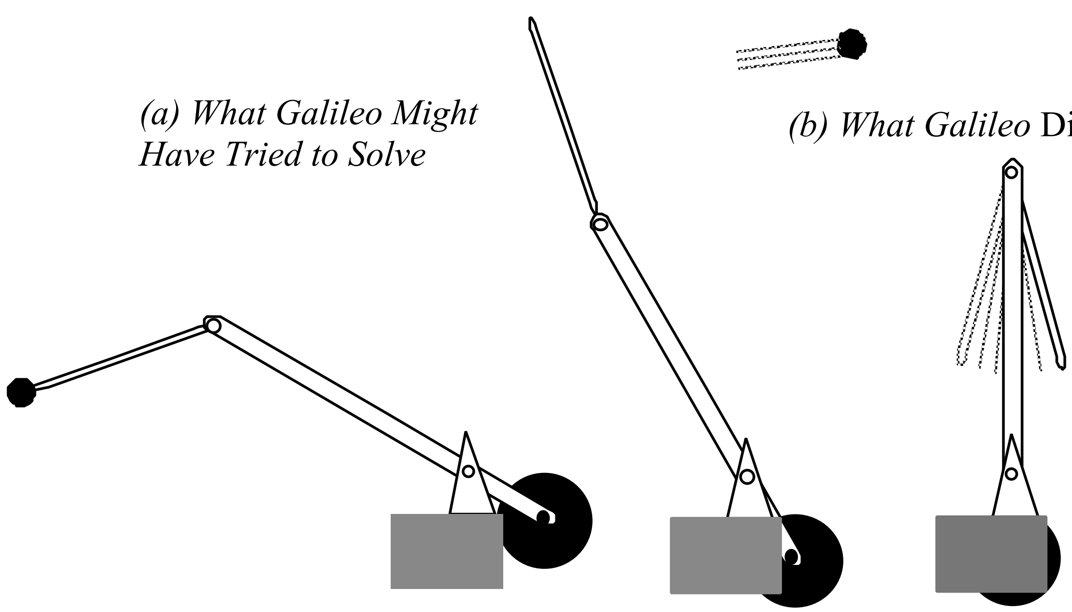


Fig. 2.1.1 An elementary ground-fixed trebuchet



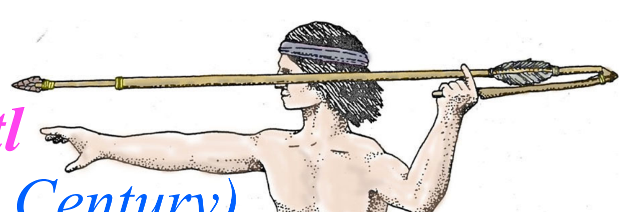
Trebuchet simulator

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem



The Atlatl
(Cahokia, IL 12th Century)

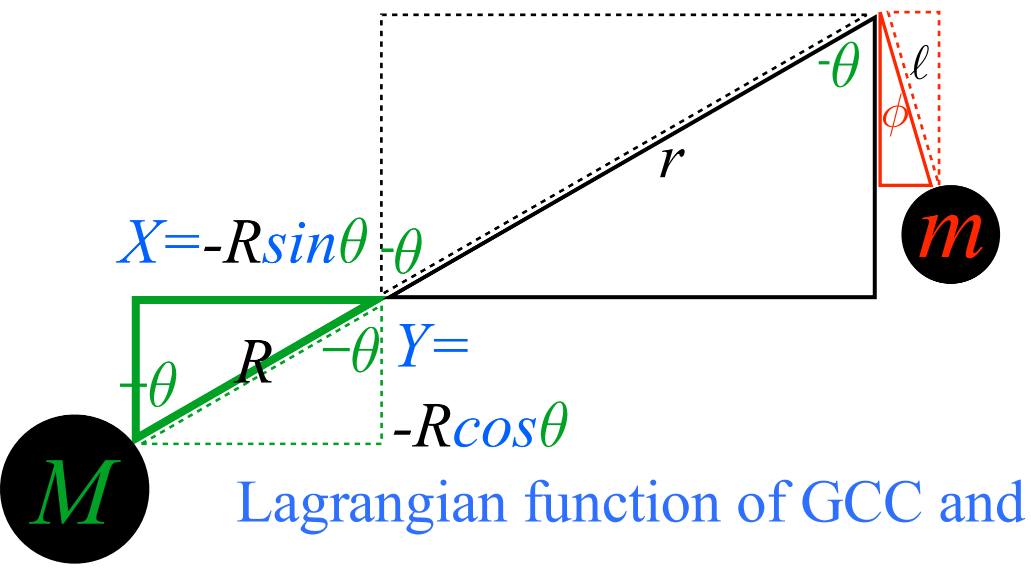
Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

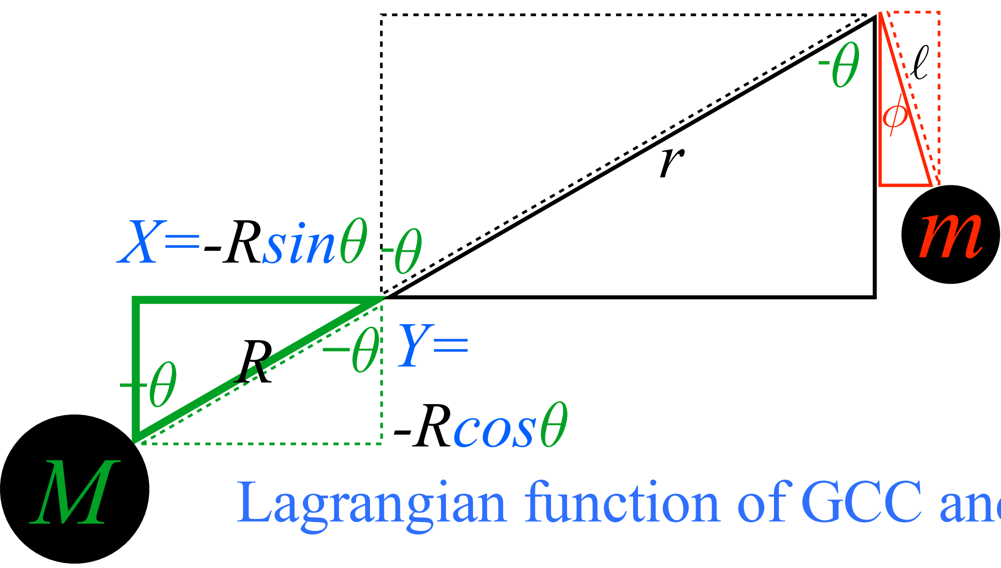
Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

1st differential chain

$$Total\ KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

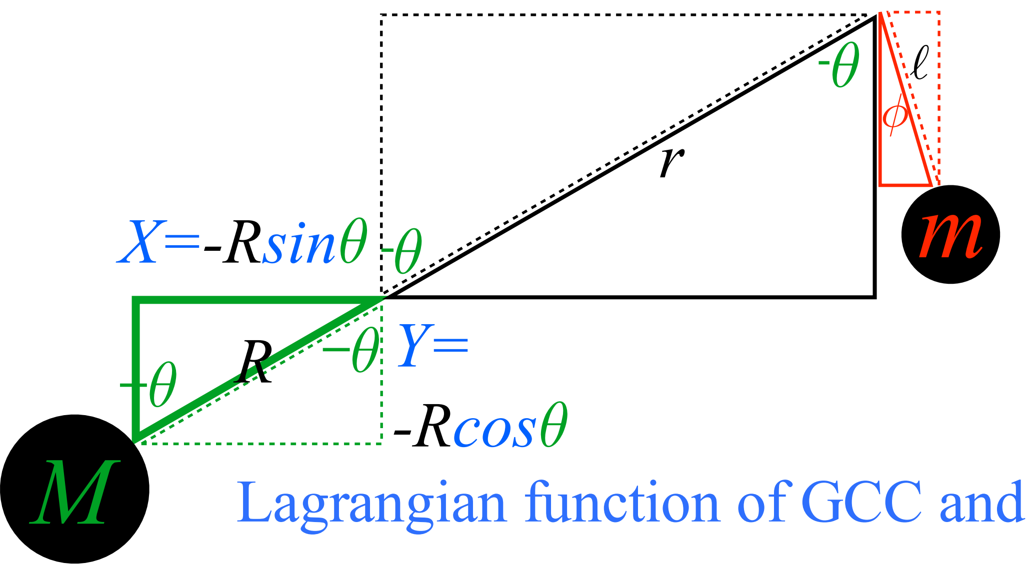
Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$ *1st differential chain*

$$\frac{dL}{dt} \equiv \dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *velocity chain*

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

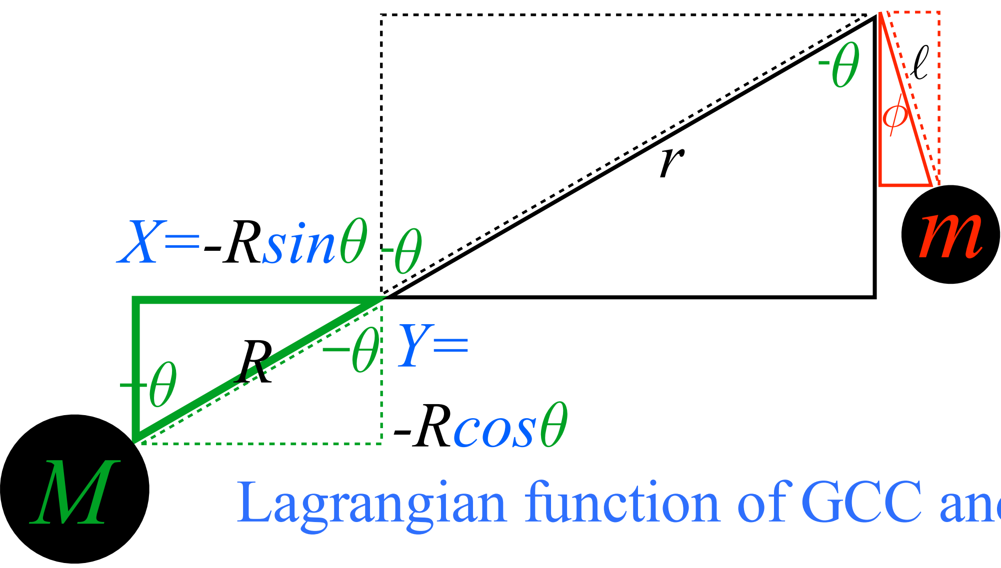
Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$
1st differential chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$
velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$
Lagrange equations 1 and 2

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}_{} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

__Covariant__
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

1st differential chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

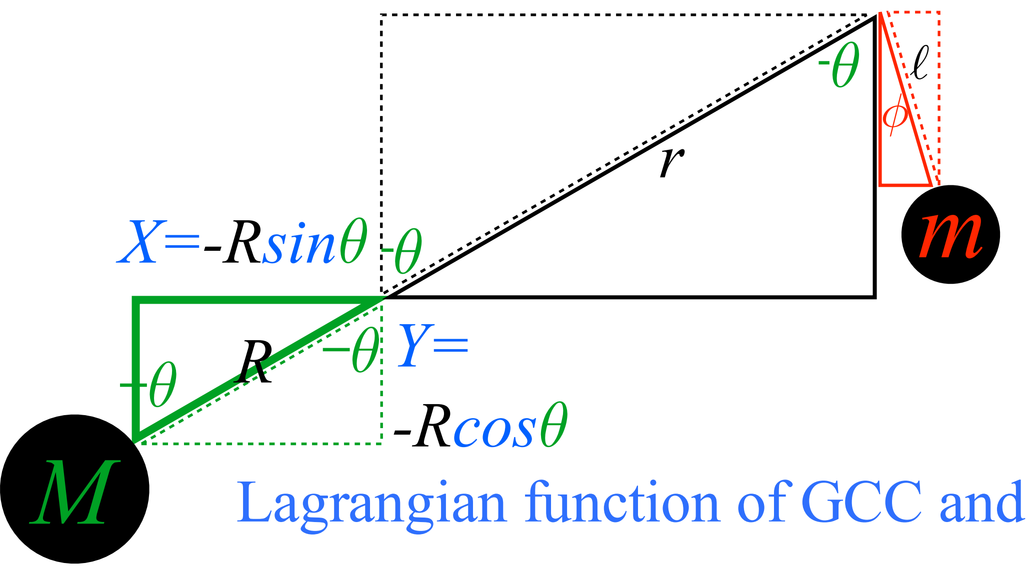
$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

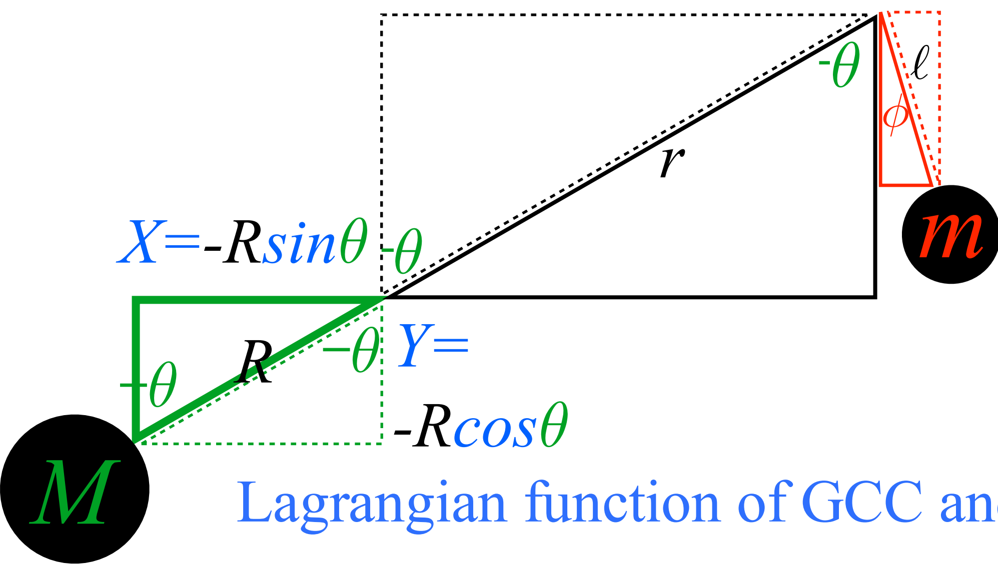
$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right) = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\textit{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)}_{\underline{H}} = -\frac{\partial L}{\partial t}$$

(Rearranging)

Defining the
Hamiltonian function H

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

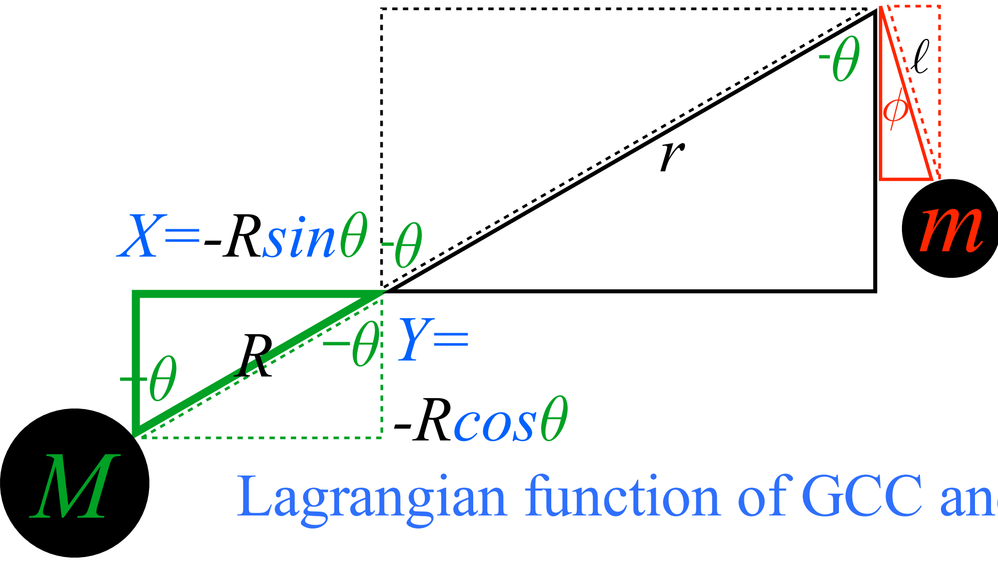
Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and covariant γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

$$Total\ KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

<u>Covariant</u>
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

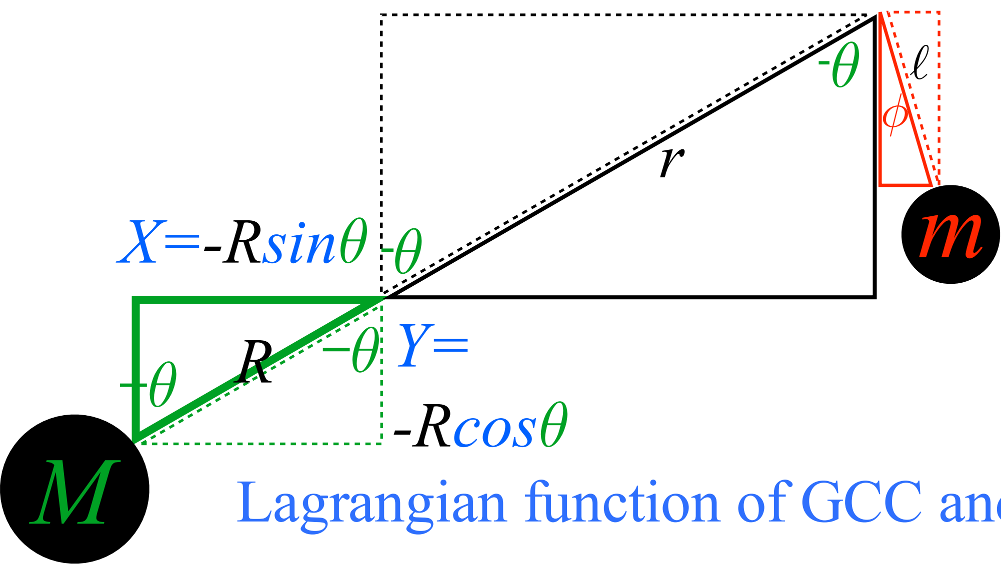
(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the
Hamiltonian function H

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial\theta}d\theta + \frac{\partial L}{\partial\phi}d\phi + \frac{\partial L}{\partial\dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial\dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial\dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial\dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

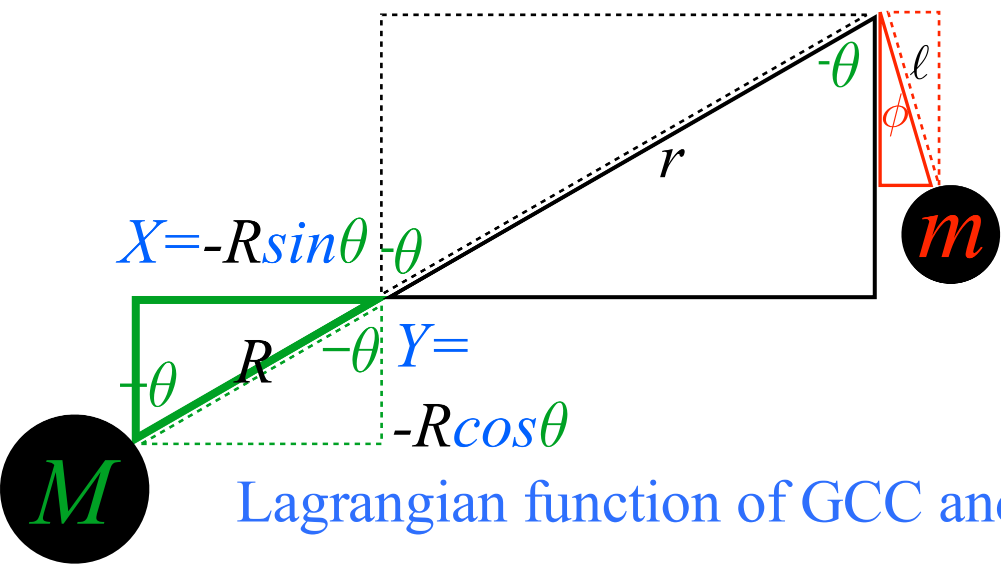
Defining the
Hamiltonian function H

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} = -\frac{\partial L}{\partial\theta} = -\dot{p}_\theta$$

by Lagrange equation 2



$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

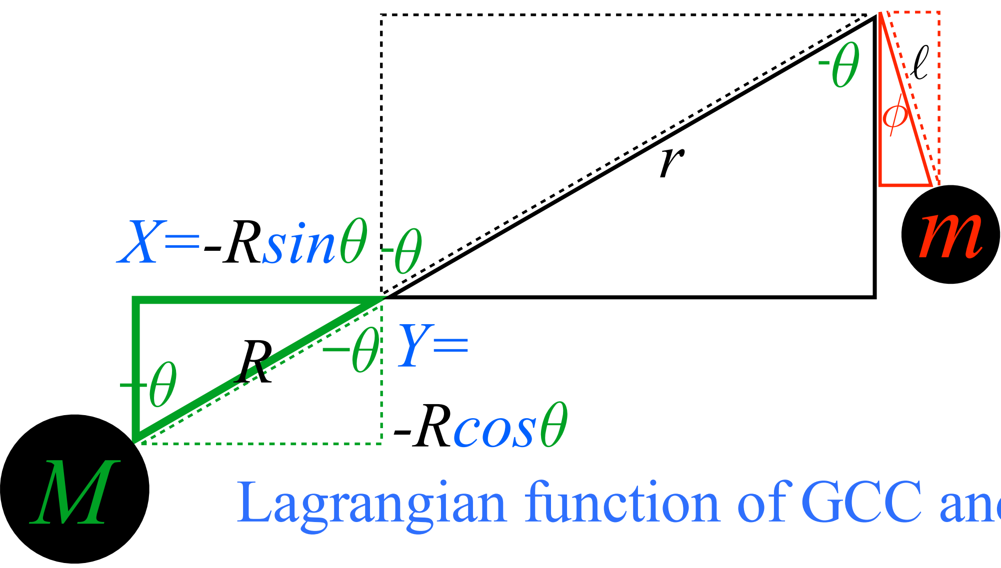
Defining the
Hamiltonian function H

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = \boxed{p_\theta}\dot{\theta} + p_\phi\dot{\phi} - L$ *by assumed Lagrange functionality*

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} = \dot{\theta}$$

Lagrange equation 0

by Lagrange equation 2



$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and φ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\underbrace{\frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)}_{\frac{dH}{dt}} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function H

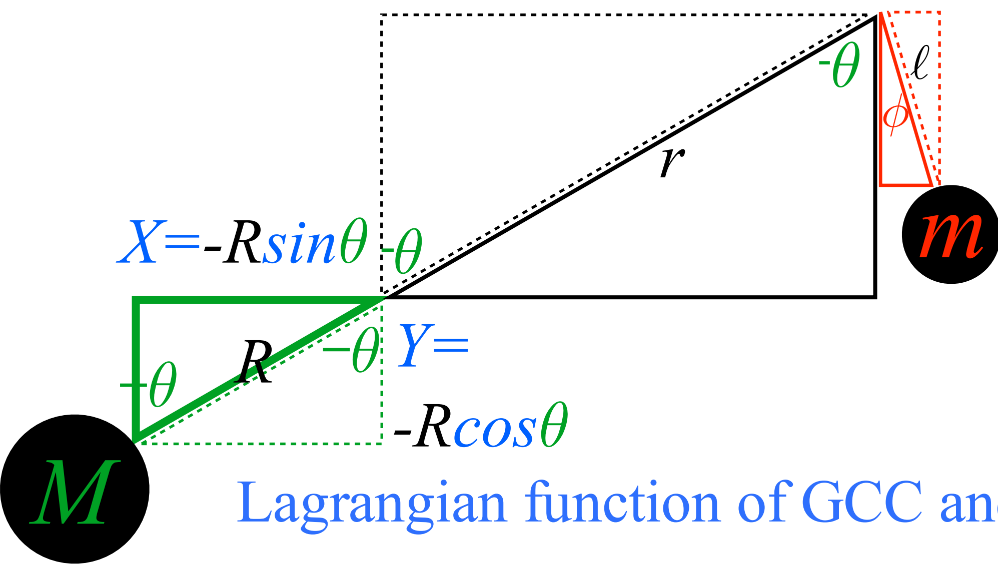
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = \boxed{p_\theta}\dot{\theta} + p_\phi\dot{\phi} - L$ *by assumed Lagrange functionality*

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \qquad \frac{\partial H}{\partial \dot{\theta}} = p_\theta - \frac{\partial L}{\partial \dot{\theta}} = 0$$

Hamilton equation 0

by Lagrange equation1

$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations 1 and 2

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)}_{\frac{dH}{dt}} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the
Hamiltonian function H

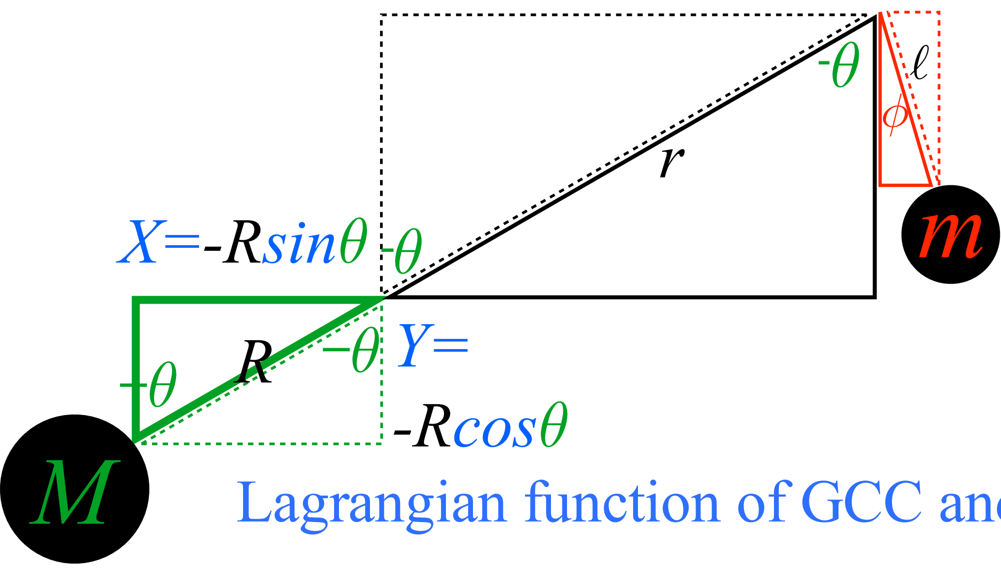
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + \boxed{p_\phi}\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \qquad \frac{\partial H}{\partial \phi} = -\frac{\partial L}{\partial \phi} = -\dot{p}_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \qquad \cancel{\frac{\partial H}{\partial \dot{\phi}}} \equiv 0$$

Hamilton's equations
by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant
Dynamic metric tensor
γ_{mn}
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *velocity chain*

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *Lagrange equations 1 and 2*

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$ *(Consolidating)*

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$ *(Rearranging)*

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$ *Defining the Hamiltonian function H*

Hamiltonian function of GCC and momenta: $\boxed{H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L}$ *Poincare-Legendre relation*

$$\boxed{\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \cancel{\frac{\partial H}{\partial \dot{\theta}}} \equiv 0 \qquad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \cancel{\frac{\partial H}{\partial \dot{\phi}}} \equiv 0}$$ *Hamilton's equations*

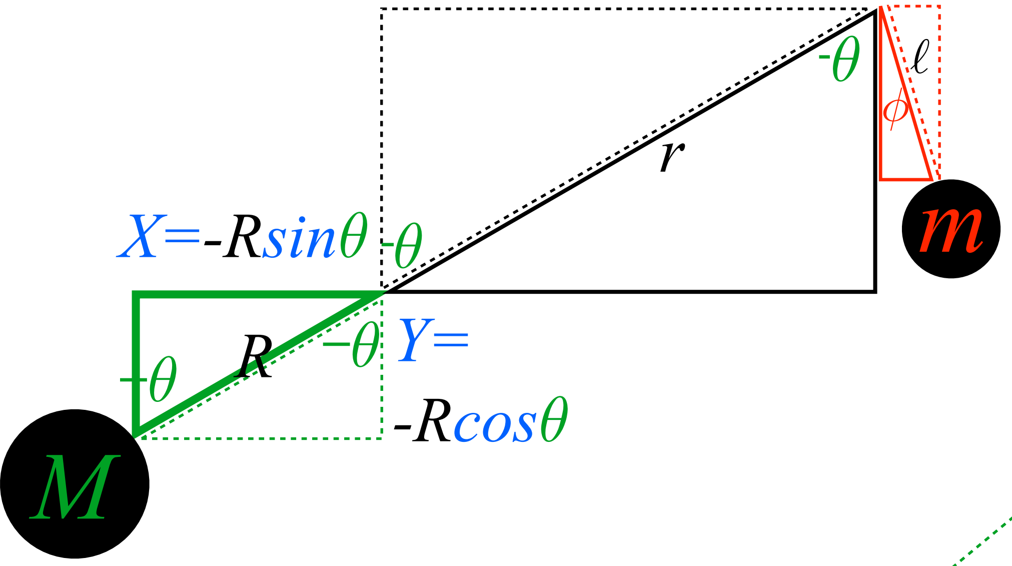
Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression with contravariant γ^{mn} tensor

Hamilton p_μ equation log-jam



$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

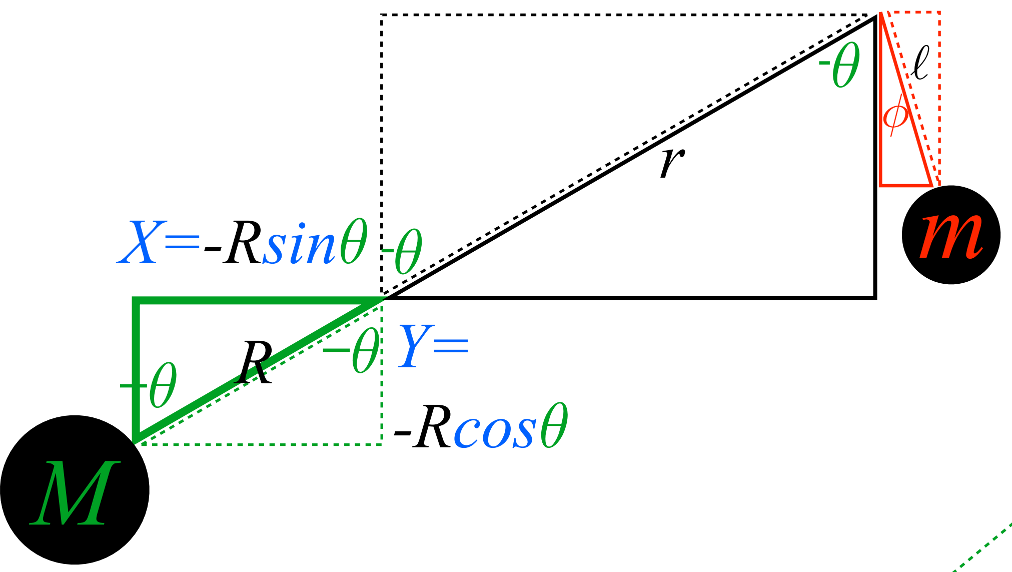
Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

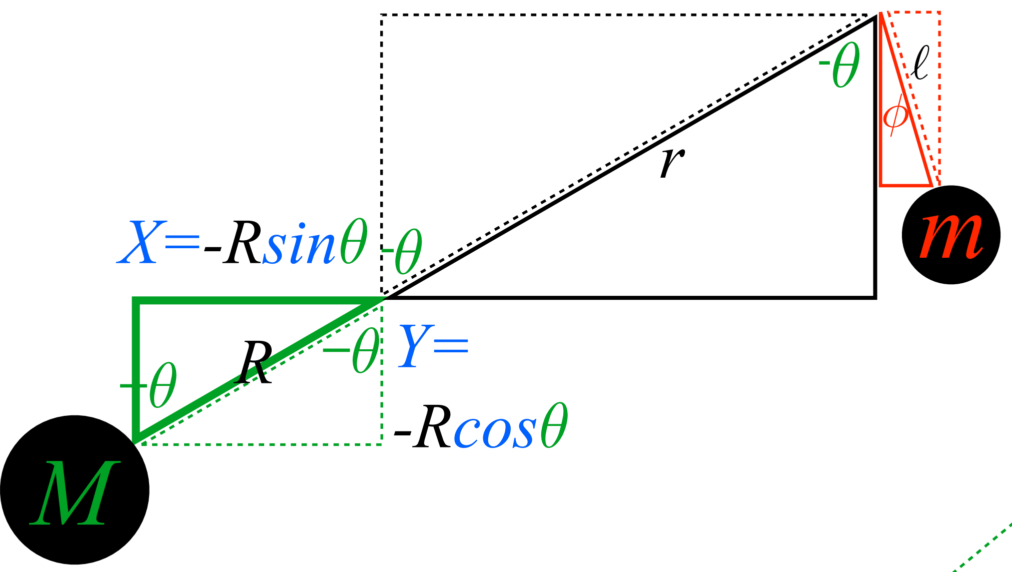
$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

$$\textit{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\,\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

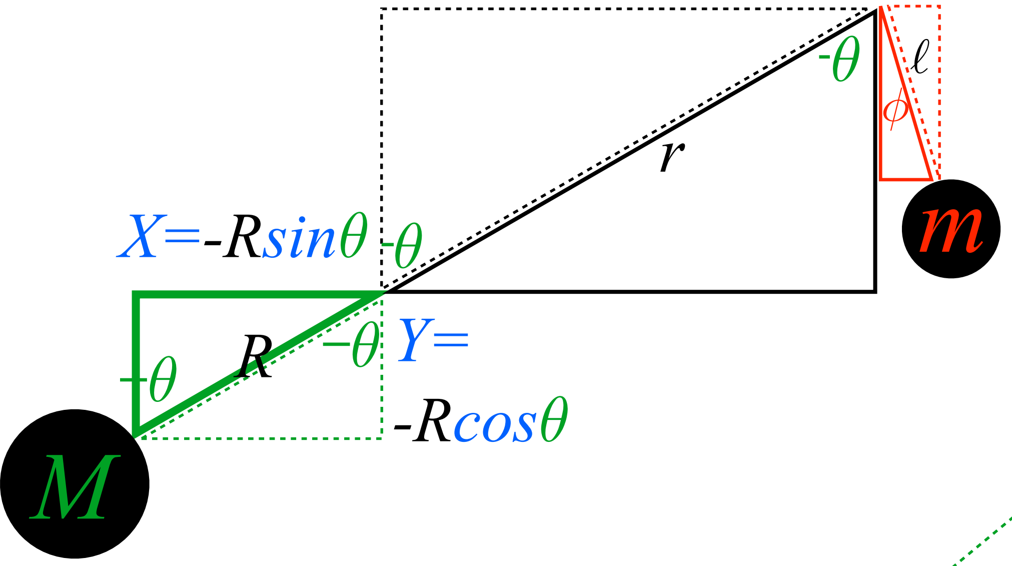
$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

Constant Energy if no explicit t-dependence

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \quad = T + V \equiv E$$

(Only correct numerically!)

Hamiltonian must be explicit function of momenta p_m <u>not</u> velocities



$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}_{} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

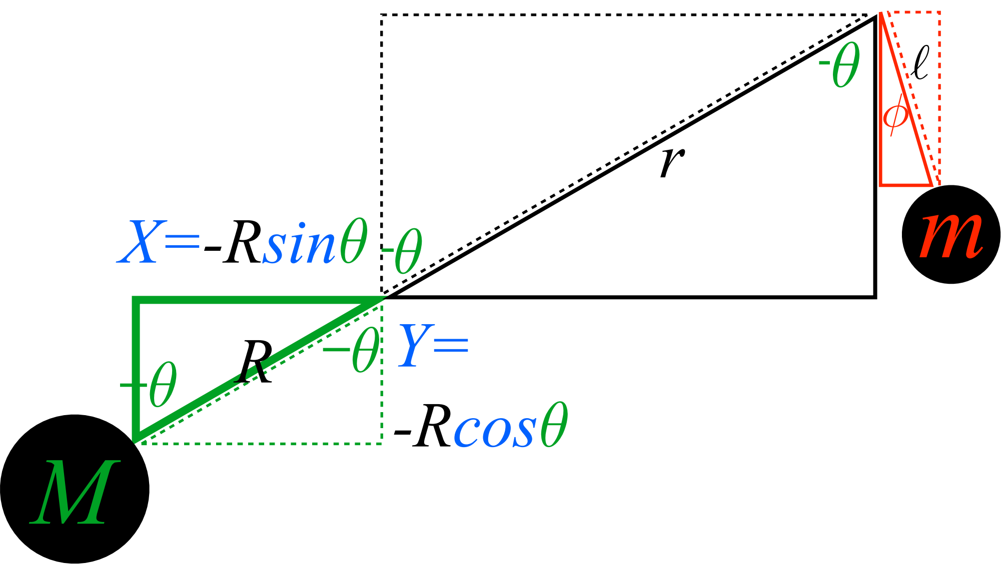
$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

Constant Energy if
no explicit t-dependence

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V \qquad = T + V \equiv E$$ (Only correct numerically!)

Hamiltonian must be explicit function
of momenta p_m not velocities



$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

Constant Energy if no explicit t-dependence

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \quad = T + V \equiv E$$

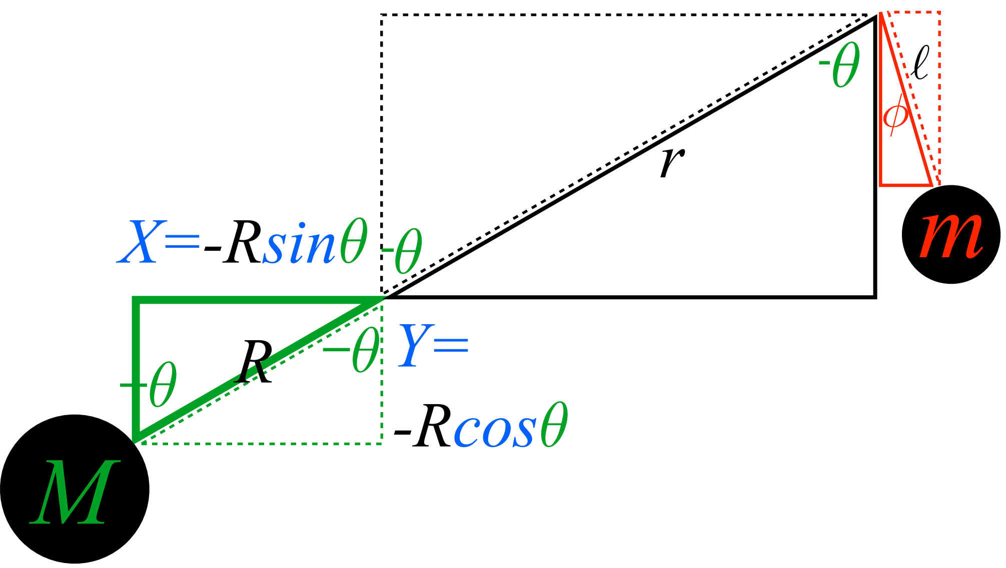
(Only correct numerically!)

Hamiltonian must be explicit function of momenta p_m not velocities

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V \quad = T + V$$

(Correct formally and numerically)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

Constant Energy if no explicit t-dependence

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V = T + V \equiv E$$

(Only correct numerically!)

Hamiltonian must be explicit function of momenta p_m not velocities

$$H = \frac{1}{2}(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi) + V = T + V$$

(Correct formally and numerically)

$$H = \frac{m\ell^2 p_\theta p_\theta + 2mr\ell\cos(\theta-\phi)p_\theta p_\phi + (MR^2 + mr^2)p_\phi p_\phi}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} + V$$

Review of Hamiltonian equation derivation (Elementary trebuchet)

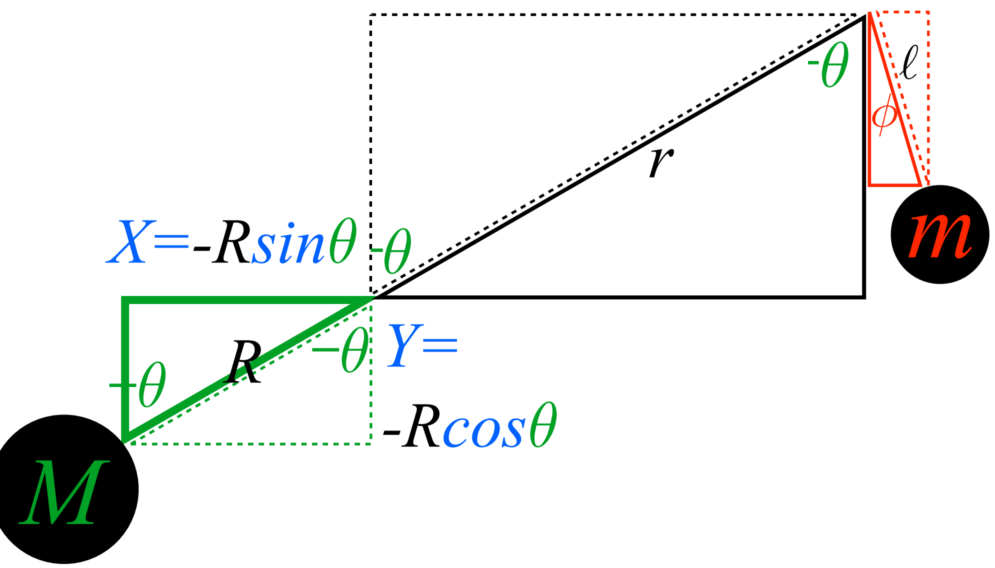
Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression with contravariant γ^{mn} tensor

→ *Hamilton p_{μ} equation log-jam*

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$\boxed{\begin{aligned} \dot\theta &= \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi \\ \dot\phi &= \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \end{aligned}}$$

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta-\phi) \right]} = \frac{1}{2} \gamma^{mn} p_m p_n$$

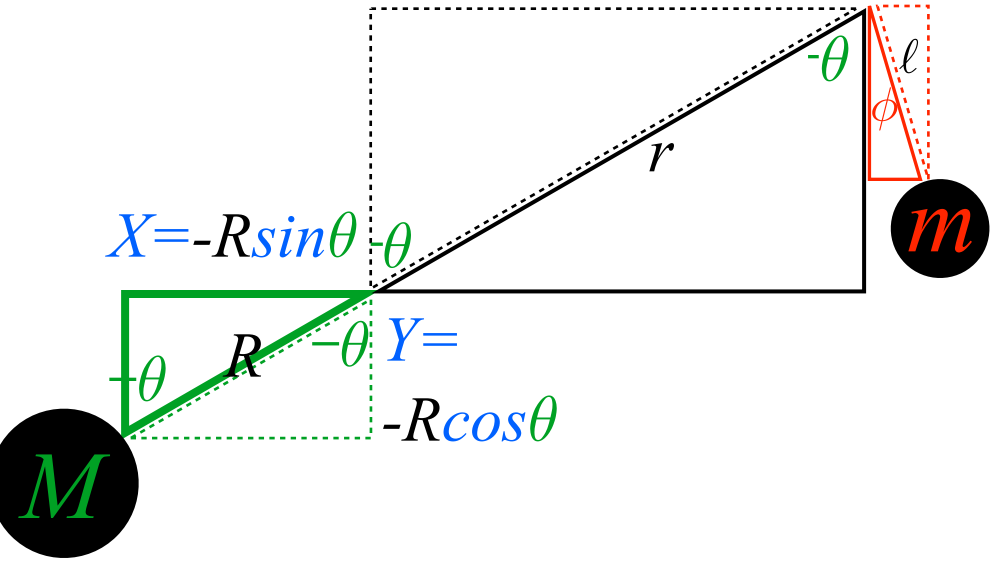
$$\boxed{\begin{aligned} p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi \\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi \end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot\theta&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot\phi&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot\theta+\gamma_{\theta\phi}\dot\phi\\ p_\phi&=\gamma_{\phi\theta}\dot\theta+\gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

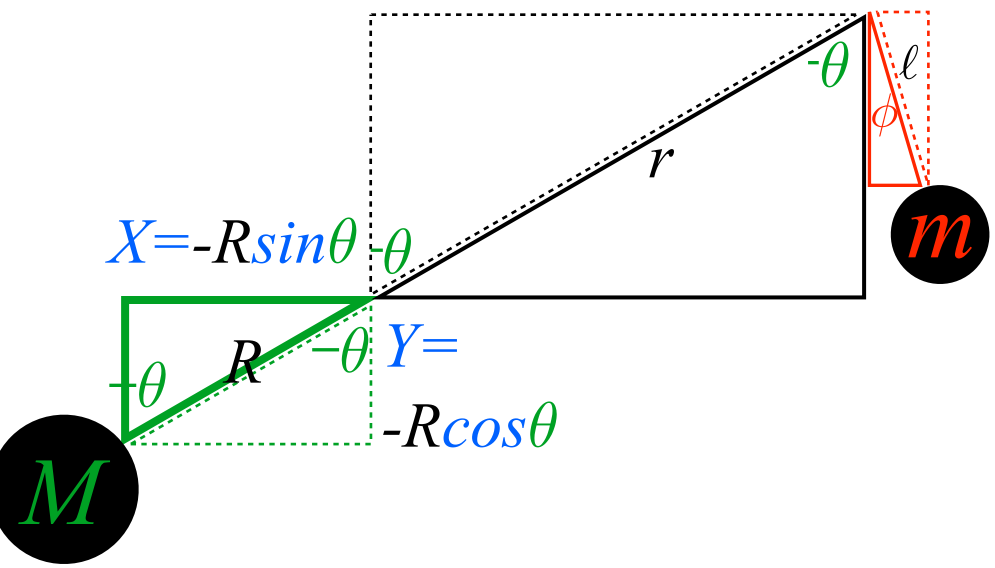
Momentum/force equations

$$\dot p_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$
$$=mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$
$$=\ -mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned} \dot\theta &= \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi \\ \dot\phi &= \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi \end{aligned}}$$

$$\boxed{\begin{aligned} p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi \\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi \end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

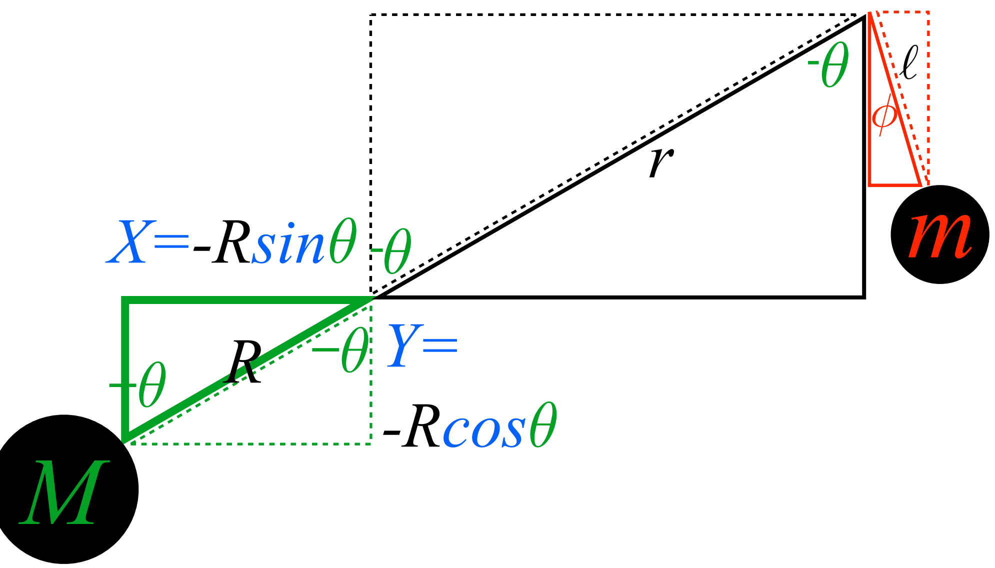
Momentum/force equations

$$\begin{aligned} \dot p_\theta &= -\frac{\partial H}{\partial\theta} = \frac{\partial L}{\partial\theta} = \frac{\partial T}{\partial\theta} - \frac{\partial V}{\partial\theta} \\ &= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta \\ &= mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\theta \end{aligned}$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\begin{aligned} \dot p_\phi &= -\frac{\partial H}{\partial\phi} = \frac{\partial L}{\partial\phi} = \frac{\partial T}{\partial\phi} - \frac{\partial V}{\partial\phi} \\ &= -mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi \\ &= -mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\phi \end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$\boxed{\begin{aligned}\dot\theta &= \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi\\ \dot\phi &= \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$T = \frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi\\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi \qquad\qquad \frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

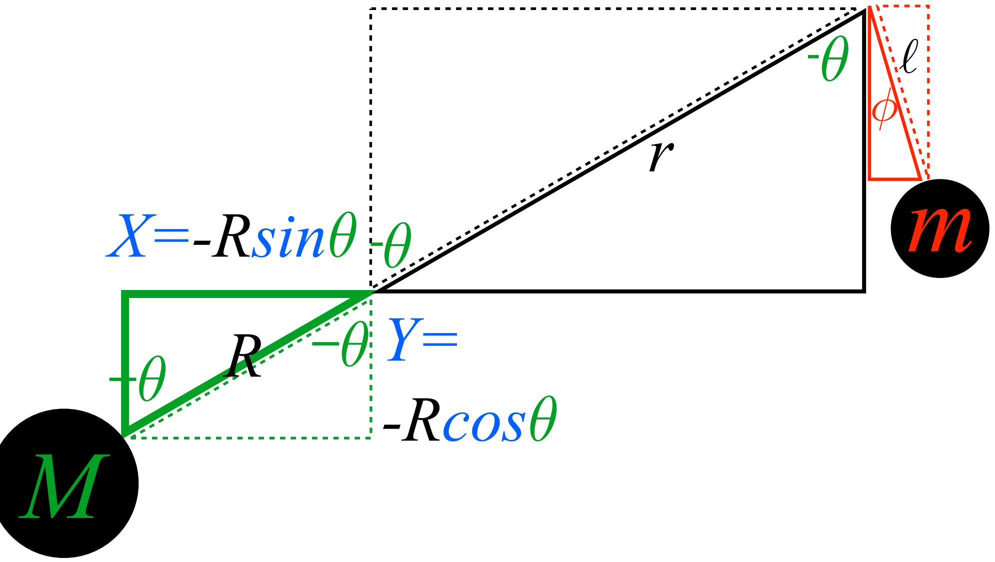
Momentum/force equations

$$\begin{aligned}\dot p_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}\\ &= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta\\ &= mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\theta\\ &= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2]p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi) + F_\theta\end{aligned}$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\begin{aligned}\dot p_\phi &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}\\ &= -mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi\\ &= -mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\phi\\ &= -[\overset{\text{same}}{\text{messy}}\text{ factor}]\sin(\theta-\phi) + F_\phi\end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\,\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot{\theta} &= \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi\\ \dot{\phi} &= \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta &= \gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\\ p_\phi &= \gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Momentum/force equations

$$\begin{aligned}\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}\\ &= mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + F_\theta\\ &= mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\theta\\ &= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2]p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi) + F_\theta\end{aligned}$$

$$\begin{aligned}\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}\\ &= -mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + F_\phi\\ &= -mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\phi\\ &= -[\overset{\text{same}}{\text{messy factor}}]\sin(\theta-\phi) + F_\phi\end{aligned}$$

A lesson in Hamiltonian "elegance"...
...may be elegant formally...but not be so elegant algebraically!

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta + \pi/2$

or $\phi_B = -\theta + \phi - \pi/2$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

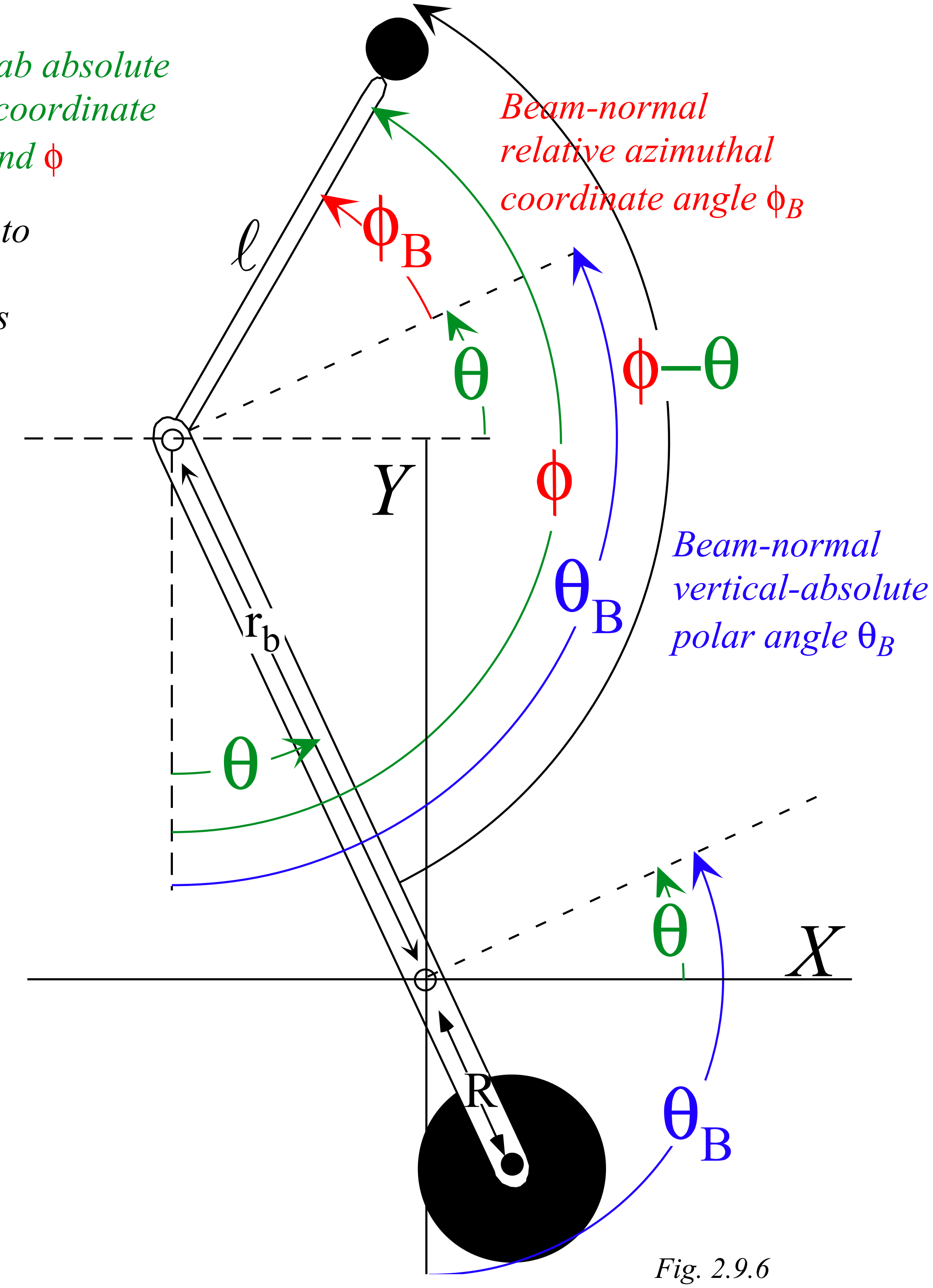


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Lemma-1 from Lect.9 p.13 used in Lect.14 p.63

$$\begin{pmatrix} \frac{\partial\dot{\theta}_B}{\partial\dot{\theta}} & \frac{\partial\dot{\theta}_B}{\partial\dot{\phi}} \\ \frac{\partial\dot{\phi}_B}{\partial\dot{\theta}} & \frac{\partial\dot{\phi}_B}{\partial\dot{\phi}} \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix}$$

(Same Jacobian for coordinates *and* velocities)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

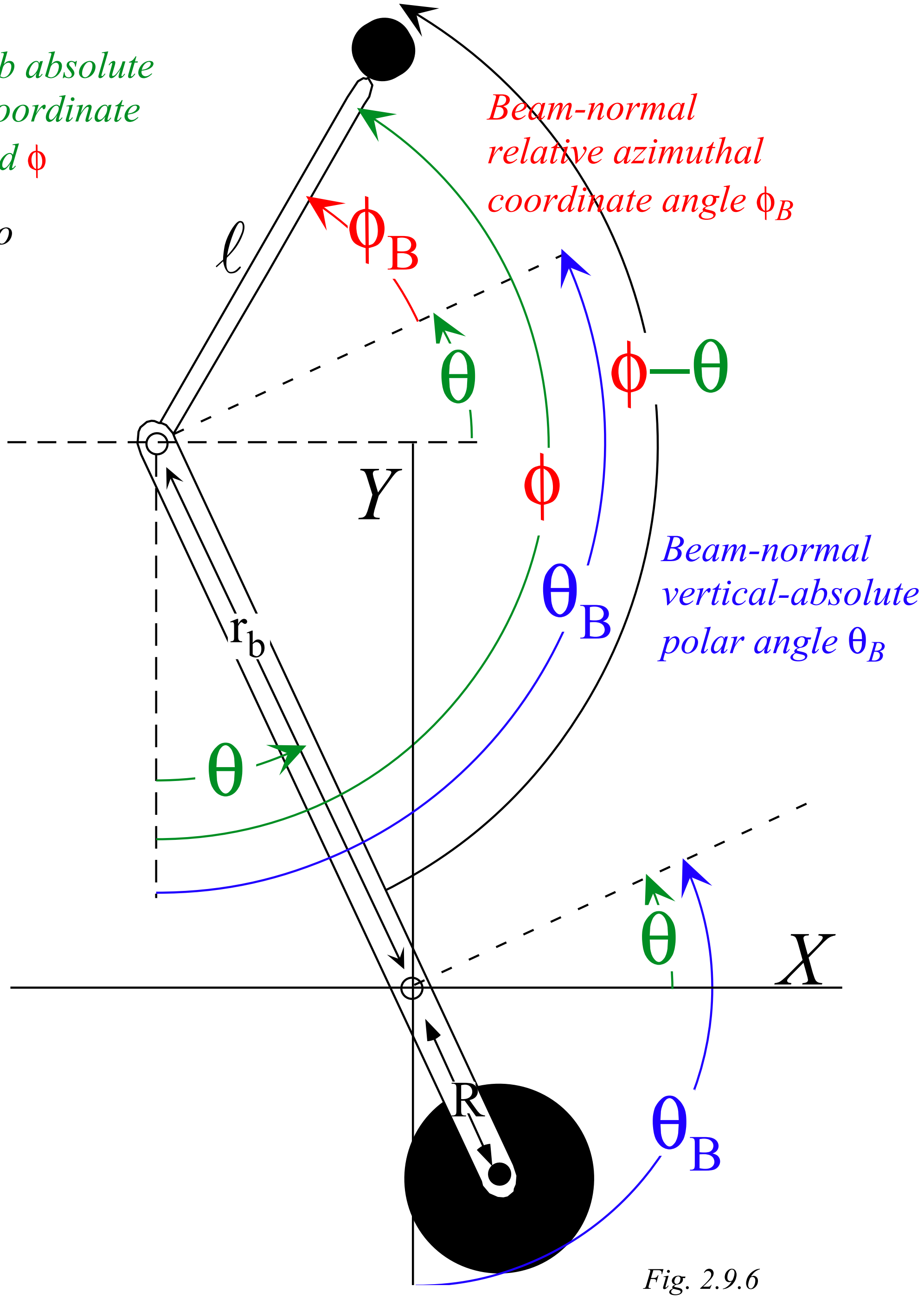


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

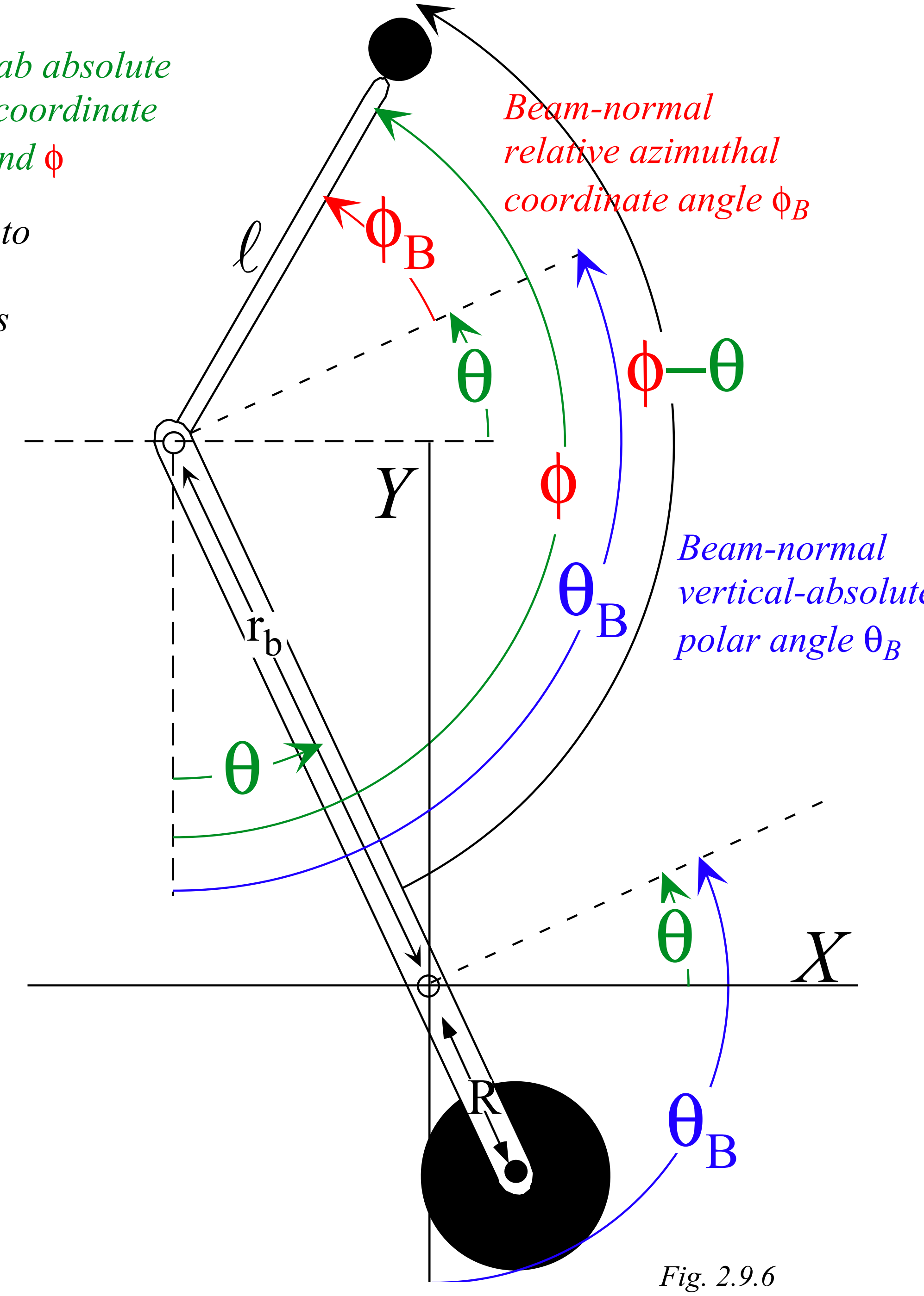


Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

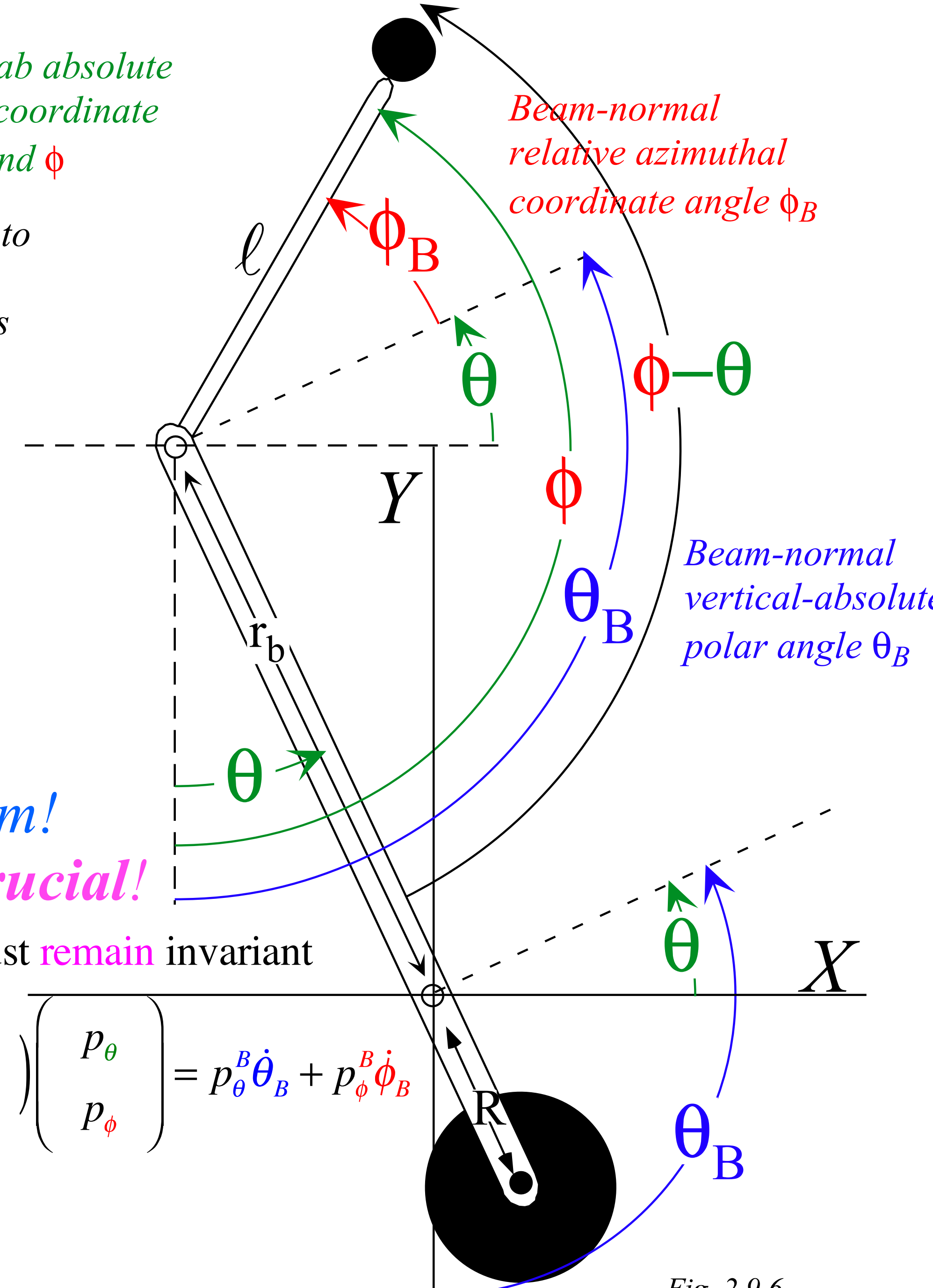
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Be careful with momentum!

Poincare invariance is ***crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m transform

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \phi}{\partial \theta_B} \\ \frac{\partial \theta}{\partial \phi_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \theta} \\ \frac{\partial \theta_B}{\partial \phi} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

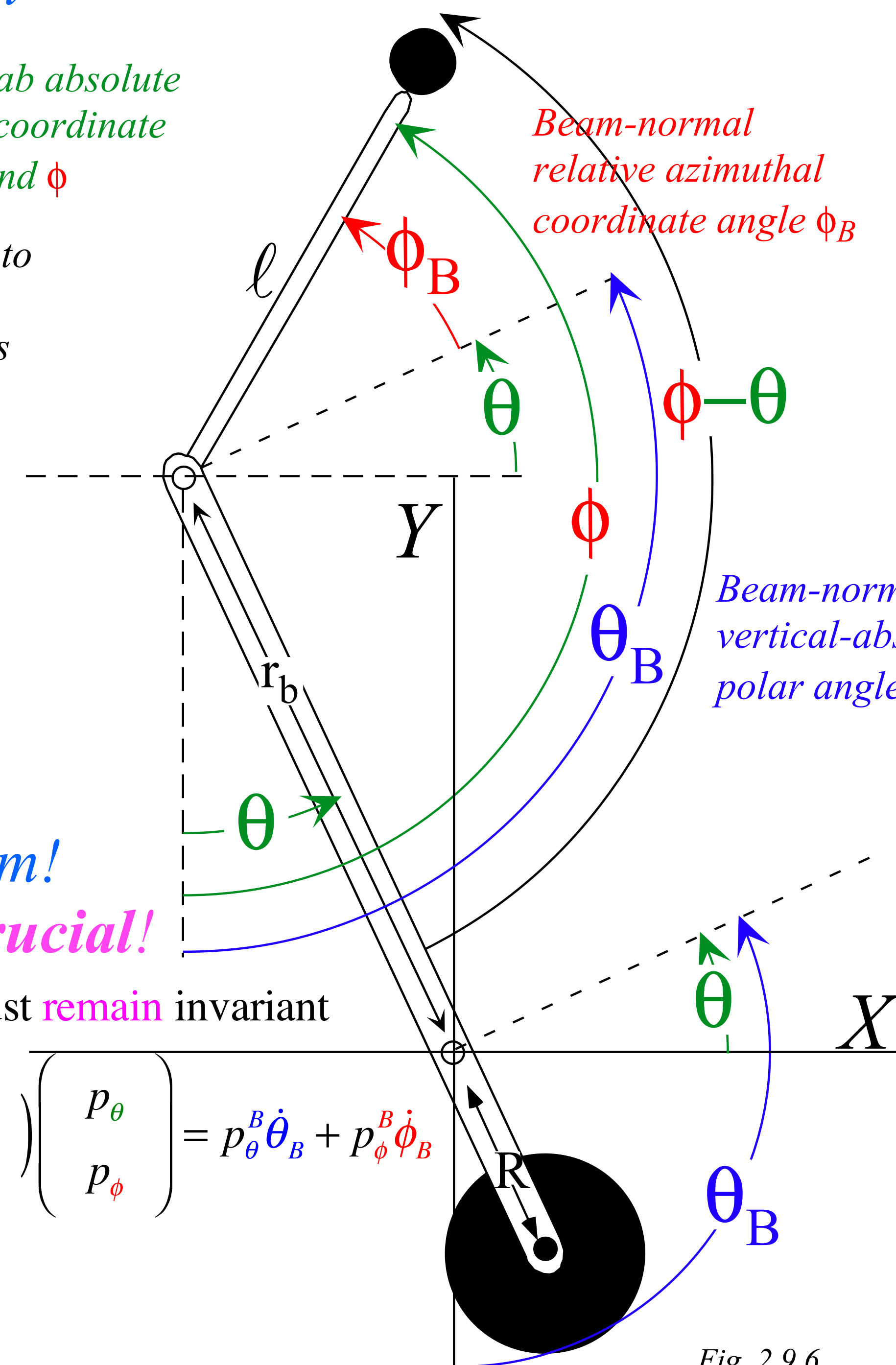


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Be careful with momentum!
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta \dot{\theta} + p_\phi \dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta \dot{\theta}_B + p^B_\phi \dot{\phi}_B$$

From Lect.14 p.98: $\quad \dot{\phi}, \dot{\theta} \qquad \dot{\phi}_B, \dot{\theta}_B$

Transform contravariant velocity $\left\{\dot{q}^1, \dot{q}^2, \cdots\right\}$ to "barred" $\left\{\dot{\bar{q}}^{\bar{1}}, \dot{\bar{q}}^{\bar{2}}, \cdots\right\}$ by *"chain-saw-sum rule"*

$$\mathbf{E}^{\mathbf{m}} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{} \frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \text{ or: } \mathbf{E}^{\mathbf{m}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \bar{\mathbf{E}}^{\bar{\mathbf{m}}}$$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

p_m transform is TRANSPOSE INVERSE to q^m transform

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Be careful with momentum!

*Poincare invariance is **crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

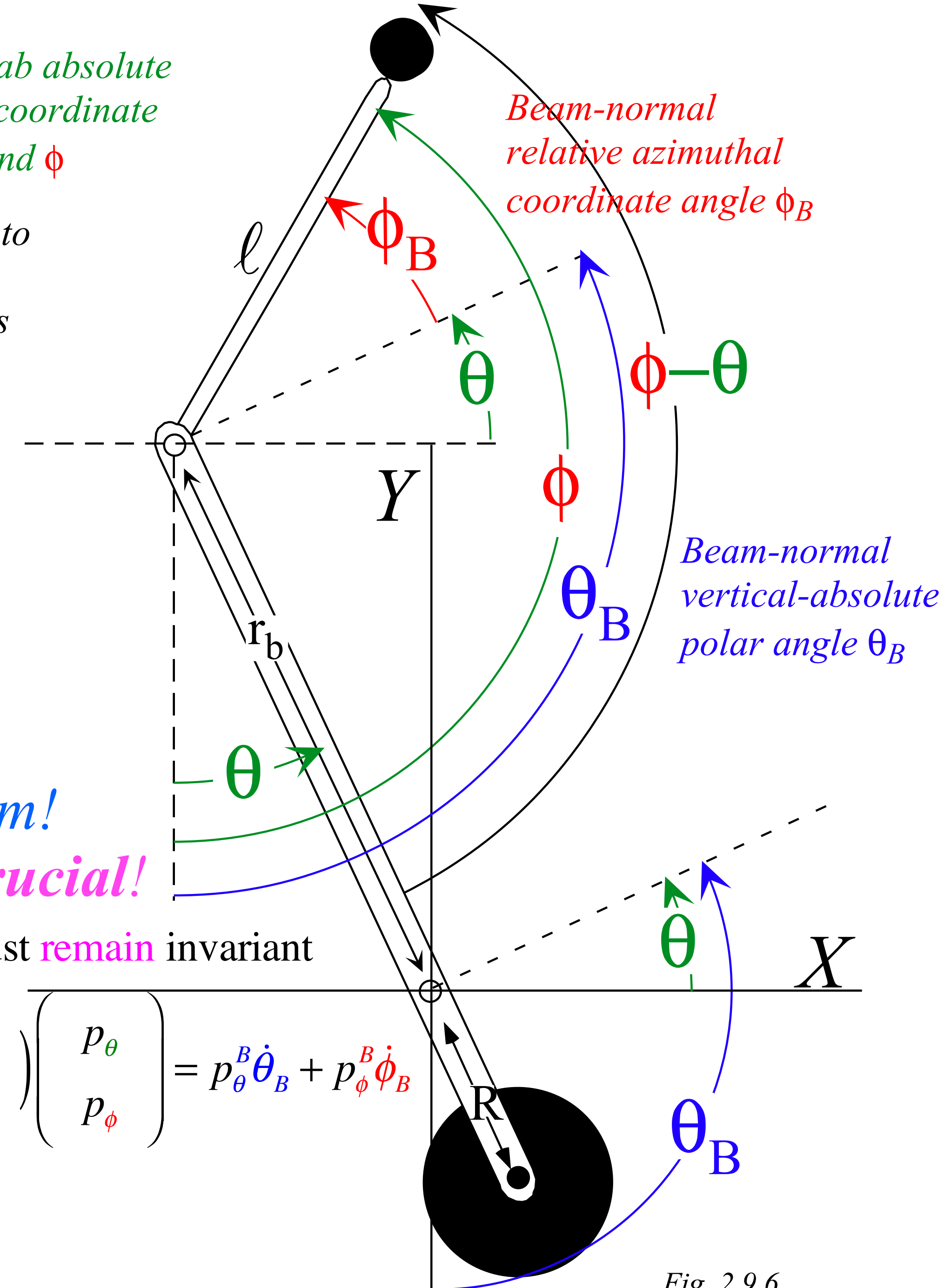


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

From Lect.14 p.98:

$\dot\phi$, $\dot\theta$ $\qquad$ $\dot\phi_B$, $\dot\theta_B$

Transform contravariant velocity $\{\dot q^1, \dot q^2, \cdots\}$ to "barred" $\{\dot{\bar q}^{\bar 1}, \dot{\bar q}^{\bar 2}, \cdots\}$ by *"chain-saw-sum rule"*

$$V^m = \mathbf{V}\bullet\mathbf{E^m} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\mathbf{V}\bullet\mathbf{\bar E^{\bar m}} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\bar V^{\bar m}$$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\frac{\partial \bar q^{\bar m}}{\partial \mathbf{r}}, \text{ or: } \mathbf{E^m} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\mathbf{\bar E^{\bar m}}$$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m transform

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Be careful with momentum!
Poincare invariance is *crucial!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

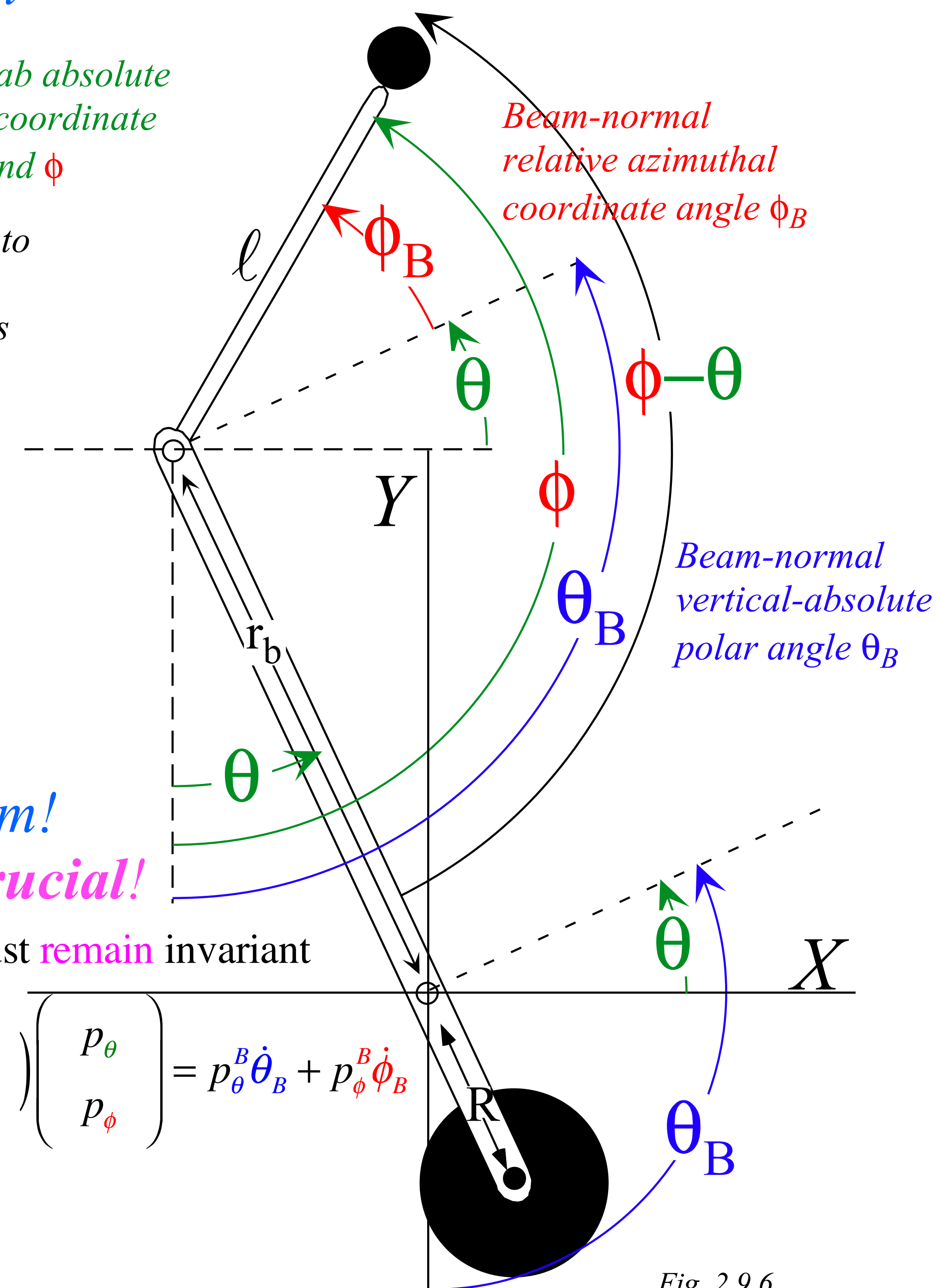


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

From Lect.14 p.98:

$\dot\phi$, $\dot\theta$ $\qquad\qquad$ $\dot\phi_B$, $\dot\theta_B$

Transform contravariant velocity $\{\dot q^1, \dot q^2, \cdots\}$ to "barred" $\{\dot{\bar q}^{\bar 1}, \dot{\bar q}^{\bar 2}, \cdots\}$ by *"chain-saw-sum rule"*

$$V^m = \mathbf{V}\bullet\mathbf{E}^m = \frac{\partial q^m}{\partial \bar q^{\bar m}}\mathbf{V}\bullet\bar{\mathbf{E}}^{\bar m} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\bar V^{\bar m}$$

$$\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\frac{\partial \bar q^{\bar m}}{\partial \mathbf{r}}, \text{ or: } \mathbf{E}^m = \frac{\partial q^m}{\partial \bar q^{\bar m}}\bar{\mathbf{E}}^{\bar m}$$

Transform covariant momentum $\{p_1, p_2, \cdots\}$ to "barred" $\{\bar p_{\bar 1}, \bar p_{\bar 2}, \cdots\}$ by *"chain-saw-sum rule"*

p_θ , p_ϕ $\qquad\qquad$ p^B_θ , p^B_ϕ

$$\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar q^{\bar m}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar q^{\bar m}}, \text{ or: } \mathbf{E}_m = \frac{\partial \bar q^{\bar m}}{\partial q^m}\bar{\mathbf{E}}_{\bar m}$$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\quad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m transform

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \phi}{\partial \theta_B} \\ \frac{\partial \theta}{\partial \phi_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \theta} \\ \frac{\partial \theta_B}{\partial \phi} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

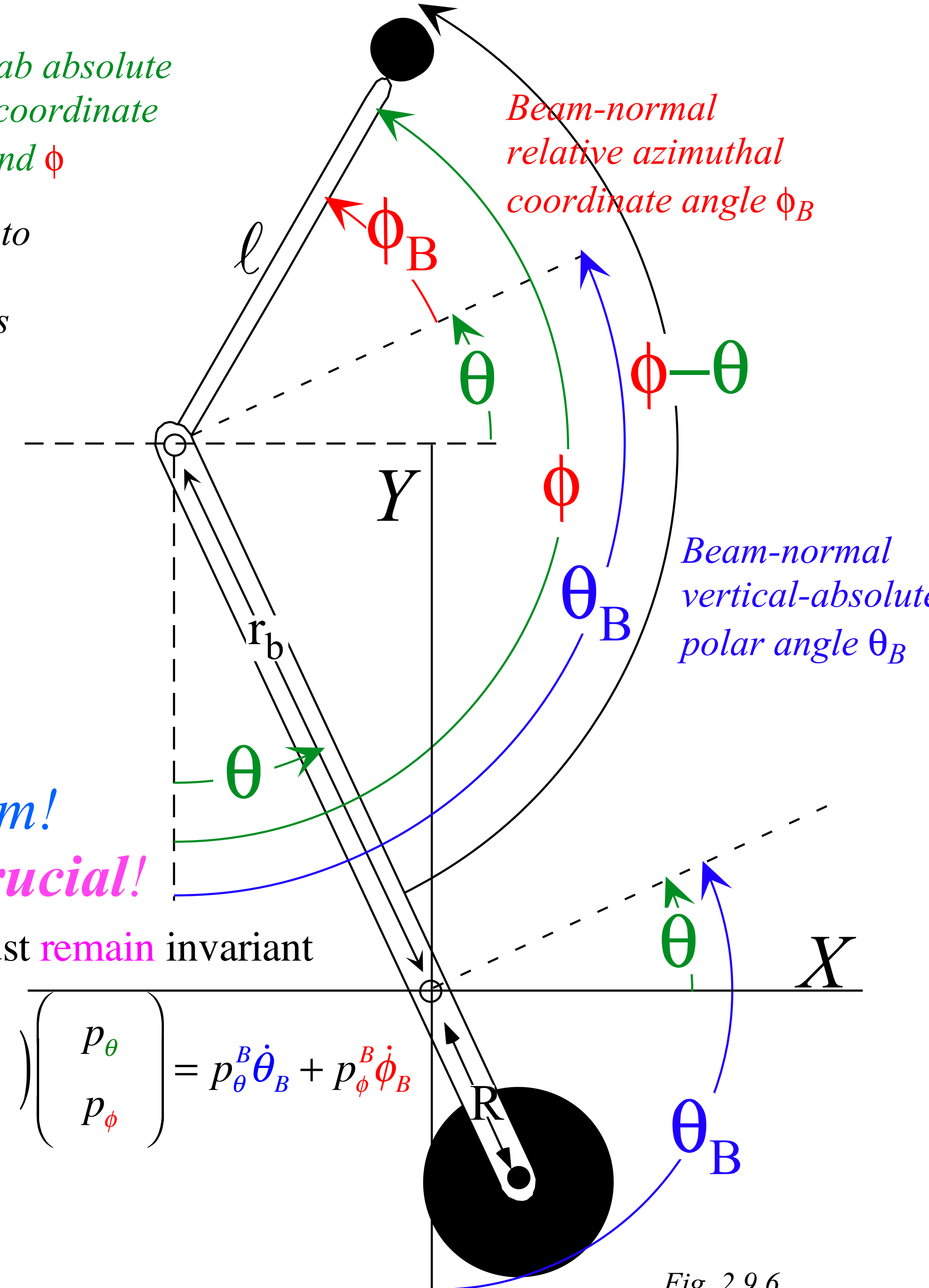


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Be careful with momentum!
Poincare invariance is *crucial!*

Poincare invariant must remain invariant

$$p_\theta \dot\theta + p_\phi \dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta \dot\theta_B + p^B_\phi \dot\phi_B$$

From Lect.14 p.98:

Transform contravariant velocity $\{\dot q^1, \dot q^2, \cdots\}$ (i.e. $\dot\phi, \dot\theta$) to "barred" $\{\dot{\bar q}^{\bar 1}, \dot{\bar q}^{\bar 2}, \cdots\}$ (i.e. $\dot\phi_B, \dot\theta_B$) by *"chain-saw-sum rule"*

$$V^m = \mathbf{V}\bullet\mathbf{E}^m = \frac{\partial q^m}{\partial \bar q^{\bar m}} \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar m} = \frac{\partial q^m}{\partial \bar q^{\bar m}} \bar V^{\bar m}$$

$$\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad} \frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar q^{\bar m}} \frac{\partial \bar q^{\bar m}}{\partial \mathbf{r}}, \text{ or: } \mathbf{E}^m = \frac{\partial q^m}{\partial \bar q^{\bar m}} \bar{\mathbf{E}}^{\bar m}$$

Transform covariant momentum $\{p_1, p_2, \cdots\}$ (i.e. p_θ, p_ϕ) to "barred" $\{\bar p_{\bar 1}, \bar p_{\bar 2}, \cdots\}$ (i.e. p^B_θ, p^B_ϕ) by *"chain-saw-sum rule"*

$$p_m = \mathbf{p}\bullet\mathbf{E}_m = \frac{\partial \bar q^{\bar m}}{\partial q^m} \mathbf{p}\bullet\bar{\mathbf{E}}_{\bar m} = \frac{\partial \bar q^{\bar m}}{\partial q^m} \bar p_{\bar m}$$

$$\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m} \frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar q^{\bar m}}{\partial q^m} \frac{\partial \mathbf{r}}{\partial \bar q^{\bar m}}, \text{ or: } \mathbf{E}_m = \frac{\partial \bar q^{\bar m}}{\partial q^m} \bar{\mathbf{E}}_{\bar m}$$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

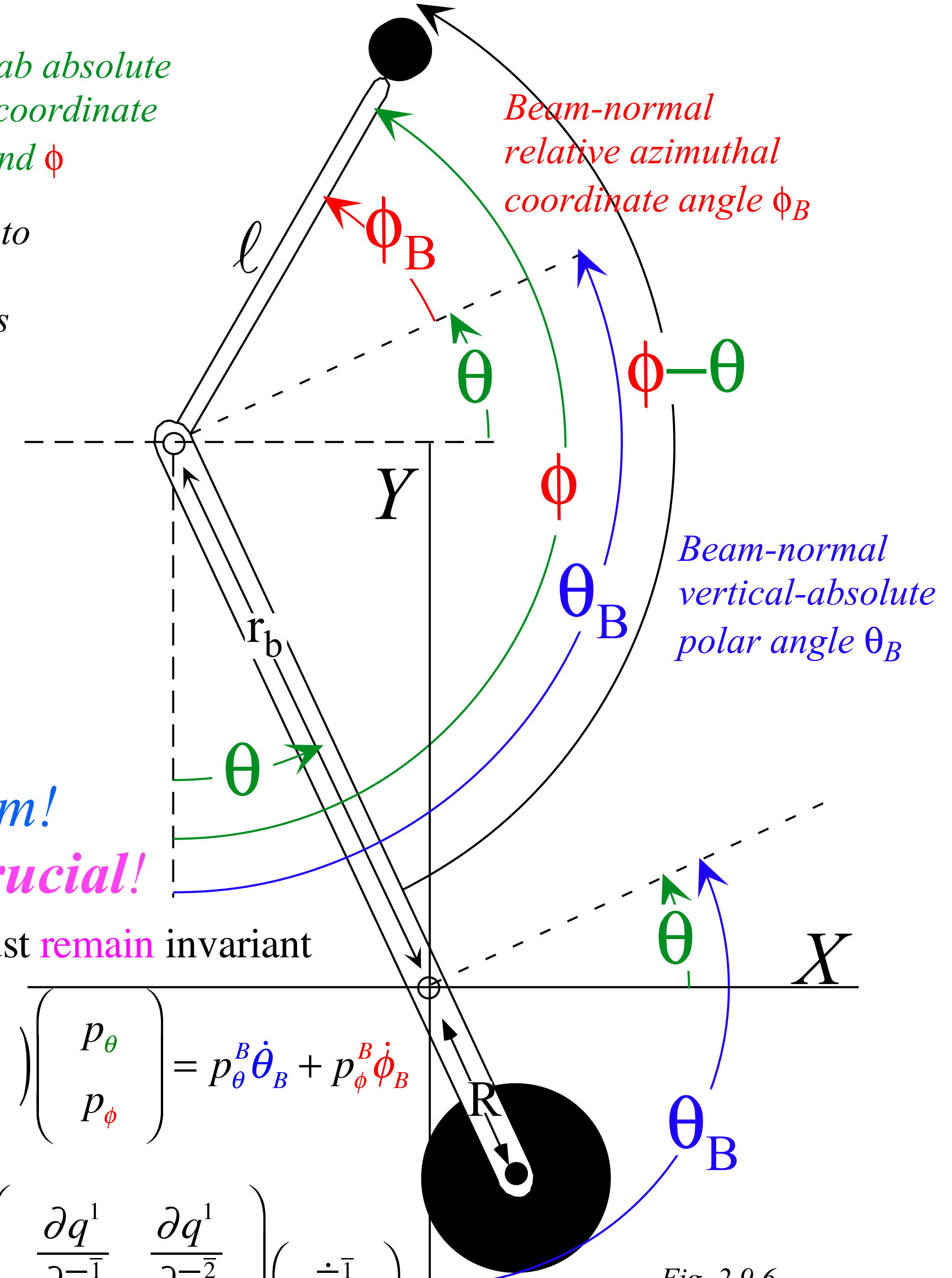
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



p_m transform is TRANSPOSE INVERSE to q^m transform

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Be careful with momentum!
Poincare invariance is ***crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

From Lect.14 p.98:

$\dot\phi$, $\dot\theta$ $\qquad$ $\dot\phi_B$, $\dot\theta_B$

Transform contravariant velocity $\{\dot q^1, \dot q^2, \cdots\}$ to "barred" $\{\dot{\bar q}^{\bar 1}, \dot{\bar q}^{\bar 2}, \cdots\}$

$$V^m = \mathbf{V}\bullet\mathbf{E^m} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\mathbf{V}\bullet\mathbf{\bar E^{\bar m}} = \frac{\partial q^m}{\partial \bar q^{\bar m}}\bar V^{\bar m}$$

by matrix:

$$\begin{pmatrix} \dot q^1 \\ \dot q^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial q^1}{\partial \bar q^{\bar 1}} & \frac{\partial q^1}{\partial \bar q^{\bar 2}} \\ \frac{\partial q^2}{\partial \bar q^{\bar 1}} & \frac{\partial q^2}{\partial \bar q^{\bar 2}} \end{pmatrix} \begin{pmatrix} \dot{\bar q}^{\bar 1} \\ \dot{\bar q}^{\bar 2} \end{pmatrix}$$

Transform covariant momentum $\{p_1, p_2, \cdots\}$ to "barred" $\{\bar p_{\bar 1}, \bar p_{\bar 2}, \cdots\}$

p_θ , p_ϕ $\qquad$ p^B_θ , p^B_ϕ

$$p_m = \mathbf{p}\bullet\mathbf{E_m} = \frac{\partial \bar q^{\bar m}}{\partial q^m}\mathbf{p}\bullet\mathbf{\bar E_{\bar m}} = \frac{\partial \bar q^{\bar m}}{\partial q^m}\bar p_{\bar m}$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar q^{\bar 1}}{\partial q^1} & \frac{\partial \bar q^{\bar 2}}{\partial q^1} \\ \frac{\partial \bar q^{\bar 1}}{\partial q^2} & \frac{\partial \bar q^{\bar 2}}{\partial q^2} \end{pmatrix} \begin{pmatrix} \bar p_{\bar 1} \\ \bar p_{\bar 2} \end{pmatrix}$$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m transform*

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

*Poincare invariance is **crucial**!*

Be careful with momentum!
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot\theta_B + p_\phi^B\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

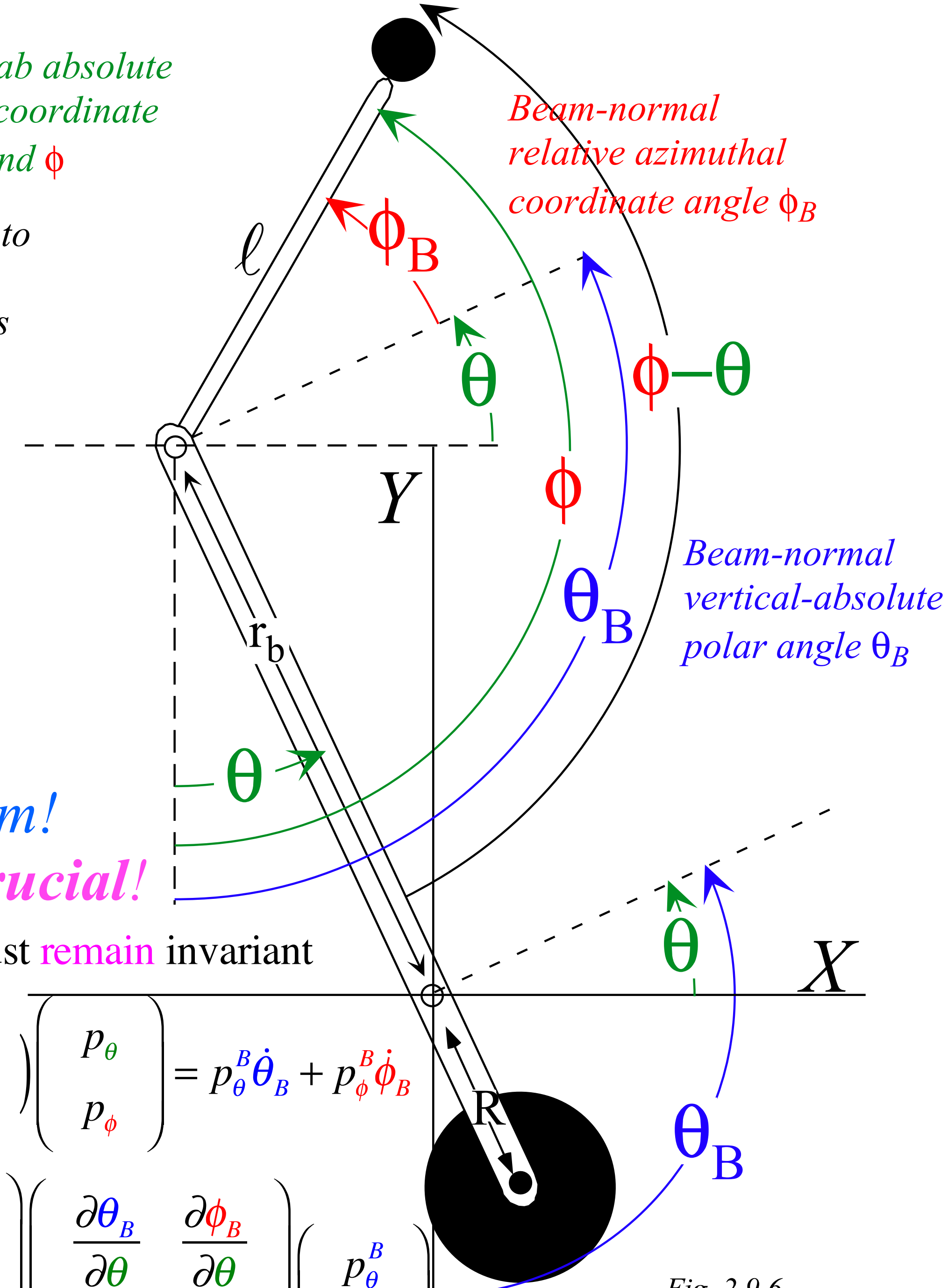


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \; -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m transform

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

$p_\phi = \quad p^B_\phi$

Be careful with momentum!

*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

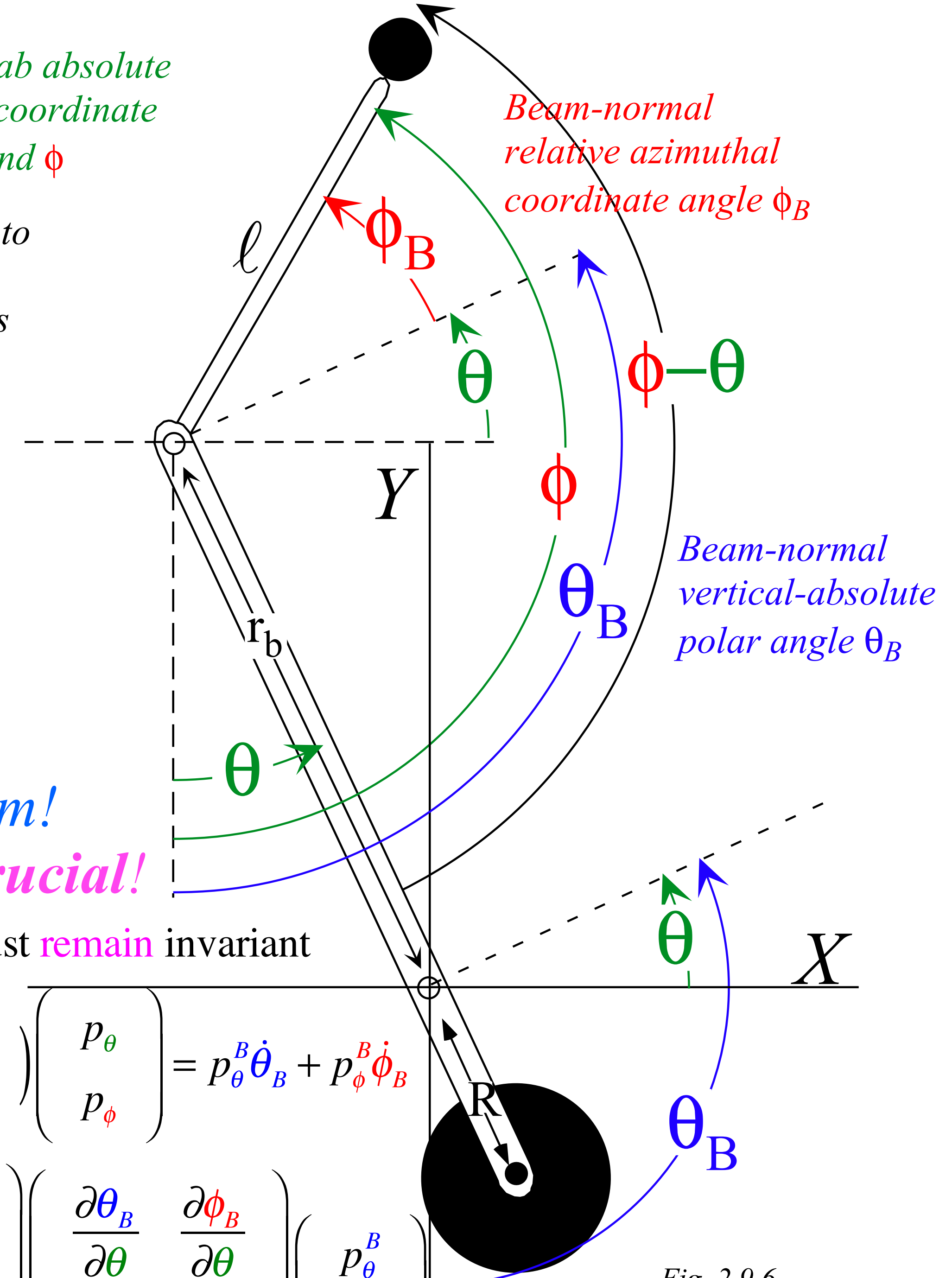


Fig. 2.9.6

Lab (θ,φ) and beam-normal (θB,φB) relative coordinates for trebuchet. (Each value is positive.)

*Poincare invariance is **crucial**!*

Recall Lect.12 p.50 discussion of differential Action S

$$dS = L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt$$

...gives quantum phase... $\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r} - H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r} - \omega\cdot t)}$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \phi}{\partial \theta_B} \\ \frac{\partial \theta}{\partial \phi_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \theta} \\ \frac{\partial \theta_B}{\partial \phi} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

Be careful with momentum!
Poincare invariance is *crucial*!

Poincare invariant must remain invariant

$$p_\theta \dot{\theta} + p_\phi \dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B \dot{\theta}_B + p_\phi^B \dot{\phi}_B$$

$$\begin{pmatrix} \dot{\theta}_B & \dot{\phi}_B \end{pmatrix} \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \phi}{\partial \theta_B} \\ \frac{\partial \theta}{\partial \phi_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \theta} \\ \frac{\partial \theta_B}{\partial \phi} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

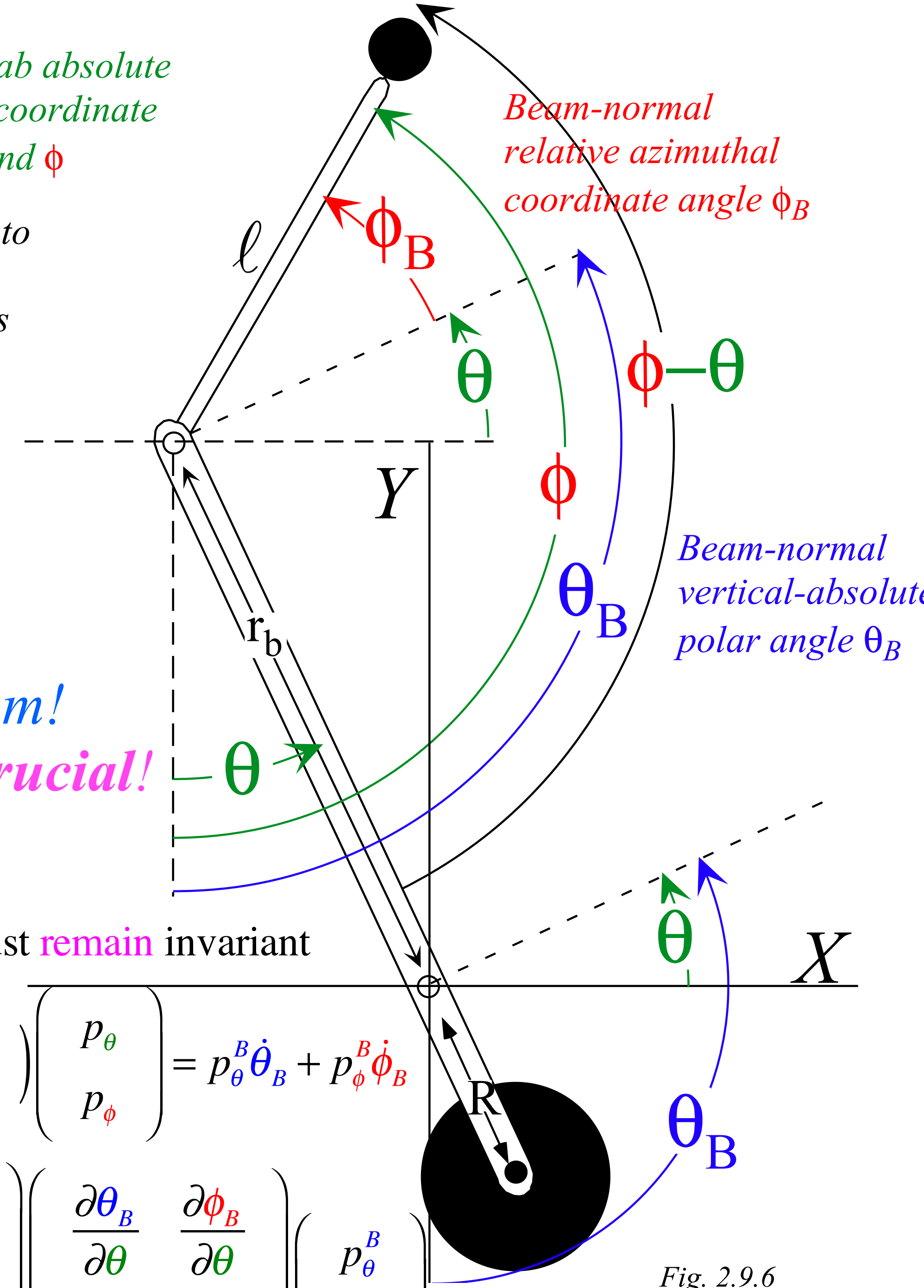


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta - \phi) \right]} + V$$

Original (φ,θ) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

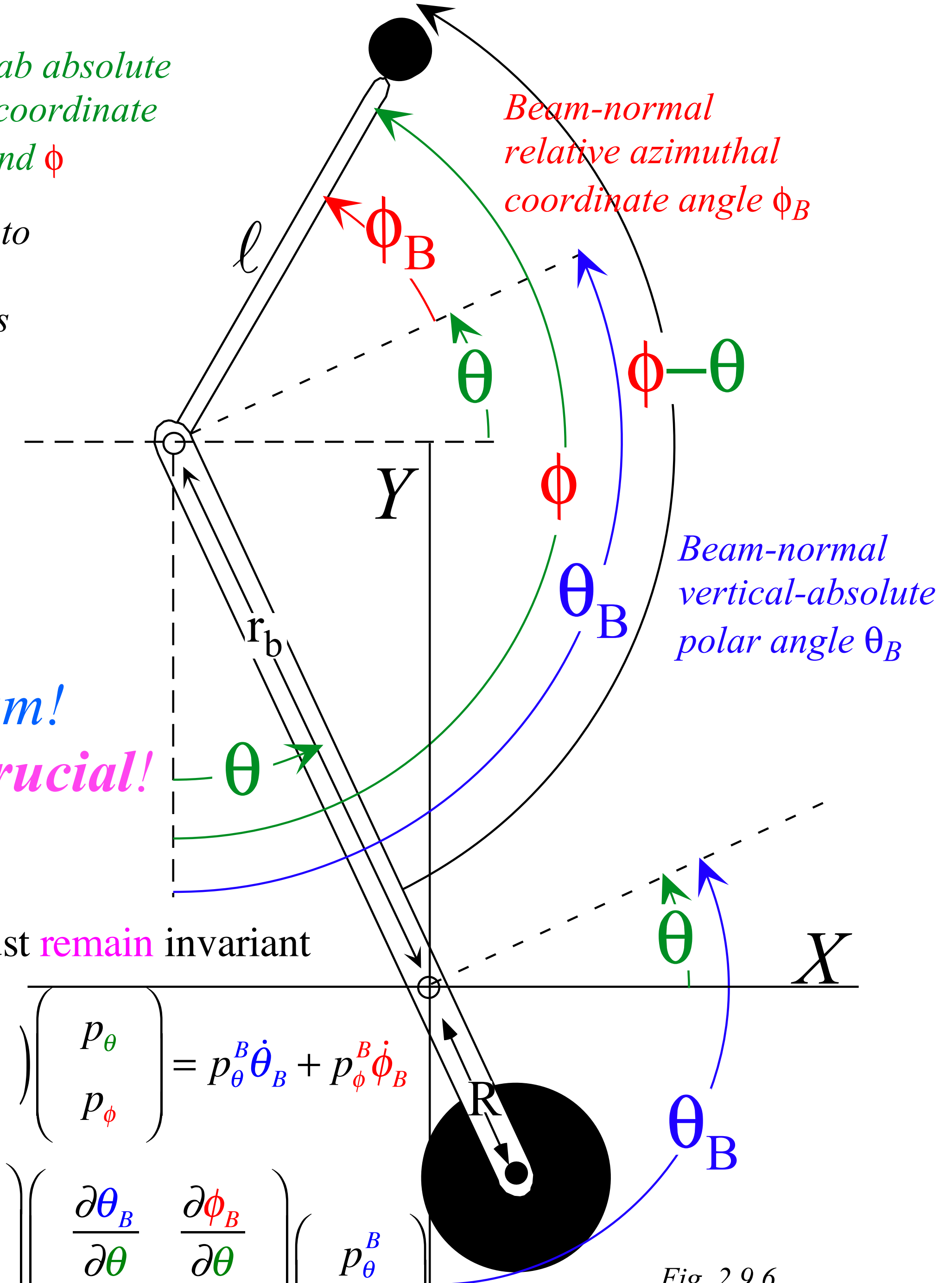
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

$p_\phi = \quad p^B_\phi$

Be careful with momentum!

*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta - \phi) \right]} + V$$

Original (φ,θ) Hamiltonian

(Use $\phi_B = \pi/2 - (\theta - \phi)$)

Transformed (ϕ_B,θ_B) Hamiltonian

$$H = \frac{m\ell^2 \left(p^B_\theta - p^B_\phi \right)^2 + \left(MR^2 + mr^2 \right)\left(p^B_\phi \right)^2 - 2mr\ell p^B_\phi \left(p^B_\theta - p^B_\phi \right) \sin\phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2\phi_B \right]} + V$$

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$$\phi = \theta_B + \phi_B$$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is **TRANSPOSE INVERSE** *to q^m*

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

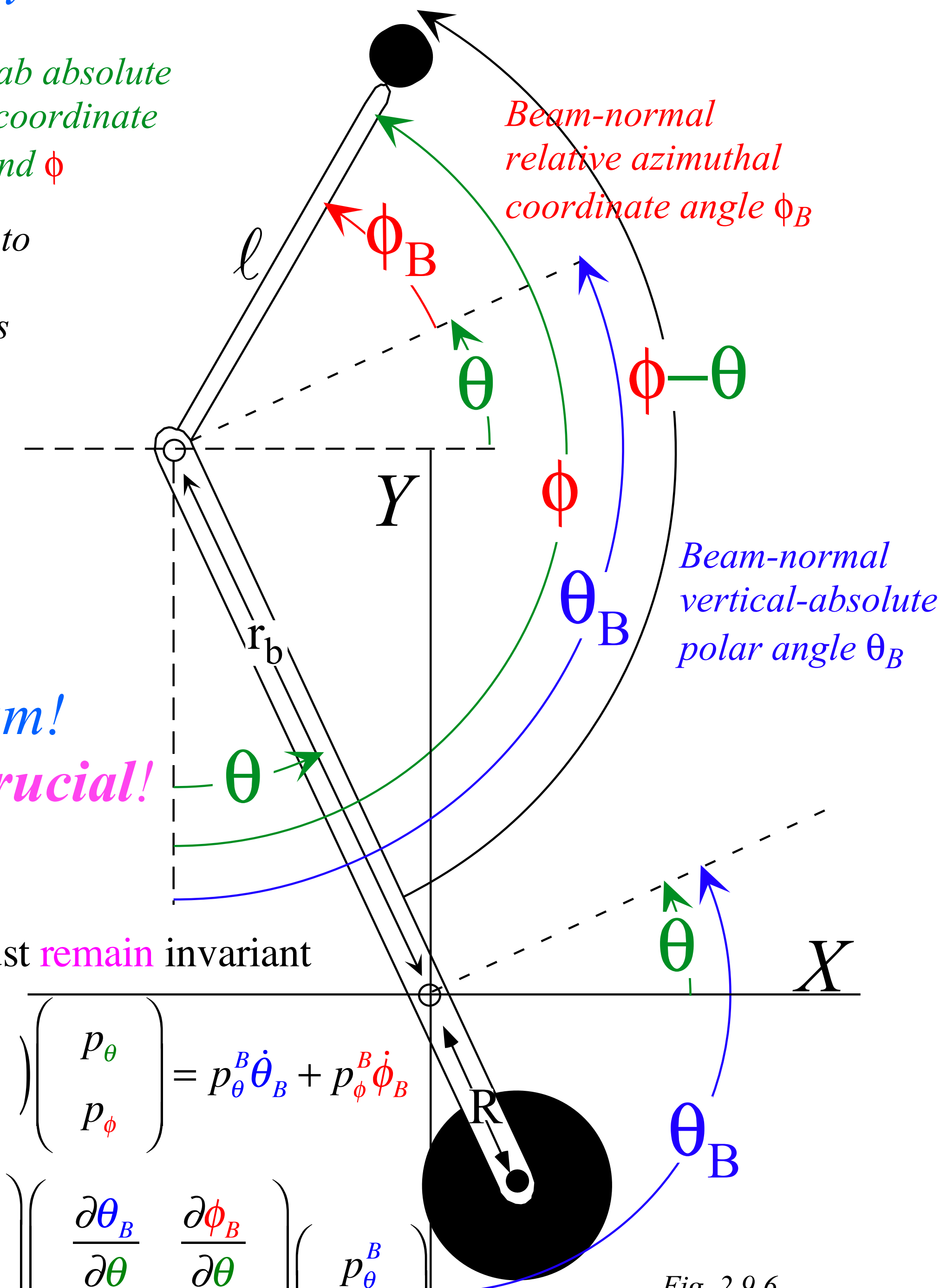
Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$$p_\phi = \quad p_\phi^B$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Be careful with momentum!
*Poincare invariance is **crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot\theta_B + p_\phi^B\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta-\phi)}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} + (MR - mr) g\cos\theta + mg\ell\cos\phi$$

$$\theta - \phi = -\pi/2 - \phi_B$$

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$
$$F_\phi = -mg\ell\sin\phi$$

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} - \left(MR - mr\right) g\sin\theta_B \; - mg\ell\cos\left(\phi_B + \theta_B\right)$$

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

Algebraic approach

Direct approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=-\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

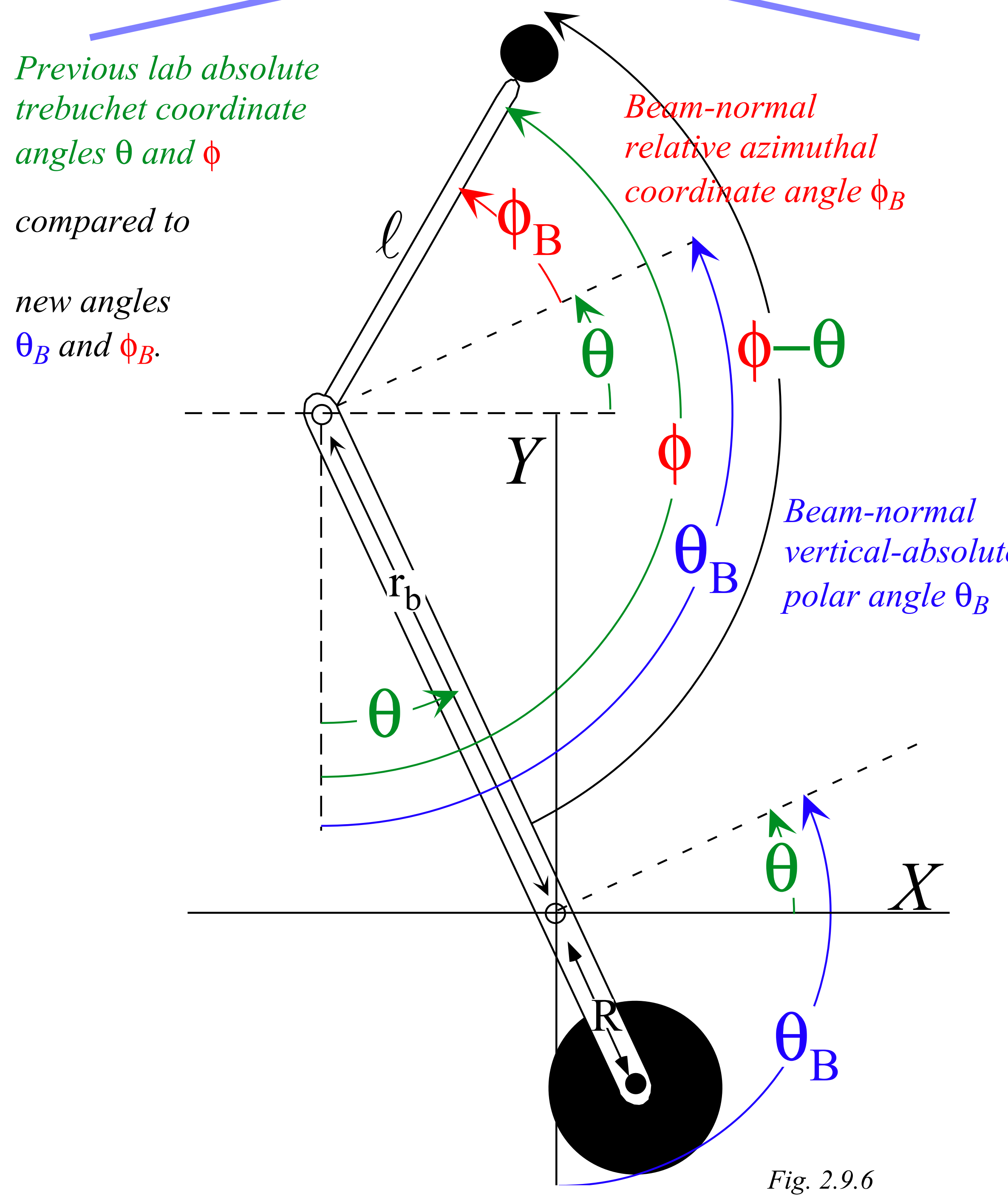


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} \cancel{-\left(MR - mr\right)g\sin\theta_B - mg\ell\cos\left(\phi_B + \theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = -\frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = const.$

H is not an explicit function of t so : $H=const.=E$

$$H = \frac{m\ell^2\left(\Lambda - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(\Lambda - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} = const. = E$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

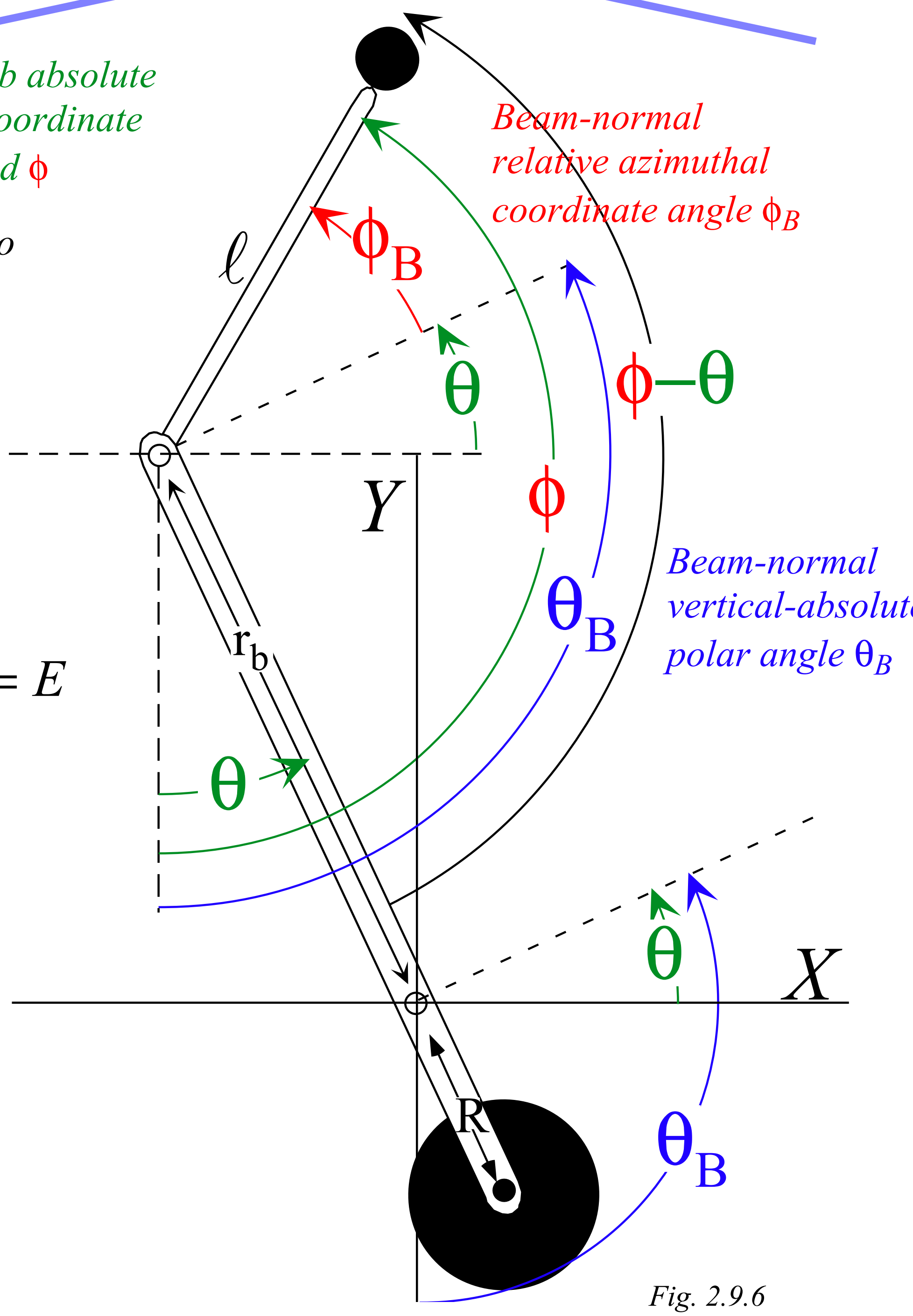


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

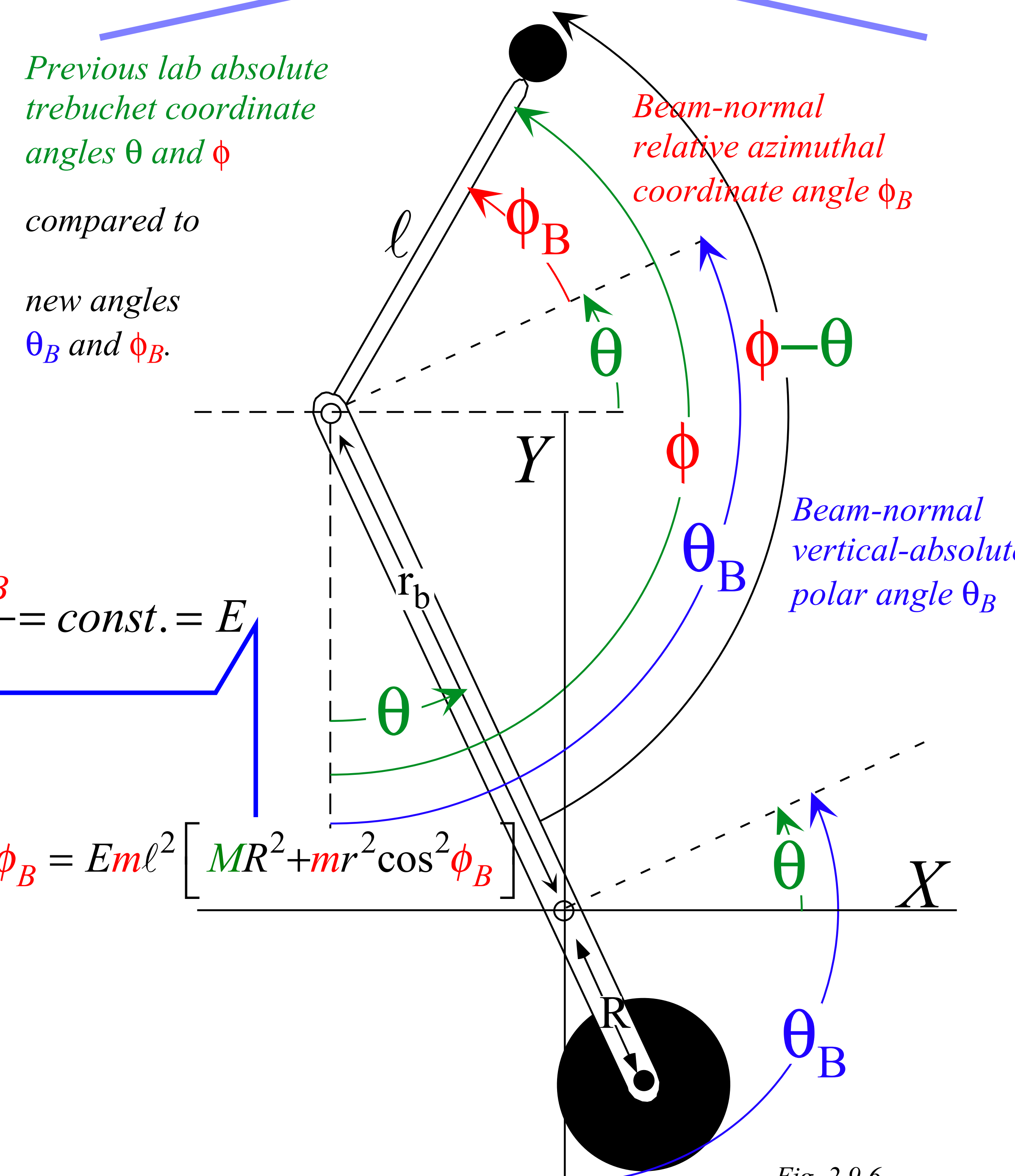
(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=-\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$



$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ *:*

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Fig. 2.9.6
Lab (θ,ϕ) *and beam-normal* (θ_B,ϕ_B) *relative coordinates for trebuchet.*
(Each value is positive.)

Throwing-momentum $p^B{}_\phi$ *is a function of beam-relative angle* ϕ_B*, total E, and* $\Lambda=p^B{}_\theta$ *.*

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}-\cancel{\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=-\dfrac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

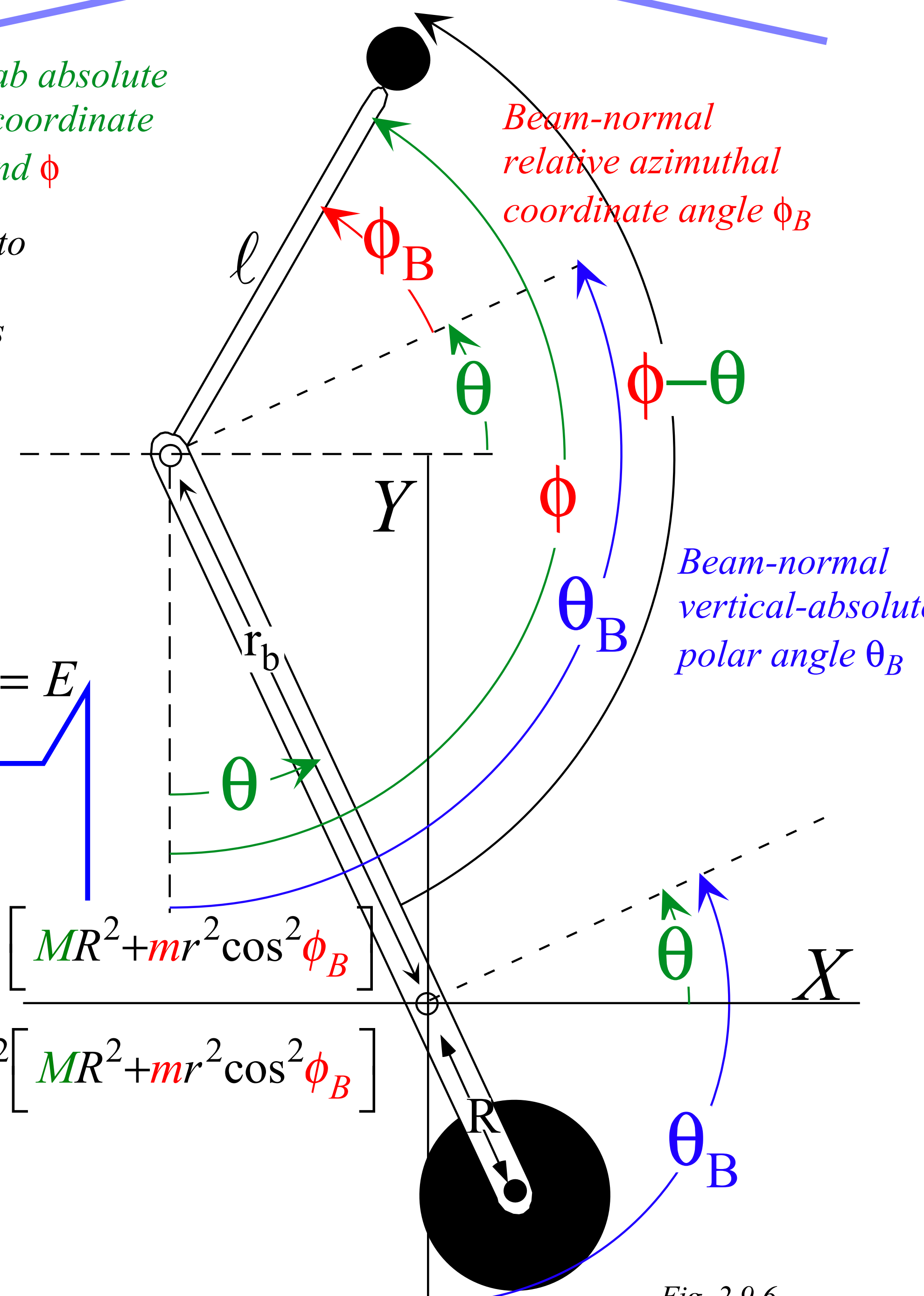


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=-\dfrac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

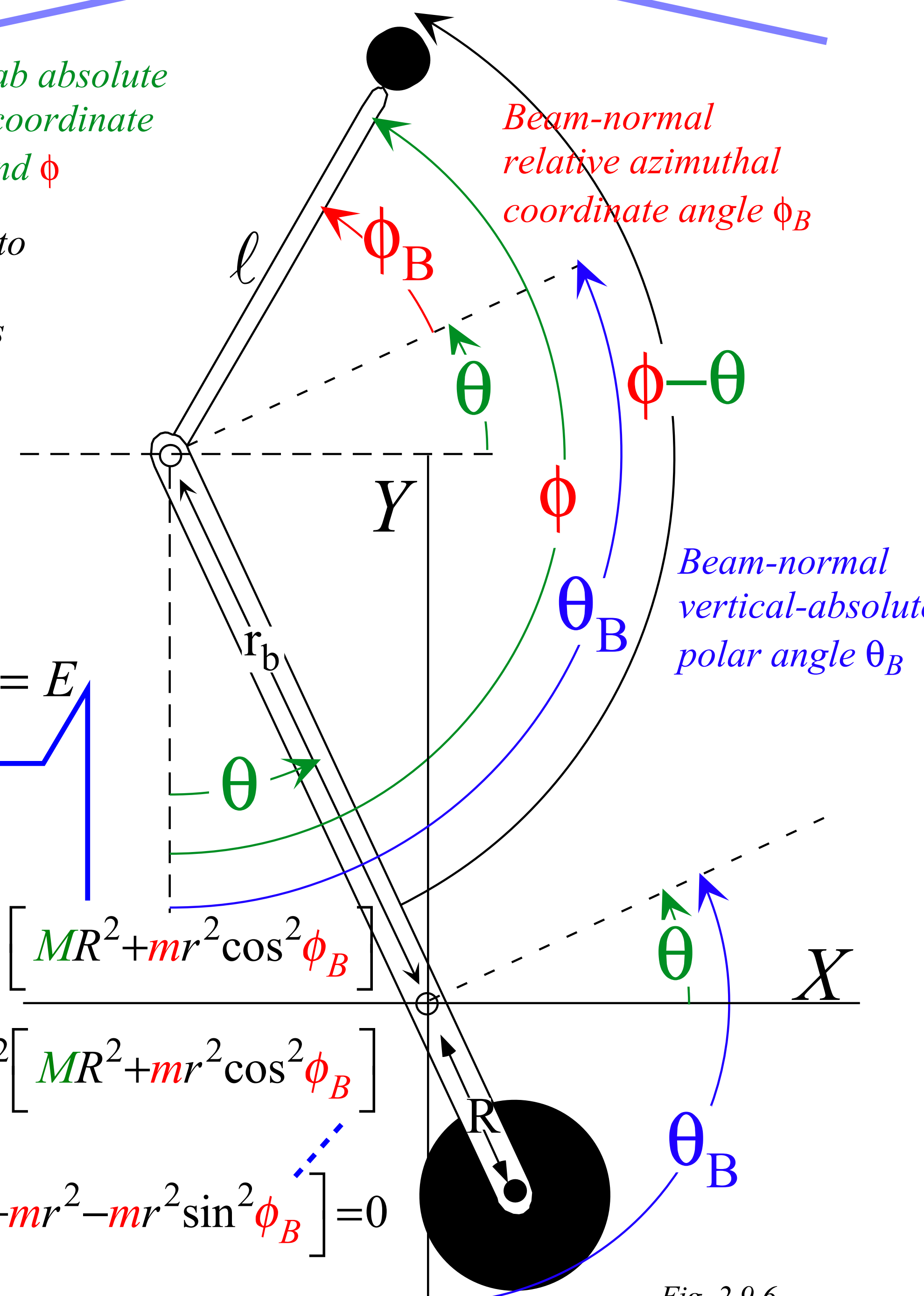


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=-\dfrac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

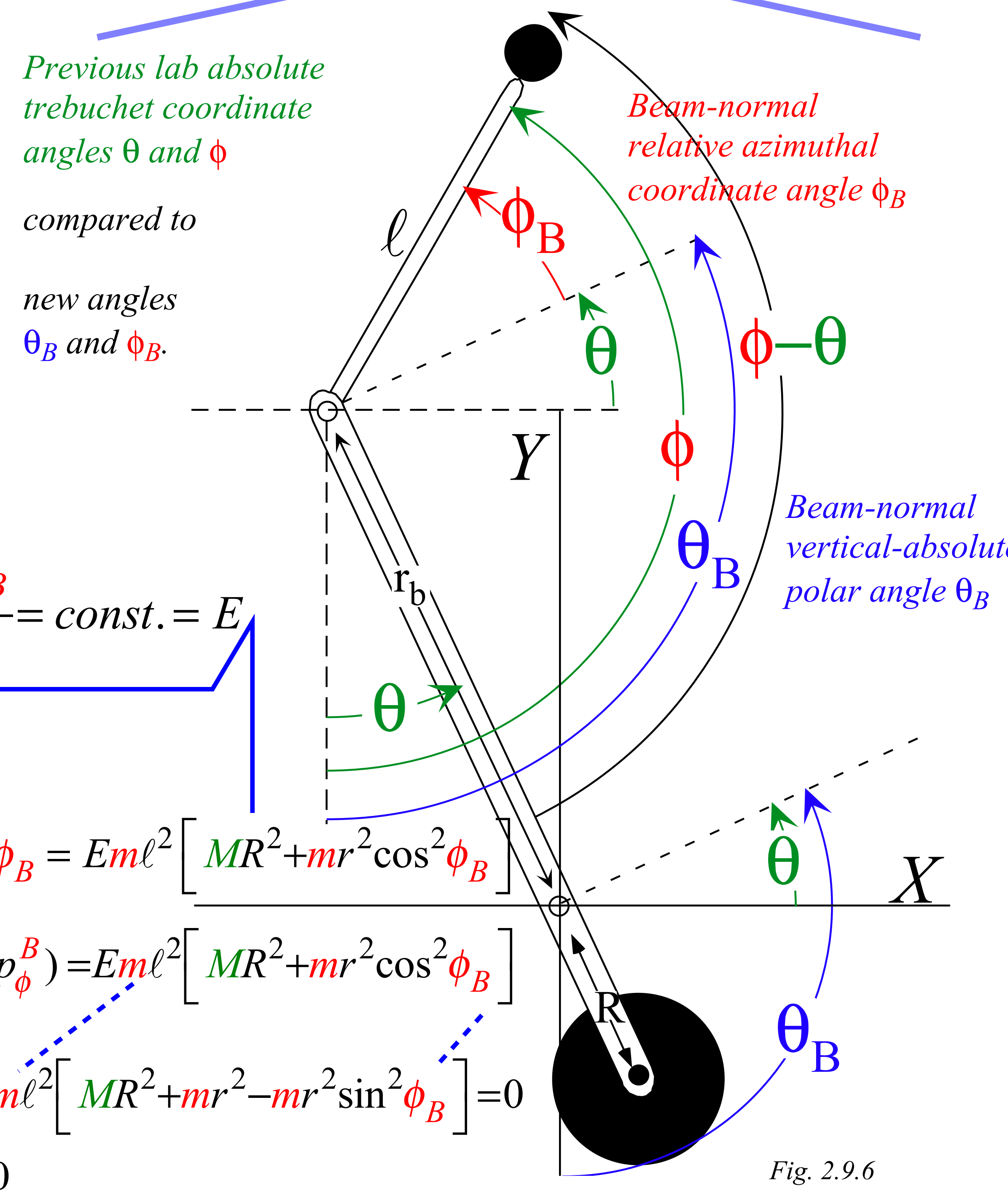


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ *:*

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

Throwing-momentum $p^B{}_\phi$ *is a function of beam-relative angle* ϕ_B*, total E, and* $\Lambda=p^B{}_\theta$ *.*

with: $J=\dfrac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

Previous lab absolute trebuchet coordinate angles θ and φ

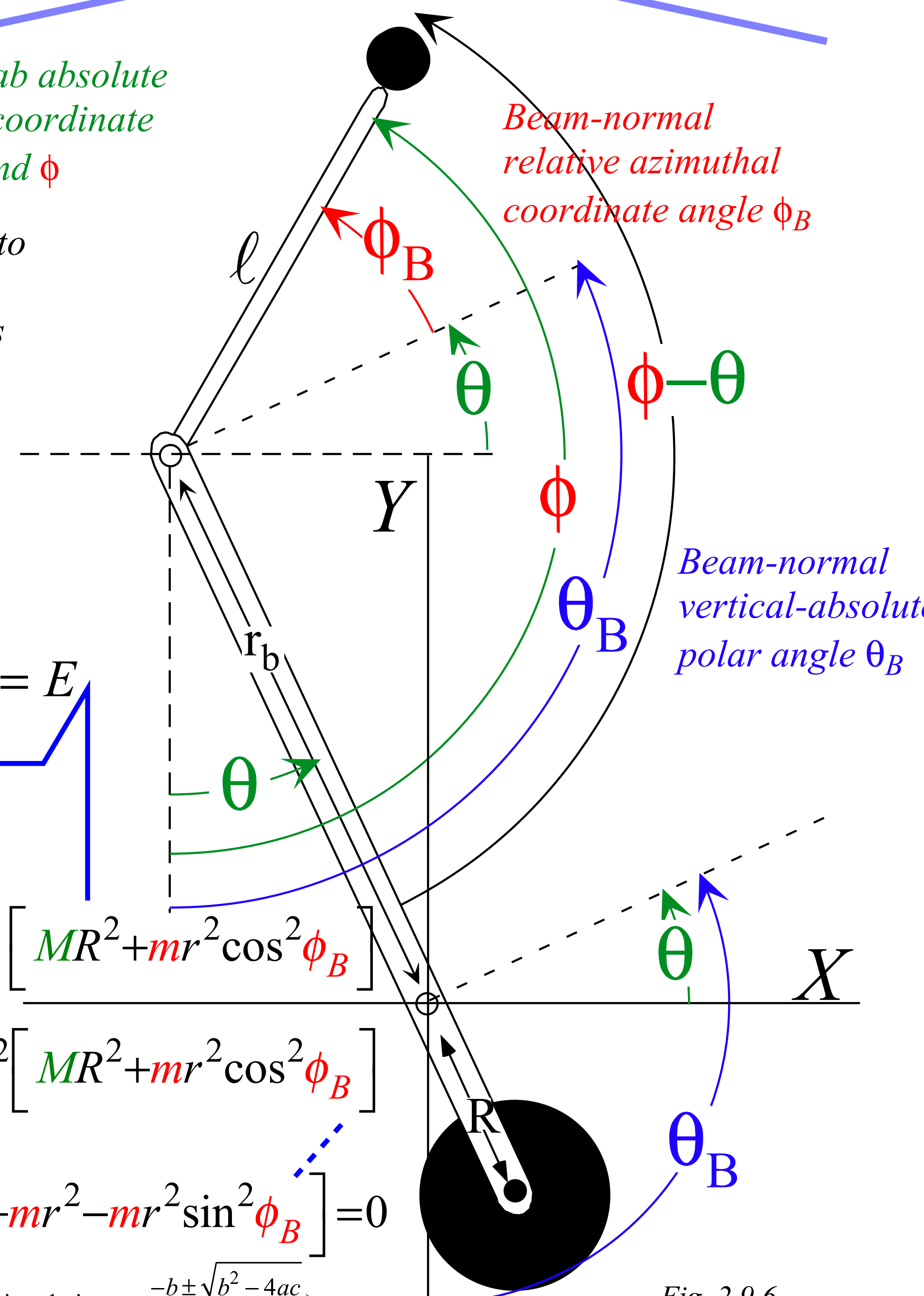
compared to

new angles θ_B *and* ϕ_B.

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=-\dfrac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E



$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ *:*

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

(using quadratic solution: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$)

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Throwing-momentum $p^B{}_\phi$ *is a function of beam-relative angle* ϕ_B*, total E, and* $\Lambda=p^B{}_\theta$ *.*

$$p_\phi^B=\frac{2\Lambda\left((r/\ell)\sin\phi_B+1\right)\pm\sqrt{4\Lambda^2\left((r/\ell)\sin\phi_B+1\right)^2-4\left(1+2(r/\ell)\sin\phi_B+J\right)\left(\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]\right)}}{2\left(1+2(r/\ell)\sin\phi_B+J\right)}$$

with: $J=\dfrac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

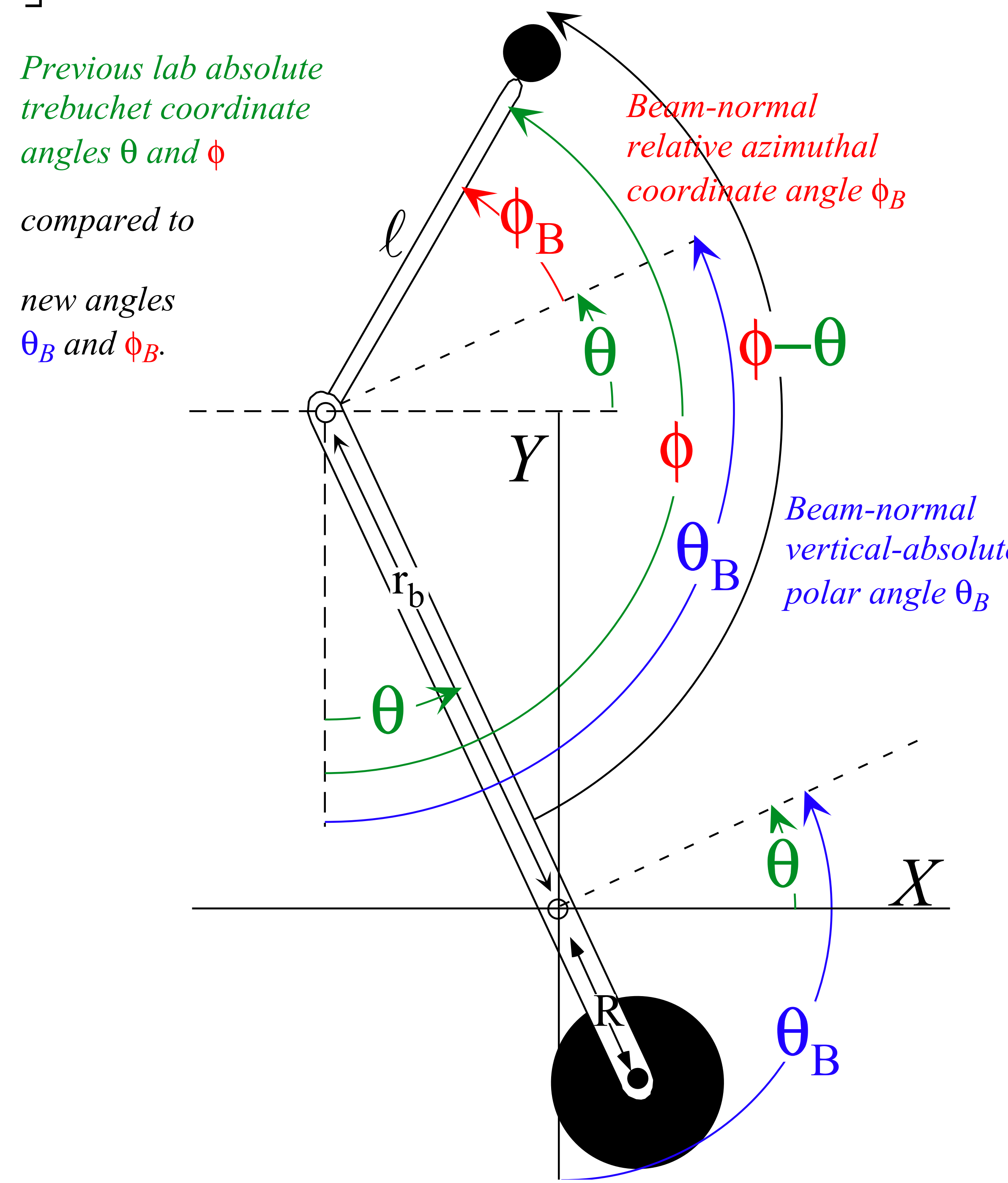
(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$\theta_B = \theta \quad +\pi/2$

$\phi_B = -\theta + \phi - \pi/2$

$p_\theta = p_\theta^B - p_\phi^B$

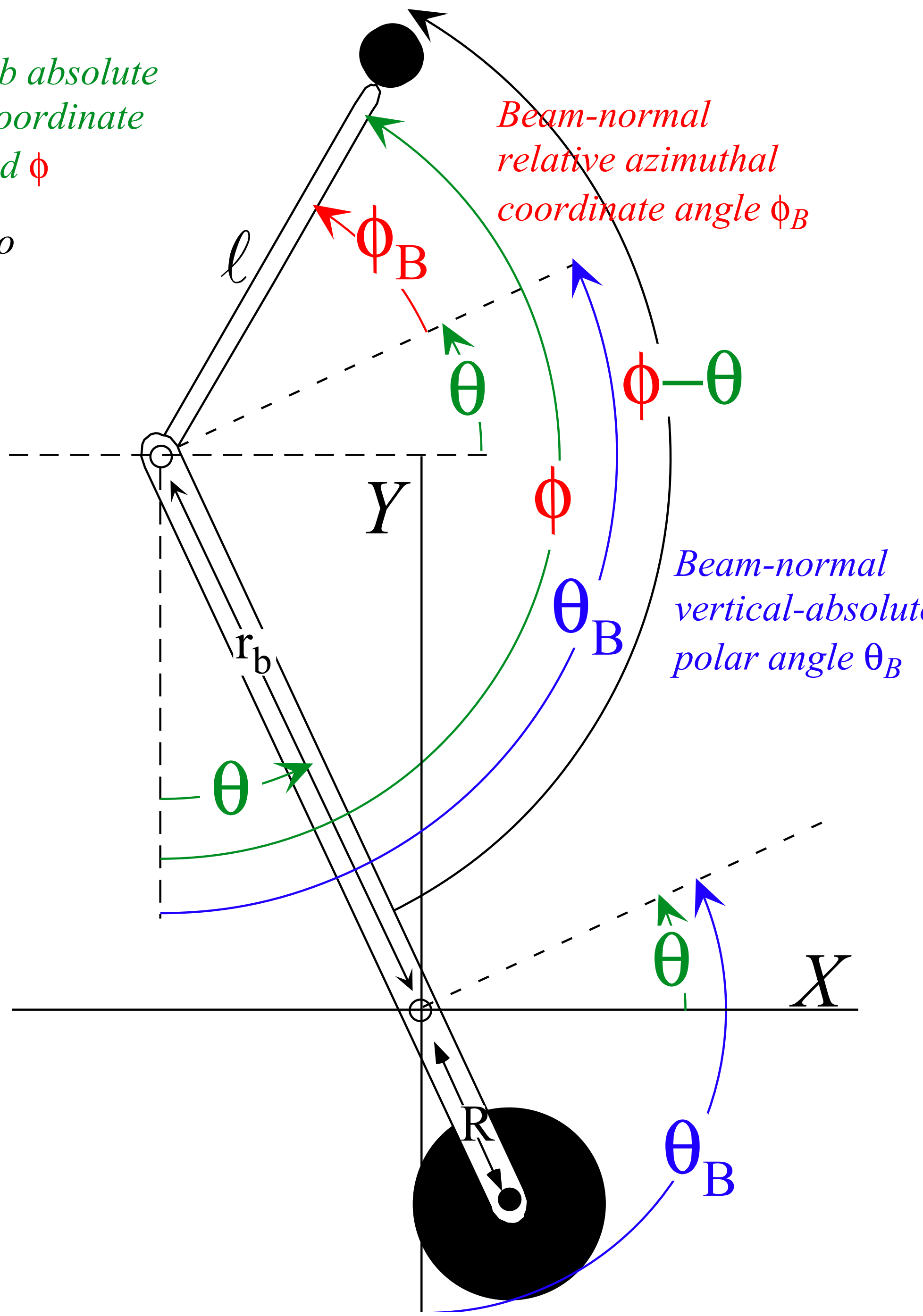
$p_\phi = \quad p_\phi^B$

$\theta - \phi = -\phi_B - \pi/2$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)$$

Transform to beam-relative coordinates and momenta

$$\theta = \theta_B \quad -\pi/2$$
$$\phi = \theta_B + \phi_B$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$\theta_B = \theta \quad +\pi/2$$
$$\phi_B = -\theta + \phi \quad -\pi/2$$

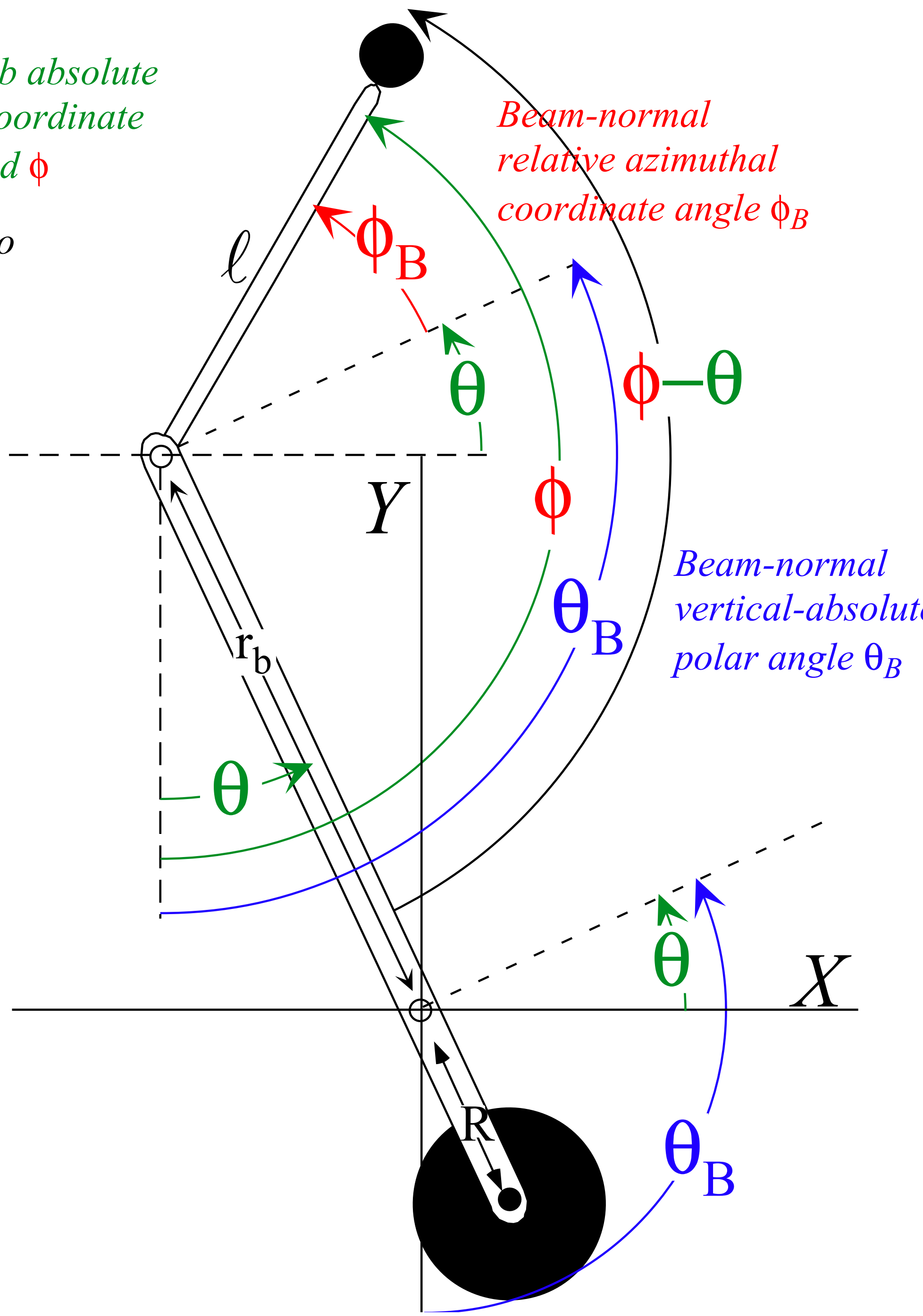
$$p_\theta = p_\theta^B - p_\phi^B$$
$$p_\phi = p_\phi^B$$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$\theta_B = \theta \quad +\pi/2$

$\phi_B = -\theta + \phi \quad -\pi/2$

$p_\theta = p^B_\theta - p^B_\phi$

$p_\phi = \quad p^B_\phi$

$\theta - \phi = -\phi_B - \pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

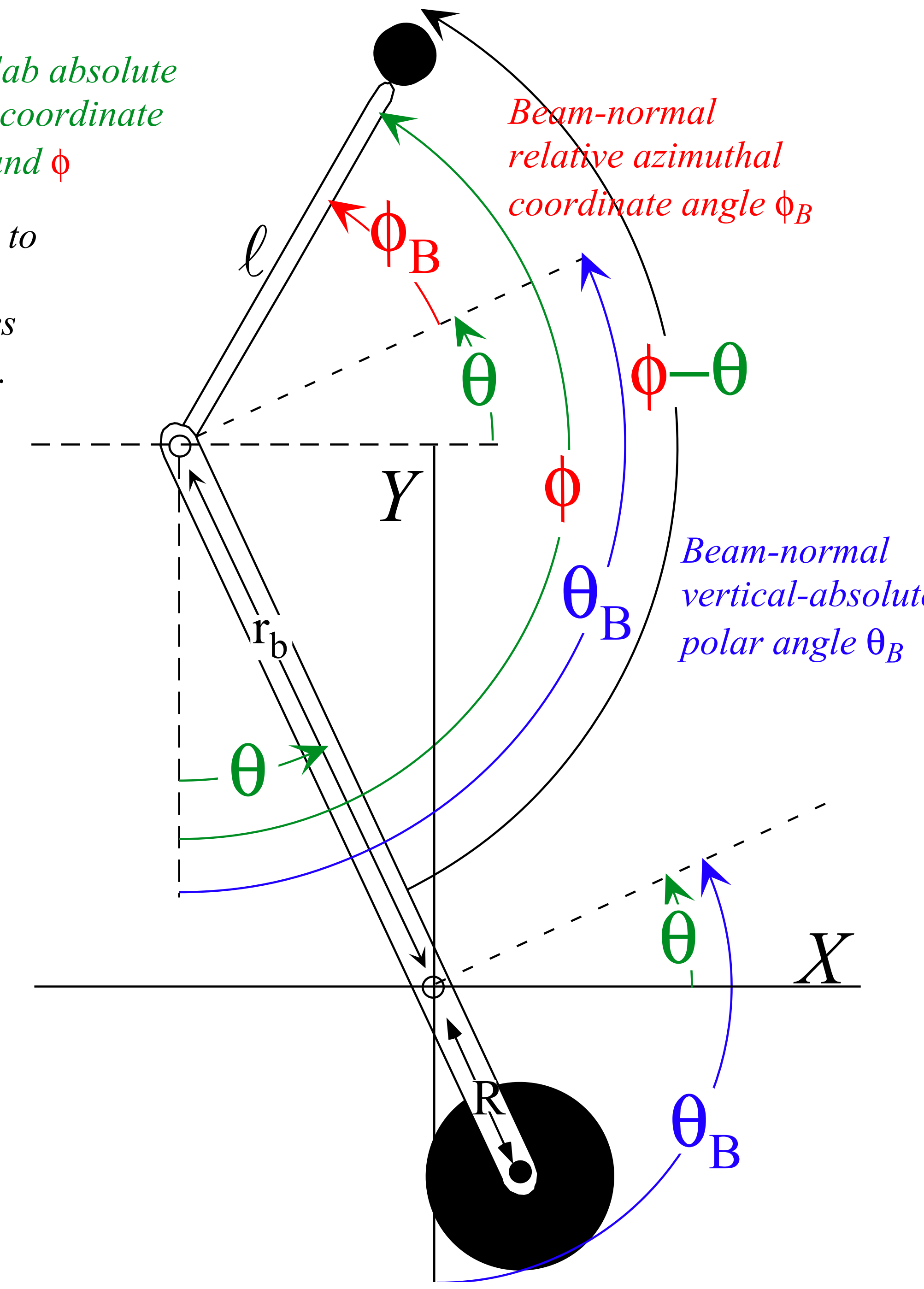
$$p^B_\theta = \Lambda = const. = \qquad p_\theta \qquad + \qquad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B \right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B \right)$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

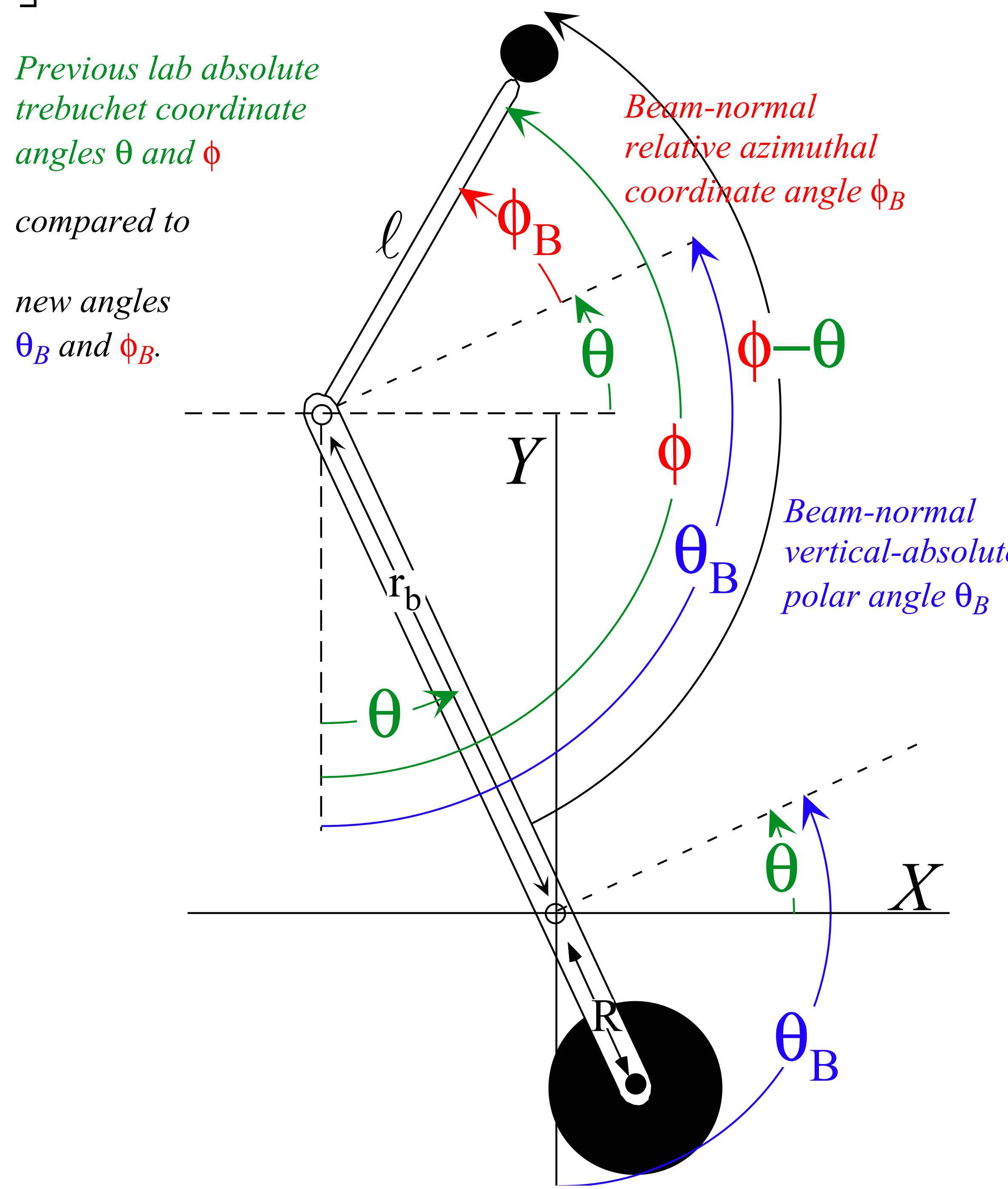
$\theta = \theta_B \quad -\pi/2$	$\theta_B = \theta \quad +\pi/2$	$p_\theta = p_\theta^B - p_\phi^B$
$\phi = \theta_B + \phi_B$	$\phi_B = -\theta + \phi \quad -\pi/2$	$p_\phi = p_\phi^B$

$\theta - \phi = -\phi_B - \pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B\right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B\right)$$



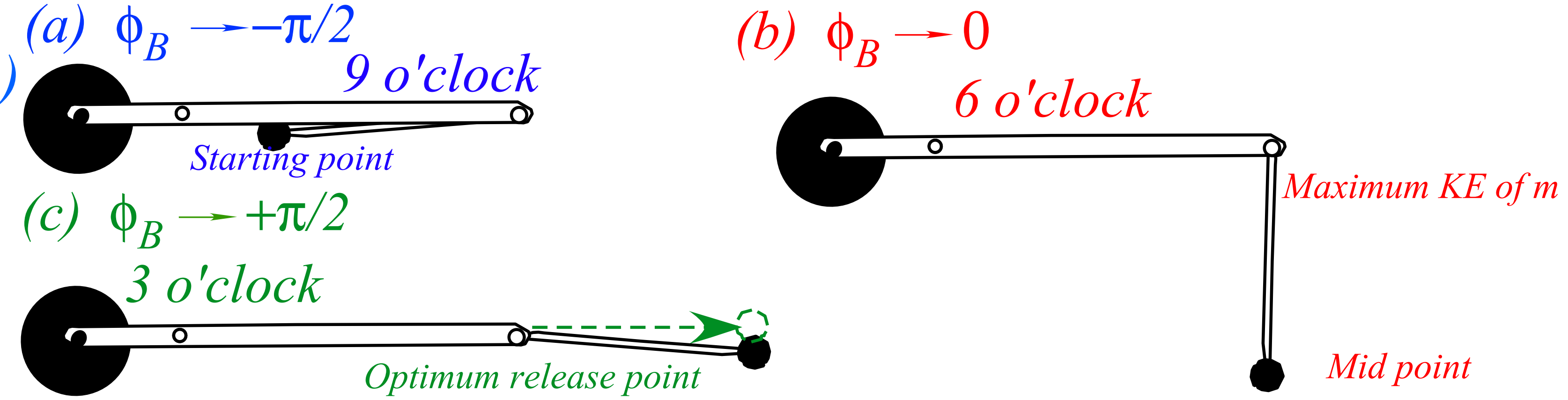
Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell\text{)}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

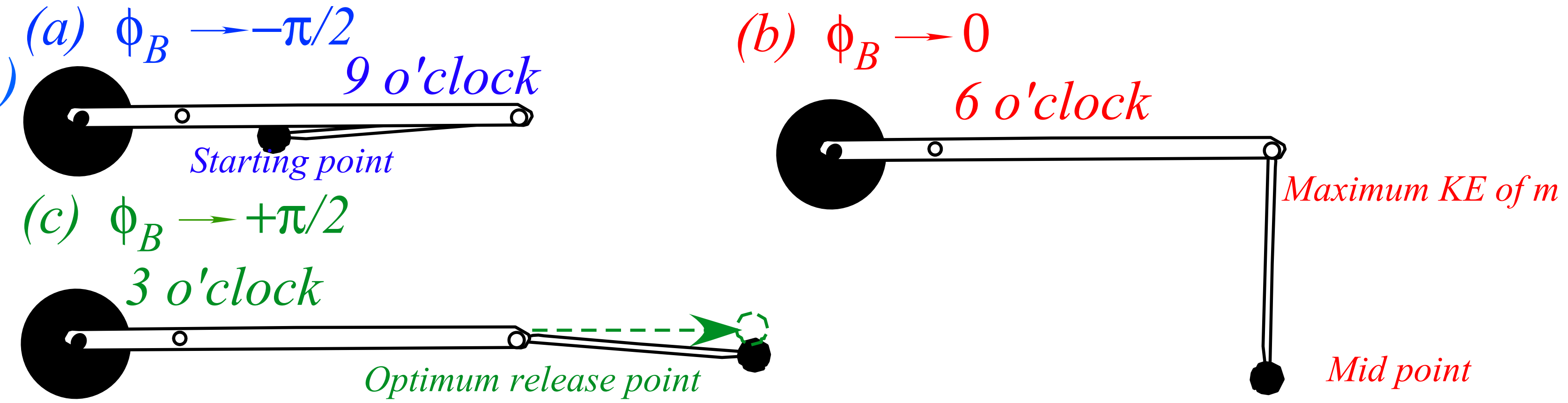
$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \begin{array}{l}\text{For:}\\ r=\ell\end{array}$$



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2}: \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$\sin\phi_B = -1$

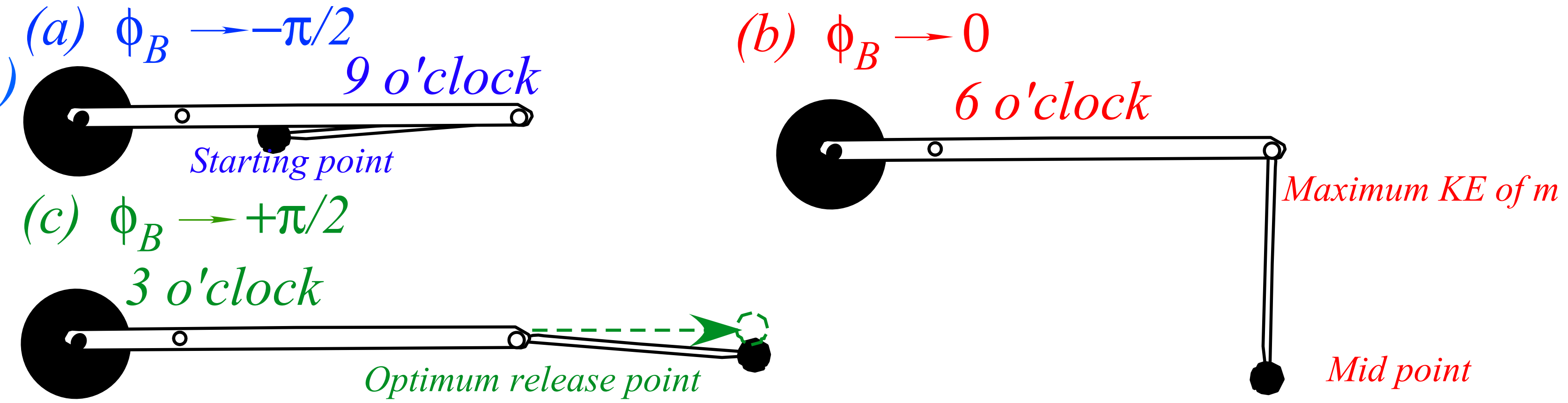
Conserved initial 2E

initial Λ

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{ For: } r=\ell$$



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right.$$

Conserved initial $2E$

$$or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \qquad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

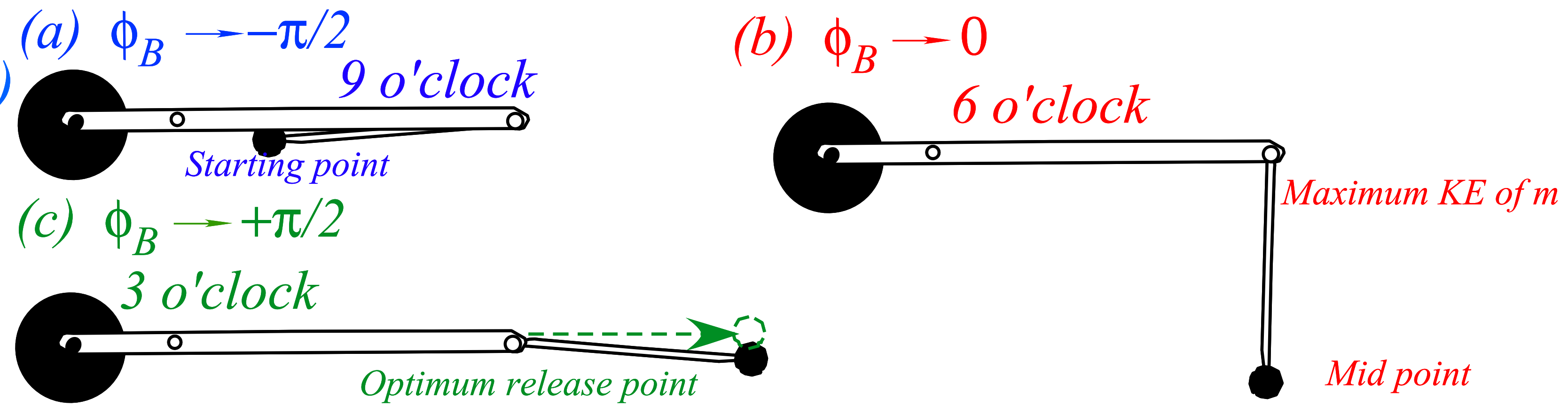
initial Λ

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)



$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \quad \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2} \\ \sin\phi_B = -1\end{matrix} : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_{-\pi/2}^2 + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} &\text{Conserved initial } 2E \\ 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \\ &\text{initial } \Lambda \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

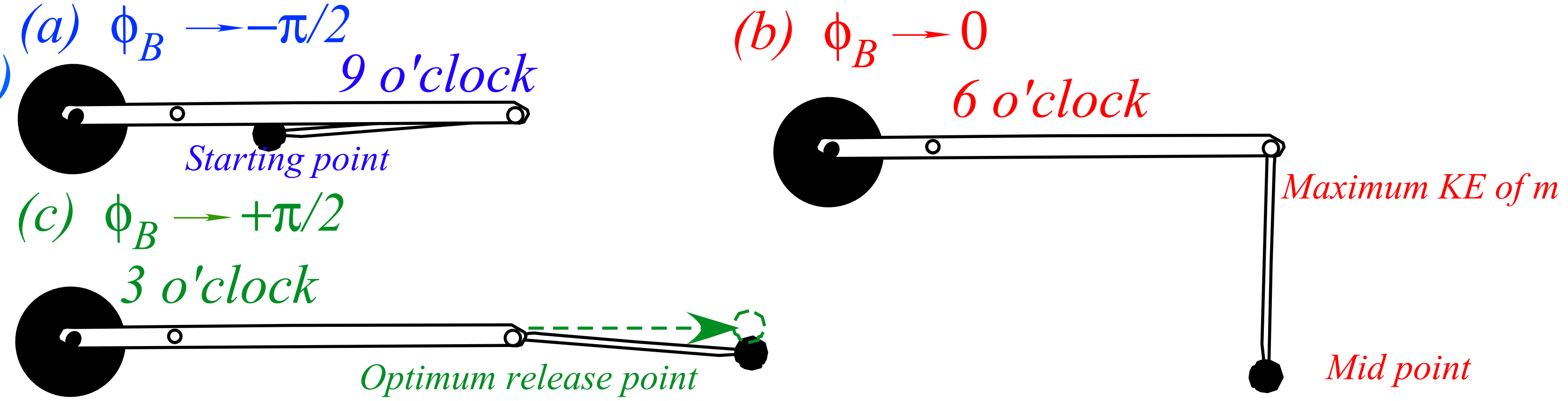
$$\begin{matrix}\phi_B = 0 \\ \sin\phi_B = 0\end{matrix} : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2 \\ \sin\phi_B = +1\end{matrix} : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_{\pi/2}^2 + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \underset{\text{initial } 2E}{\overset{\text{Conserved}}{=}} MR^2\omega^2 \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) \underset{\text{initial } \Lambda}{=} MR^2\omega \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)



$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \quad \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{ \begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.$$

$\sin\phi_B = -1$

Conserved initial 2E ... initial Λ

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$\sin\phi_B = 0$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

$\sin\phi_B = +1$

Conserved initial 2E ... initial Λ

KE of projectile m

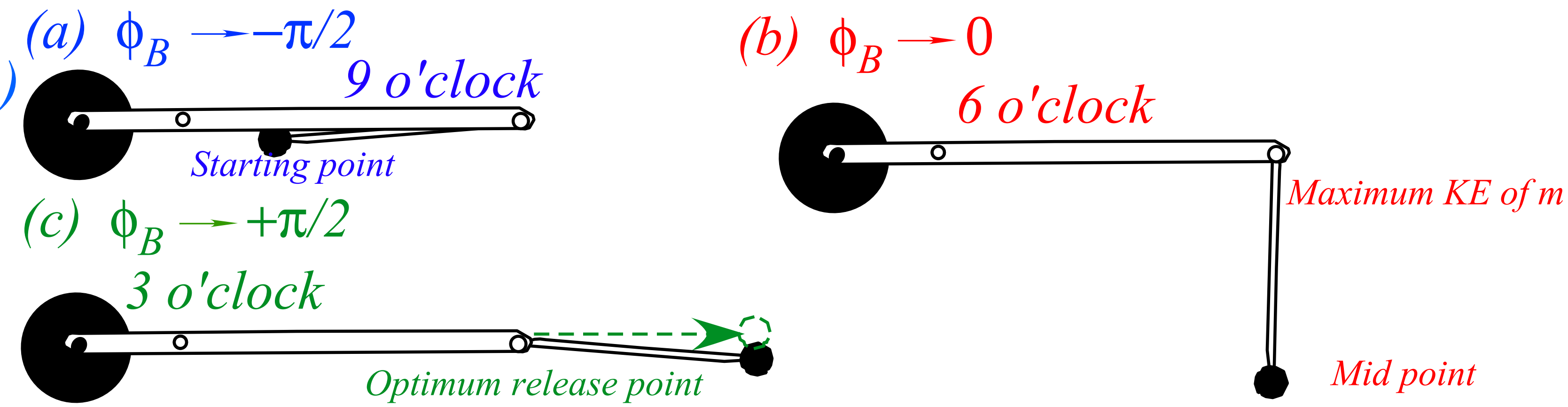
$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \left\{ \begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right.$$

$$or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \quad \text{(Conserved initial } 2E\text{)} \\ \Lambda &= MR^2\omega \quad \text{(initial } \Lambda\text{)} \end{aligned}\right. \qquad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

KE of projectile m

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

$$\phi_B = \pi/2: \quad \sin\phi_B = +1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{(Conserved initial } 2E\text{)} \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{(initial } \Lambda\text{)} \end{aligned}\right.$$

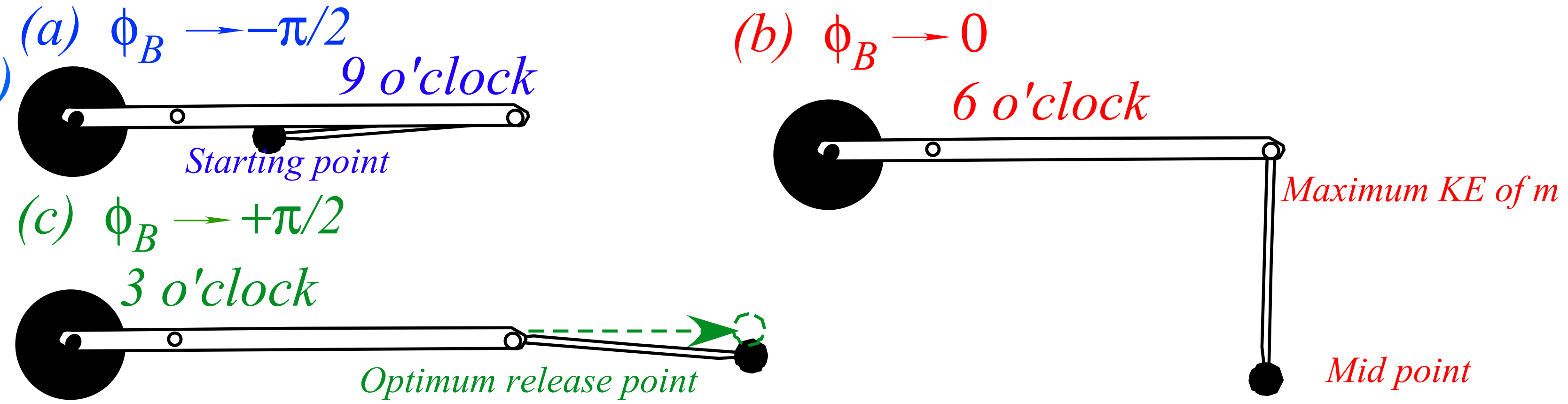
$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \left\{ \begin{aligned} &\text{Conserved initial } 2E \\ &2E = MR^2\omega^2 \\ &\Lambda = MR^2\omega \\ &\text{initial } \Lambda \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$\sin\phi_B$=-1

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

$\sin\phi_B$=0

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right) = \left\{ \begin{aligned} &\frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{Conserved initial } 2E \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right.$$

$\sin\phi_B$=+1

$$(\omega^2 - \dot{\theta}^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2$$

$$(\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

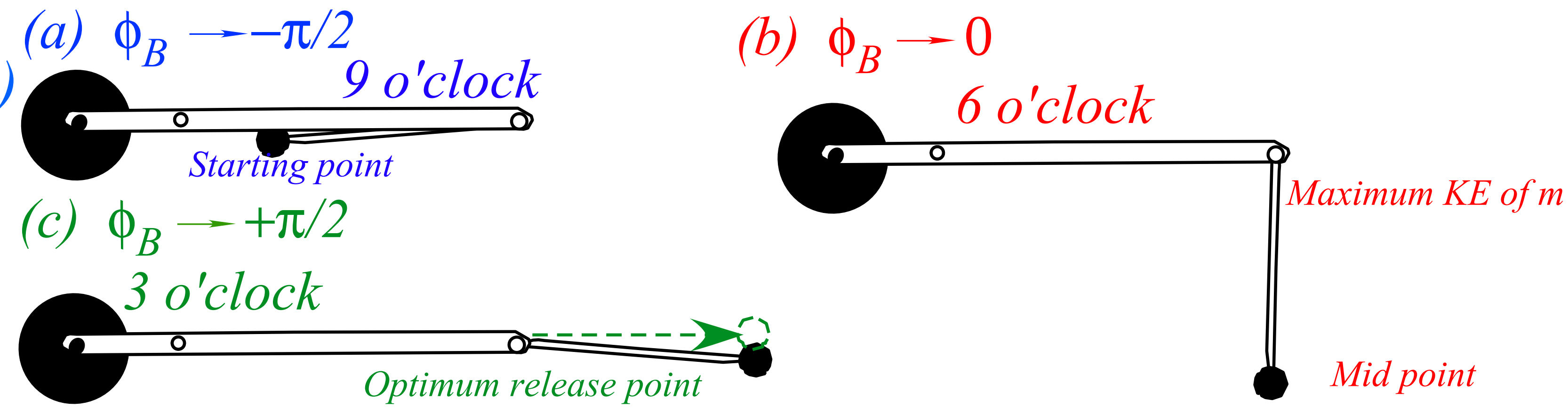
divide $2E$ by Λ

$$(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned}2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together at ω)

$$\begin{matrix}\phi_B=\dfrac{-\pi}{2}:\\ \sin\phi_B=-1\end{matrix}\left\{\begin{aligned}2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned}&\overset{\text{Conserved initial }2E}{2E=MR^2\omega^2}\\ &\underset{\text{initial }\Lambda}{\Lambda=MR^2\omega}\end{aligned}\right.\quad \text{For: } \dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned}&\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 &&\left(\text{For: } \phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) &&\left(\text{For: } \phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 &&\left(\text{For: } \phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B=0:\\ \sin\phi_B=0\end{matrix}\left\{\begin{aligned}2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

$$\begin{matrix}\phi_B=\pi/2:\\ \sin\phi_B=+1\end{matrix}\left\{\begin{aligned}2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2\overset{\text{Conserved initial }2E}{=MR^2\omega^2}\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)\underset{\text{initial }\Lambda}{=MR^2\omega}\end{aligned}\right.$$

$$(\omega^2-\dot\theta^2_{\pi/2})=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$(\omega-\dot\theta_{\pi/2})=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

divide 2E by Λ

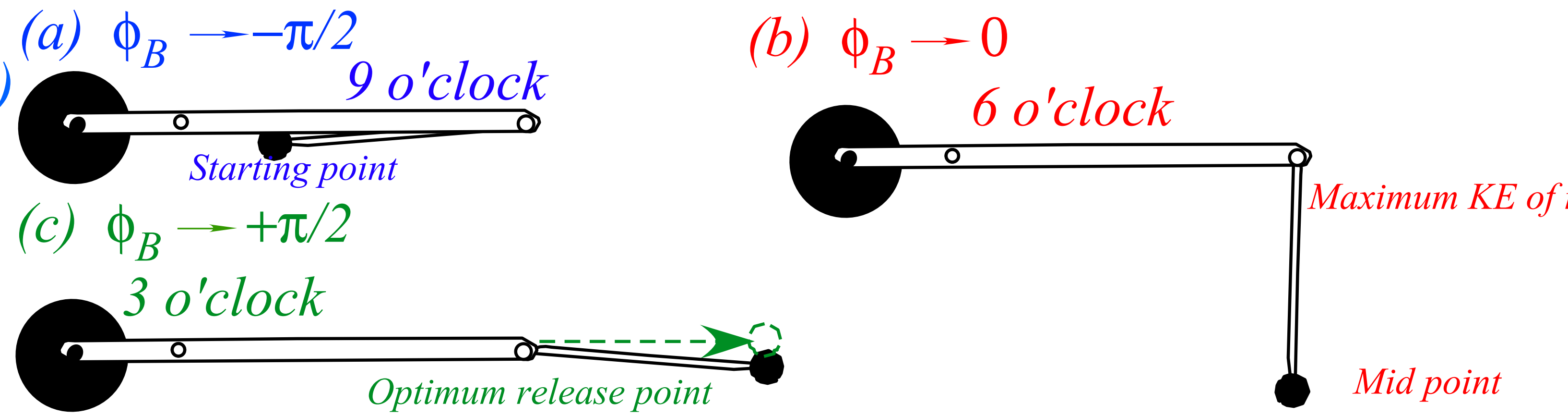
$$(\omega+\dot\theta_{\pi/2})=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2+mr^2\left(\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B+\dot{\phi}^2\right)\\ \Lambda &= MR^2\dot{\theta}+mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right)\end{aligned}\right\}\ \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together at ω)

$$\begin{matrix}\phi_B=\dfrac{-\pi}{2}:\\ \sin\phi_B=-1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2}+mr^2\left(\dot{\phi}_{-\pi/2}-\dot{\phi}_{-\pi/2}\right)^2\\ \Lambda &= MR^2\dot{\theta}_{-\pi/2}\end{aligned}\right.\quad or: \left\{\begin{aligned} &\text{Conserved initial } 2E\\ 2E &= MR^2\omega^2\\ \Lambda &= MR^2\omega\\ &\text{initial } \Lambda\end{aligned}\right.\quad \text{For: } \dot{\theta}_{-\pi/2}=\omega=\dot{\phi}_{-\pi/2}$$

$$KE(m)=\frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B\right)=\begin{cases}\dfrac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B=-\dfrac{\pi}{2}\right)\\ \dfrac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B=0\right)\\ \dfrac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B=\dfrac{\pi}{2}\right)\end{cases}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B=0:\\ \sin\phi_B=0\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_0+mr^2\left(\dot{\phi}^2_0+\dot{\theta}^2_0\right)\\ \Lambda &= MR^2\dot{\theta}_0+mr^2\left(\dot{\phi}_0+\dot{\theta}_0\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

$$\begin{matrix}\phi_B=\pi/2:\\ \sin\phi_B=+1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2}+mr^2\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{(Conserved initial } 2E)\\ \Lambda &= MR^2\dot{\theta}_{\pi/2}+2mr^2\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right) = MR^2\omega \quad \text{(initial } \Lambda)\end{aligned}\right.$$

$$(\omega^2-\dot{\theta}^2_{\pi/2})=\frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)^2$$

$$(\omega-\dot{\theta}_{\pi/2})=\frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)$$

divide 2E by Λ

$$(\omega+\dot{\theta}_{\pi/2})=\frac{1}{2}\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)$$

$$\boxed{\dot{\phi}_{\pi/2}=\dot{\theta}_{\pi/2}+2\omega}$$

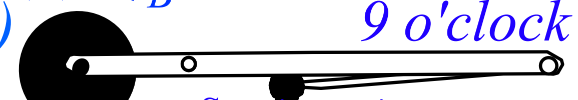
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$

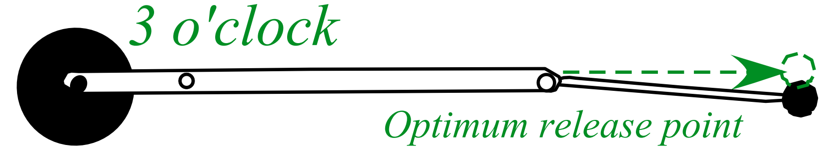
(a) $\phi_B \rightarrow -\pi/2$

9 o'clock



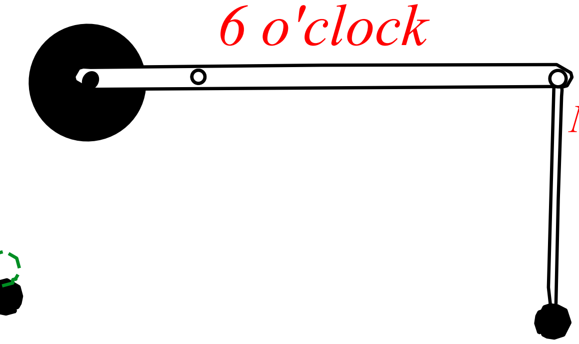
(c) $\phi_B \rightarrow +\pi/2$

3 o'clock



(b) $\phi_B \rightarrow 0$

6 o'clock



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$\sin\phi_B = -1$

Conserved initial 2E

initial Λ

$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right)$

$$= \begin{cases} \frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{cases}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

$\sin\phi_B = 0$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

Conserved initial 2E

$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

$\sin\phi_B = +1$

initial Λ

$$(\omega^2 - \dot{\theta}^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2$$

divide 2E by Λ

$$(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$(\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

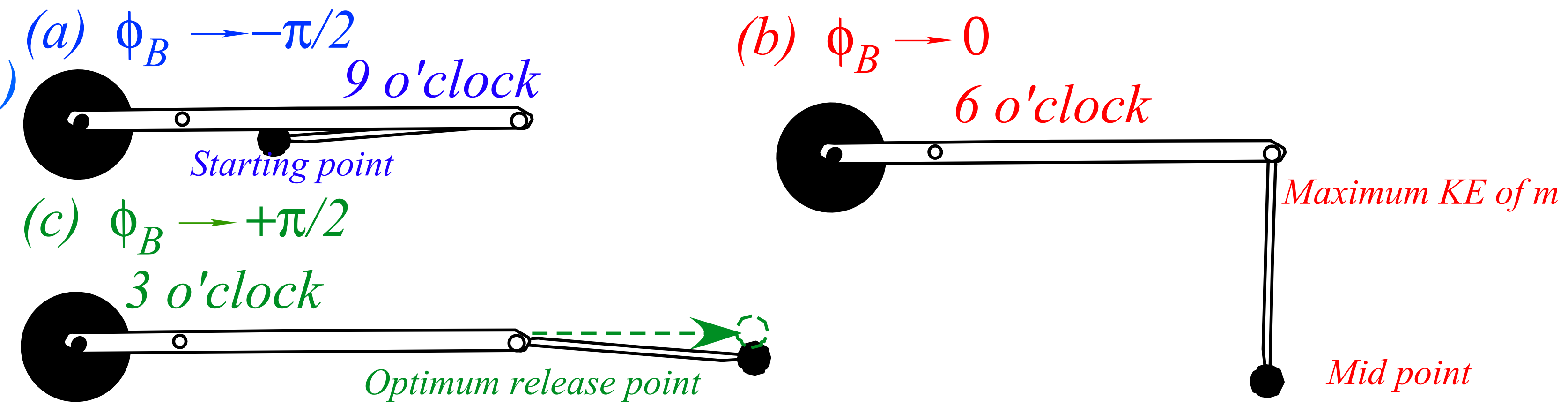
$$\boxed{\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot{\theta}_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)



$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2}\end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega\end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$\sin\phi_B = -1$

Conserved initial 2E

initial Λ

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \begin{cases} \frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right)\end{cases}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{aligned}\right.$$

$\sin\phi_B = 0$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

Conserved initial 2E

$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega\end{aligned}\right.$$

initial Λ

$\sin\phi_B = +1$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

divide 2E by Λ

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

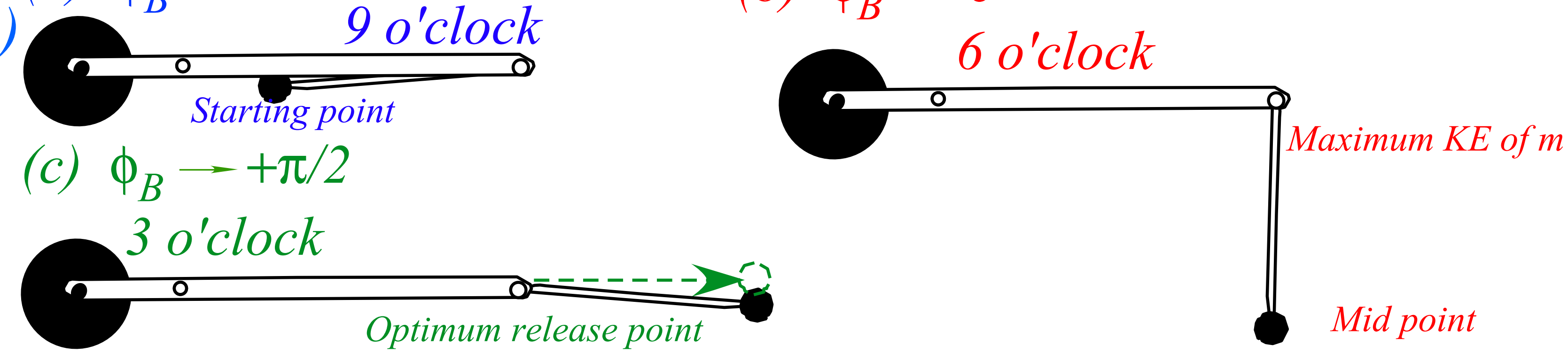
Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \quad \text{For: } r=\ell$$

(a) $\phi_B \to -\pi/2$

(b) $\phi_B \to 0$

(c) $\phi_B \to +\pi/2$



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \quad \text{Conserved initial } 2E \\ \Lambda &= MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right.$$

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) \quad \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

$$\phi_B = \pi/2: \quad \sin\phi_B = +1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{Conserved initial } 2E \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide $2E$ by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

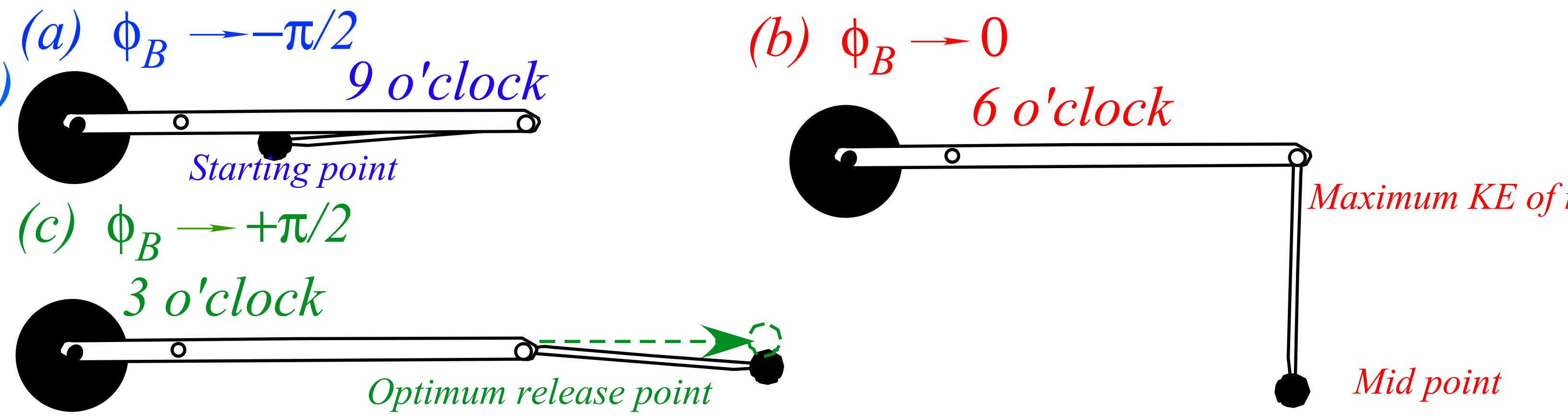
solve

$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\,\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together at ω)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2} \\ \sin\phi_B = -1\end{matrix} : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \quad \text{Conserved initial } 2E \\ \Lambda &= MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0 \\ \sin\phi_B = 0\end{matrix} : \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Divide by MR^2 and subtract

$$\begin{matrix}\phi_B = \pi/2 \\ \sin\phi_B = +1\end{matrix} : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{Conserved initial } 2E \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

solve

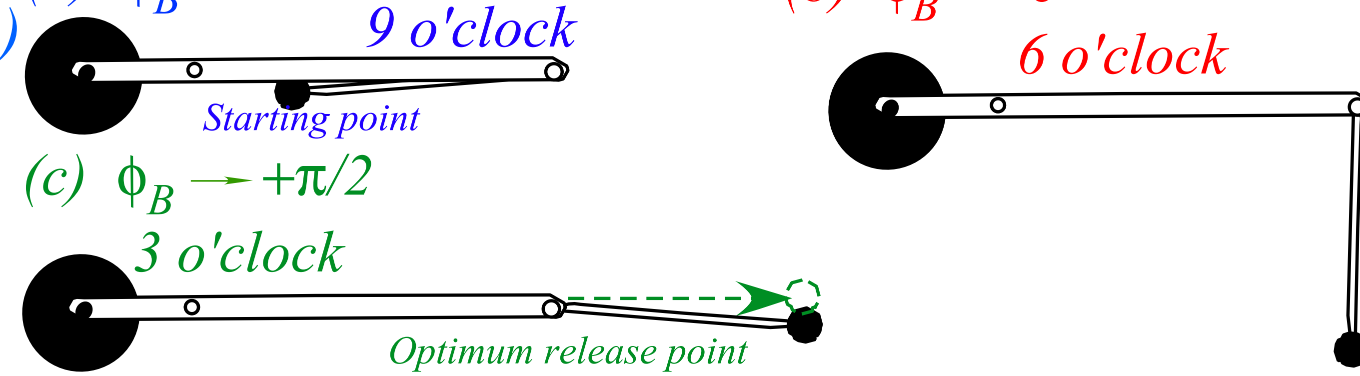
$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\omega}$$

Large $M \gg m$ case (Beam nearly constant ω)

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together at ω)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \begin{cases} 2E = MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{cases} \quad or: \begin{cases} 2E = MR^2\omega^2 & \text{Conserved initial } 2E \\ \Lambda = MR^2\omega & \text{initial } \Lambda \end{cases}$$

For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right) = \begin{cases} \frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{cases}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \begin{cases} 2E = MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda = MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2: \quad \sin\phi_B = +1 \quad \begin{cases} 2E = MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 & \text{Conserved initial } 2E \\ \Lambda = MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega & \text{initial } \Lambda \end{cases}$$

$$\left(\omega^2 - \dot{\theta}^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2$$

$$\left(\omega - \dot{\theta}_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot{\theta}_{\pi/2}\right) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$\boxed{\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot{\theta}_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2}\dot{\theta}_{\pi/2}$$

solve

$$\boxed{\dot{\theta}_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\omega}$$

Large $M \gg m$ case

$$\boxed{\dot{\phi}_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot{\theta}_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Optimum $MR^2 = 4mr^2$ case

$$\boxed{\dot{\phi}_{\pi/2} = 0 + 2\omega = 2\omega}$$

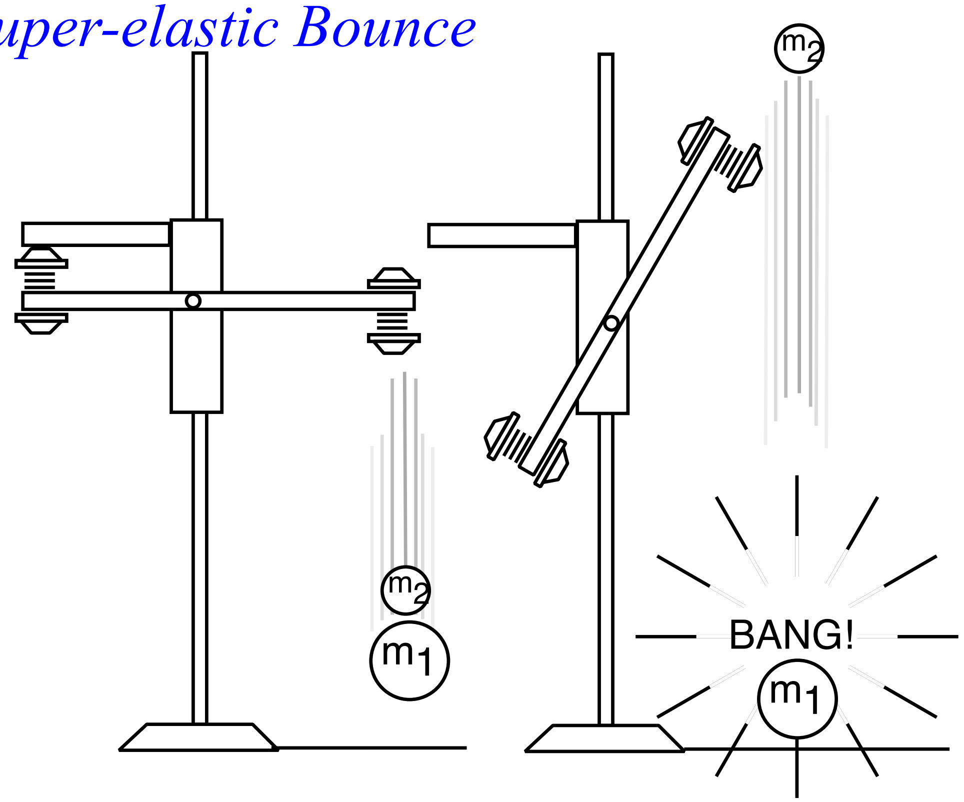
$$\boxed{\dot{\theta}_{\pi/2} = \frac{1-1}{1+1}\omega = 0}$$

Analogous Superball Models

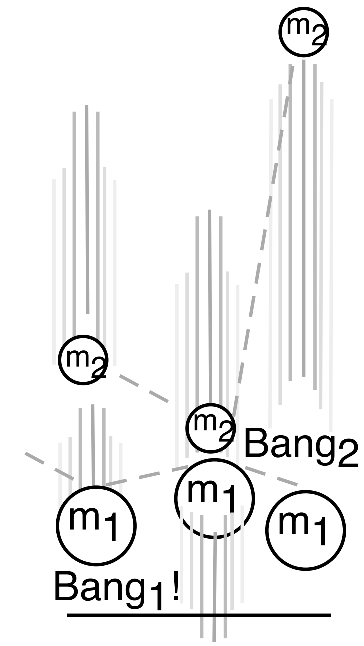
Similar in some ways to trebuchet models

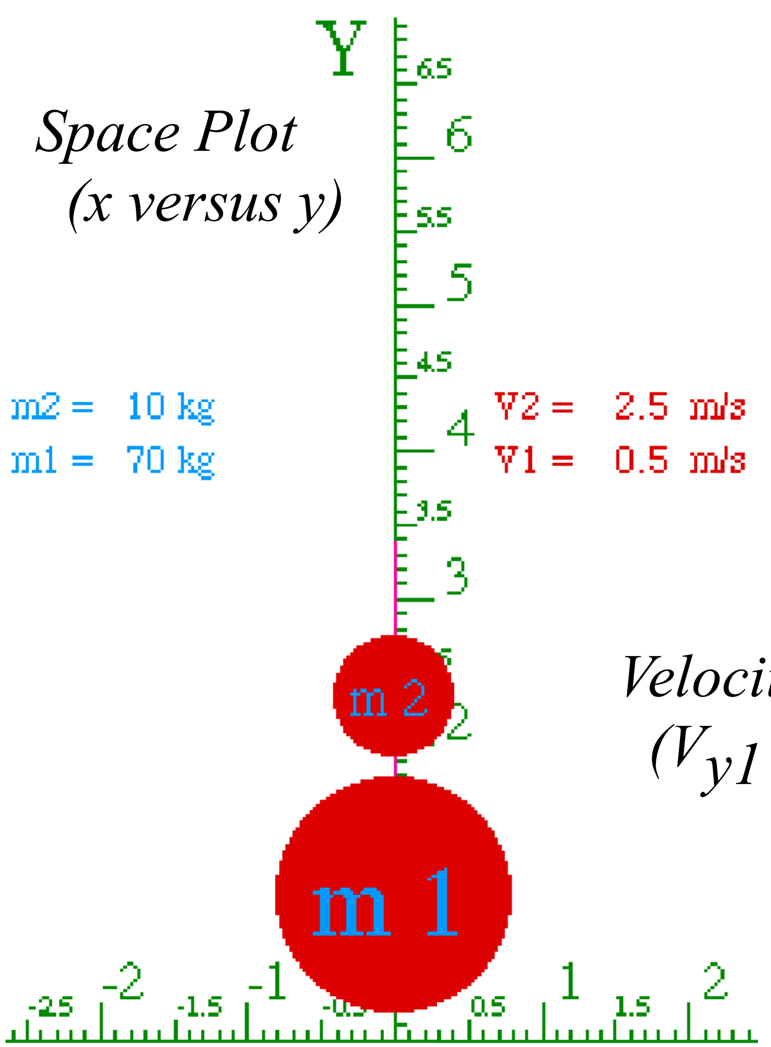
Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," Am. J. Phys. **39**, 656 (1971) (A class project)

Super-elastic Bounce

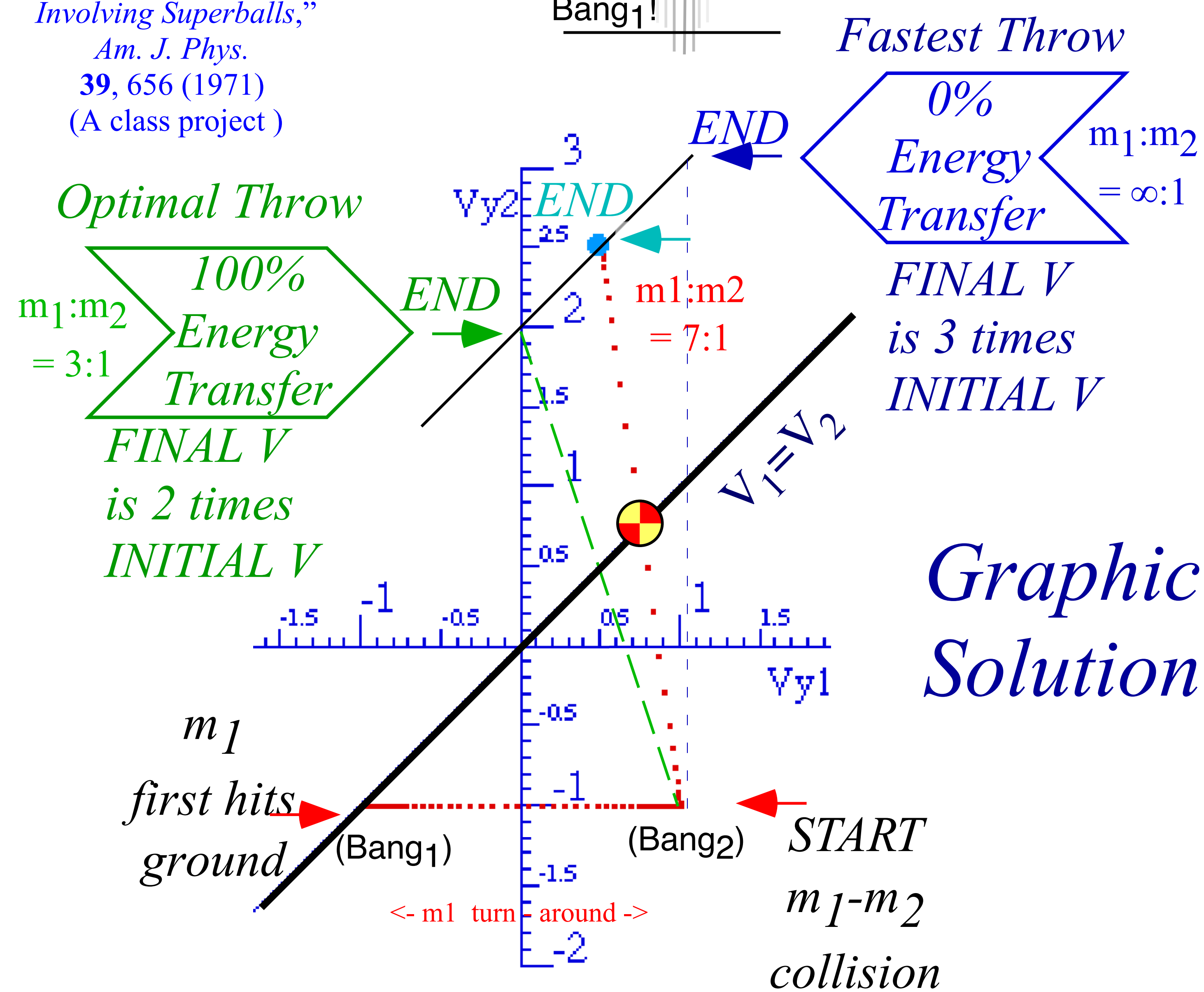


2-Bang Model





Velocity Plot (V_{y1} versus V_{y2})



NOVA trebuchet kinematics compared in Unit 2 p. 66: Direct approach and Superball analogy

The reproductions of 13th-century trebuchets in Ref. [2] were based on drawings from ingeniums used in the *1215 Siege of Kenilworth*. The modern-day ingeniators used twelve to thirteen thousand pounds of lead and sand to throw a 250 pound stone ball, that is, $M/m \sim 50$. Had they used the criterion $MR^2=4mr^2$ for *100%*-energy transfer from (2.9.24) their trebuchet would have a ratio $r/R = \sqrt{50}/2=3.5$ of the throwing beam to driving beam radii. Instead, the drawings show that $\ell \sim r \sim 42$ *ft.* is about 3.7 times the driving radius R of the sand box M. Using (2.9.22) to (2.9.25) with 100% energy transfer assumption we find the following approximate launch velocity.

$$v_{final} \cong 2\omega r = 2\sqrt{\frac{g}{R}}\, r = 2\sqrt{\frac{32}{42/3.5}}\, 42 = 137\ ft/sec. = 94\ mph \qquad (2.9.26)$$

As one might expect, this 100% estimate is below the *100* to *120 mph.* velocities achieved in Ref. [2]. By assuming the correct ratio of radii in using (2.9.22) to (2.9.25), one can bring the estimate closer.

PBS PBS.org

The plan: NOVA and a team of master builders from England, Germany, France and the United States will reconstruct one of the most destructive of medieval weapons ever made: a giant trebuchet. They will raise the weapon in the shadows of Castle Urquhart, located on the shores of Loch Ness in the Scottish Highlands.

This is one of the castles that English armies attacked during Edward I's Scottish campaign 700 years ago. As part of the campaign, the army was said to have built one of the most monstrous trebuchets ever. Only its name survives: Warwolf.

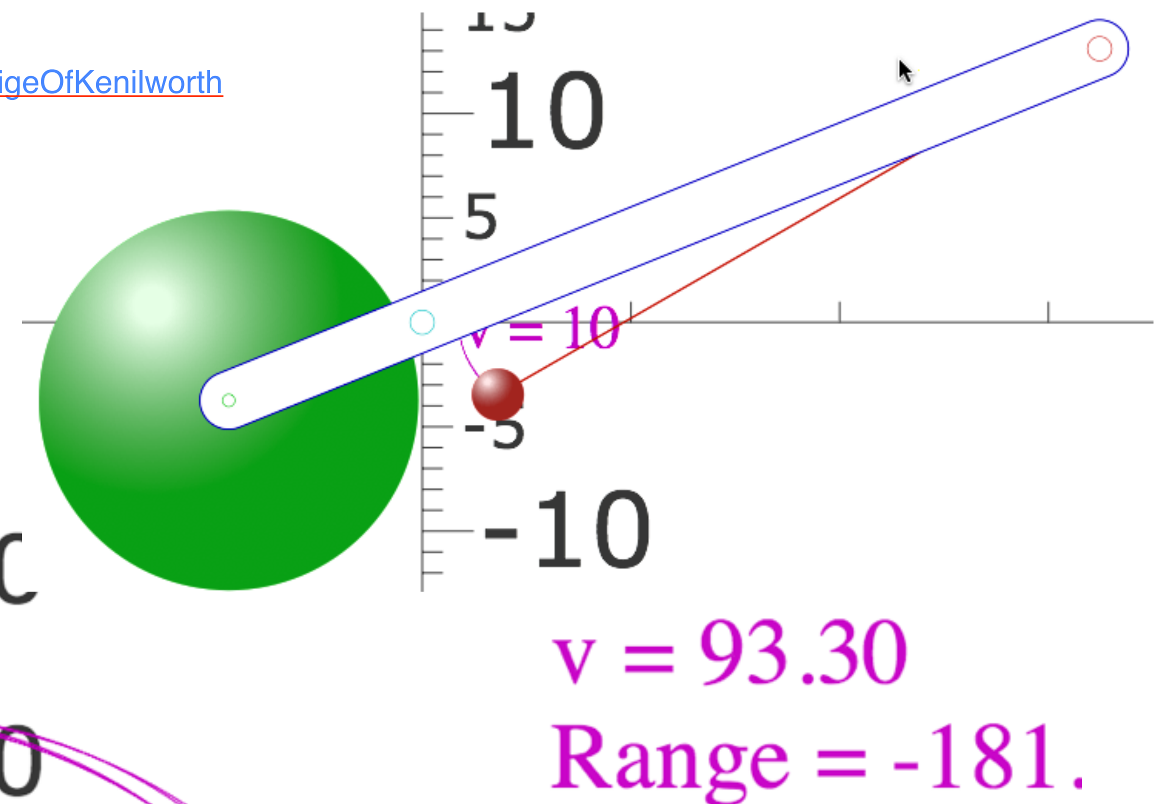
References

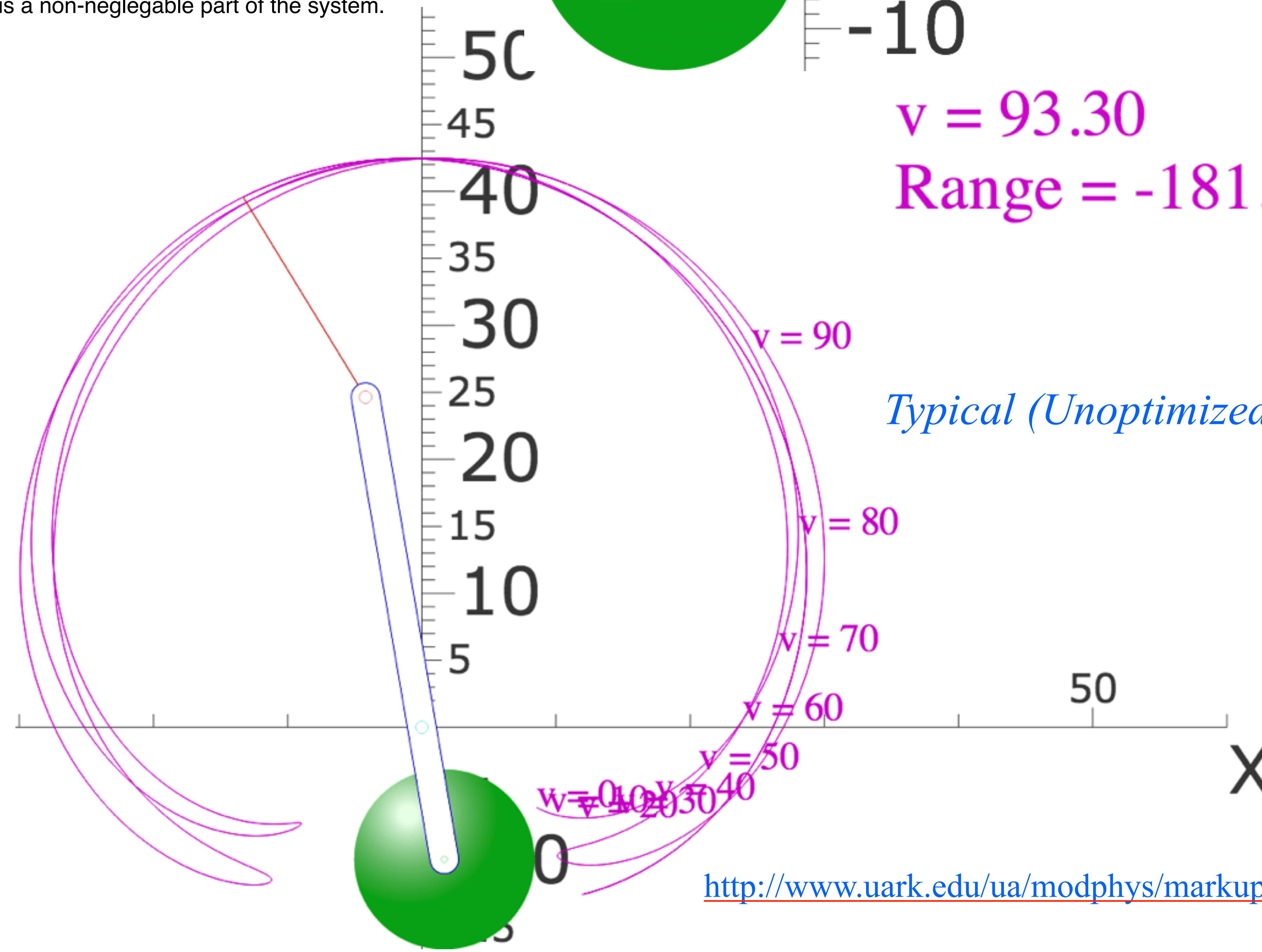
1. Paul E. Chevedden, Les Eigenbrod, Vernard Foley, and Werner Soedel, *"The Trebuchet"*, *Scientific American* **273**, 66-71 (July 1995).
2. Evan Hadingham and Patrick Ward, *"Ready, Aim, Fire!"*, *Smithsonian* (January 2000) p. 80.

NOVA links from beginning of Lect. 14

https://www.pbs.org/wgbh/nova/lostempires/trebuchet/builds.html http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth

Here is the system, again, at NCAR. Four pivot
https://www.youtube.com/watch?v=zvIY1z0xcek
Another rig:
https://www.youtube.com/watch?v=foZHjI8Lydo
Ha! the clock face is a non-neglegable part of the system.

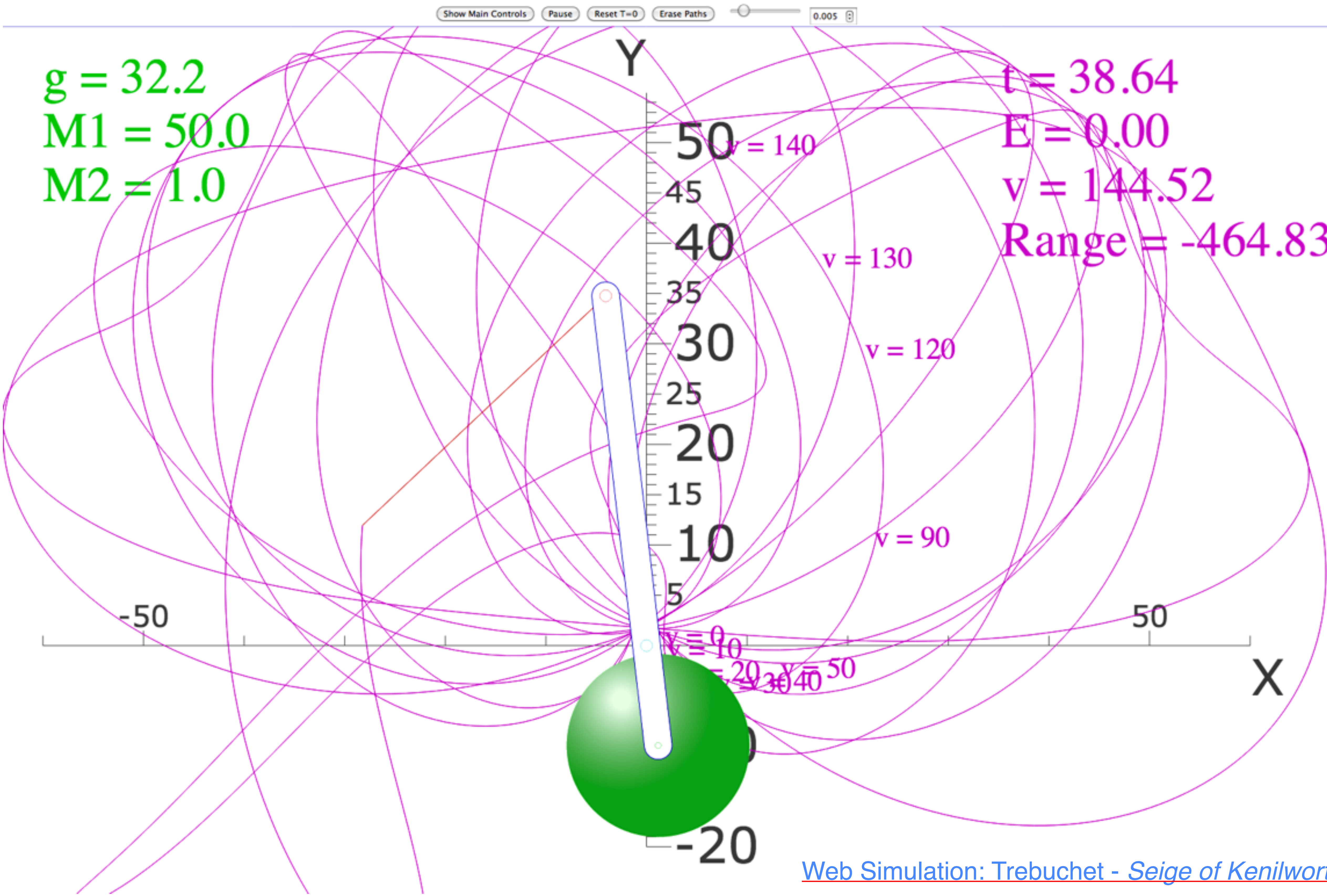




Typical (Unoptimized) Trebuchet

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

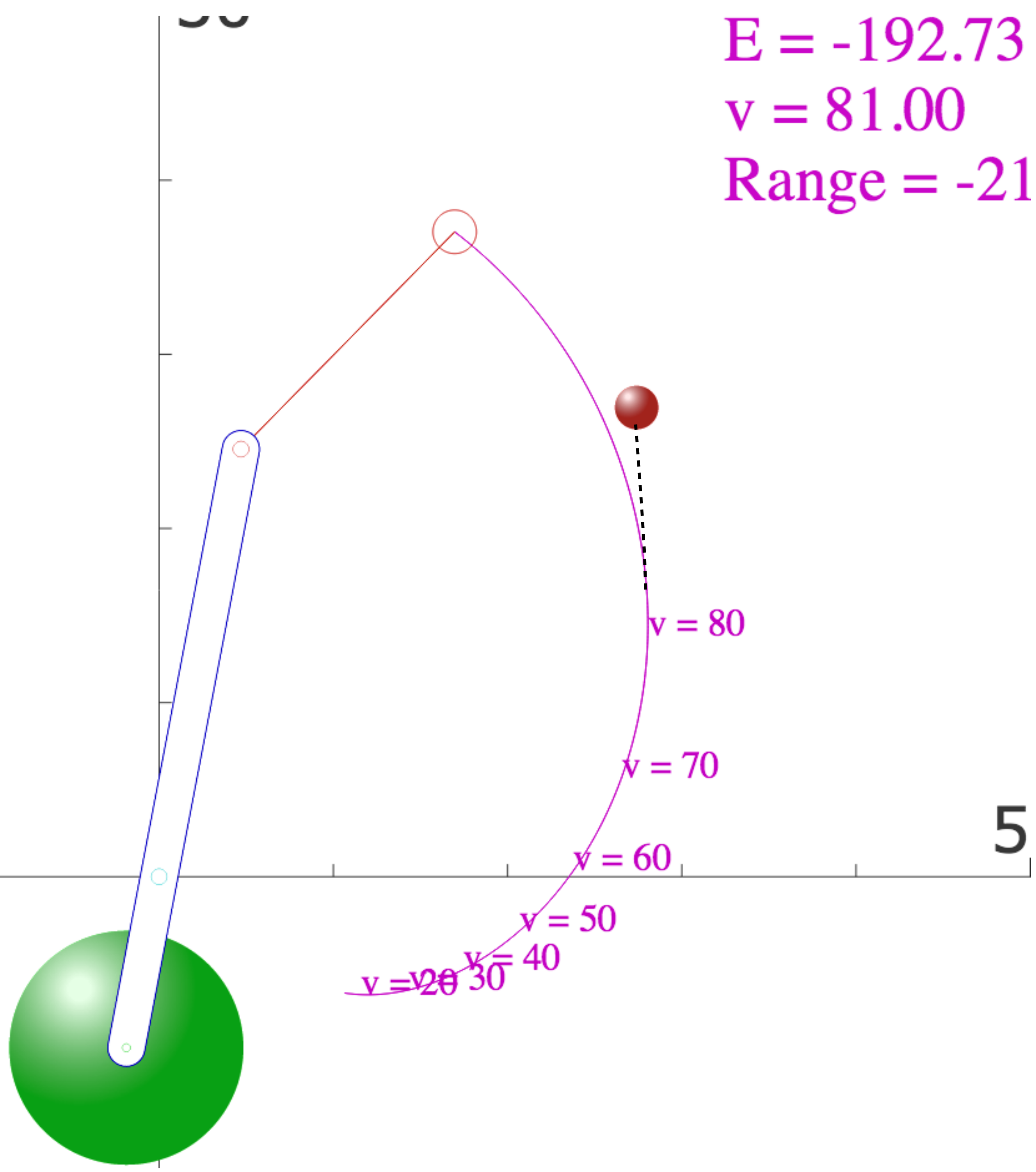
Trebuchet in Siege of Kenilworth 1215 ACE
(Re-enactment shown on NOVA-TV 2005)



Web Simulation: Trebuchet - *Seige of Kenilworth*

There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...

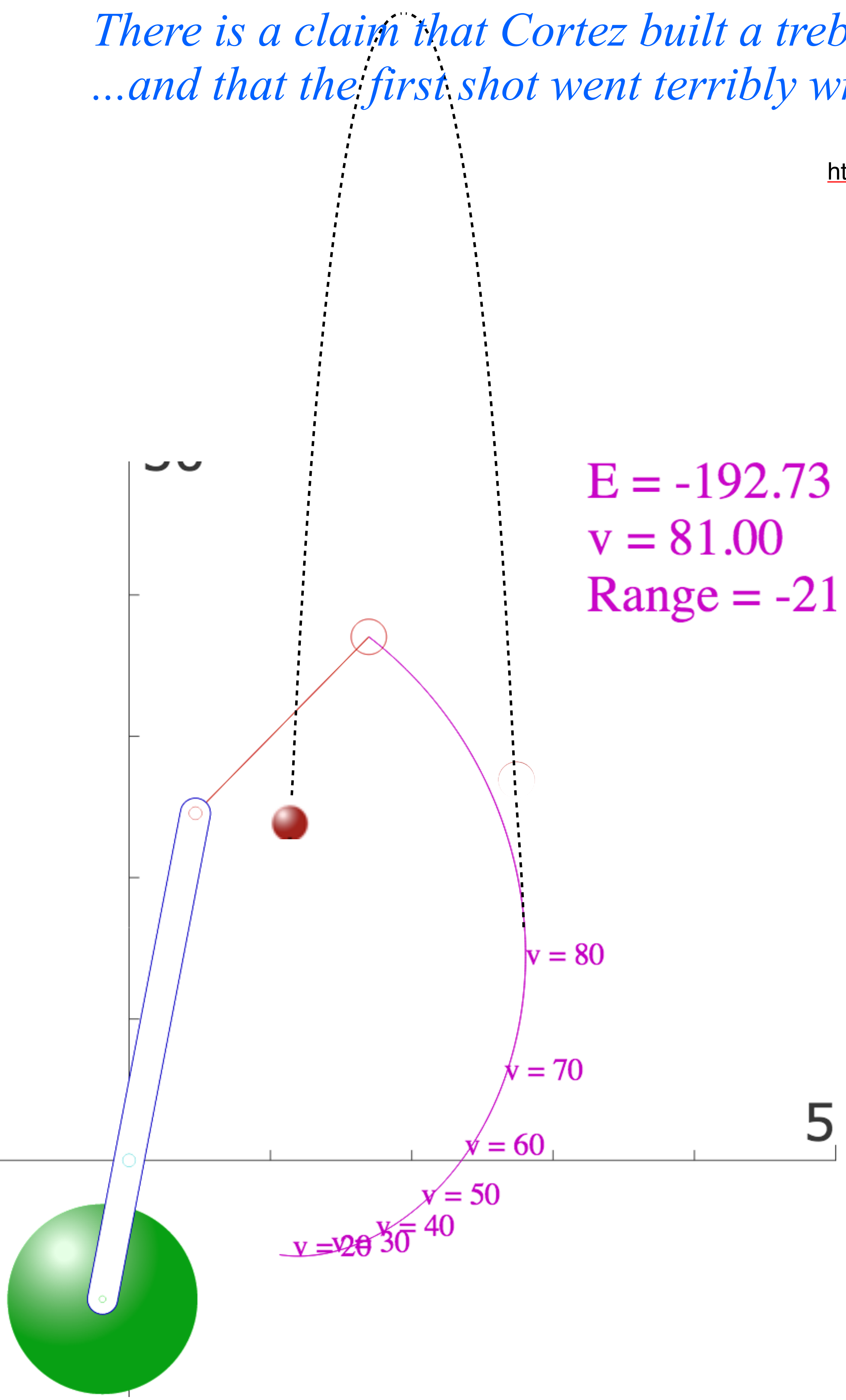
http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge



Web Simulation: Trebuchet - *Montezuma's Revenge*

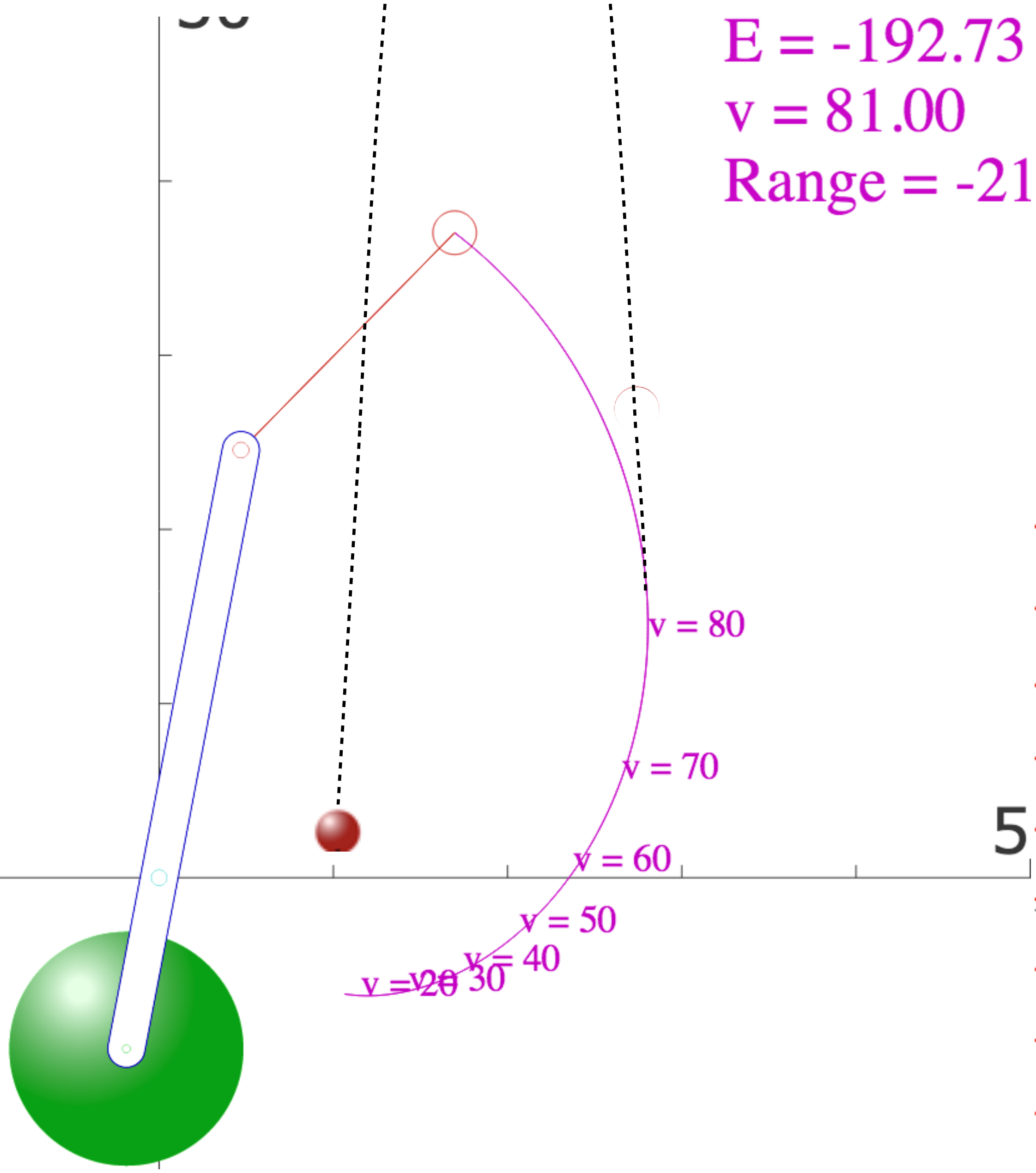
There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

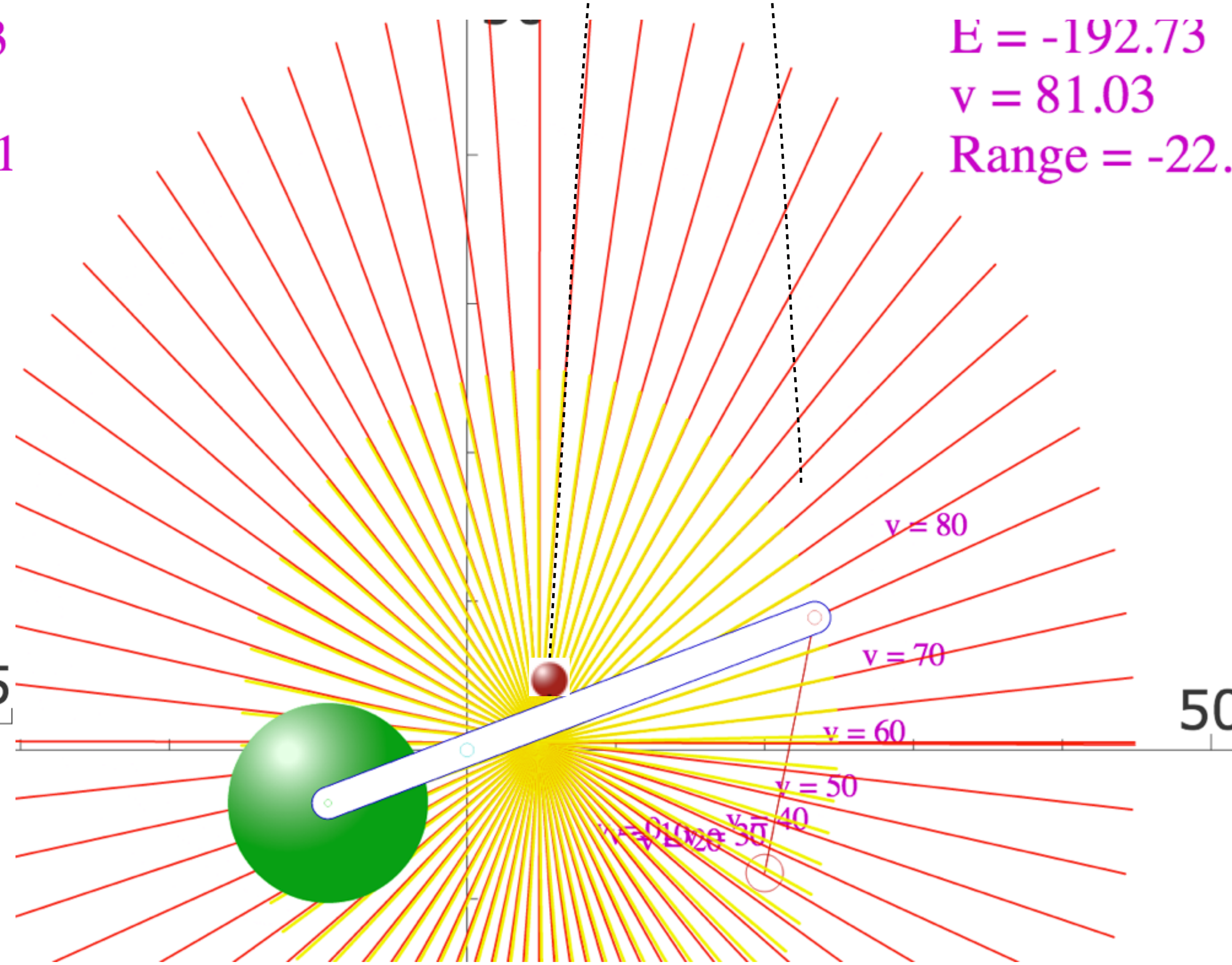
http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge



There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

...if this story is true, then it gives new meaning to the expression "Montezuma's Revenge"...





[Web Simulation: Trebuchet - *Montezuma's Revenge*]

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

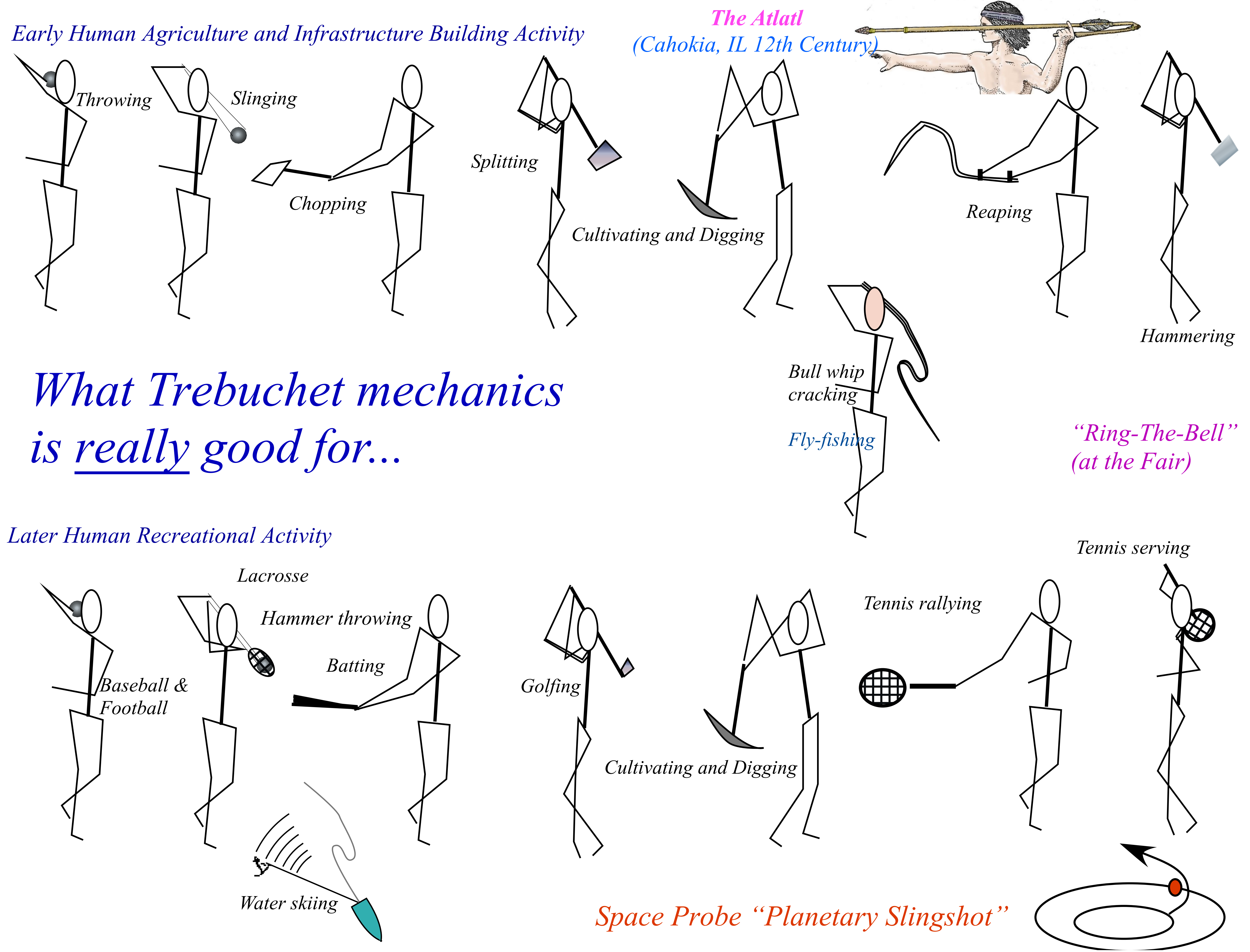
Free-space trebuchet kinematics by symmetry

Algebraic approach

Direct approach and Superball analogy

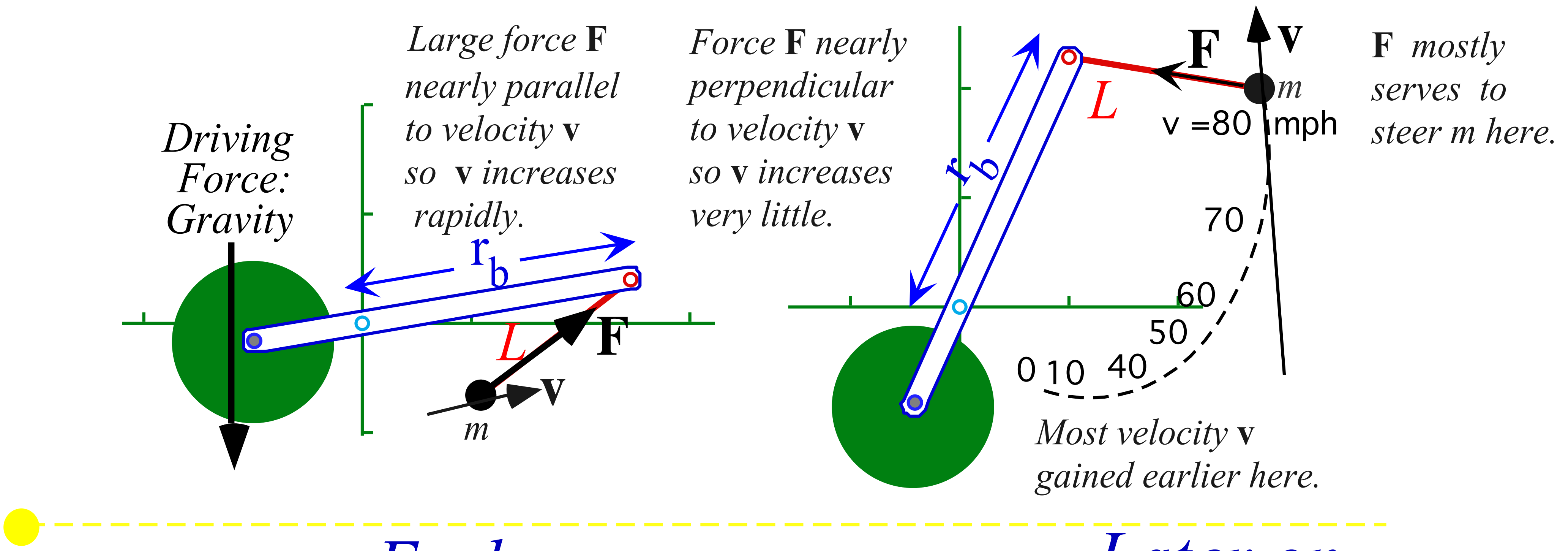
Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics



Early Human Agriculture and Infrastructure Building Activity
The Atlatl
(Cahokia, IL 12th Century)
Throwing
Slinging
Chopping
Splitting
Cultivating and Digging
Reaping
Hammering
Bull whip
cracking
Fly-fishing
What Trebuchet mechanics
is really good for...
"Ring-The-Bell"
(at the Fair)
Later Human Recreational Activity
Lacrosse
Hammer throwing
Batting
Baseball &
Football
Golfing
Cultivating and Digging
Tennis rallying
Tennis serving
Water skiing
Space Probe "Planetary Slingshot"

Trebuchet analogy with racquet swing - What we learn



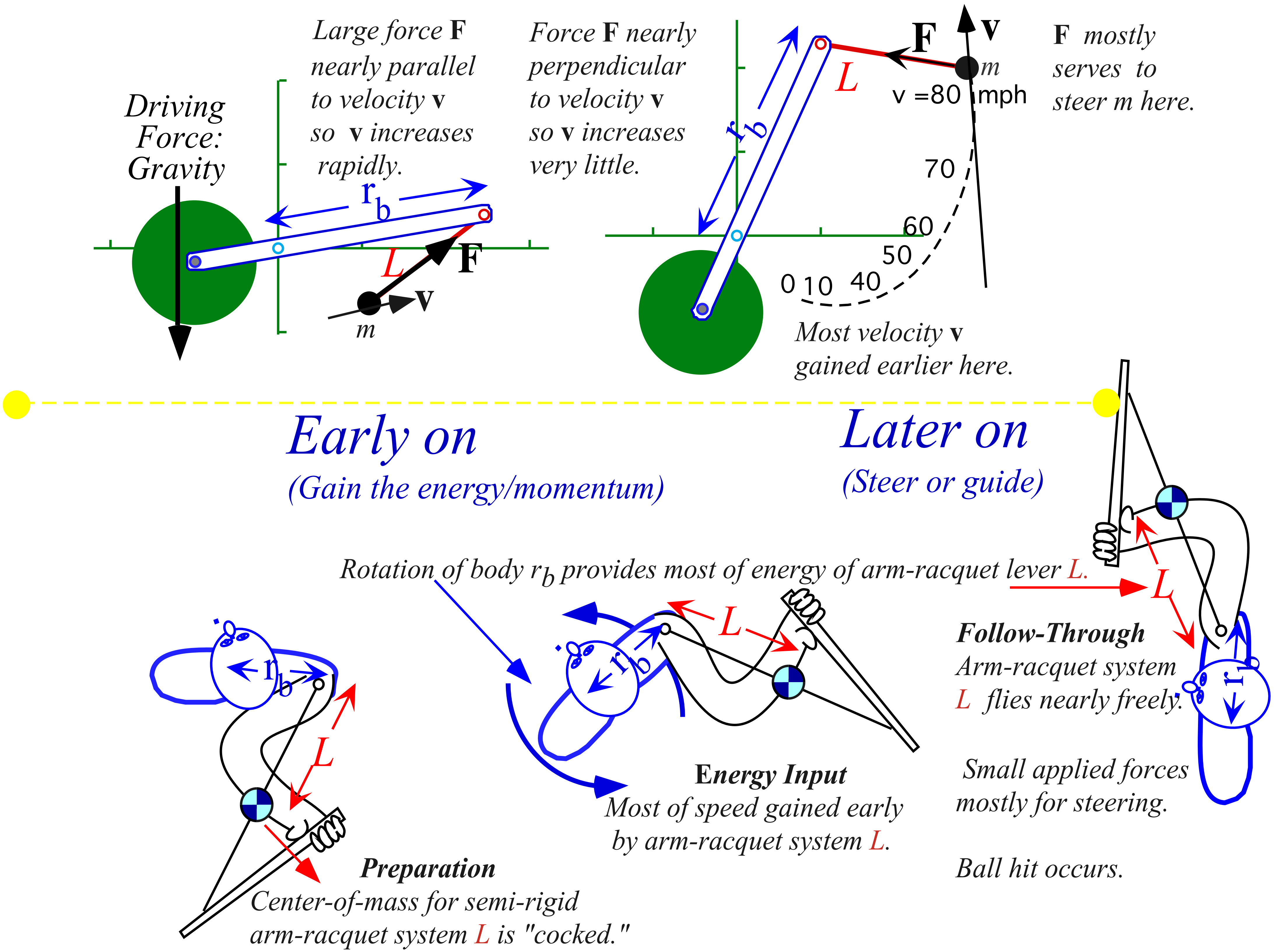
Early on

(Gain the energy/momentum)

Later on

(Steer or guide)

Trebuchet analogy with racquet swing - What we learn

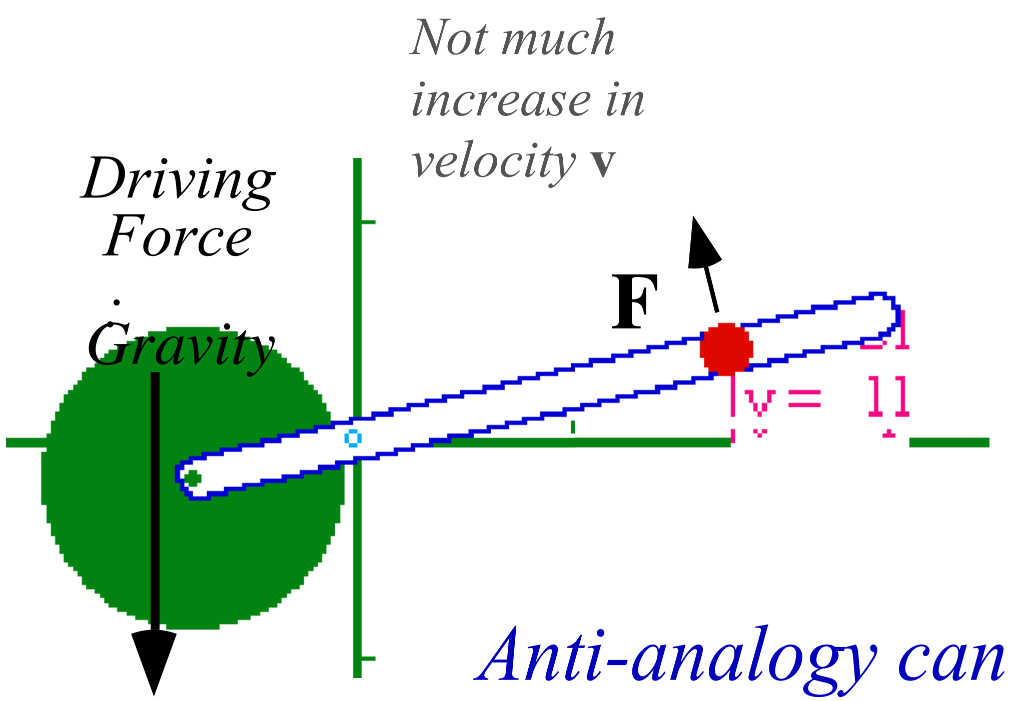


An Opposite to Trebuchet Mechanics- The “Flinger”

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=AnimateFlinger

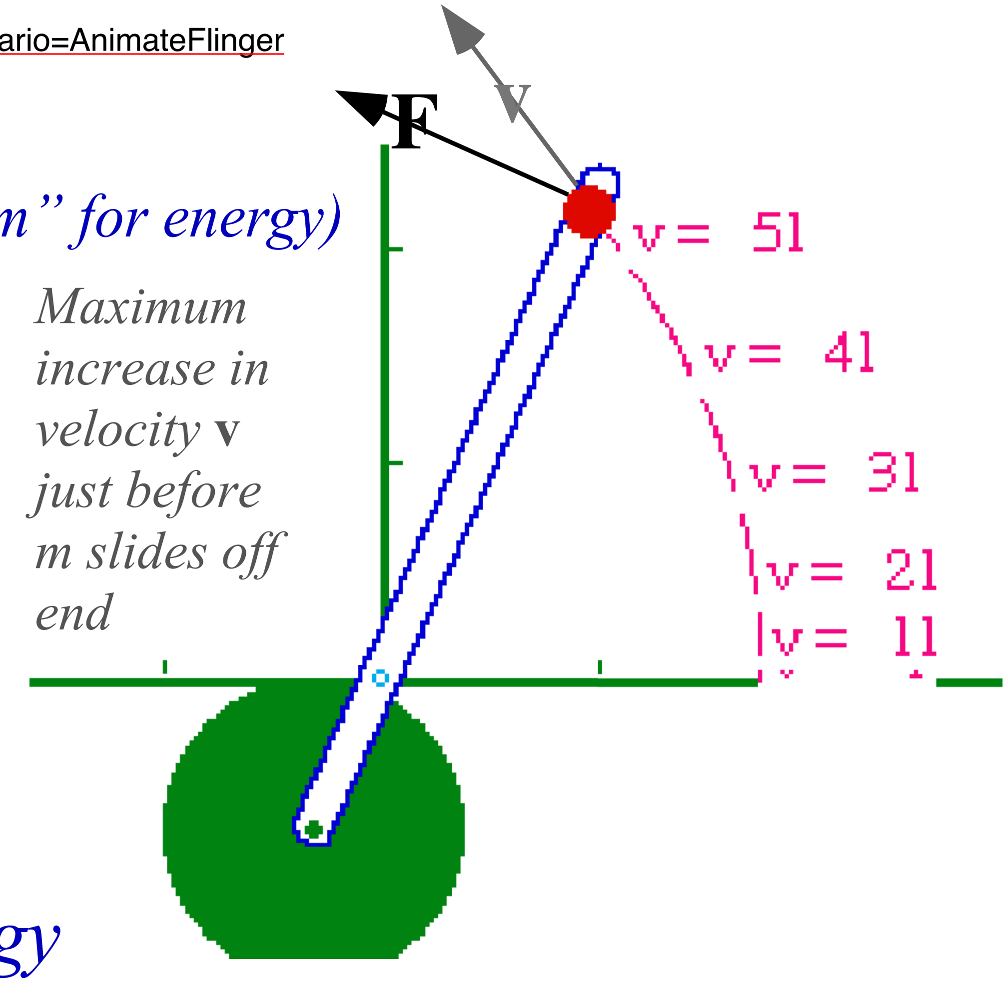
Early on

(Not much happening)



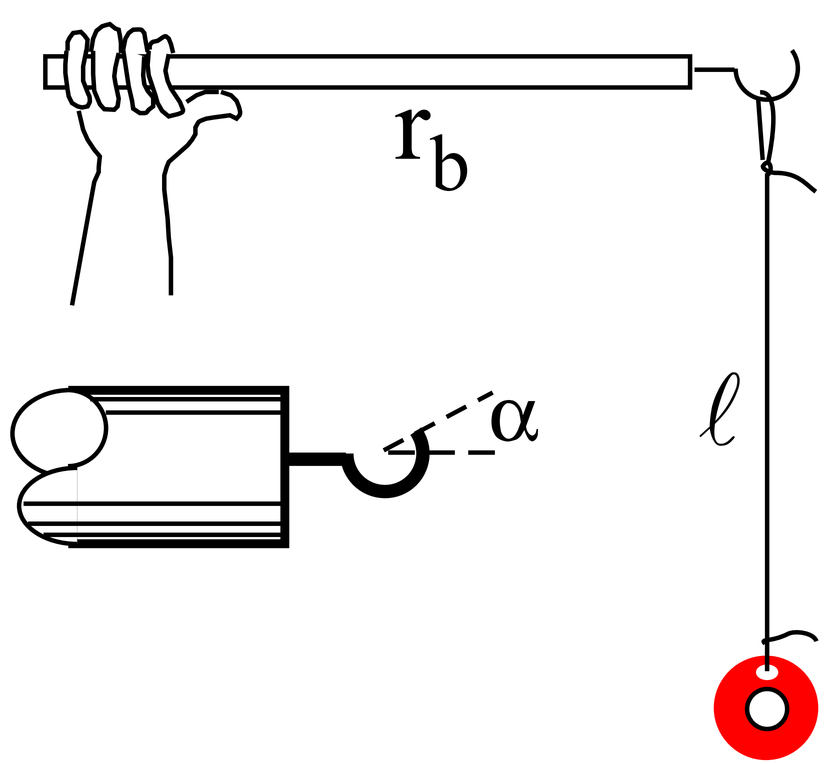
Later on

(Last-minute “cram” for energy)



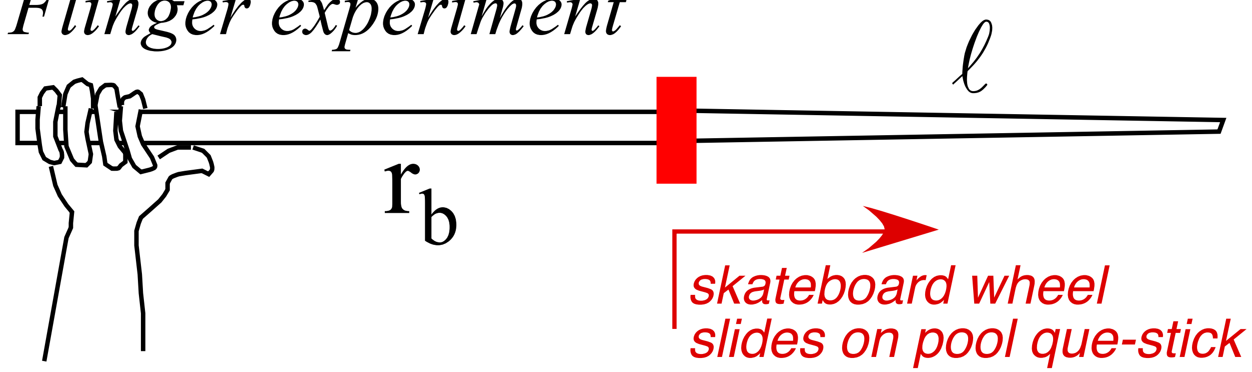
Anti-analogy can be useful pedagogy

Trebuchet-like experiment



skateboard wheel swings

Flinger experiment

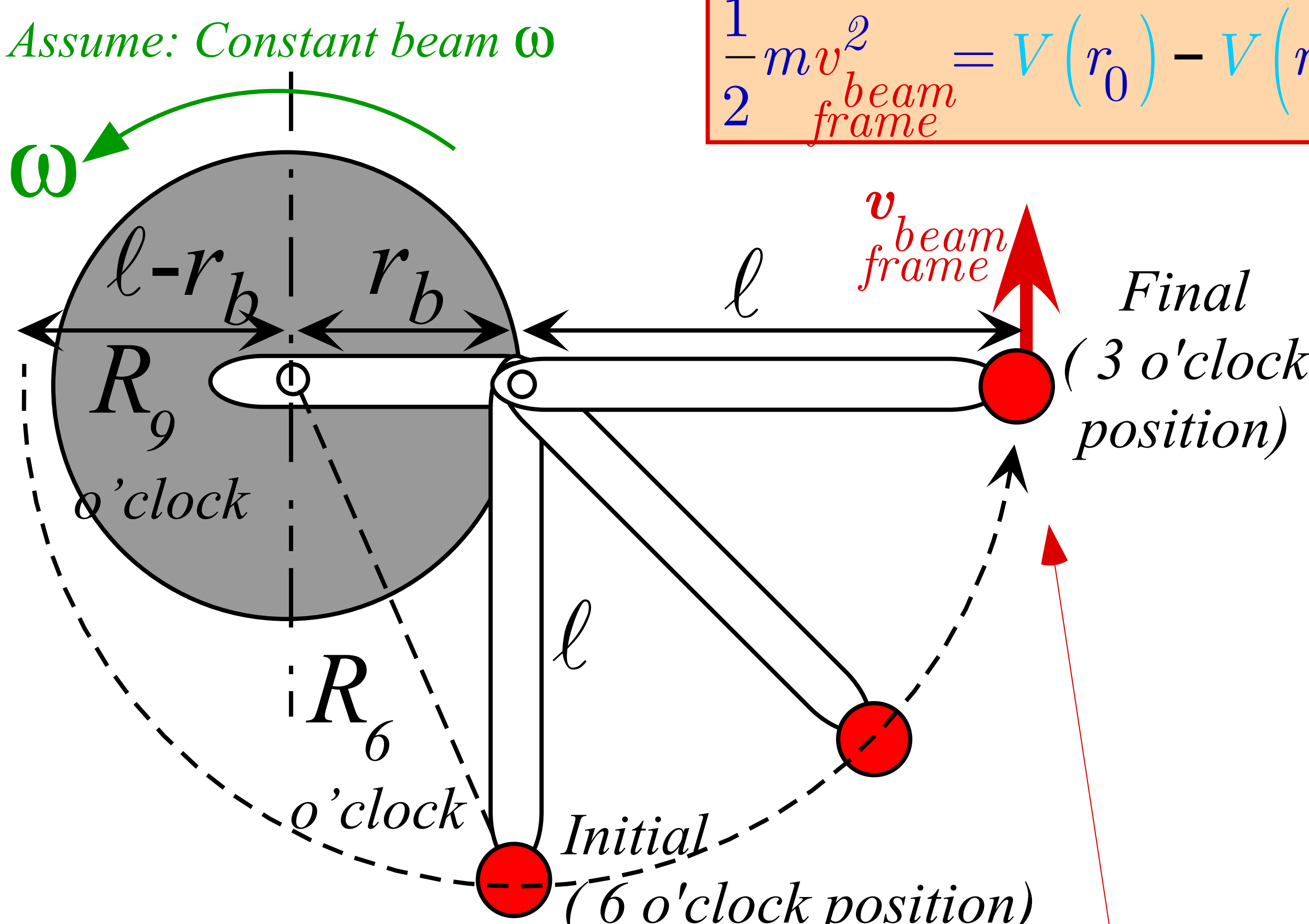


Generic URL - http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

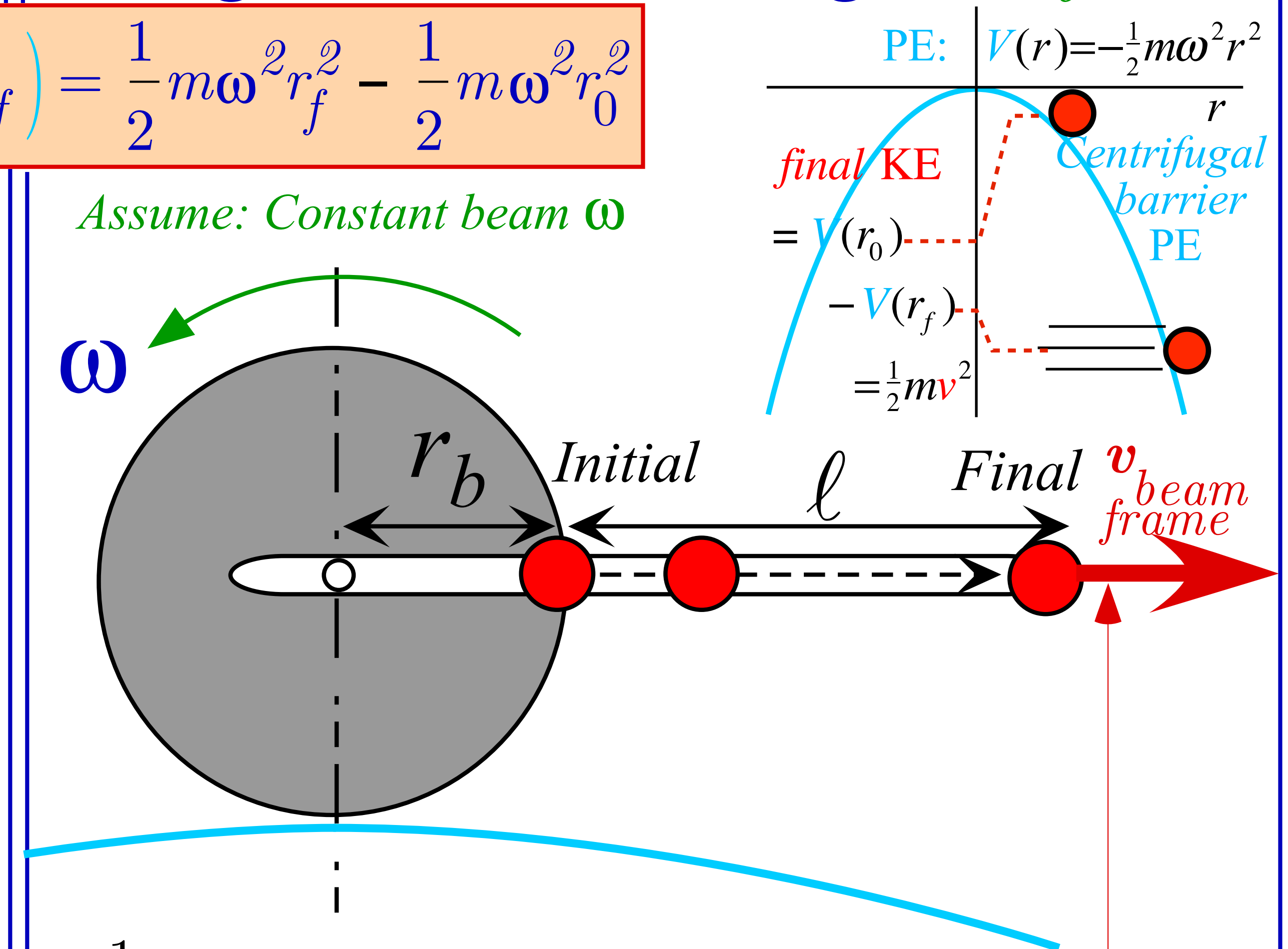
Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$



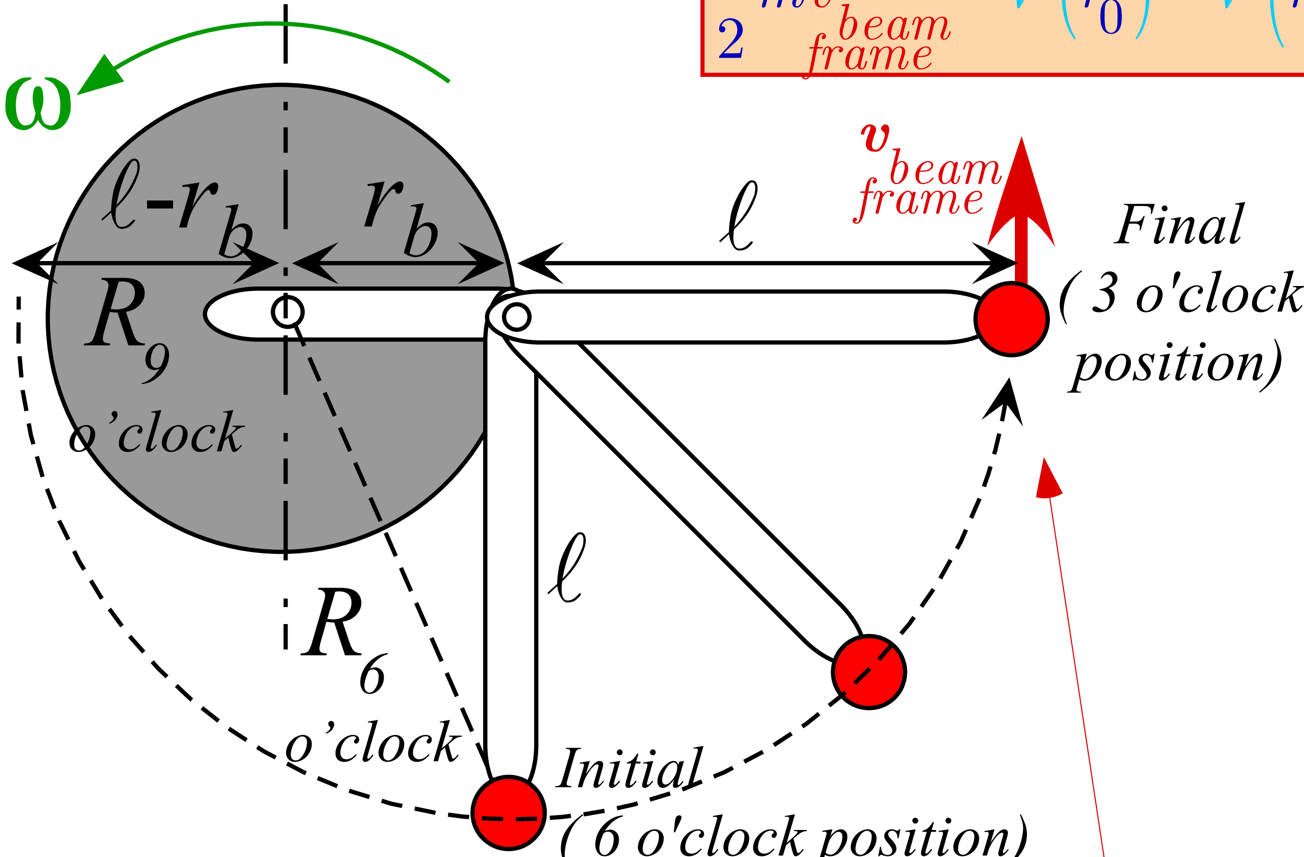
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

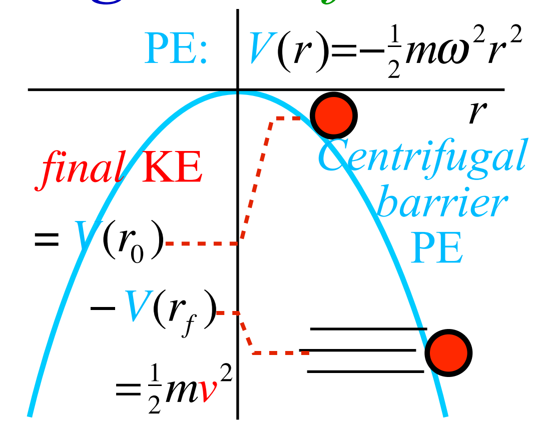
$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω

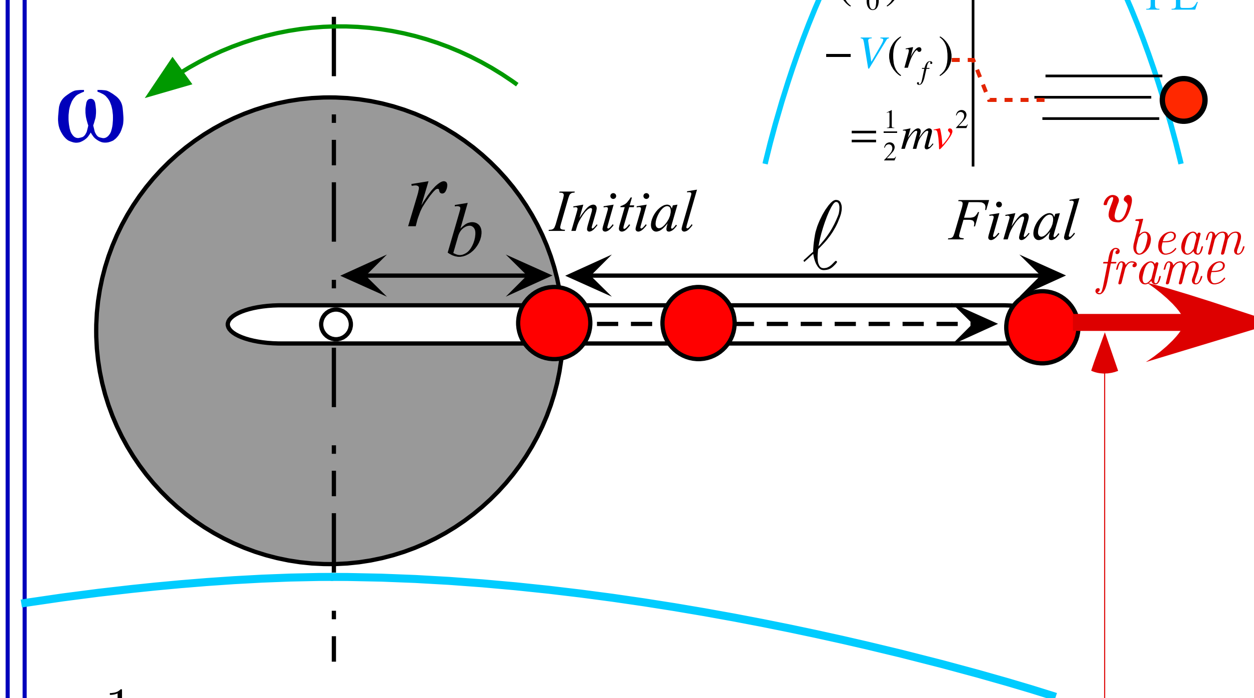


$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \textit{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{Final} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)}_{Initial} = \frac{1}{2}m\omega^2\left(2r_b\,\ell\right)$$



Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \textit{Final Flinger } KE_{\substack{beam\\frame}}$$

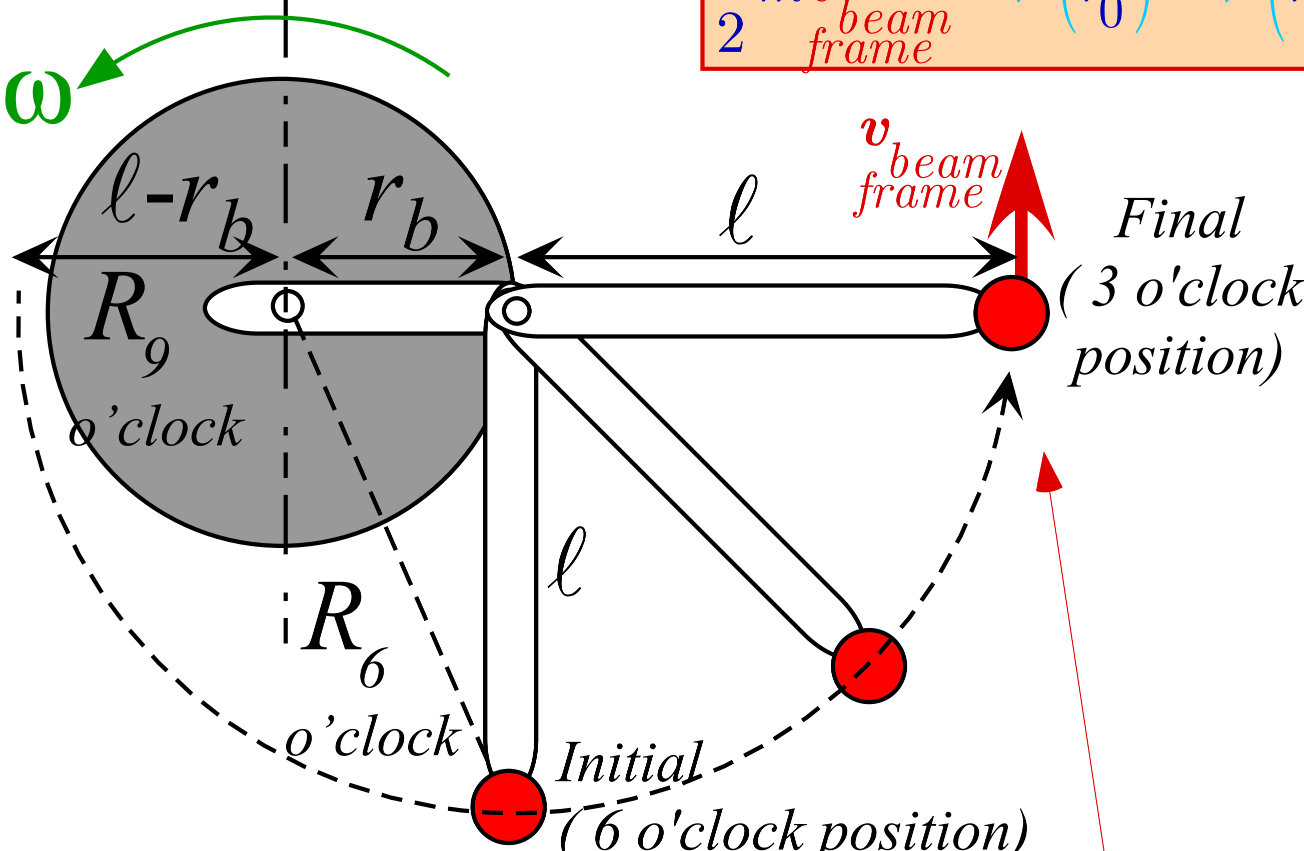
$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{Final} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{Initial} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

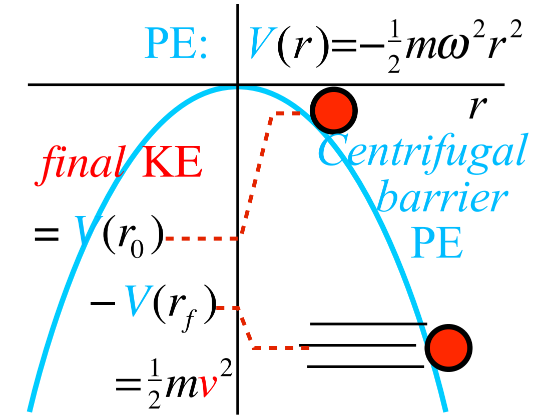
Assume: Constant beam ω



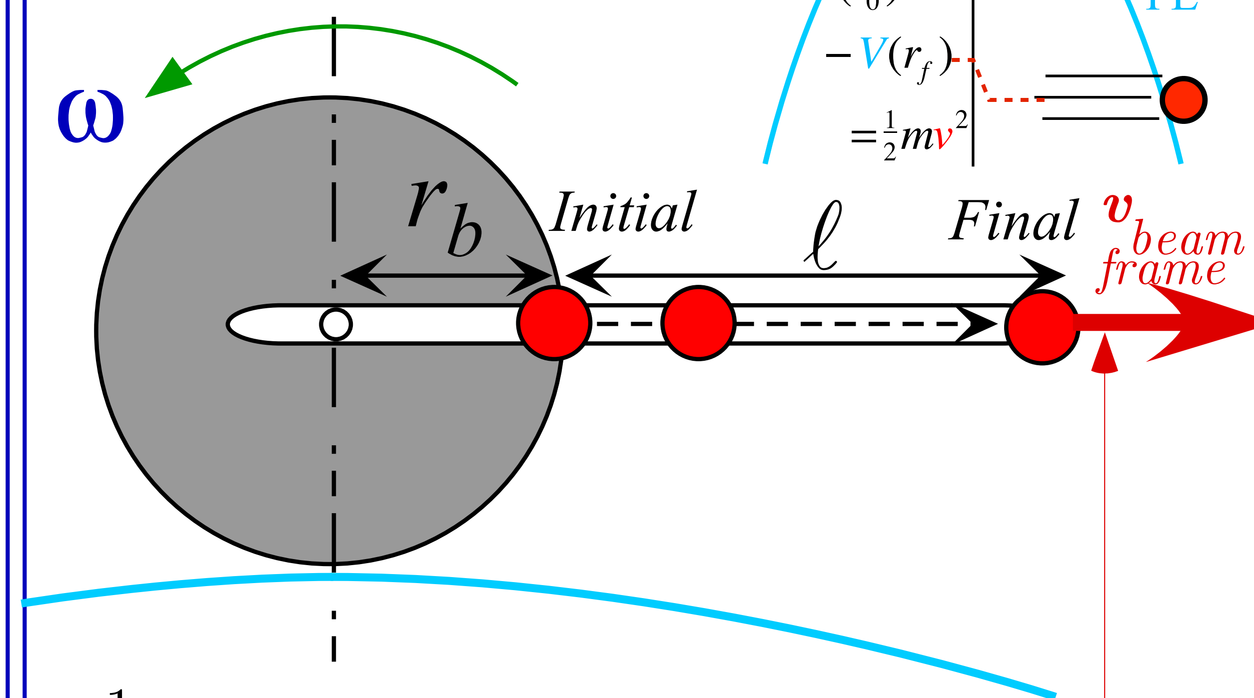
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \color{red}{\textit{Final Trebuchet } KE_{\substack{beam\\frame}}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2}_{\substack{Final\\3\ o'clock}} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2 + \ell^2\right)}_{\substack{Initial\\6\ o'clock}} = \frac{1}{2}m\omega^2\left(2r_b\,\ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2 + \ell^2$$



Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \color{red}{\textit{Final Flinger } KE_{\substack{beam\\frame}}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b + \ell\right)^2}_{\substack{Final\\3\ o'clock}} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{\substack{Initial\\3\ o'clock}} = \frac{1}{2}m\omega^2\ell\left(2r_b + \ell\right)$$

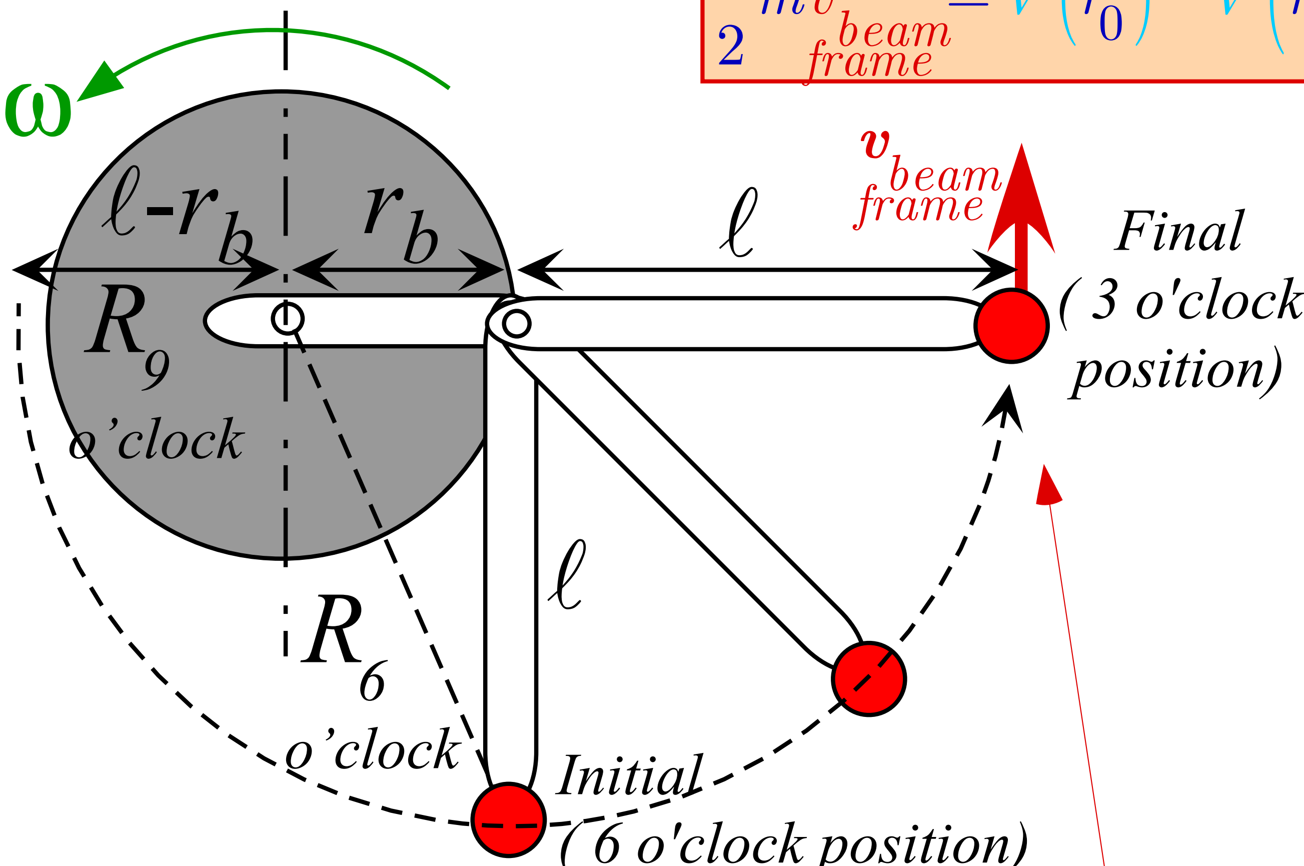
Flinger KE is $\frac{m\omega^2}{2}\ell^2$ *more than 6 o'clock trebuchet but* *misdirected*

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) \quad V(r_f) = \frac{1}{2}m\omega^2 r_f^2 \quad \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω

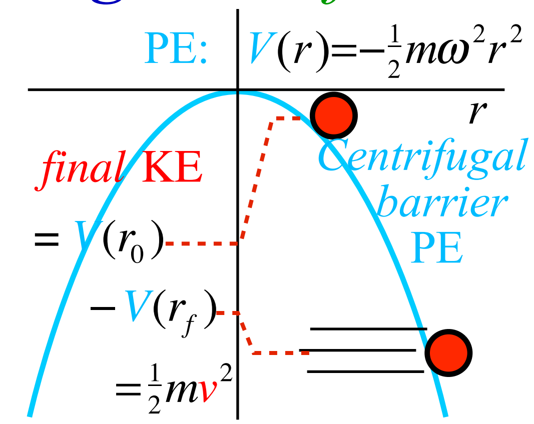


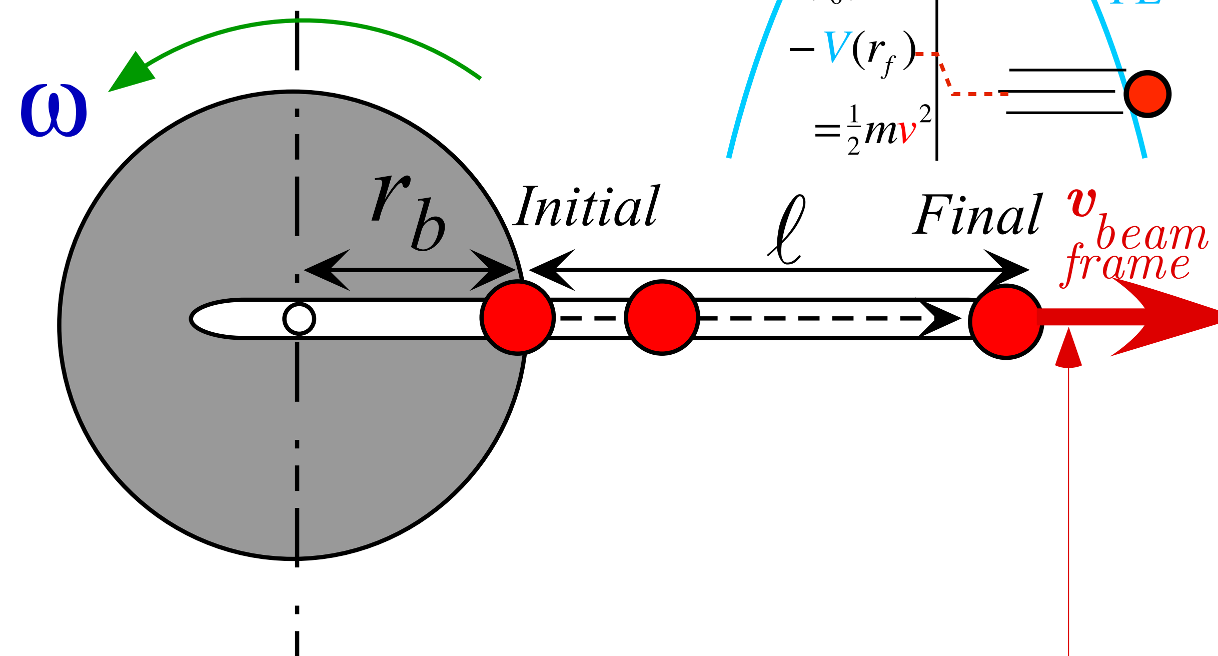
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{\substack{Final\\3\ o'clock}} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)}_{\substack{Initial\\6\ o'clock}} = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2+\ell^2$$

Assume: Constant beam ω





$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{\substack{Final\\3\ o'clock}} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{\substack{Initial\\3\ o'clock}} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

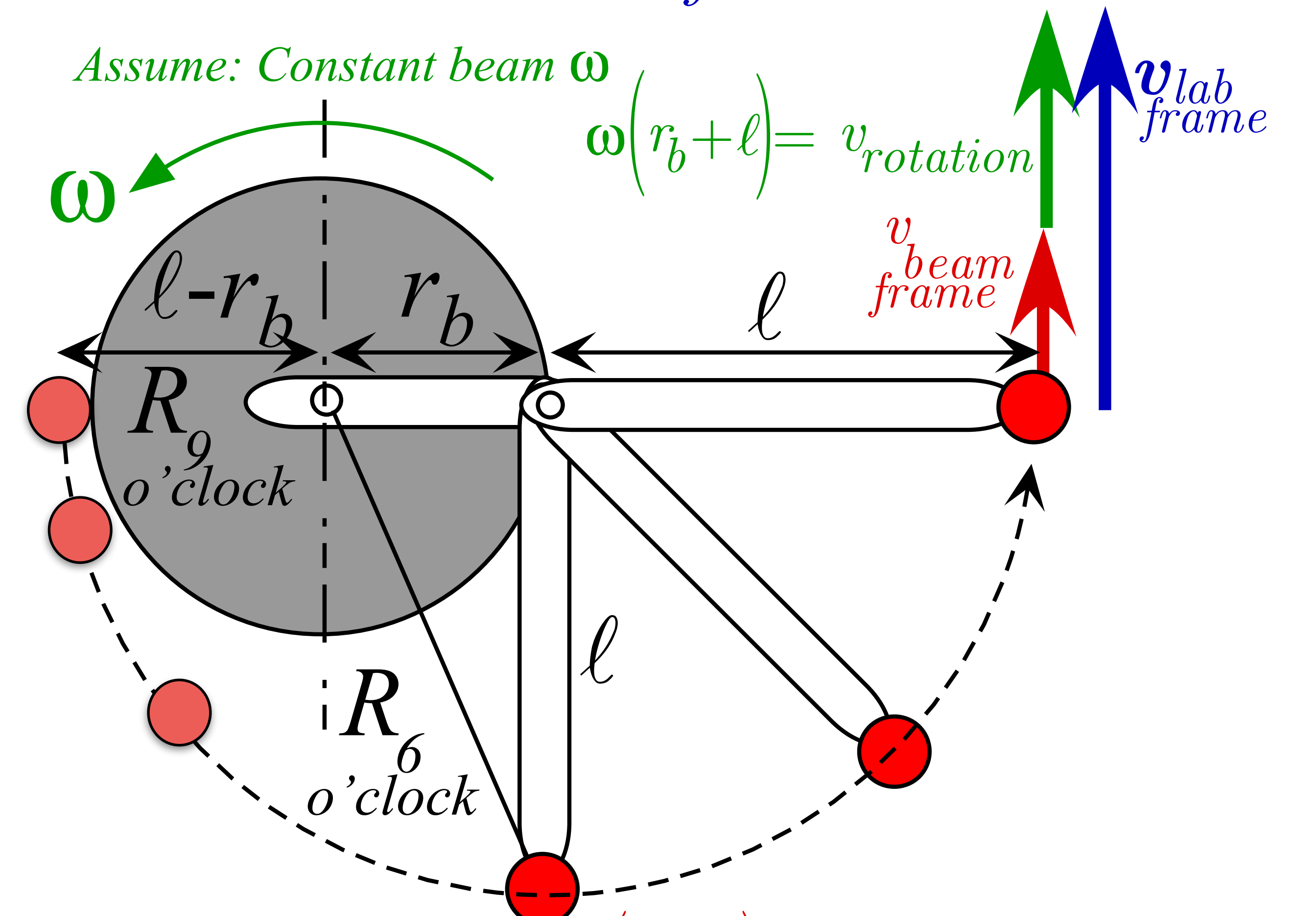
Flinger KE is $\frac{m\omega^2}{2}\ell^2$ *more than 6 o'clock trebuchet but* *misdirected*

$$\underset{\substack{Initial\\9\ o'clock}}{} = \frac{1}{2}m\omega^2\left(4r_b\ \ell\right)$$

$$R^2_{\substack{9\\o'clock}} = r_b^2+\ell^2-2r_b\ell$$

Flinger KE is $\frac{m\omega^2}{2}\left(2r_b\ell-\ell^2\right)$ *less than 9 o'clock trebuchet* *and misdirected*

Trebuchet model in lab frame



$$v^2_{beam\ frame}(\text{trebuchet}) = \begin{cases} \omega^2(2r_b\,\ell) & \text{half-cocked 6 o'clock} \\ \omega^2(4r_b\,\ell) & \text{fully-cocked 9 o'clock} \end{cases}$$

$$v_{lab\ frame}(\text{trebuchet}) =$$

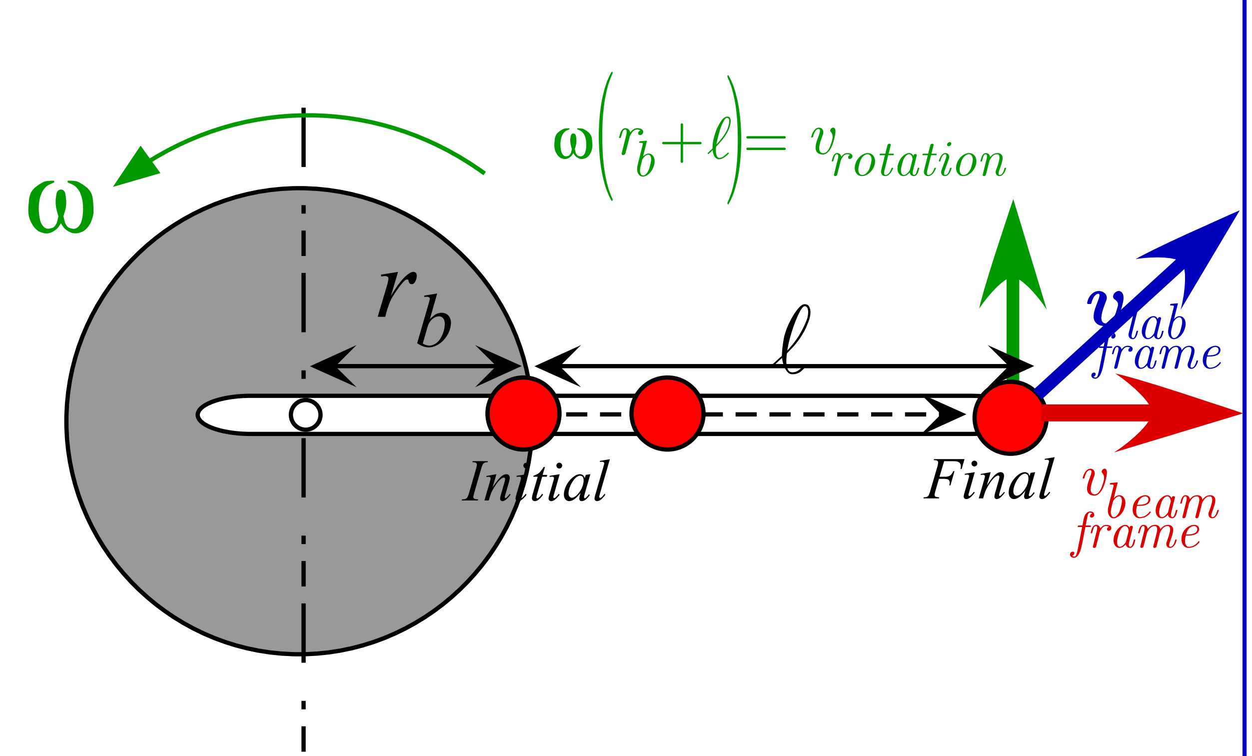
$$\begin{cases} \omega\left(r_b + \ell + \sqrt{2\ell r_b}\right) & \text{half-cocked 6 o'clock} \\ \omega\left(r_b + \ell + 2\sqrt{\ell r_b}\right) & \text{fully-cocked 9 o'clock} \end{cases}$$

$$= \begin{cases} 5.00\omega \\ 5.82\omega \end{cases} \qquad = \begin{cases} 5.16\omega \\ 6.00\omega \end{cases} \qquad = \begin{cases} 5.00\omega \\ 5.82\omega \end{cases}$$

$$(r_b = 2,\ \ell = 1),\ (r_b = 1.5,\ \ell = 1.5),\ (r_b = 1,\ \ell = 2)$$

(compare)

Flinger model in lab frame



$$v^2_{beam\ frame}(\text{flinger}) = \omega^2\ell(2r_b + \ell)$$

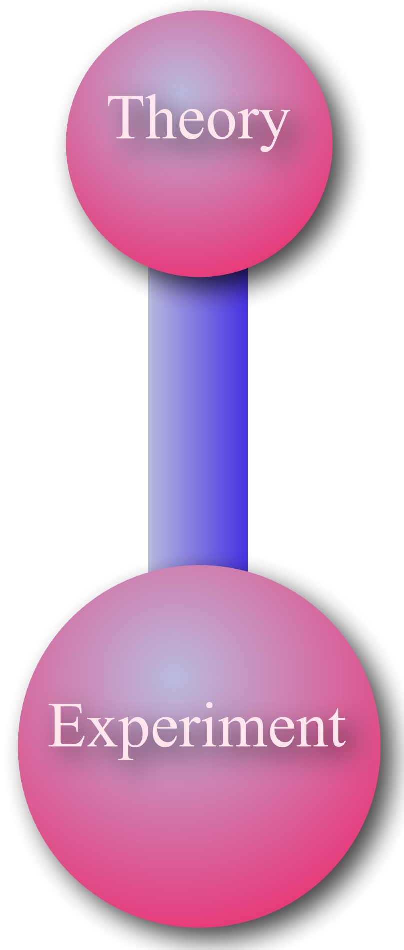
$$v_{lab\ frame}(\text{flinger}) =$$

$$= \omega\sqrt{(r_b + \ell)^2 + \ell(2r_b + \ell)} = \omega\sqrt{2(r_b + \ell)^2 - r_b^2}$$

$$= 3.74\omega \qquad = 3.96\omega \qquad = 4.12\omega$$

$$(r_b = 2,\ \ell = 1),\ (r_b = 1.5,\ \ell = 1.5),\ (r_b = 1,\ \ell = 2)$$

Physics used to be pretty much bi-polar...



Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

Quick'n dirty

Newton F=Ma Equations

Cartesian coordinates

- French Approach

Tres elegant

Lagrange Equations

in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

- German Approach

Pride and Precision

Riemann Christoffel Equations

in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

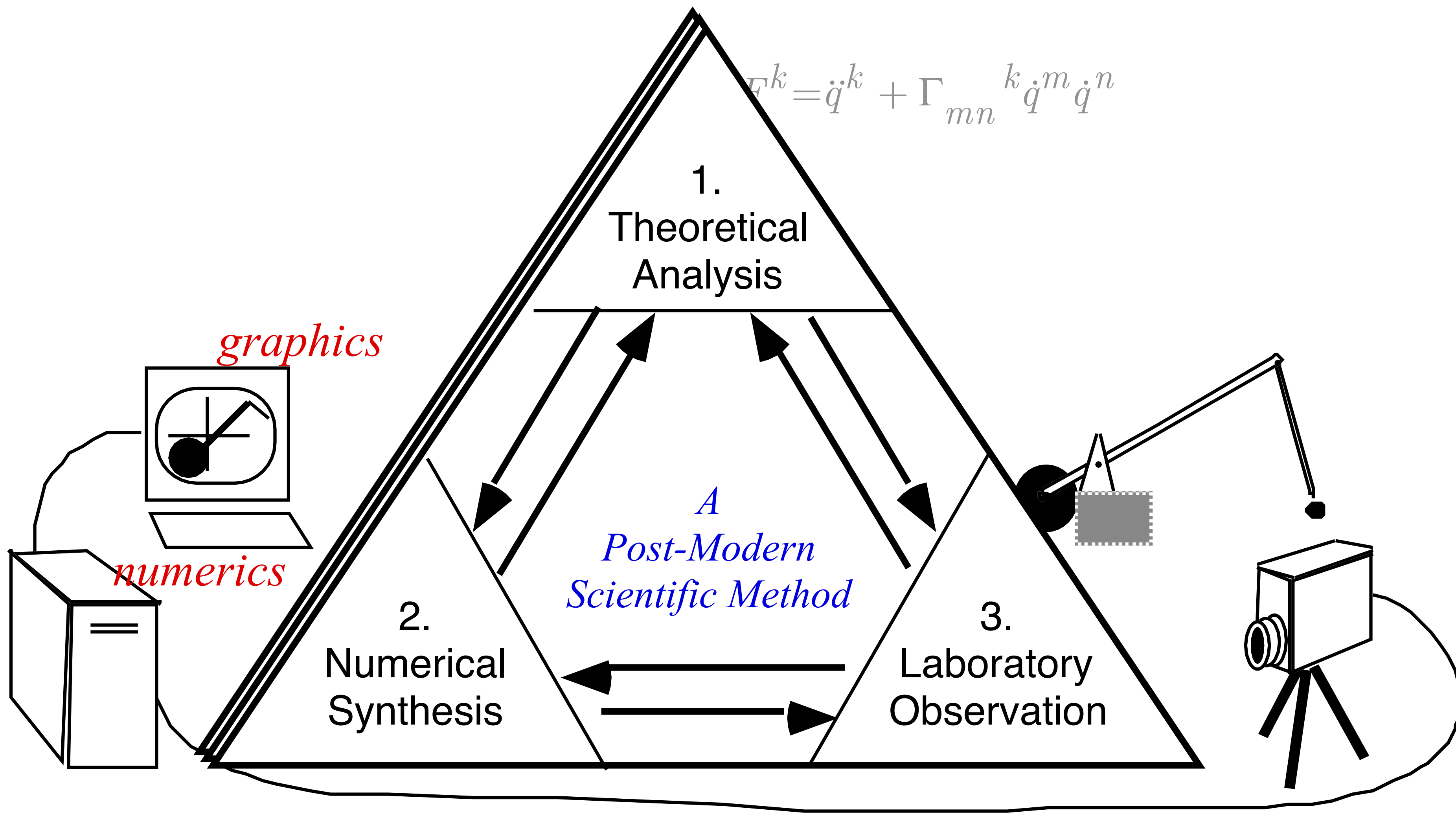
- Anglo-Irish Appproach

Powerfully Creative

Hamilton's Equations

Phase Space $\quad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$

- Unified Approach



All approaches have one thing in common:

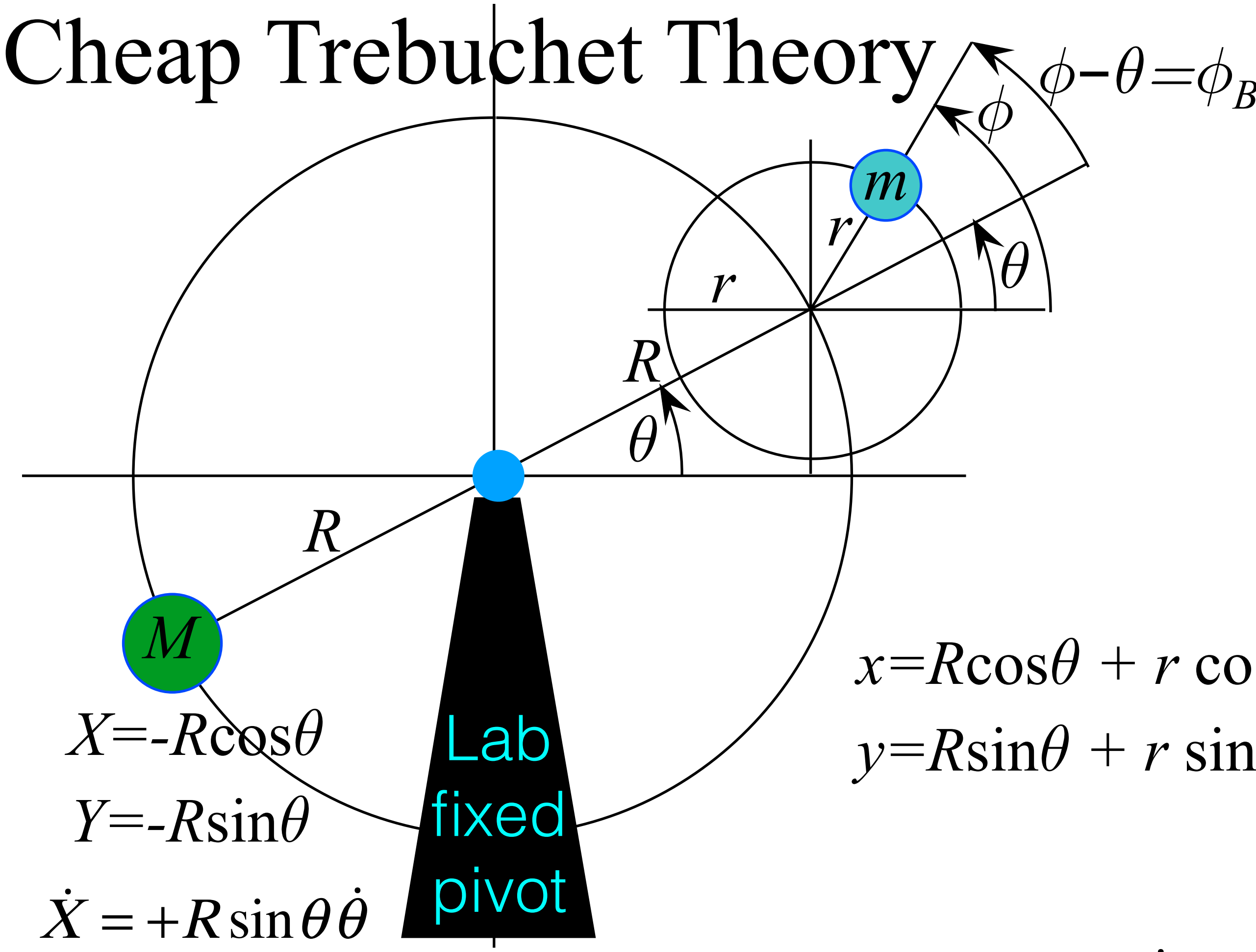
The *Art of Approximation*

Physics lives and dies by the art of approximate models and analogs.

$dS = Ldt = p_\mu dq^\mu - Hdt$

Hamilton-Jacobi-Poincare: $p_\mu = \frac{\partial S}{\partial q^\mu}, \; -H = \frac{\partial S}{\partial t}$

Cheap Trebuchet Theory

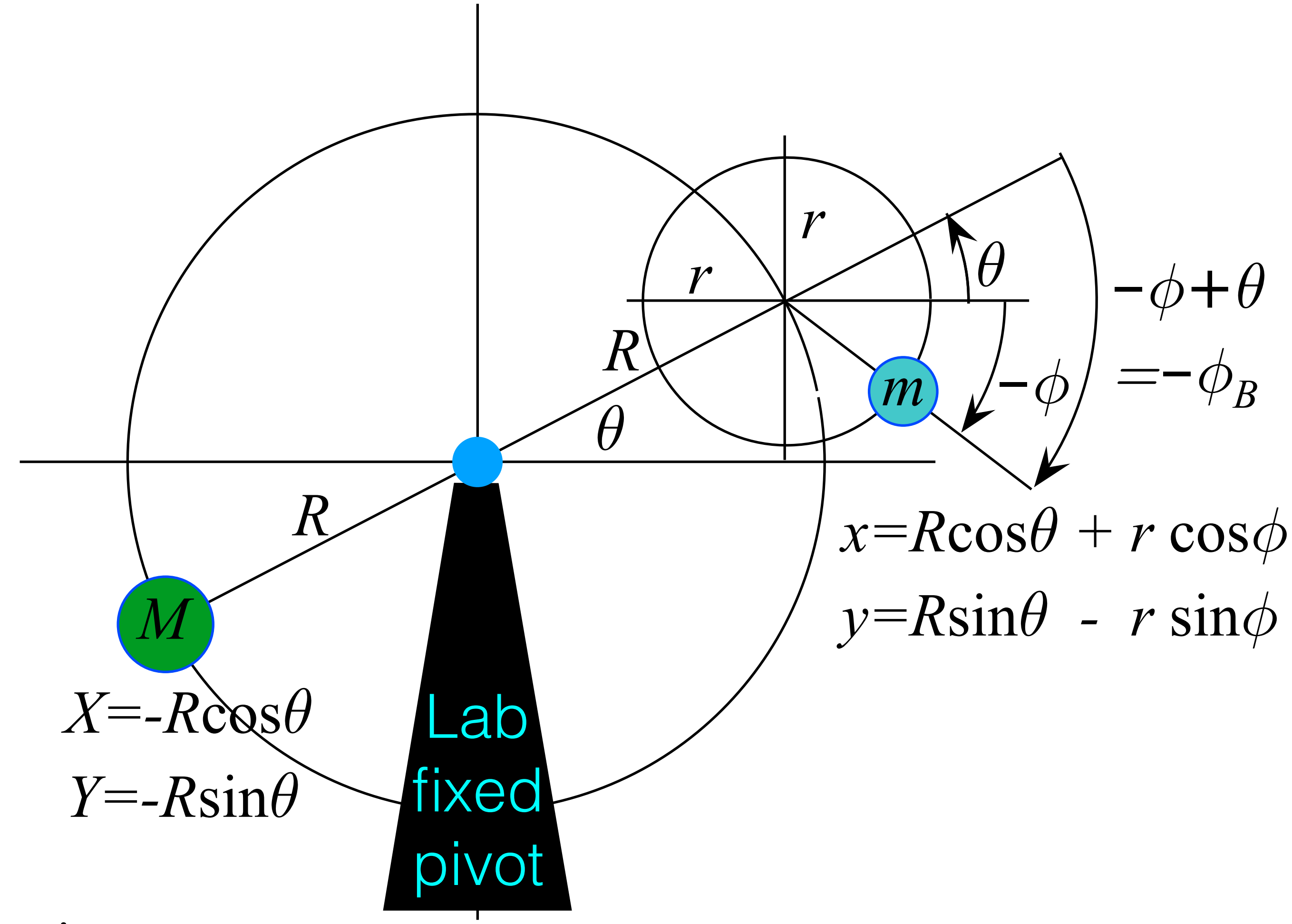


$X = -R\cos\theta$

$Y = -R\sin\theta$

$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta + r\sin\phi$



$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta - r\sin\phi$

$X = -R\cos\theta$

$Y = -R\sin\theta$

$$\dot{X} = +R\sin\theta\,\dot{\theta}$$

$$\dot{Y} = +R\cos\theta\,\dot{\theta}$$

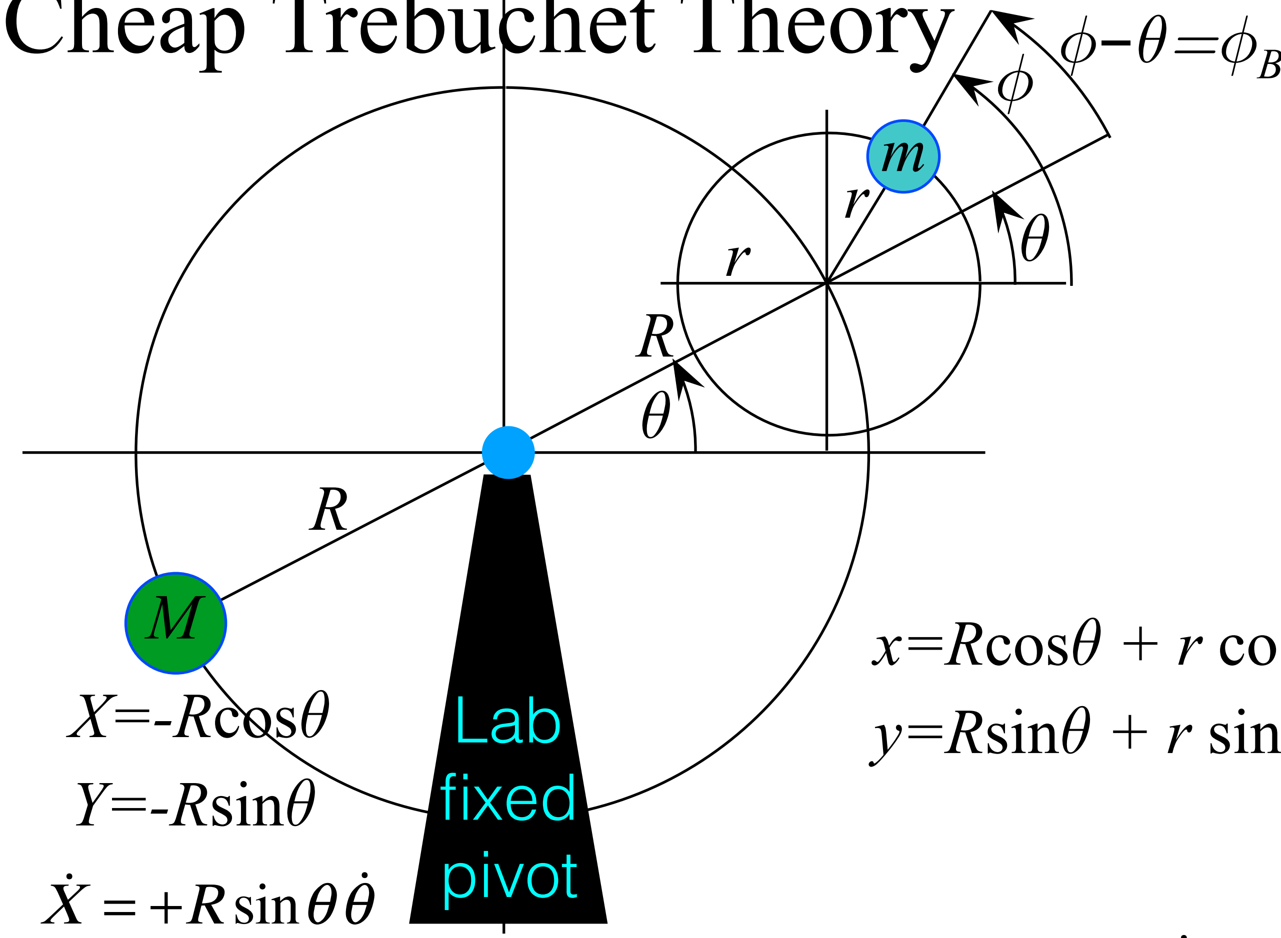
$$\dot{x} = -R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}$$

$$\dot{y} = +R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}$$

$$\dot{X}^2 = +R^2\sin^2\theta\,\dot{\theta}^2$$

$$\dot{Y}^2 = +R^2\cos^2\theta\,\dot{\theta}^2$$

Cheap Trebuchet Theory



$X = -R\cos\theta$

$Y = -R\sin\theta$

$\dot{X} = +R\sin\theta\,\dot{\theta}$

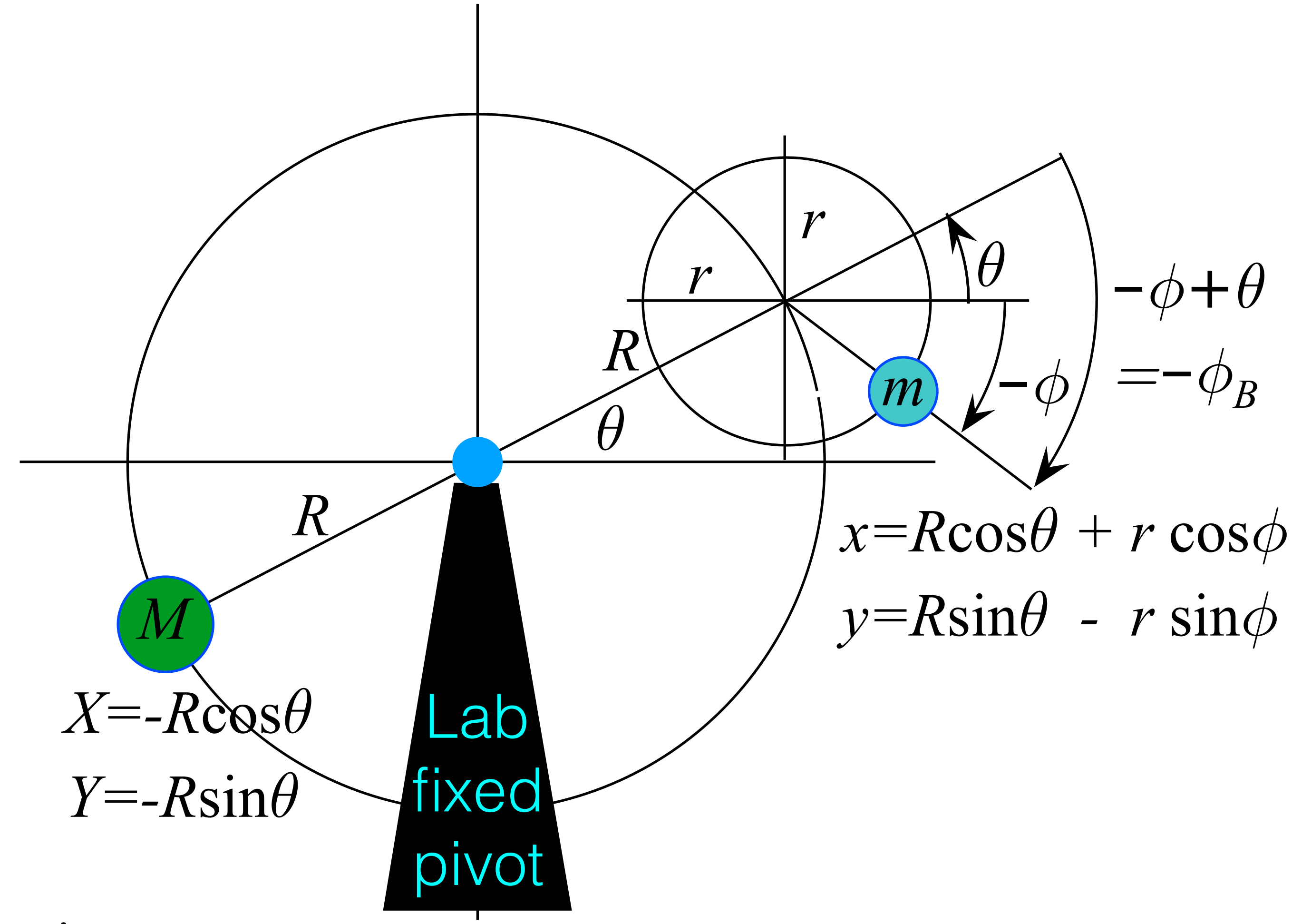
$\dot{Y} = +R\cos\theta\,\dot{\theta}$

$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta + r\sin\phi$

$\dot{x} = -R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}$

$\dot{y} = +R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}$



$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta - r\sin\phi$

$X = -R\cos\theta$

$Y = -R\sin\theta$

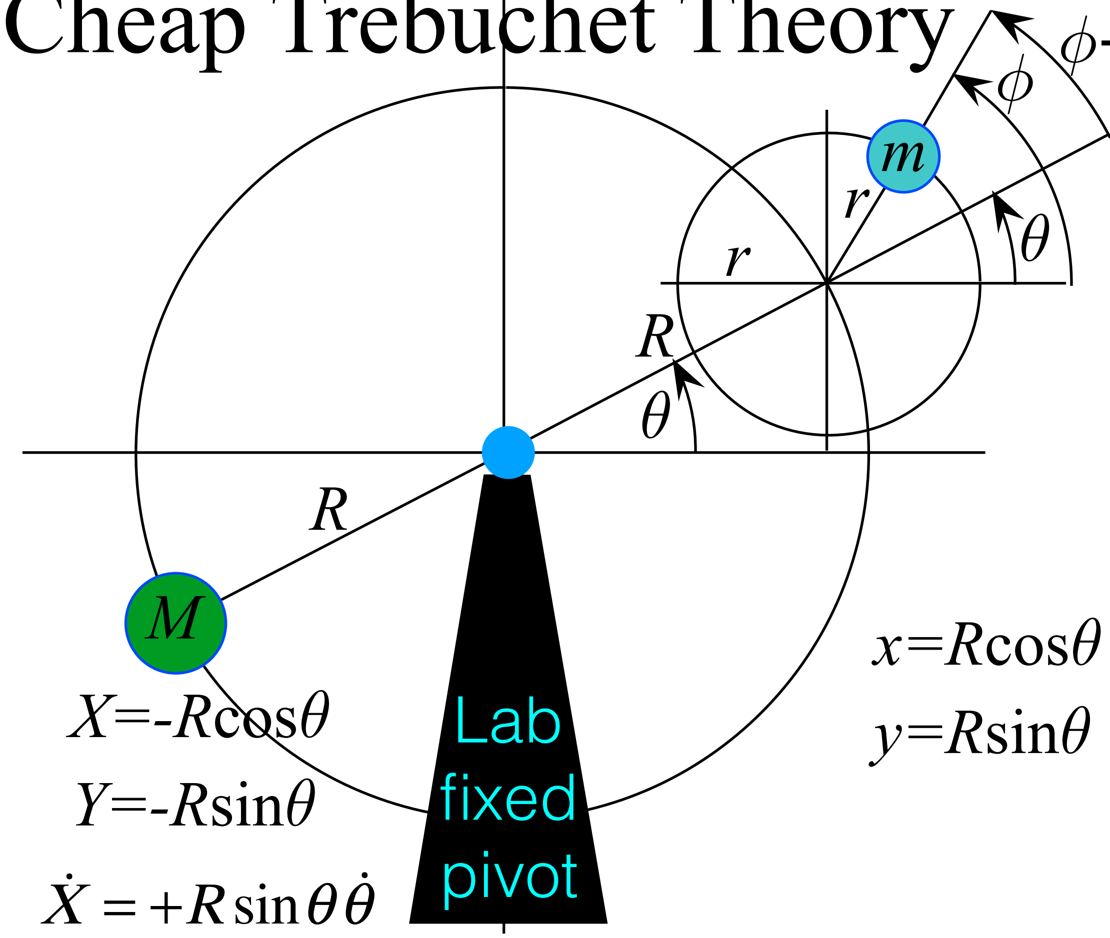
$\dot{X}^2 = +R^2\sin^2\theta\,\dot{\theta}^2$

$\dot{Y}^2 = +R^2\cos^2\theta\,\dot{\theta}^2$

$$\dot{x}^2 = \left(-R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}\right)\left(-R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}\right) = R^2\sin^2\theta\,\dot{\theta}^2 + 2Rr\sin\theta\sin\phi\,\dot{\theta}\dot{\phi} + r^2\sin^2\phi\,\dot{\phi}^2$$

$$\dot{y}^2 = \left(R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}\right)\left(+R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}\right) = R^2\cos^2\theta\,\dot{\theta}^2 + 2Rr\cos\theta\cos\phi\,\dot{\theta}\dot{\phi} + r^2\cos^2\phi\,\dot{\phi}^2$$

Cheap Trebuchet Theory



$X = -R\cos\theta$

$Y = -R\sin\theta$

$\dot{X} = +R\sin\theta\,\dot{\theta}$

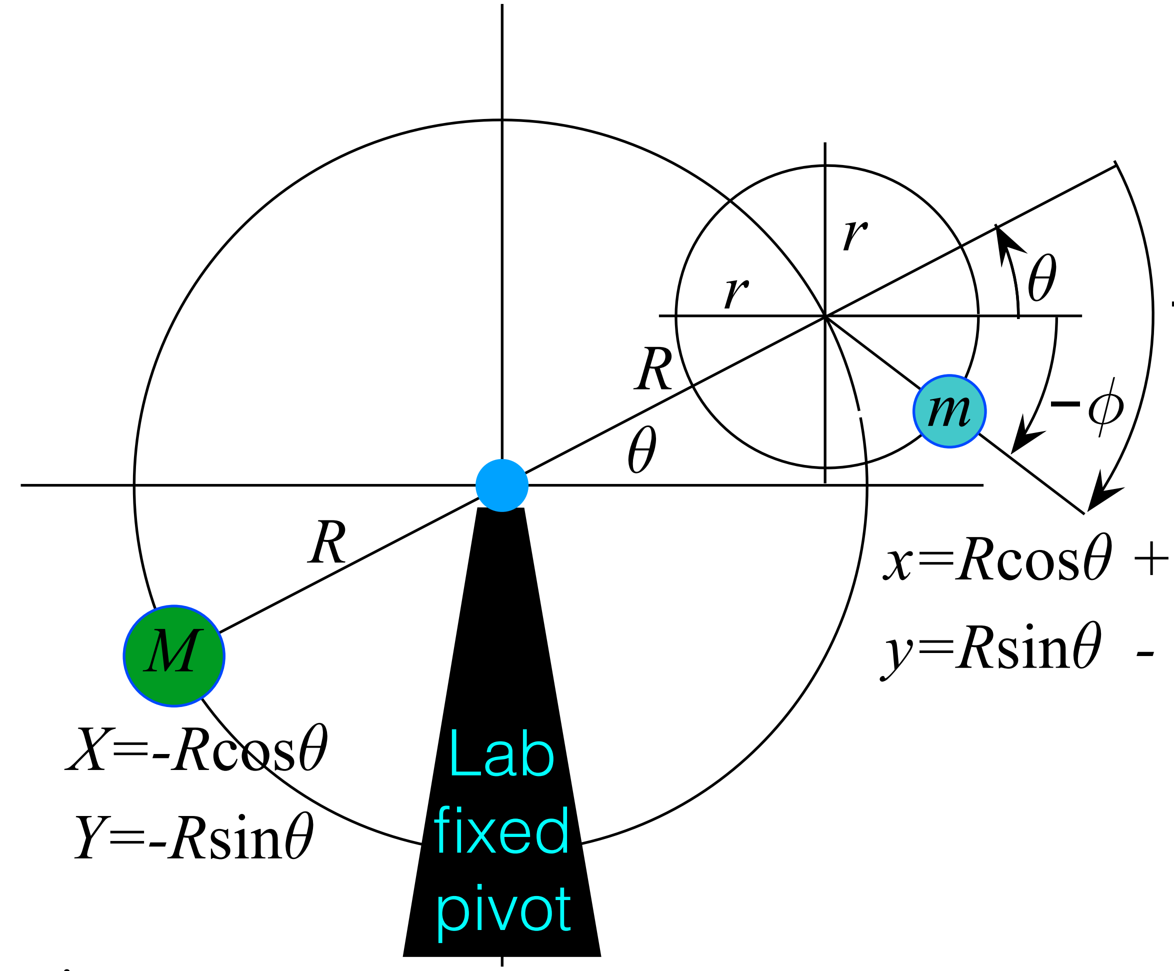
$\dot{Y} = +R\cos\theta\,\dot{\theta}$

$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta + r\sin\phi$

$\dot{x} = -R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}$

$\dot{y} = +R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}$



$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta - r\sin\phi$

$X = -R\cos\theta$

$Y = -R\sin\theta$

$$\dot{X}^2 = +R^2\sin^2\theta\,\dot{\theta}^2$$

$$\dot{Y}^2 = +R^2\cos^2\theta\,\dot{\theta}^2$$

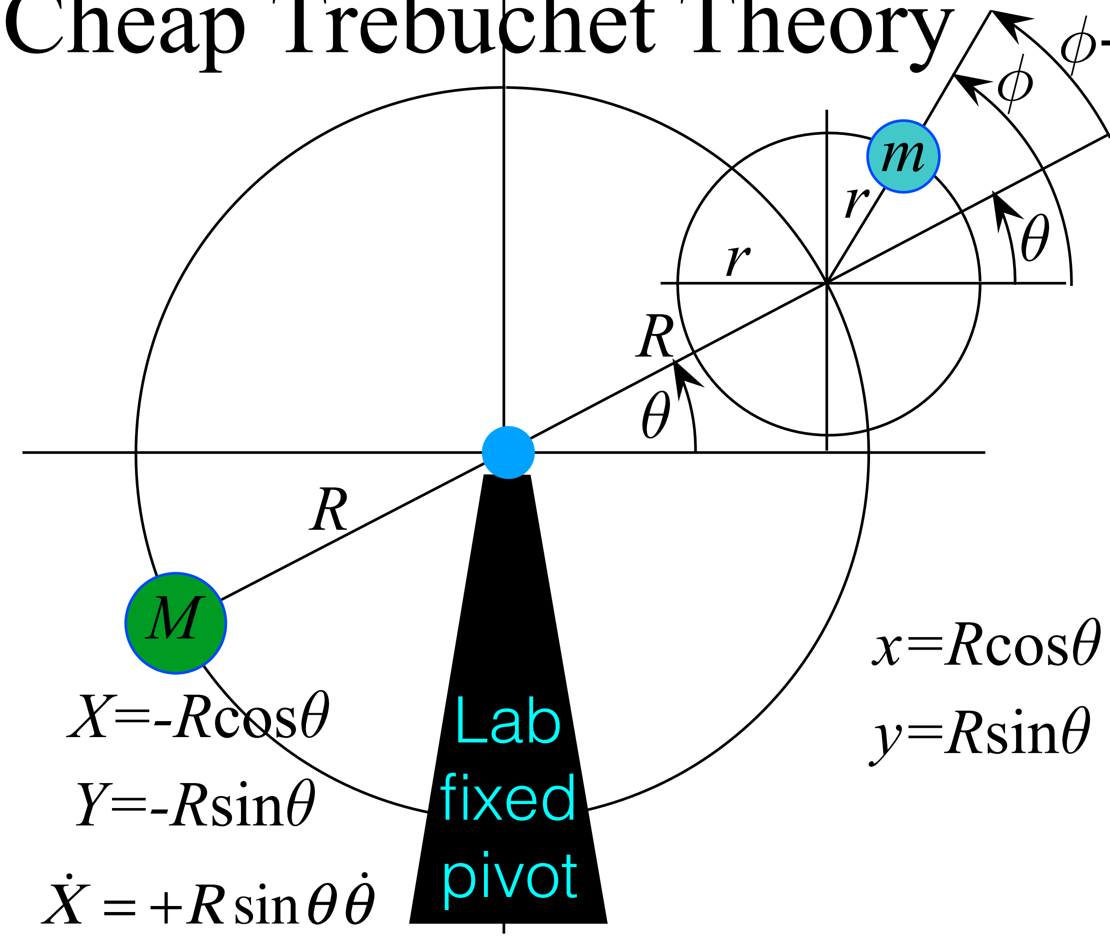
$$\dot{X}^2 + \dot{Y}^2 = R^2\dot{\theta}^2$$

$$\dot{x}^2 = \left(-R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}\right)\left(-R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}\right) = R^2\sin^2\theta\,\dot{\theta}^2 + 2Rr\sin\theta\sin\phi\,\dot{\theta}\dot{\phi} + r^2\sin^2\phi\,\dot{\phi}^2$$

$$\dot{y}^2 = \left(R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}\right)\left(+R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}\right) = R^2\cos^2\theta\,\dot{\theta}^2 + 2Rr\cos\theta\cos\phi\,\dot{\theta}\dot{\phi} + r^2\cos^2\phi\,\dot{\phi}^2$$

$$\dot{x}^2 + \dot{y}^2 = \quad R^2\dot{\theta}^2 \quad + 2Rr\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + r^2\dot{\phi}^2$$

Cheap Trebuchet Theory



X=-$R\cos\theta$

Y=-$R\sin\theta$

$\dot{X}=+R\sin\theta\dot{\theta}$

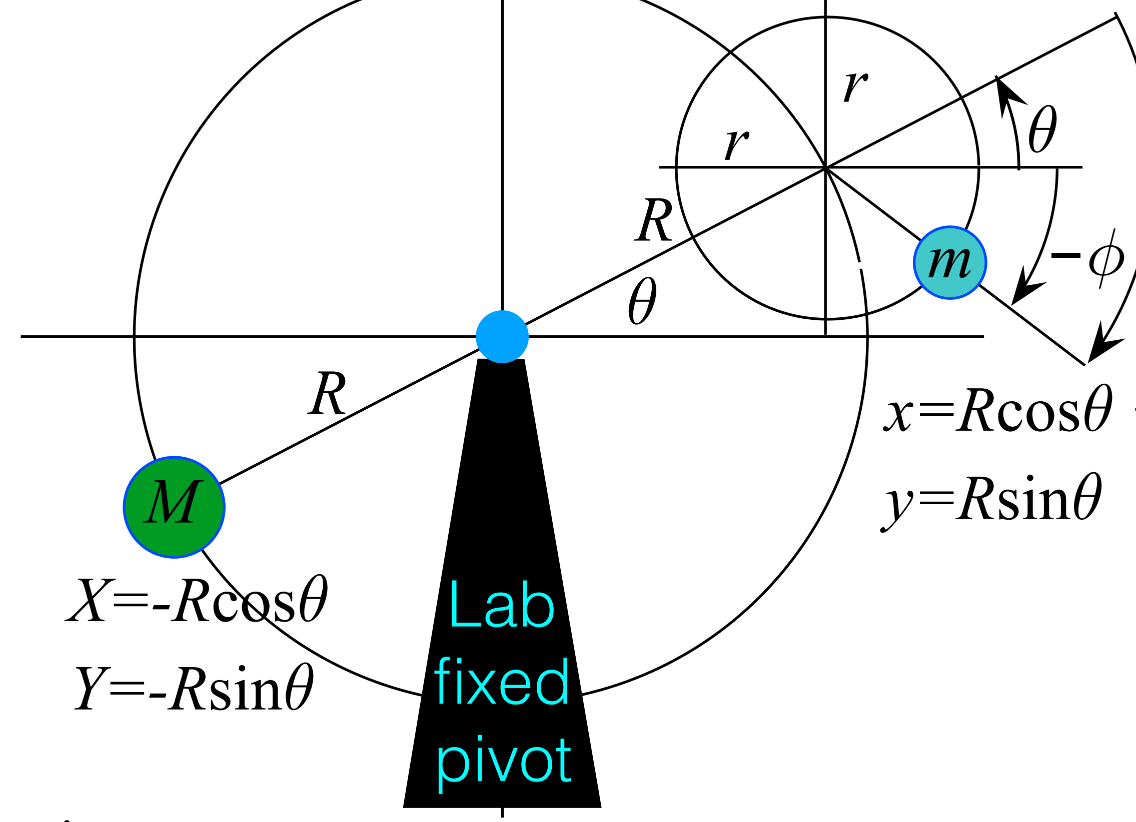
$\dot{Y}=+R\cos\theta\dot{\theta}$

$x=R\cos\theta + r\cos\phi$

$y=R\sin\theta + r\sin\phi$

$\dot{x}=-R\sin\theta\dot{\theta}-r\sin\phi\dot{\phi}$

$\dot{y}=+R\cos\theta\dot{\theta}+r\cos\phi\dot{\phi}$



$x=R\cos\theta + r\cos\phi$

$y=R\sin\theta - r\sin\phi$

X=-$R\cos\theta$

Y=-$R\sin\theta$

$\dot{X}^2=+R^2\sin^2\theta\dot{\theta}^2$

$\dot{Y}^2=+R^2\cos^2\theta\dot{\theta}^2$

$\dot{X}^2+\dot{Y}^2=R^2\dot{\theta}^2$

$\dot{x}^2=\left(-R\sin\theta\dot{\theta}-r\sin\phi\dot{\phi}\right)\left(-R\sin\theta\dot{\theta}-r\sin\phi\dot{\phi}\right)=R^2\sin^2\theta\dot{\theta}^2+2Rr\sin\theta\sin\phi\dot{\theta}\dot{\phi}+r^2\sin^2\phi\dot{\phi}^2$

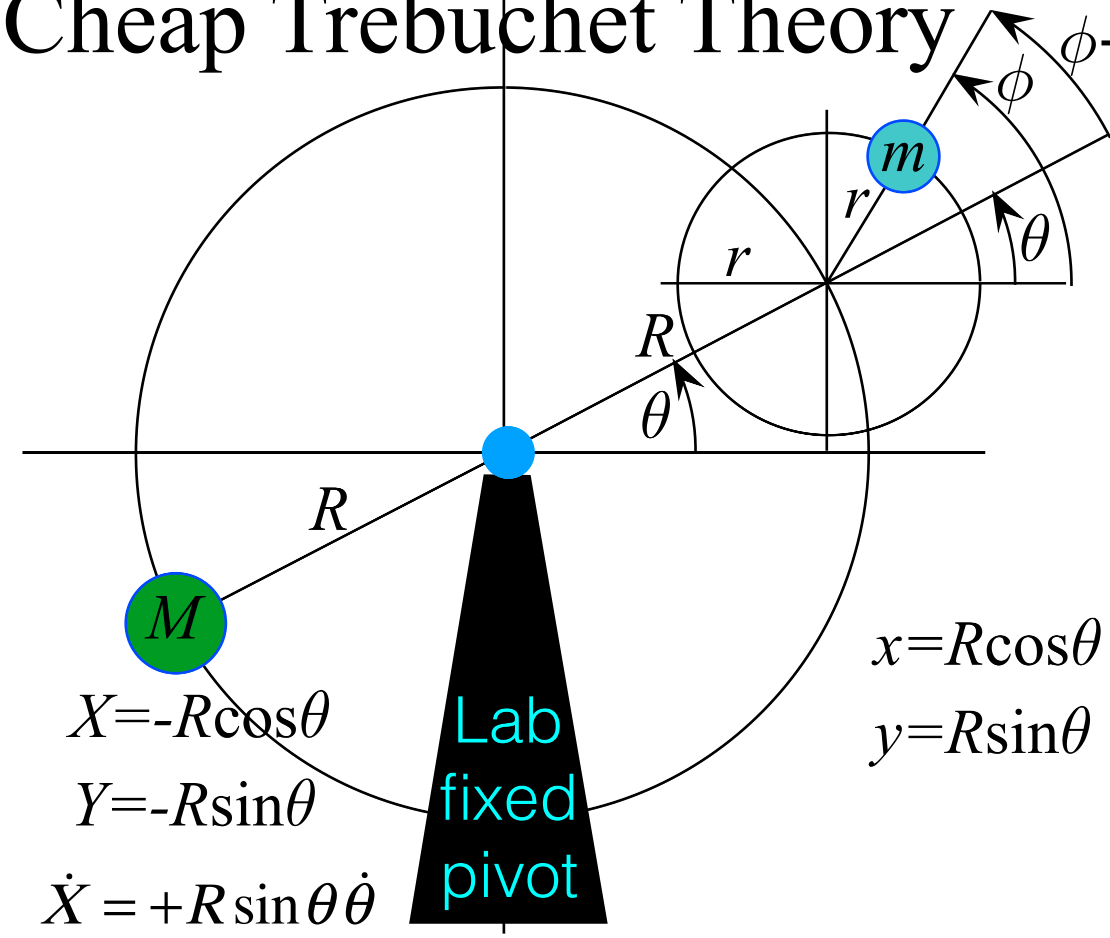
$\dot{y}^2=\left(R\cos\theta\dot{\theta}+r\cos\phi\dot{\phi}\right)\left(+R\cos\theta\dot{\theta}+r\cos\phi\dot{\phi}\right)=R^2\cos^2\theta\dot{\theta}^2+2Rr\cos\theta\cos\phi\dot{\theta}\dot{\phi}+r^2\cos^2\phi\dot{\phi}^2$

$\dot{x}^2+\dot{y}^2= \quad R^2\dot{\theta}^2 \quad +2Rr\cos(\theta-\phi)\dot{\theta}\dot{\phi}+r^2\dot{\phi}^2$

KE of driver M plus projectile m

$$T=\tfrac{1}{2}M(\dot{X}^2+\dot{Y}^2)+\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)=\tfrac{1}{2}M(R^2\dot{\theta}^2)+\tfrac{1}{2}m\left(R^2\dot{\theta}^2 \quad +2Rr\cos(\theta-\phi)\dot{\theta}\dot{\phi}+r^2\dot{\phi}^2\right)$$

Cheap Trebuchet Theory



X=-$R\cos\theta$

Y=-$R\sin\theta$

$\dot{X} = +R\sin\theta\,\dot{\theta}$

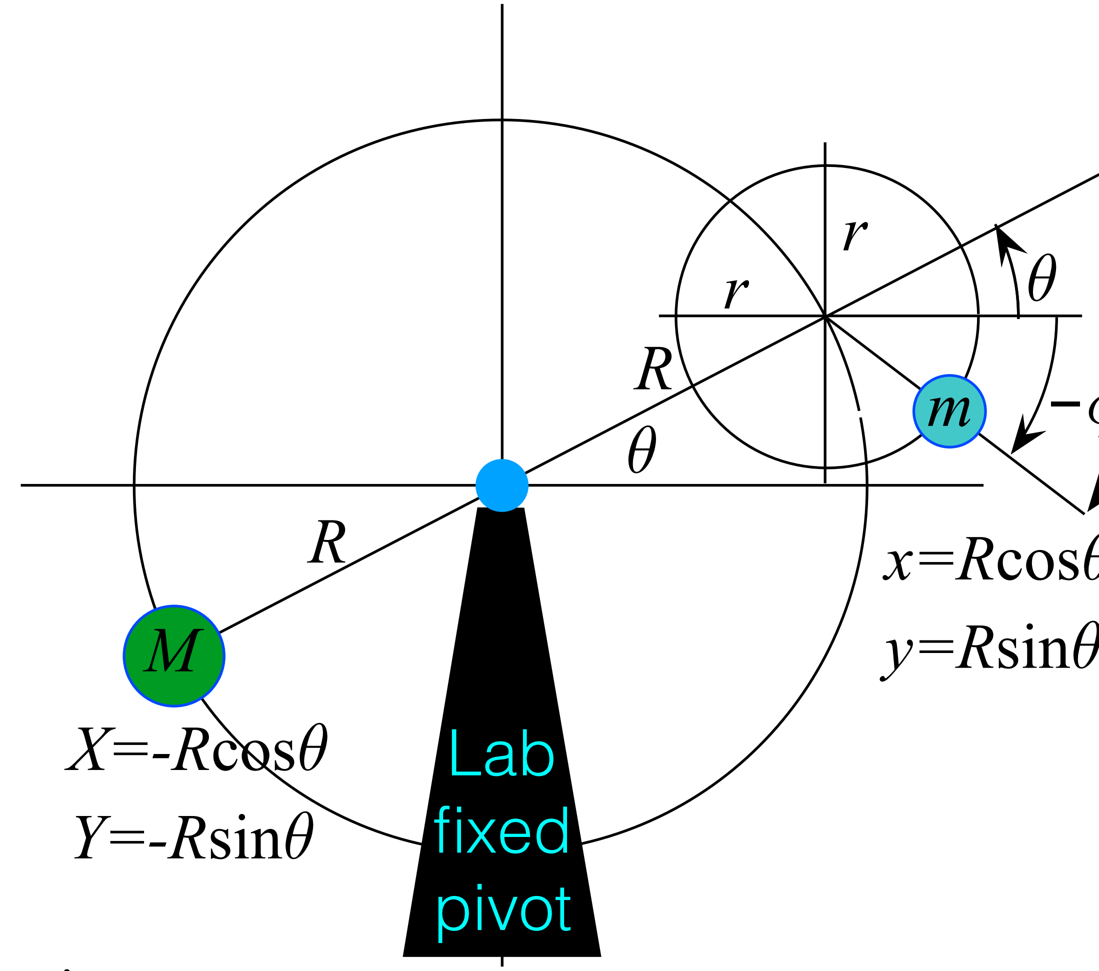
$\dot{Y} = +R\cos\theta\,\dot{\theta}$

x=$R\cos\theta$ + $r\cos\phi$

y=$R\sin\theta$ + $r\sin\phi$

$\dot{x} = -R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}$

$\dot{y} = +R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}$



x=$R\cos\theta$ + $r\cos\phi$

y=$R\sin\theta$ - $r\sin\phi$

X=-$R\cos\theta$

Y=-$R\sin\theta$

$\dot{X}^2 = +R^2\sin^2\theta\,\dot{\theta}^2$

$\dot{Y}^2 = +R^2\cos^2\theta\,\dot{\theta}^2$

$\dot{X}^2 + \dot{Y}^2 = R^2\dot{\theta}^2$

$\dot{x}^2 = \left(-R\sin\theta\dot{\theta} - r\sin\phi\dot{\phi}\right)\left(-R\sin\theta\dot{\theta} - r\sin\phi\dot{\phi}\right) = R^2\sin^2\theta\dot{\theta}^2 + 2Rr\sin\theta\sin\phi\dot{\theta}\dot{\phi} + r^2\sin^2\phi\dot{\phi}^2$

$\dot{y}^2 = \left(R\cos\theta\dot{\theta} + r\cos\phi\dot{\phi}\right)\left(+R\cos\theta\dot{\theta} + r\cos\phi\dot{\phi}\right) = R^2\cos^2\theta\dot{\theta}^2 + 2Rr\cos\theta\cos\phi\dot{\theta}\dot{\phi} + r^2\cos^2\phi\dot{\phi}^2$

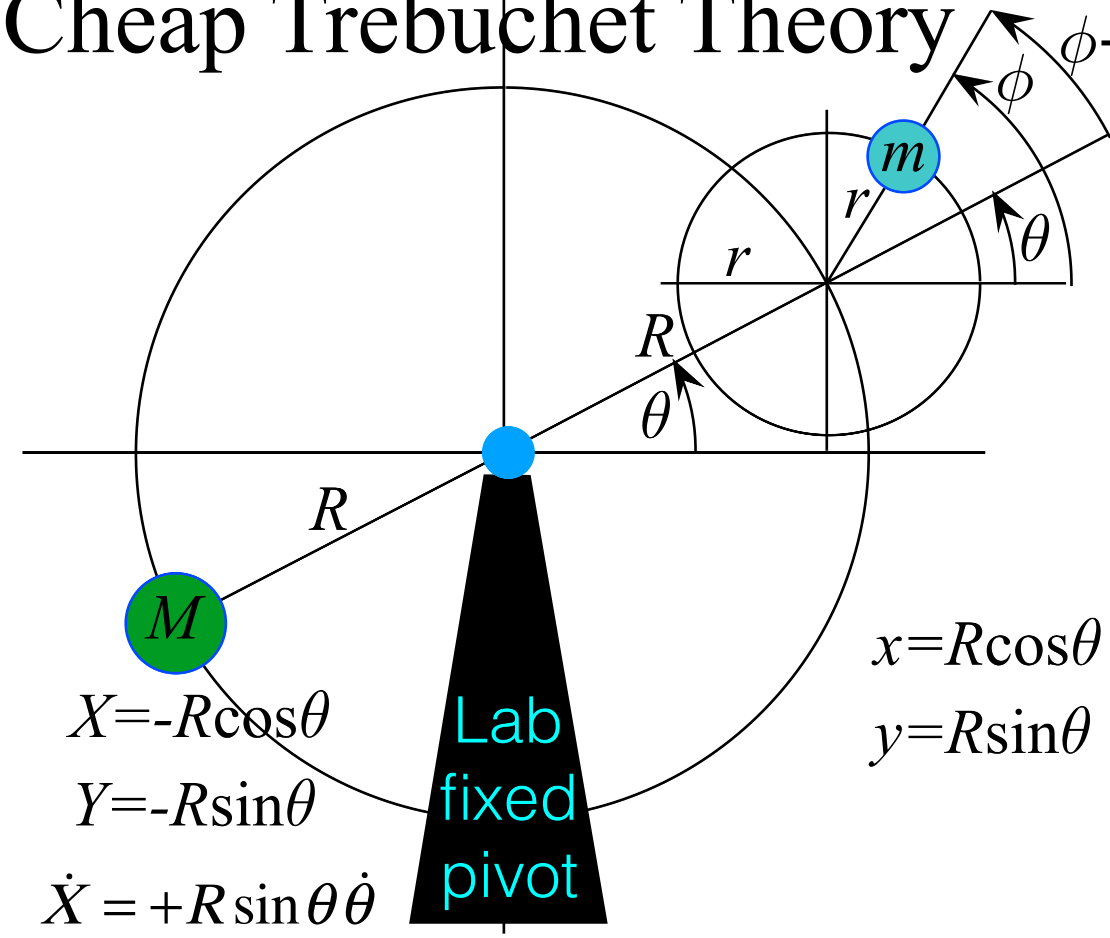
$\dot{x}^2 + \dot{y}^2 = R^2\dot{\theta}^2 + 2Rr\cos(\theta-\phi)\dot{\theta}\dot{\phi} + r^2\dot{\phi}^2$

KE of driver M plus projectile m is explicit function of beam-relative coordinate -$\phi_B = (\theta-\phi)$ only

$$T = \tfrac{1}{2}M(\dot{X}^2+\dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}M(R^2\dot{\theta}^2) + \tfrac{1}{2}m\left(R^2\dot{\theta}^2 + 2Rr\cos(\theta-\phi)\dot{\theta}\dot{\phi} + r^2\dot{\phi}^2\right)$$

$$T = \tfrac{1}{2}M(\dot{X}^2+\dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}(M+m)(R^2\dot{\theta}^2) + mRr\cos(\theta-\phi)\dot{\theta}\dot{\phi} + \tfrac{1}{2}mr^2\dot{\phi}^2$$

Cheap Trebuchet Theory



$X = -R\cos\theta$

$Y = -R\sin\theta$

$\dot{X} = +R\sin\theta\,\dot{\theta}$

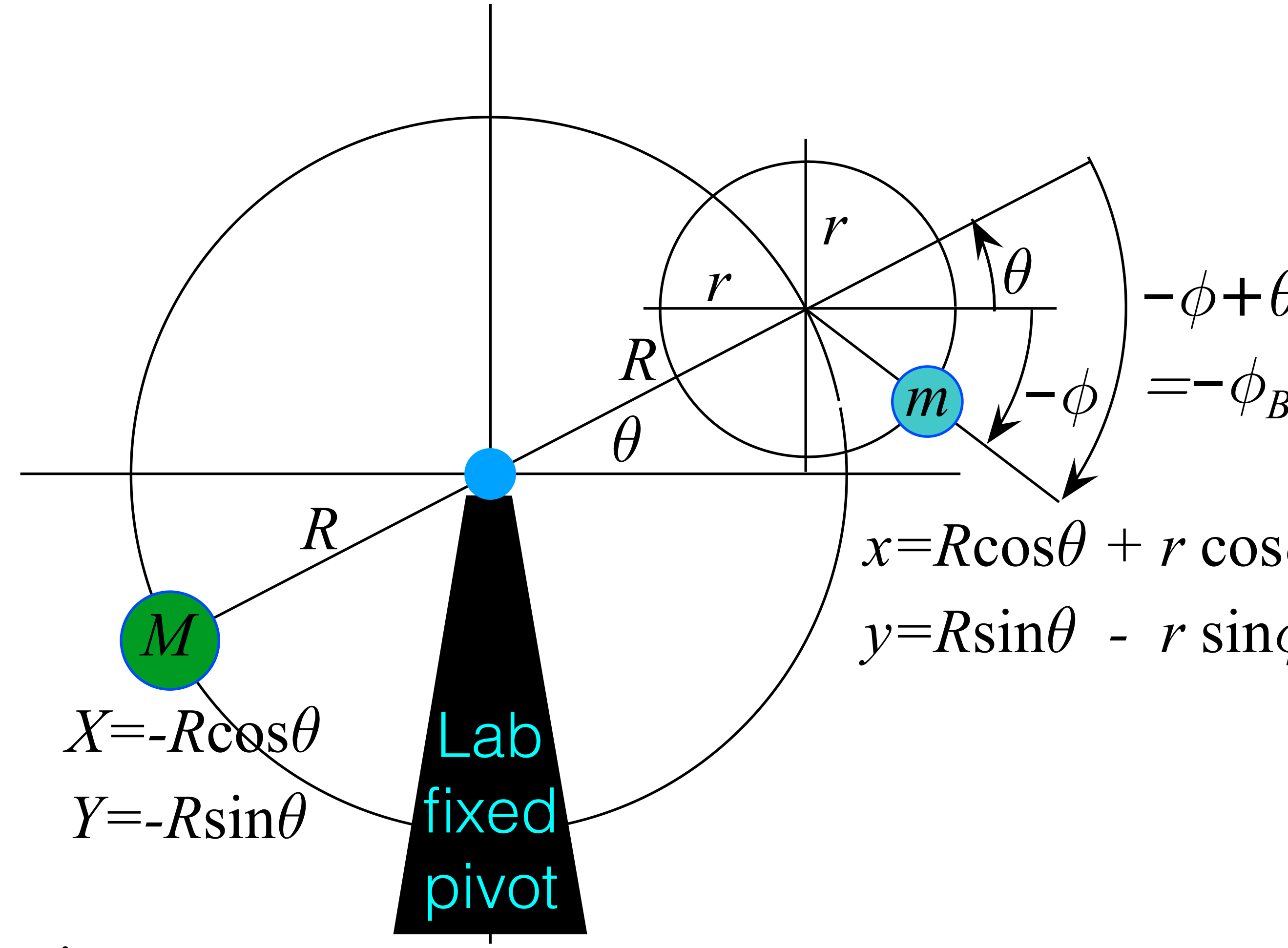
$\dot{Y} = +R\cos\theta\,\dot{\theta}$

$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta + r\sin\phi$

$\dot{x} = -R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}$

$\dot{y} = +R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}$



$x = R\cos\theta + r\cos\phi$

$y = R\sin\theta - r\sin\phi$

$X = -R\cos\theta$

$Y = -R\sin\theta$

$$\dot{X}^2 = +R^2\sin^2\theta\,\dot{\theta}^2$$
$$\dot{Y}^2 = +R^2\cos^2\theta\,\dot{\theta}^2$$
$$\dot{X}^2 + \dot{Y}^2 = R^2\dot{\theta}^2$$

$$\dot{x}^2 = \left(-R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}\right)\left(-R\sin\theta\,\dot{\theta} - r\sin\phi\,\dot{\phi}\right) = R^2\sin^2\theta\,\dot{\theta}^2 + 2Rr\sin\theta\sin\phi\,\dot{\theta}\dot{\phi} + r^2\sin^2\phi\,\dot{\phi}^2$$
$$\dot{y}^2 = \left(R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}\right)\left(+R\cos\theta\,\dot{\theta} + r\cos\phi\,\dot{\phi}\right) = R^2\cos^2\theta\,\dot{\theta}^2 + 2Rr\cos\theta\cos\phi\,\dot{\theta}\dot{\phi} + r^2\cos^2\phi\,\dot{\phi}^2$$
$$\dot{x}^2 + \dot{y}^2 = \quad R^2\dot{\theta}^2 \quad + 2Rr\cos(\theta-\phi)\dot{\theta}\dot{\phi} + r^2\dot{\phi}^2$$

KE of driver M plus projectile m is explicit function of beam-relative coordinate $-\phi_B = (\theta - \phi)$ only

$$T = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}M(R^2\dot{\theta}^2) + \tfrac{1}{2}m\left(R^2\dot{\theta}^2 \quad + 2Rr\cos(\theta-\phi)\dot{\theta}\dot{\phi} + r^2\dot{\phi}^2\right)$$
$$T = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}(M+m)(R^2\dot{\theta}^2) + mRr\cos(\theta-\phi)\dot{\theta}\dot{\phi} + \tfrac{1}{2}mr^2\dot{\phi}^2$$

For symmetric ($R=r$) trebuchet: $T = R^2\left[\frac{M+m}{2}\dot{\theta}^2 + \frac{m}{2}\dot{\phi}^2 + m\dot{\theta}\dot{\phi}\cos\phi_B\right]$

Force, Work, and Acceleration (compare to Cheap Trebuchet Theory)

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_y\,dy$$

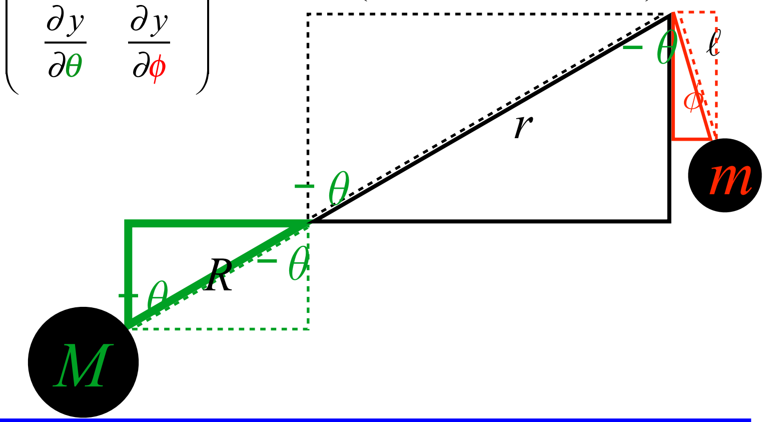
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$



Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta}$$

$$\text{Let}: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given
(F_X =0 , F_Y =-Mg)
(F_x =0 , F_y =-mg)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

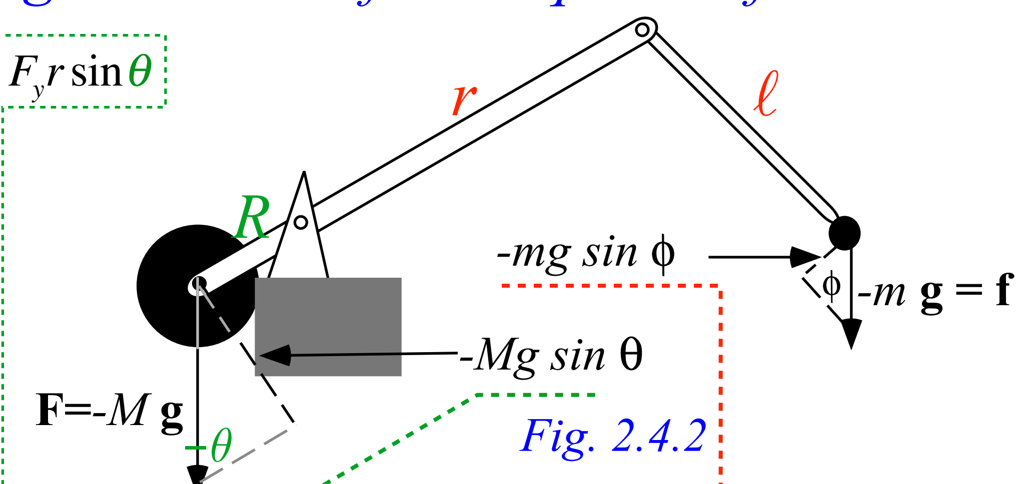


Fig. 2.4.2

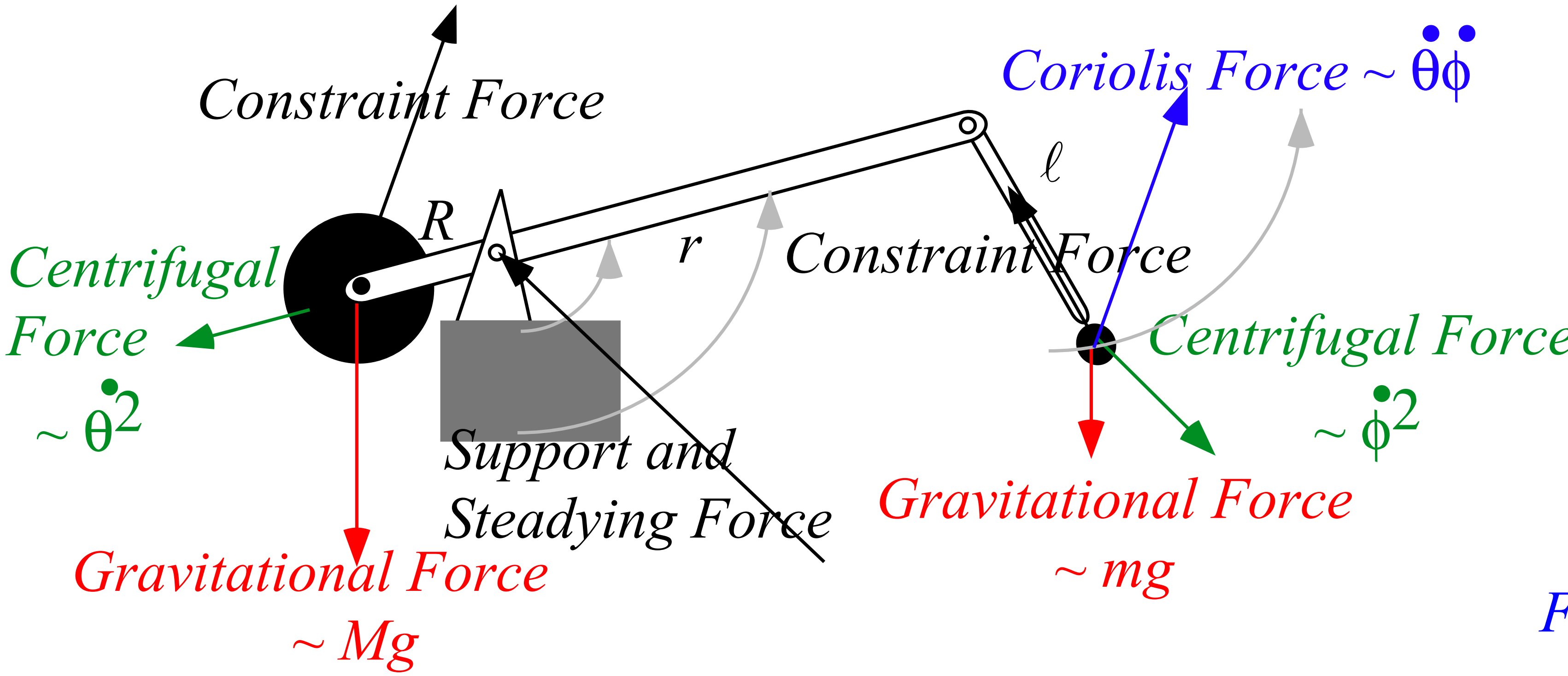
$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given
(F_X =0 , F_Y =-Mg)
(F_x =0 , F_y =-mg)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... *... and a torque on throwing lever ℓ*

Forces: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\dfrac{\partial V}{\partial \theta}$ and: $\dfrac{\partial V}{\partial \dot\theta} = 0$

$F_\phi = -\dfrac{\partial V}{\partial \phi}$ and: $\dfrac{\partial V}{\partial \dot\phi} = 0$

Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot p_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot p_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations

(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot\theta} \qquad \dot p_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} \qquad \dot p_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations

$L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.