Lecture 15

Wed.10.23.2019

*treb-yew-shay

GCC Lagrange and Riemann Equations for Trebuchet
(Ch. 1-5 of Unit 2 and Unit 3)

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 82)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 74)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #15

In reverse order

***Trebuchet* Web Animations:**

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Excerpts (Page 44-47 in *Preliminary Draft*) from the
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019

Select, exciting, and related Research & Articles of Interest:

*(Many of these may be a bit beyond this course,
but are included to lend added insight)*:

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces - Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.

CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,

RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

An assist from *Physics Girl* (YouTube Channel):

Posted this year:
How to Make VORTEX RINGS in a Pool

Crazy pool vortex (new inclusion with more background)
Crazy pool vortex - pg-yt-2014

Posting with the best visuals:
Fun with Vortex Rings in the Pool - pg-yt-2014

She covers it beautifully!

An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019

Rovibrational_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:

Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74, Page=75, Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted, FVPlot*

CMwithBang Lecture 8, page=20

WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:

2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)

CoulIt Web Simulation of the Volcanoes of Io

BohrIt Multi-Panel Plot:

Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:

Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:

Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**

Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**

Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**

Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:

Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,
Nuclear_Collisions - Mishustin-PhysRevC-2007, APS Link & Abstract
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0)
y_2 vs y_1 plots: (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0), (v_1, v_2)=(1, -1)
Estrangian plot *V_2 vs V_1: (v_1, v_2)=(0, 1), (v_1, v_2)=(1, -1)*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *(v_1, v_2)=(1, 0): V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

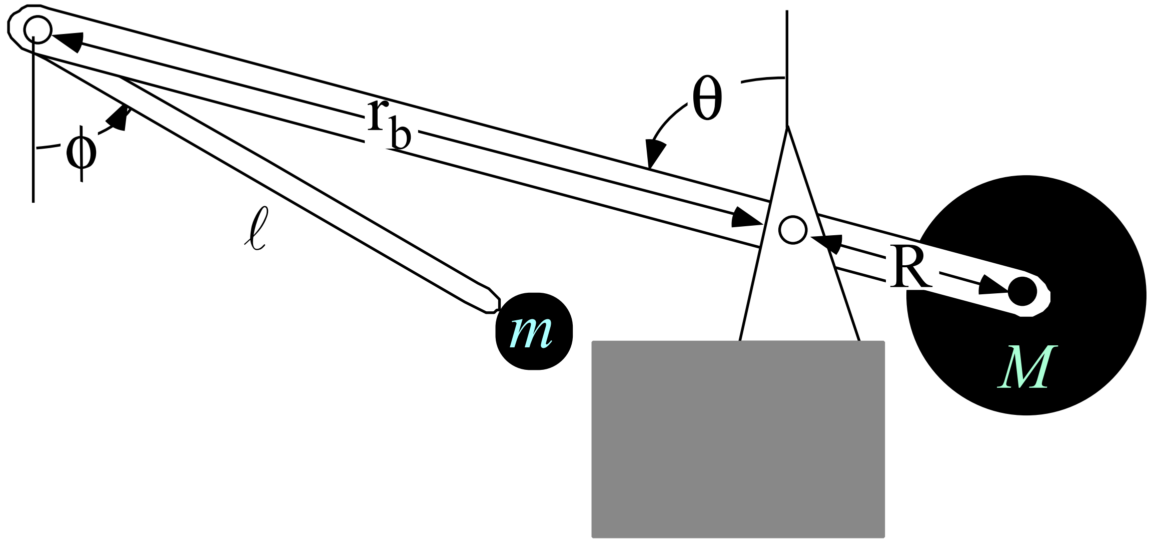
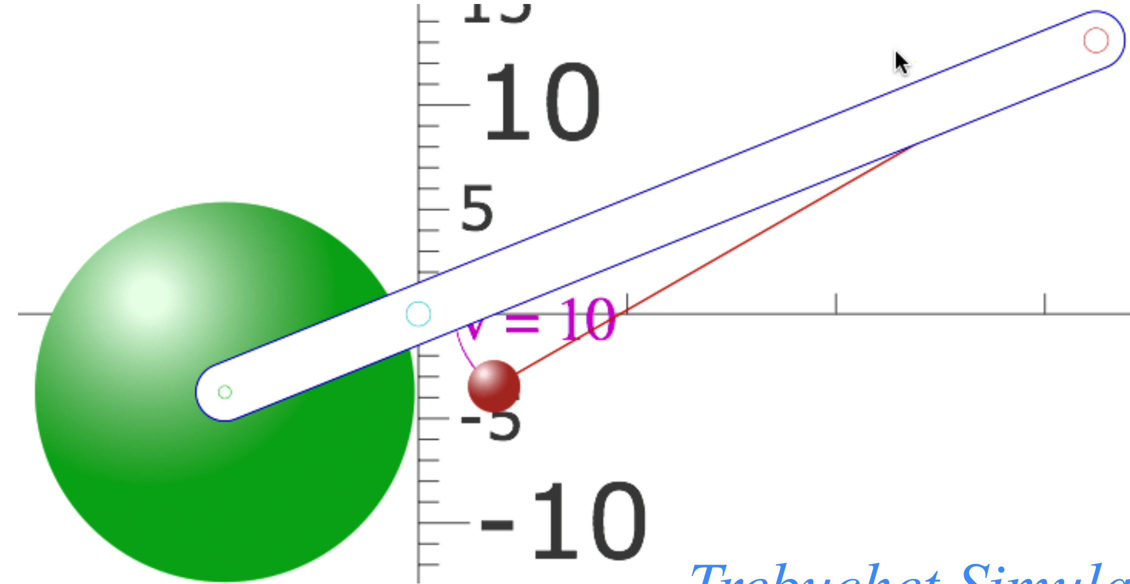


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet Simulator

Default URL: https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html

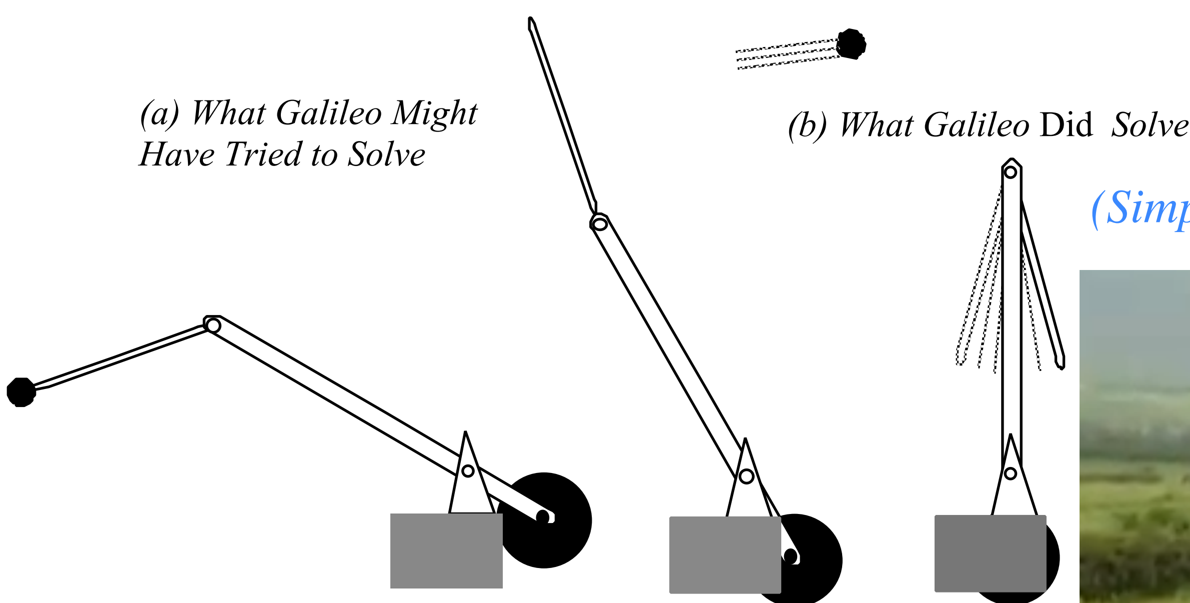


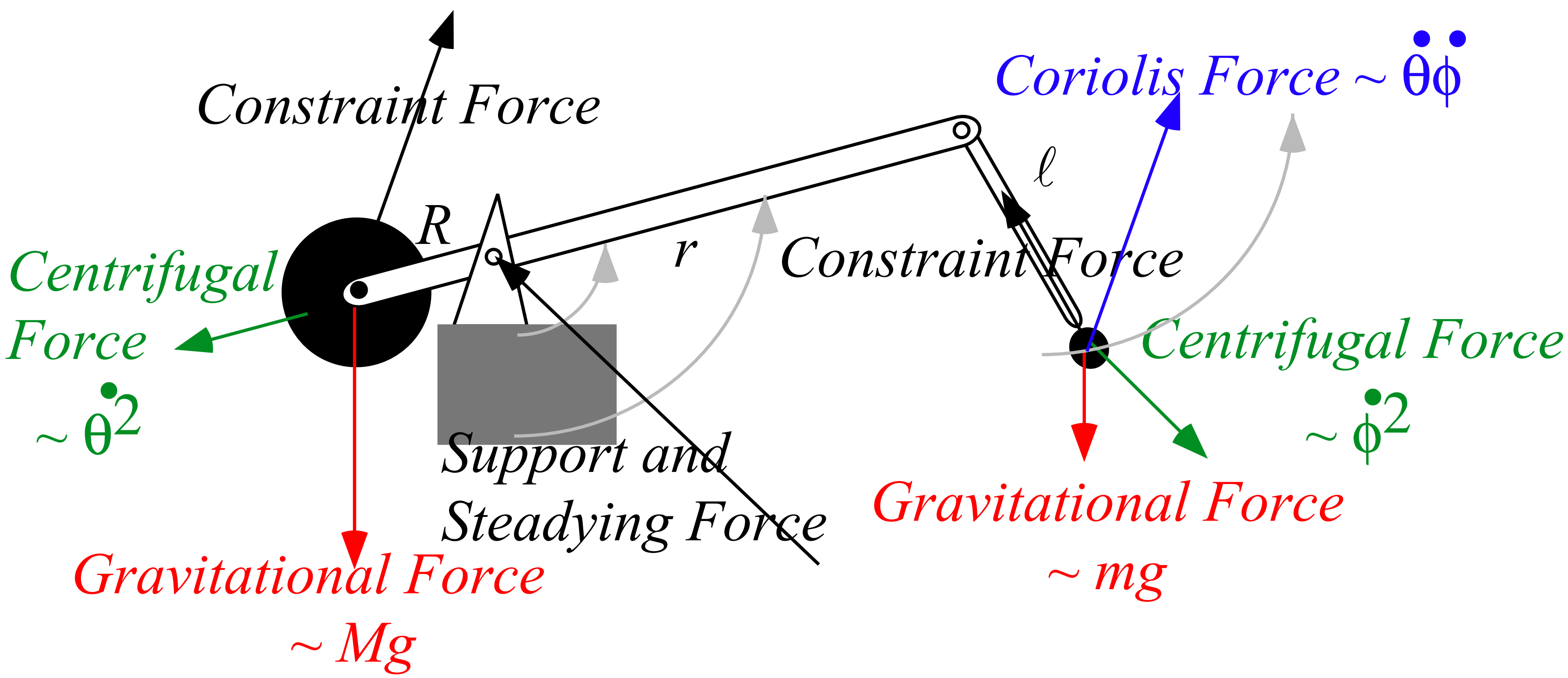
Fig. 2.1.2 Galileo's (supposed) problem

(Simple pendulum dynamics)

***Trebuchet* Web Simulations:**

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute.
Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

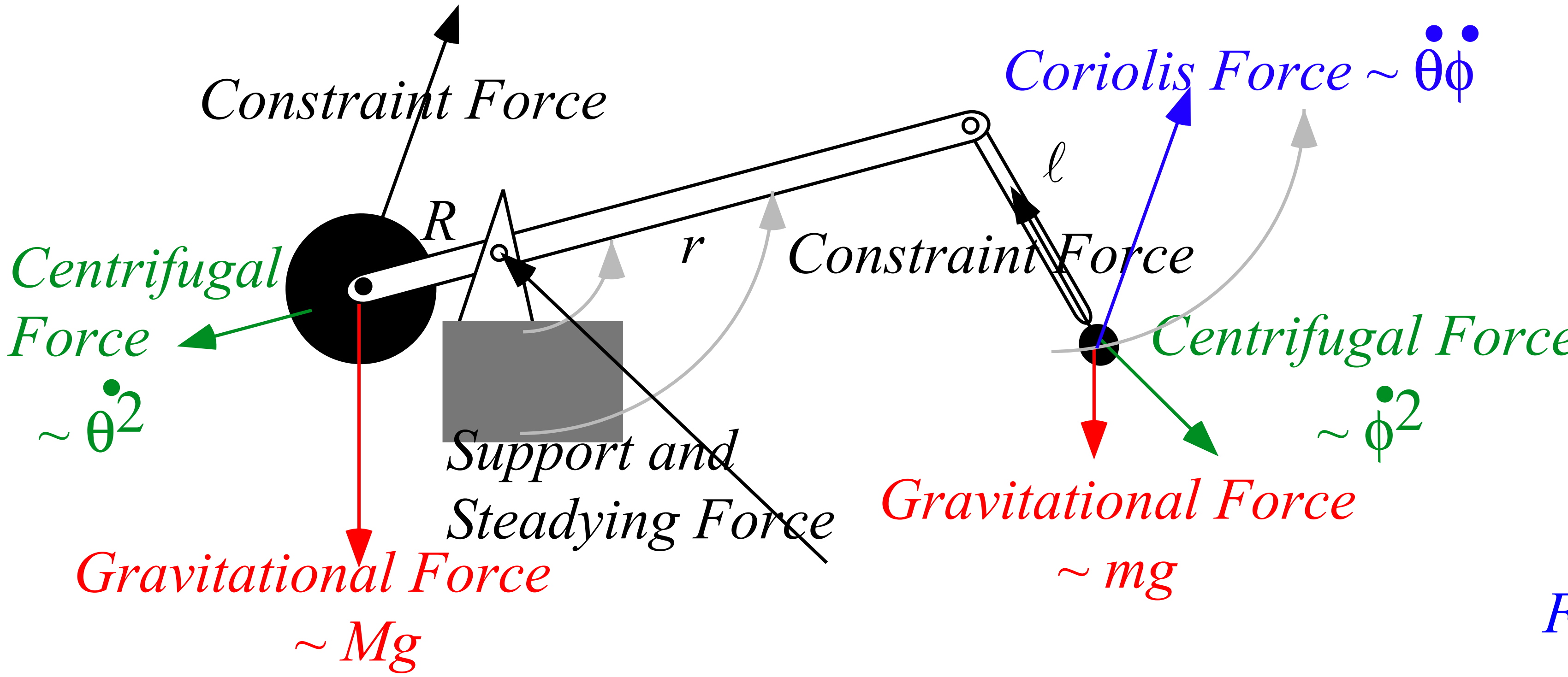
$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

Fig. 2.5.2
(modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\dfrac{\partial V}{\partial \theta}$ and: $\dfrac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\dfrac{\partial V}{\partial \phi}$ and: $\dfrac{\partial V}{\partial \dot{\phi}} = 0$

Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations

$$L = T - V$$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ *and contravariant* $\mathbf{E}^m$ *vectors: Jacobian J vs. Kajobian K*

Covariant metric g_{mn} *vs. contravariant metric* g^{mn} *(Lect. 9 p.53)*

Tangent $\{\mathbf{E}_n\}$*space vs. Normal* $\{\mathbf{E}^m\}$*space*

Covariant vs. contravariant coordinate transformations

Metric g_{mn} *tensor geometric relations to length, area, and volume*

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} *(Lect. 14 p. 77)*

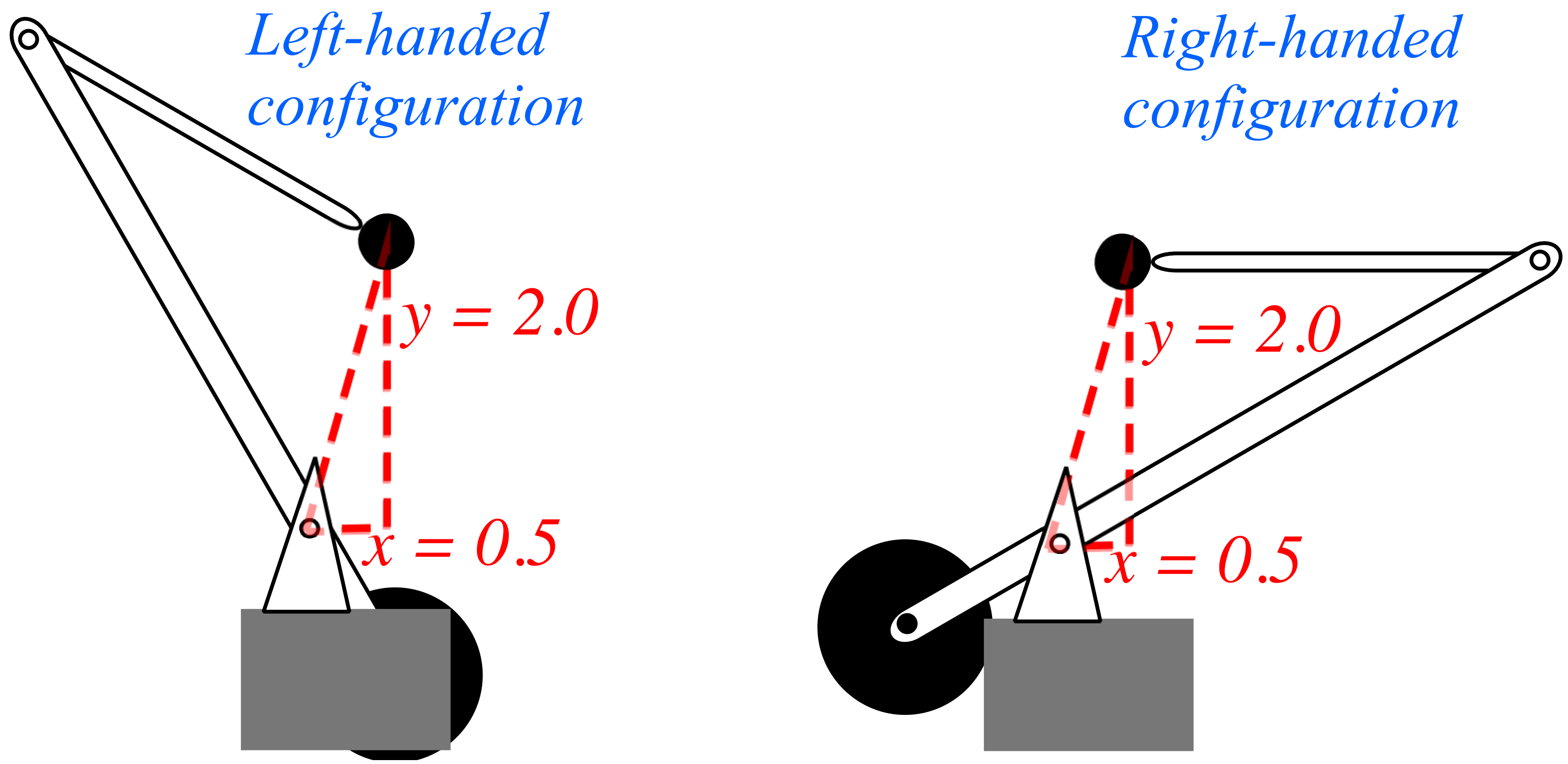
Review and application of trebuchet covariant forces F_θ *and* F_ϕ *(Lect. 14 p. 69)*

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Trebuchet Cartesian projectile coordinates are double-valued



from p. 85 of Lect. 14

Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m.

Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)

from p. 86 of Lect. 14

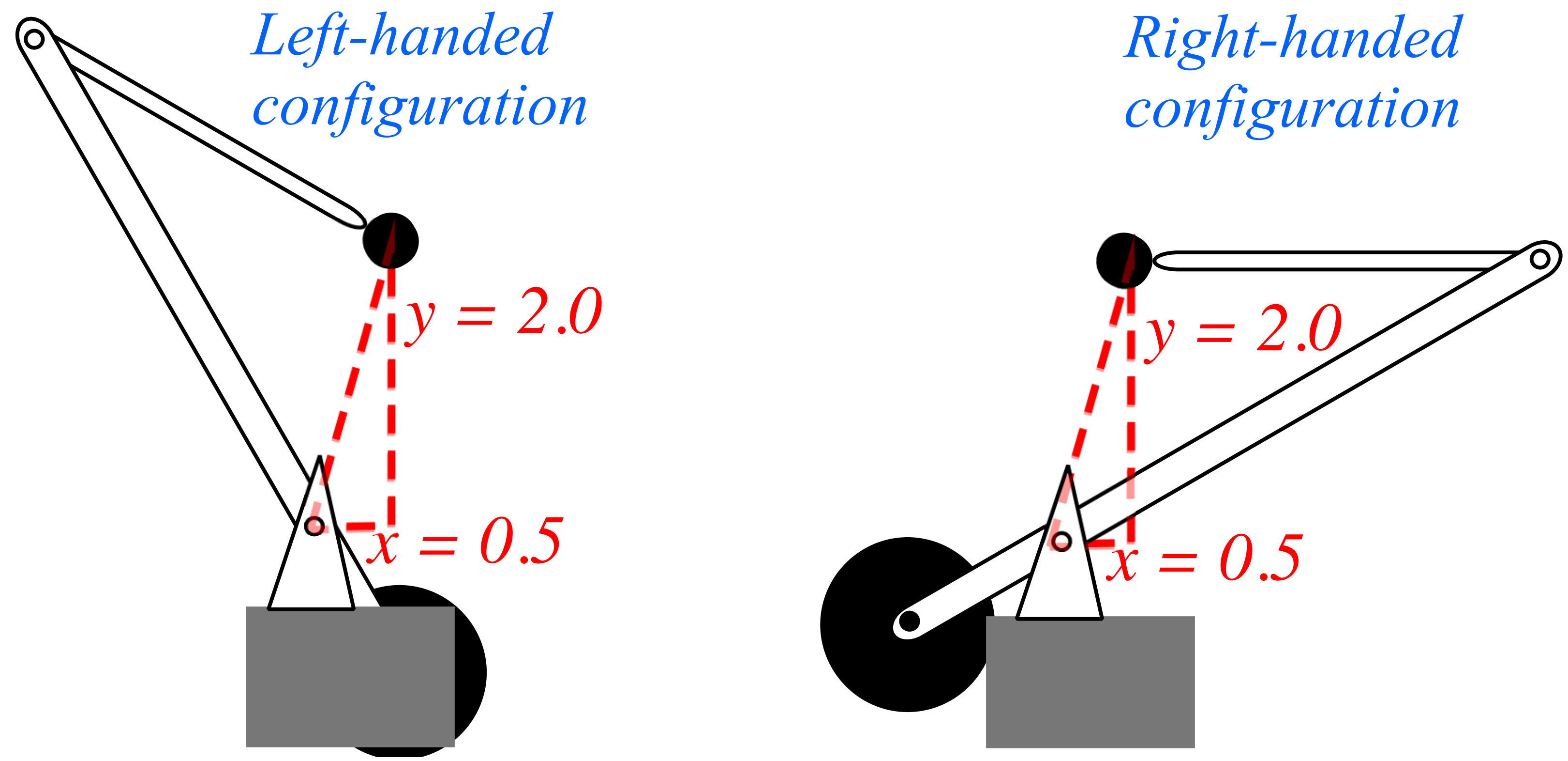


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

So, for example, are polar coordinates … (for each angle there are two r-values)

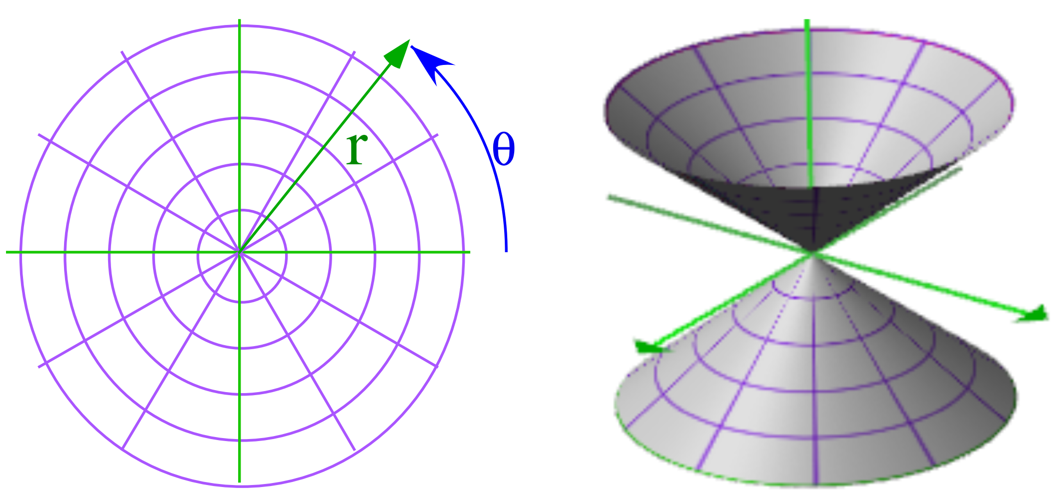


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

from p. 87 of Lect. 14

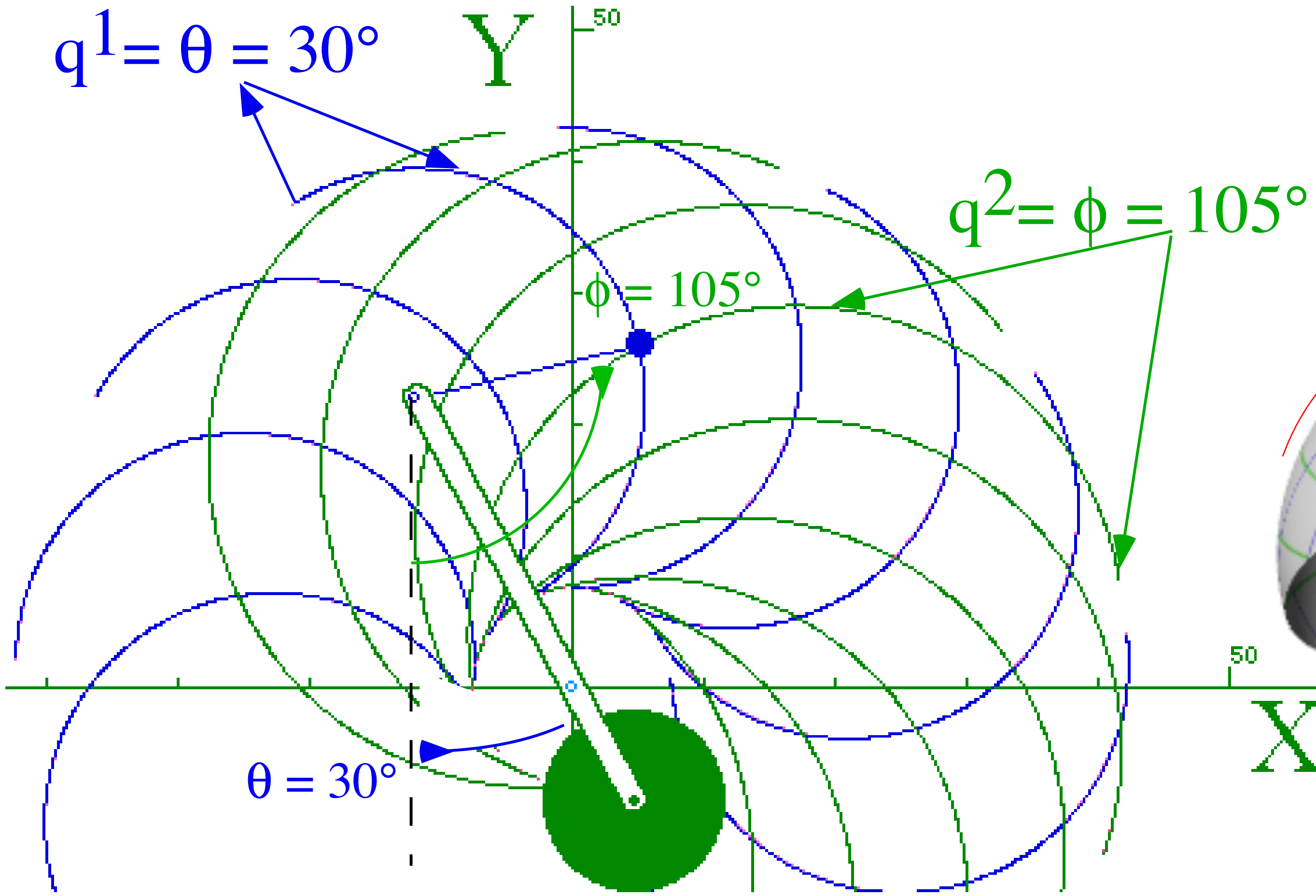


Fig. 3.1.1a (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Left handed sheet.)

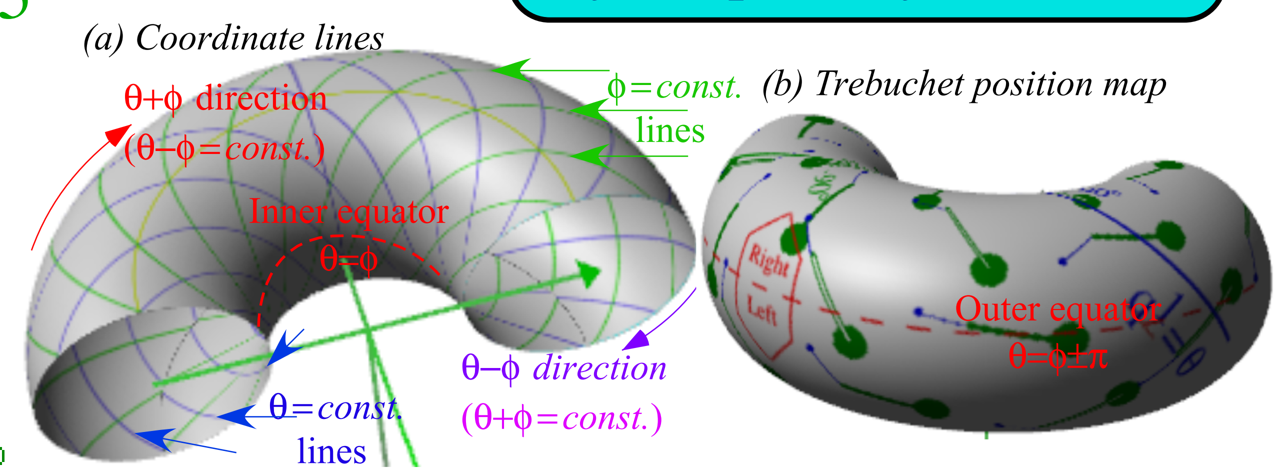


Fig. 3.1.2 Trebuchet torus.

(a) (q^1=θ, q^2=ϕ) coordinate lines. (b)Trebuchet position map and equators.

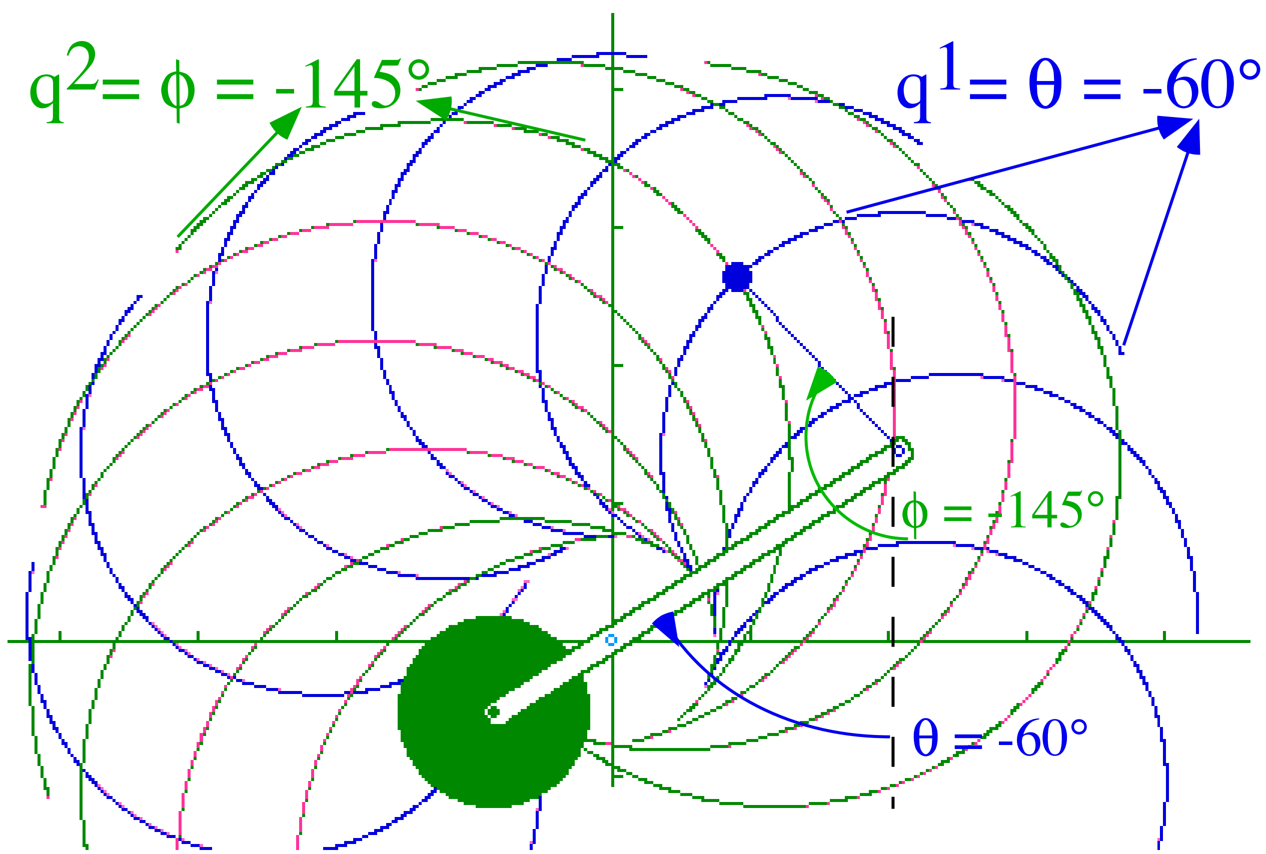


Fig. 3.1.1b (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Right handed sheet.)

“Flat” ($q^1=\theta$, $q^2=\phi$)-graph of trebuchet loci compared to “rolled-up” toroidal manifold

from p. 88 of Lect. 14

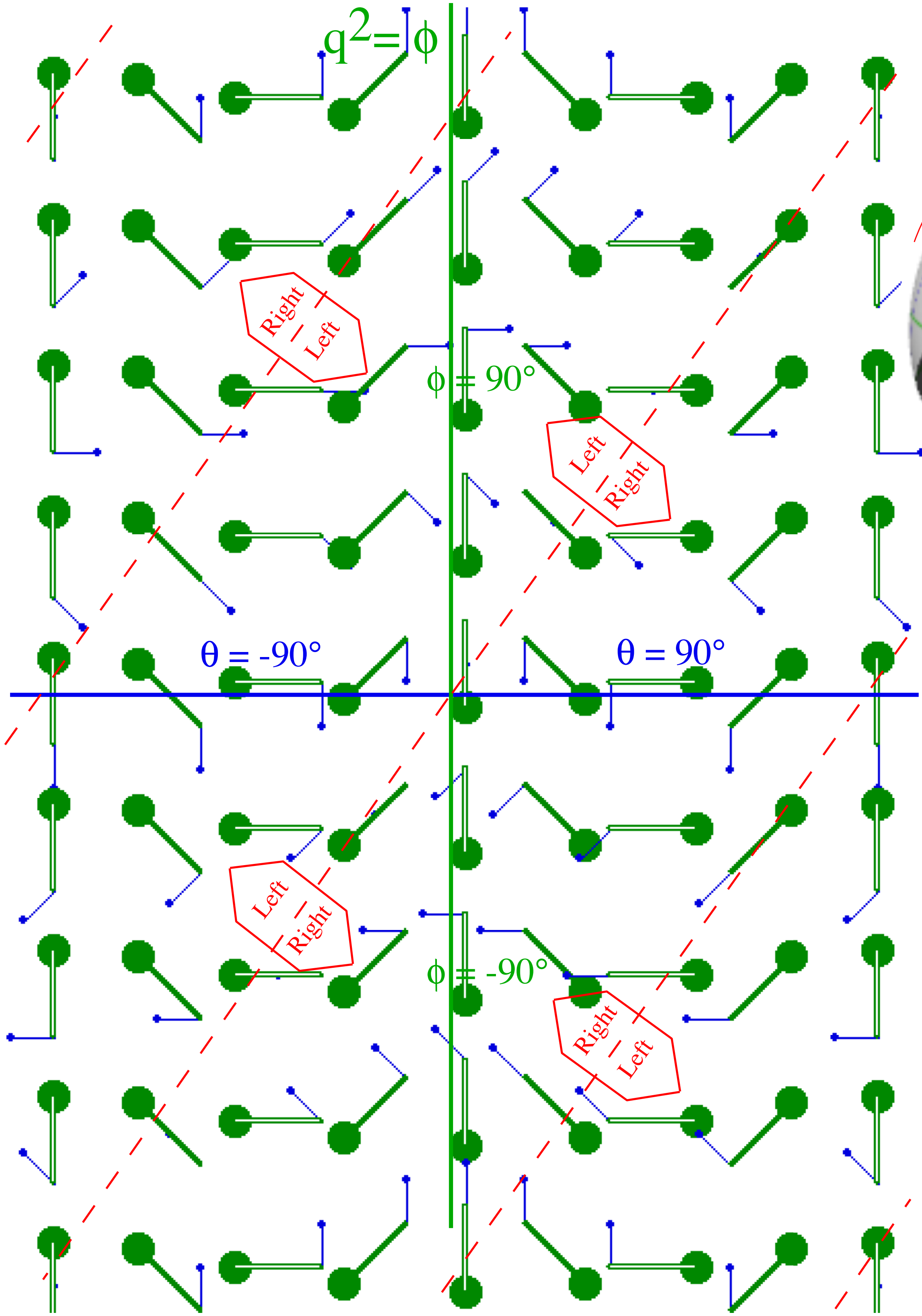


Fig. 3.1.3 "Flattened" ($q^1=\theta$, $q^2=\phi$) coordinate manifold for trebuchet

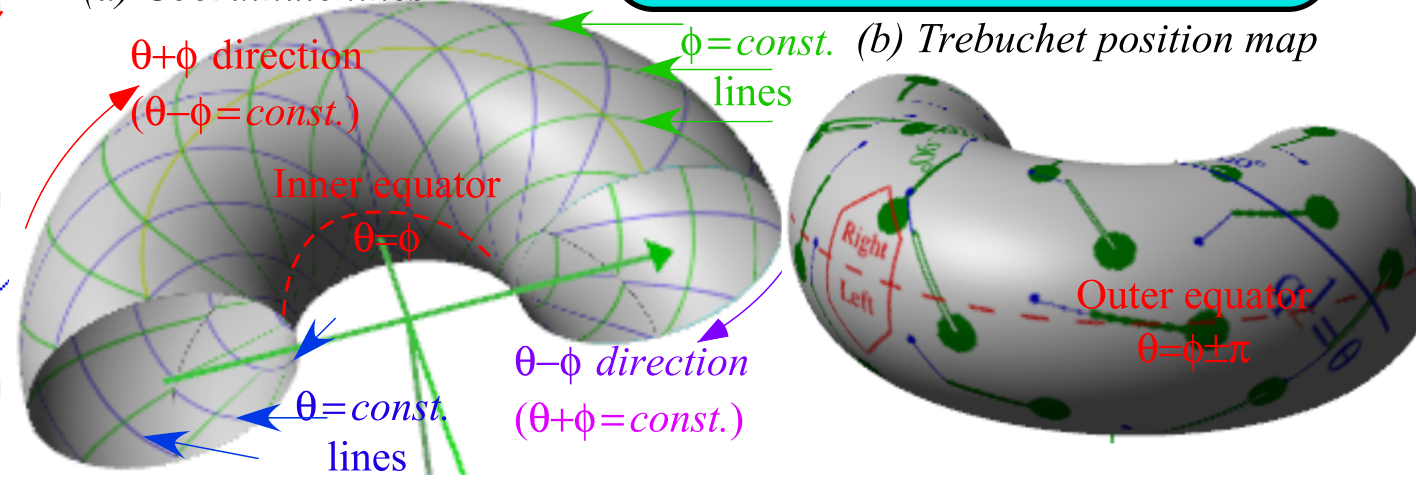


Fig. 3.1.2 Trebuchet torus.
(a) ($q^1=\theta$, $q^2=\phi$) coordinate lines.(b)Trebuchet position map and equators.

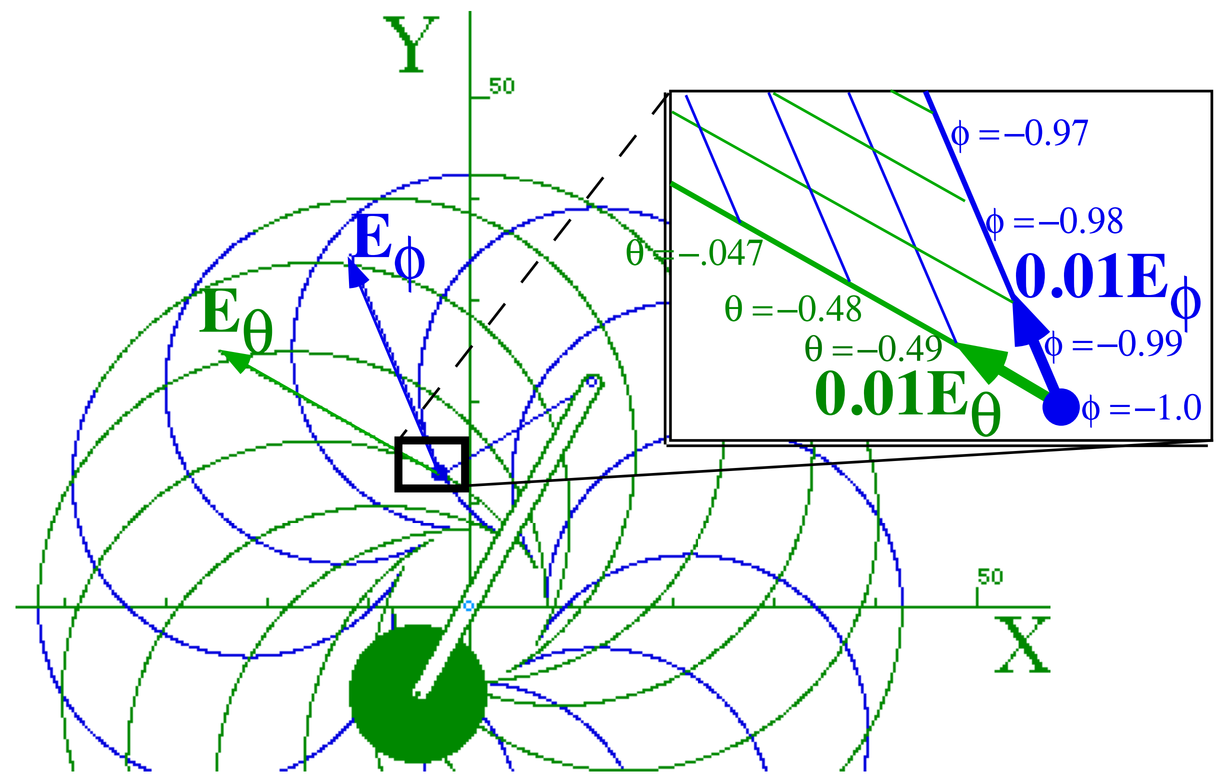


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

***Trebuchet* Web Simulations:**

[Default/Generic URL], [Montezuma's Revenge], [Seige of Kenilworth], ["Flinger"],
[Position Space (Course)], [Position Space (Fine)]

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold of trebuchet positions

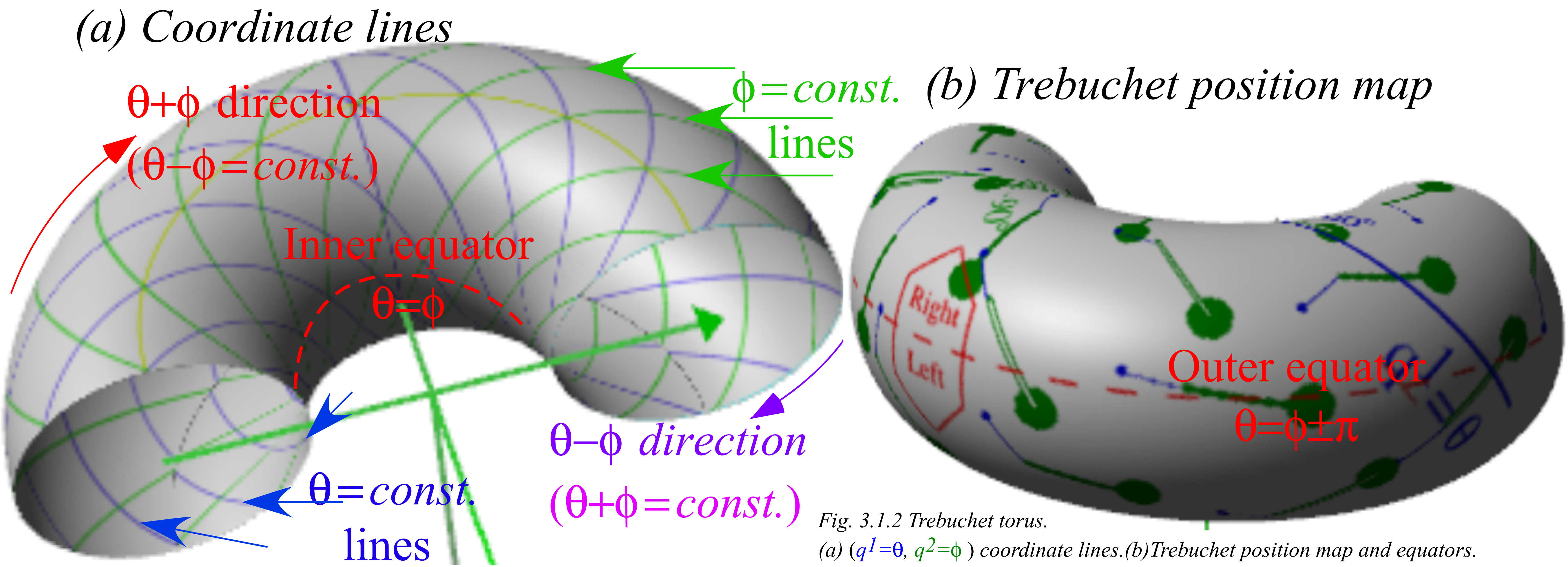


Fig. 3.1.2 Trebuchet torus.
(a) ($q^1=\theta$, $q^2=\phi$) coordinate lines.(b)Trebuchet position map and equators.

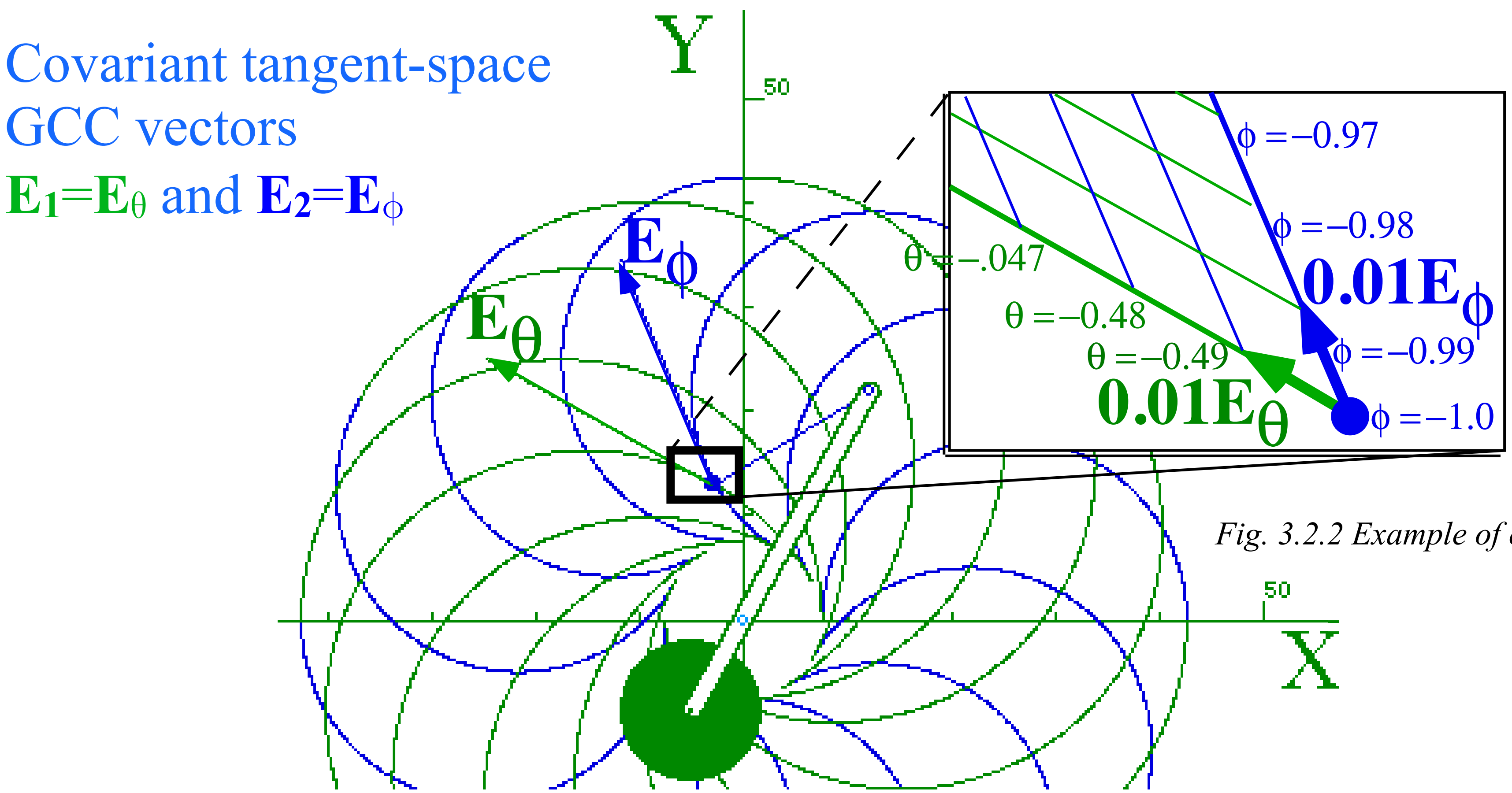


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Toroidal “rolled-up” (q^1=θ, q^2=ϕ)-manifold of trebuchet positions and “Flat” (q^1=θ, q^2=ϕ)-graph

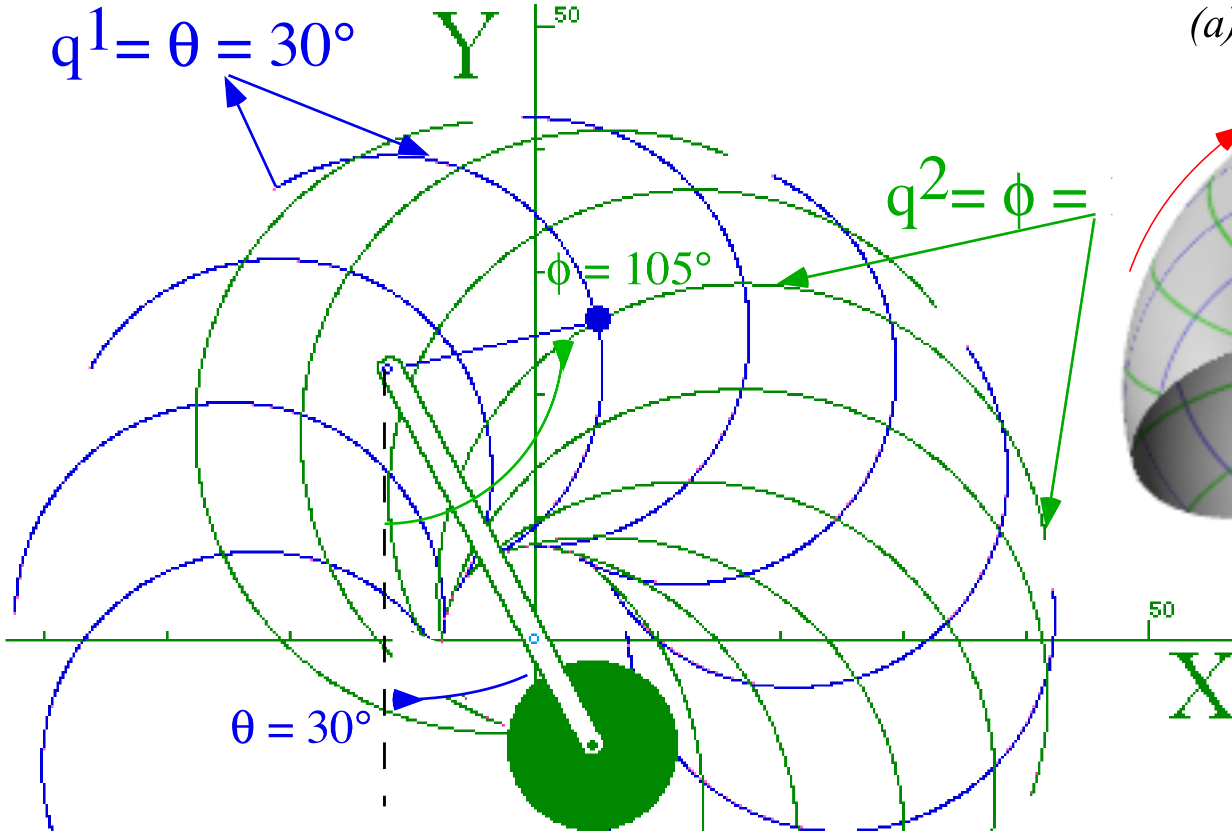


Fig. 3.1.1a (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Left handed sheet.)

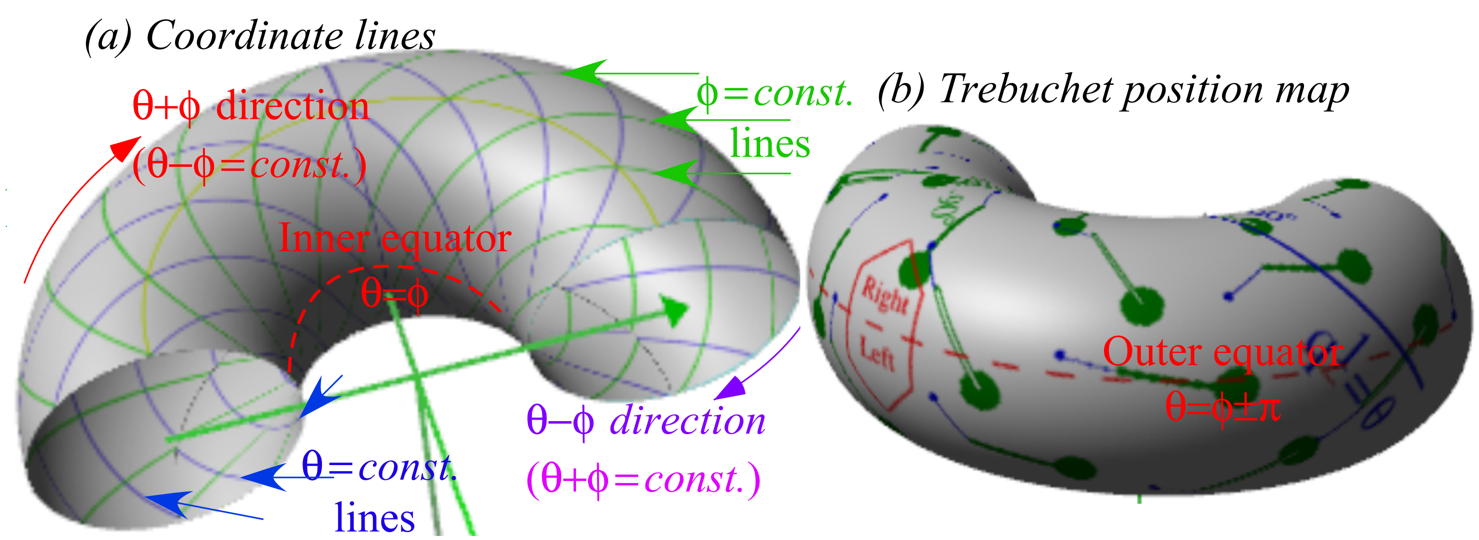
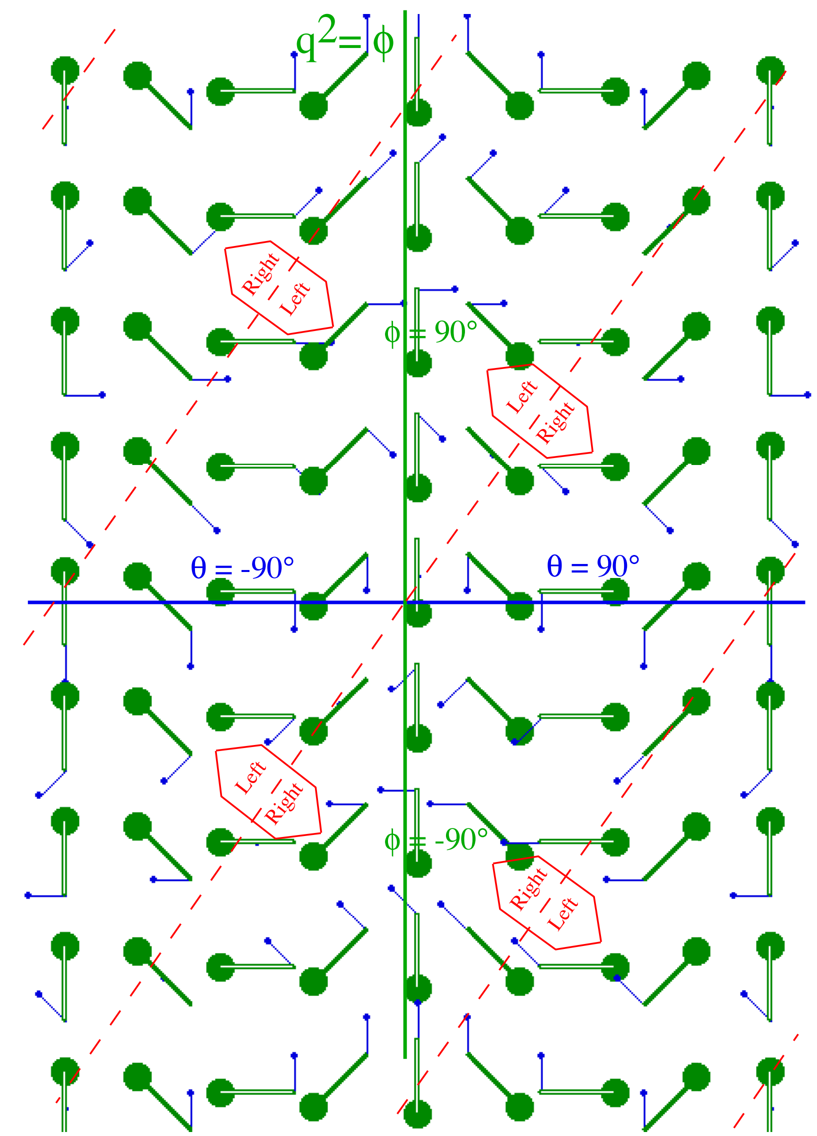


Fig. 3.1.2 Trebuchet torus.

(a) (q^1=θ, q^2=ϕ) coordinate lines. (b)Trebuchet position map and equators.



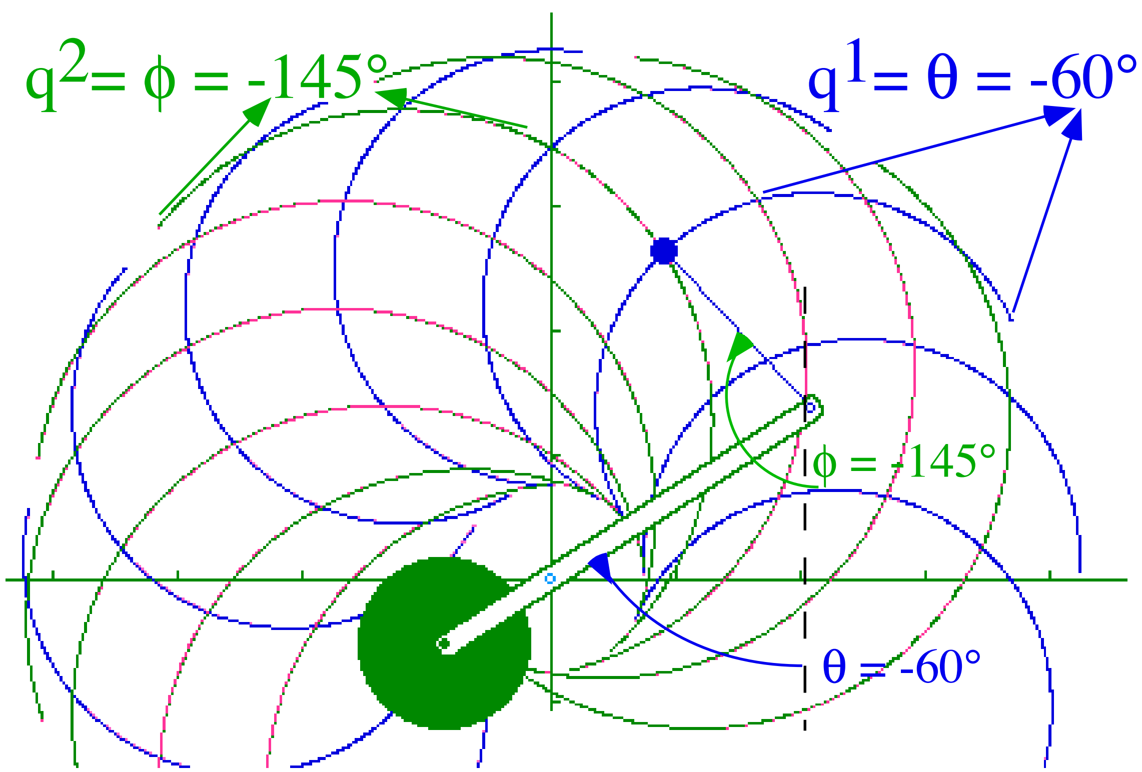


Fig. 3.1.1b (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Right handed sheet.)

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
→ *Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K*
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

from p. 45 of Lect. 9

J-Columns are *covariant vectors* $\{\mathbf{E}_1 = \mathbf{E}_r \;\; \mathbf{E}_2 = \mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1 = \mathbf{E}^r \;\; \mathbf{E}^2 = \mathbf{E}^\phi\}$

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow \mathbf{E}_1 & \uparrow \mathbf{E}_2 & \uparrow \mathbf{E}_r & \uparrow \mathbf{E}_\phi \end{array}$$

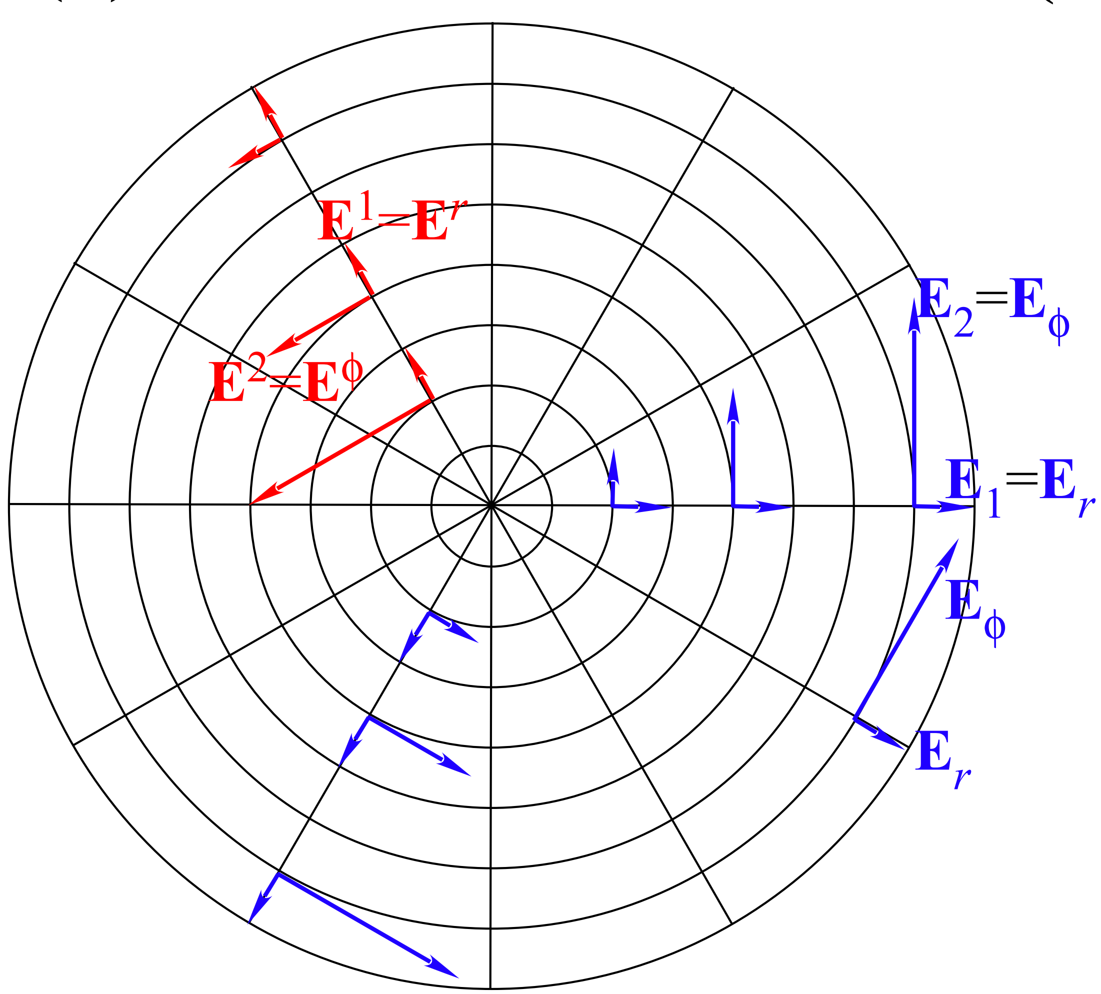
Derived from polar definition: $x = r\cos\phi$ *and* $y = r\sin\phi$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Inverse polar definition:

$r^2 = x^2 + y^2$ *and* $\phi = \mathrm{atan2}(y,x)$

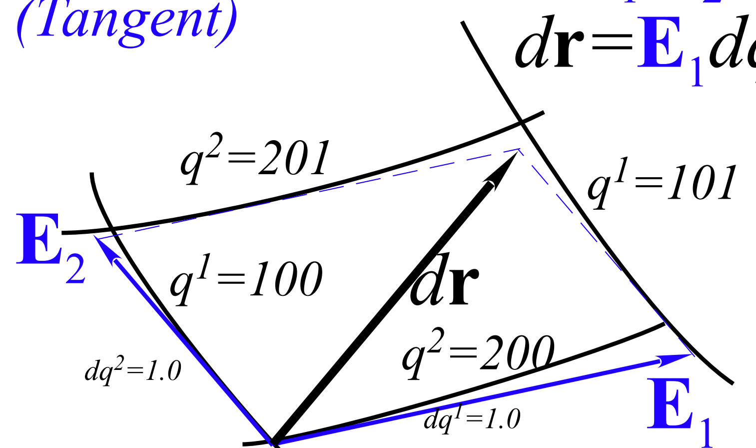
(a) Polar coordinate bases



(b) *Co*variant bases $\{\mathbf{E}_1 \; \mathbf{E}_2\}$

(Tangent)

$d\mathbf{r} = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$

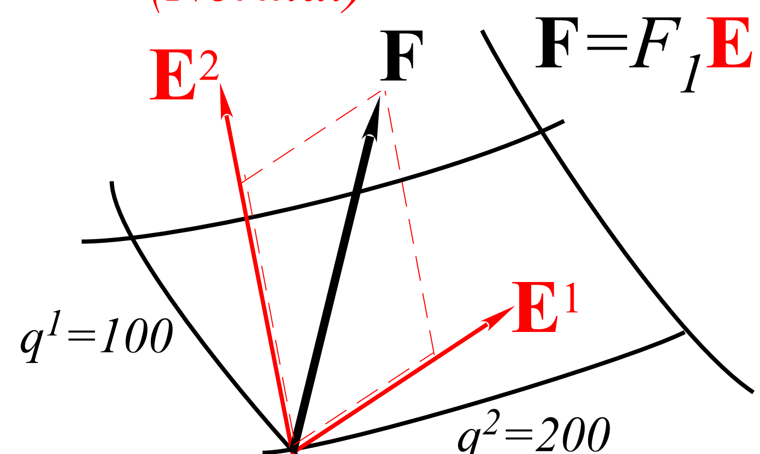


NOTE:These
are 2D drawings!
No 3D *perspective*

(c) *Contra*variant bases $\{\mathbf{E}^1 \; \mathbf{E}^2\}$

(Normal)

$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$



Unit 1
Fig. 12.10

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

from p. 50 of Lect. 9

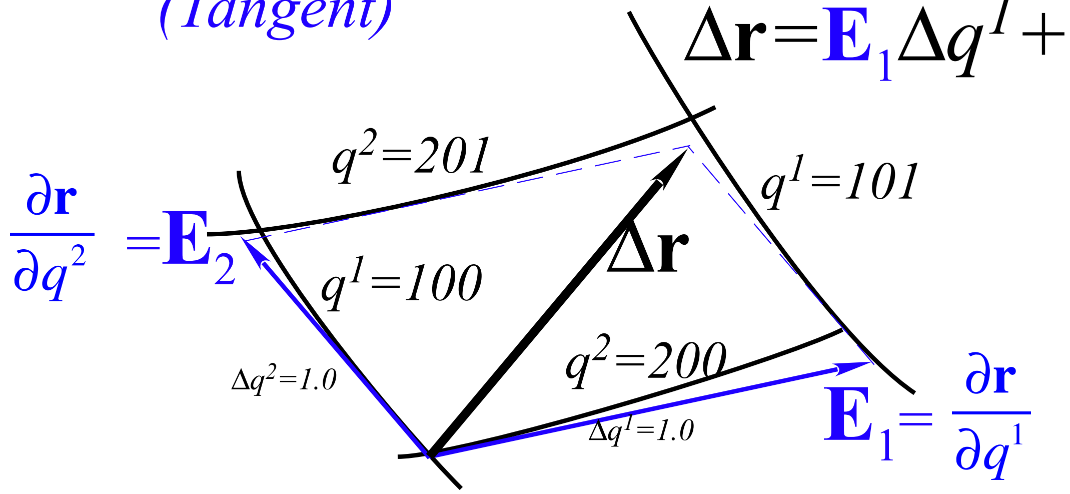
*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls

(Tangent)

geometric unit

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



$\mathbf{E}_1$ *follows tangent to* q^2*=const. ...*
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3*,... remain constant*

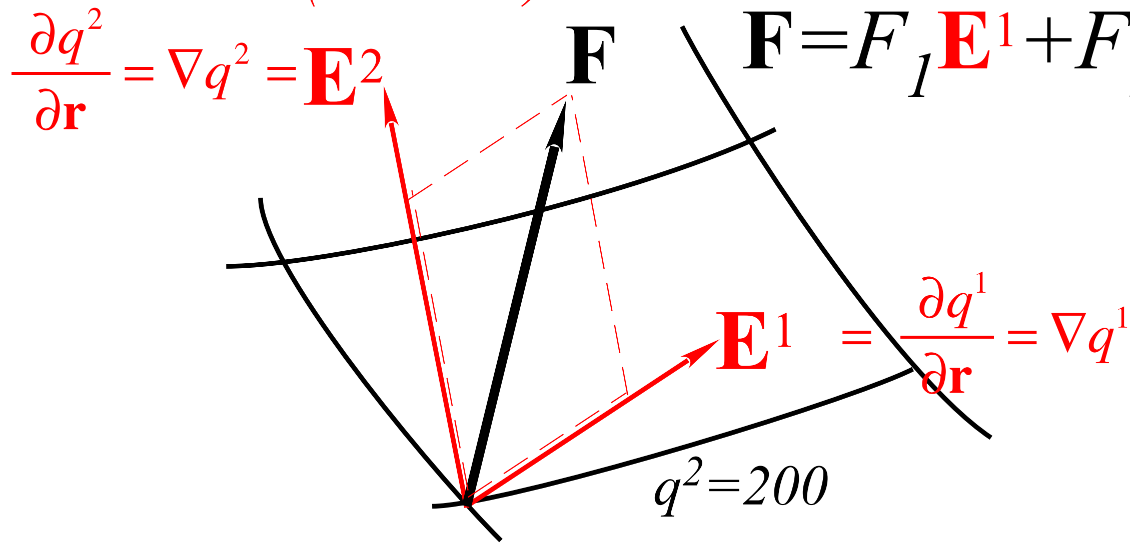
$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1\mathbf{E}_1 + V^2\mathbf{E}_2 = V^1\frac{\partial \mathbf{r}}{\partial q^1} + V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2$$



NOTE: These are 2D drawings! *No* 3D *perspective*

$\mathbf{E}^1$ *is normal to* q^1*=const. since* **gradient** *of* q^1 *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$ *of all its partial derivatives*

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2 = F_1\frac{\partial q^1}{\partial \mathbf{r}} + F_2\frac{\partial q^2}{\partial \mathbf{r}} = F_1\nabla q^1 + F_2\nabla q^2$$

Comparison: Covariant $\mathbf{E}_m=\frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^n=\frac{\partial q^n}{\partial \mathbf{r}}=\nabla q^n$

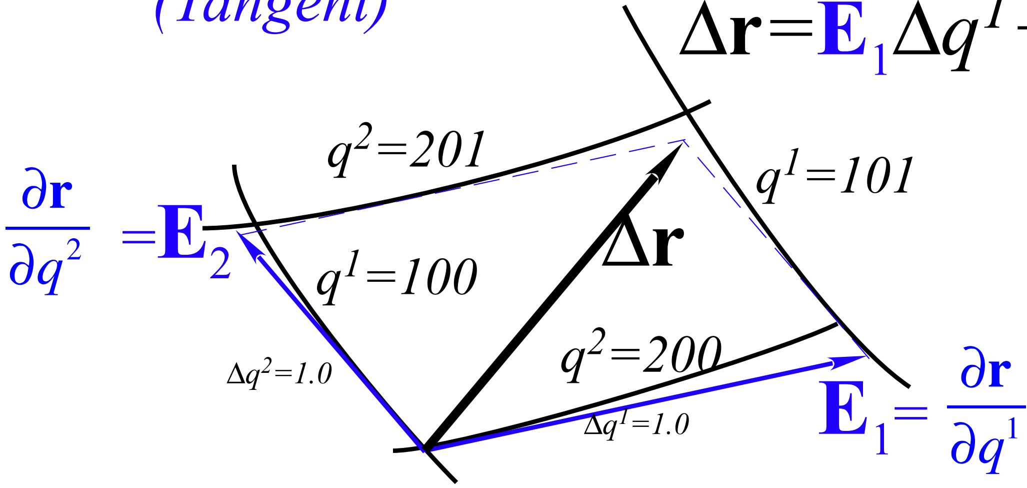
from p. 50 of Lect. 9

*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls (*geometric unit*)

(Tangent)

$$\Delta\mathbf{r}=\mathbf{E}_1\Delta q^1+\mathbf{E}_2\Delta q^2$$

is based on chain rule: $d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial q^1}dq^1+\frac{\partial \mathbf{r}}{\partial q^2}dq^2=\mathbf{E}_1dq^1+\mathbf{E}_2dq^2$



$\mathbf{E}_1$ *follows tangent to* q^2=*const. ...*
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while q^2, q^3,*... remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V}=V^1\mathbf{E}_1+V^2\mathbf{E}_2=V^1\frac{\partial \mathbf{r}}{\partial q^1}+V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

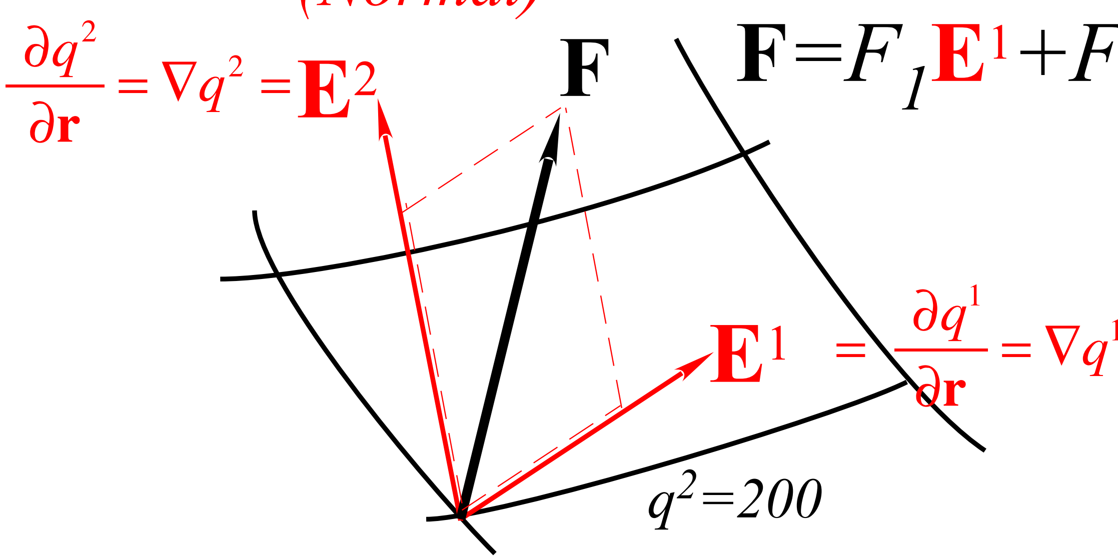
Co-Contra dot products $\mathbf{E}_m\cdot\mathbf{E}^n$ *are orthonormal:*

$$\mathbf{E}_m\bullet\mathbf{E}^n=\frac{\partial \mathbf{r}}{\partial q^m}\bullet\frac{\partial q^n}{\partial \mathbf{r}}=\delta_m^n$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2$$



$\mathbf{E}^1$ *is normal to* q^1=*const. since* **gradient** *of* q^1 *is vector sum* $\nabla q^1=\begin{pmatrix}\frac{\partial q^1}{\partial x}\\ \frac{\partial q^1}{\partial y}\end{pmatrix}$
of all its partial derivatives

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2=F_1\frac{\partial q^1}{\partial \mathbf{r}}+F_2\frac{\partial q^2}{\partial \mathbf{r}}=F_1\nabla q^1+F_2\nabla q^2$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

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Riemann equation derivation for trebuchet model

Riemann equation force analysis

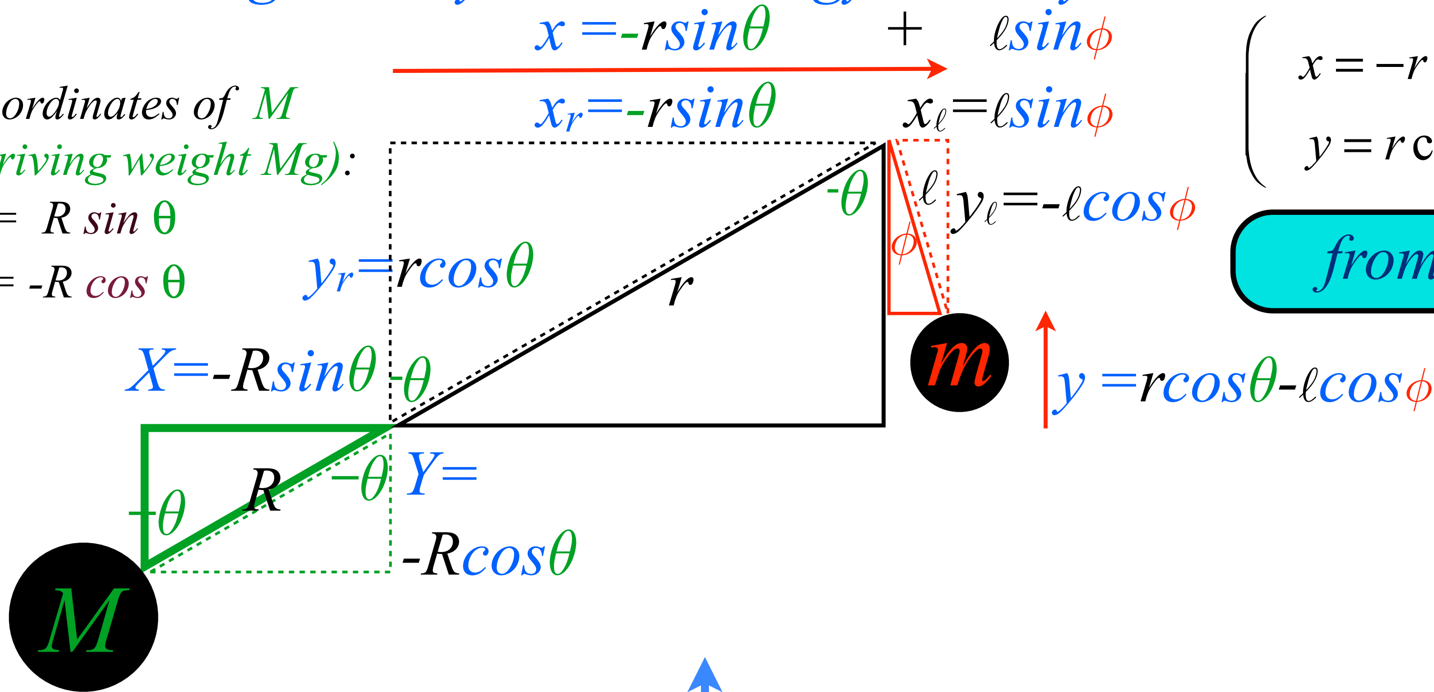
2nd-guessing Riemann equation?

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M *(Driving weight Mg)*:

$X = R \sin\theta$

$Y = -R\cos\theta$



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

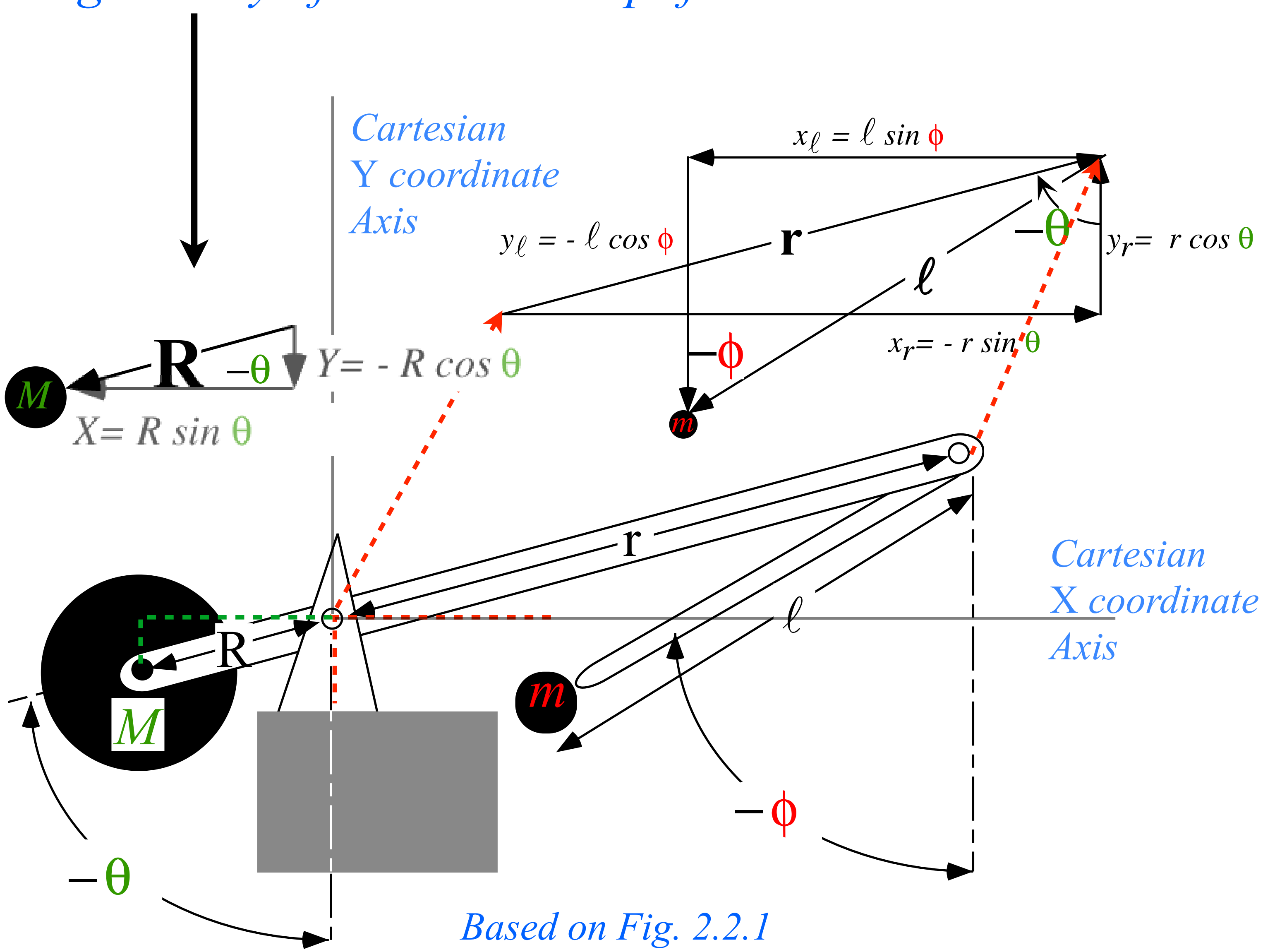
from p. 20 of Lect. 14

Coordinates of mass m *(Payload or projectile)*:

$x = x_r + x_\ell = -\, r \sin\theta + \ell \sin\phi$

$y = y_r + y_\ell = \; r\cos\theta - \ell\cos\phi$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Jacobian transformation matrix

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

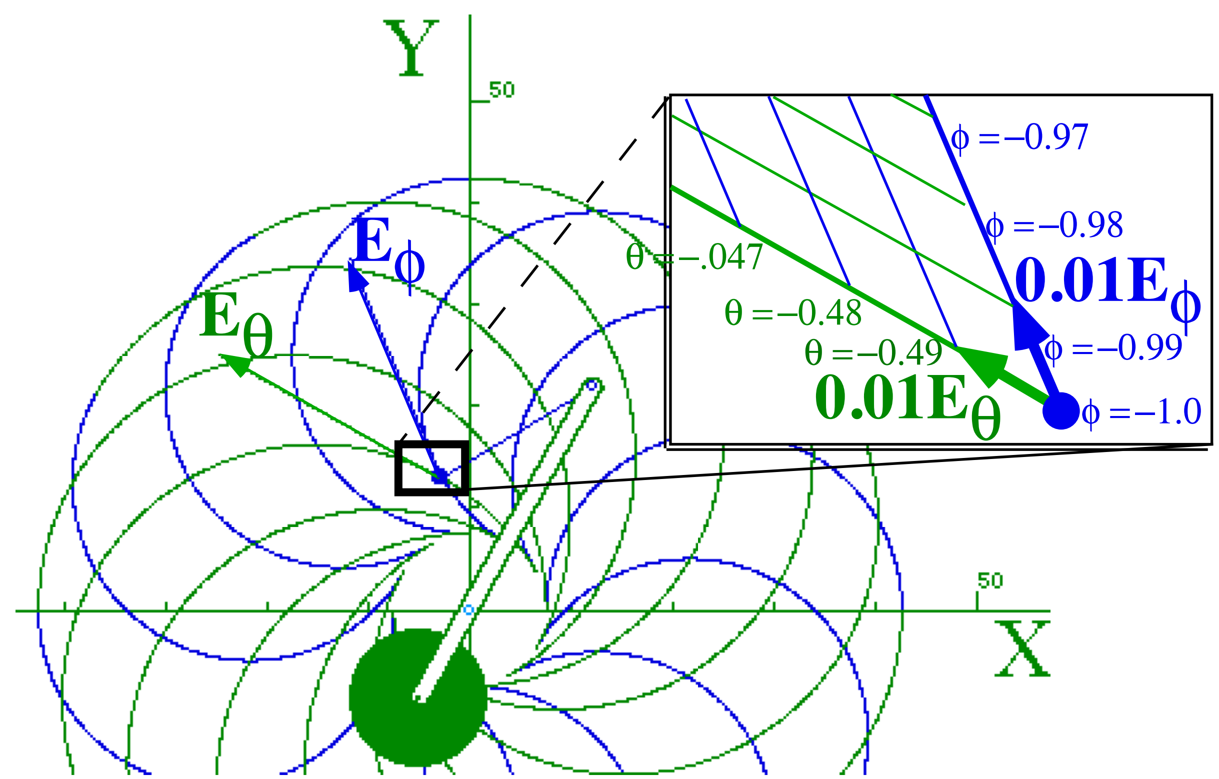


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

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Riemann equation force analysis

2nd-guessing Riemann equation?

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

Using 2x2 inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

$$\begin{array}{c|c} \begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} & \begin{array}{c} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{array} \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{c|c} \begin{vmatrix} \ell\sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} & \begin{array}{c} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{array} \end{array}}{\ell r \sin\theta\cos\phi - \ell r \sin\phi\cos\theta}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

$$\begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \end{array}$$
$$\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline \end{array} \begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix}$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

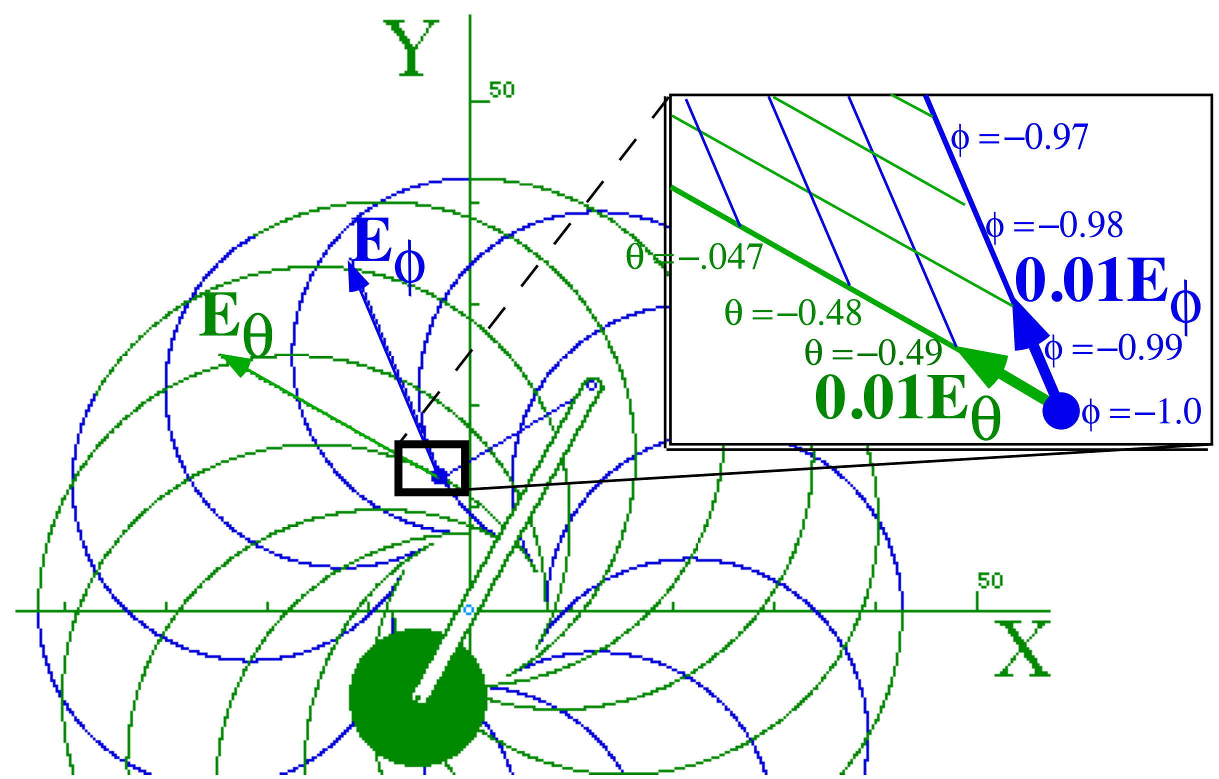

Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

$$\left| \begin{array}{ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} \right| = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{array}{cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array} \right.}{r\ell\sin(\theta-\phi)}$$

$$\left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

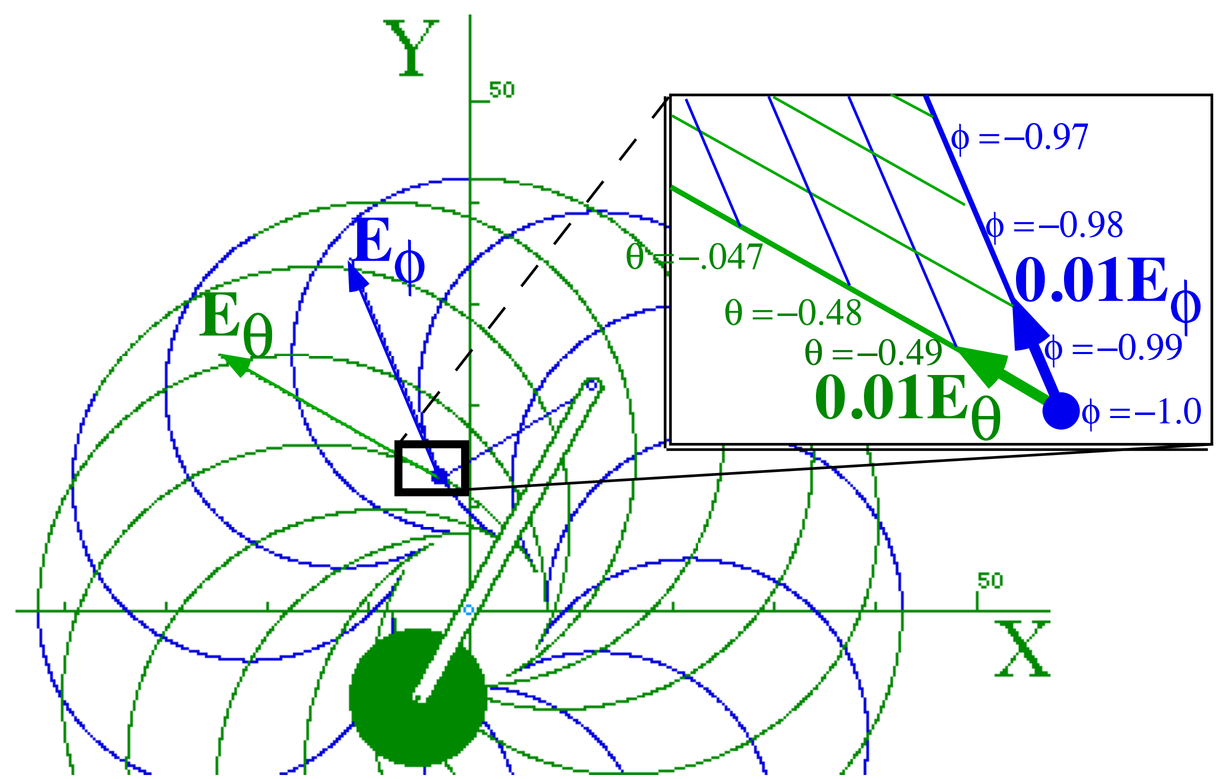

Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell \sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

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Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

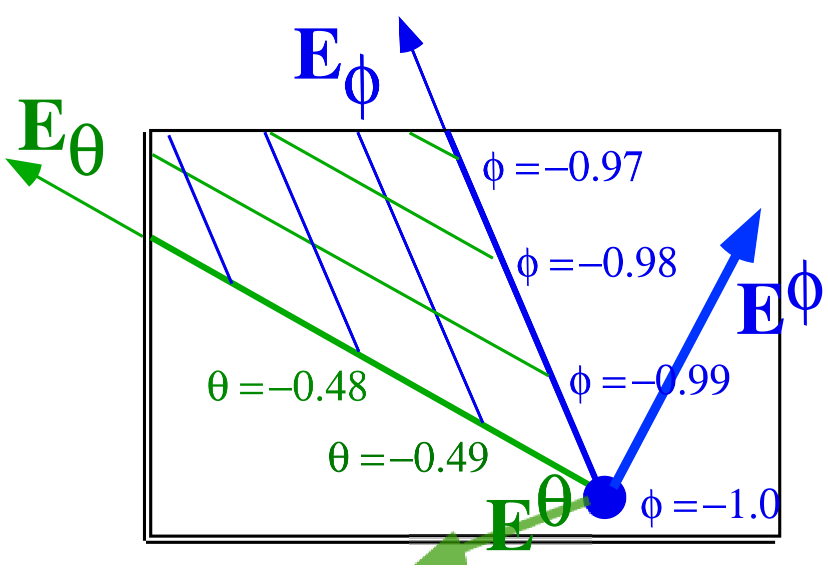
$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Covariant tangent-space GCC vectors

$\mathbf{E}_1 = \mathbf{E}_\theta$ and $\mathbf{E}_2 = \mathbf{E}_\phi$



Contravariant normal-space GCC vectors

$\mathbf{E}^1 = \mathbf{E}^\theta$ and $\mathbf{E}^2 = \mathbf{E}^\phi$

Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

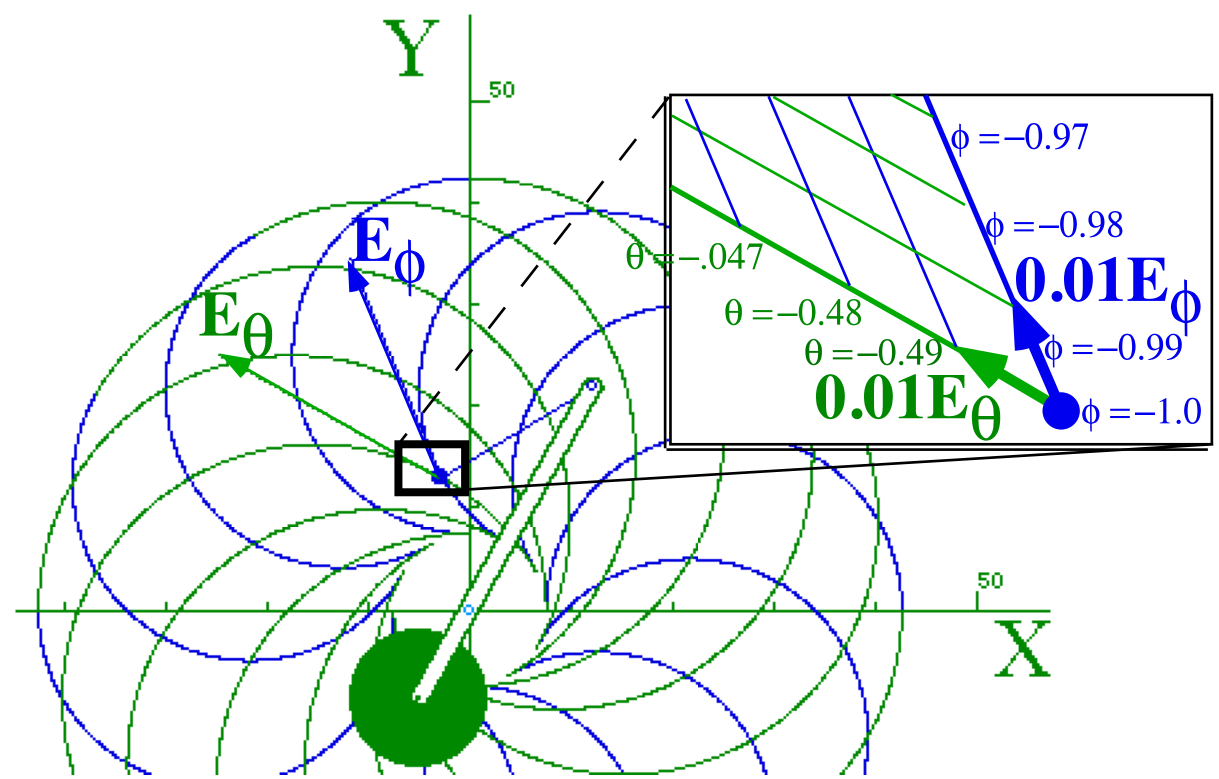


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} \ell \sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{r\ell \sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

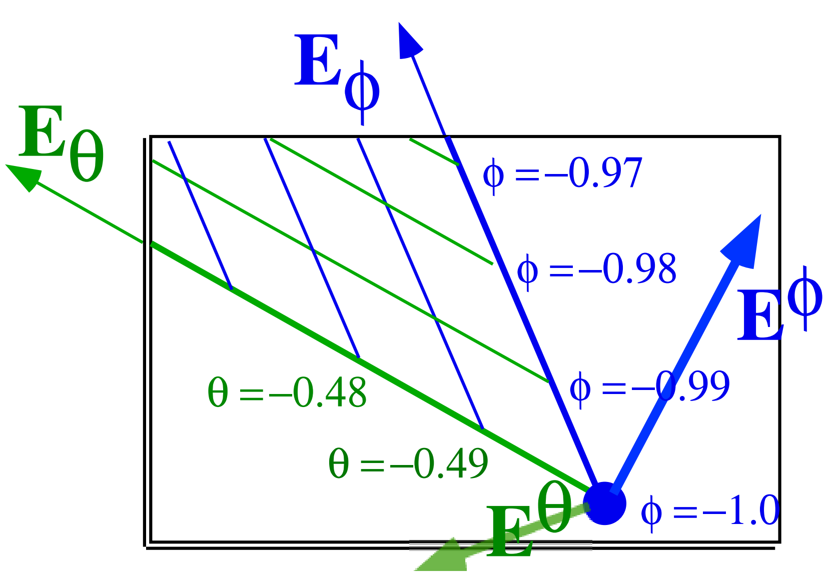
$$\mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Covariant tangent-space GCC vectors

$\mathbf{E}_1 = \mathbf{E}_\theta$ and $\mathbf{E}_2 = \mathbf{E}_\phi$



Contravariant normal-space GCC vectors

$\mathbf{E}^1 = \mathbf{E}^\theta$ and $\mathbf{E}^2 = \mathbf{E}^\phi$

Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

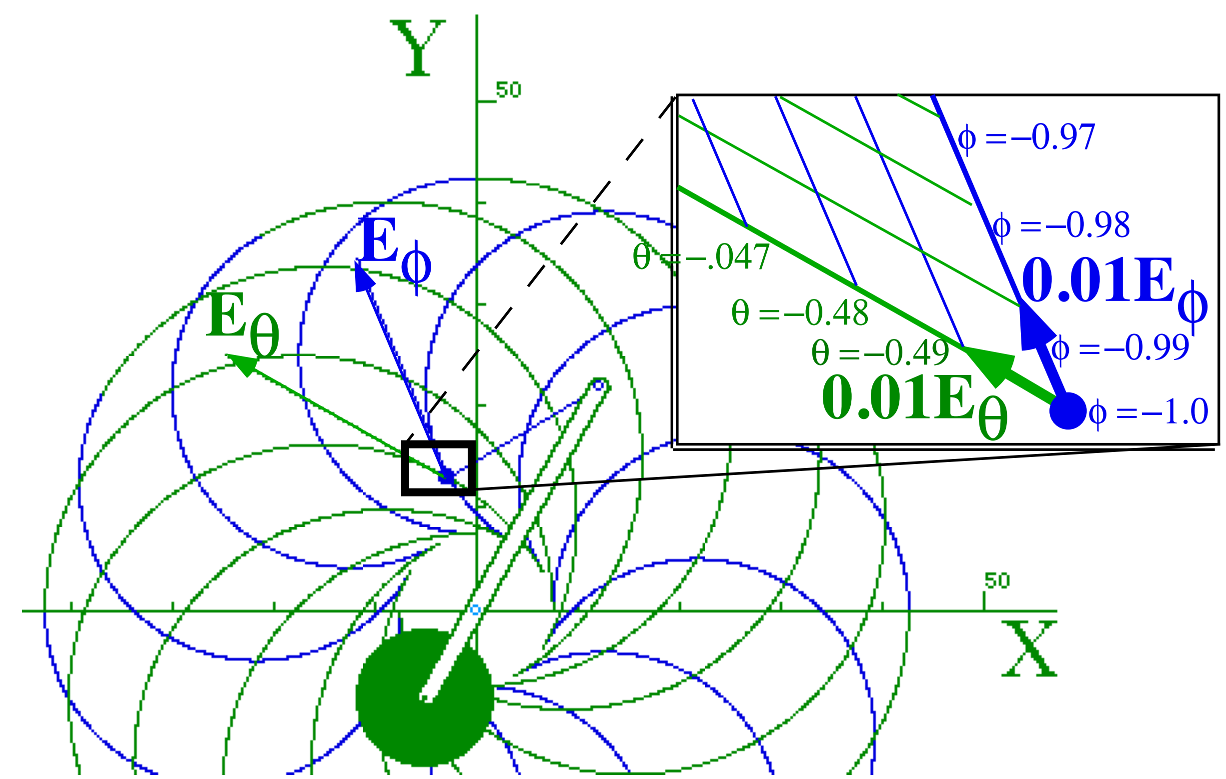


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors. Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant metric tensor g_{mn}

Invariant Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant metric tensor g^{mn}

from p. 55 of Lect. 9

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow \mathbf{E}_1 & \uparrow \mathbf{E}_2 & \uparrow \mathbf{E}_r & \uparrow \mathbf{E}_\phi \end{array}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \qquad\qquad \left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 93 of Lect. 14

$$\begin{array}{|ccc|c} \dfrac{\partial q^1}{\partial x^1} & \dfrac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \dfrac{\partial q^2}{\partial x^1} & \dfrac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\theta}{\partial y} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \dfrac{\partial x^1}{\partial q^1} & \dfrac{\partial x^1}{\partial q^2} & \cdots \\ \dfrac{\partial x^2}{\partial q^1} & \dfrac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \dfrac{\partial x}{\partial\theta} & \dfrac{\partial x}{\partial\phi} \\ \dfrac{\partial y}{\partial\theta} & \dfrac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi) \qquad \mathbf{E}^\theta\bullet\mathbf{E}_\phi = 0 = \mathbf{E}_\theta\bullet\mathbf{E}^\phi$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi) \qquad \mathbf{E}^\theta\bullet\mathbf{E}_\theta = 1 = \mathbf{E}_\phi\bullet\mathbf{E}^\phi$$

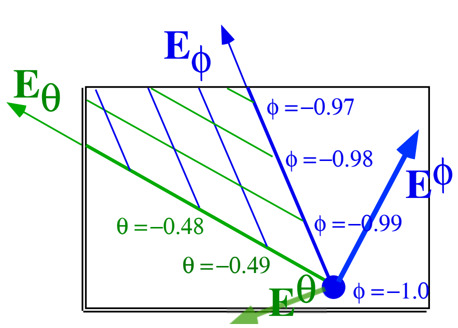
$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \qquad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

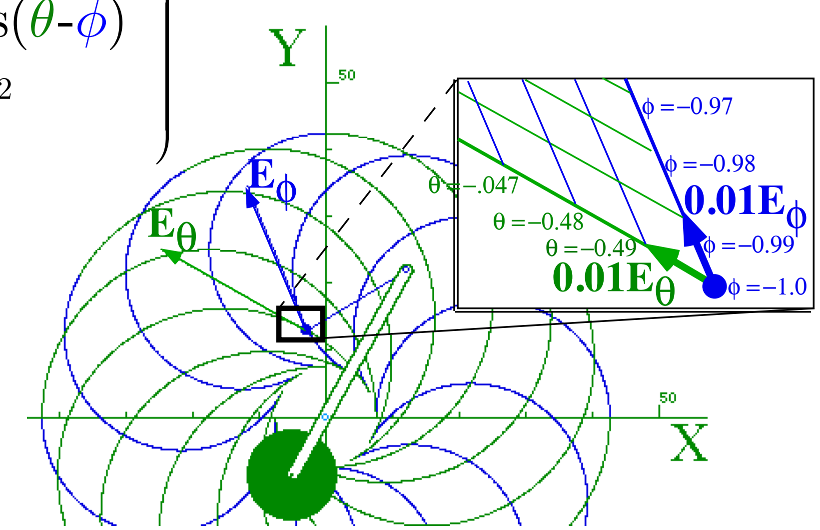
Contravariant metric $g^{mn} = \mathbf{E}^m\bullet\mathbf{E}^n = g^{nm}$ *versus* *Covariant metric* $g_{mn} = \mathbf{E}_m\bullet\mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta\bullet\mathbf{E}_\theta & \mathbf{E}_\theta\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_\theta & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta-\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$





Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

- *Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

- *Trebuchet Cartesian projectile coordinates are double-valued*
    - *Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph*
- *Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K*
    - *Covariant metric g_{mn} vs contravariant metric g^{mn} (Lect. 10 p.43-49)*
    - *Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space*
    - *Covariant vs. contravariant coordinate transformations*
- *Metric g_{mn} tensor geometric relations to length, area, and volume*

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

- *Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 15 p. 77)*
- *Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 15 p. 69)*
- *Riemann equation derivation for trebuchet model*
- *Riemann equation force analysis*
    - *2nd-guessing Riemann equation?*

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \qquad \textit{Using 2x2 inverse} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD\text{-}BC}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\left| \begin{matrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \right| \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{matrix} \ell\sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{matrix} \right| \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{r\ell\sin(\theta-\phi)}$$

$$\left| \begin{matrix} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \right| = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \left| \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{matrix} \right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi) \qquad \mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi) \qquad \mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Contravariant metric $g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm}$ *versus* *Covariant metric* $g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \bullet \mathbf{E}^\theta & \mathbf{E}^\theta \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^\theta & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix}$$

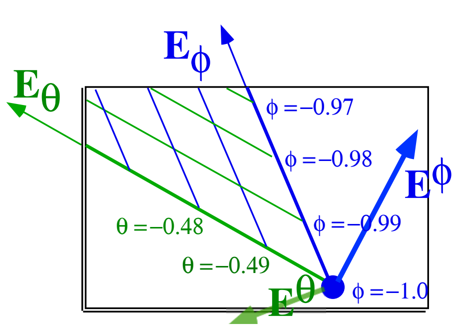
$$= \begin{pmatrix} \ell^2 & r\ell(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

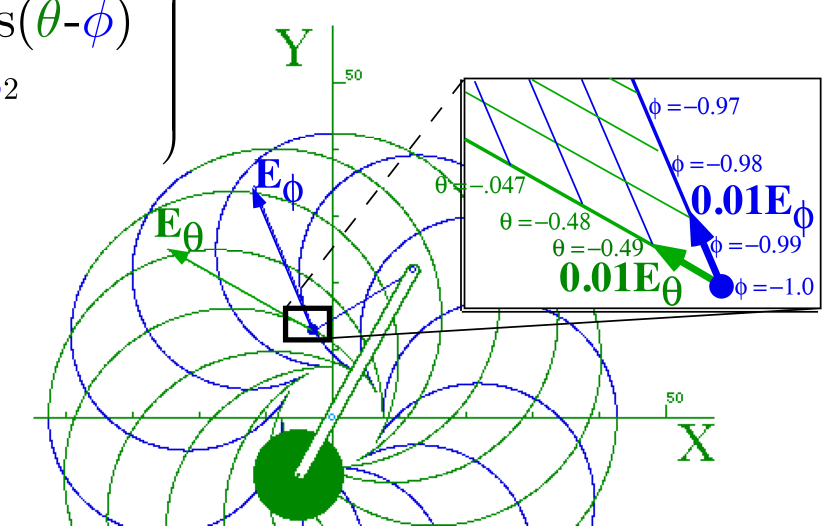
$$= \begin{pmatrix} \ell^2 & r\ell\cos(\theta\text{-}\phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \bullet \mathbf{E}_\theta & \mathbf{E}_\theta \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_\theta & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & \text{-}r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & \text{-}r\ell\cos(\theta\text{-}\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$





Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \qquad \left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Using 2x2 inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} \ell\sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{r\ell\sin(\theta-\phi)}$$

$$\begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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Contravariant metric $g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm}$ *versus* *Covariant metric* $g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \bullet \mathbf{E}^\theta & \mathbf{E}^\theta \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^\theta & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix}$$

$$= \begin{pmatrix} \ell^2 & r\ell(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

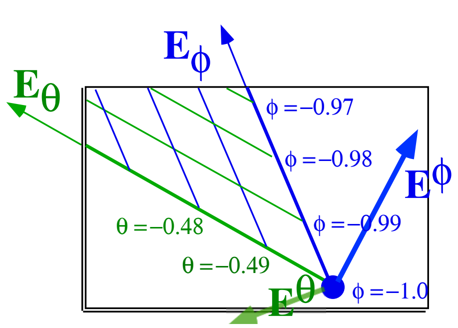
$$= \begin{pmatrix} \ell^2 & r\ell\cos(\theta\text{-}\phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \bullet \mathbf{E}_\theta & \mathbf{E}_\theta \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_\theta & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix}$$

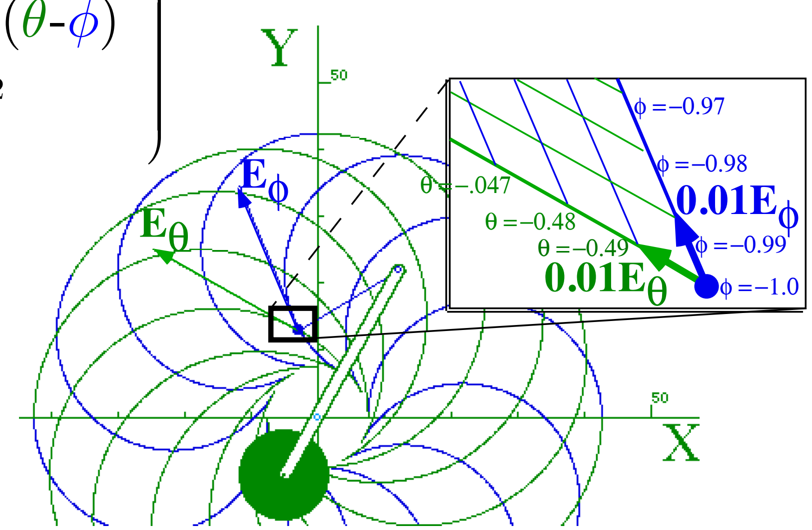
$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta\text{-}\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

Jacobian $J^T J$*-product gives* g_{mn}



$$J^T J = \begin{array}{c|cc} \mathbf{E}_\theta & -r\cos\theta & -r\sin\theta \\ \mathbf{E}_\phi & \ell\cos\phi & \ell\sin\phi \end{array} \quad \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$



Kajobian transformation matrix *versus* Jacobian transformation matrix

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \qquad\qquad \left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ *versus* Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi) \qquad \mathbf{E}^\theta\bullet\mathbf{E}_\phi = 0 = \mathbf{E}_\theta\bullet\mathbf{E}^\phi$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi) \qquad \mathbf{E}^\theta\bullet\mathbf{E}_\theta = 1 = \mathbf{E}_\phi\bullet\mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Contravariant metric $g^{mn} = \mathbf{E}^m\bullet\mathbf{E}^n = g^{nm}$ *versus* Covariant metric $g_{mn} = \mathbf{E}_m\bullet\mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta\bullet\mathbf{E}^\theta & \mathbf{E}^\theta\bullet\mathbf{E}^\phi \\ \mathbf{E}^\phi\bullet\mathbf{E}^\theta & \mathbf{E}^\phi\bullet\mathbf{E}^\phi \end{pmatrix}$$

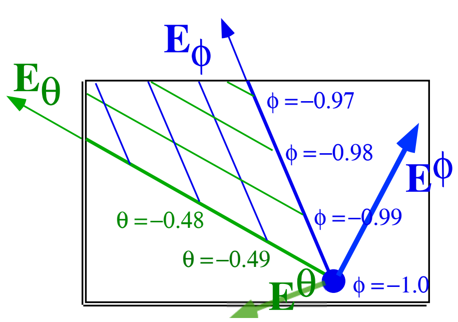
$$= \begin{pmatrix} \ell^2 & r\ell(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$= \begin{pmatrix} \ell^2 & r\ell\cos(\theta\text{-}\phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2\ell^2\sin^2(\theta\text{-}\phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta\bullet\mathbf{E}_\theta & \mathbf{E}_\theta\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_\theta & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix}$$

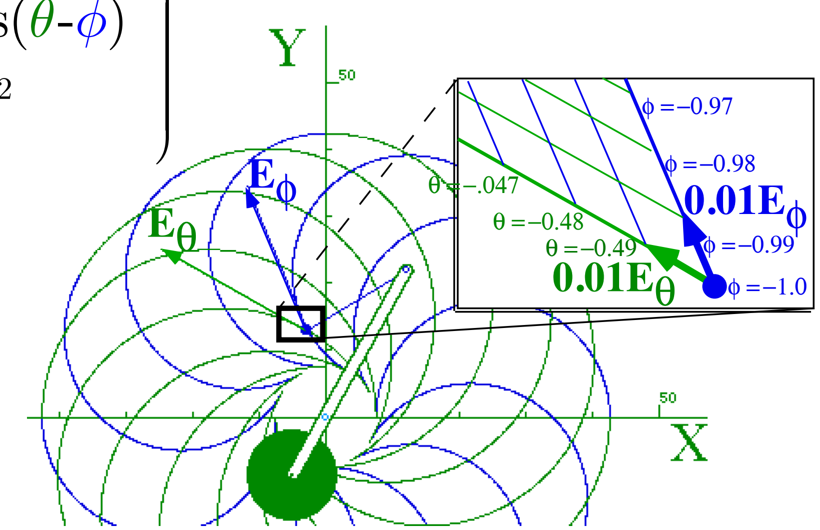
$$= \begin{pmatrix} r^2 & -r\ell(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -r\ell\cos(\theta\text{-}\phi) \\ g_{\phi\theta} & \ell^2 \end{pmatrix}$$



Jacobian J^TJ-product gives g_{mn}

$$J^TJ = \begin{array}{c|cc} \mathbf{E}_\theta & -r\cos\theta & -r\sin\theta \\ \mathbf{E}_\phi & \ell\cos\phi & \ell\sin\phi \end{array} \;\; \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$



Kajobian KK^T-product would give g^{mn}

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
→ *Tangent {$\mathbf{E}_n$}space vs. Normal {$\mathbf{E}^m$}space*
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ V^m = \mathbf{V}\bullet\mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}},\ ,$$

$$U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}},\ etc.$$

Normal space (Contravariant)

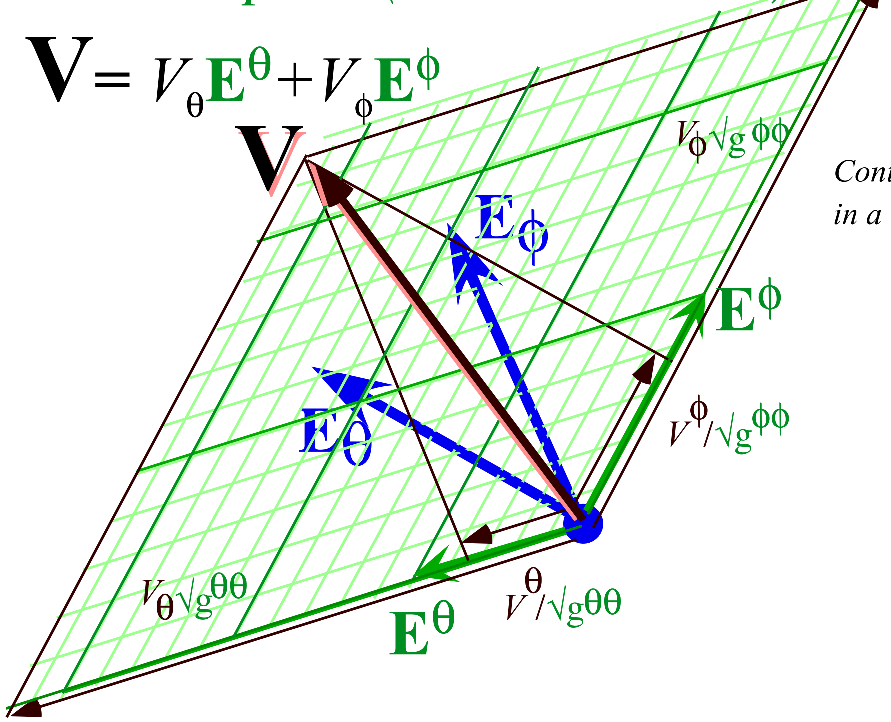


Fig. 3.3.2
Contravariant vector geometry in a normal space $(\mathbf{E}^\theta,\mathbf{E}^\phi)$.

Tangent space (Covariant)

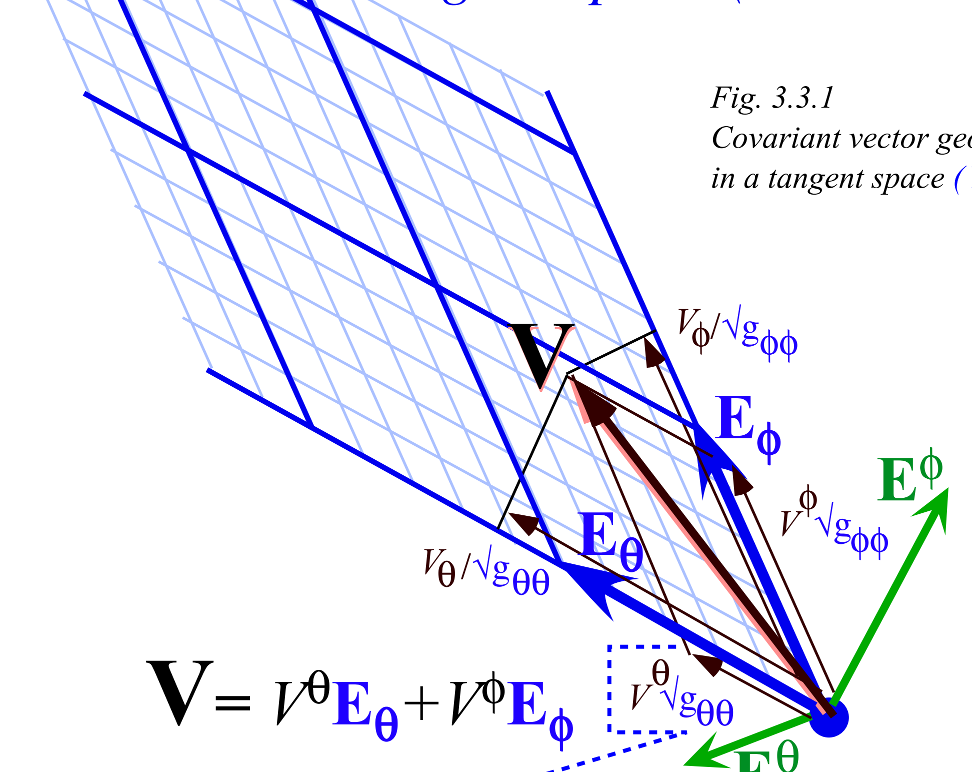


Fig. 3.3.1
Covariant vector geometry in a tangent space $(\mathbf{E}_\theta,\mathbf{E}_\phi)$.

covariant parallelogram side

$$\left|V^\theta\,\mathbf{E}_\theta\right| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Metric g_{mn} *or* g^{mn} *tensor geometric relations to length, area, and volume*

from p. 95 of Lect. 14

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector $\mathbf{U},\mathbf{V},...$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E}_\mathbf{m} = U_n\,\mathbf{E}^\mathbf{n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E}_\mathbf{m} = V_n\,\mathbf{E}^\mathbf{n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E}^\mathbf{m}$, $V^m = \mathbf{V}\bullet\mathbf{E}^\mathbf{m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\quad U_n = \mathbf{U}\bullet\mathbf{E}_\mathbf{n}$, $V_n = \mathbf{V}\bullet\mathbf{E}_\mathbf{n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

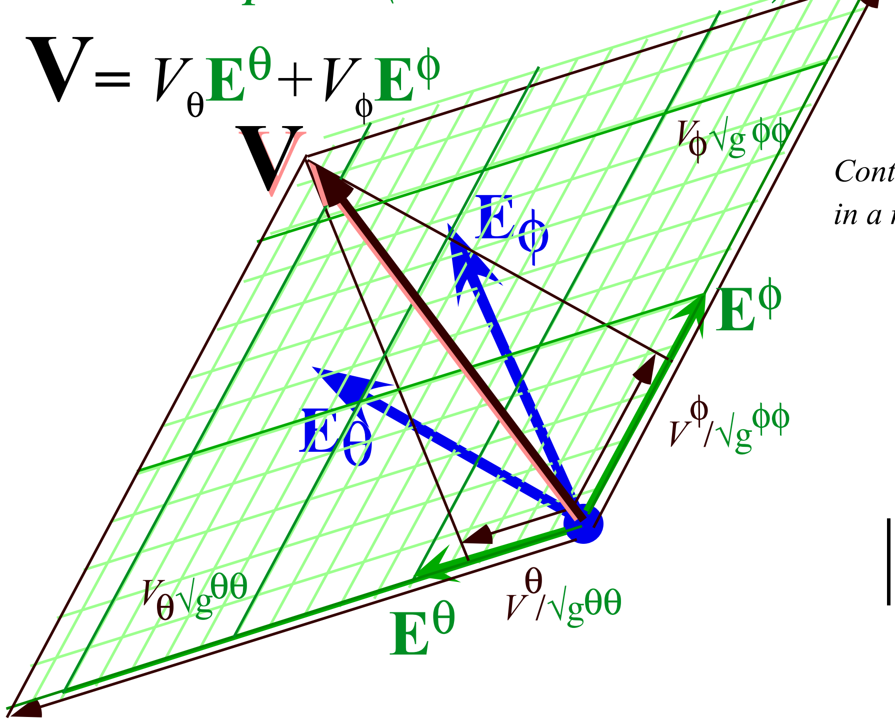


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

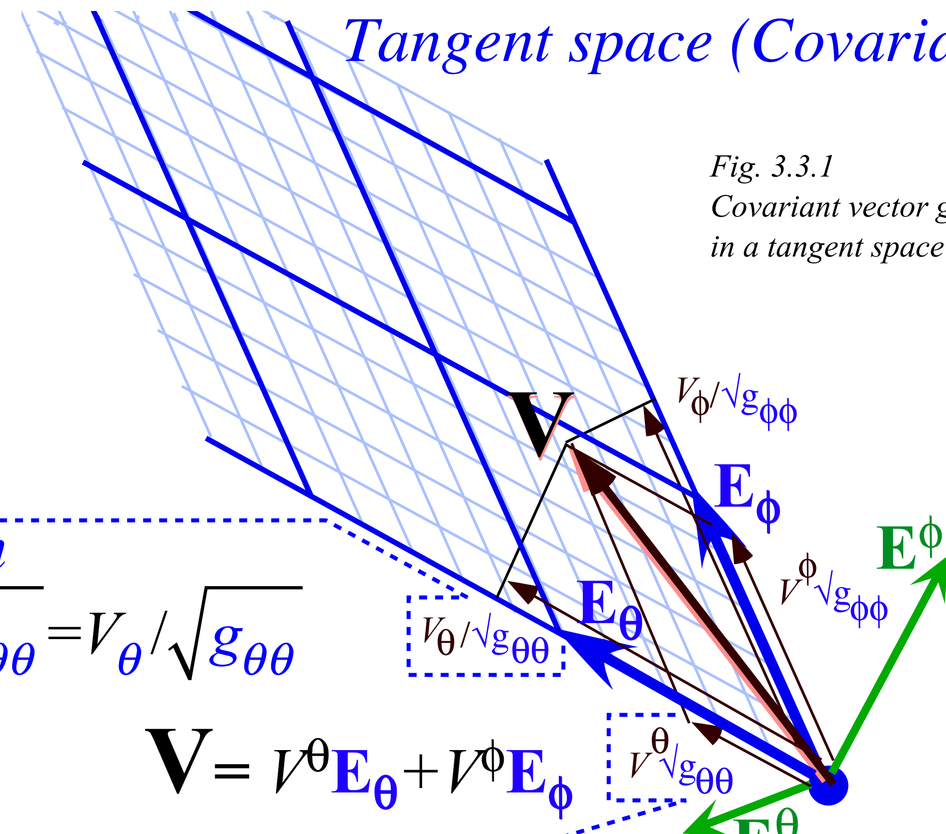


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\hat{\mathbf{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

covariant parallelogram side

$$\left|V^\theta\,\mathbf{E}_\theta\right| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Contravariant vector $\mathbf{E}^m$ is written in terms of covariant vectors $\mathbf{E}_n$ as would any vector $\mathbf{V}= V^n\,\mathbf{E}_n$ using dot product $V^n=\mathbf{V}\bullet\mathbf{E}^n$ and metric g_{mn} or g^{mn}...

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Contravariant vectors $\mathbf{E}^m$ versus Covariant vectors $\mathbf{E}_n$

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Normal space (Contravariant)

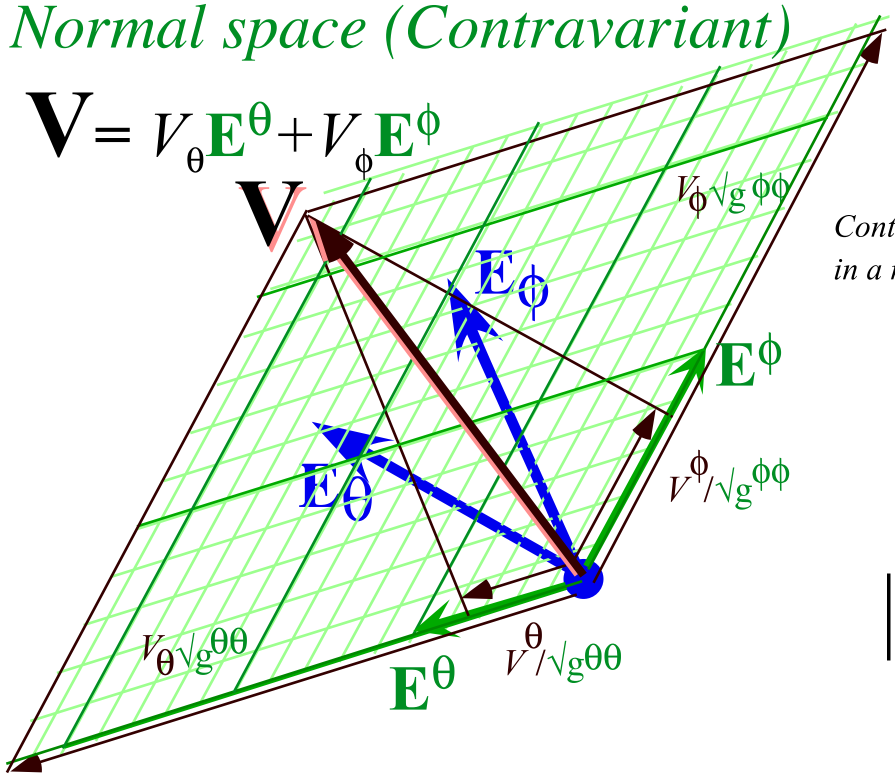

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

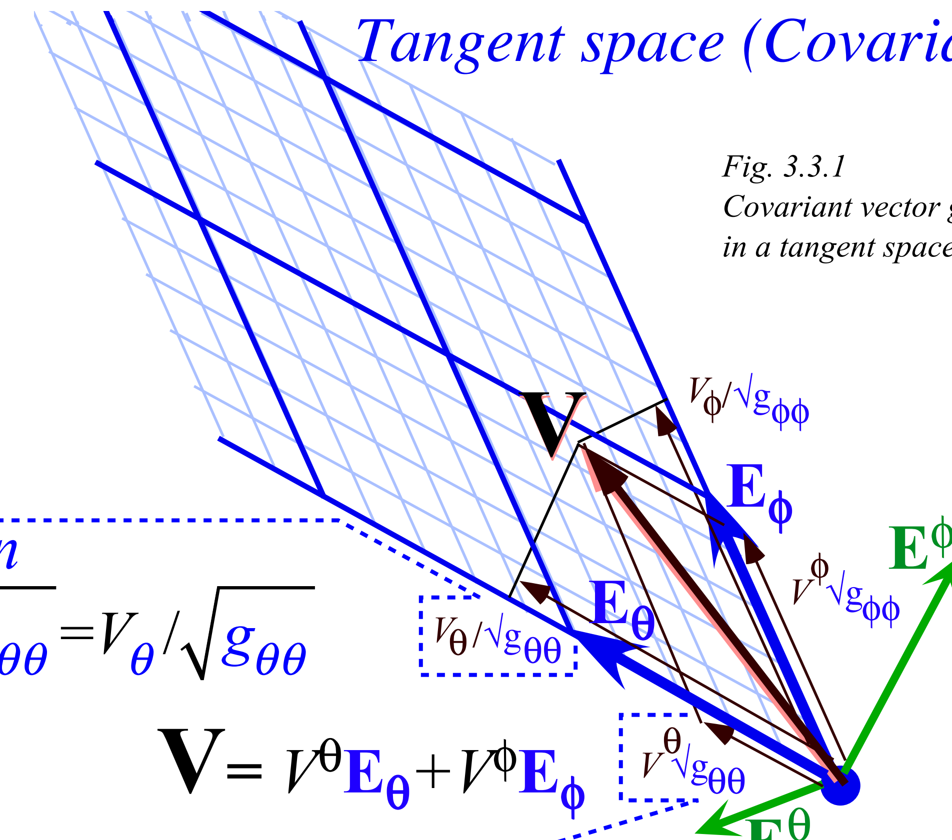

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\mathbf{\hat{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

covariant parallelogram side

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$$\mathbf{E}^m = (\mathbf{E}^m)^n\mathbf{E}_n \text{ implies: } (\mathbf{E}^m)^n = \mathbf{E}^m\bullet\mathbf{E}^n = g^{mn}$$

so: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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Normal space (Contravariant)

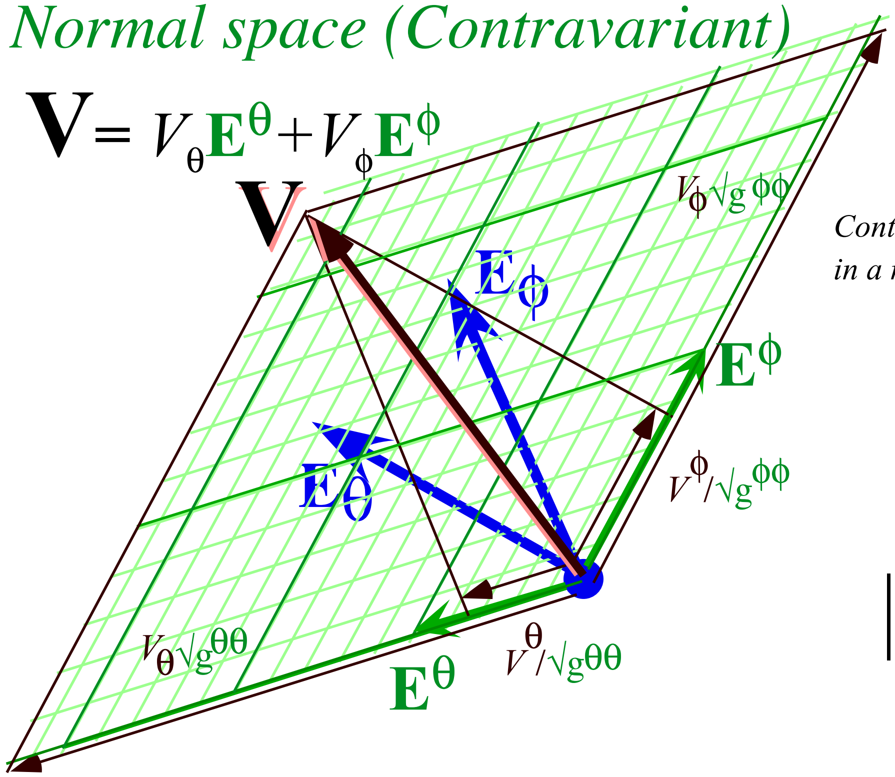

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

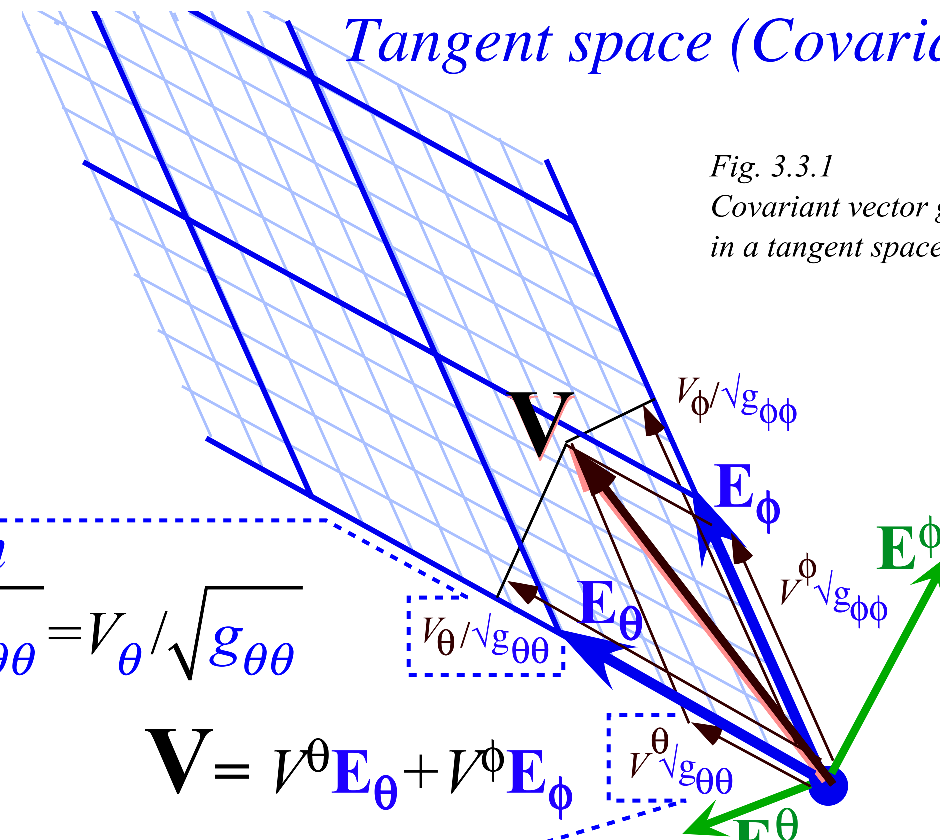

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

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$\mathbf{E}^m = (\mathbf{E}^m)^n\mathbf{E}_n$ implies: $(\mathbf{E}^m)^n = \mathbf{E}^m\bullet\mathbf{E}^n = g^{mn}$

so: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$

and: $\mathbf{E}_n = g_{mn}\,\mathbf{E}^m$ ...the same for covariant vectors

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
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Normal space (Contravariant)

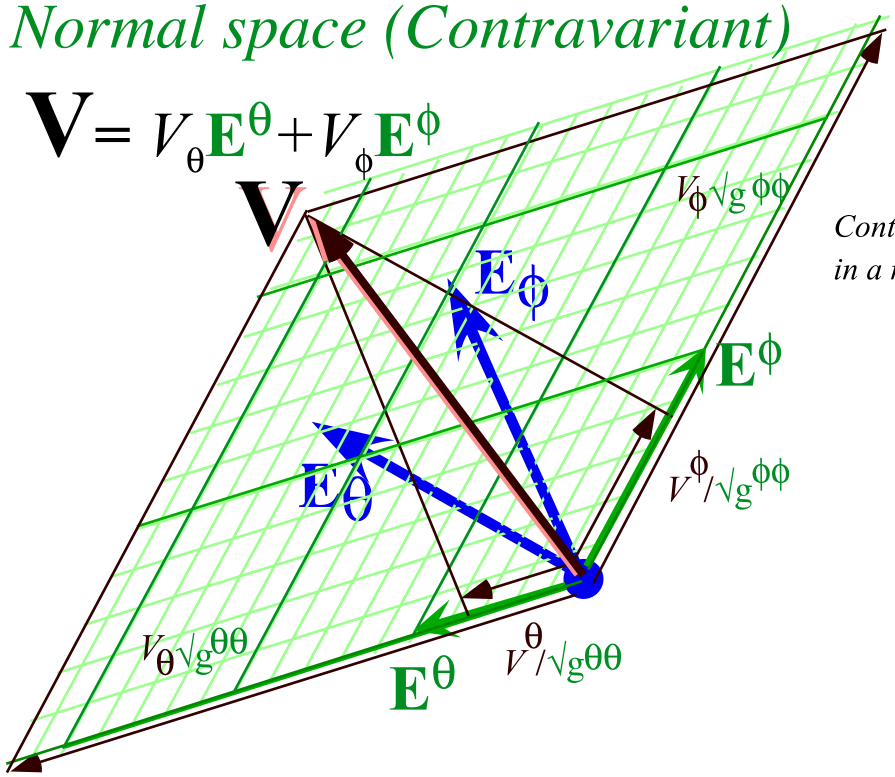

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

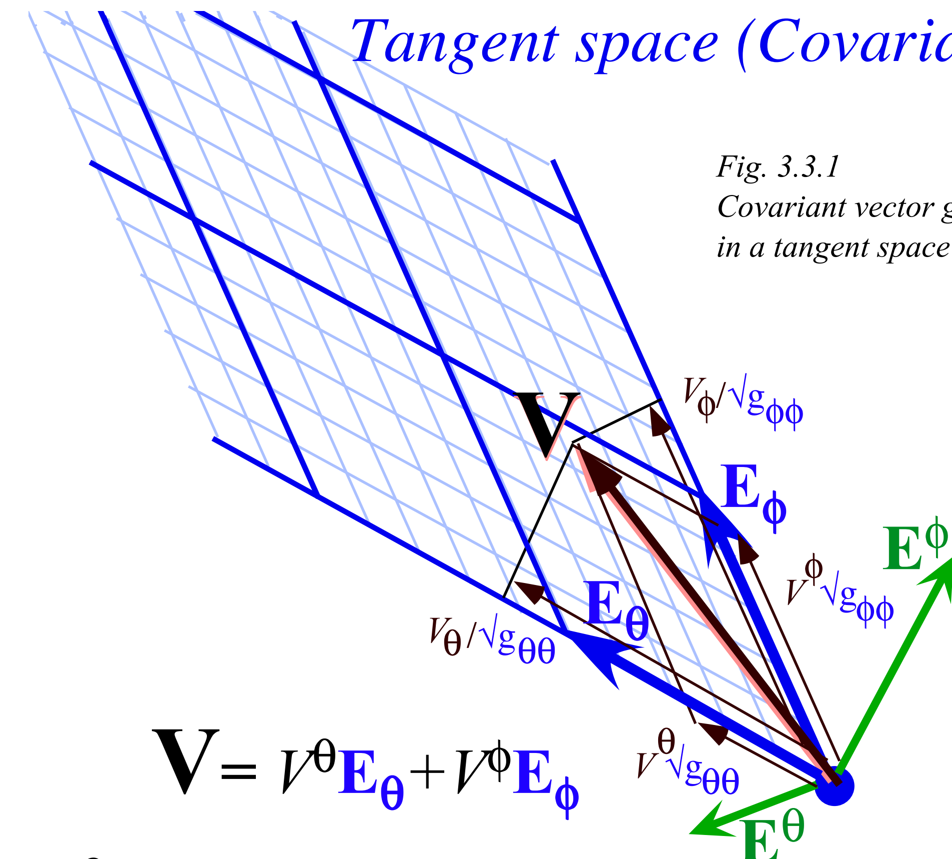

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E}^\mathbf{m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*....

$$\mathbf{E}^\mathbf{m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E}^\mathbf{m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

from p. 96 of Lect. 14

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

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Normal space (Contravariant)

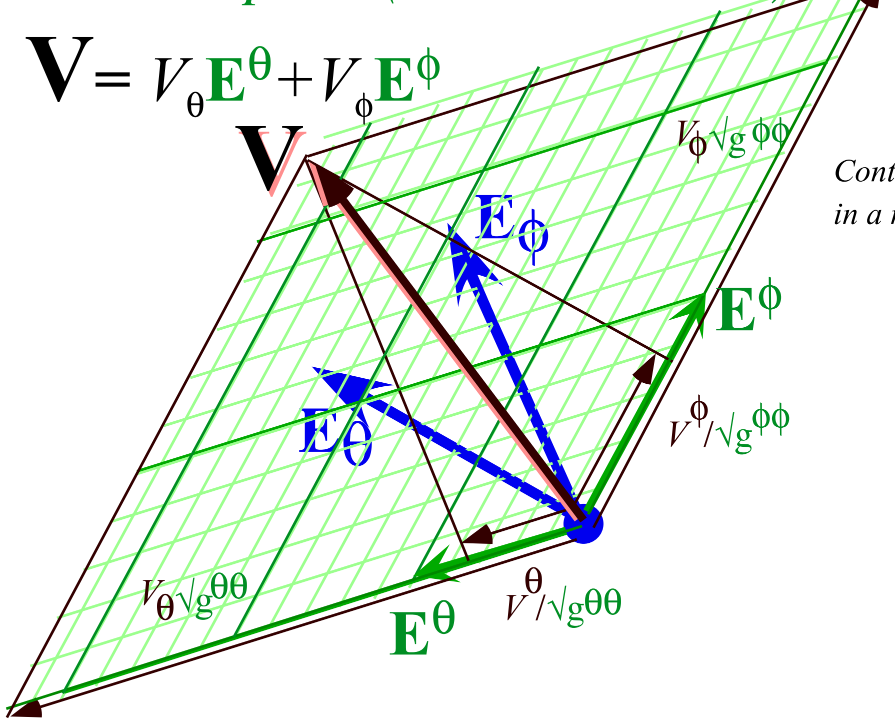
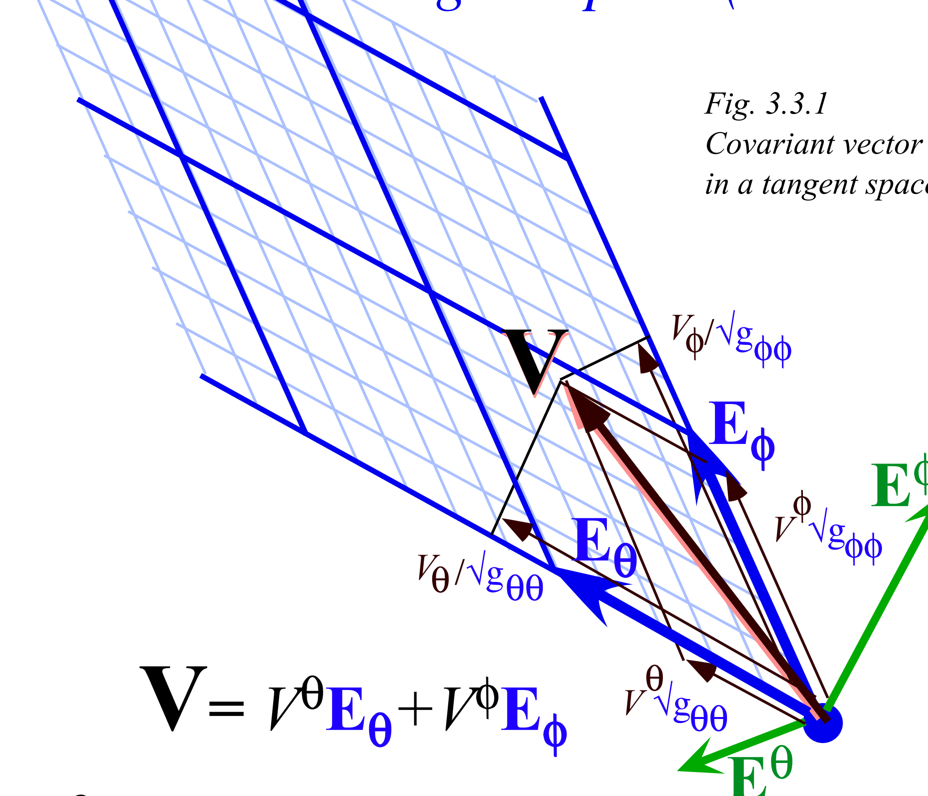


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)



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Contravariant vector $\mathbf{E}^\mathbf{m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*....

...and the same for covariant vectors $\mathbf{E}_\mathbf{m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E}^\mathbf{m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E}^\mathbf{m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

$$\mathbf{E}_\mathbf{m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E}_\mathbf{m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E_m} = U_n \mathbf{E^n} = \bar{U}^{\bar{m}} \mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}} \mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m \mathbf{E_m} = V_n \mathbf{E^n} = \bar{V}^{\bar{m}} \mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}} \mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$$U^m = \mathbf{U} \bullet \mathbf{E^m},\ V^m = \mathbf{V} \bullet \mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U} \bullet \mathbf{\bar{E}^{\bar{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V} \bullet \mathbf{\bar{E}^{\bar{m}}},\ , \qquad U_n = \mathbf{U} \bullet \mathbf{E_n},\ V_n = \mathbf{V} \bullet \mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U} \bullet \mathbf{\bar{E}_{\bar{n}}},\ etc.$$

Normal space (Contravariant)

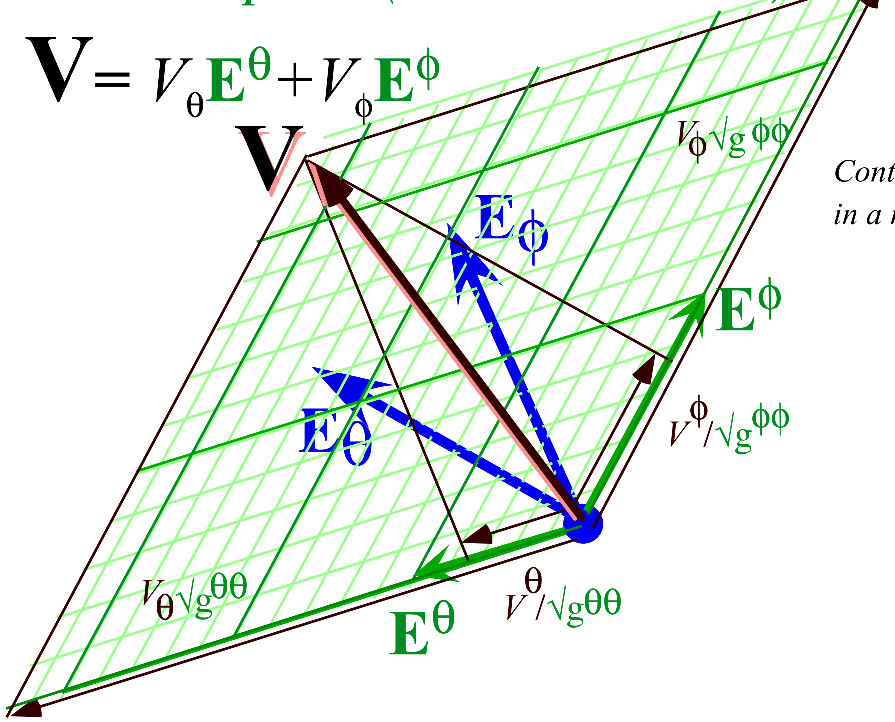
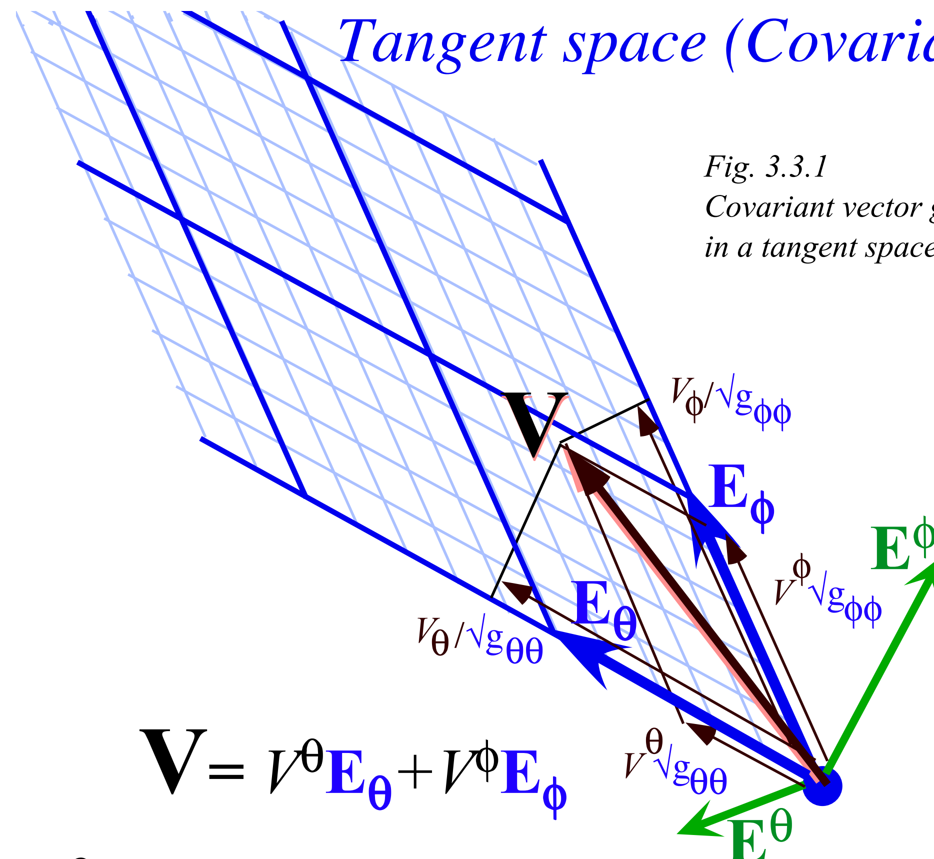


Fig. 3.3.2
Contravariant vector geometry in a normal space $(\mathbf{E}^\theta, \mathbf{E}^\phi)$.

Tangent space (Covariant)

Fig. 3.3.1
Covariant vector geometry in a tangent space $(\mathbf{E}_\theta, \mathbf{E}_\phi)$.



Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}},\ \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

implies: $V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}},\ \text{or:}\ \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

implies: $V_m = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{V}_{\bar{m}}$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\quad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

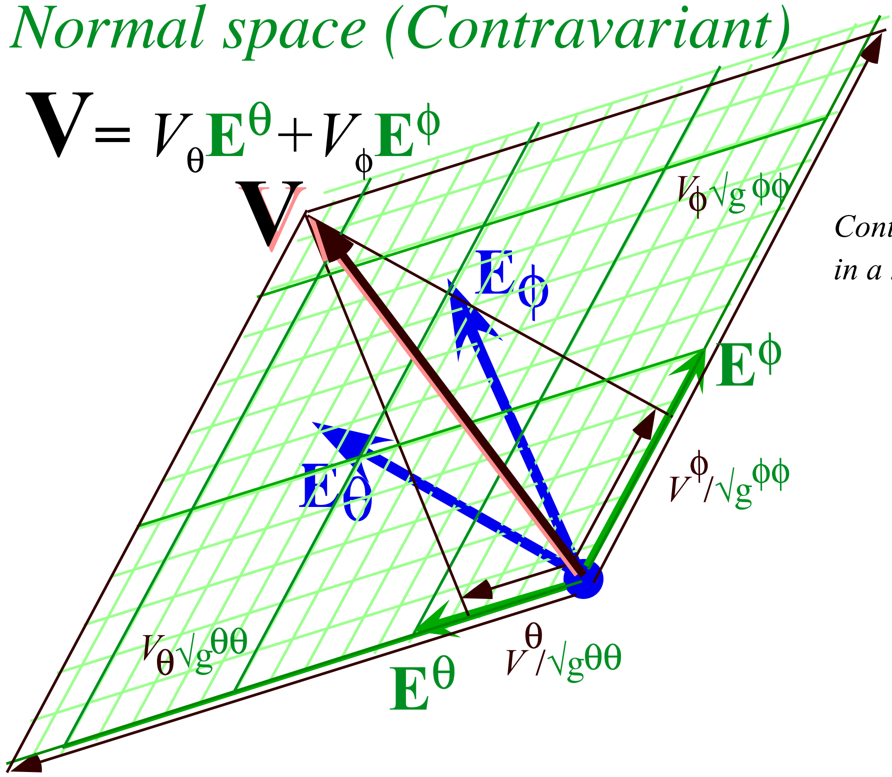


Fig. 3.3.2
Contravariant vector geometry in a normal space $(\mathbf{E}^\theta,\mathbf{E}^\phi)$.

Tangent space (Covariant)

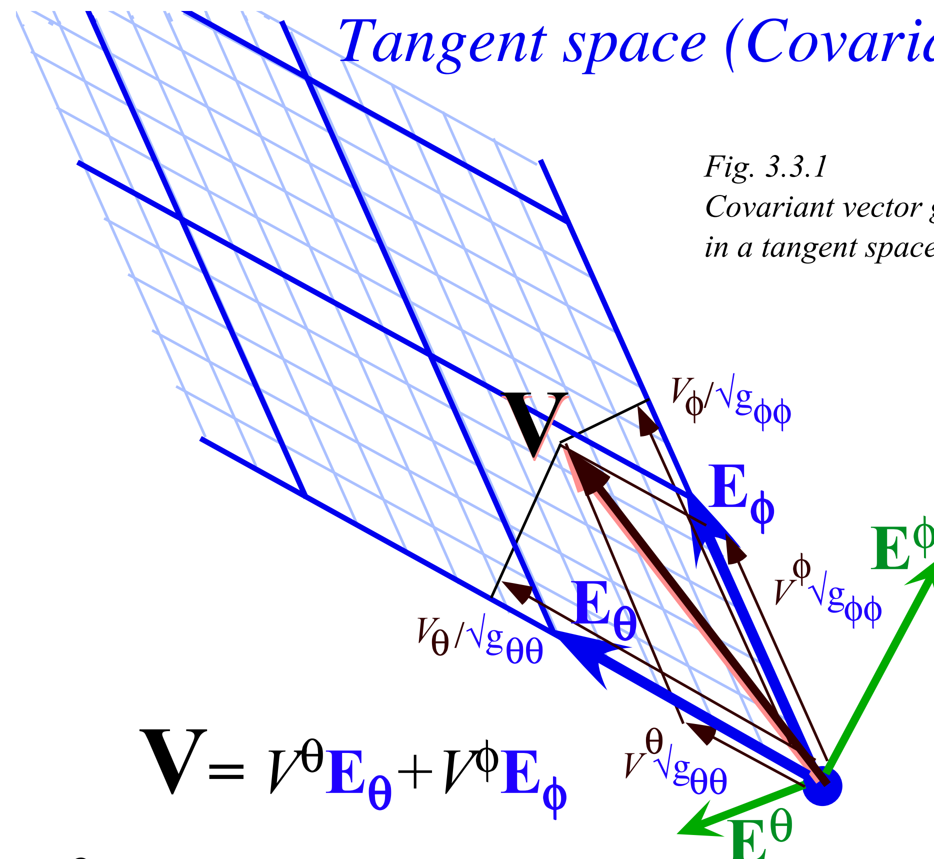


Fig. 3.3.1
Covariant vector geometry in a tangent space $(\mathbf{E}_\theta,\mathbf{E}_\phi)$.

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*….

…and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|$$

implies: $\langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U**,**V**,**...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

Normal space (Contravariant)

$\mathbf{V} = V_\theta \mathbf{E}^\theta + V_\phi \mathbf{E}^\phi$

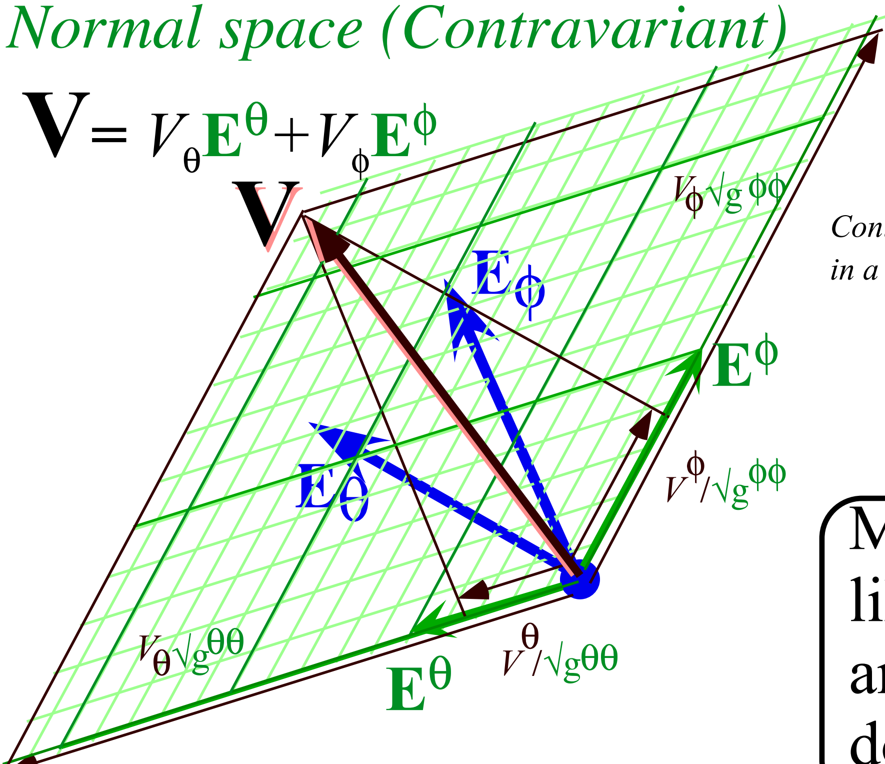
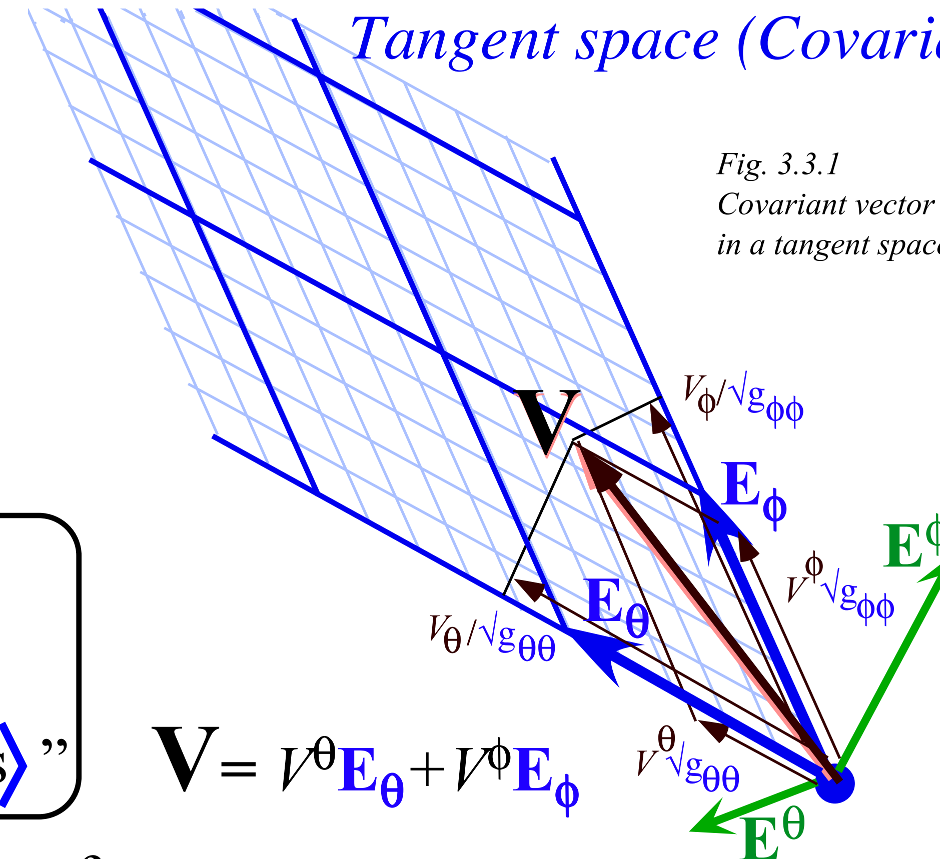


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)



$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Metric relations
like: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$
and: $\mathbf{E}_n = g_{mn}\,\mathbf{E}^m$
don't exist for "$\langle$bra-kets$\rangle$"

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ *(not drawn here)* using a *"chain-saw-sum rule"*....

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Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}| \quad \text{implies: } \langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

→ *Metric g_{mn} tensor geometric relations to length, area, and volume*

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

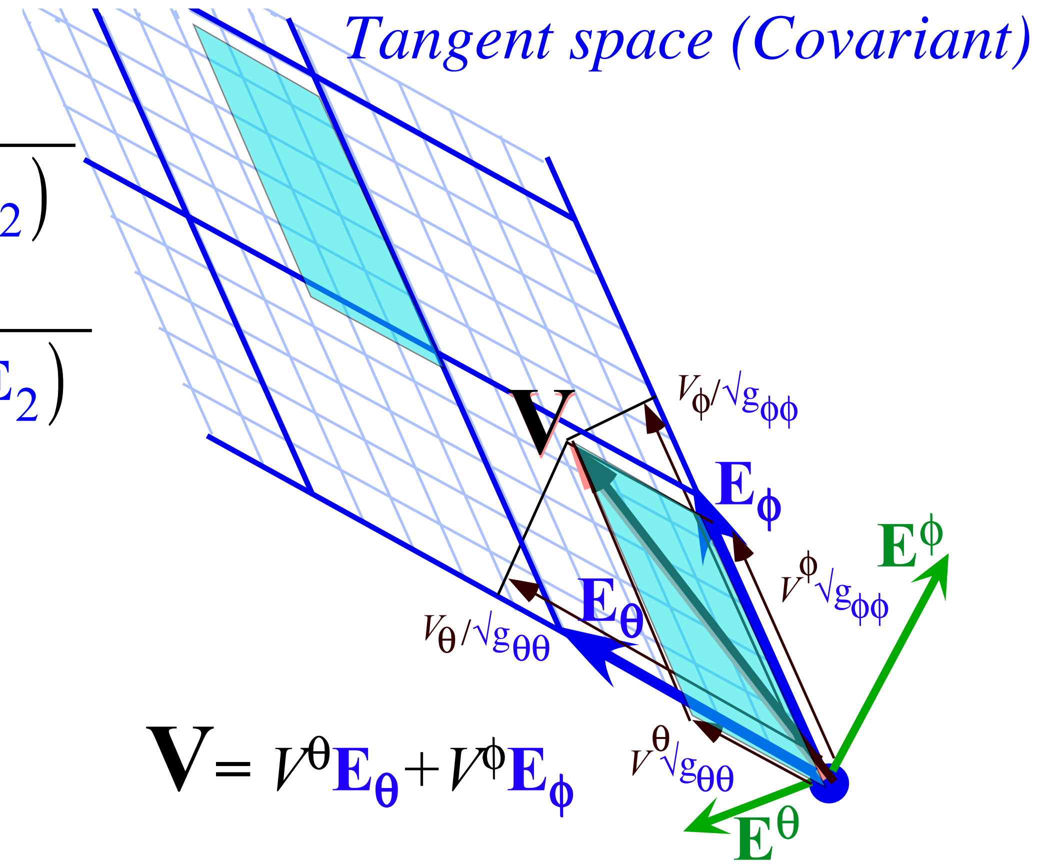
Riemann equation force analysis

2nd-guessing Riemann equation?

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

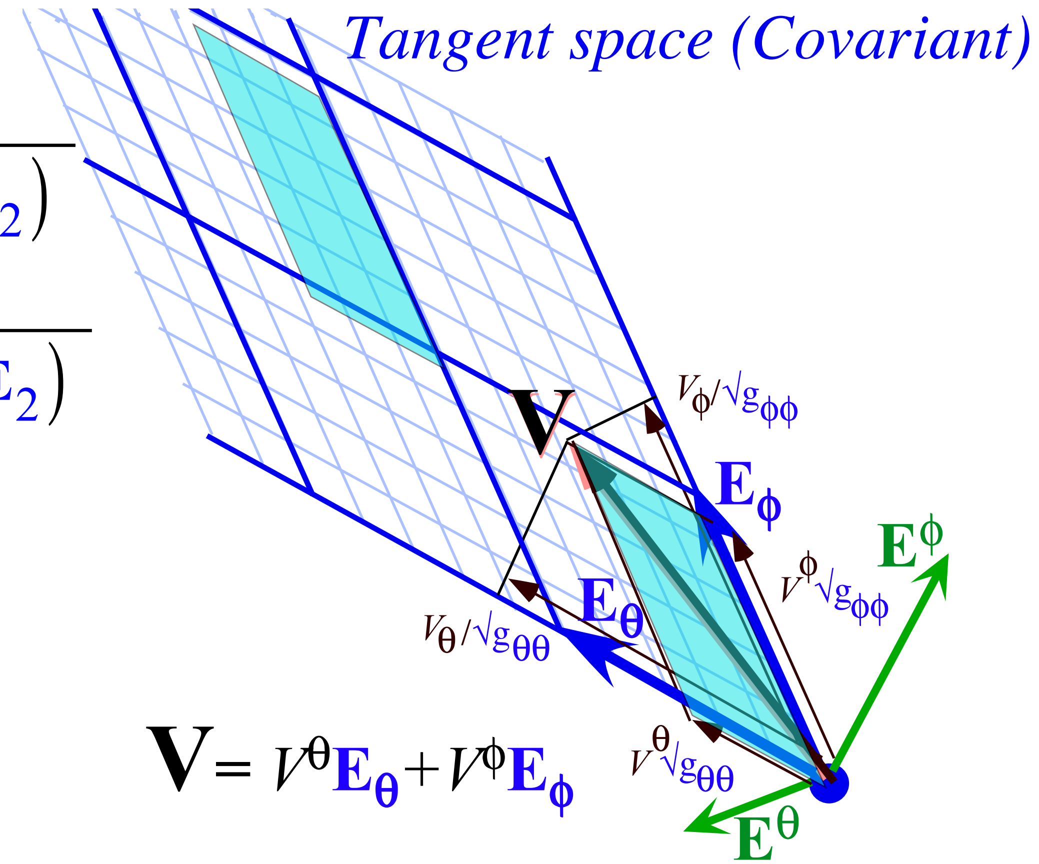
$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$



Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$



$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

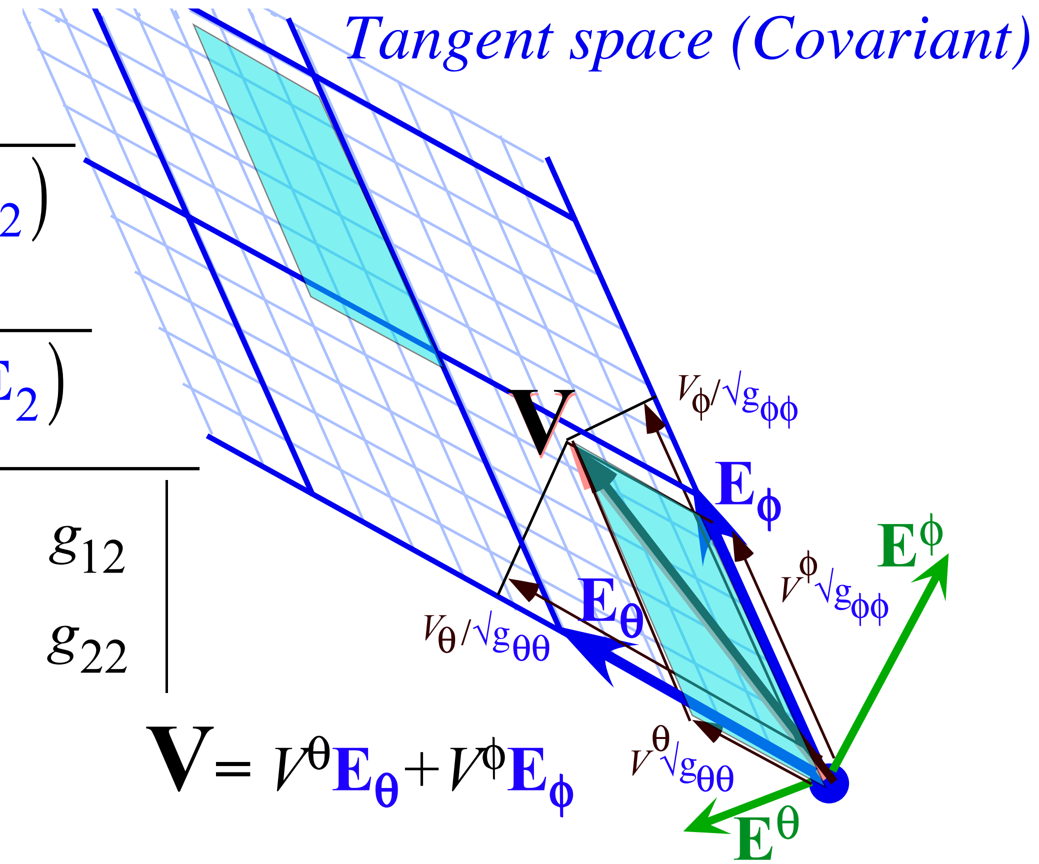
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2\right)=V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right|=V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2\right)=V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

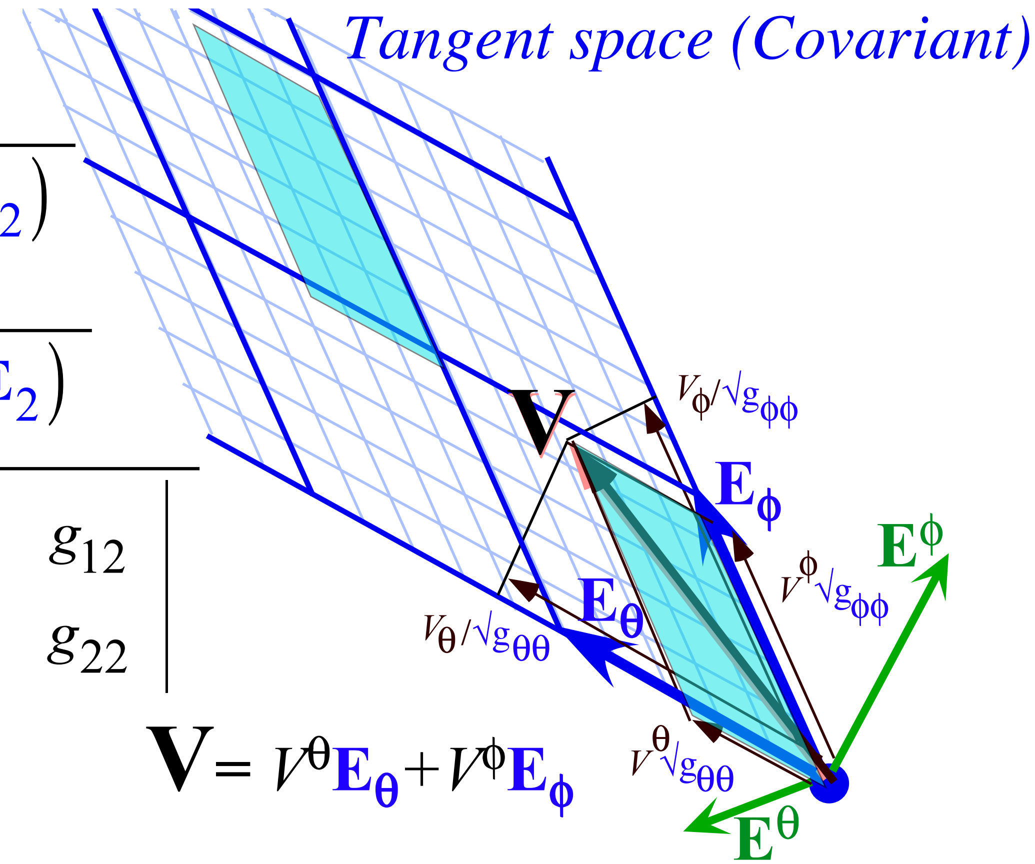
$$=V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}}=V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12}=\mathbf{E}_1\bullet\mathbf{E}_2=g_{21}$



Metric $\mathbf{g}_{mn}$ or g^{mn} tensor geometric relations to length, <u>area</u>, and volume

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$



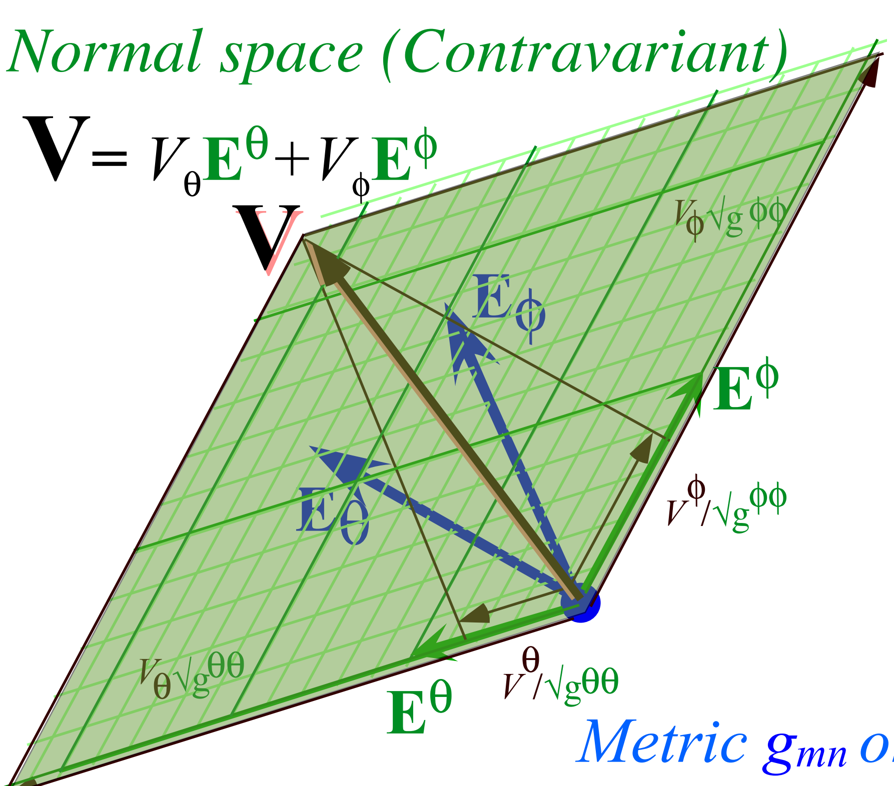
$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$

Normal space (Contravariant) area spanned by $V_1\mathbf{E}^1$ and $V_2\mathbf{E}^2$



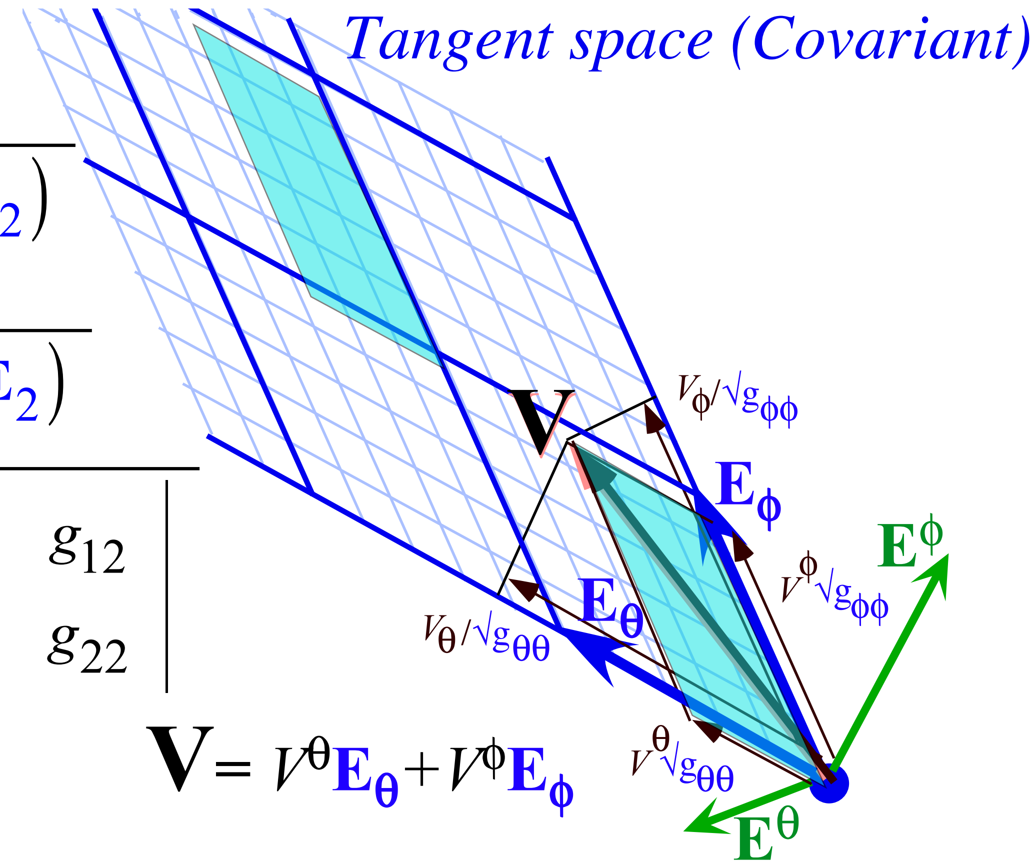
$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\left|\mathbf{E}^1\times\mathbf{E}^2\right| = V_1V_2\sqrt{\left(\mathbf{E}^1\times\mathbf{E}^2\right)\bullet\left(\mathbf{E}^1\times\mathbf{E}^2\right)}$$

$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\sqrt{\left(\mathbf{E}^1\bullet\mathbf{E}^1\right)\left(\mathbf{E}^2\bullet\mathbf{E}^2\right)-\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)}$$

$$= V_1V_2\sqrt{g^{11}g^{22}-g^{12}g^{21}} = V_1V_2\sqrt{\det\begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12} = \mathbf{E}^1\bullet\mathbf{E}^2 = g^{21}$



$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

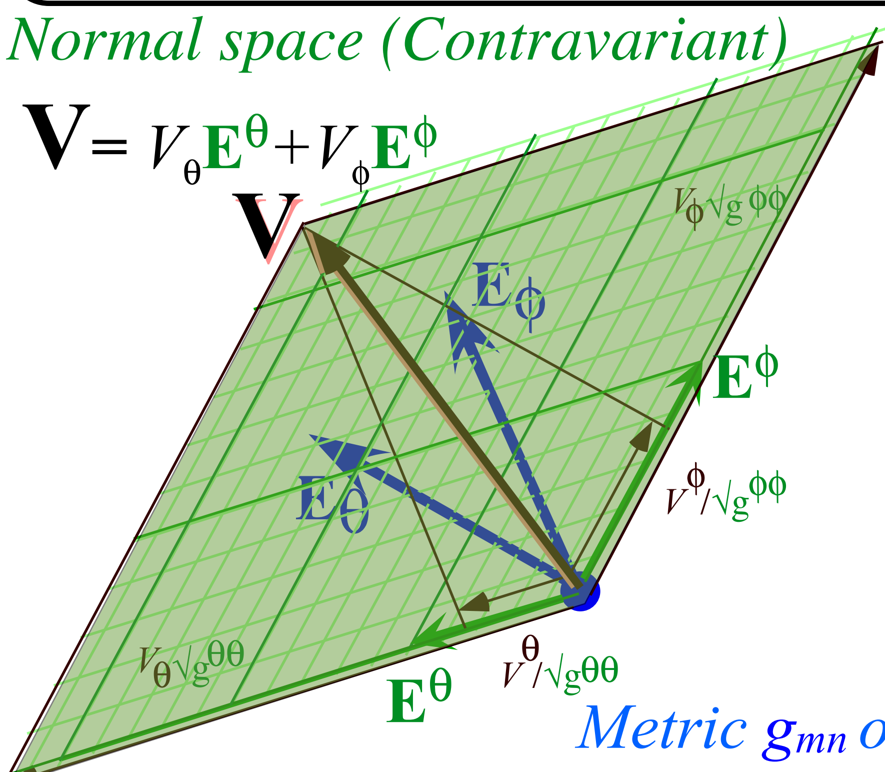
$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Determinant product rule: $det|\mathbf{g}_{cov}|\cdot det|\mathbf{g}^{cont}| = 1$ since $(\mathbf{g}_{cov})^{-1} = \mathbf{g}^{cont}$ or : $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}\cdot\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$

(Recall: $\mathbf{E}_n\bullet\mathbf{E}^n = 1$) $(\mathbf{g}_{cov}) \cdot (\mathbf{g}^{cont}) = (\mathbf{1})$



$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\left|\mathbf{E}^1\times\mathbf{E}^2\right| = V_1V_2\sqrt{\left(\mathbf{E}^1\times\mathbf{E}^2\right)\bullet\left(\mathbf{E}^1\times\mathbf{E}^2\right)}$$

$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\sqrt{\left(\mathbf{E}^1\bullet\mathbf{E}^1\right)\left(\mathbf{E}^2\bullet\mathbf{E}^2\right)-\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)}$$

$$= V_1V_2\sqrt{g^{11}g^{22}-g^{12}g^{21}} = V_1V_2\sqrt{\det\begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12} = \mathbf{E}^1\bullet\mathbf{E}^2 = g^{21}$

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and <u>volume</u>

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1\times\mathbf{E}_2\bullet\mathbf{E}_3\right| = V^1V^2V^3\det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J-matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T\bullet J$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J-matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Then determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives:

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3 \det\left|J\right| = V^1V^2V^3\sqrt{\det\left|\mathbf{g}_{cov}\right|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Contravariant Kajobian determinant K-rows are $\mathbf{E}^1$, $\mathbf{E}^2$ and $\mathbf{E}^3$.

$$Volume\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3\right) = V_1V_2V_3\left|\mathbf{E}^1 \times \mathbf{E}^2 \bullet \mathbf{E}^3\right| = V_1V_2V_3 \det\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{vmatrix}$$

Contravariant metric matrix is product of K-matrix and its transpose K^T

$$\mathbf{g}^{cont} \equiv \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^2}{\partial x^1} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial q^1}{\partial x^2} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^3}{\partial x^2} \\ \frac{\partial q^1}{\partial x^3} & \frac{\partial q^2}{\partial x^3} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} = K \bullet K^T$$

Then determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives:

$$Volume\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3\right) = V_1V_2V_3 \det\left|K\right| = V_1V_2V_3\sqrt{\det\left|\mathbf{g}^{cont}\right|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 82)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 74)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Canonical momentum and γ_{mn} tensor

Review of p_θ , p_ϕ vs γ_{mn} from p. 79 of Lect. 14

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem:

(matrix-proof on page 80 of Lect.14)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm}$$

QED

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Dot means *total* differentiation

<u>Everything that can move</u> contributes. (<u>Very</u> easy to miss a term!)

$$\dot{p}_{\theta} = \frac{d}{dt}p_{\theta} = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

p-dot part of Lagrange 2^{nd} equations

$$\dot{p}_{\phi} = \frac{d}{dt}p_{\phi} = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$
$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

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$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

From preceding Lagrange 1^{st} equations

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

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$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

p-dot part of Lagrange 2^{nd} equations

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$$p_{\theta} = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

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Lagrange equation force analysis

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$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)}$$

p-dot part of Lagrange 2nd equations

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

From preceding Lagrange 1st equations

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

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Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent {$\mathbf{E}_n$}space vs. Normal {$\mathbf{E}^m$}space

Covariant vs. contravariant coordinate transformations

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Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 82)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 74)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

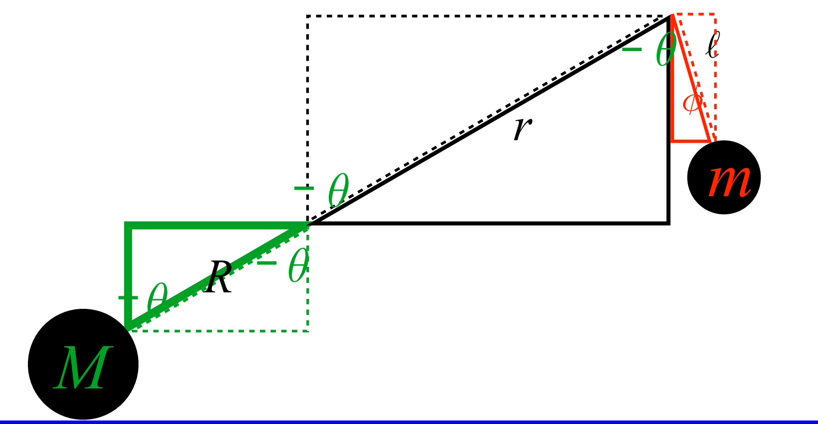
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Review of F_θ, F_ϕ vs F_x, F_y, F_X, F_Y from p. 74 of Lect. 14

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$
\begin{aligned}
dW = \;& F_X\,dX &=\;& M\ddot{X}\,dX &=\;& F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &=\;& M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi\\
&+F_Y\,dY && +M\ddot{Y}\,dY && +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi\\
&+F_x\,dx && +m\ddot{x}\,dx && +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi\\
&+F_y\,dy && +m\ddot{y}\,dy && +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}
$$



Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

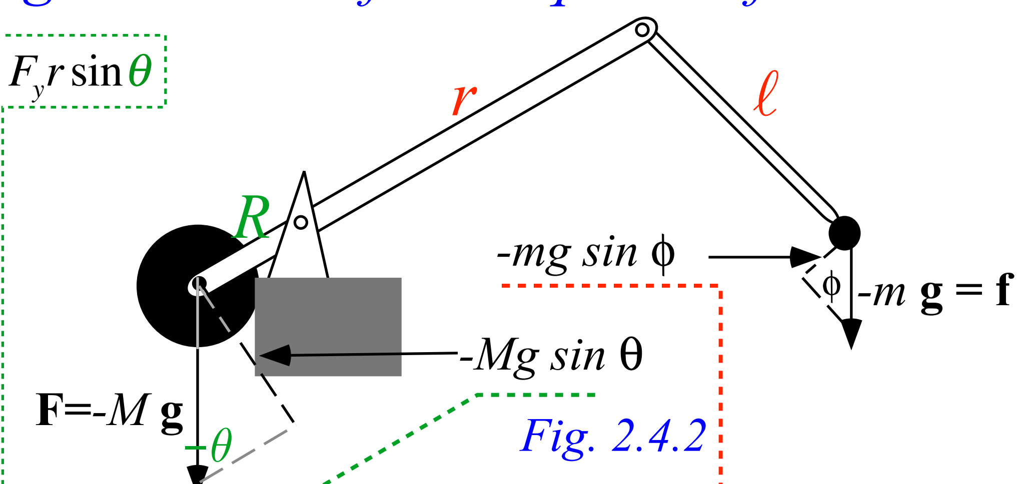


Fig. 2.4.2

Q: Are there
± sign errors here?
A: No. Beam in $-\theta$ position.

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... ... and a torque on throwing lever ℓ

Used on p.67-71

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt}p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$ [$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

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p-dot part of Lagrange 2nd equations

$$\dot{p}_\phi = \frac{d}{dt}p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

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The rest of Lagrange 2nd equations

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

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Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
- *Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
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Riemann equation force analysis

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Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi)} = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

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In matrix form:

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In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} *tensor :* $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}}-\frac{\partial T}{\partial q^{\mu}}=\dot{p}_{\mu}-\frac{\partial T}{\partial q^{\mu}}=F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta}=\boxed{\left(MR^2+mr^2\right)\ddot{\theta}-mr\ell\ddot{\phi}\cos(\theta-\phi)-mr\ell\dot{\phi}^2\sin(\theta-\phi)}=F_{\theta}=-MgR\sin\theta+mgr\sin\theta$$

$$\dot{p}_{\phi}=\boxed{m\ell^2\ddot{\phi}\quad-mr\ell\ddot{\theta}\cos(\theta-\phi)+mr\ell\dot{\theta}^2\sin(\theta-\phi)}=F_{\phi}=-mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix}\dot{p}_{\theta}\\ \dot{p}_{\phi}\end{pmatrix}=\begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}-\begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}=\begin{pmatrix}F_{\theta}\\ F_{\phi}\end{pmatrix}$$

This uses the γ_{mn} tensor :
$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}=\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$$

$$=\begin{pmatrix}-MgR\sin\theta+mgr\sin\theta\\ -mg\ell\sin\phi\end{pmatrix}$$

$$\begin{pmatrix}\dot{p}_{\theta}\\ \dot{p}_{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}F_{\theta}+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_{\phi}-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix… Let's consolidate …

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu}-\frac{\partial T}{\partial q^\mu}=\dot{p}_\mu-\frac{\partial T}{\partial q^\mu}=F_\mu$$

$$\dot{p}_\theta=\boxed{\left(MR^2+mr^2\right)\ddot{\theta}-mr\ell\ddot{\phi}\cos(\theta-\phi)-mr\ell\dot{\phi}^2\sin(\theta-\phi)=F_\theta=-MgR\sin\theta+mgr\sin\theta}$$

$$\dot{p}_\phi=\boxed{m\ell^2\ddot{\phi}-mr\ell\ddot{\theta}\cos(\theta-\phi)+mr\ell\dot{\theta}^2\sin(\theta-\phi)=F_\phi=-mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}-\begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}=\begin{pmatrix}F_\theta\\ F_\phi\end{pmatrix}$$

$$=\begin{pmatrix}-MgR\sin\theta+mgr\sin\theta\\ -mg\ell\sin\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}=\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta \\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta} \\ \ddot{\phi}\end{pmatrix} - \begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix} = \begin{pmatrix}F_\theta \\ F_\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix} = \begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$= \begin{pmatrix}-MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi\end{pmatrix}$$

$$\begin{pmatrix}\dot{p}_\theta \\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta} \\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1} = \frac{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu$ ...

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} - \begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix} = \begin{pmatrix}F_\theta\\ F_\phi\end{pmatrix} = \begin{pmatrix}-MgR\sin\theta + mgr\sin\theta\\ -mg\ell\sin\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix} = \begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1} = \frac{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

... and apply it...

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Riemann equation form

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu$ ...

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} - \begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix} = \begin{pmatrix}F_\theta\\ F_\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix} = \begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$= \begin{pmatrix}-MgR\sin\theta + mgr\sin\theta\\ -mg\ell\sin\phi\end{pmatrix}$$

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1} = \frac{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$ $\quad I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)$

First trebuchet (~3000BCE in China) was Gravity-free...
.. powered by many Chinese warriors!

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu$ ...

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_S \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

→ *Riemann equation force analysis*

2nd-guessing Riemann equation?

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$ *becomes* $\gamma^{\mu\nu}\dot{p}_{\mu} = \ddot{q}^{\nu}\ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_{\theta} + mr\ell\dot{\phi}^{2}\sin(\theta-\phi) \\ F_{\phi} - mr\ell\dot{\theta}^{2}\sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$F_{\theta}=0=F_{\phi}$

$$I_{s}=m\ell^{2}\left[MR^{2}+mr^{2}\sin^{2}(\theta-\phi)\right]$$

("Super-Inertia")

$$I_{S}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_{S}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^{2} \\ -\dot{\theta}^{2} \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^{2} & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^{2}+mr^{2} \end{pmatrix} \begin{pmatrix} \dot{\phi}^{2} \\ -\dot{\theta}^{2} \end{pmatrix} mr\ell\sin(\theta-\phi)$$

$Let: (\theta-\phi) = -\frac{\pi}{2}$ so: $I_{S} = m\ell^{2}\left[MR^{2}+mr^{2}\right]$ and let: $\omega \equiv \dot{\theta} = \dot{\phi}$

$$I_{S}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_{S}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^{2} \\ \dot{\theta}^{2} \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^{2} & 0 \\ 0 & MR^{2}+mr^{2} \end{pmatrix} \begin{pmatrix} -\omega^{2} \\ \omega^{2} \end{pmatrix} mr\ell$$

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^{2} \\ \dot{\theta}^{2} \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^{2} & 0 \\ 0 & MR^{2}+mr^{2} \end{pmatrix}}{m\ell^{2}\left[MR^{2}+mr^{2}\right]} \begin{pmatrix} -mr\ell\omega^{2} \\ mr\ell\omega^{2} \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^{2}}{MR^{2}+mr^{2}} \\ \omega^{2}r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results

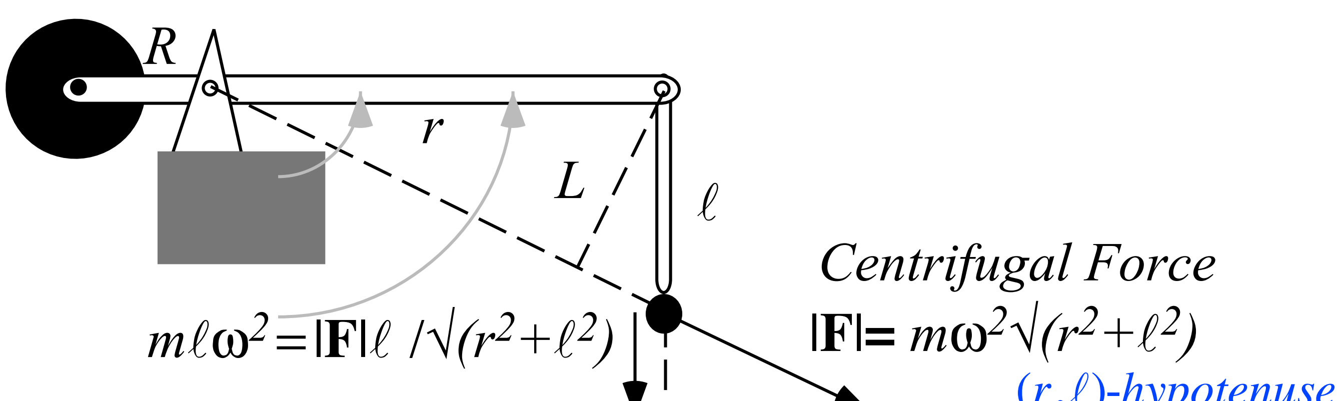

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi=0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$ becomes $\gamma^{\mu\nu}\dot{p}_{\mu} = \ddot{q}^{\nu}\ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_{\theta} + mr\ell\dot{\phi}^2 \sin(\theta-\phi) \\ F_{\phi} - mr\ell\dot{\theta}^2 \sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$$I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right] \quad \text{(“Super-Inertia”)}$$

$F_{\theta}=0=F_{\phi}$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

$$Let:(\theta-\phi) = -\frac{\pi}{2} \qquad \text{so:} \qquad I_S = m\ell^2\left[MR^2+mr^2\right] \qquad \text{and let: } \omega \equiv \dot{\theta} = \dot{\phi}$$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix}\begin{pmatrix} -\omega^2 \\ \omega^2 \end{pmatrix} mr\ell$$

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Trying to 2nd-guess Riemann results

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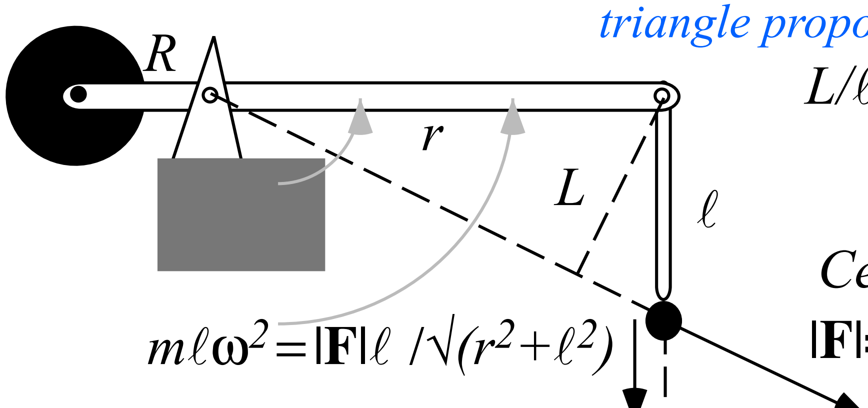

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(r,ℓ)-hypotenuse

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi=0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu}-\frac{\partial T}{\partial q^\mu}=\dot{p}_\mu-\frac{\partial T}{\partial q^\mu}=F_\mu$ *becomes* $\gamma^{\mu\nu}\dot{p}_\mu=\ddot{q}^\nu \ldots$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Riemann equation form

Gravity-free case:

$I_s=m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]$ *("Super-Inertia")*

$F_\theta=0=F_\phi$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)=\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)$$

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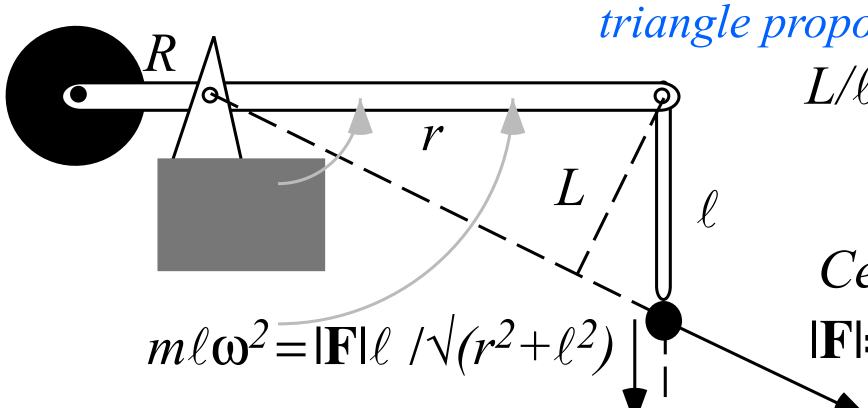

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Trying to 2nd-guess Riemann results (Gravity-free case)

Move to top of page...

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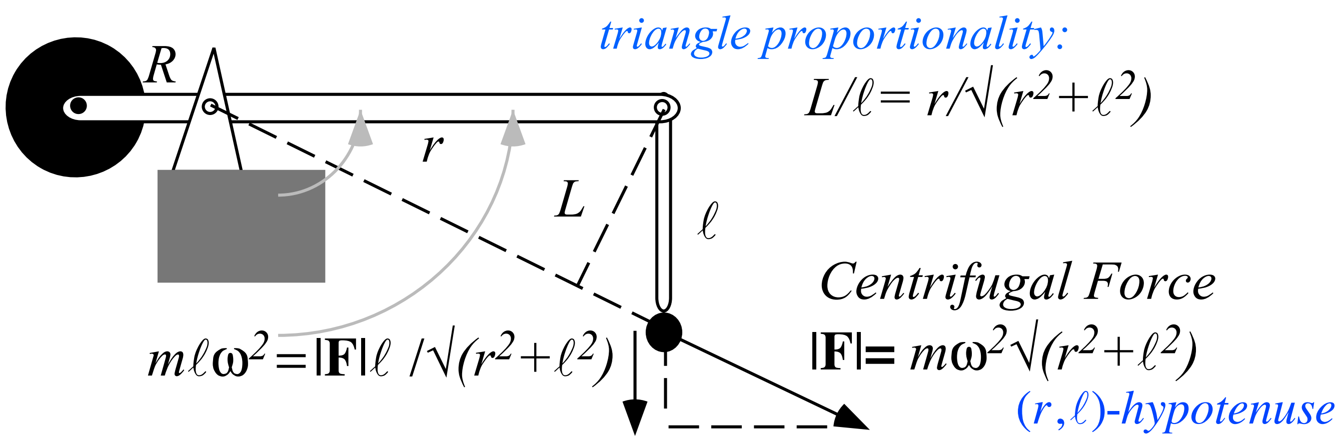

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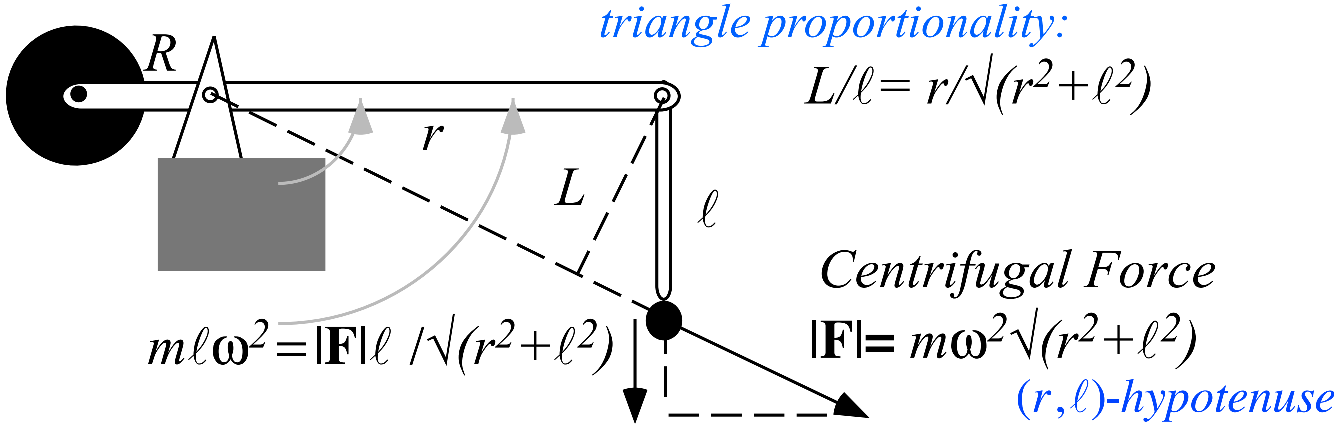

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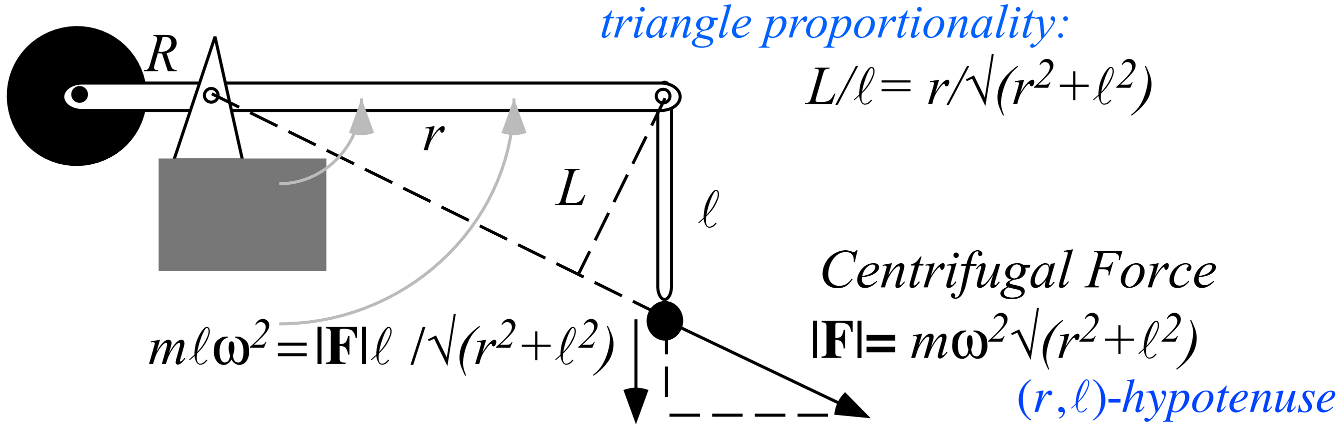

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Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent {$\mathbf{E}_n$}space vs. Normal {$\mathbf{E}^m$}space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

→ *2nd-guessing Riemann equation?*

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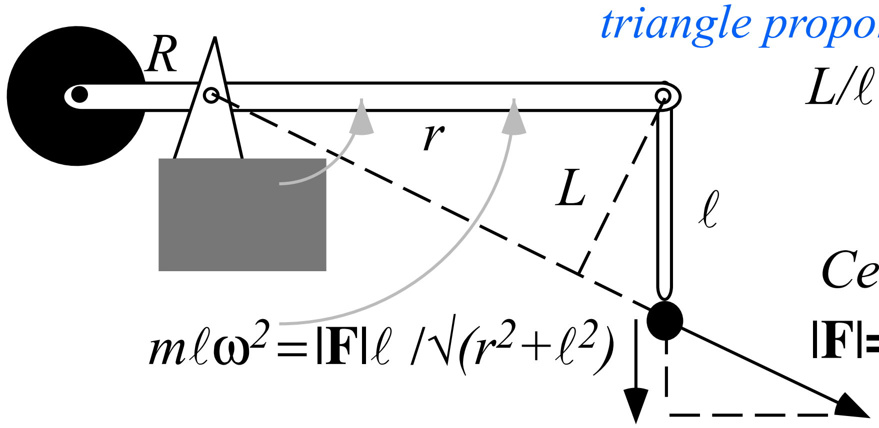

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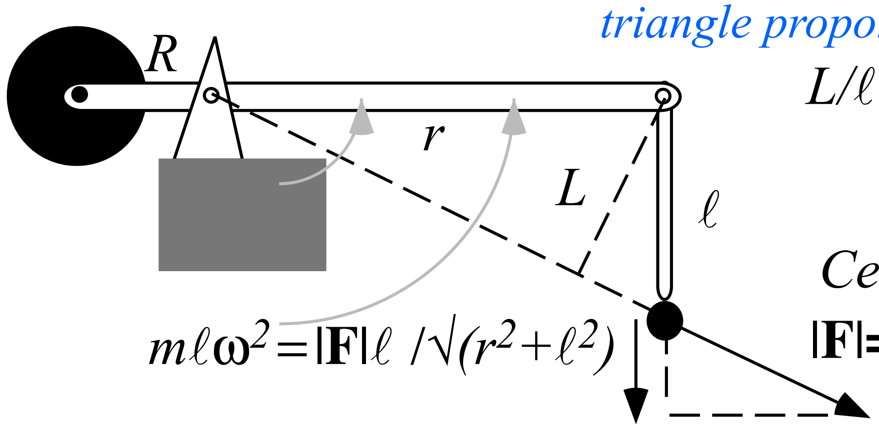

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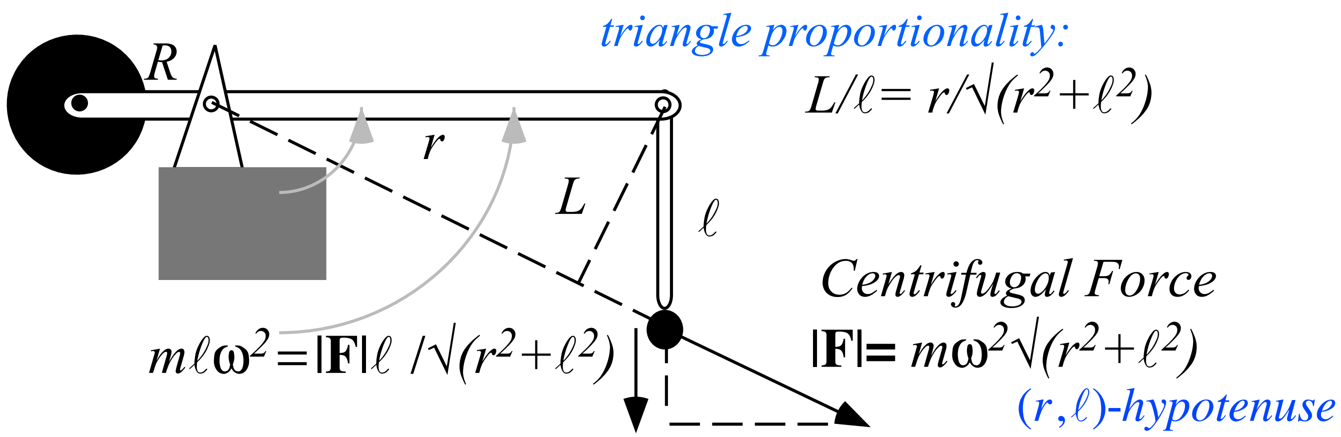

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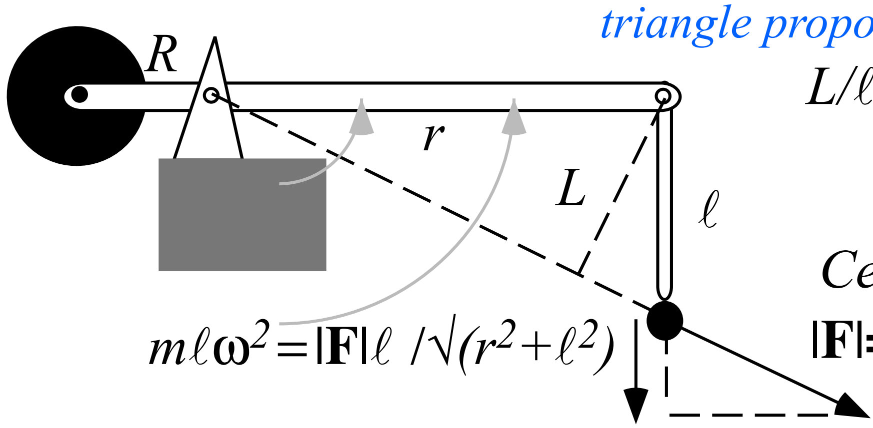

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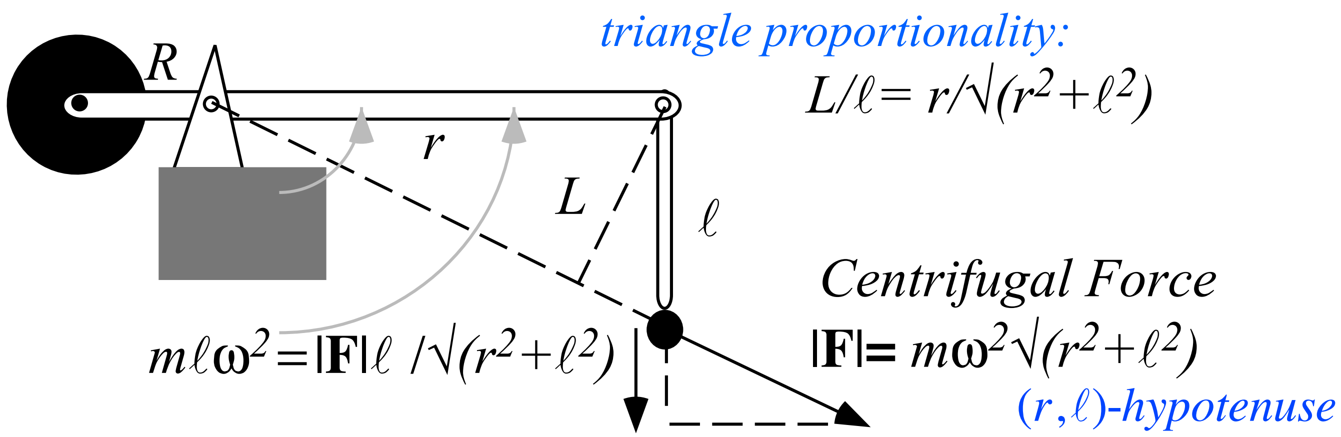

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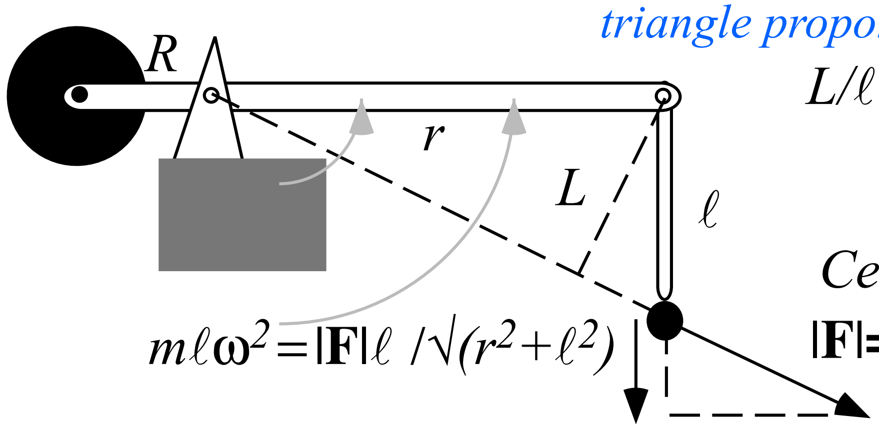

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Checks with $\ddot{\theta}$ Riemann equation

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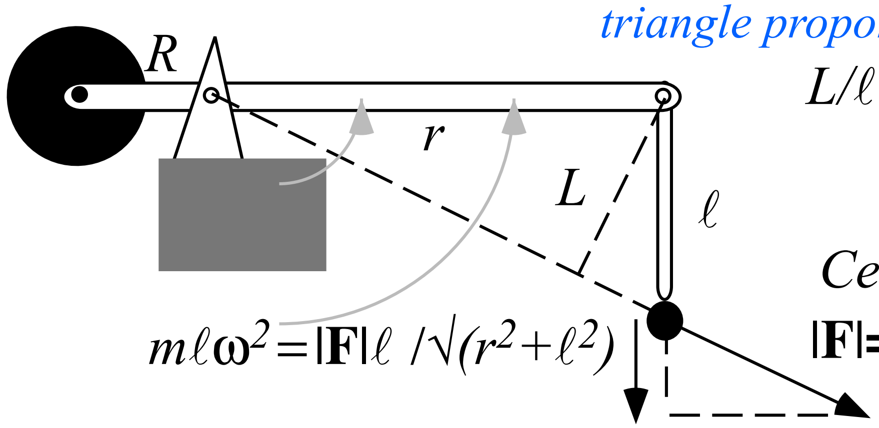

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Checks with $\ddot{\theta}$ Riemann equation

Note the time derivative of total momentum is zero if outside torques are zero.(twirling skater analogy)

$$\dot{p}_\theta + \dot{p}_\phi = 0,\ \text{if}\ F_\theta = 0 = F_\phi$$

End of Lect. 15 *(Finally!)*