Lecture 14
Wed. 10.16.2019

*treb-yew-shay

Introducing GCC Lagrangian `a la Trebuchet Dynamics*

Ch. 1-3 of Unit 2 and Unit 3 (Mostly Unit 2.)

*The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. **273**, 66 (July 1995))*
- *The medieval ingenium (9th to 14th century) and modern re-enactments*
- *Human kinesthetics and sports kinesiology*

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)
- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}*
- *Structure of dynamic metric tensor γ_{mn}*
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and γ_{mn} tensor*

Summary of Lagrange equations and force analysis (Mostly from Unit 2.)
- *Forces: total, genuine, potential, and/or fictitious*

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
- *Multivalued functionality and connections*
- *Covariant and contravariant relations*
  - *Tangent space vs. Normal space*
- *Metric g_{mn} tensor geometric relations to length, area, and volume*

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #14

In reverse order

Trebuchet **Web Animations:**
Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Excerpts (Page 44-47 in *Preliminary Draft*) from the
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019

Select, exciting, and related Research & Articles of Interest:
*(Many of these may be a bit beyond this course, but are included to lend added insight)***:**

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces - Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*
Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,

RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

An assist from *Physics Girl* (YouTube Channel):

Posted this year:
How to Make VORTEX RINGS in a Pool

Crazy pool vortex (new inclusion with more background)
Crazy pool vortex - pg-yt-2014

Posting with the best visuals:
Fun with Vortex Rings in the Pool - pg-yt-2014

She covers it beautifully!

An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019

Rovibrational_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74*, *Page=75*, *Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20

WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)

CoulIt Web Simulation of the Volcanoes of Io

BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
- *Generic/Default*
- *Most Basic A-Type*
- *Basic A-Type w/reference lines*
- *Basic A-Type A-Type with Potential energy*
- *A-Type with Potential energy and Stokes Plot*
- *A-Type w/3 time rates of change*
- *A-Type w/3 time rates of change with Stokes Plot*
- *B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)*

RelaWavity Web Elliptical Motion Simulations:
- *Orbits with b/a=0.125*
- *Orbits with b/a=0.5*
- *Orbits with b/a=0.7*
- *Exegesis with b/a=0.125*
- *Exegesis with b/a=0.5*
- *Exegesis with b/a=0.7*
- *Contact Ellipsometry*

***CoulIt* Web Simulations:**
- *Basic/Generic*
- *Exploding Starlet*
- *Volcanoes of Io (Color Quantized)*

***JerkIt* Web Simulations:**
- *Basic/Generic*
- *Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot*

***OscillatorPE* Web Simulation:**
- *Coulomb-Newton-Inverse_Square*,
- *Hooke-Isotropic Harmonic*,
- *Pendulum-Circular_Constraint*

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:

Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019

A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018

*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,
Nuclear_Collisions - Mishustin-PhysRevC-2007, *APS Link & Abstract*
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot *V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *$(v_1, v_2)=(1, 0)$: V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

Chapter 1. The Trebuchet: A dream problem for Galileo?

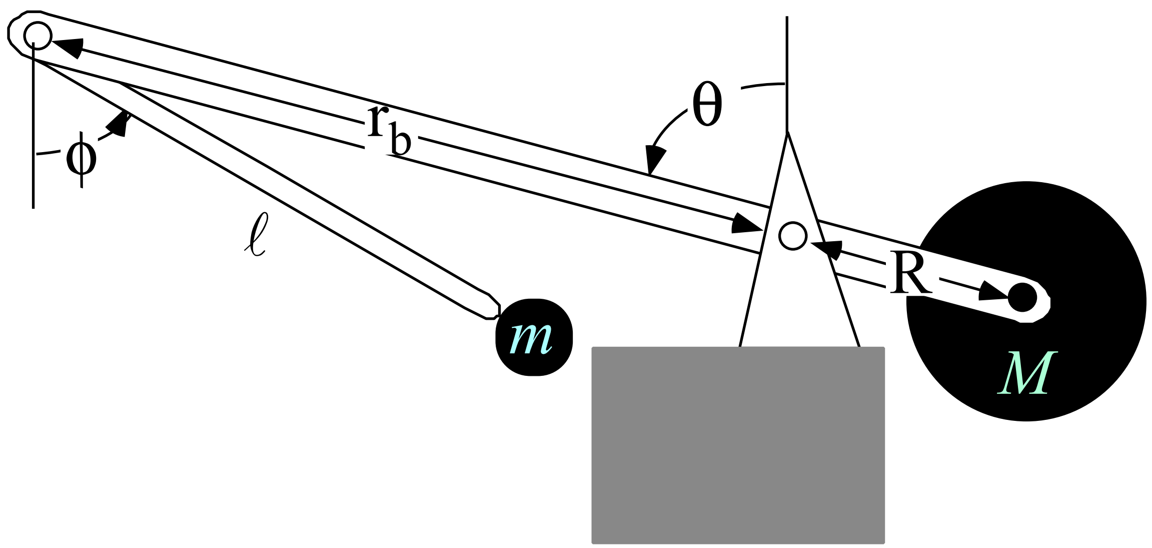
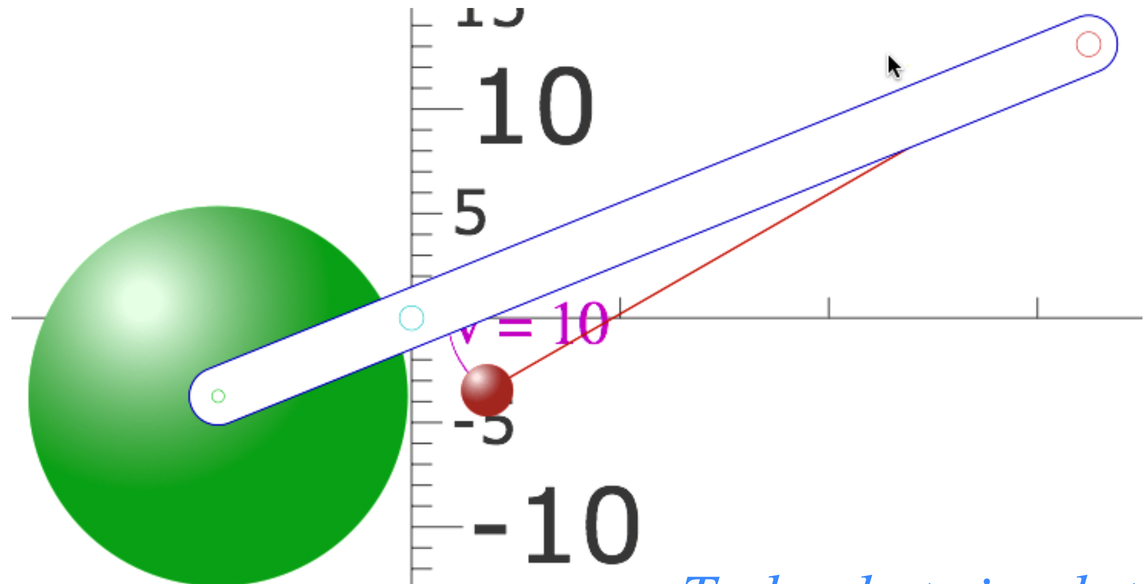


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html

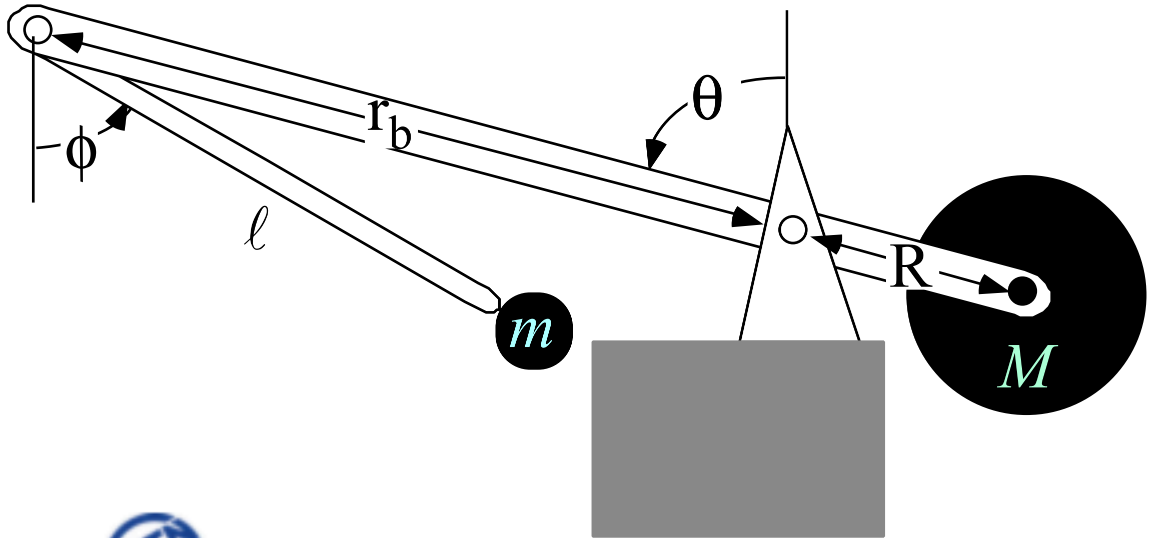
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth

file:///Users/williamharter/Documents/CMwBang%202019/modphys/markup/TrebuchetWeb.html

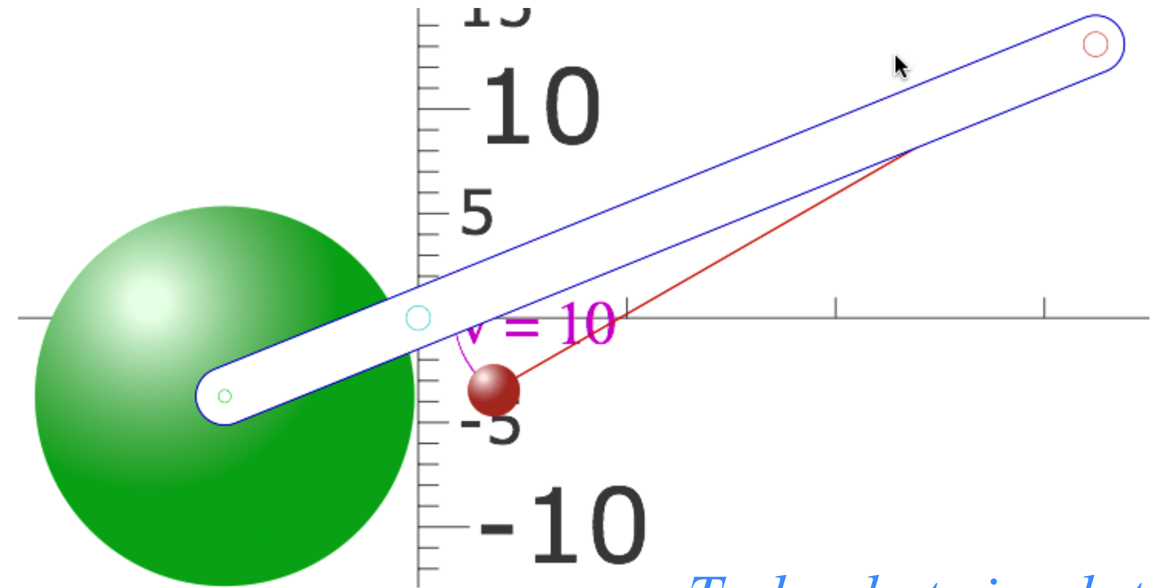
file:///Users/williamharter/Documents/CMwBang 2019/modphys/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth

file:///Users/williamharter/Documents/CMwBang 2019/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge

Chapter 1. The Trebuchet: A dream problem for Galileo?



PBS PBS.org



Trebuchet simulator

https://www.pbs.org/wgbh/nova/lostempires/trebuchet/builds.html

The plan: NOVA and a team of master builders from England, Germany, France and the United States will reconstruct one of the most destructive of medieval weapons ever made: a giant trebuchet. They will raise the weapon in the shadows of Castle Urquhart, located on the shores of Loch Ness in the Scottish Highlands.

Siege of Kenilworth 1264-1267

***Trebuchet* Web Animations:**
Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth, "Flinger",
Position Space (Course), Position Space (Fine)

Trebuchet Web App with Passed Parameters,
use a URL with the string after the equal sign replaced with the desired scenario.

PlotPosSpaceCourse
PlotPosSpaceFine
AnimateFlinger
AnimateTrebuchet
MontezumasRevenge
SeigeOfKenilworth

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=XXX =>

Chapter 1. The Trebuchet: A dream problem for Galileo?

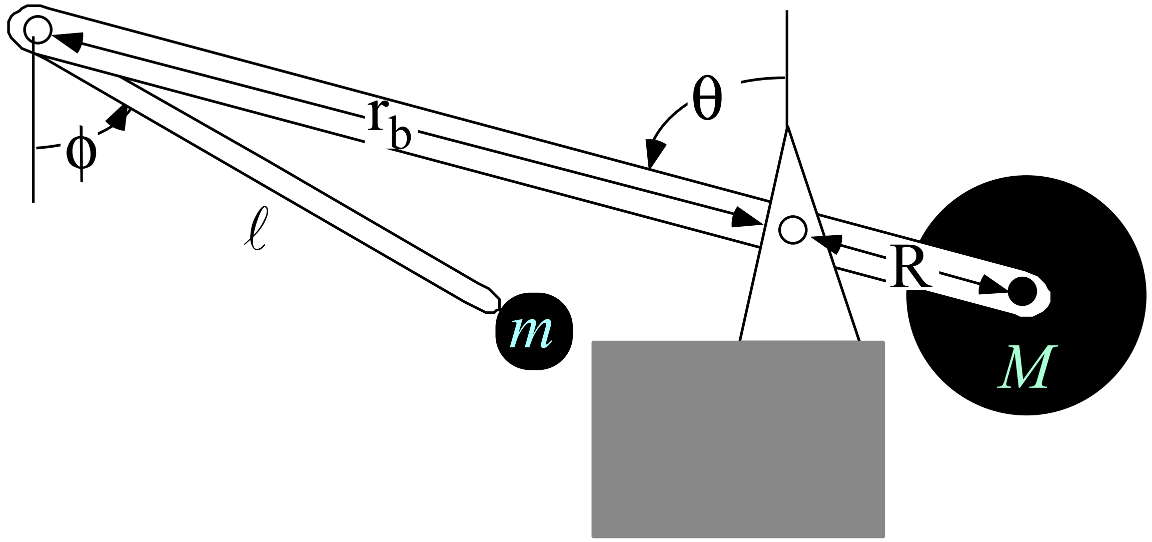
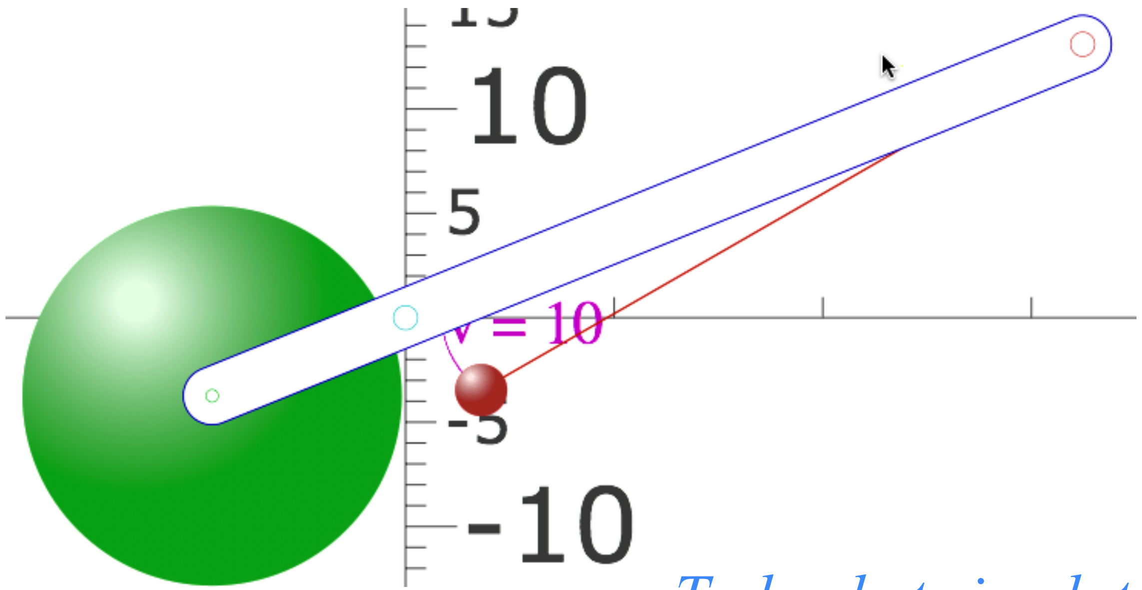


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html

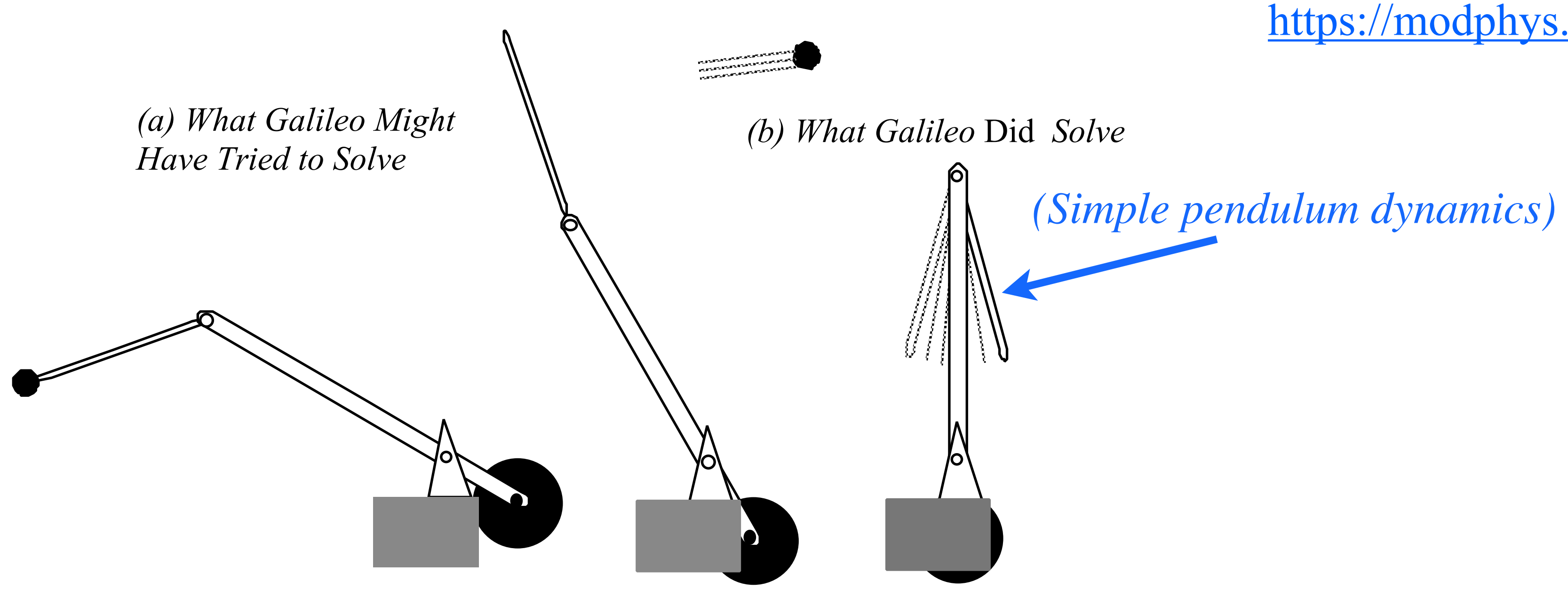
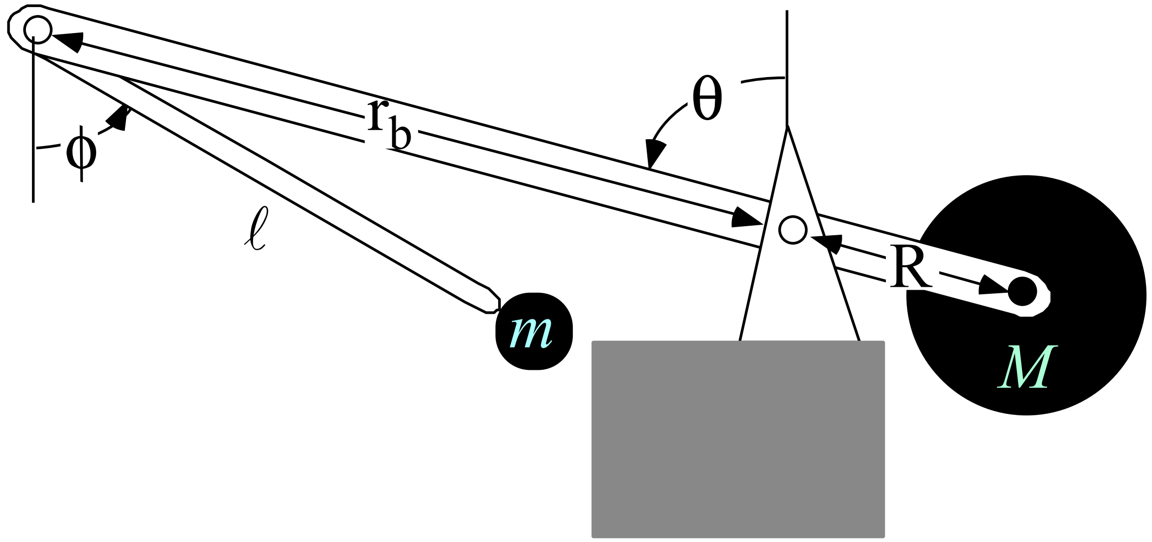


Fig. 2.1.2 Galileo's (supposed fictitious) problem

Chapter 1. The Trebuchet: A dream problem for Galileo?



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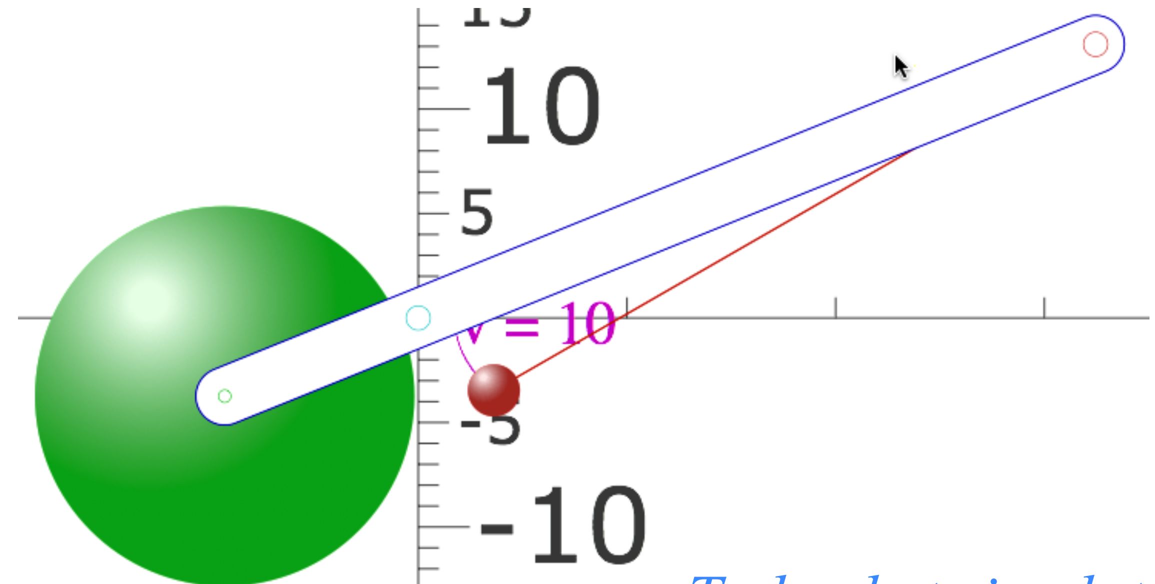


Trebuchet simulator

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html

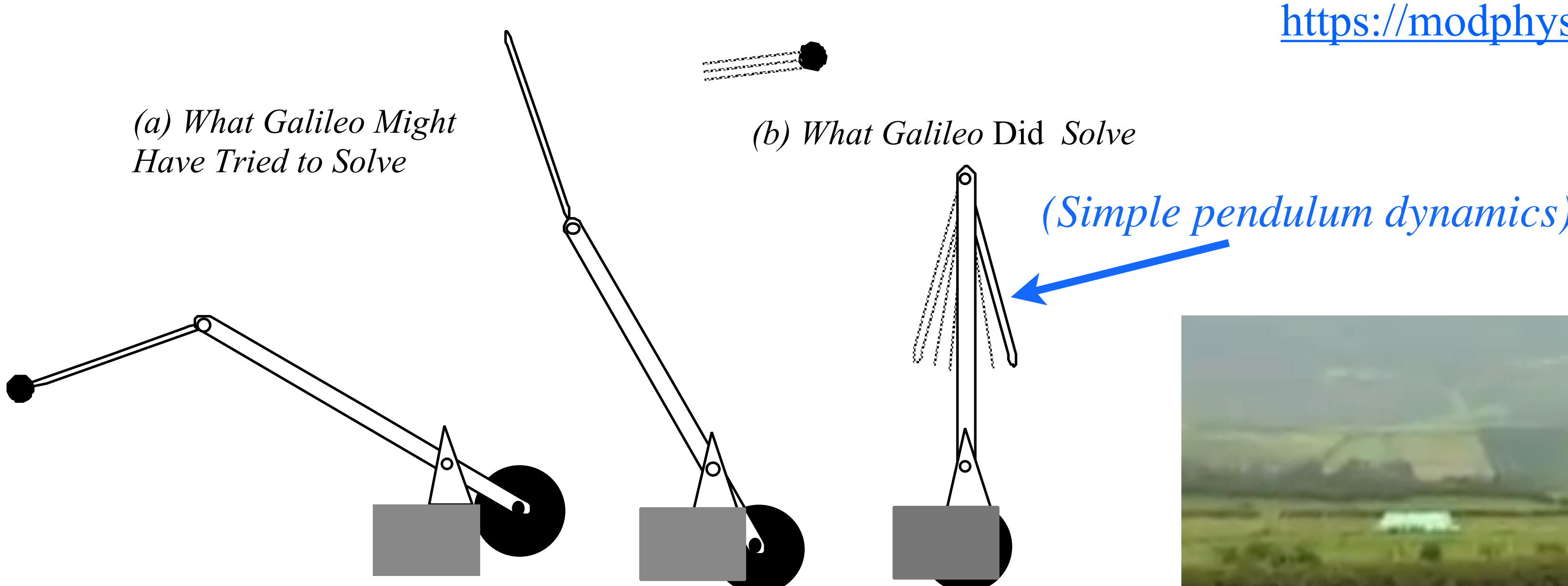


Fig. 2.1.2 Galileo's (supposed fictitious) problem

Chapter 1. The Trebuchet: A dream problem for Galileo?

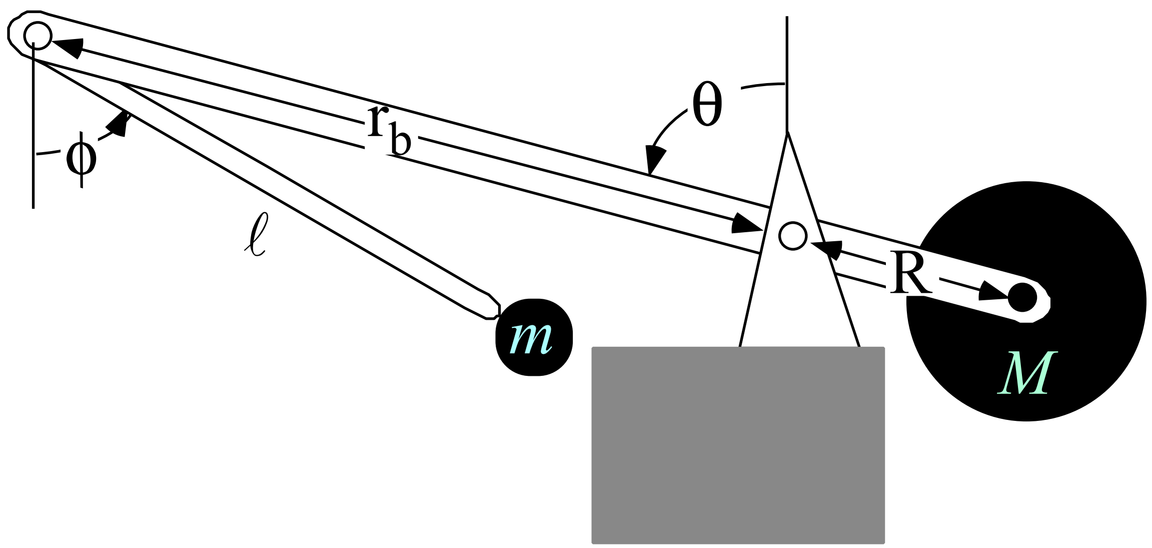
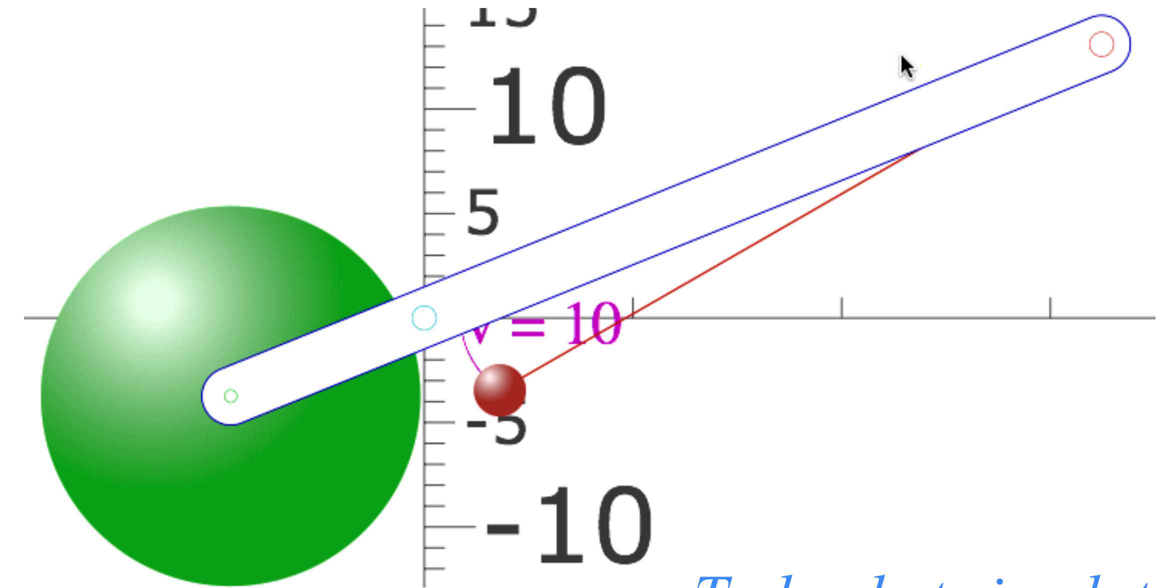
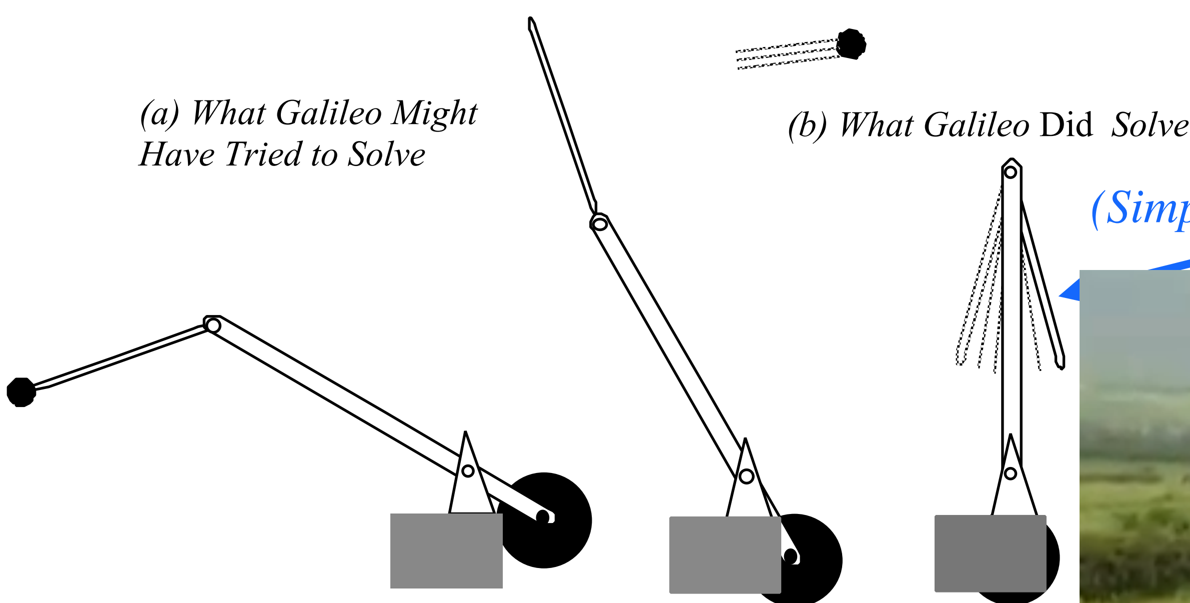


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem

It’s Halloween!...and time for Punkin’ Chunkin’ Trebuchets

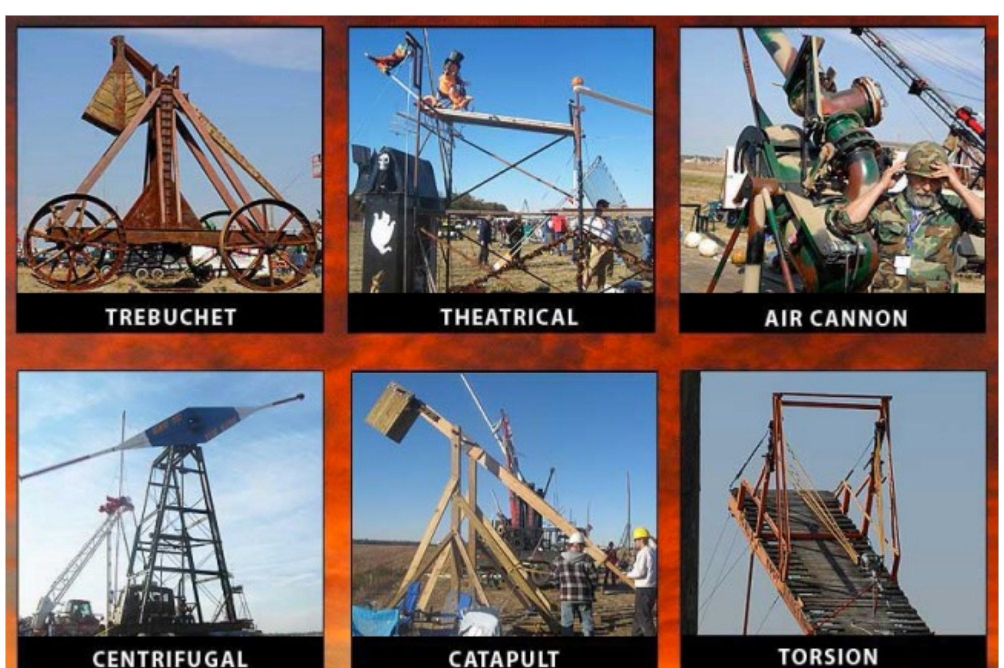
http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html

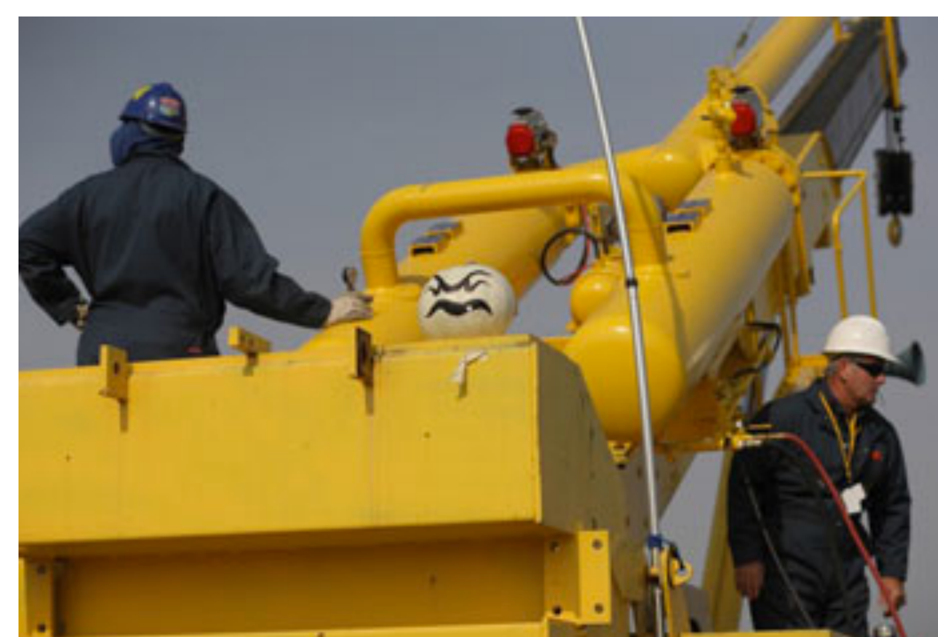
As happened in history…Trebuchet is replaced by higher-tech (or lower tech)

Giant cannons can chunk-a-punkin over 4,000 ft. *Trebuchet range max ~1,200ft.*

http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows

http://www.sussexcountyonline.com/news/photos/punkinchunkin.html

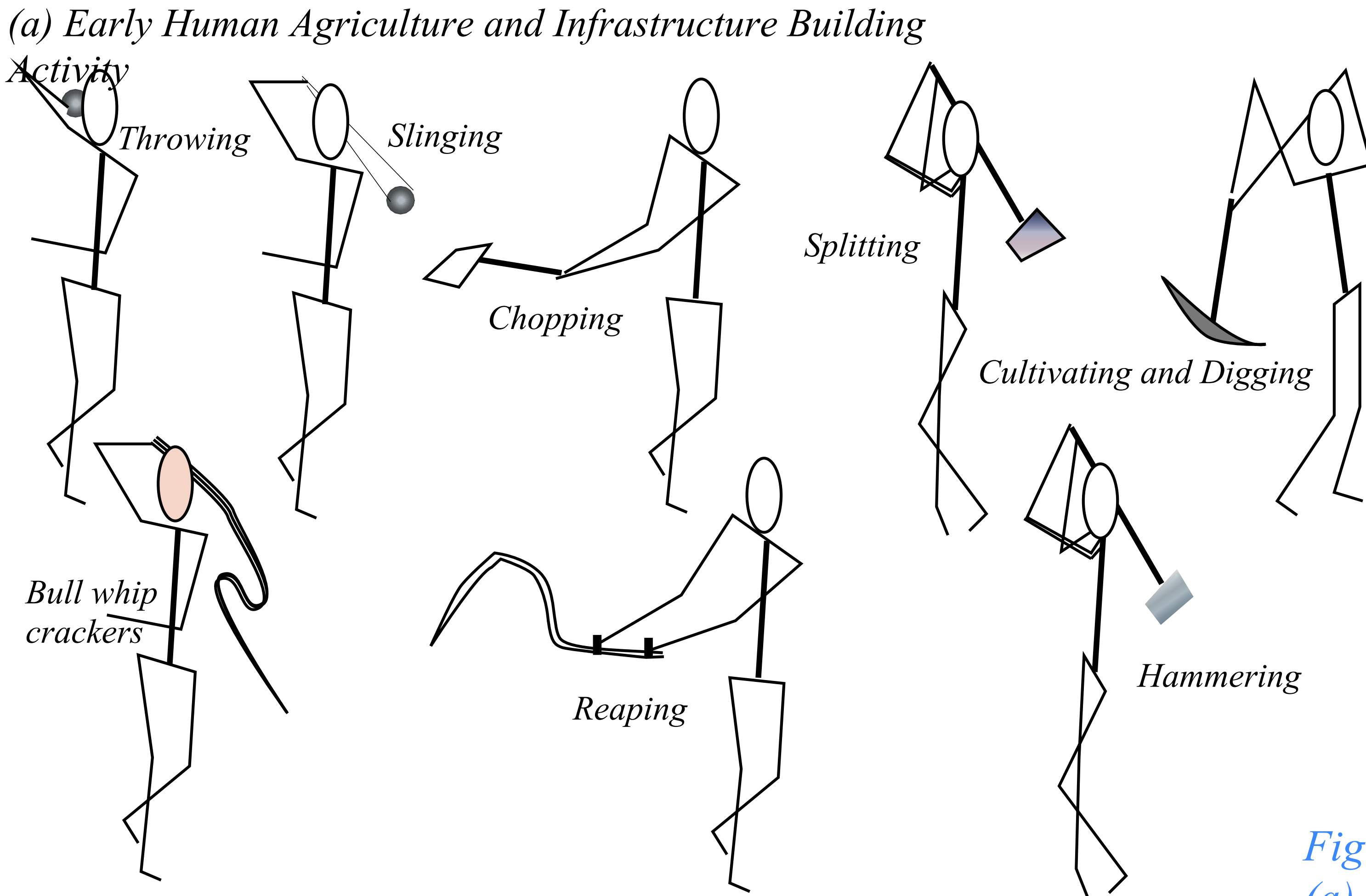




The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. ***273****, 66 (July 1995))*

The medieval ingenium (9th to 14th century) and modern re-enactments

→ *Human kinesthetics and sports kinesiology*

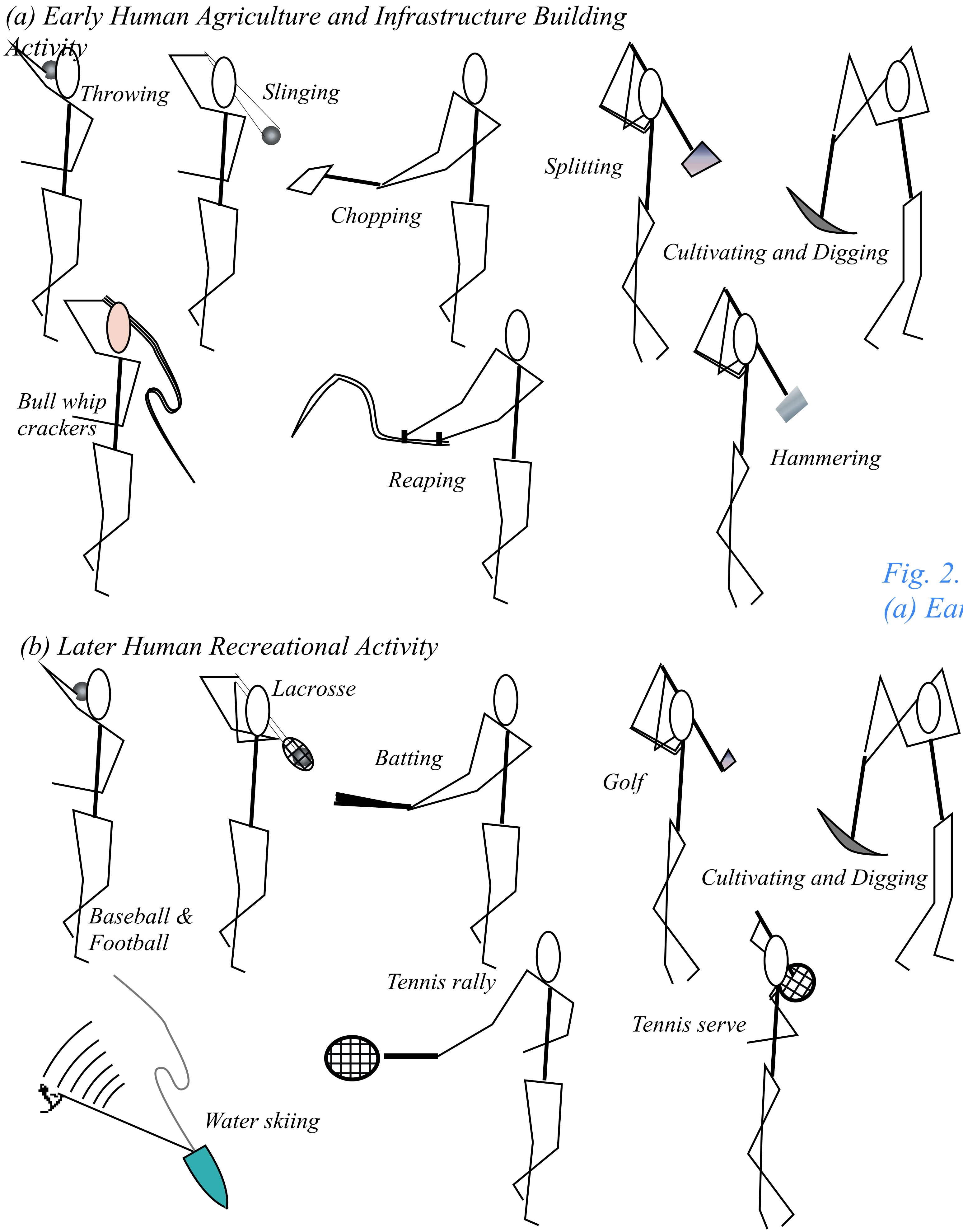


Some technique required!

KE achieved by non-linear whip action

Must avoid injury

Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work.



Some technique required!
KE achieved by non-linear whip action
Must avoid injury

Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work. (b) Later recreational kinesthetics.

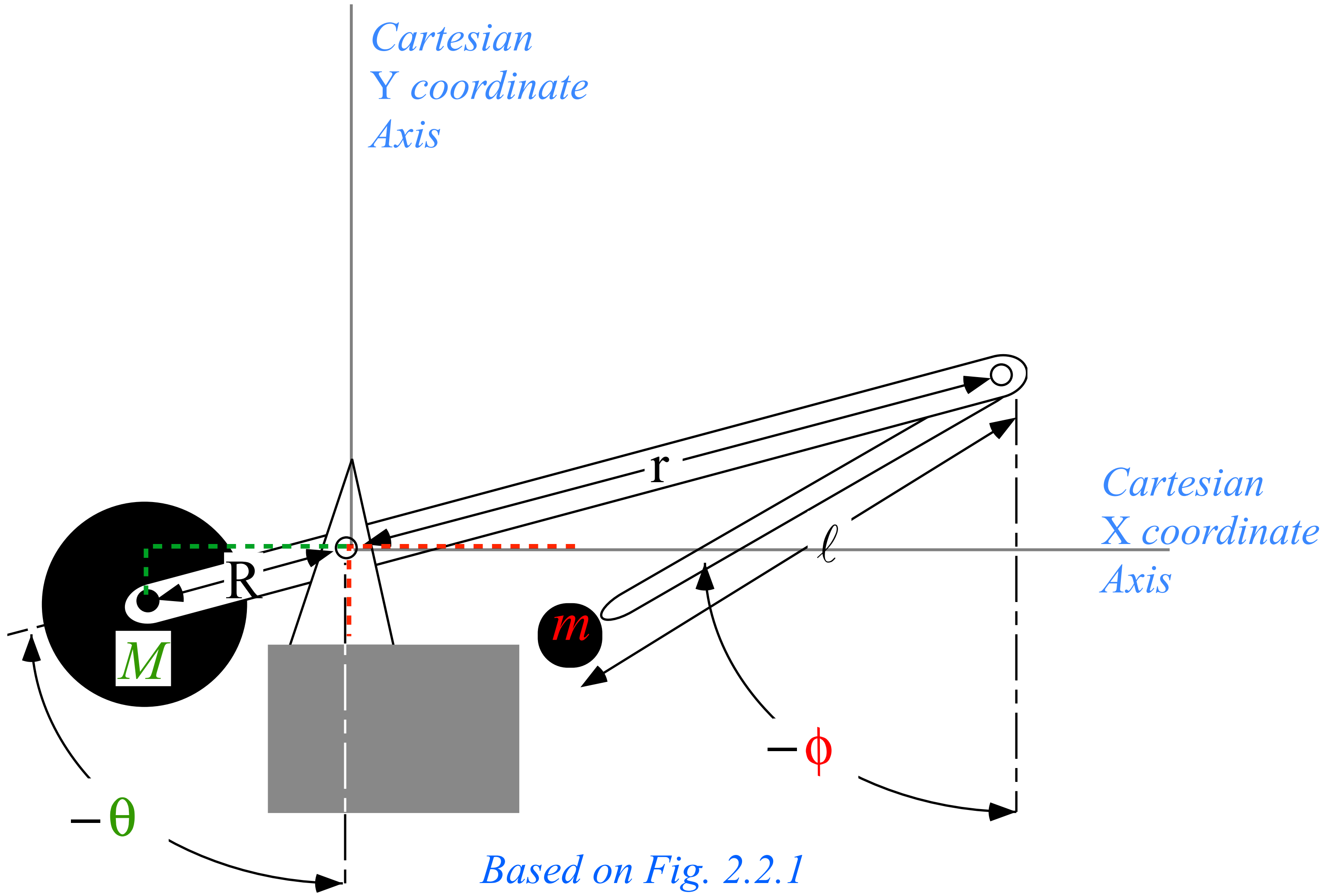
Some technique required!
KE achieved by non-linear whip action
Must avoid injury

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

geometry of trebuchet

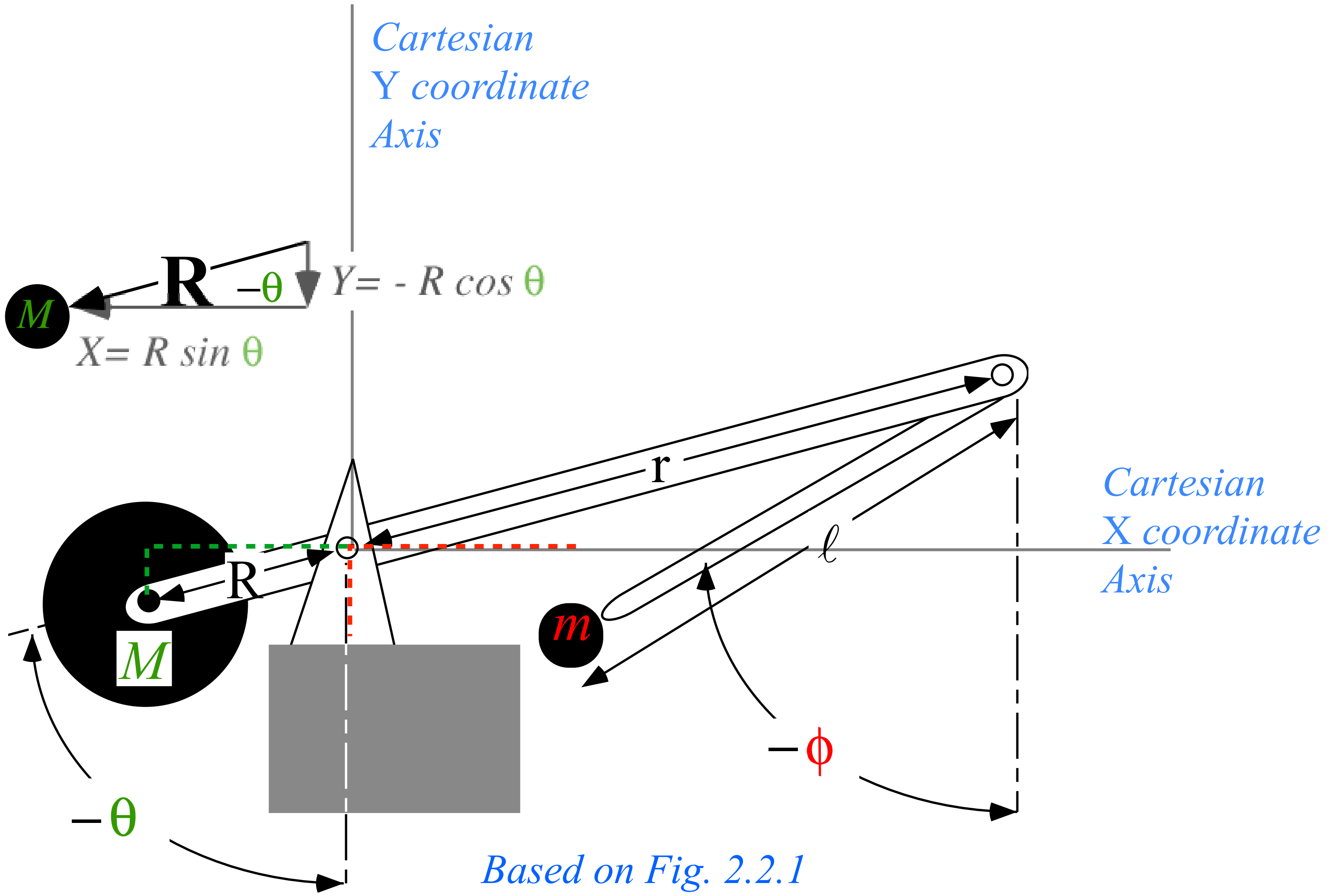


Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):
$X = R \sin \theta$
$Y = -R \cos \theta$

geometry of trebuchet



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M (Driving weight Mg):

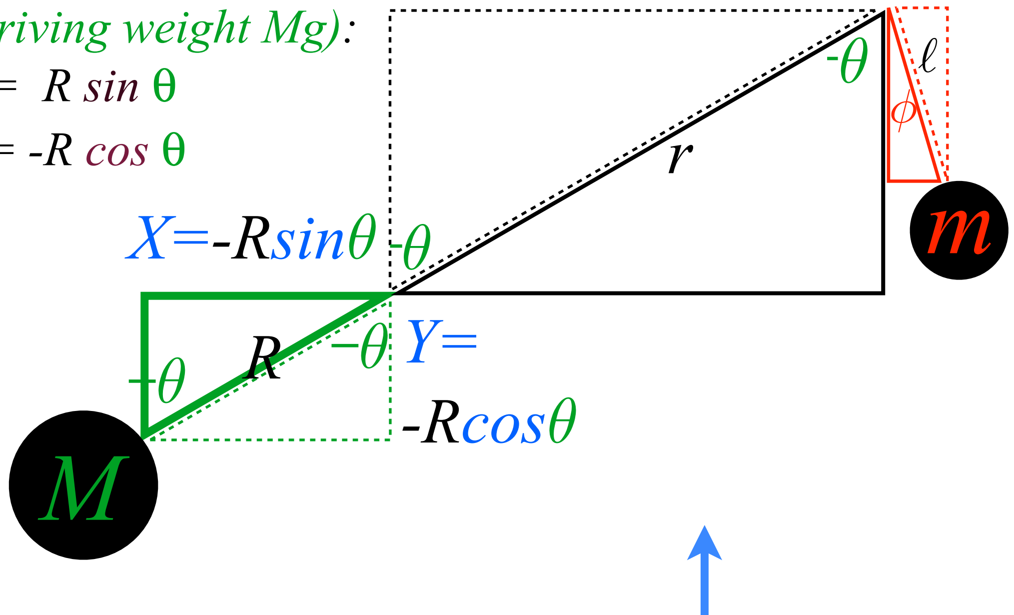
$X = R \sin \theta$

$Y = -R \cos \theta$

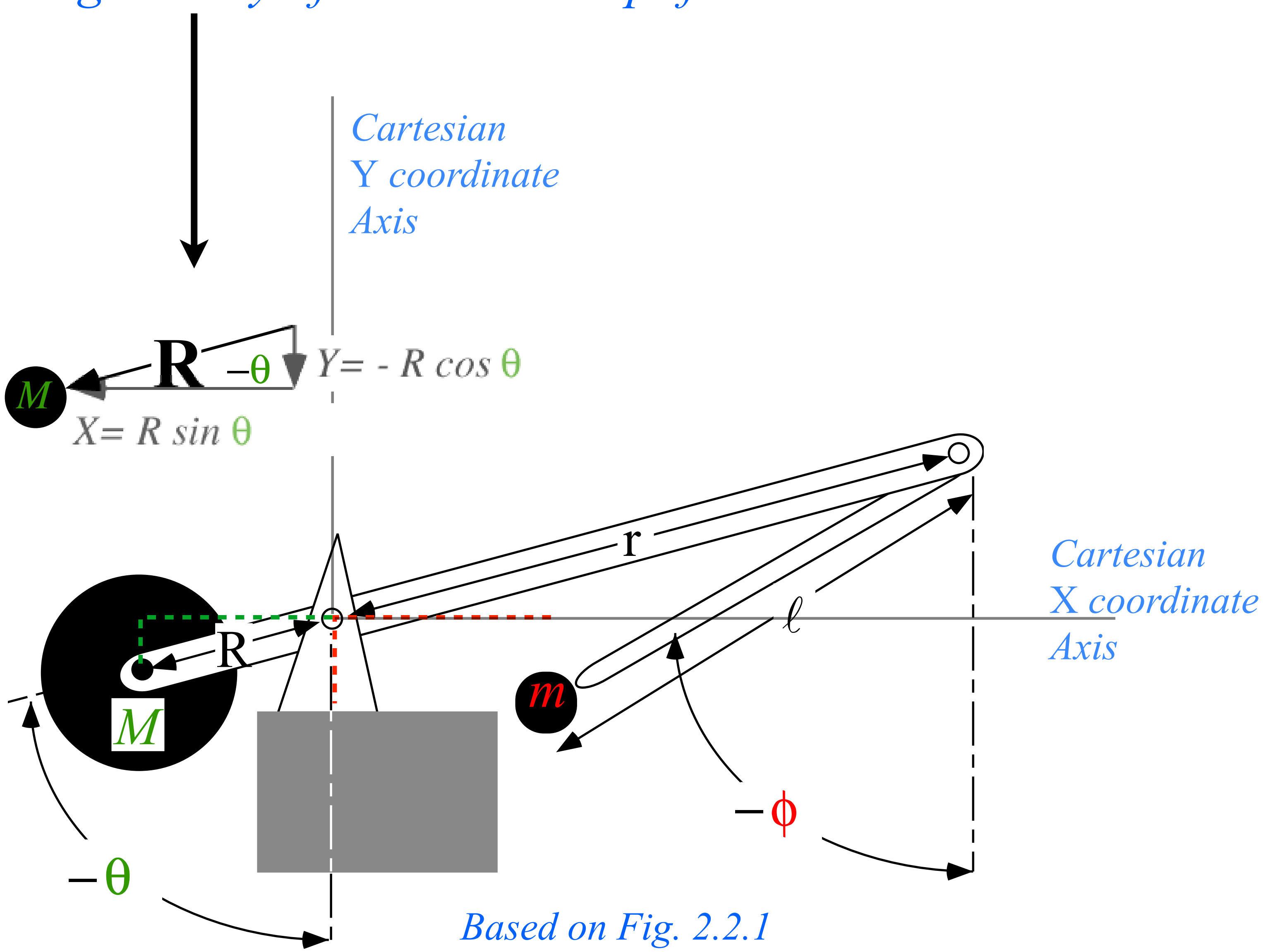
Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = - r \sin \theta + \ell \sin \phi$

$y = y_r + y_\ell = r \cos \theta - \ell \cos \phi$



geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of mass *m* (Payload or projectile):

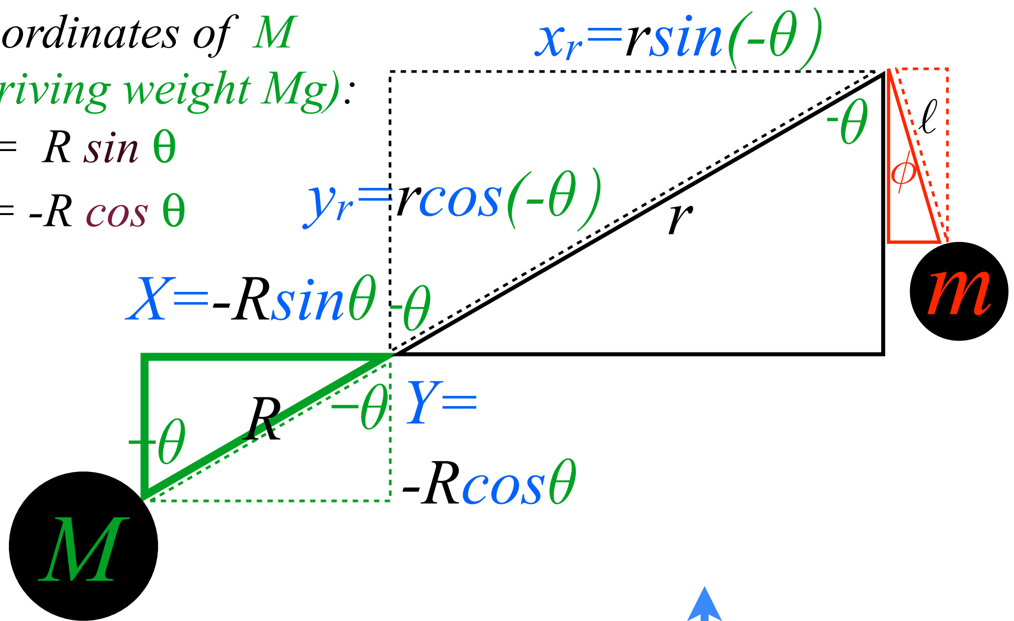
$x = x_r + x_\ell = - r \sin\theta + \ell \sin\phi$

$y = y_r + y_\ell = \quad r \cos\theta - \ell \cos\phi$

Coordinates of *M* (Driving weight Mg):

$X = \; R \sin\theta$

$Y = -R \cos\theta$



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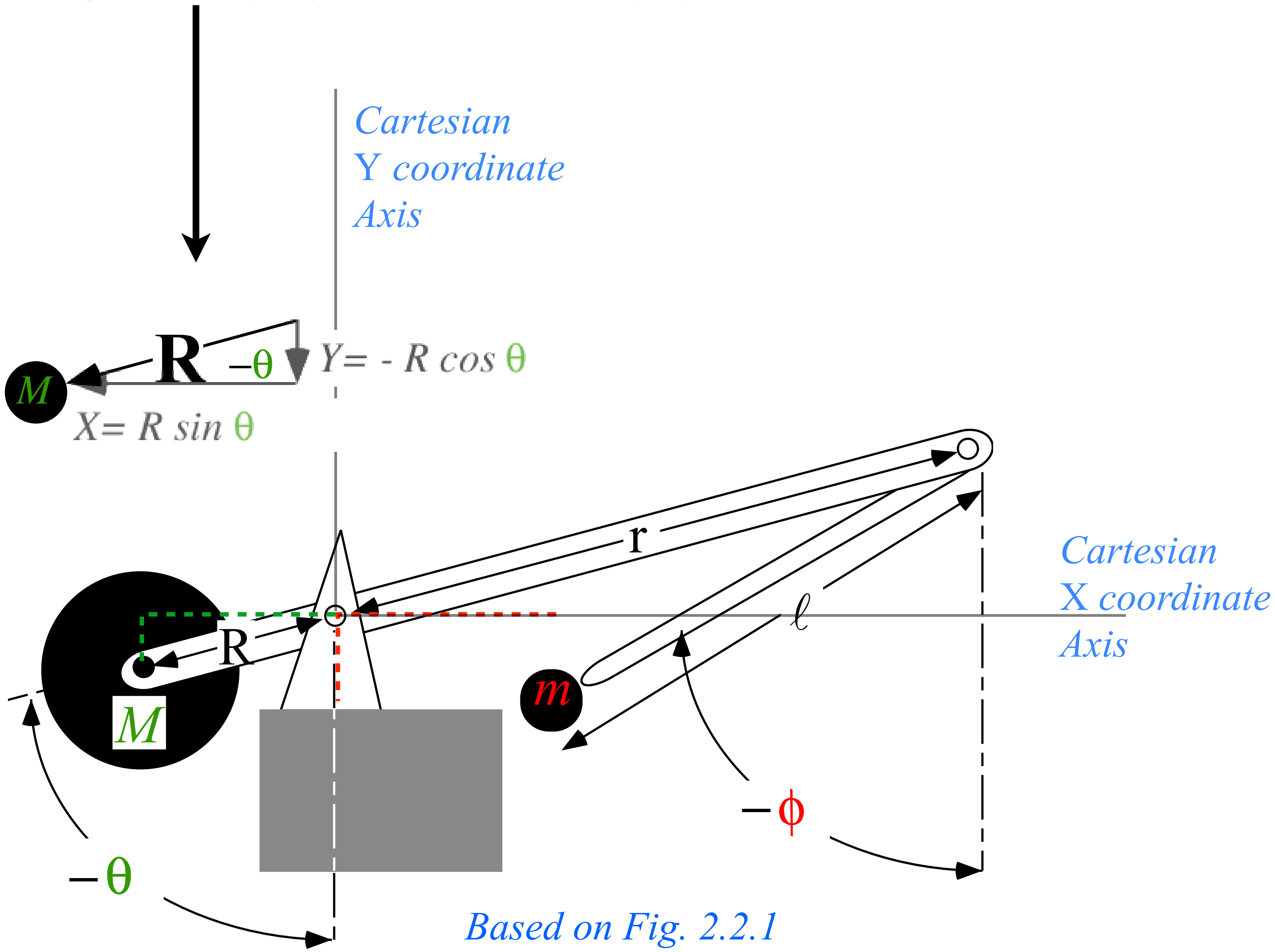


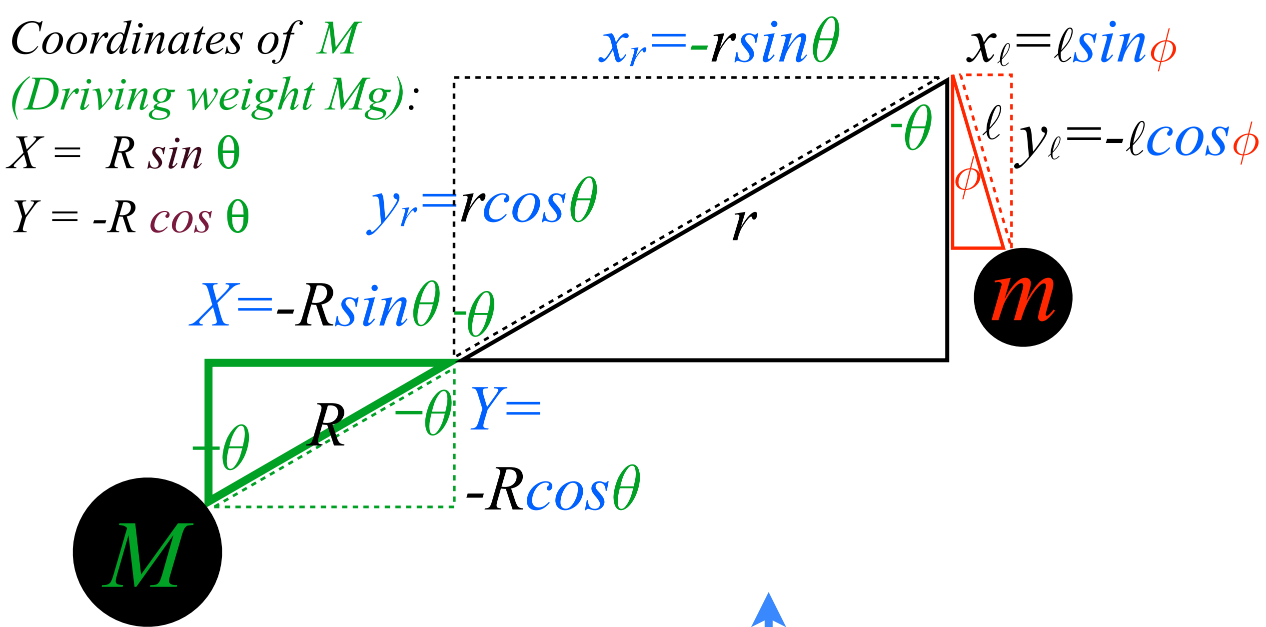
Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

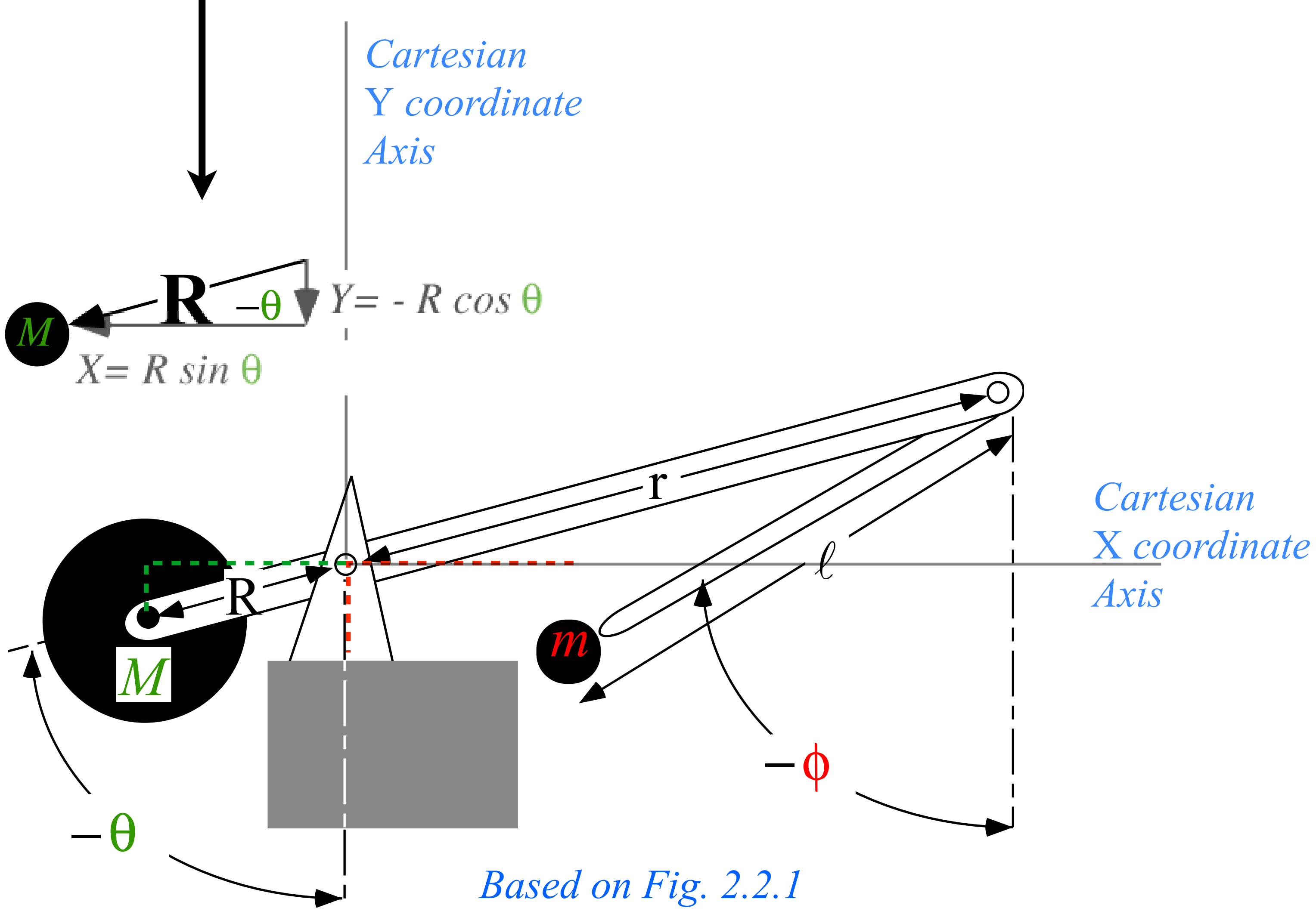
Coordinates of mass *m* (Payload or projectile):

$x = x_r + x_\ell = - r \sin\theta + \ell \sin\phi$

$y = y_r + y_\ell = r \cos\theta - \ell \cos\phi$



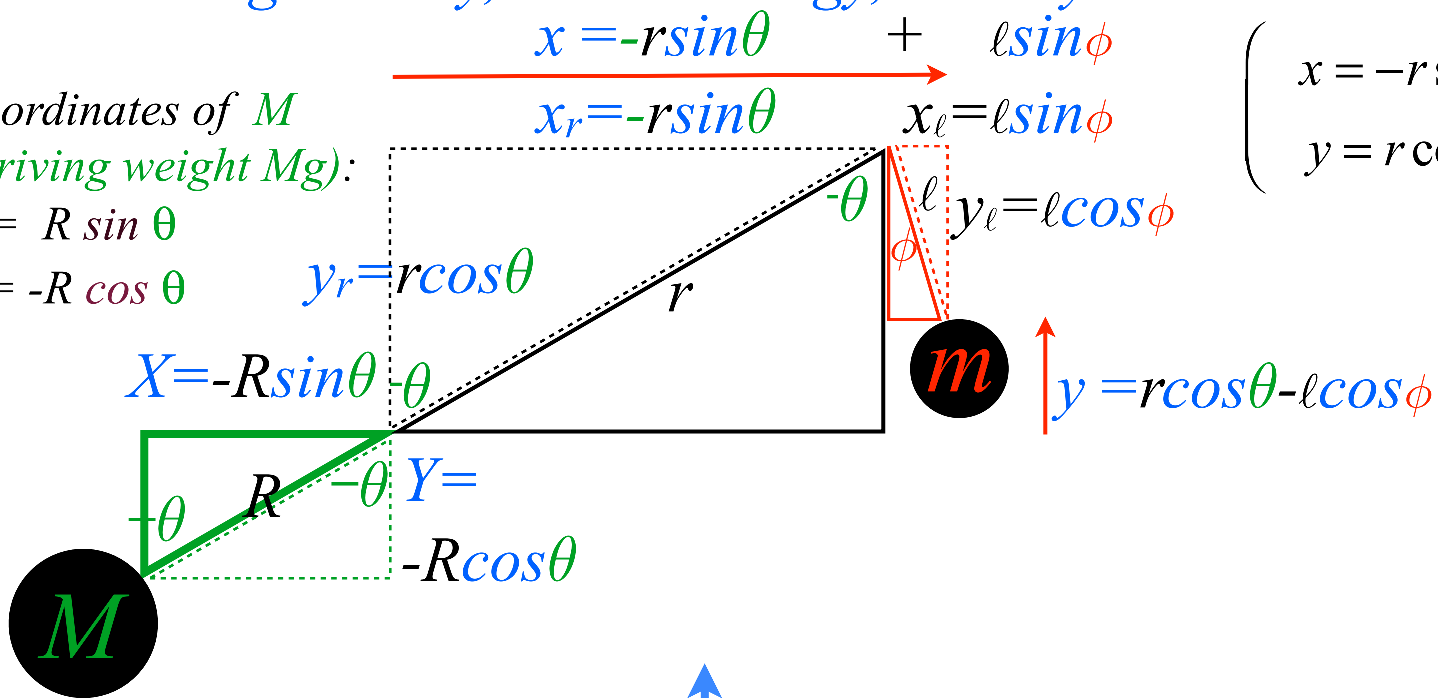
geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

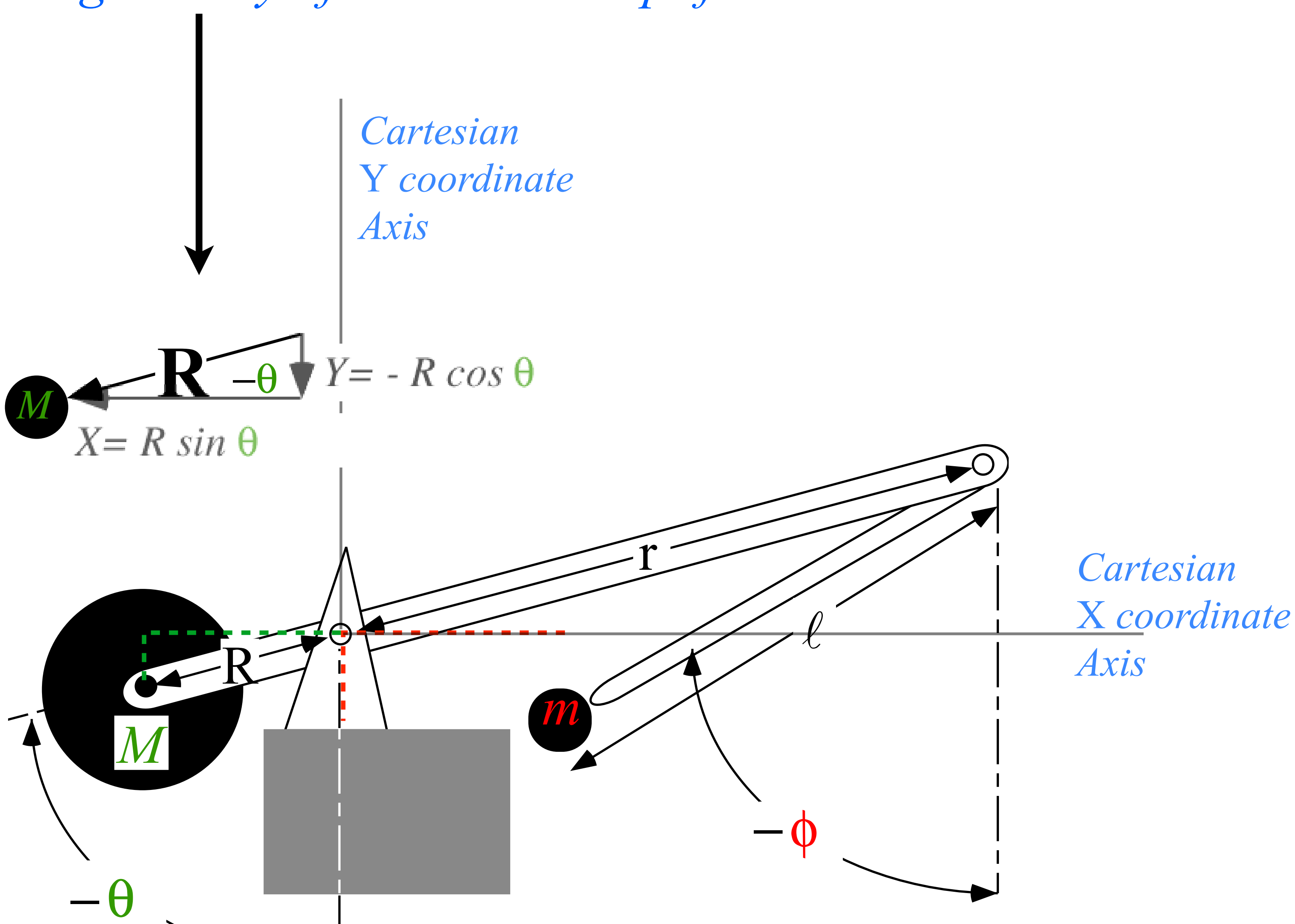
Coordinates of M
(Driving weight Mg):
$X = R \sin \theta$
$Y = -R \cos \theta$



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

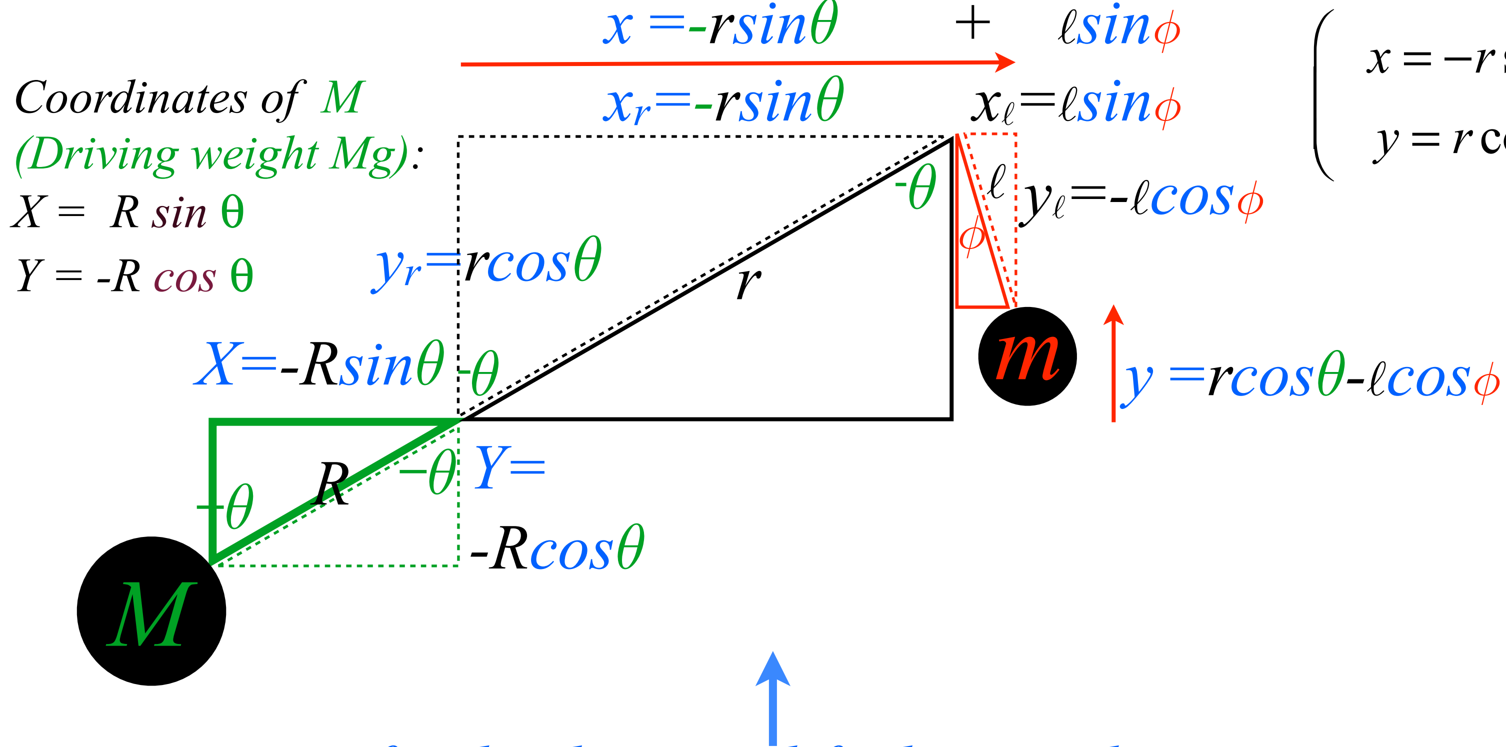
Coordinates of mass m
(Payload or projectile):
$x = x_r + x_\ell = - r \sin \theta + \ell \sin \phi$
$y = y_r + y_\ell = \ r \cos \theta - \ell \cos \phi$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



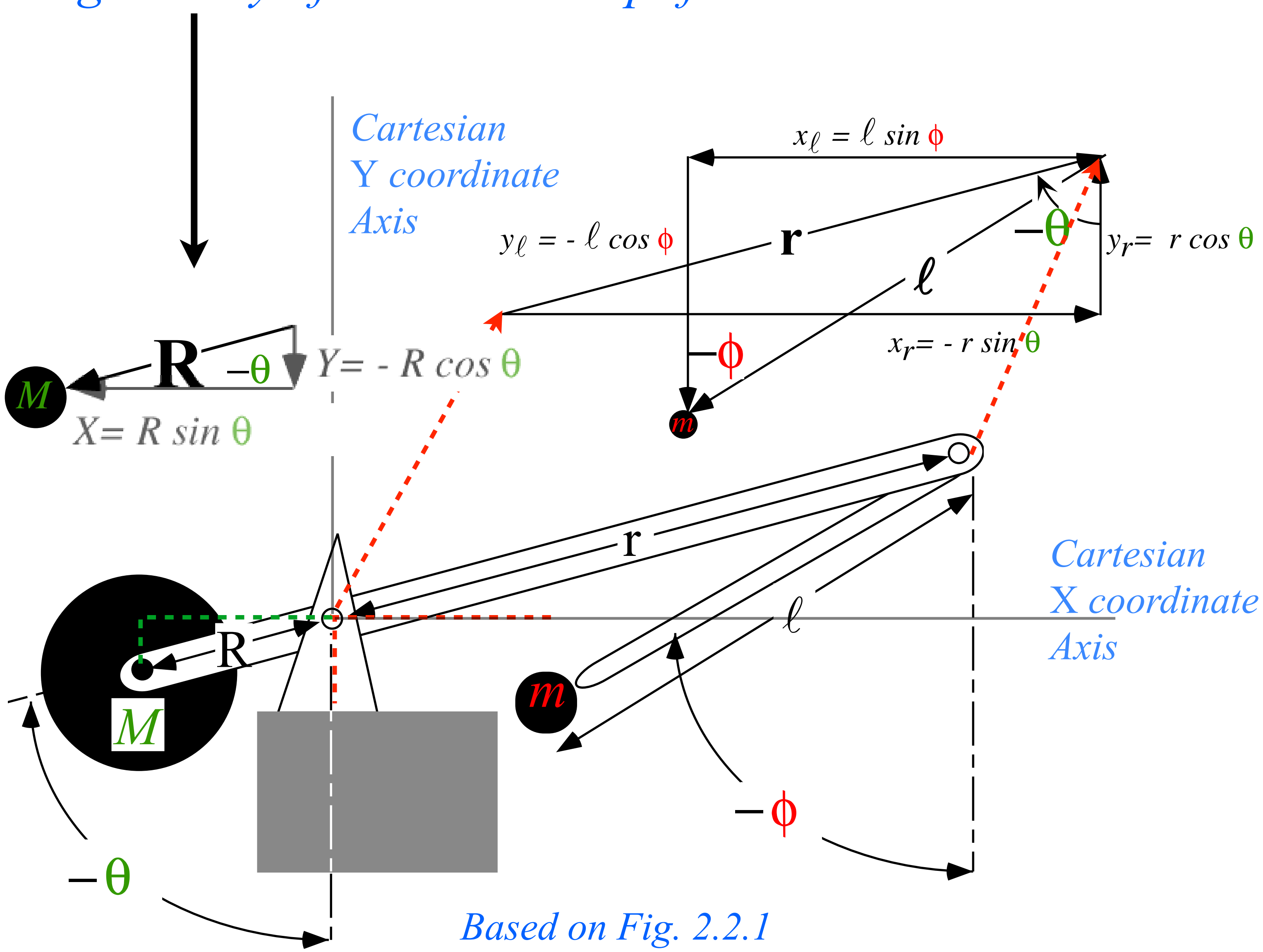
$$
\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}
$$

Coordinates of mass *m* (*Payload or projectile*):

$x = x_r + x_\ell = - r \sin \theta + \ell \sin \phi$

$y = y_r + y_\ell = \ r \cos \theta - \ell \cos \phi$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

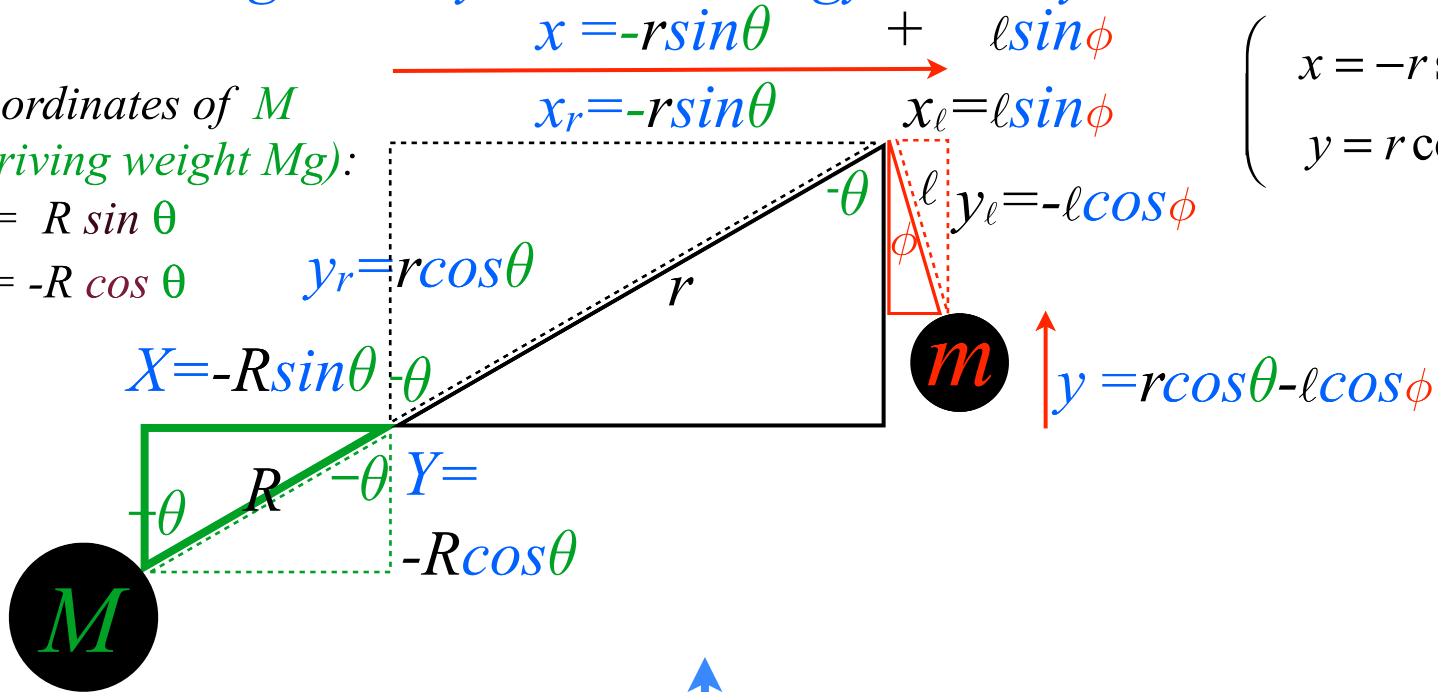
- → *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M (Driving weight Mg):

$X = R \sin\theta$

$Y = -R \cos\theta$



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = - r \sin\theta + \ell \sin\phi$

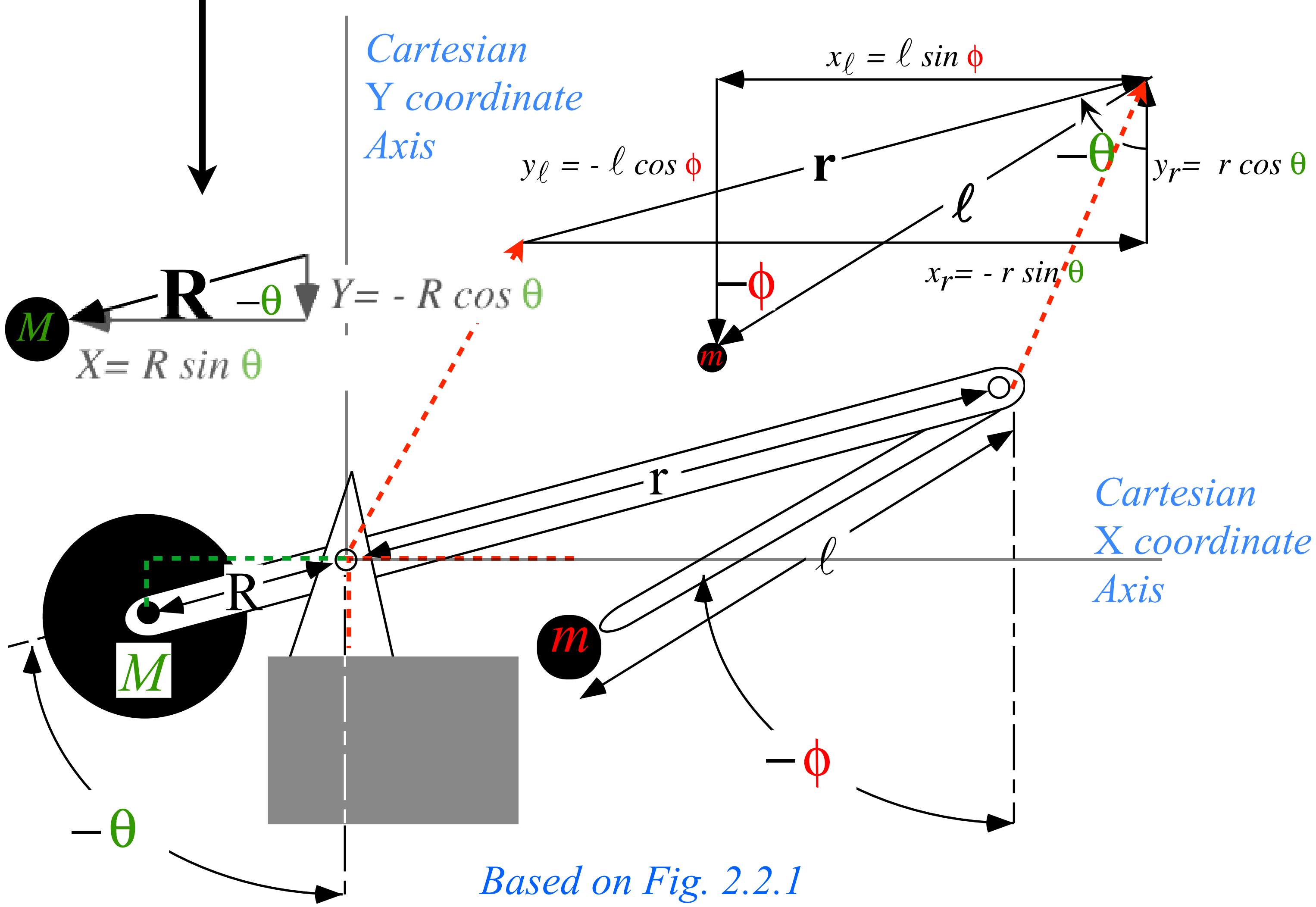
$y = y_r + y_\ell = r \cos\theta - \ell \cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

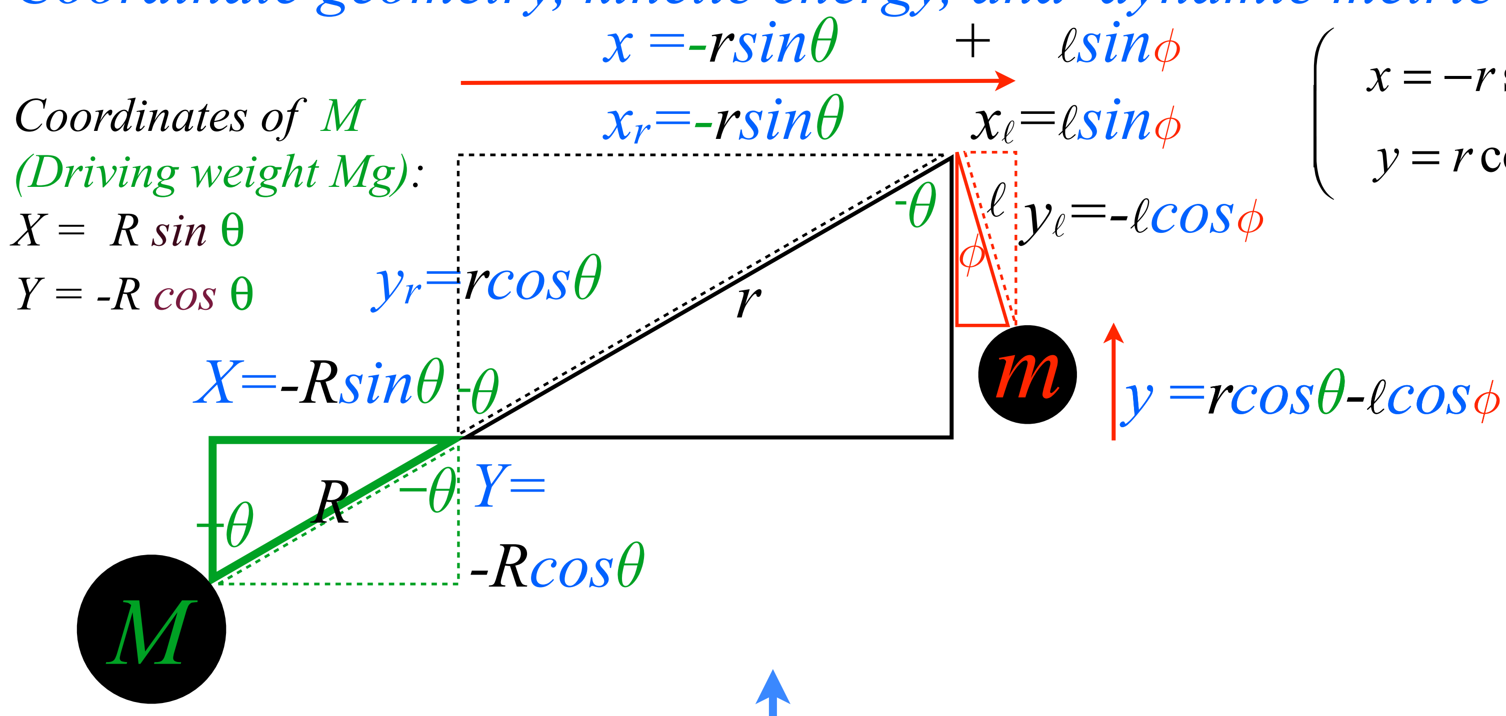
$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass *m* (*Payload or projectile*):

$$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$$

$$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$$

1st differential and Jacobian relations:

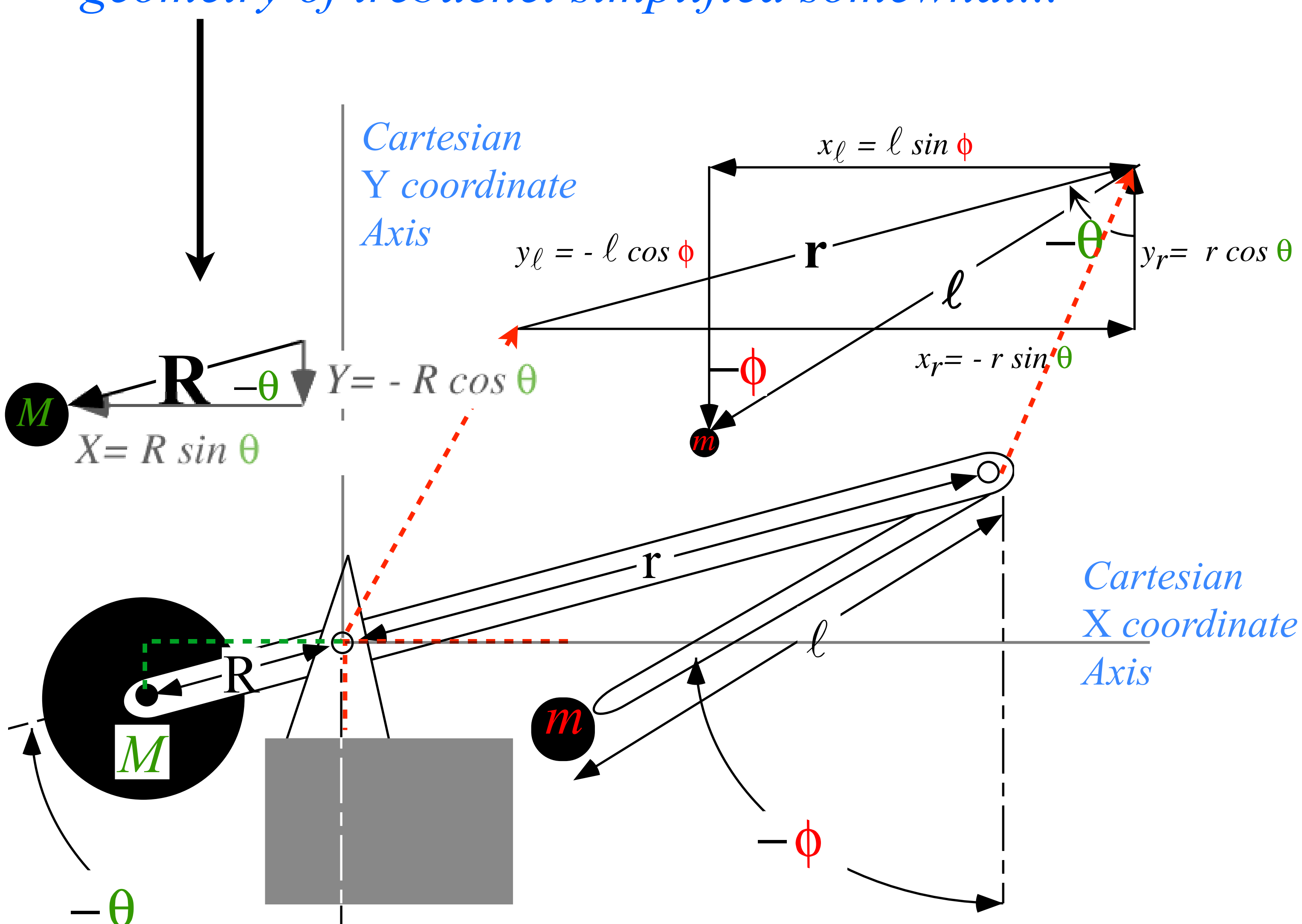
$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

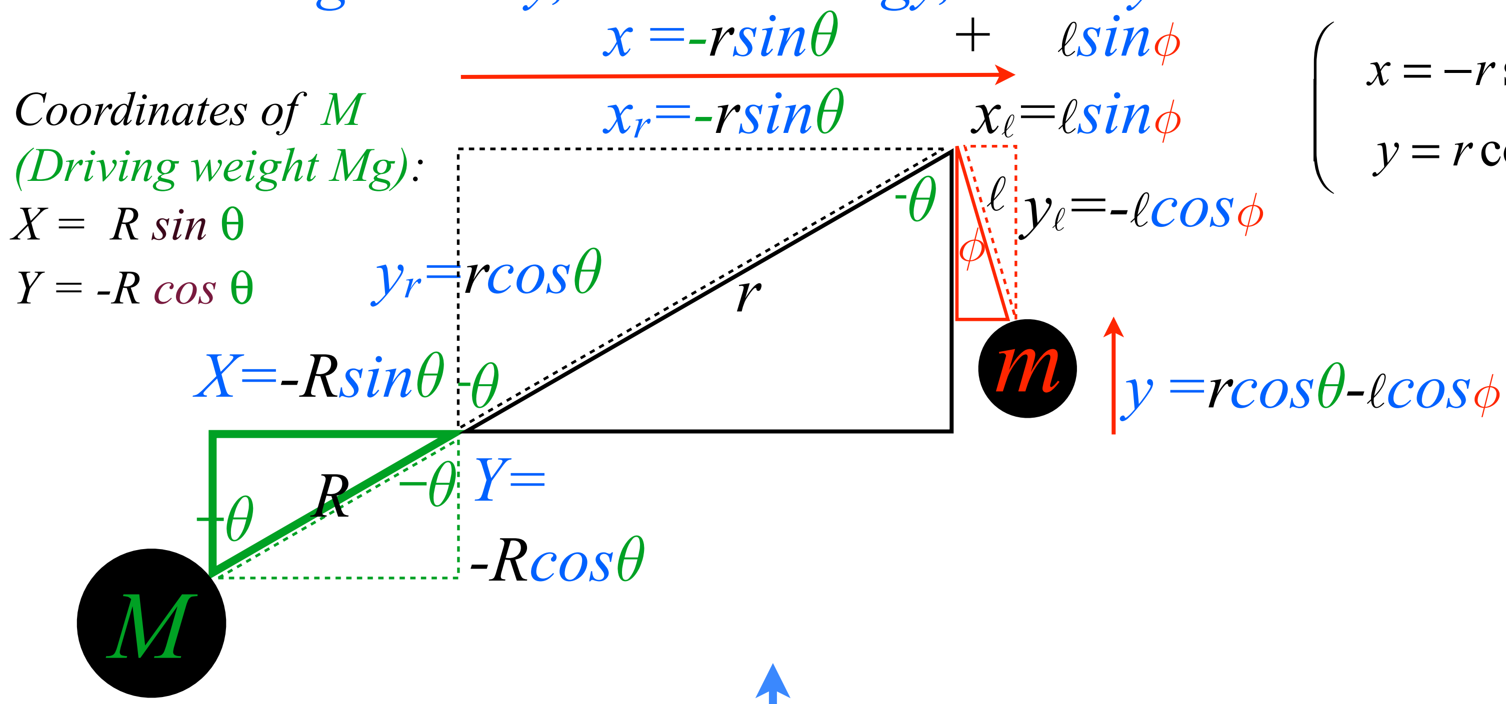
$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
(Payload or projectile):

$x = x_r + x_\ell = - r \sin \theta + \ell \sin \phi$

$y = y_r + y_\ell = \; r \cos \theta - \ell \cos \phi$

geometry of trebuchet simplified somewhat...

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \qquad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \qquad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

$$dX = R\cos\theta \; d\theta + 0, \qquad dx = -r\cos\theta \; d\theta + \ell\cos\phi \; d\phi$$

$$dY = R\sin\theta \; d\theta + 0, \qquad dy = -r\sin\theta \; d\theta + \ell\sin\phi \; d\phi$$

Constraint relations:

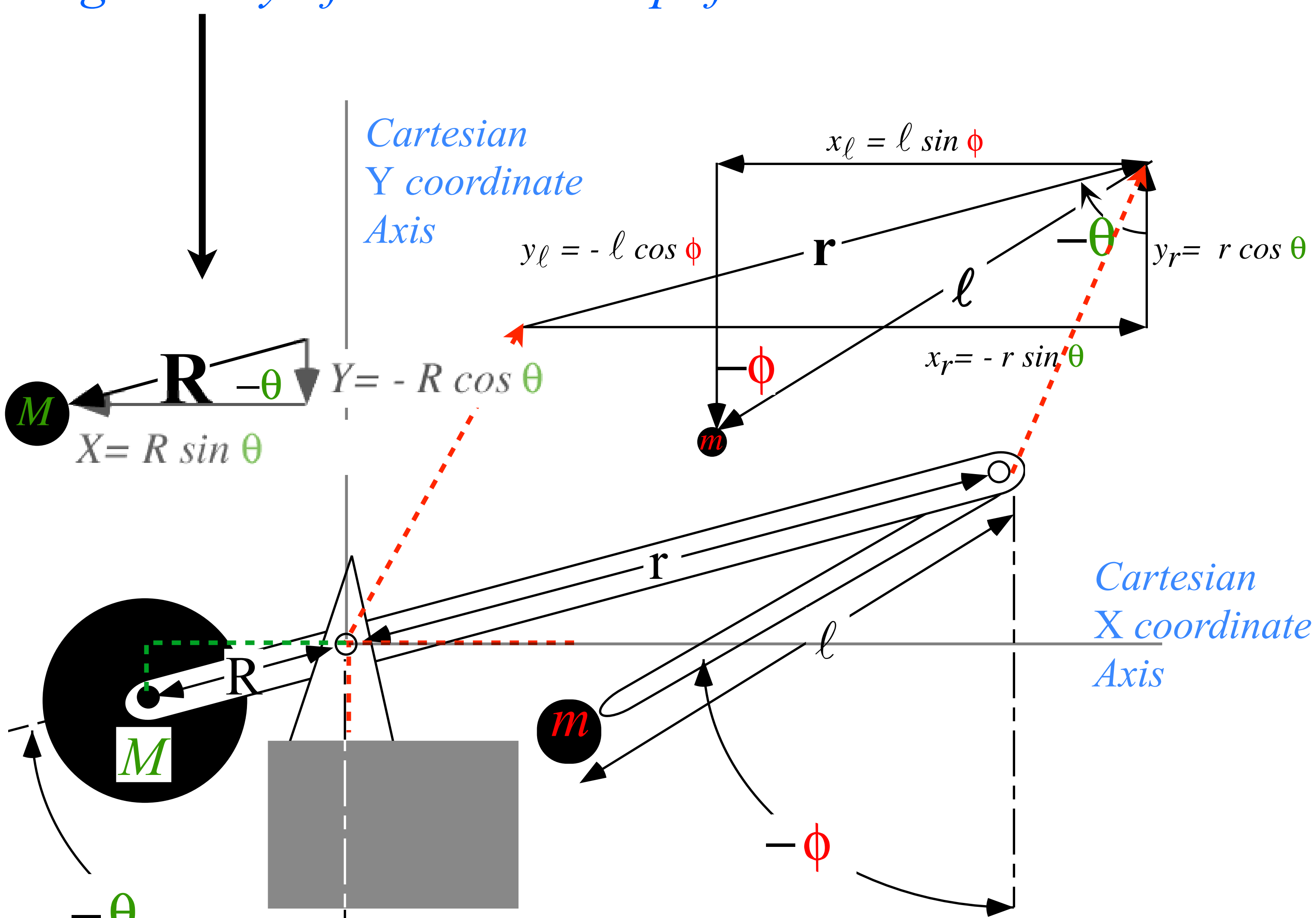
$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$

$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$

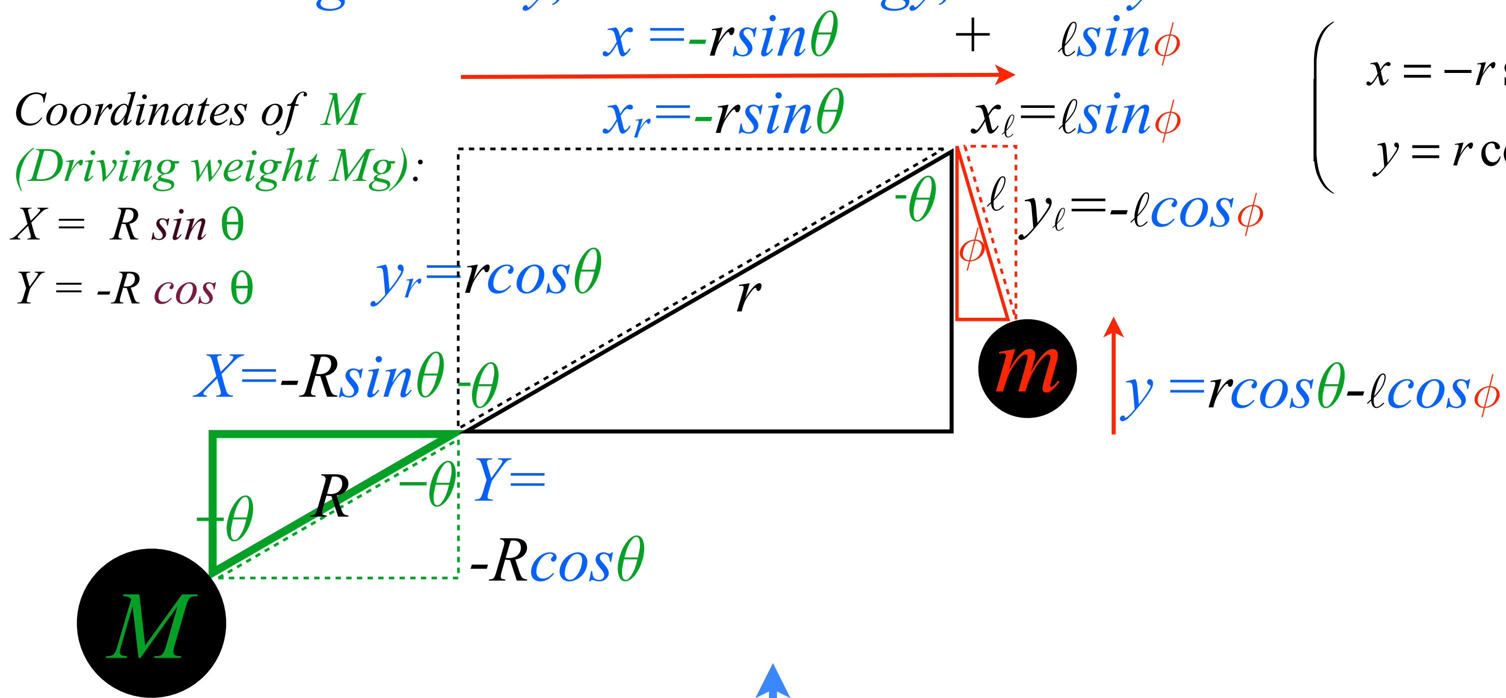
'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass *m* (Payload or projectile):

$$x = x_r + x_\ell = -\,r\sin\theta + \ell\sin\phi$$

$$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

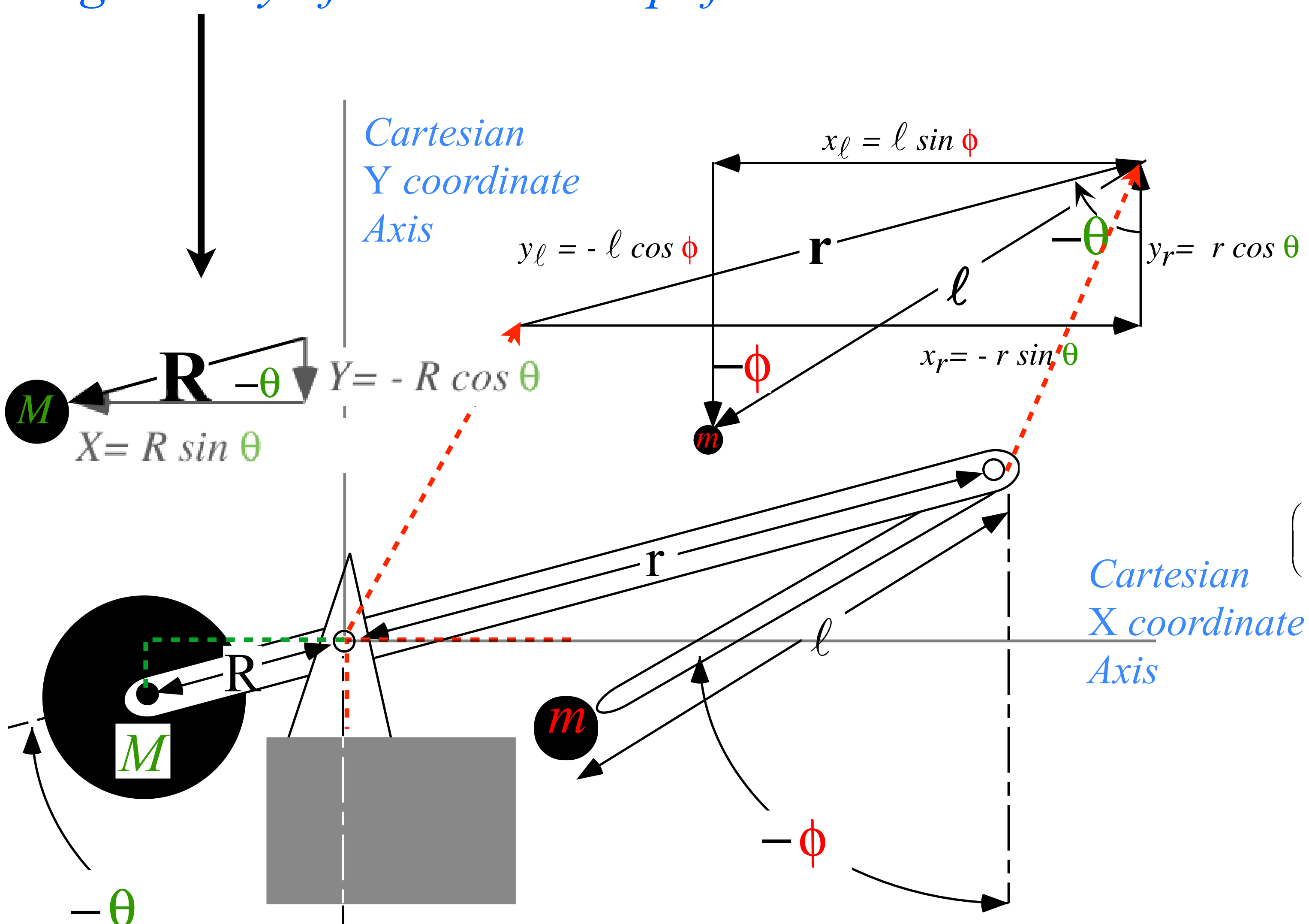
Constraint relations:

$$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$$

$$c_\ell(x_\ell,y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$$

$$c_r(x_r,y_r) = x_r^2 + y_r^2 = r^2 = const.$$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

'Raw' Jacobian form

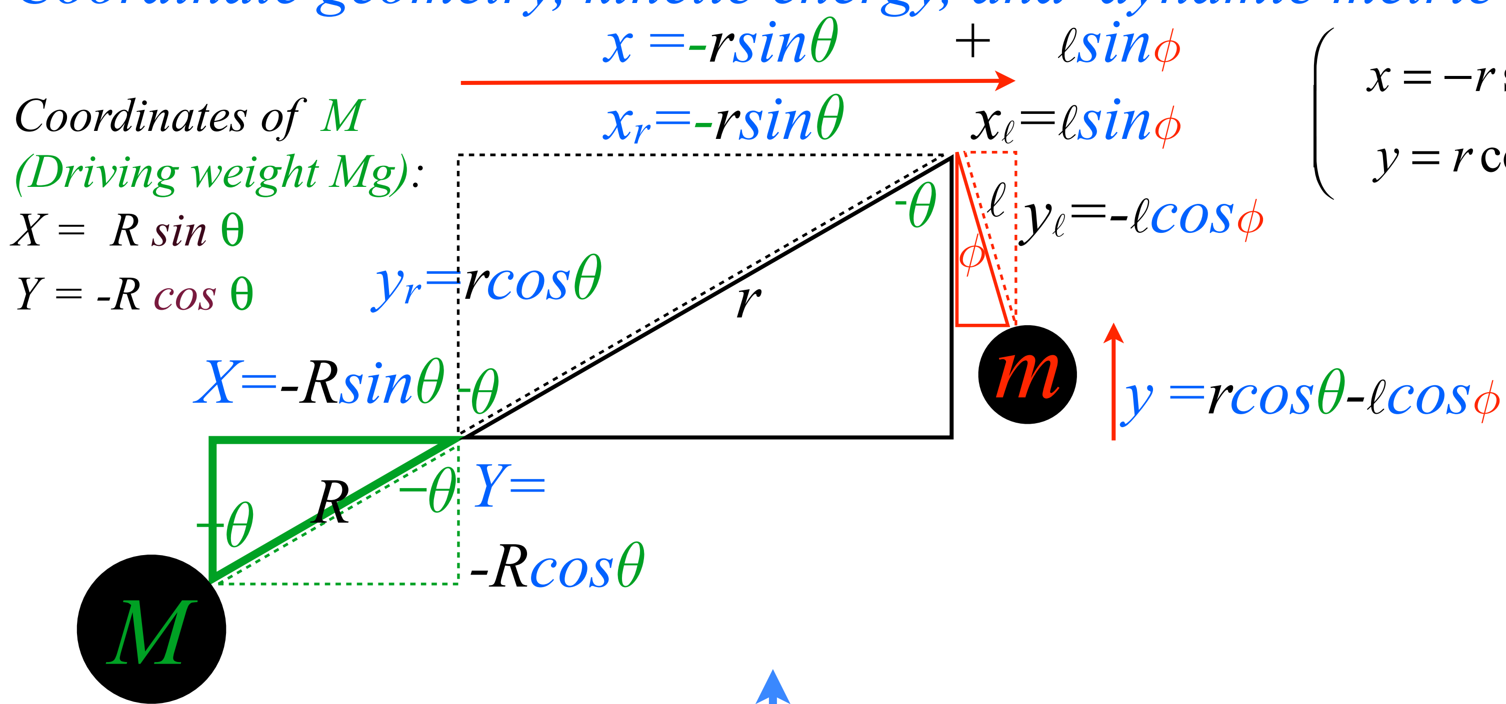
$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass *m*
(Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

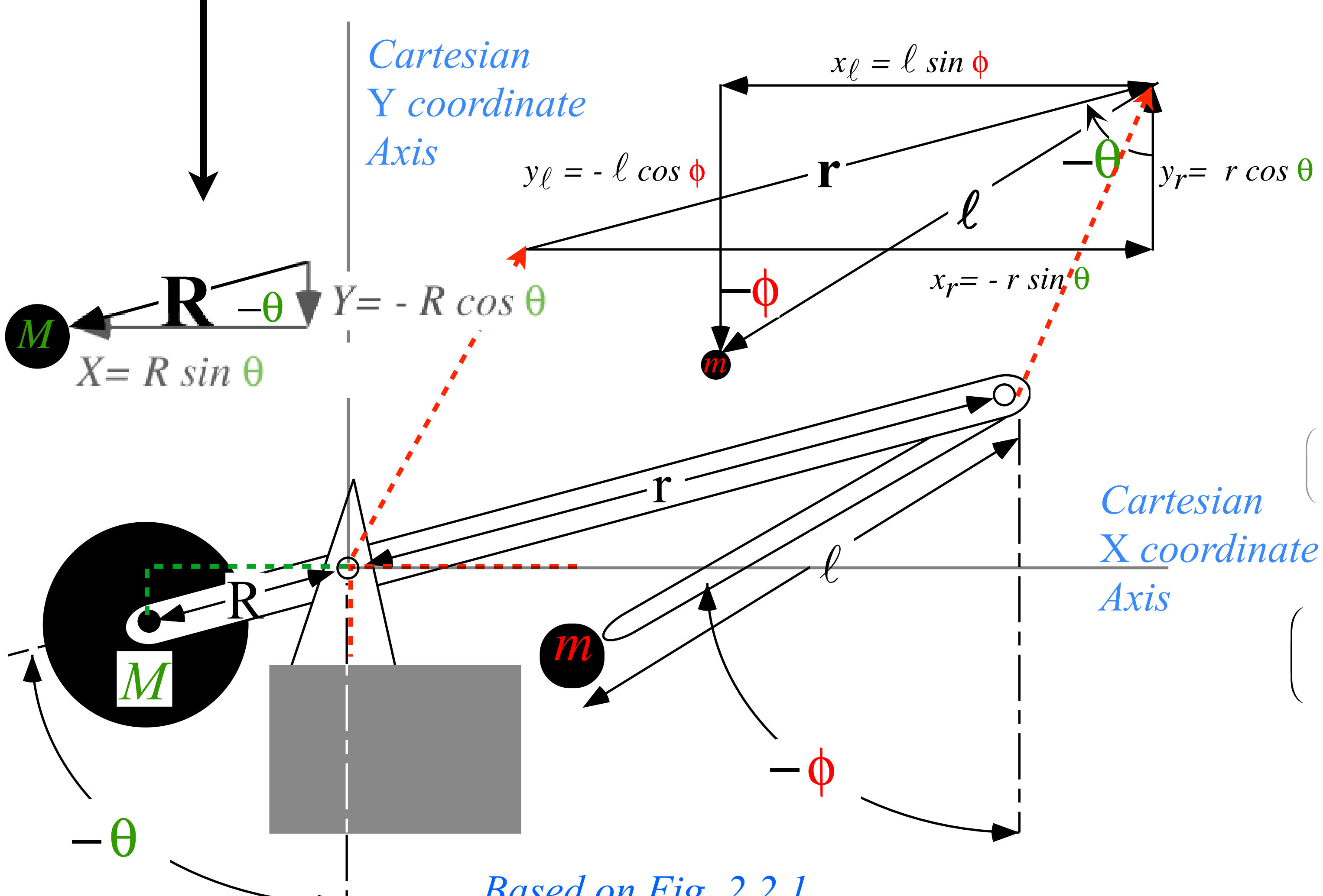
geometry of trebuchet simplified somewhat...

Constraint relations:

$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$

$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$



Based on Fig. 2.2.1

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

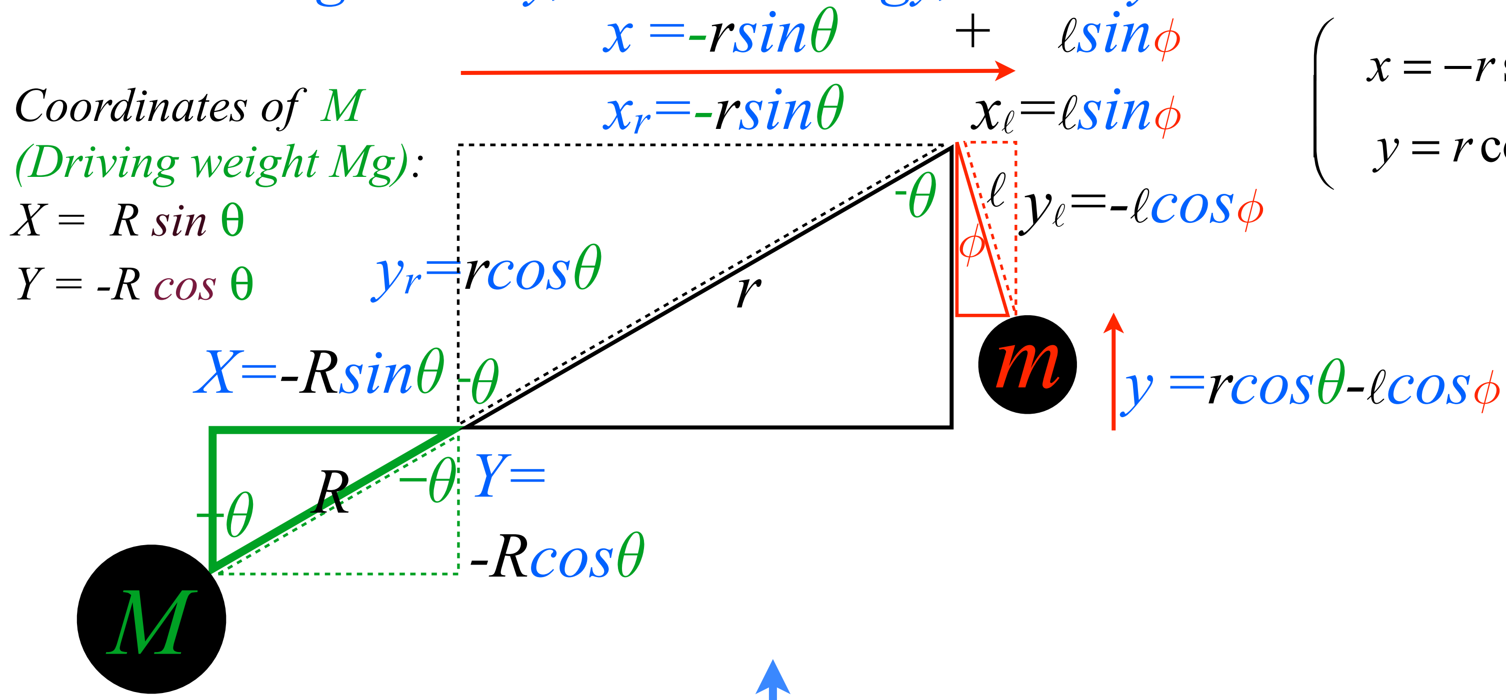
Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

geometry of trebuchet simplified somewhat...

Constraint relations:

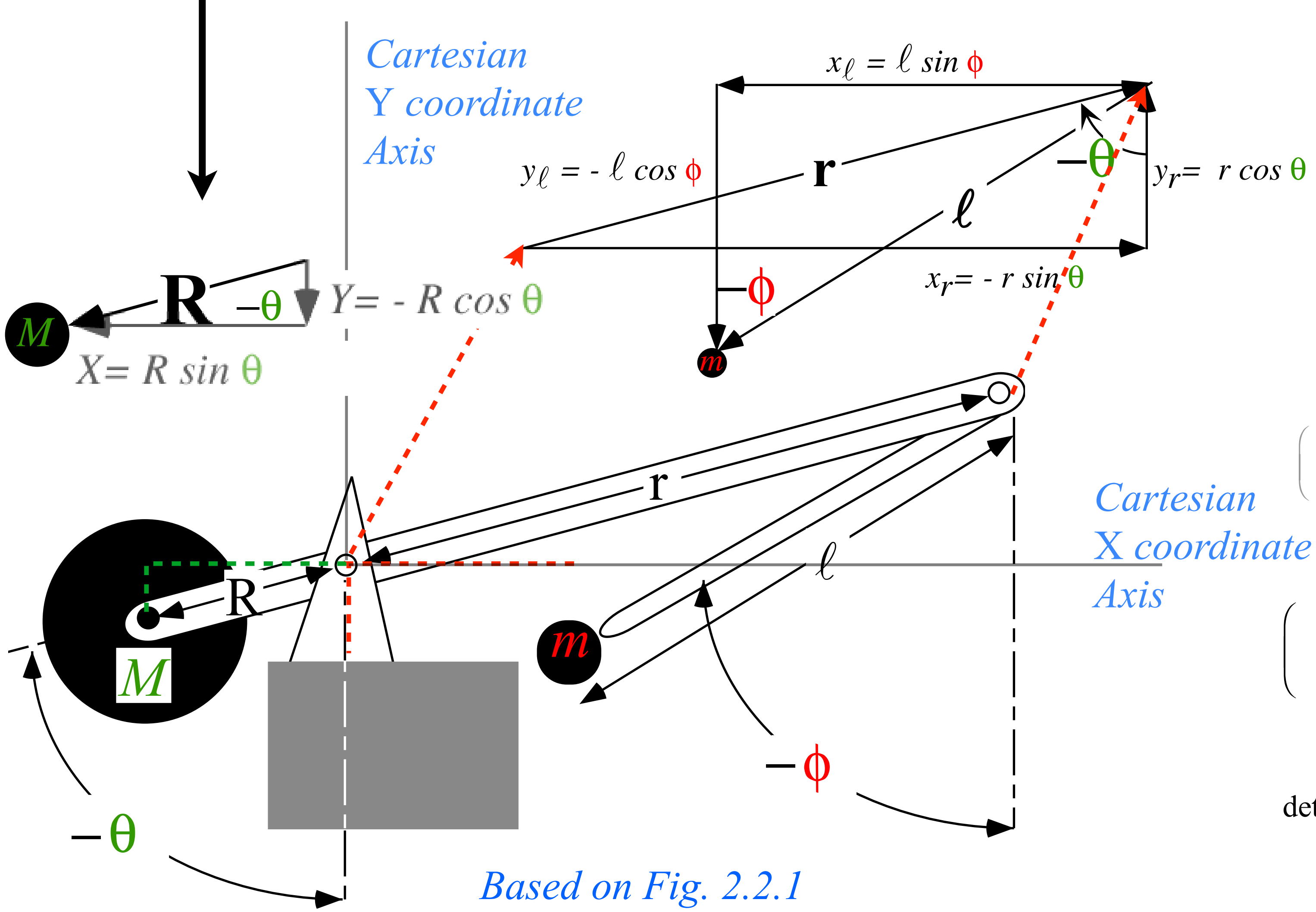
$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$

$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Based on Fig. 2.2.1

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$ FAILS since: $\det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$

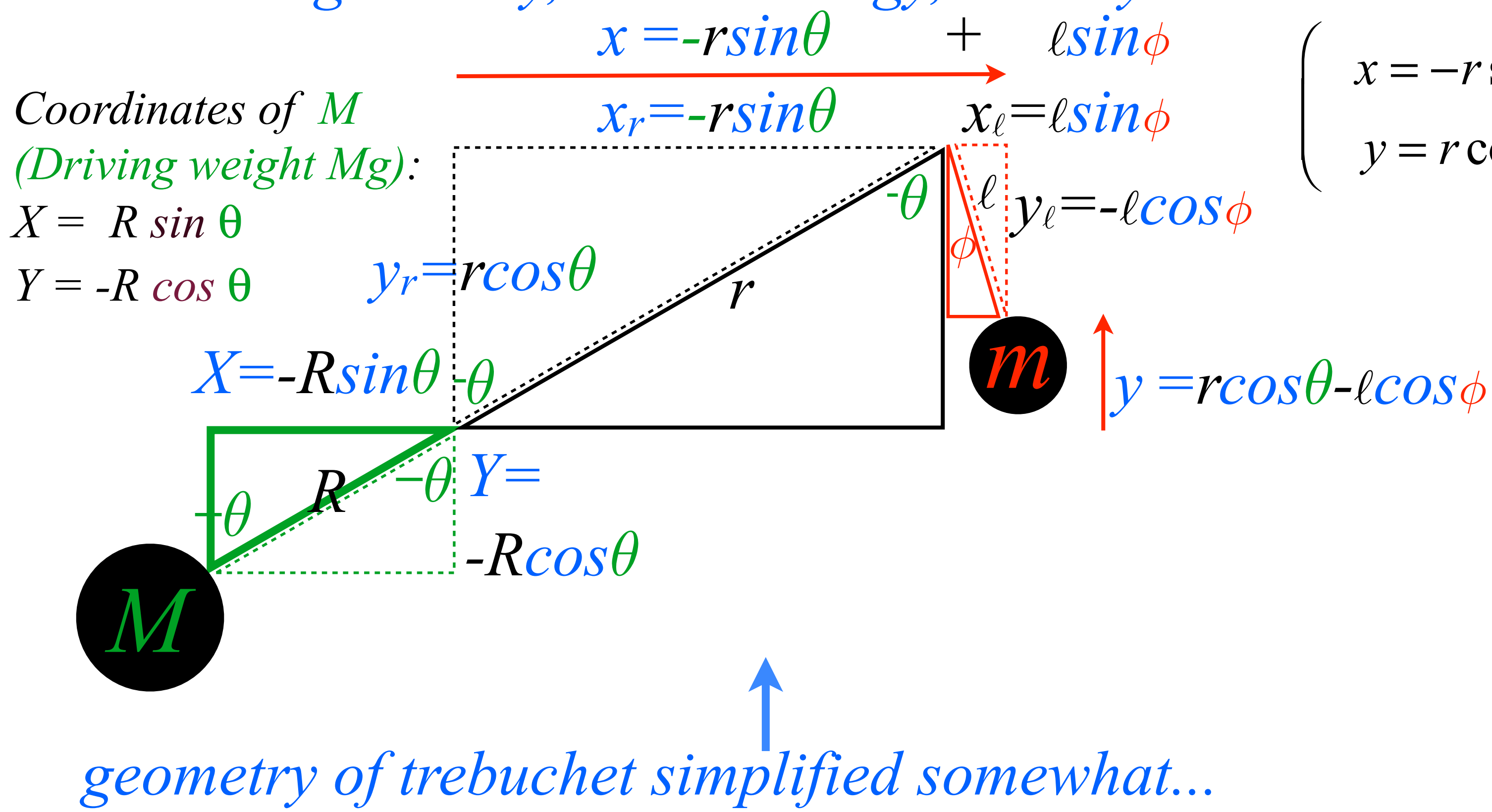
FAILS! (Always singular)

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta - \phi)$$

SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

Constraint relations:

$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

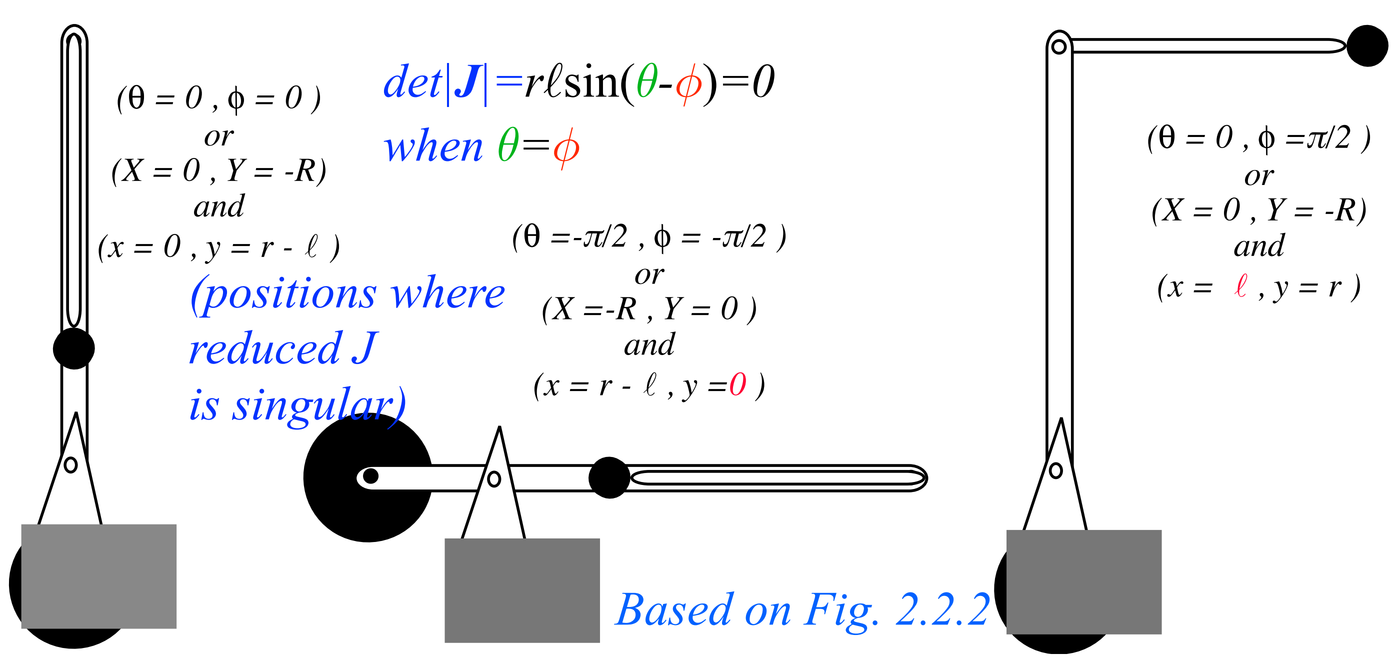
$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$

$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Fig. 2.2.2 Singular positions of the trebuchet



Based on Fig. 2.2.2

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: det} \begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

Jacobian *J*-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

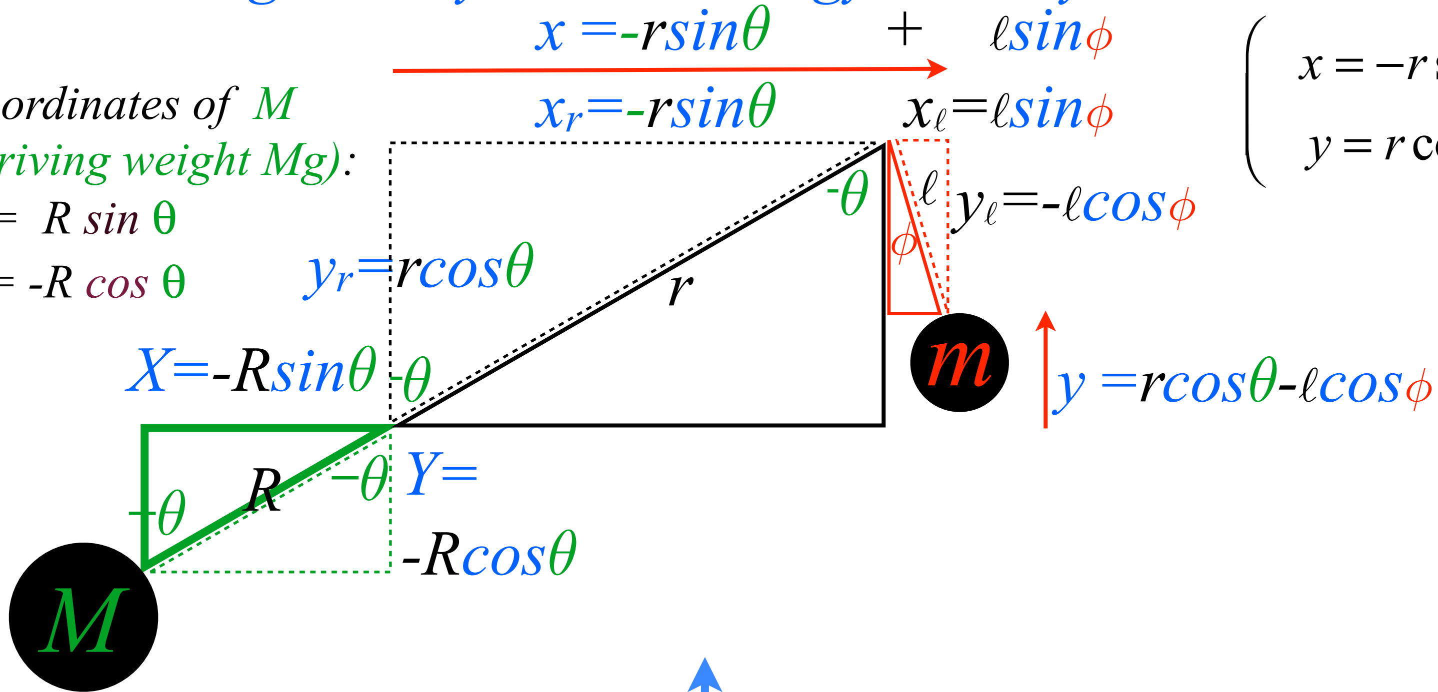
SUCCESS! (Usually non-singular)

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):
$X = R\sin\theta$
$Y = -R\cos\theta$



geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
(Payload or projectile):
$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$
$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

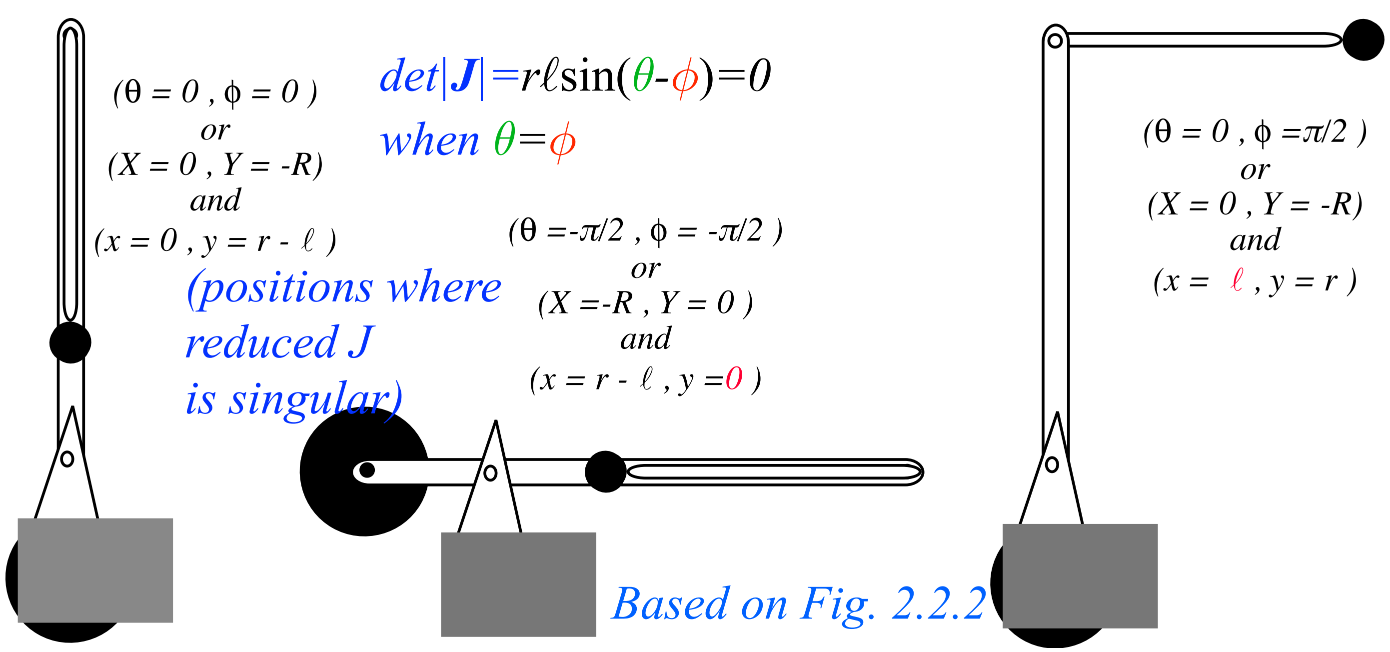
$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

GCC Velocity relations:

$$\dot{X} = R\cos\theta\ \dot{\theta} + 0, \qquad \dot{x} = -r\cos\theta\ \dot{\theta} + \ell\cos\phi\ \dot{\phi}$$

$$\dot{Y} = R\sin\theta\ \dot{\theta} + 0, \qquad \dot{y} = -r\sin\theta\ \dot{\theta} + \ell\sin\phi\ \dot{\phi}$$

Fig. 2.2.2 Singular positions of the trebuchet



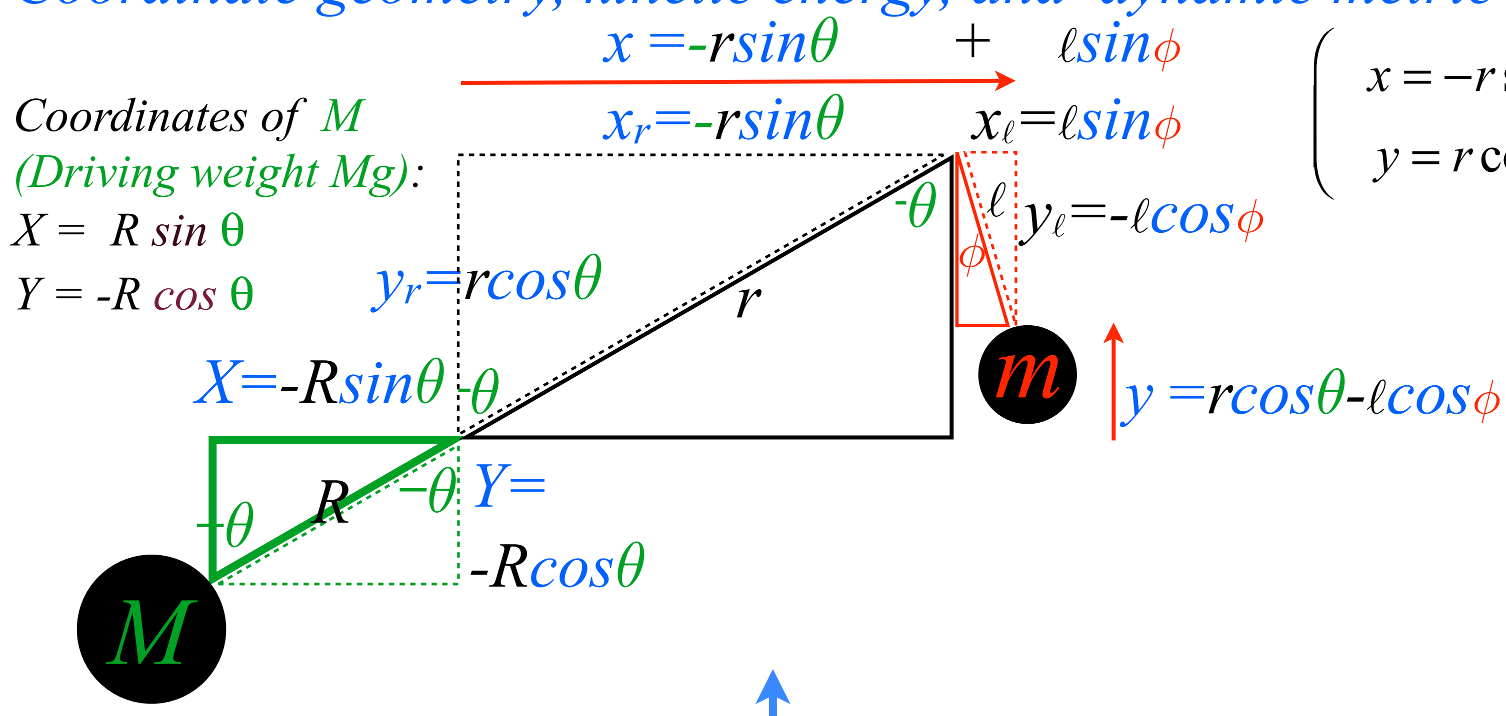
Jacobian
J-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
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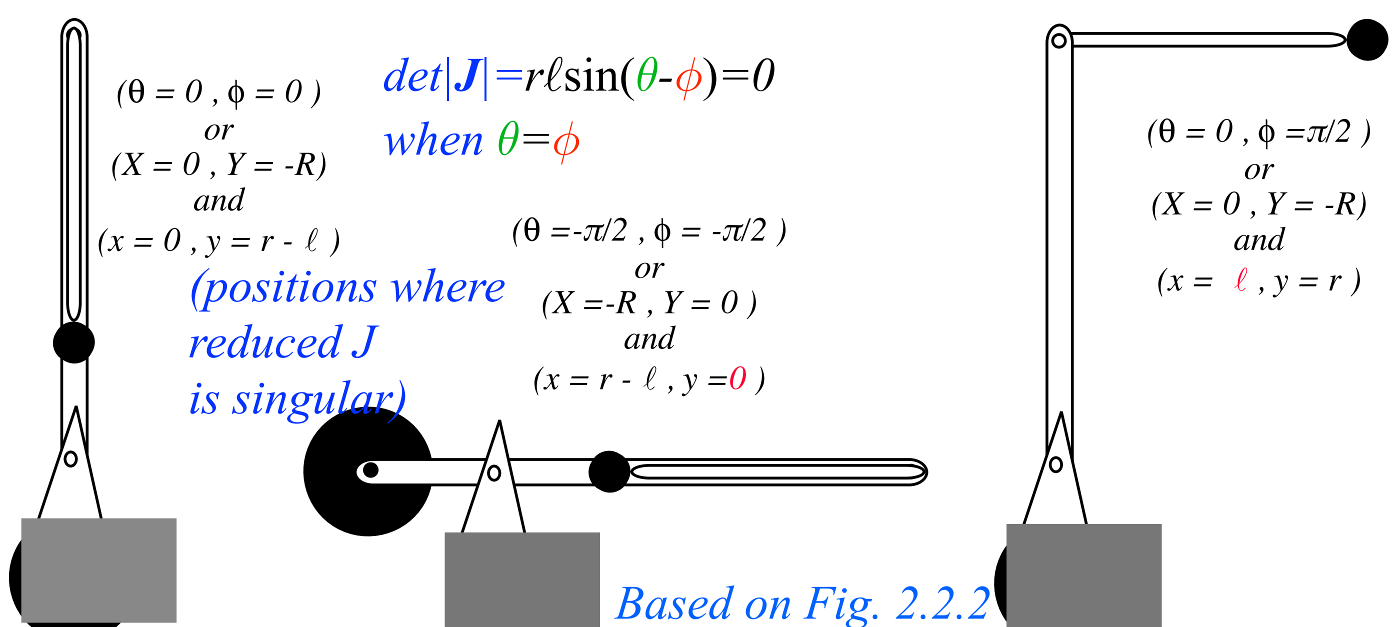
*Jacobian **J**-matrix velocity relations:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

***J**-matrix*

Fig. 2.2.2 Singular positions of the trebuchet



$\det|\boldsymbol{J}| = r\ell\sin(\theta-\phi) = 0$

when $\theta = \phi$

(positions where reduced J is singular)

Based on Fig. 2.2.2

*Jacobian **J**-matrix*

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

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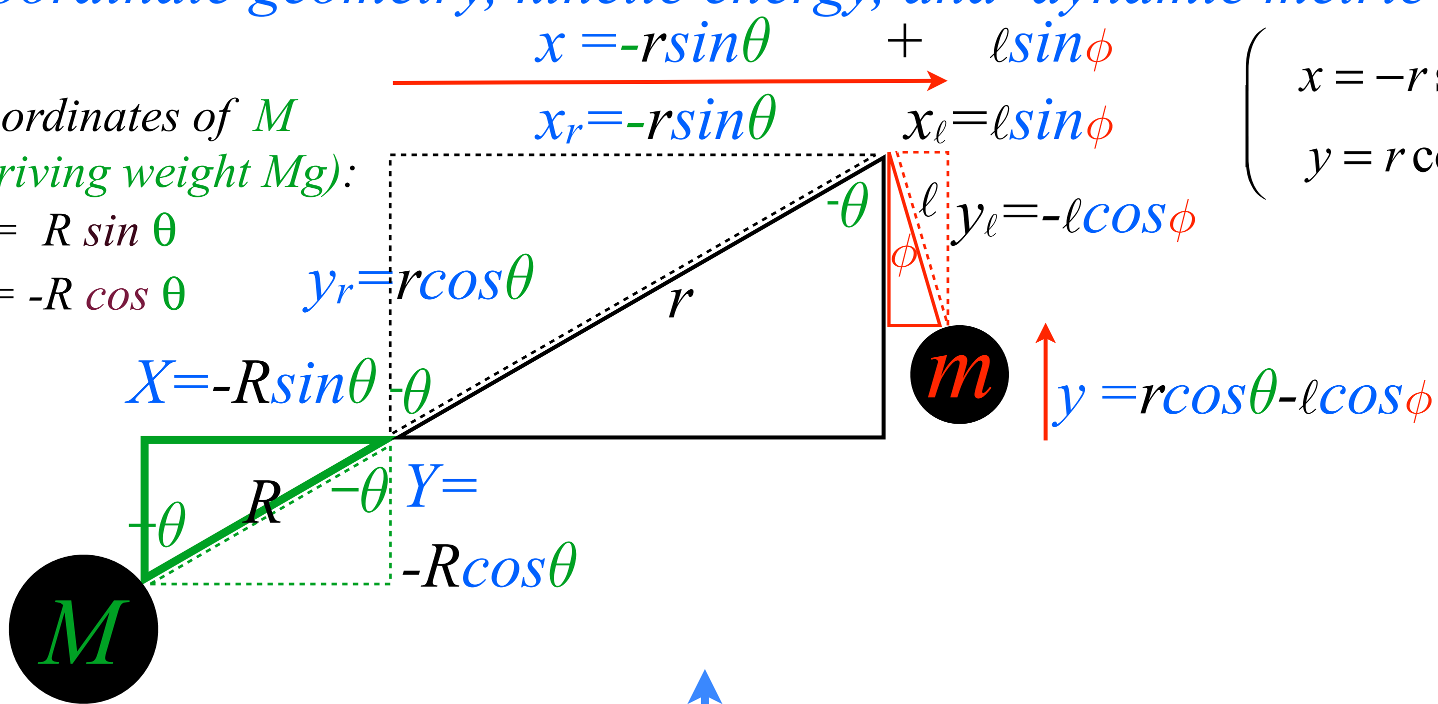

geometry of trebuchet simplified somewhat...

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$$\dot{X} = R\cos\theta \;\; \dot{\theta} + 0, \qquad \dot{x} = -r\cos\theta \;\; \dot{\theta} + \ell\cos\phi \;\; \dot{\phi}$$

$$\dot{Y} = R\sin\theta \;\; \dot{\theta} + 0, \qquad \dot{y} = -r\sin\theta \;\; \dot{\theta} + \ell\sin\phi \;\; \dot{\phi}$$

*Jacobian **J**-matrix velocity relations:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

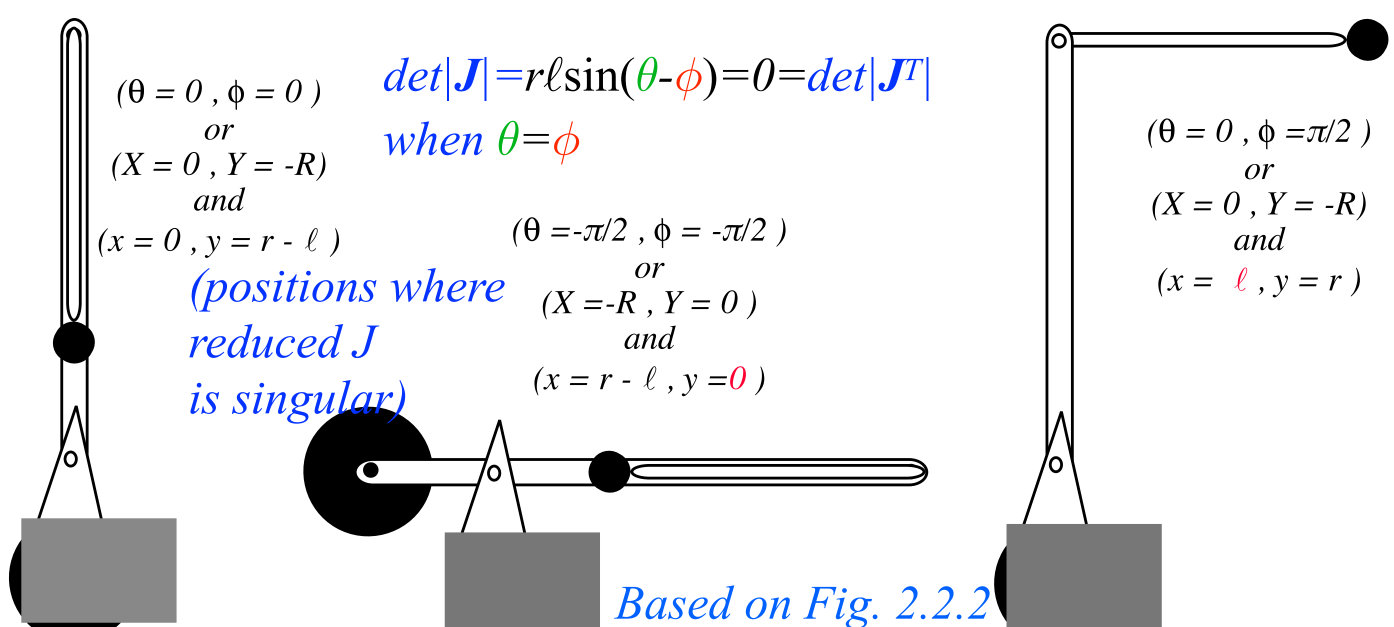
***J**-matrix*

*Transpose Jacobian **J**T-matrix velocity relations:*

***J**T-matrix*

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

Fig. 2.2.2 Singular positions of the trebuchet



*Jacobian **J**-matrix*

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta - \phi)$$

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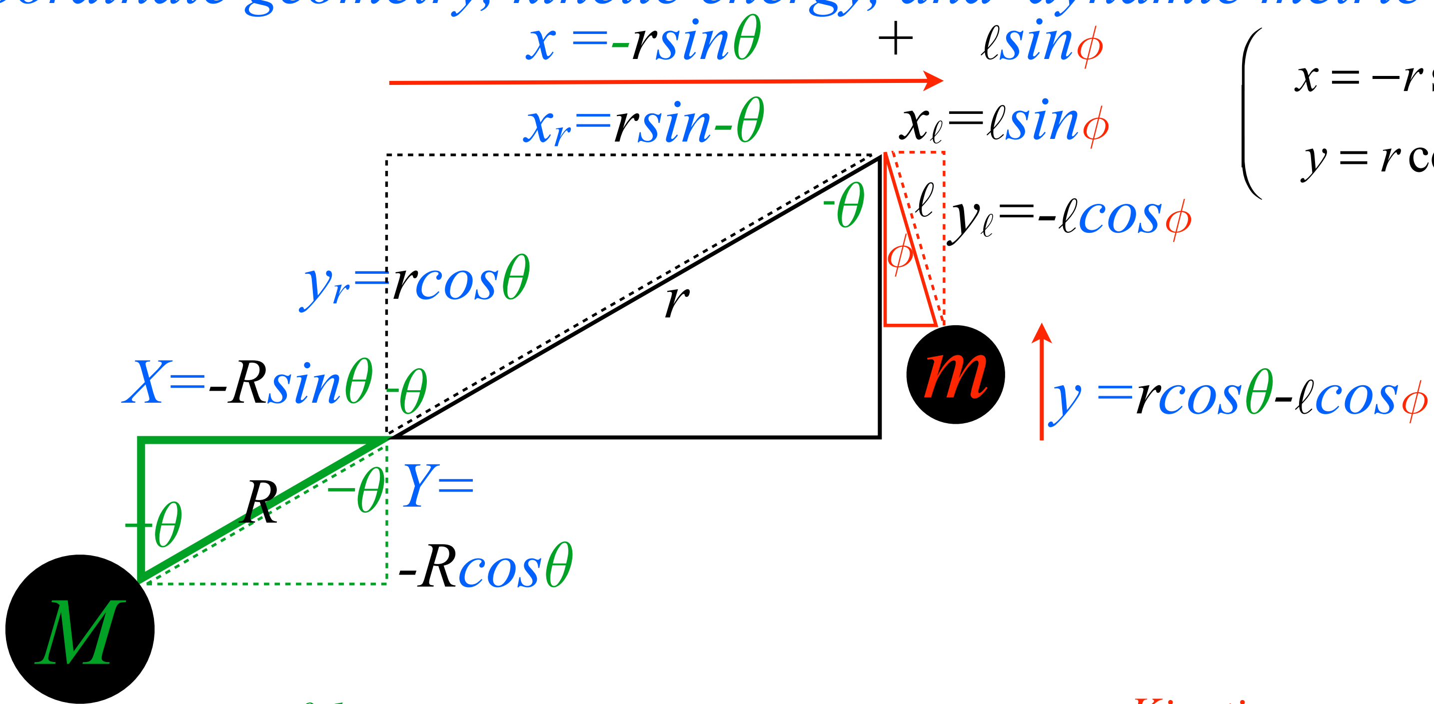
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

***J*-matrix**

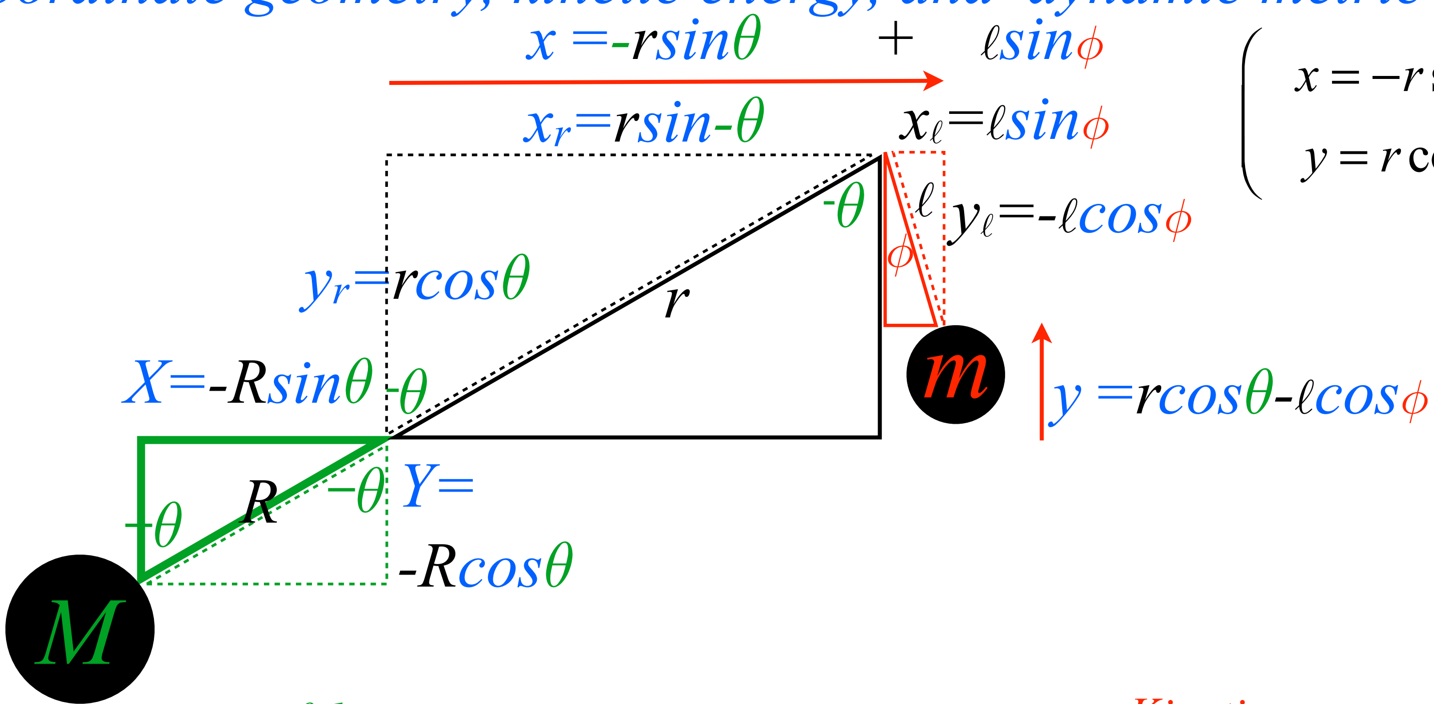
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} *Velocity, Jacobian, and KE*



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

J-*matrix*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J^T-*matrix*

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

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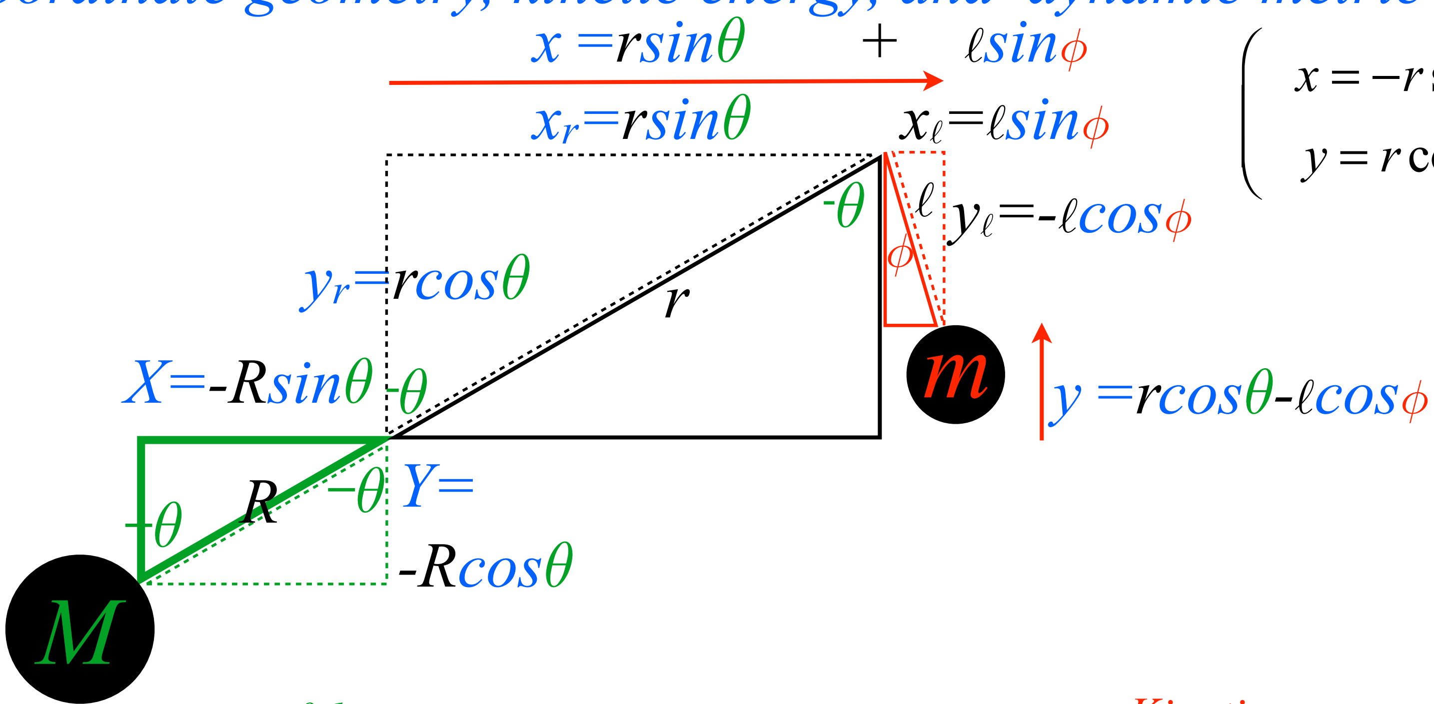
$$T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

J-matrix

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J^T-matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

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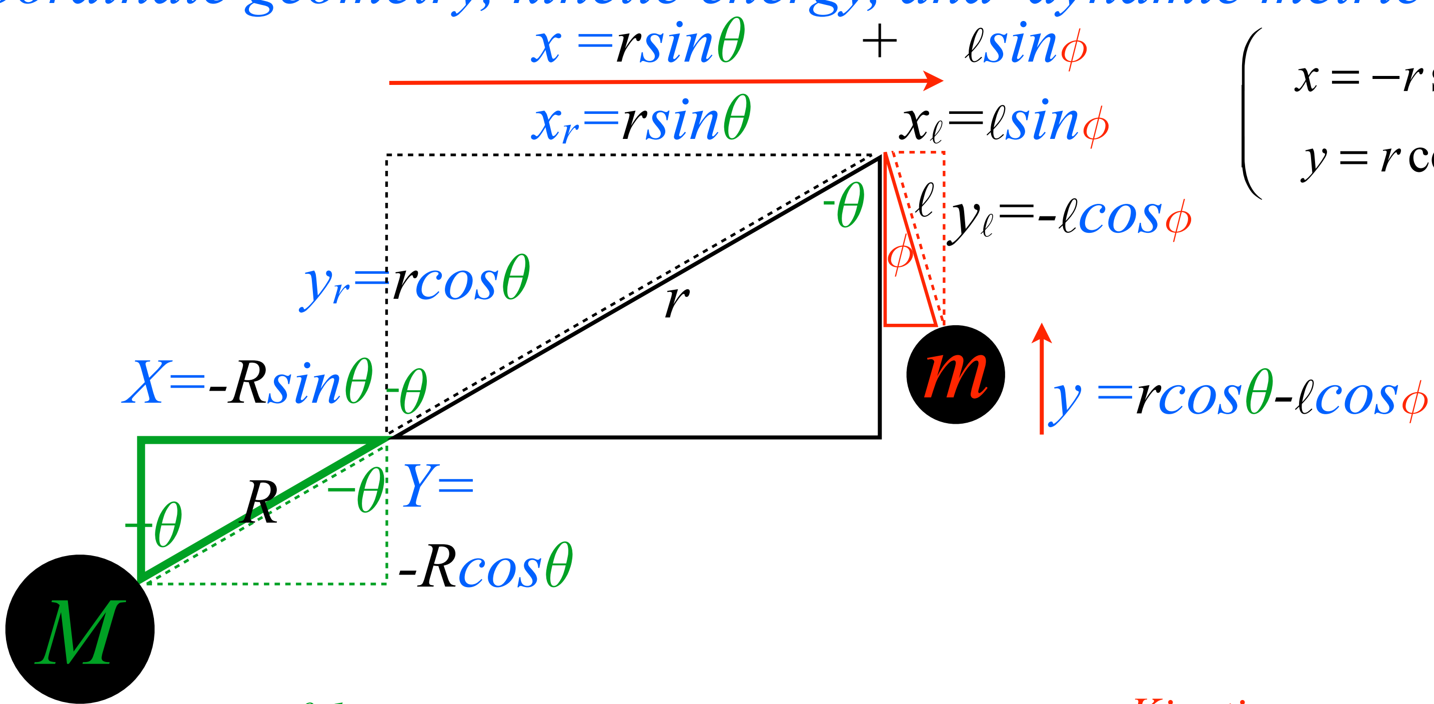
$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}_{J^T\text{-matrix}} \underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{J\text{-matrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

J-matrix

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

J^T-matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

$$= \frac{1}{2} MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

J^T-matrix, **J**-matrix

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\textit{Total KE} = T = T(M) + T(m) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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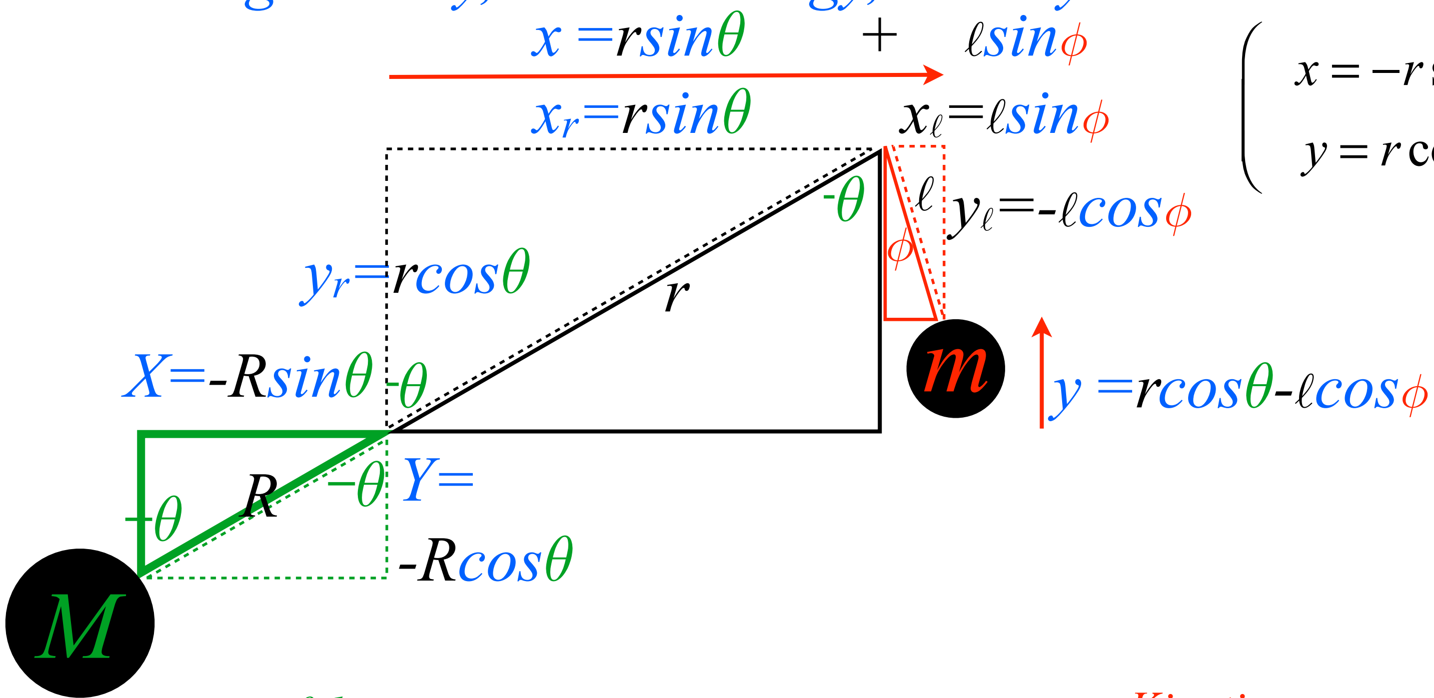
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

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'Raw' Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2 = \frac{1}{2}MR^2\dot{\theta}^2$$

J^T-matrix, *J-matrix*

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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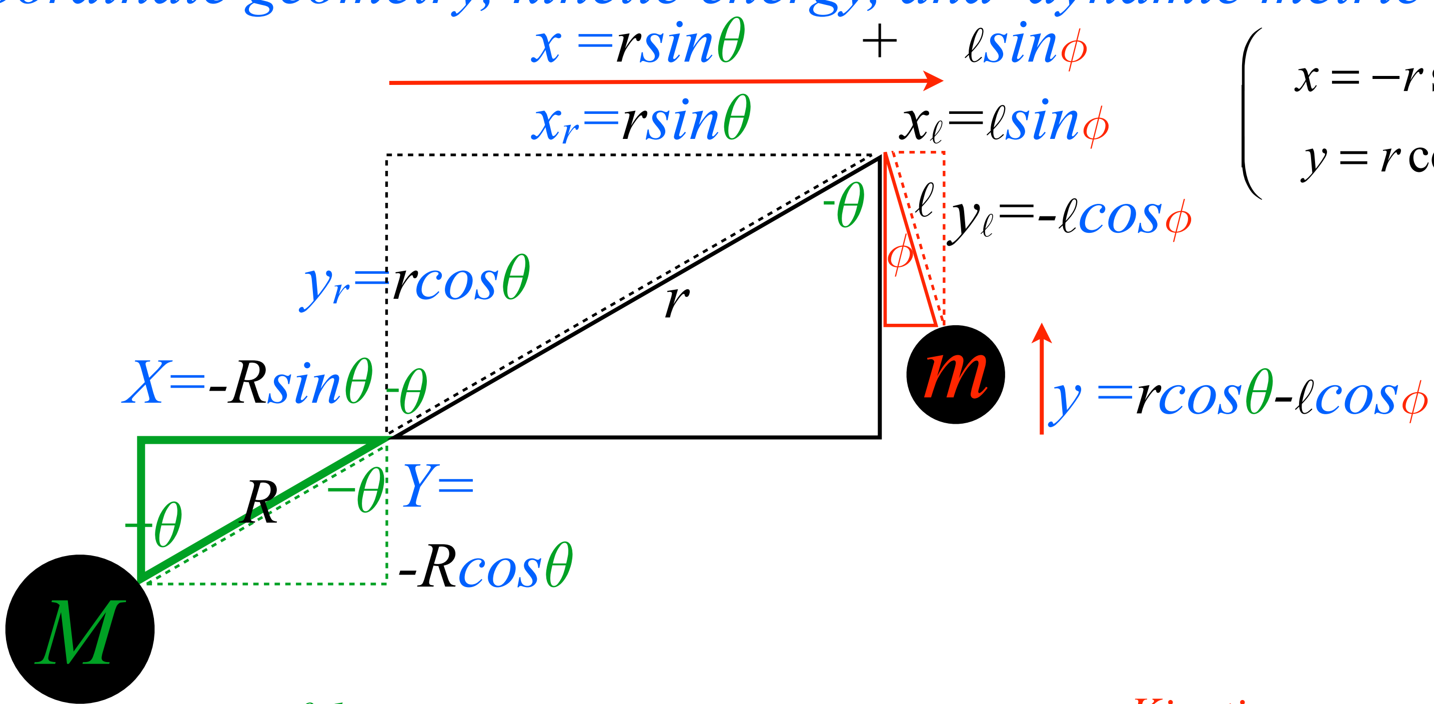
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $\quad T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2 \right]$

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

$$= \frac{1}{2} MR^2\dot{\theta}^2$$

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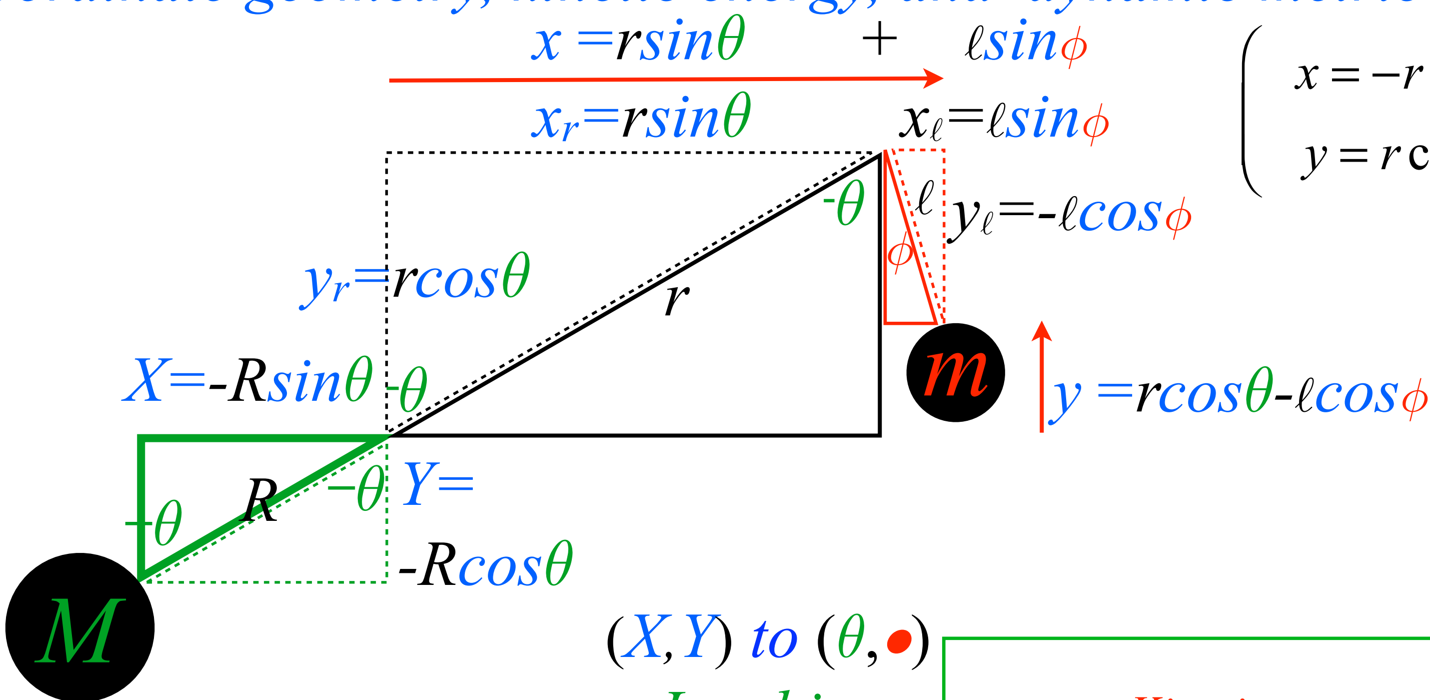
$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (θ,ϕ) Jacobian

(X,Y) to $(\theta,\bullet)$ Jacobian

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

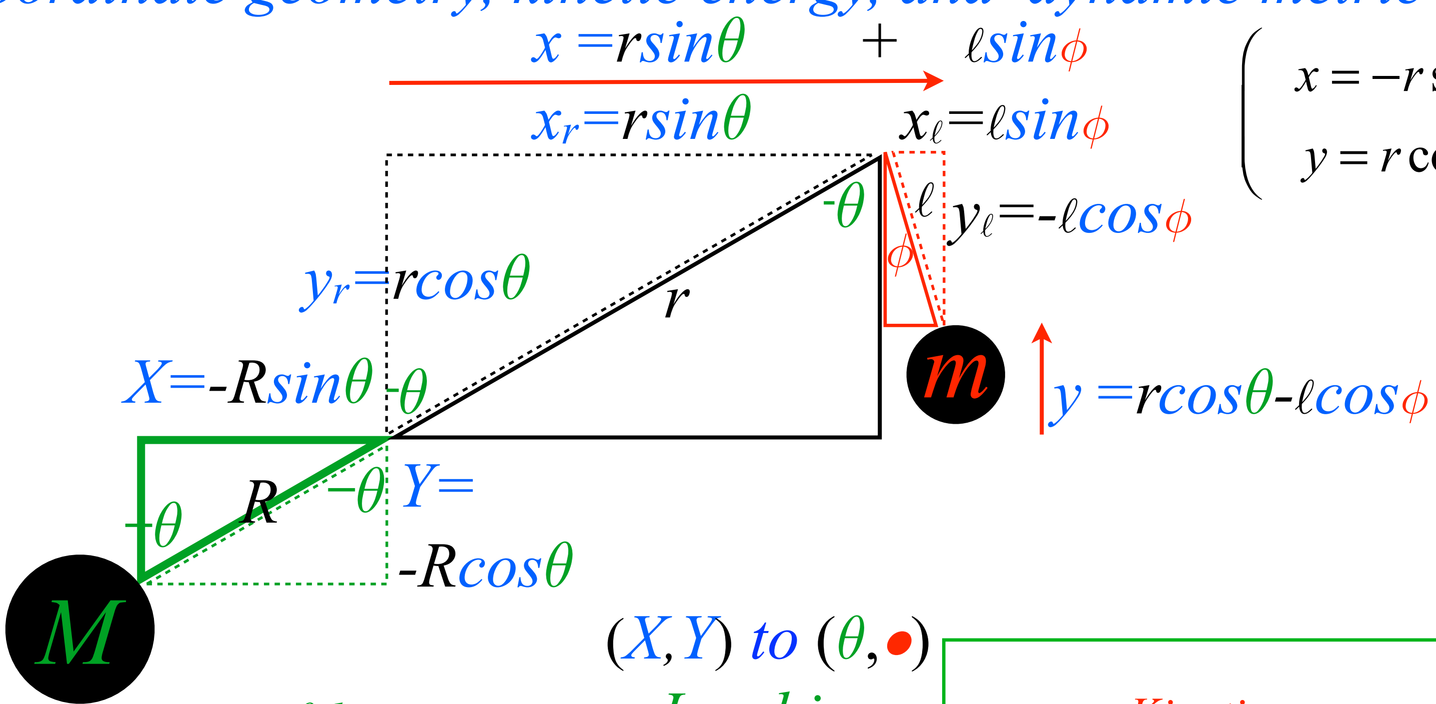
$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(X,Y) to $(\theta,\bullet)$ Jacobian

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2 = \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $\quad T(m)=\frac{1}{2}\left[m\dot{x}^2+m\dot{y}^2\right]$

J^T-matrix $\qquad$ J-matrix

$$T(m)=\frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

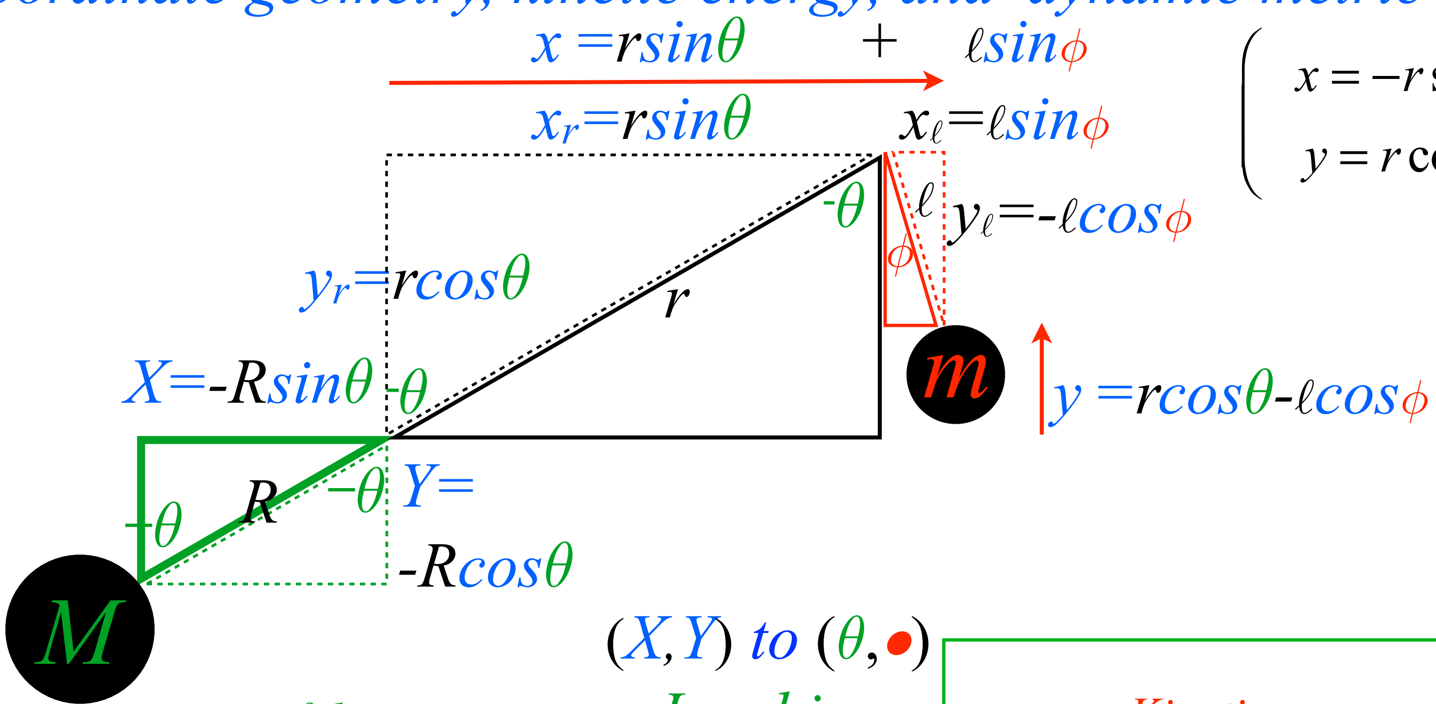
$$\text{Total KE} = T = T(M)+T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2+M\dot{Y}^2+m\dot{x}^2+m\dot{y}^2\right]$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix}$$

Velocity, Jacobian, and KE

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

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(X,Y) to $(\theta,\bullet)$ Jacobian

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Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot X^2+\frac{1}{2}M\dot Y^2$$

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Kinetic energy of projectile m $T(m)=\frac{1}{2}\left[m\dot x^2+m\dot y^2\right]$

J^T-matrix, J-matrix

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot x & \dot y\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

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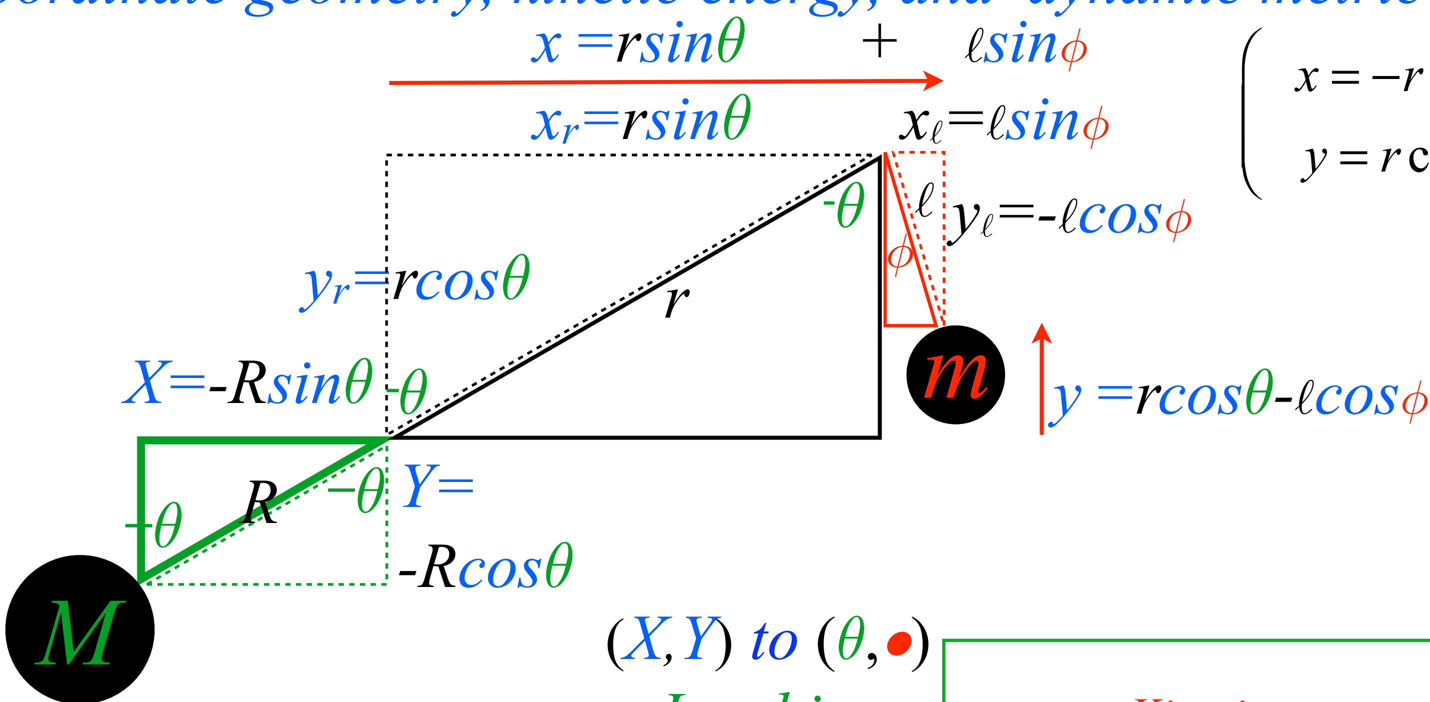
$$\text{Total KE}=T=\frac{1}{2}\left[M\dot X^2+M\dot Y^2+m\dot x^2+m\dot y^2\right]$$

$$T=\frac{1}{2}\left[(MR^2+mr^2)\dot\theta^2-2mr\ell\cos(\theta-\phi)\,\dot\theta\dot\phi+m\ell^2\dot\phi^2\right]$$

$$\begin{pmatrix}\gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y}\end{pmatrix}=\begin{pmatrix}M & & & \\ & M & & \\ & & m & \\ & & & m\end{pmatrix}$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Velocity, Jacobian, and KE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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(X,Y) to $(\theta,\bullet)$ Jacobian

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Kinetic energy of driver M

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J^T-matrix　　J-matrix

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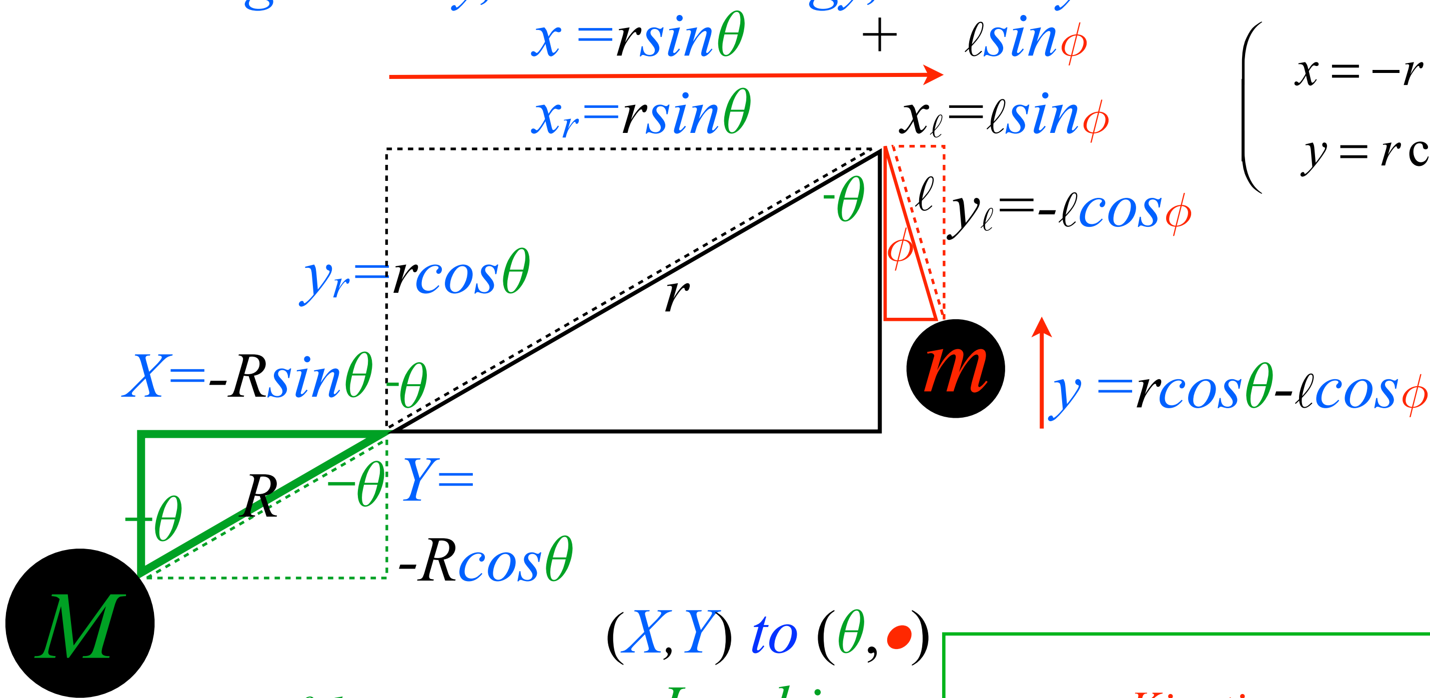
Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Velocity, Jacobian, and KE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(X,Y) to $(\theta,\bullet)$ Jacobian

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

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$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

$$T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right] = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

J^T-matrix, J-matrix

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta\text{-}\phi) \\ -mr\ell\cos(\theta\text{-}\phi) & m\ell^2 \end{pmatrix} = \begin{pmatrix} MR^2 & 0 \\ 0 & 0 \end{pmatrix} + m\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}$$

Dynamic metric tensor γ_{mn} in GCC θ and ϕ

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$= \frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

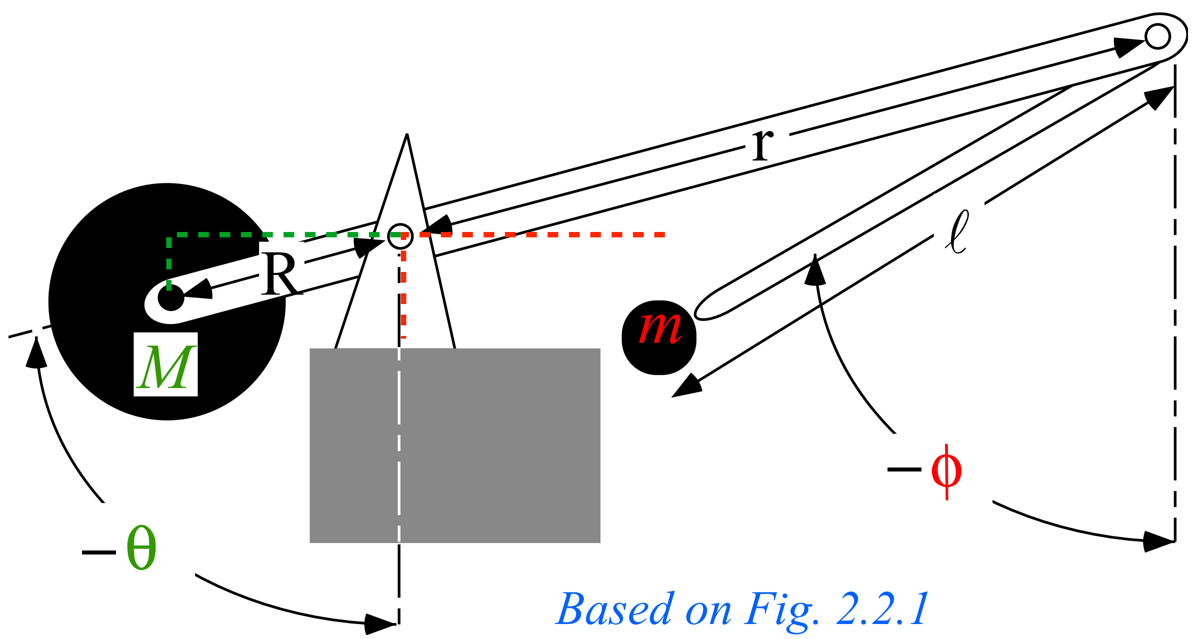
$$= \frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta + r^2\sin^2\theta & -r\ell\cos\theta\cos\phi - r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta - r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi + \ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total } KE = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2 + mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$



Based on Fig. 2.2.1

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

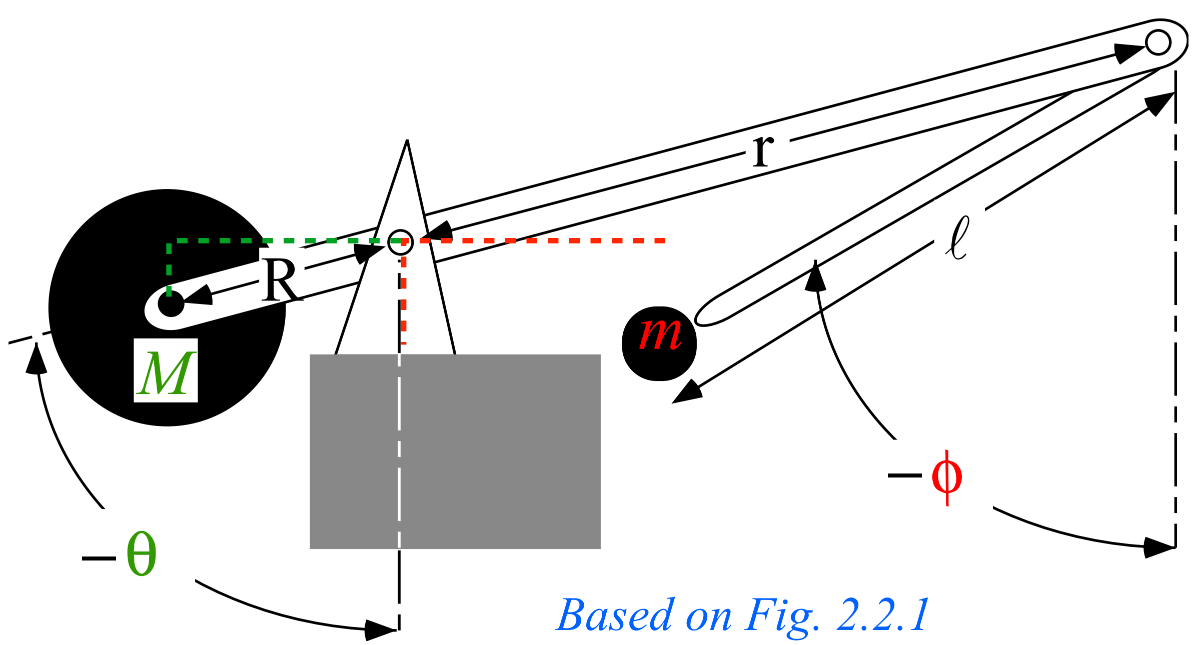
$$\text{Total } KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor $\gamma_{mn}=\sum_{mass\ \mu}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

$$=\sum_{mass\ \mu}m(\mu)\frac{\partial\mathbf{r}(\mu)}{\partial q^m}\bullet\frac{\partial\mathbf{r}(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\mathbf{E_m}(\mu)\bullet\mathbf{E_n}(\mu)$$



$$KE=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\dot{x}^j(\mu)\dot{x}^j(\mu)=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}\dot{q}^m\dot{q}^n$$

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Based on Fig. 2.2.1

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

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Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

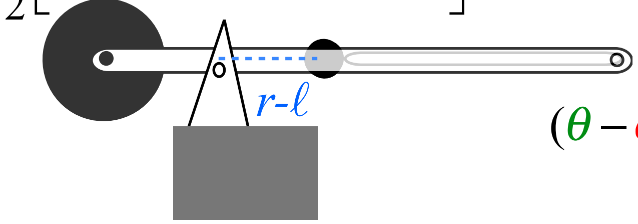
$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

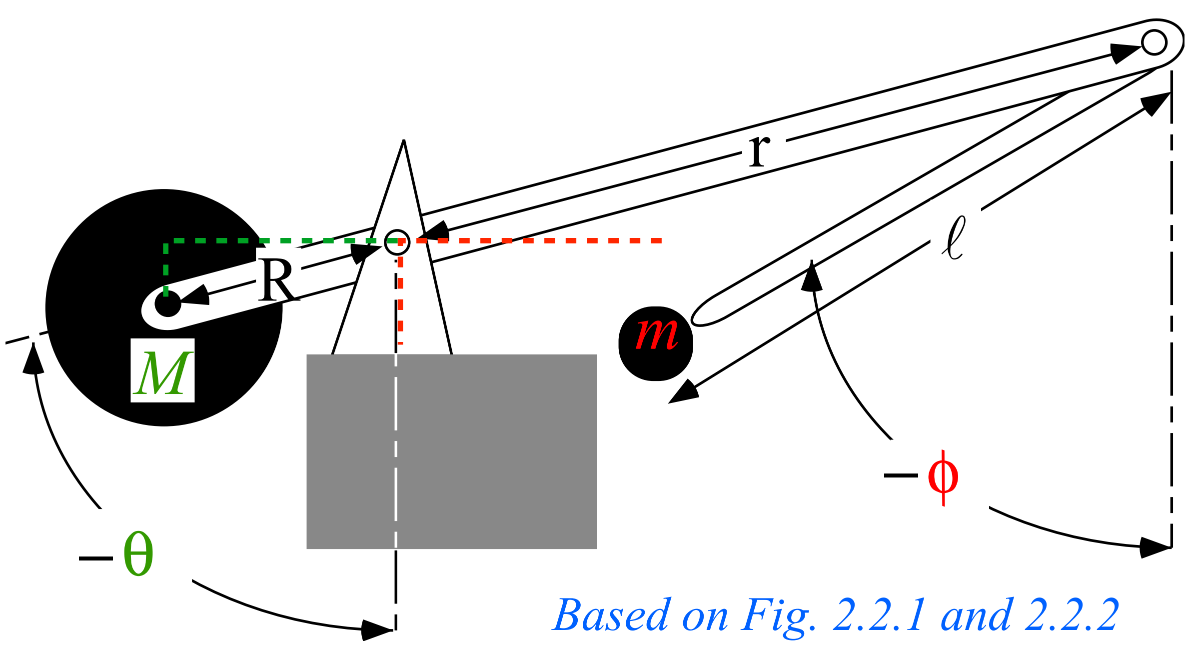
Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$



(J is Singular)

$(\theta-\phi)=0$



Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

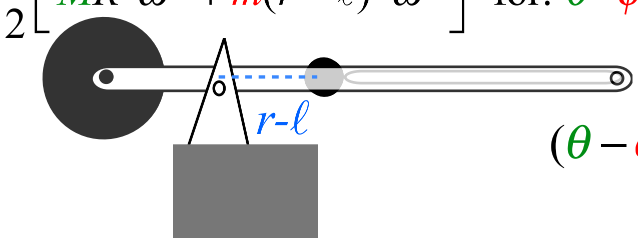
$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

Special cases (rigid rotation)

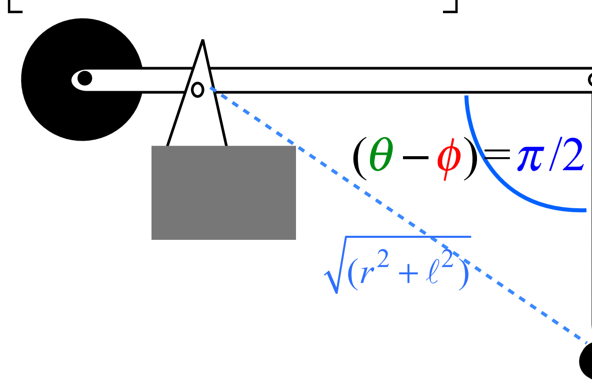
$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$



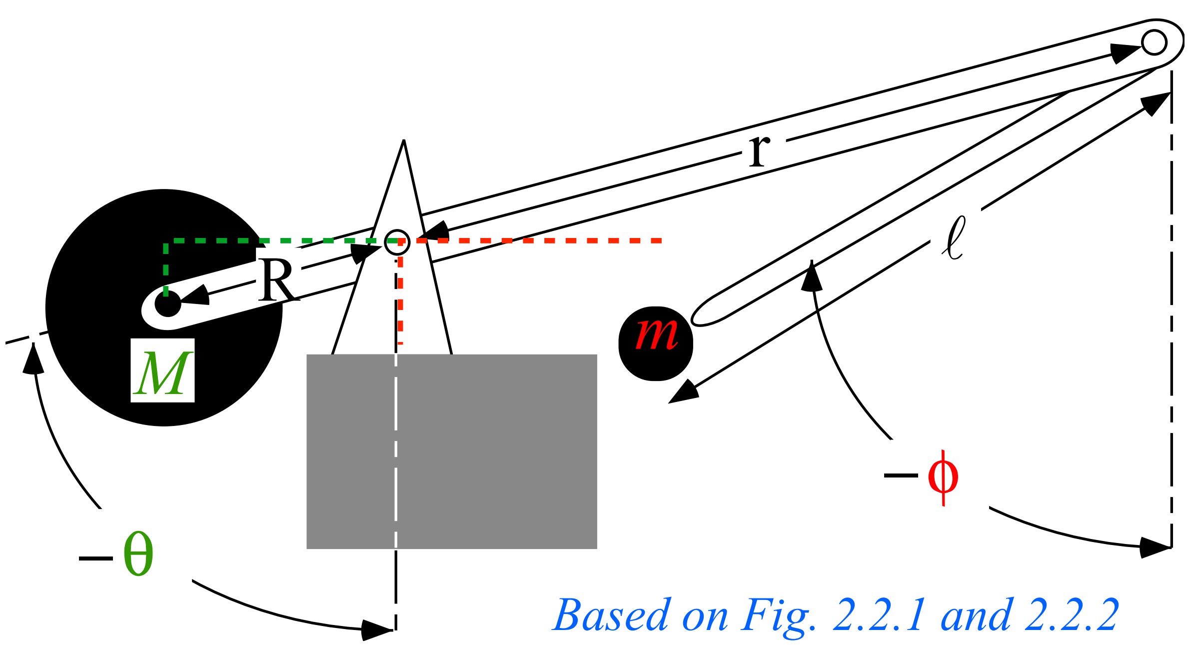
(J is Singular)

$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=\pi/2$$



(J is Orthogonal)



Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

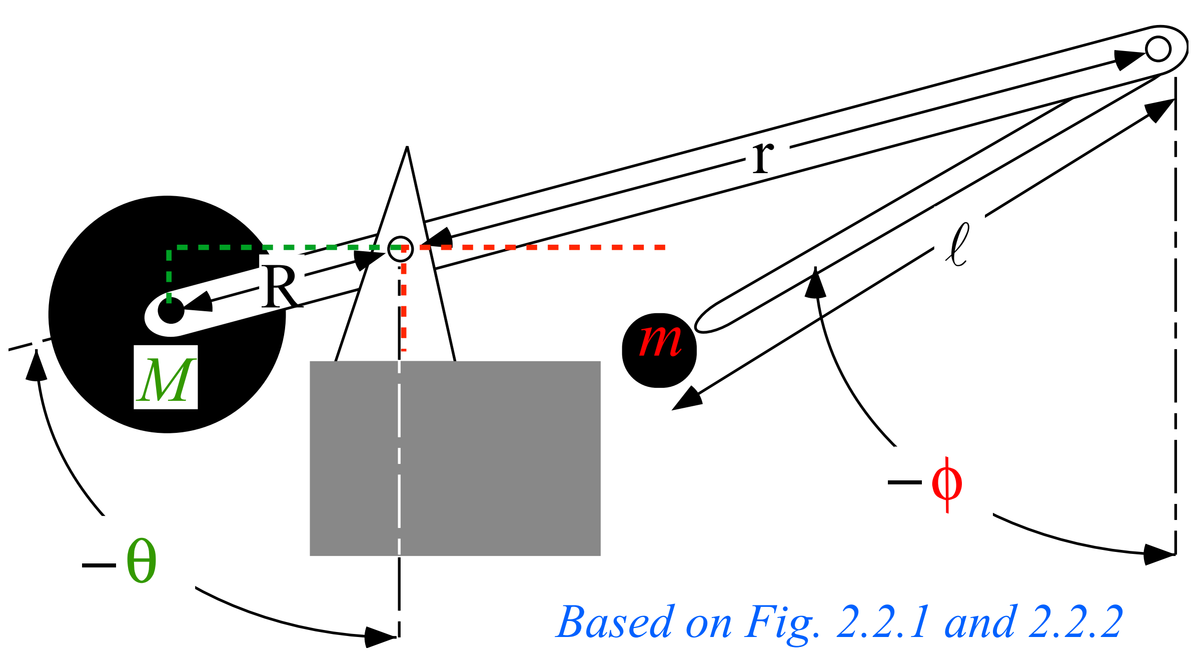
$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total } KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

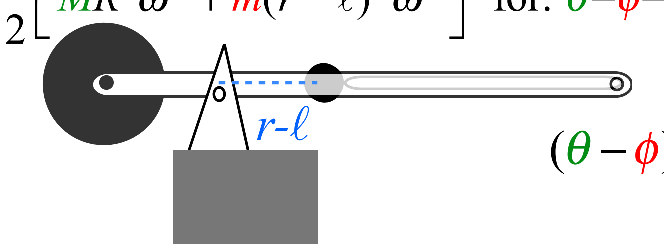
$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$



Based on Fig. 2.2.1 and 2.2.2

Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

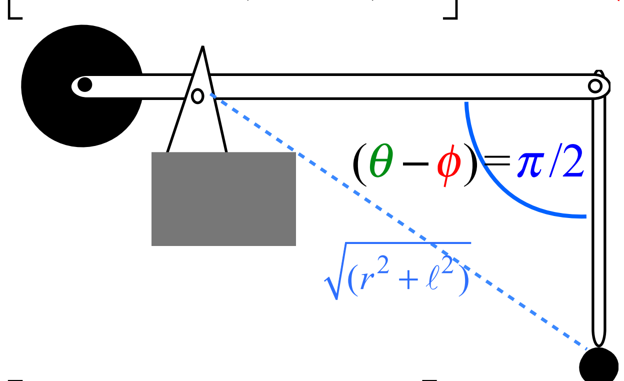


(J is Singular)

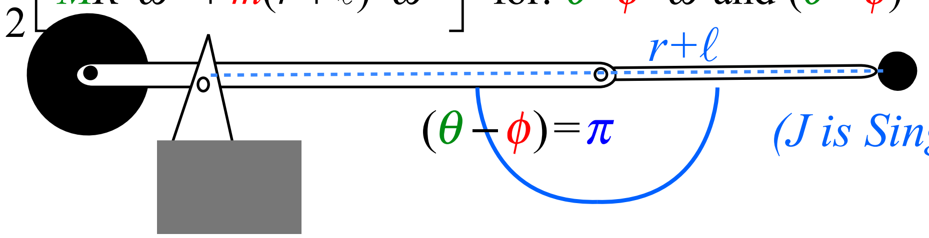
$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

(J is Orthogonal)



$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$



(J is Singular)

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

SPECIAL CASE

(a) When (θ,ϕ) coordinates are orthogonal

USUAL CASE

(b) When (θ,ϕ) coordinates are not orthogonal

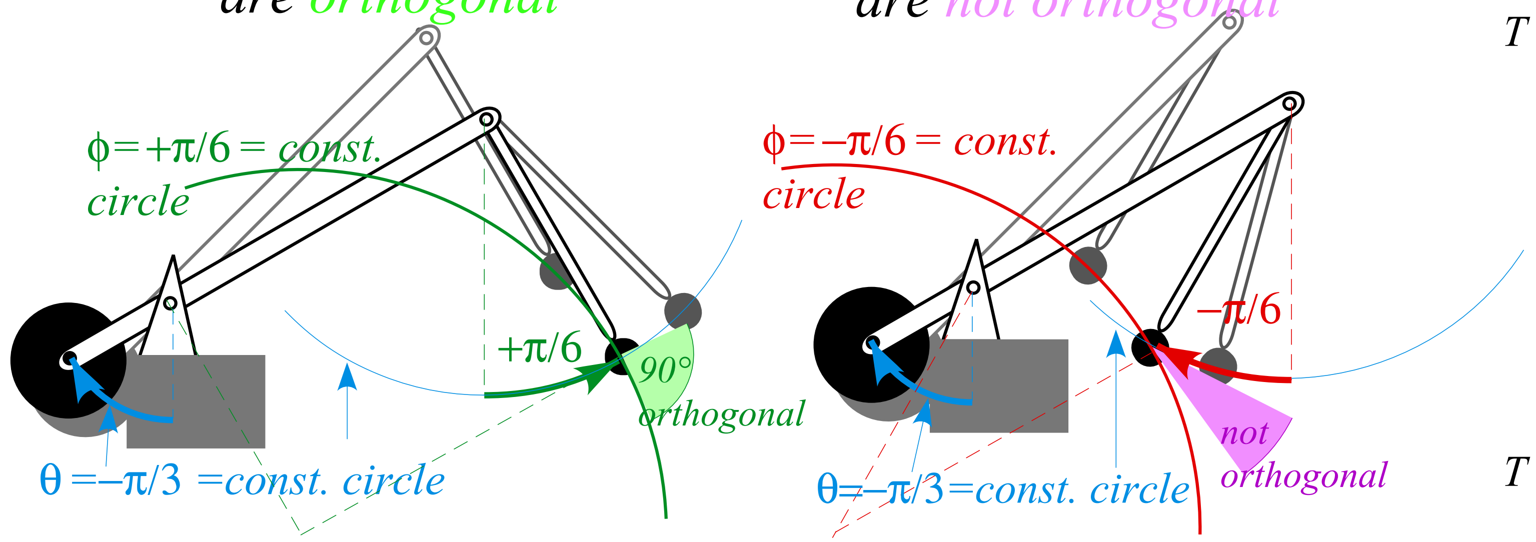


Fig. 2.3.1 Examples of (θ,ϕ) intersections (a) othogonal (special case), (b) non-orthogonal (typical).

Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

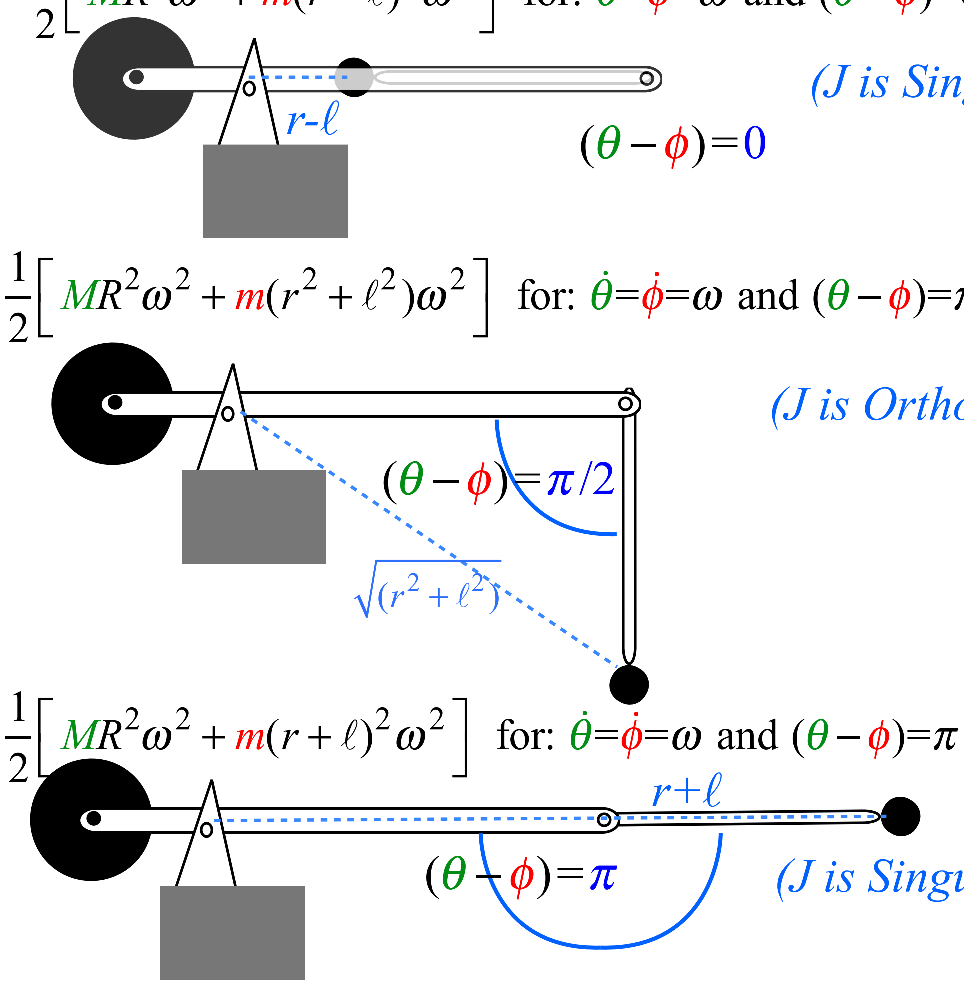
(J is Singular)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

(J is Orthogonal)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$

(J is Singular)



Based on Fig. 2.3.1 and 2.2.2

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- → *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

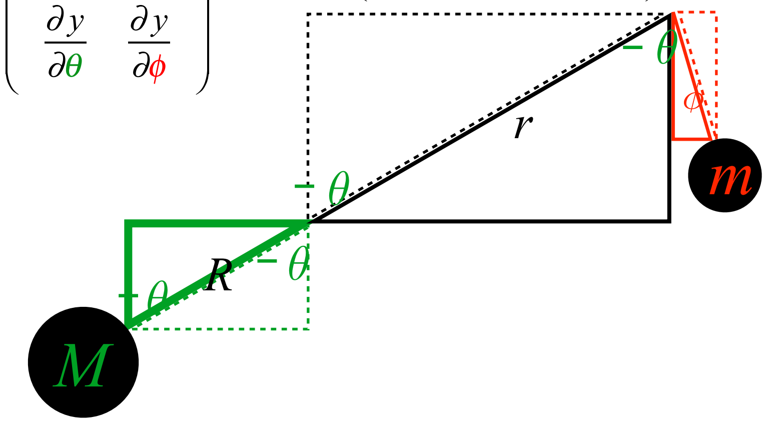
Write work-sums in columns:

$$dW = F_X\ dX \quad = \quad M\ddot{X}\,dX$$

$$+ F_Y\ dY \qquad + M\ddot{Y}\,dY$$

$$+ F_x\ dx \qquad + m\ddot{x}\,dx$$

$$+ F_y\ dy \qquad + m\ddot{y}\,dy$$



Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

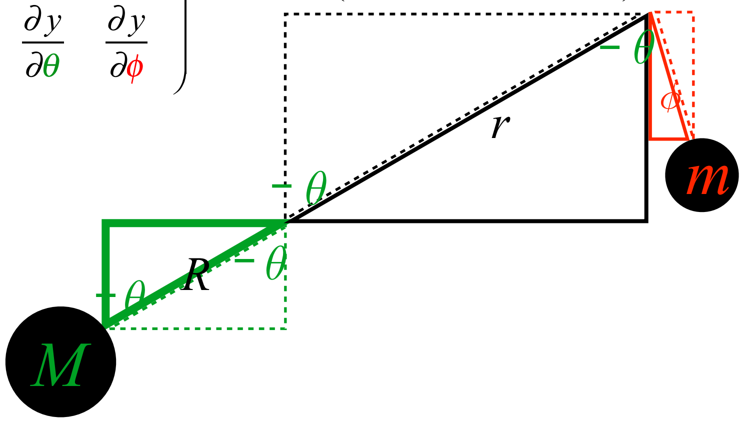
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \ & F_X\ dX && = M\ddot{X}\,dX && = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi && = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
& + F_Y\ dY && + M\ddot{Y}\,dY && + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
& + F_x\ dx && + m\ddot{x}\,dx && + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
& + F_y\ dy && + m\ddot{y}\,dy && + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$



Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

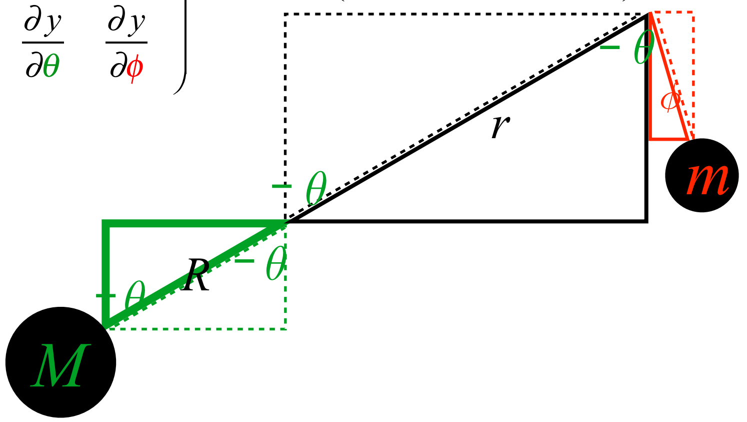
$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX \quad &= \quad M\ddot{X}\, dX \quad = \quad F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi \quad = \quad M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
+ F_Y\, dY \quad & \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
+ F_x\, dx \quad & \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
+ F_y\, dy \quad & \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$



Assuming variables θ and ϕ are independent...

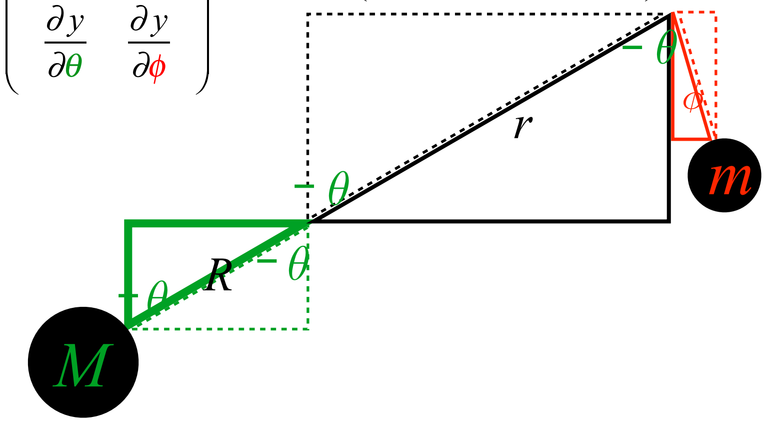
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

Assuming variables θ and ϕ *are independent...*

Set: $d\theta=1 \quad d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} \quad + m\ddot{x} \frac{\partial x}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} \quad + m\ddot{y} \frac{\partial y}{\partial \theta}$$

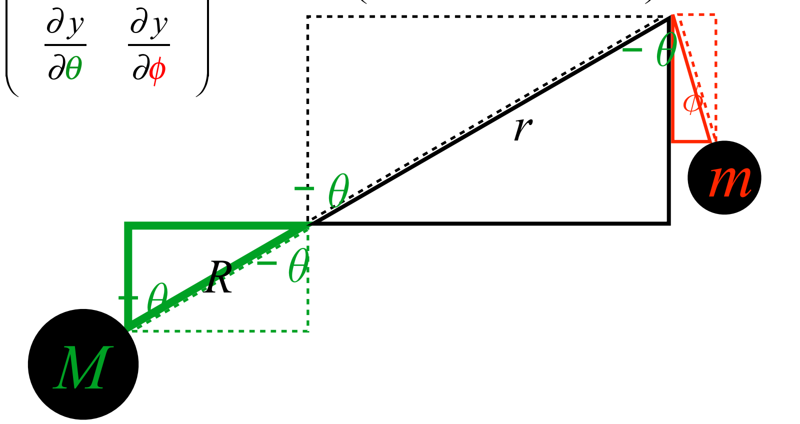
Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \ & F_X\ dX & = \ & M\ddot{X}\,dX & = \ & F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi & = \ & M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
& +F_Y\ dY & & +M\ddot{Y}\,dY & & +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi & & +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
& +F_x\ dx & & +m\ddot{x}\,dx & & +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi & & +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
& +F_y\ dy & & +m\ddot{y}\,dy & & +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi & & +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Assuming variables θ and ϕ are independent...

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} & = M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} & \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} & \quad +m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} & \quad +m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} & = M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} & \quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} & \quad +m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} & \quad +m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

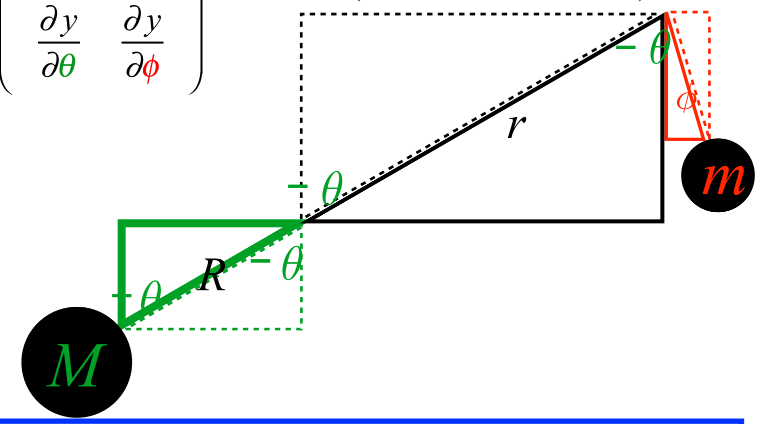
Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC dθ and dφ in Jacobian)

$$\begin{aligned}
dW = F_X\ dX &= M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\ dY &\quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\ dx &\quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\ dy &\quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

$$\text{(using } \frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}\text{)}$$

STEP **A**

Set: dθ=1 dφ=0

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad + m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad + m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: dθ=0 dφ=1

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad + m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad + m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

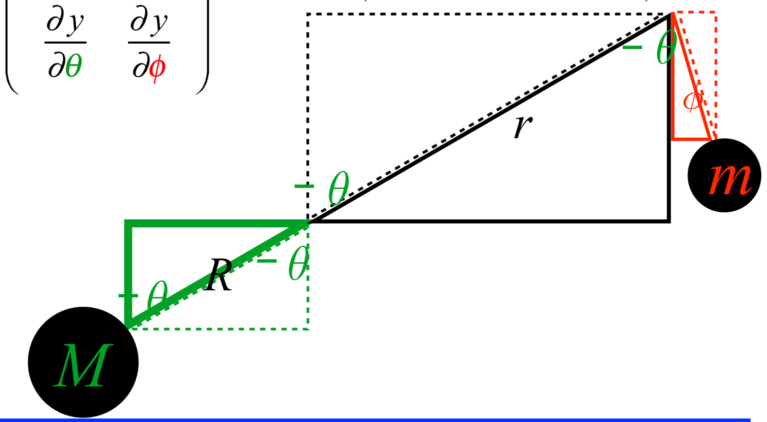
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC dθ and dϕ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\, dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\, dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\, dy \quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial\dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial\dot{X}}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **A**

STEP **B**

by *lemma 1*: $\frac{\partial\dot{X}}{\partial\dot{q}} = \frac{\partial X}{\partial q}$

and *lemma 2*: $\frac{\partial\dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

Lemmas from Lect.9

lemma 1: p.9.13

lemma 2: p.9.24

Set: dθ=1 dϕ=0

$$F_X\frac{\partial X}{\partial\theta} = M\ddot{X}\frac{\partial X}{\partial\theta}$$

$$+F_Y\frac{\partial Y}{\partial\theta} \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}$$

$$+F_x\frac{\partial x}{\partial\theta} \quad +m\ddot{x}\frac{\partial x}{\partial\theta}$$

$$+F_y\frac{\partial y}{\partial\theta} \quad +m\ddot{y}\frac{\partial y}{\partial\theta}$$

Set: dθ=0 dϕ=1

$$F_X\frac{\partial X}{\partial\phi} = M\ddot{X}\frac{\partial X}{\partial\phi}$$

$$+F_Y\frac{\partial Y}{\partial\phi} \quad +M\ddot{Y}\frac{\partial Y}{\partial\phi}$$

$$+F_x\frac{\partial x}{\partial\phi} \quad +m\ddot{x}\frac{\partial x}{\partial\phi}$$

$$+F_y\frac{\partial y}{\partial\phi} \quad +m\ddot{y}\frac{\partial y}{\partial\phi}$$

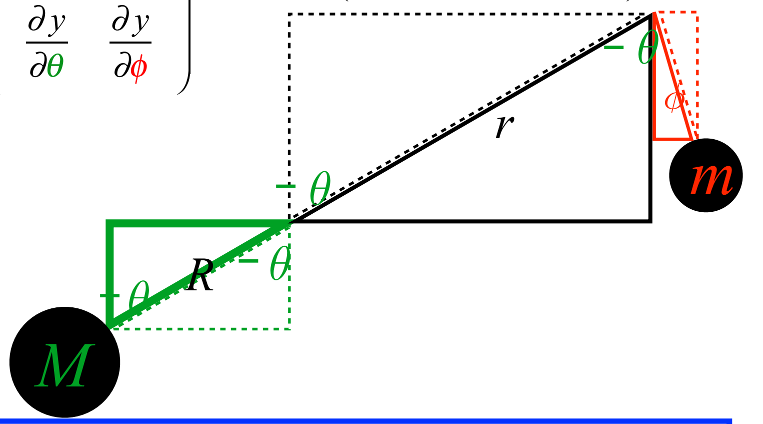
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\, dY \quad +M\ddot{Y}\, dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\, dx \quad +m\ddot{x}\, dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\, dy \quad +m\ddot{y}\, dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Lagrange trickery:

STEP **A**

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **B**

$$= \frac{d}{dt}\left(\dot{X}\frac{\partial\dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial\dot{X}}{\partial\theta}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial\dot{X}}{\partial\dot{q}}$

and lemma 2: $\frac{\partial\dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP **C**

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial\dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial\theta}$$

(using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

Set: $d\theta=1 \quad d\phi=0$

$$F_X\frac{\partial X}{\partial\theta} = M\ddot{X}\frac{\partial X}{\partial\theta}$$

$$+F_Y\frac{\partial Y}{\partial\theta} \qquad +M\ddot{Y}\frac{\partial Y}{\partial\theta}$$

$$+F_x\frac{\partial x}{\partial\theta} \qquad +m\ddot{x}\frac{\partial x}{\partial\theta}$$

$$+F_y\frac{\partial y}{\partial\theta} \qquad +m\ddot{y}\frac{\partial y}{\partial\theta}$$

Set: $d\theta=0 \quad d\phi=1$

$$F_X\frac{\partial X}{\partial\phi} = M\ddot{X}\frac{\partial X}{\partial\phi}$$

$$+F_Y\frac{\partial Y}{\partial\phi} \qquad +M\ddot{Y}\frac{\partial Y}{\partial\phi}$$

$$+F_x\frac{\partial x}{\partial\phi} \qquad +m\ddot{x}\frac{\partial x}{\partial\phi}$$

$$+F_y\frac{\partial y}{\partial\phi} \qquad +m\ddot{y}\frac{\partial y}{\partial\phi}$$

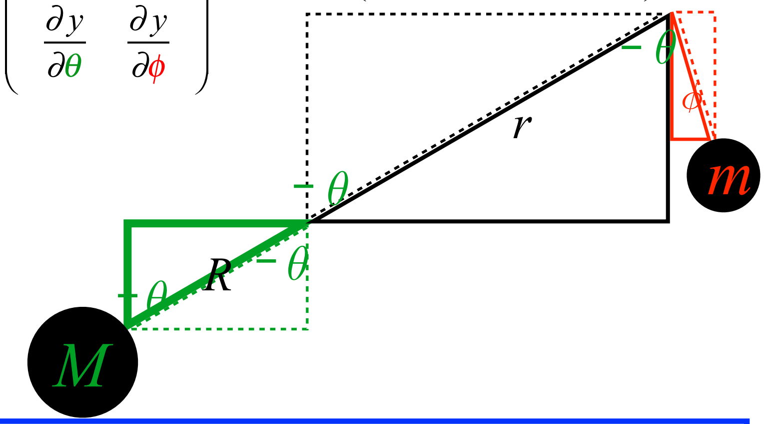
Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\ dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial \theta}d\theta + F_X\frac{\partial X}{\partial \phi}d\phi = M\ddot{X}\frac{\partial X}{\partial \theta}d\theta + M\ddot{X}\frac{\partial X}{\partial \phi}d\phi$$

$$+F_Y\ dY \quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial \theta}d\theta + F_Y\frac{\partial Y}{\partial \phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi}d\phi$$

$$+F_x\ dx \quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial \theta}d\theta + F_x\frac{\partial x}{\partial \phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial \theta}d\theta + m\ddot{x}\frac{\partial x}{\partial \phi}d\phi$$

$$+F_y\ dy \quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial \theta}d\theta + F_y\frac{\partial y}{\partial \phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial \theta}d\theta + m\ddot{y}\frac{\partial y}{\partial \phi}d\phi$$

Lagrange trickery:

STEP **A**

$$\ddot{X}\frac{\partial X}{\partial \theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial \theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial \theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **B**

$$= \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial \dot{\theta}}\right) - \dot{X}\frac{\partial \dot{X}}{\partial \theta}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial \dot{X}}{\partial \dot{q}}$

and lemma 2: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP **C**

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial \dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial \theta}$$

(using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

Set: $d\theta$=1 $d\phi$=0

$$F_X\frac{\partial X}{\partial \theta} = M\ddot{X}\frac{\partial X}{\partial \theta} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+F_Y\frac{\partial Y}{\partial \theta} \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+F_x\frac{\partial x}{\partial \theta} \quad + m\ddot{x}\frac{\partial x}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+F_y\frac{\partial y}{\partial \theta} \quad + m\ddot{y}\frac{\partial y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: $d\theta$=0 $d\phi$=1

$$F_X\frac{\partial X}{\partial \phi} = M\ddot{X}\frac{\partial X}{\partial \phi} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+F_Y\frac{\partial Y}{\partial \phi} \quad + M\ddot{Y}\frac{\partial Y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+F_x\frac{\partial x}{\partial \phi} \quad + m\ddot{x}\frac{\partial x}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+F_y\frac{\partial y}{\partial \phi} \quad + m\ddot{y}\frac{\partial y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

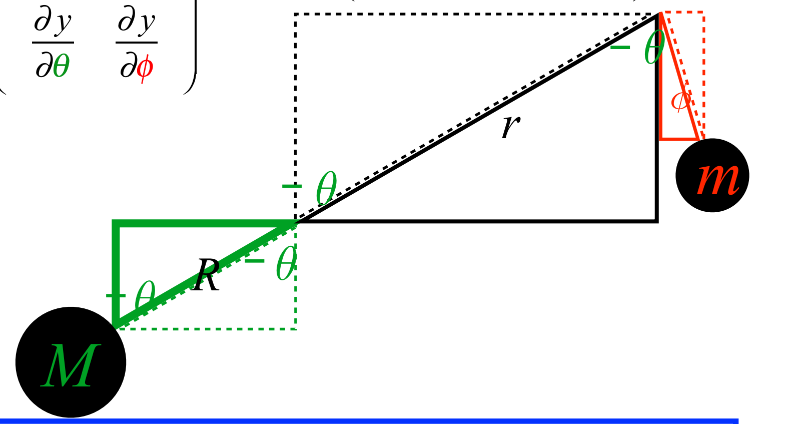
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC dθ and dφ in Jacobian)

$$dW = F_X\, dX \;=\; M\ddot{X}\,dX \;=\; F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi \;=\; M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \qquad + M\ddot{Y}\,dY \qquad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \qquad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \qquad + m\ddot{x}\,dx \qquad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \qquad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \qquad + m\ddot{y}\,dy \qquad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \qquad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and φ

Set: dθ=1 dφ=0

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} \qquad + M\ddot{Y} \frac{\partial Y}{\partial \theta} \qquad + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} \qquad + m\ddot{x} \frac{\partial x}{\partial \theta} \qquad + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} \qquad + m\ddot{y} \frac{\partial y}{\partial \theta} \qquad + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: dθ=0 dφ=1

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} \qquad + M\ddot{Y} \frac{\partial Y}{\partial \phi} \qquad + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} \qquad + m\ddot{x} \frac{\partial x}{\partial \phi} \qquad + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} \qquad + m\ddot{y} \frac{\partial y}{\partial \phi} \qquad + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

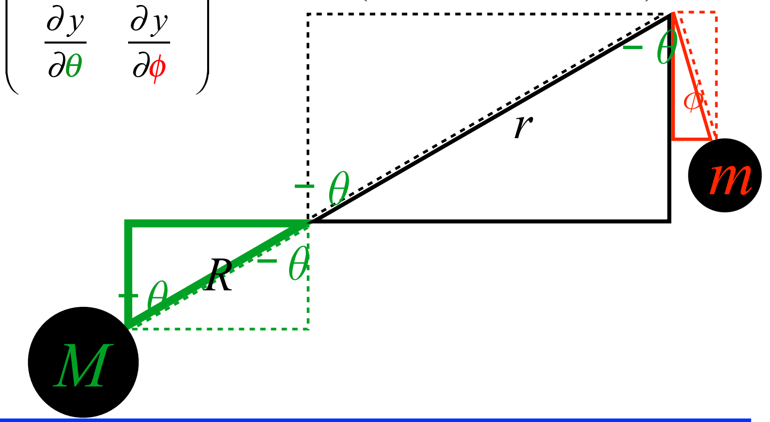
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\, dY \quad +M\ddot{Y}\, dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\, dx \quad +m\ddot{x}\, dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\, dy \quad +m\ddot{y}\, dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2}$

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta$$

Defines F_θ

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Set: $d\theta=1$ $d\phi=0$

$$F_X\frac{\partial X}{\partial\theta} = M\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta}$$

$$+F_Y\frac{\partial Y}{\partial\theta} \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} \quad +\frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta}$$

$$+F_x\frac{\partial x}{\partial\theta} \quad +m\ddot{x}\frac{\partial x}{\partial\theta} \quad +\frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta}$$

$$+F_y\frac{\partial y}{\partial\theta} \quad +m\ddot{y}\frac{\partial y}{\partial\theta} \quad +\frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta}$$

Set: $d\theta=0$ $d\phi=1$

$$F_X\frac{\partial X}{\partial\phi} = M\ddot{X}\frac{\partial X}{\partial\phi} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi}$$

$$+F_Y\frac{\partial Y}{\partial\phi} \quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} \quad +\frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi}$$

$$+F_x\frac{\partial x}{\partial\phi} \quad +m\ddot{x}\frac{\partial x}{\partial\phi} \quad +\frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi}$$

$$+F_y\frac{\partial y}{\partial\phi} \quad +m\ddot{y}\frac{\partial y}{\partial\phi} \quad +\frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi}$$

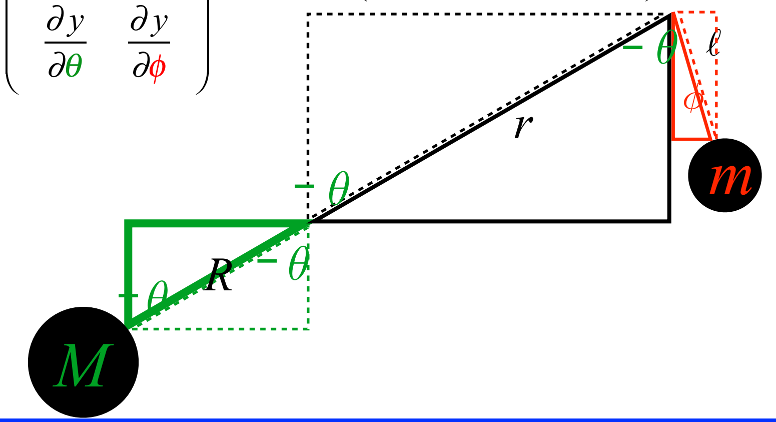
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{array}{lllll}
dW = F_X\, dX & = M\ddot{X}\, dX & = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi & = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
\quad + F_Y\, dY & + M\ddot{Y}\, dY & + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi & + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
\quad + F_x\, dx & + m\ddot{x}\, dx & + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi & + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
\quad + F_y\, dy & + m\ddot{y}\, dy & + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi & + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{array}$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:*

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$$

Defines F_θ

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$$

Defines F_ϕ

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Set: $d\theta=1 \quad d\phi=0$

$$\begin{array}{lll}
F_X \frac{\partial X}{\partial \theta} & = M\ddot{X} \frac{\partial X}{\partial \theta} & = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta} \\
+ F_Y \frac{\partial Y}{\partial \theta} & + M\ddot{Y} \frac{\partial Y}{\partial \theta} & + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta} \\
+ F_x \frac{\partial x}{\partial \theta} & + m\ddot{x} \frac{\partial x}{\partial \theta} & + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta} \\
+ F_y \frac{\partial y}{\partial \theta} & + m\ddot{y} \frac{\partial y}{\partial \theta} & + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}
\end{array}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{array}{lll}
F_X \frac{\partial X}{\partial \phi} & = M\ddot{X} \frac{\partial X}{\partial \phi} & = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi} \\
+ F_Y \frac{\partial Y}{\partial \phi} & + M\ddot{Y} \frac{\partial Y}{\partial \phi} & + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi} \\
+ F_x \frac{\partial x}{\partial \phi} & + m\ddot{x} \frac{\partial x}{\partial \phi} & + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi} \\
+ F_y \frac{\partial y}{\partial \phi} & + m\ddot{y} \frac{\partial y}{\partial \phi} & + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}
\end{array}$$

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- → *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

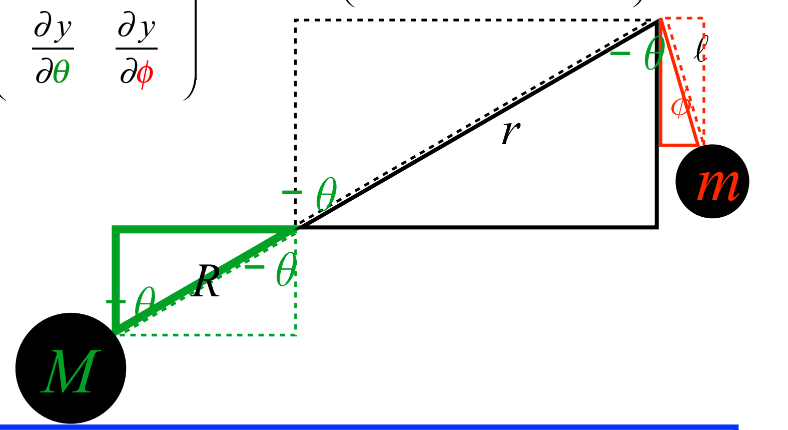
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \; & F_X\, dX && = M\ddot{X}\,dX && = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi && = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
& + F_Y\, dY && + M\ddot{Y}\,dY && + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
& + F_x\, dx && + m\ddot{x}\,dx && + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
& + F_y\, dy && + m\ddot{y}\,dy && + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

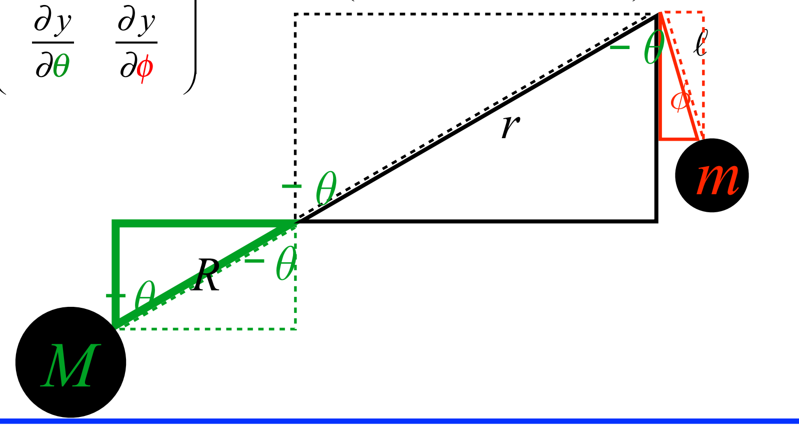
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC dθ and dϕ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta \quad \text{Defines } F_\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi \quad \text{Defines } F_\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

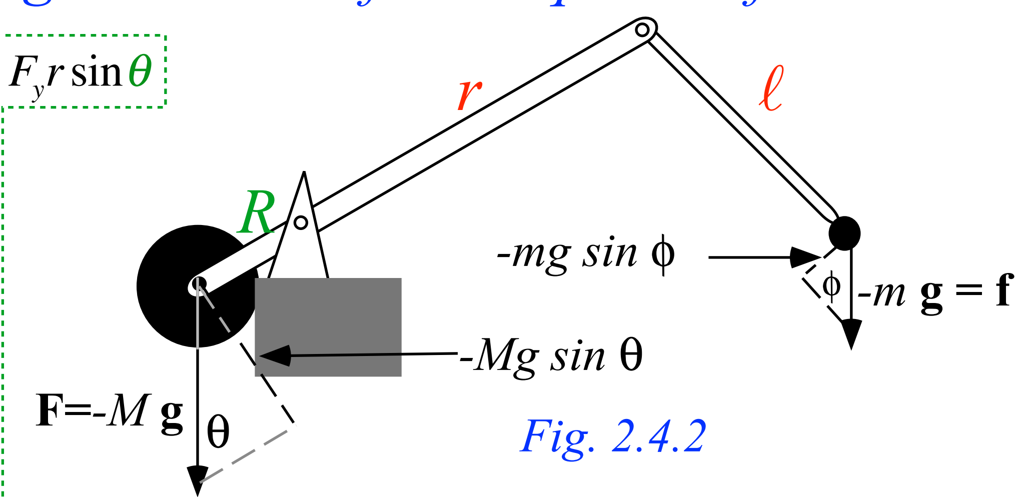


Fig. 2.4.2

Add F_θ gravity given

($F_X = 0$, $F_Y = -Mg$)

($F_x = 0$, $F_y = -mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR\sin\theta + mgr\sin\theta$$

These are competing torques on main beam R

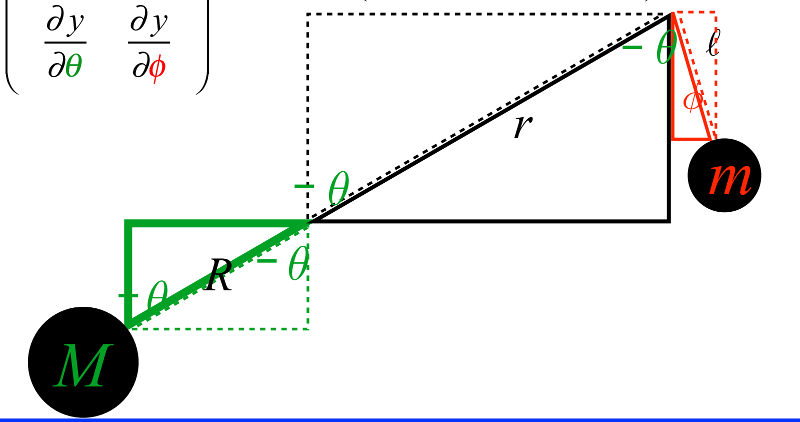
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
+ F_Y\, dY &\quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
+ F_x\, dx &\quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
+ F_y\, dy &\quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:*

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$$

Defines F_θ

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$$

Defines F_ϕ

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

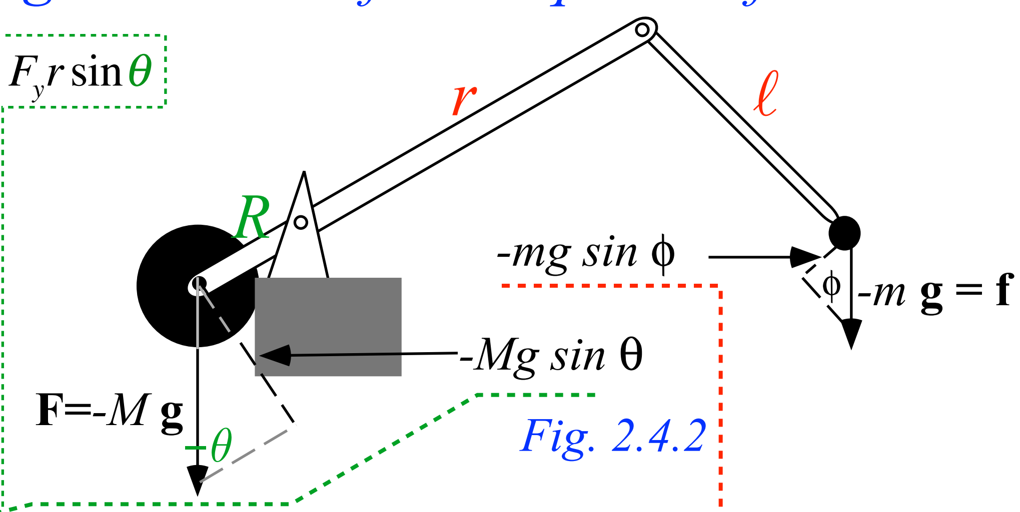


Fig. 2.4.2

Add F_θ gravity given

$(F_X = 0,\ F_Y = -Mg)$

$(F_x = 0,\ F_y = -mg)$

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR\sin\theta + mgr\sin\theta$$

Add F_ϕ gravity given

$(F_X = 0,\ F_Y = -Mg)$

$(F_x = 0,\ F_y = -mg)$

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = -mg\ell\sin\phi$$

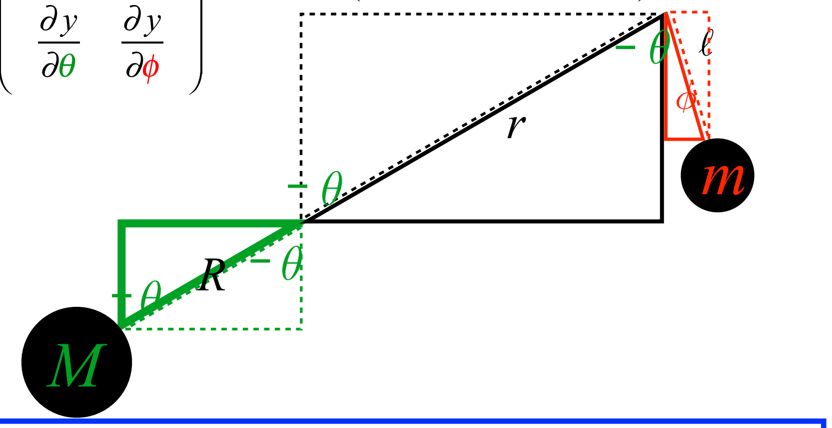
These are competing torques on main beam R... *... and a torque on throwing lever ℓ*

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$
$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$
$$+F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$
$$+F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$
$$+F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$\text{Let}: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

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$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

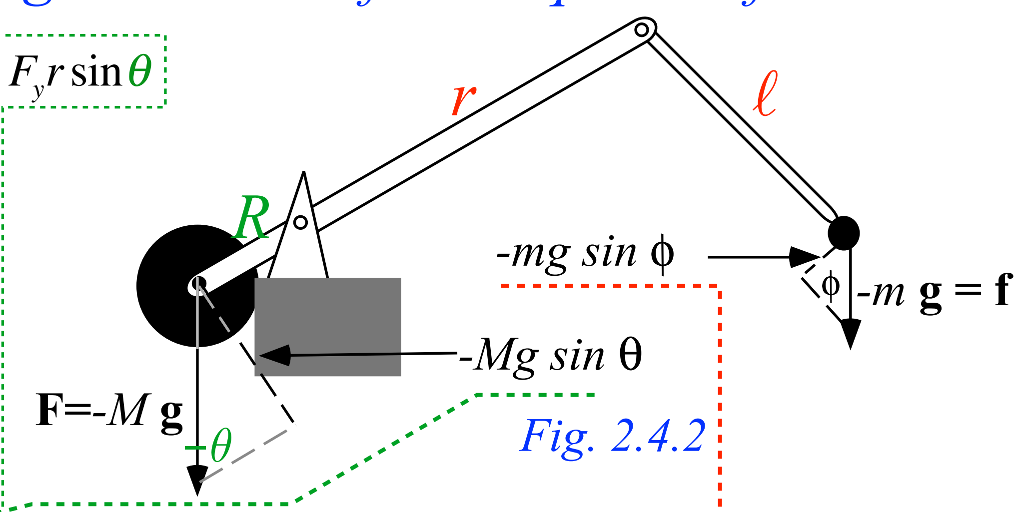

Fig. 2.4.2

Q: Are there ± sign errors here?

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... ... and a torque on throwing lever ℓ

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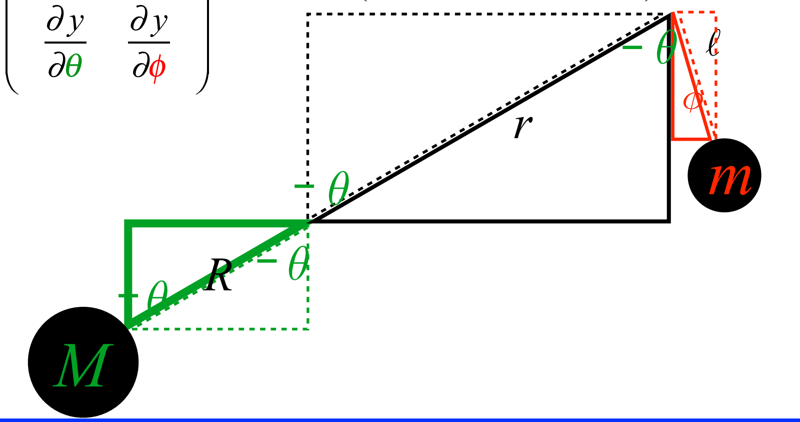

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$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

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$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

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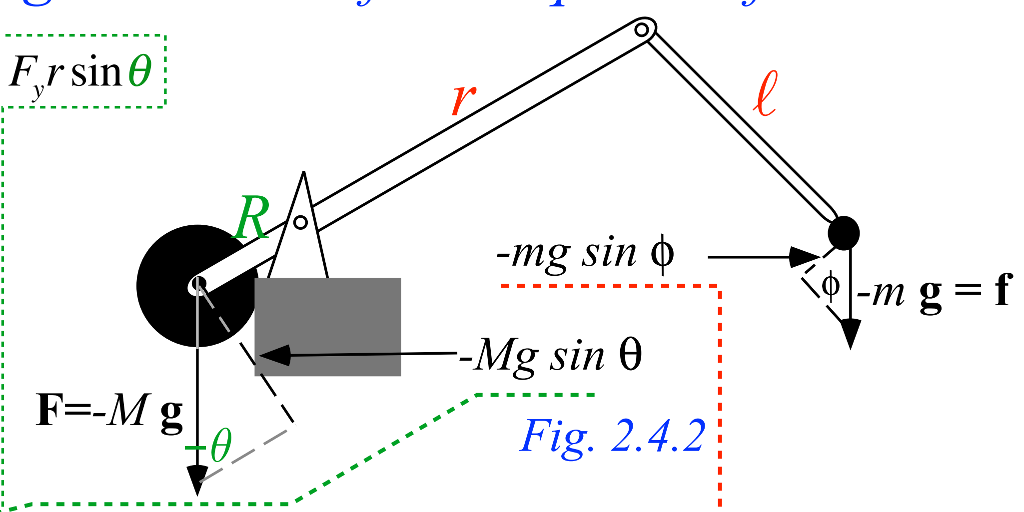

Fig. 2.4.2

Q: Are there ± sign errors here?

A: No. Beam in $-\theta$ position.

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given

($F_X = 0$, $F_Y = -Mg$)

($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... *... and a torque on throwing lever ℓ*

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

Total KE $= T = T(M) + T(m)$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

The γ_{mn} tensor/matrix formulation

Total KE $= T = T(M) + T(m)$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where:
γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

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$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

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$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

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$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

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$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 78)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

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$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

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$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix} = \begin{pmatrix}MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix}p_\theta \\ p_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix}MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

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$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total } KE = T = T(M) + T(m)$$

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$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix} = \begin{pmatrix}MR^2 + mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial T}{\partial \dot{\theta}}\\ \frac{\partial T}{\partial \dot{\phi}}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

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Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given}: p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where}: \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then}: \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

Then:
$$p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm}$$

QED

Momentum γ_{mn}-matrix theorem: *(matrix-proof here on page 43)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot\theta} \\ \frac{\partial T}{\partial \dot\phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial \dot\theta}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial \dot\theta}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \\ \frac{\partial}{\partial \dot\phi}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial \dot\phi}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\phi,\theta} \\ \gamma_{\theta,\phi} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

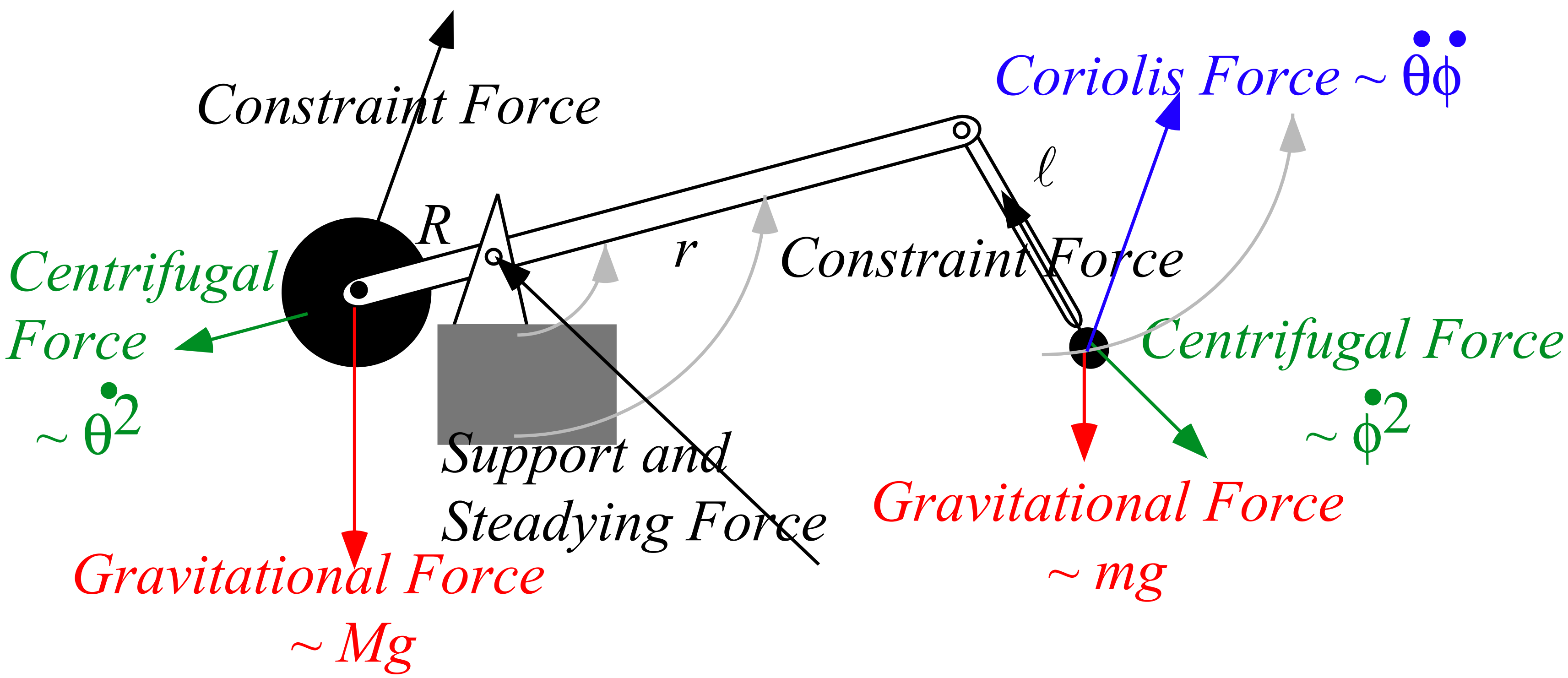
$$= \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \qquad QED$$

Summary of Lagrange equations and force analysis (Mostly Unit 2.)

→ *Forces: total, genuine, potential, and/or fictitious*

Forces: total, genuine, potential, and/or fictitious



Acceleration and *'Fictitious'* *Forces:*
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot p_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot p_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

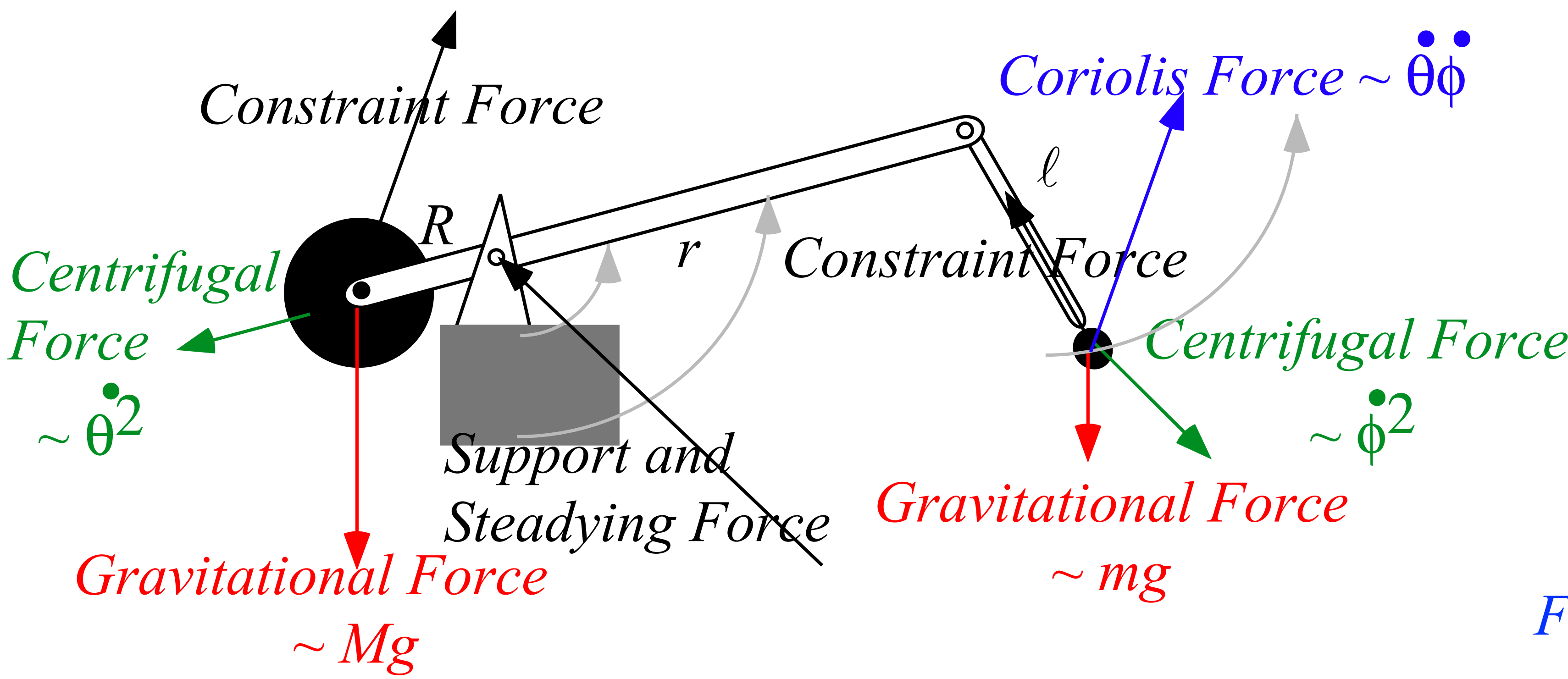
Lagrange Force equations

(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

For conservative forces

where: $F_\theta = -\frac{\partial V}{\partial\theta}$ and: $\frac{\partial V}{\partial\dot\theta} = 0$

$F_\phi = -\frac{\partial V}{\partial\phi}$ and: $\frac{\partial V}{\partial\dot\phi} = 0$

$$\dot p_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot\theta} = \frac{\partial T}{\partial\theta} + F_\theta + 0$$

$$\dot p_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot\phi} = \frac{\partial T}{\partial\phi} + F_\phi + 0$$

Lagrange Force equations

(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial\dot\theta} \qquad \dot p_\theta = \frac{\partial L}{\partial\theta}$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} \qquad \dot p_\phi = \frac{\partial L}{\partial\phi}$$

Lagrange Potential equations

$L=T-V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

→ *Multivalued functionality and connections*

Covariant and contravariant relations

Tangent space vs. Normal space

Metric g_{mn} tensor geometric relations to length, area, and volume

Trebuchet Cartesian projectile coordinates are double-valued

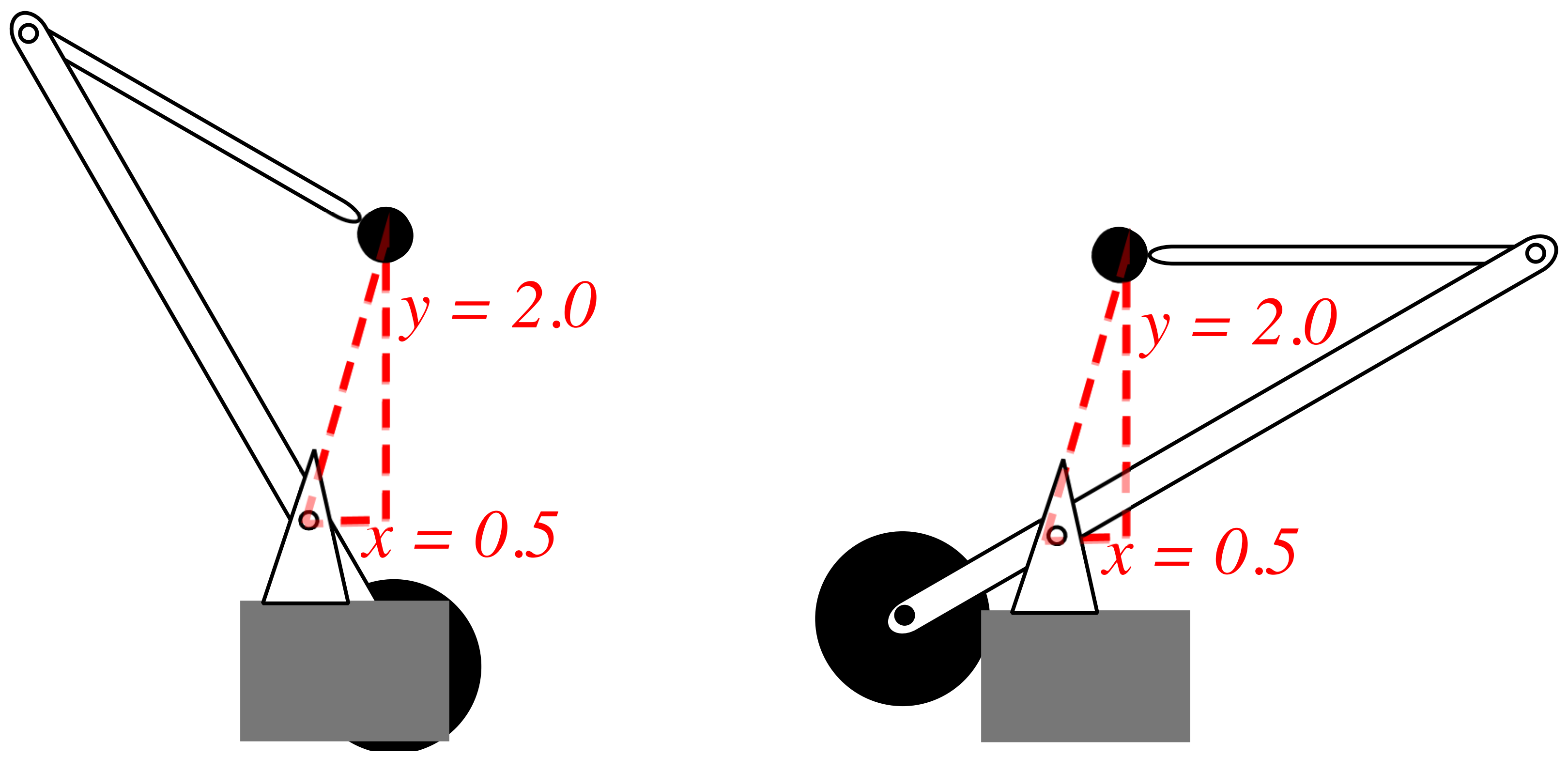


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m.

Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)

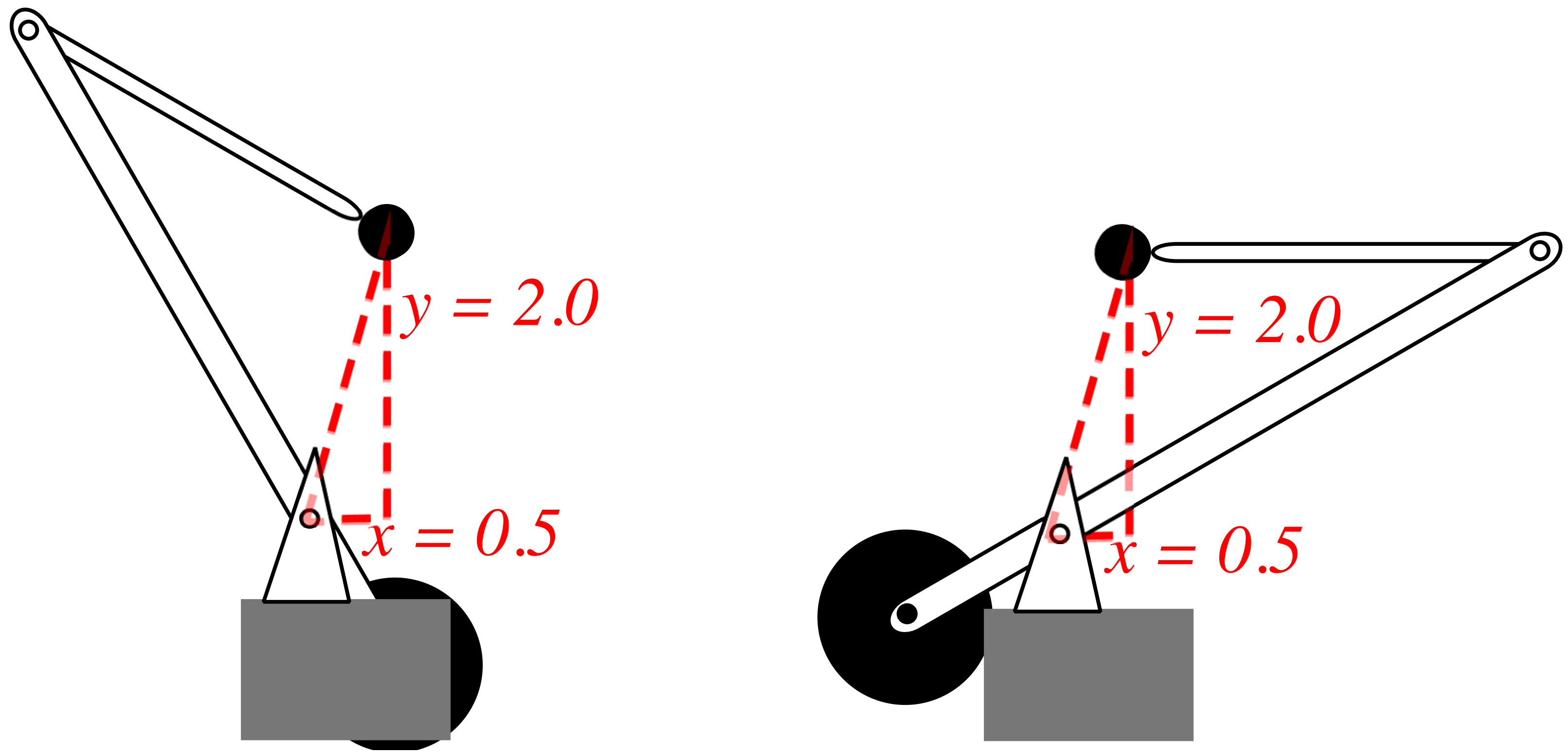


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

So, for example, are polar coordinates … (for each angle there are two r-values)

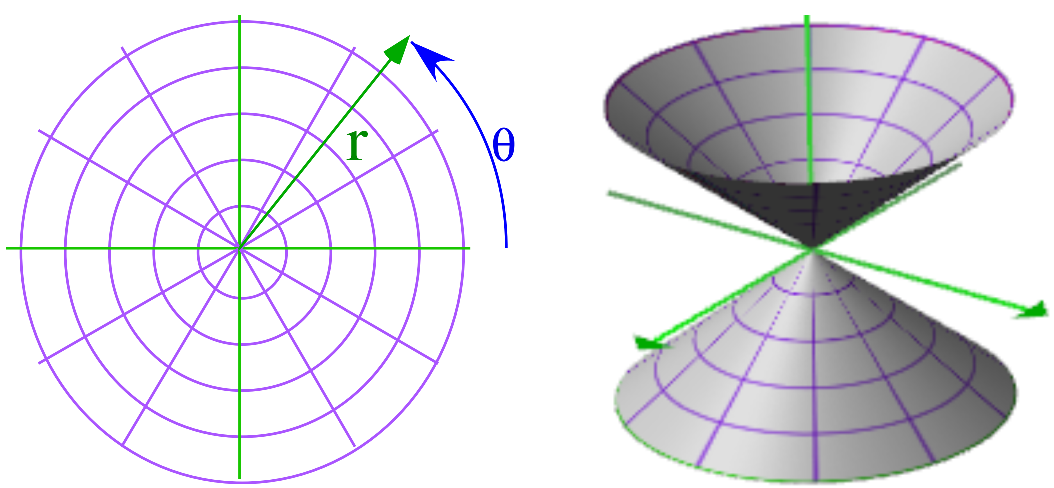


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

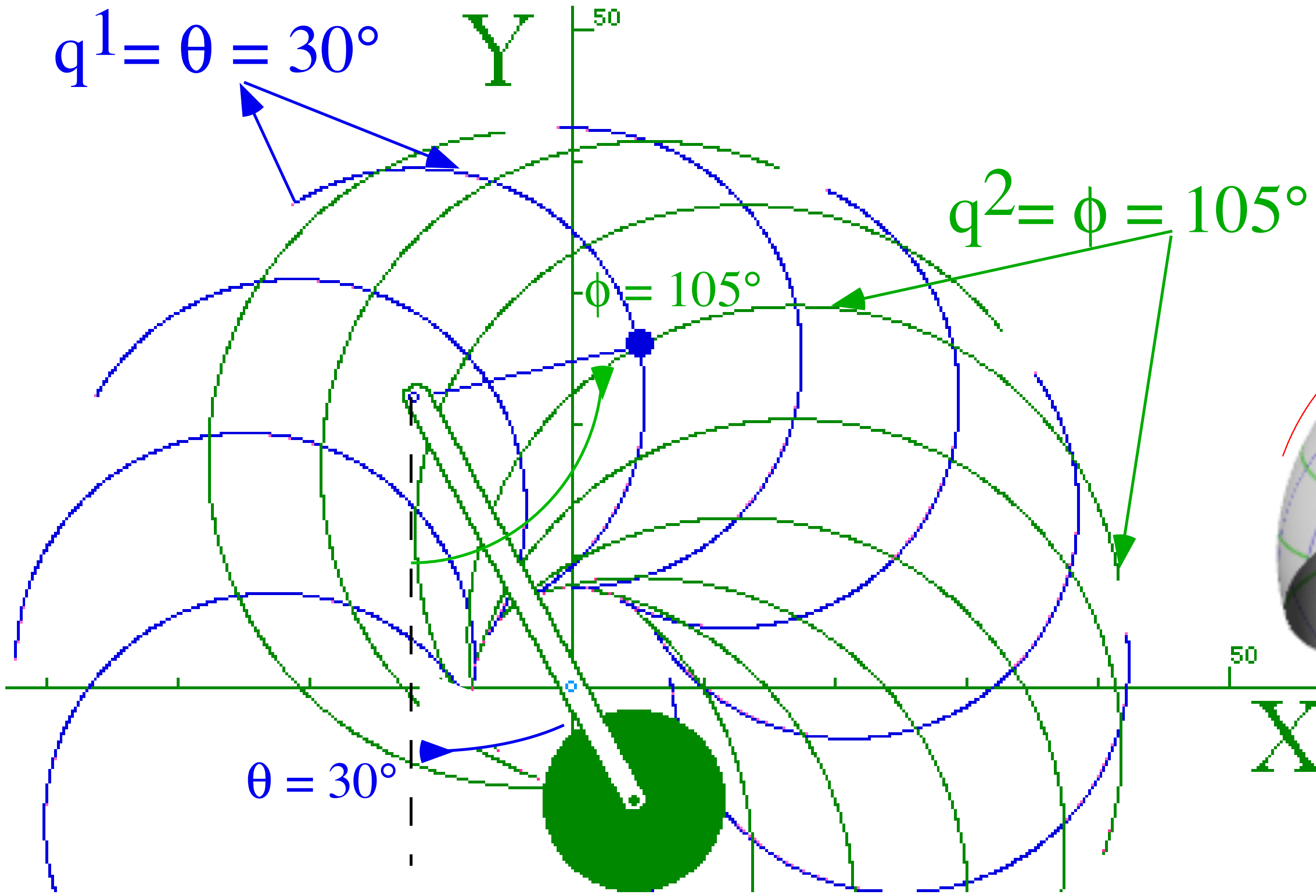


Fig. 3.1.1a ($q^1=\theta$, $q^2=\phi$)Coordinate manifold for trebuchet (Left handed sheet.)

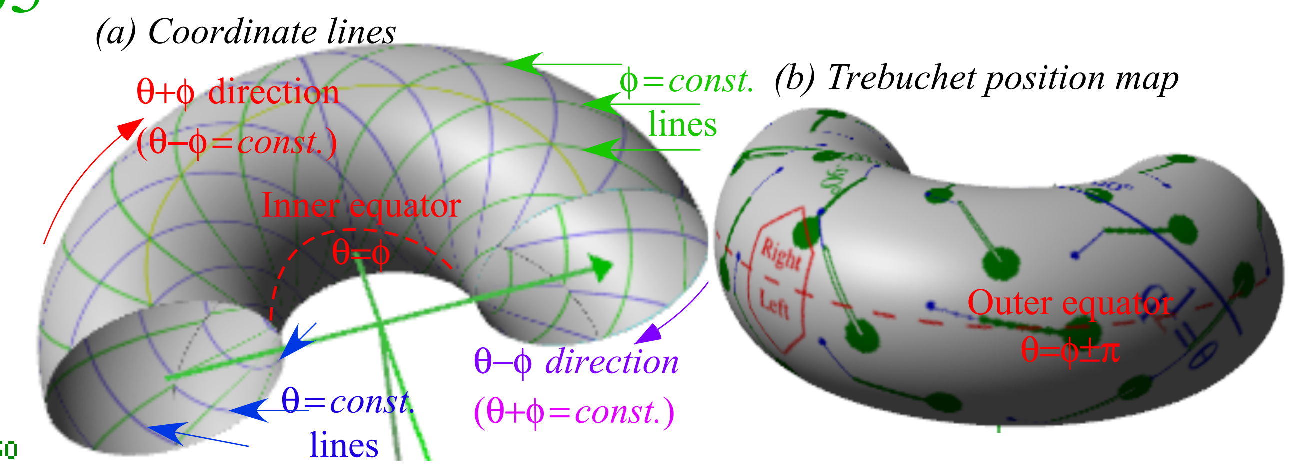


Fig. 3.1.2 Trebuchet torus.

(a) ($q^1=\theta$, $q^2=\phi$) coordinate lines. (b)Trebuchet position map and equators.

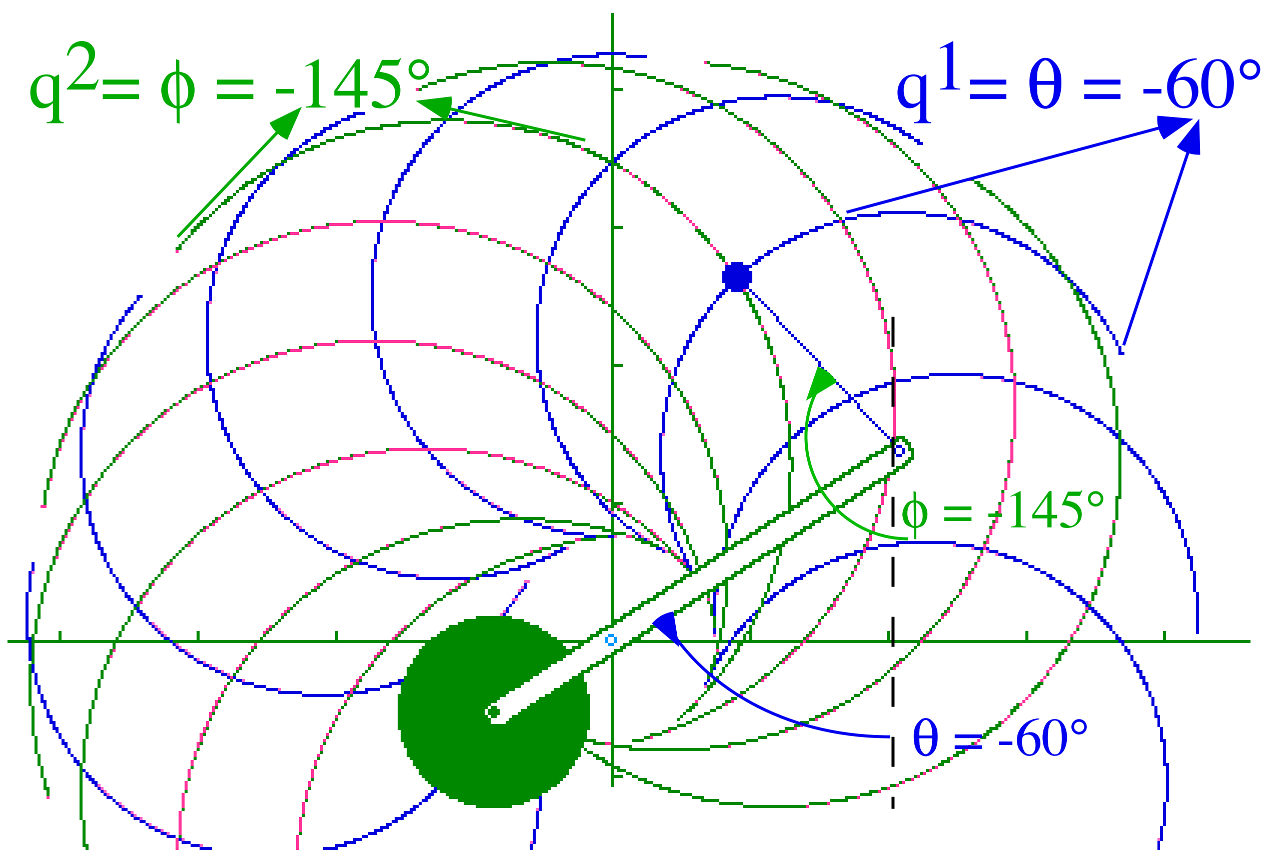


Fig. 3.1.1b ($q^1=\theta$, $q^2=\phi$)Coordinate manifold for trebuchet (Right handed sheet.)

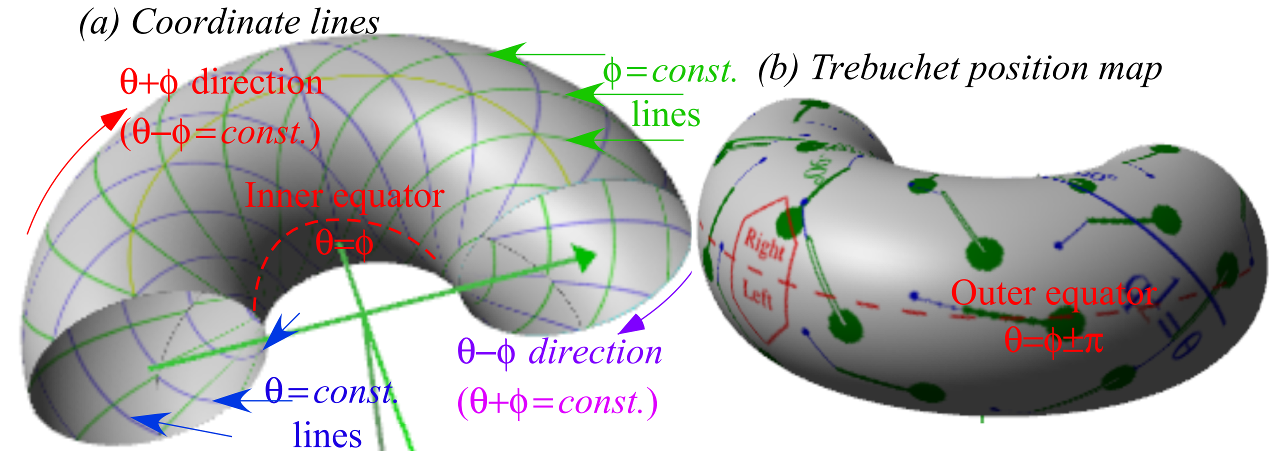


Fig. 3.1.2 Trebuchet torus.
(a) (q^1=θ, q^2=ϕ) coordinate lines.(b)Trebuchet position map and equators.

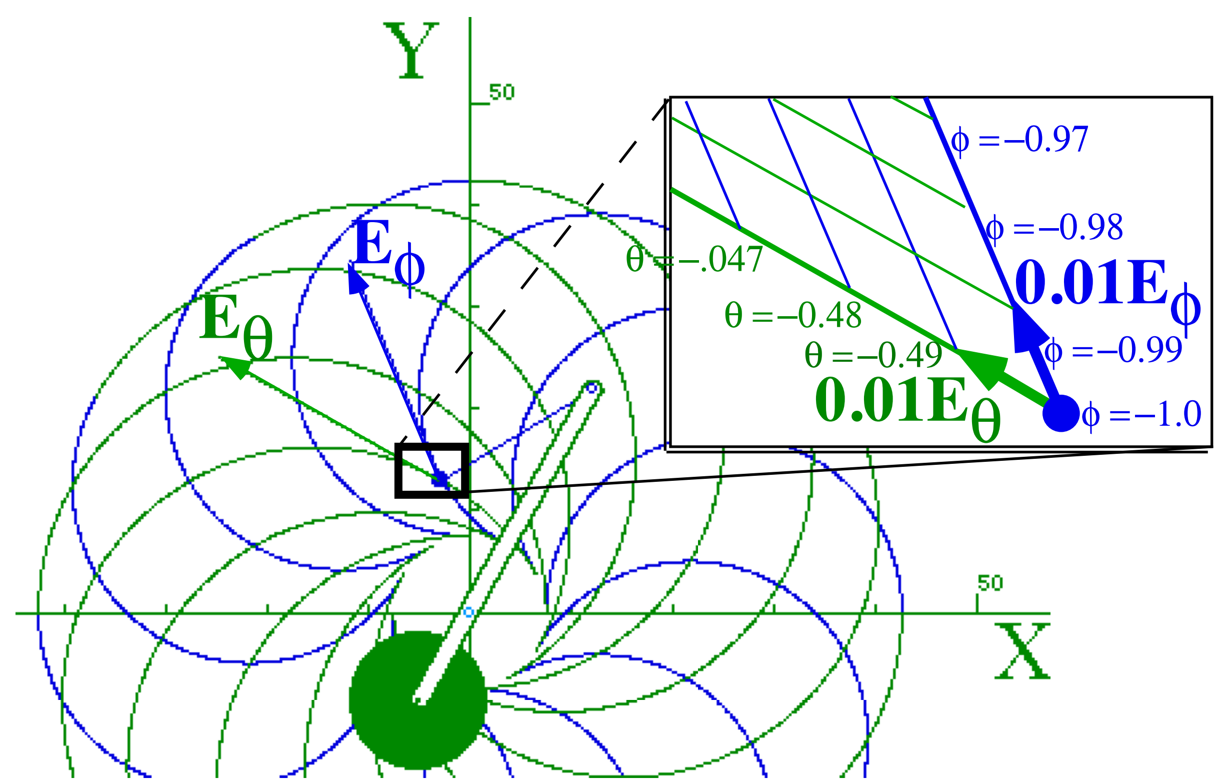


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

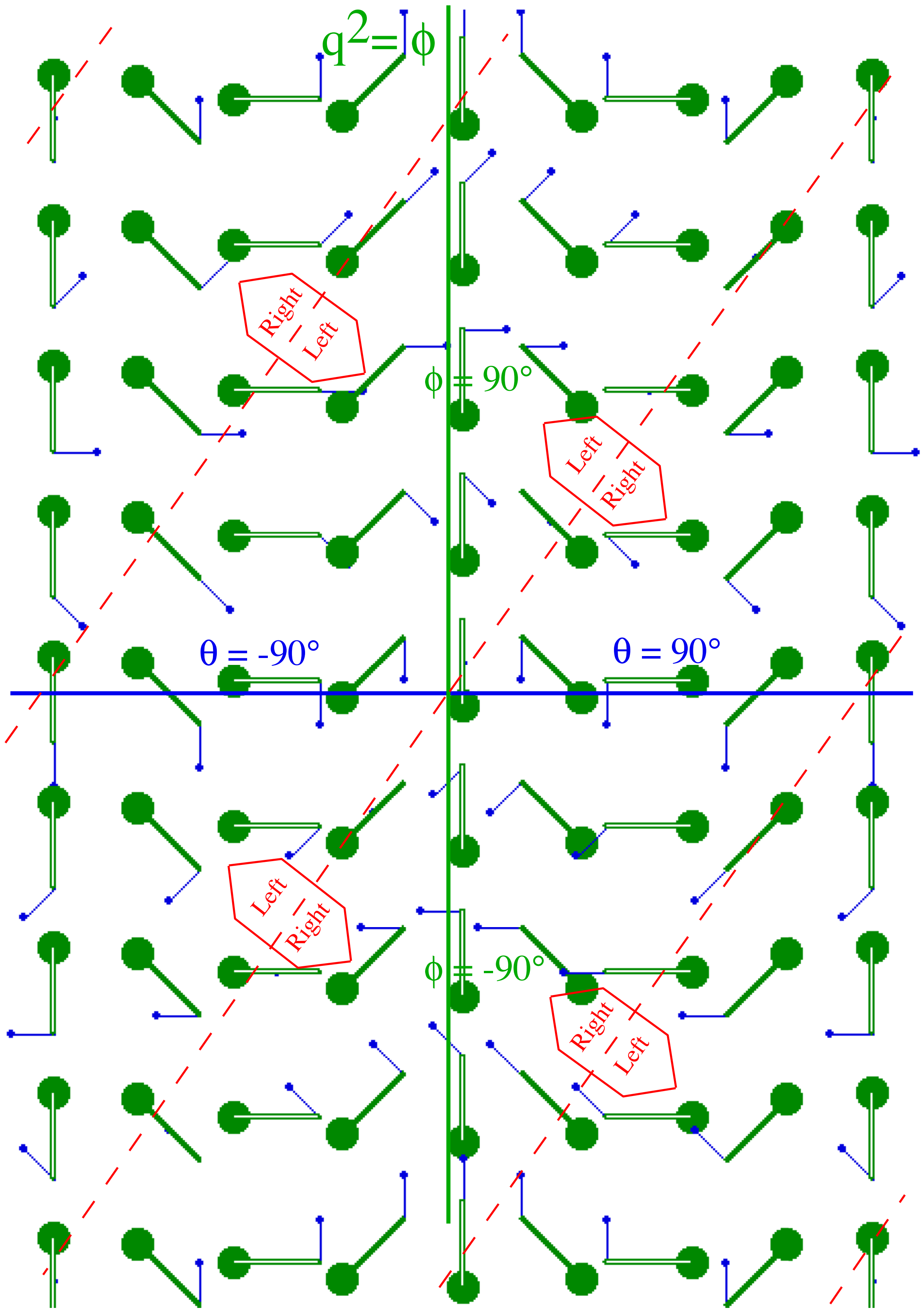


Fig. 3.1.3 "Flattened" (q^1=θ, q^2=ϕ) coordinate manifold for trebuchet

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

- *Multivalued functionality and connections*
- → *Covariant and contravariant relations*
  - *Tangent space vs. Normal space*
- *Metric g_{mn} tensor geometric relations to length, area, and volume*

Kajobian transformation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

$$\left| \begin{array}{ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} \right| = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{array}{cc|c} \ell \sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array} \right|}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

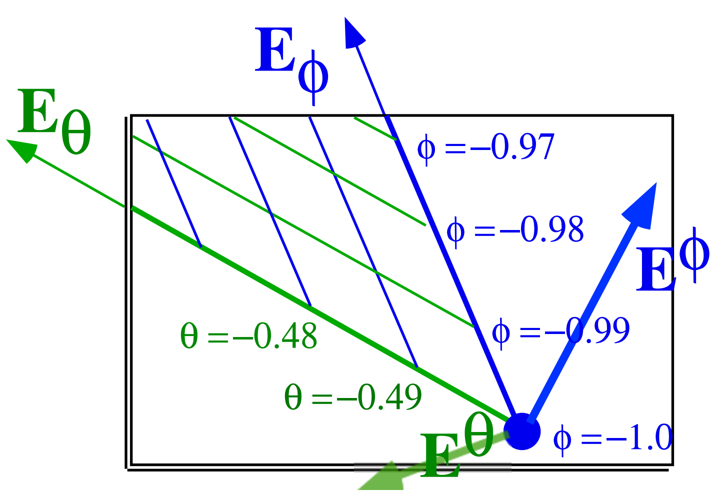


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

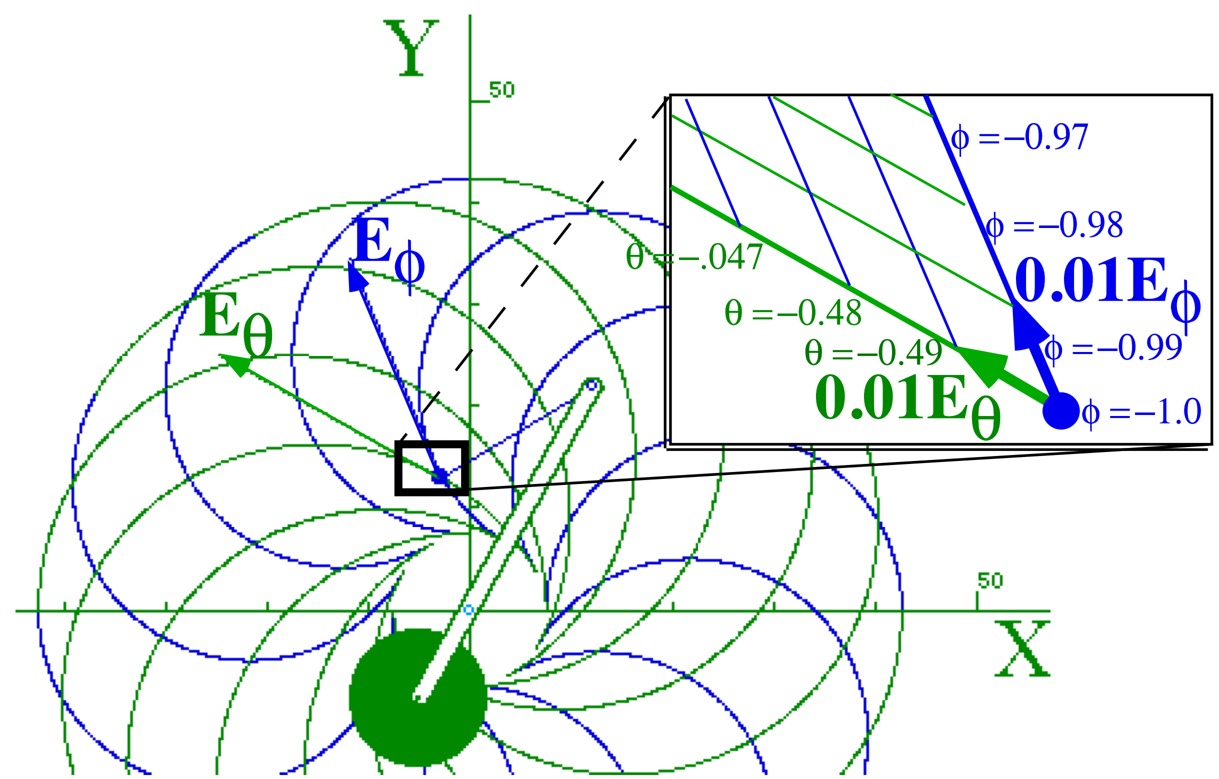


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

→ *Tangent space vs. Normal space*

Metric g_{mn} tensor geometric relations to length, area, and volume

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E_m} = U_n \mathbf{E^n} = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

$$\mathbf{V} = V^m \mathbf{E_m} = V_n \mathbf{E^n} = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, ,

$U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

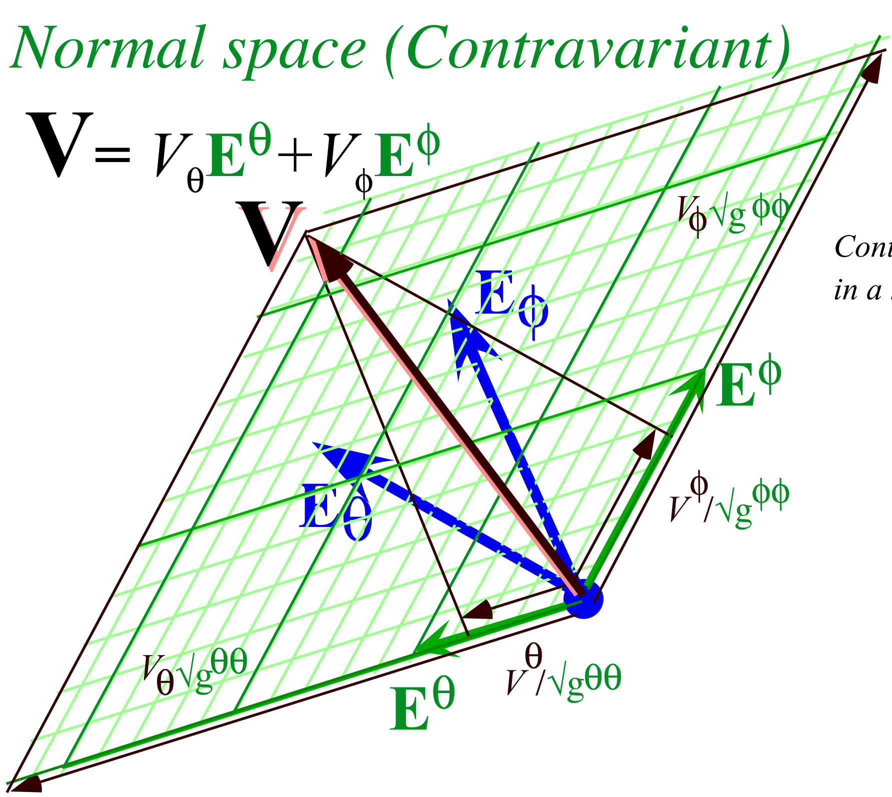


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

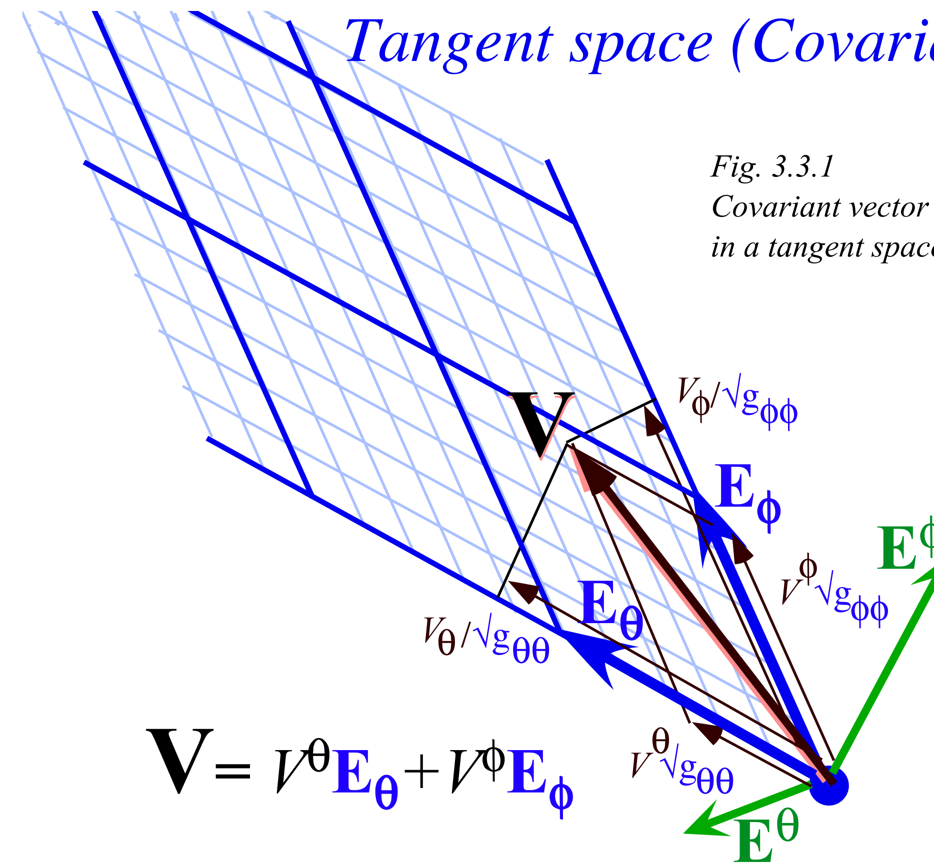


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector $\mathbf{U},\mathbf{V},...$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

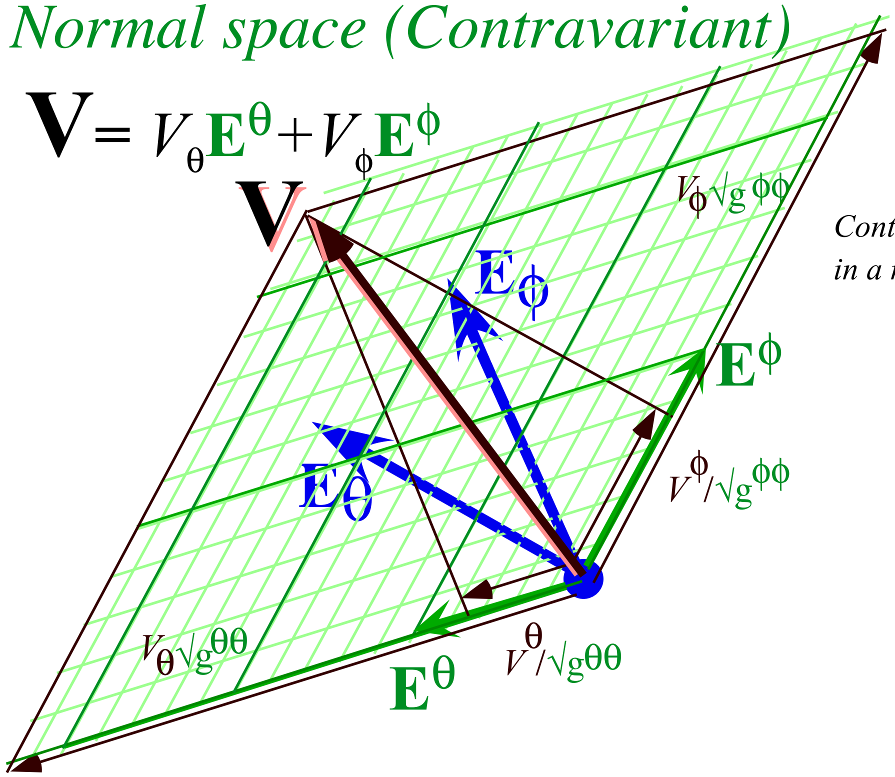


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

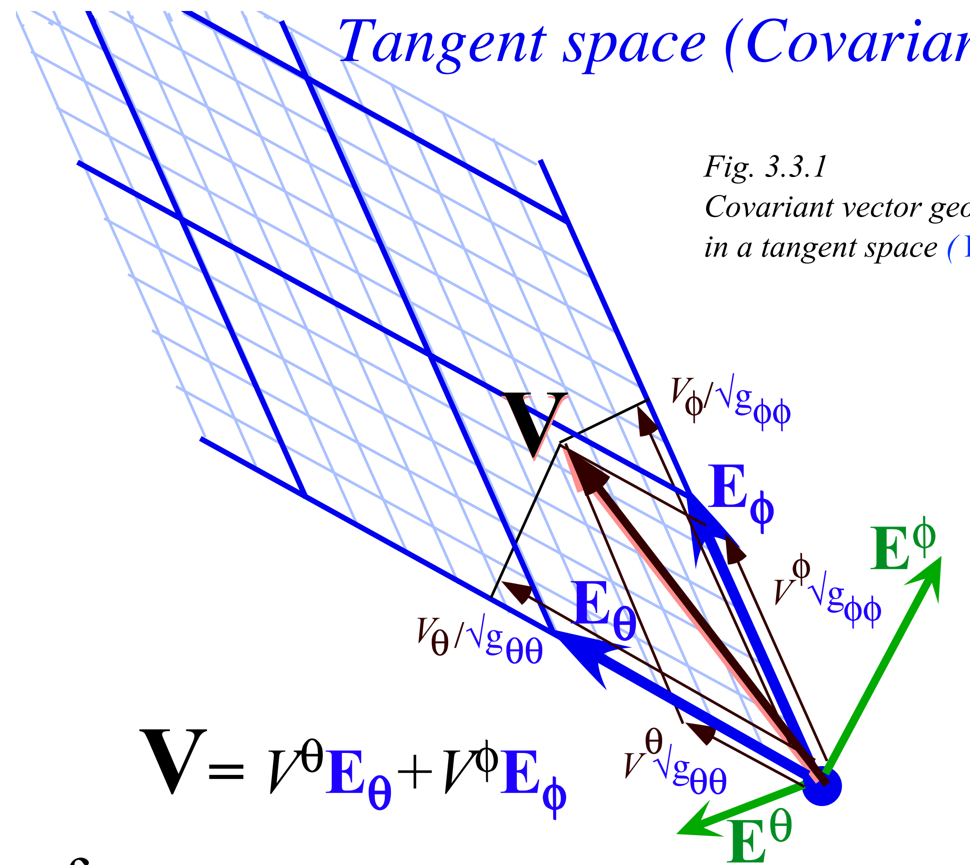


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*….

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\qquad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ , \quad \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U**,**V**,... is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

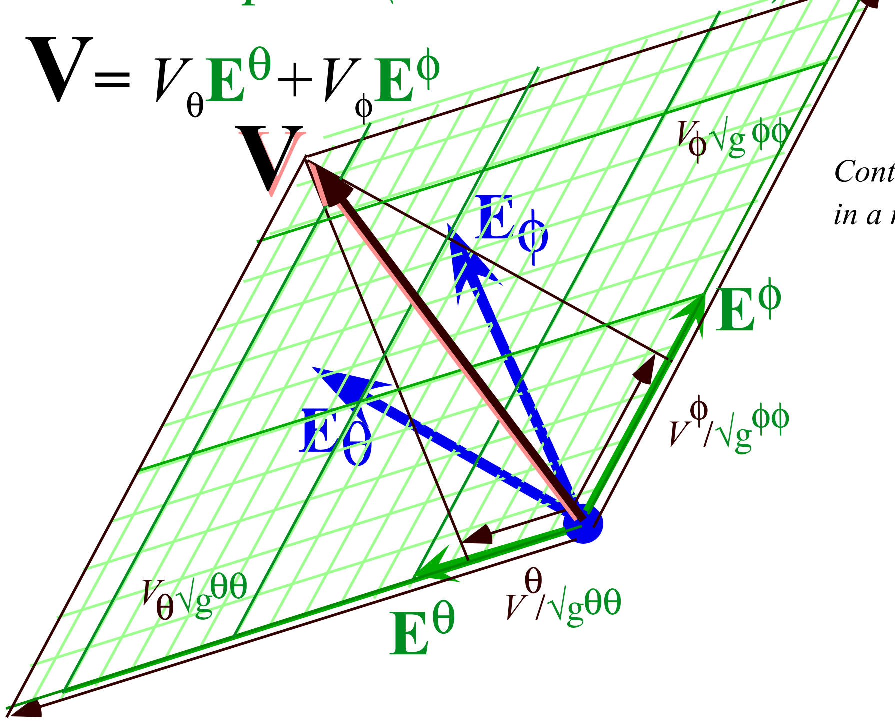


Fig. 3.3.2
Contravariant vector geometry
in a normal space $(\mathbf{E}^\theta,\mathbf{E}^\phi)$.

Tangent space (Covariant)

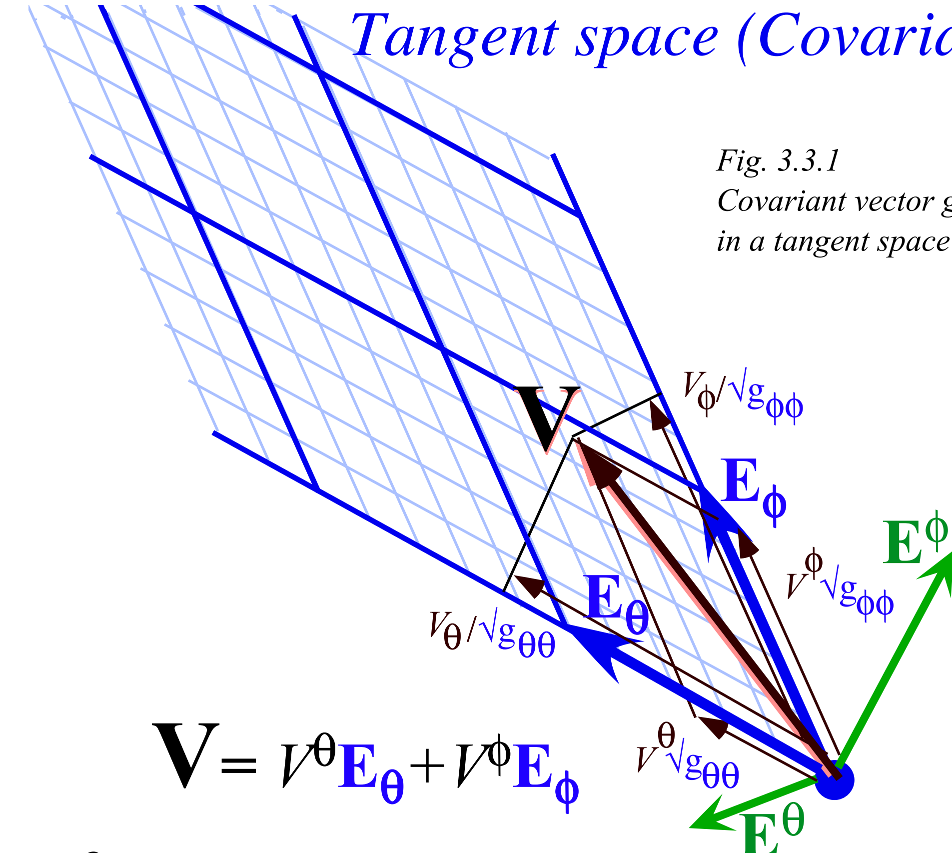


Fig. 3.3.1
Covariant vector geometry
in a tangent space $(\mathbf{E}_\theta,\mathbf{E}_\phi)$.

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1,q^2,\cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}},\bar{q}^{\bar{2}},\cdots\}$ using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

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Normal space (Contravariant)

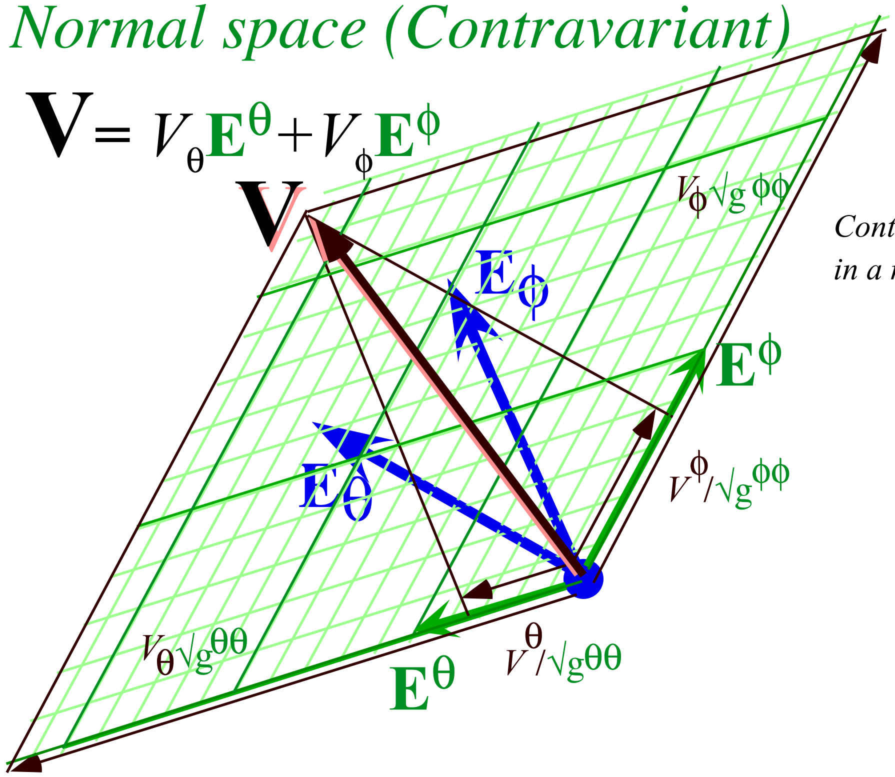
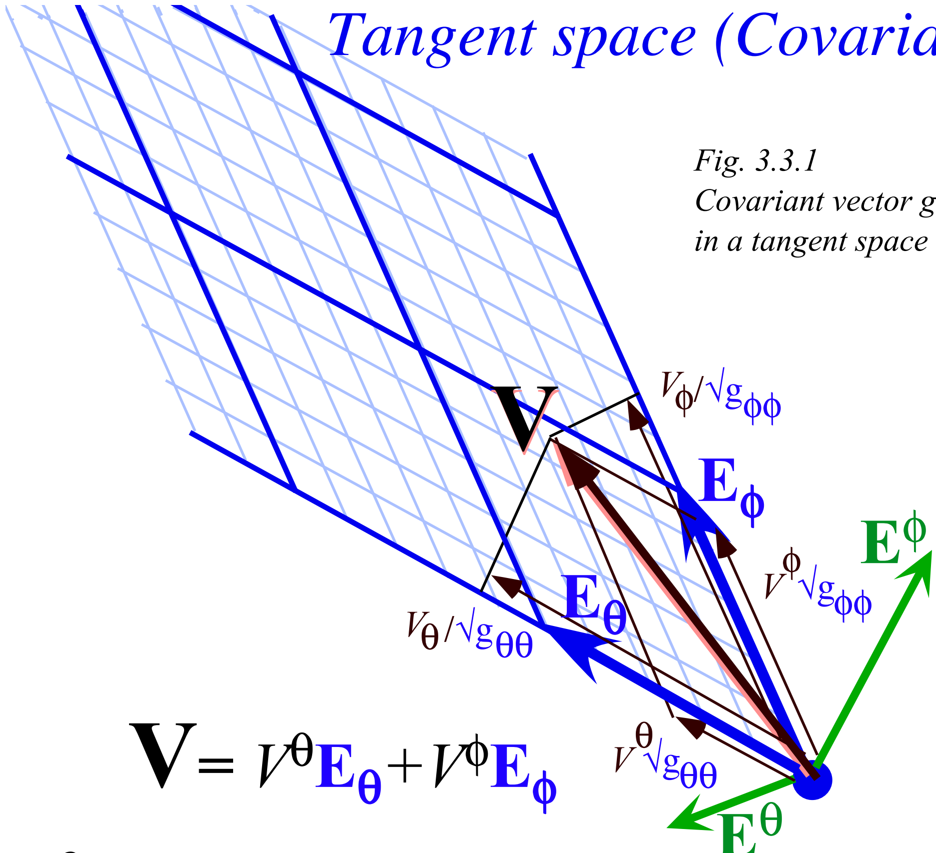


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)



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implies: $V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

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Normal space (Contravariant)

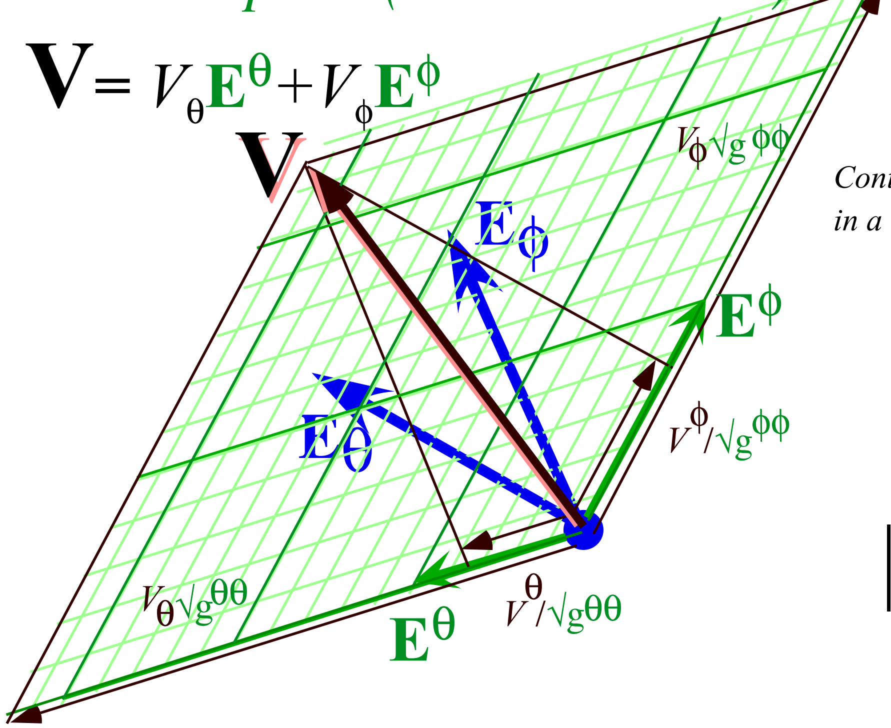

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

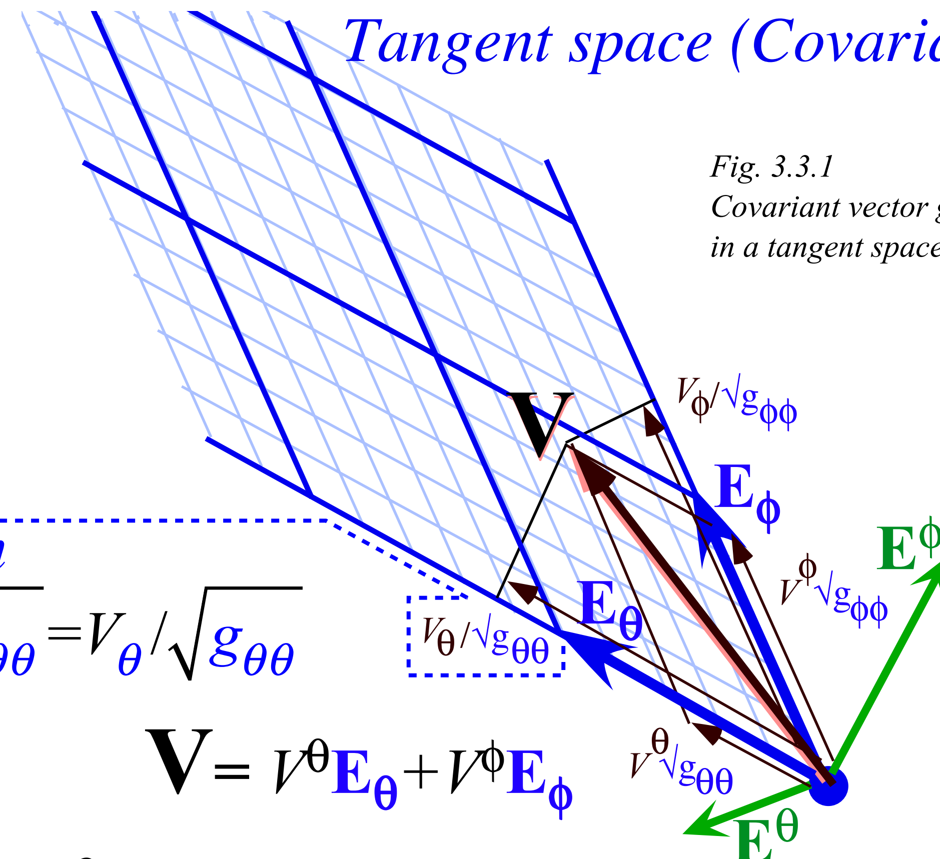

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\mathbf{\hat{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

$$\mathbf{V} = V_\theta\mathbf{E}^\theta + V_\phi\mathbf{E}^\phi \qquad \mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*....

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Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|$$

implies: $\langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

Tangent space vs. Normal space

→ *Metric g_{mn} tensor geometric relations to length, area, and volume*

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \ , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \ .$$

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \;, \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \;.$$

"Mixed" covariant-contravariant metric components

$$g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = \delta_m^n = \begin{cases} 0 & \text{if: } m \neq n \\ 1 & \text{if: } m = n \end{cases}$$

Caution: δ_{mn} is g_{mn} and <u>not</u> δ_n^m in GCC.

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Metric coefficients express covariant unitary vectors in terms of contras and vice-versa

$$\mathbf{E_m} = g_{mn}\mathbf{E^n} \ , \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n} \ .$$

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Co-and-Contra vector and tensor components are related by g-transformation. (So are g's themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \ , \quad T^{mm'} = g^{mn}g^{m'n'}T_{nn'} \ , \ etc.$$

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \ , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \ .$$

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Co-and-Contra vector and tensor components are related by g-transformation. (So are g's themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \ , \quad T^{mm'} = g^{mn}g^{m'n'}V_{nn'} \ , \ \ etc.$$

Diagonal square roots $\sqrt{g_{mm}}$ are the lengths of the covariant unitary vectors. $\left|\mathbf{E_m}\right| = \sqrt{\mathbf{E_m} \bullet \mathbf{E_m}} = \sqrt{g_{mm}}$

$$\left|\mathbf{E^m}\right| = \sqrt{\mathbf{E^m} \bullet \mathbf{E^m}} = \sqrt{g^{mm}}$$

tangent space area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1E_1,V^2E_2\right)=V^1V^2\left|\mathbf{E_1}\times\mathbf{E_2}\right|=V^1V^2\sqrt{\left(\mathbf{E_1}\times\mathbf{E_2}\right)\bullet\left(\mathbf{E_1}\times\mathbf{E_2}\right)}$$

$$Area\left(V^1E_1,V^2E_2\right)=V^1V^2\sqrt{\left(\mathbf{E_1}\bullet\mathbf{E_1}\right)\left(\mathbf{E_2}\bullet\mathbf{E_2}\right)-\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)}$$

$$=V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{12}}=V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

3D Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2,V^3\mathbf{E}_3\right)=V^1V^2V^3\left|\mathbf{E_1}\times\mathbf{E_2}\bullet\mathbf{E_3}\right|=V^1V^2V^3\det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3}\end{vmatrix}$$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33}\end{pmatrix}=\begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3}\end{pmatrix}\bullet\begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3}\end{pmatrix}=J^T\bullet J$$

Determinant product ($det|A|\ det|B| = det|A\bullet B|$) and symmetry ($det|A^T| = det|A|$) gives

$$Volume\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2,V^3\mathbf{E}_3\right)=V^1V^2V^3\det\left|J\right|=V^1V^2V^3\sqrt{\det\left|g\right|}$$