Lecture 12
Mon. 10.07.2019

Complex Variables, Series, and Field Coordinates I.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
How good are those power series?
Taylor-Maclaurin series, imaginary interest, and complex exponentials

Lecture 14 Tue. 10.15 starts here

2. What good are complex exponentials?
Easy trig
Easy 2D vector analysis
Easy oscillator phase analysis
Easy rotation and "dot" or "cross" products

3. Easy 2D vector calculus
Easy 2D vector derivatives
Easy 2D source-free field theory
Easy 2D vector field-potential theory

4. Riemann-Cauchy relations (What's analytic? What's not?)
Easy 2D curvilinear coordinate discovery
Easy 2D circulation and flux integrals
Easy 2D monopole, dipole, and 2^n-pole analysis
Easy 2^n-multipole field and potential expansion
Easy stereo-projection visualization
Cauchy integrals, Laurent-Maclaurin series

Lect. 12 ends around here

What good are complex quantities?

1. Complex numbers provide "automatic trigonometry"

2. Complex numbers add like vectors.

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

4. Complex products provide 2D rotation operations.

5. Complex products provide 2D "dot"(•) and "cross"(x) products.

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
The half-n'-half results: (Riemann-Cauchy Derivative Relations)

9. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

10. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)

11. Complex integrals define 2D monopole fields and potentials

12. Complex derivatives give 2D dipole fields

Lecture 15 Thur. 10.17 starts here

13. More derivatives give 2D 2^N-pole fields…

14. ...and 2^N-pole multipole expansions of fields and potentials...

15. ...and Laurent Series...

16. Mapping and non-analytic source analysis.

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lecture #12

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*
Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10: Calculus of exponentials, logarithms, and complex fields, page=131 pdf_page=135
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)

Select, exciting, and related Research & Articles of Interest

(Many of these may be just beyond this course, but are included to lend added insight):

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces_-_Foundations_-_Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019

Excerpts from the Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019- Page 44-47 (Preliminary Draft)

Past Articles of Interest:

An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_ Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
Rovibrational_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74*, *Page=75*, *Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20

WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

“RelaWavity” Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:

Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:

Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:

Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
*Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_*,
Nuclear_Collisions - Mishustin-PhysRevC-2007, *APS Link & Abstract*
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938

Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971

WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots

$m_1:m_2 = 3:1$
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot *V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$*

$m_1:m_2 = 4:1$
v2 vs v1, *y2 vs y1*

$m_1:m_2 = 100:1$, *$(v_1, v_2)=(1, 0)$: V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

The Story of e (A Tale of Great $Interest$)

British spelling: *intrest*

Simple *interest* at some rate r based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time t later they would pay you $p(t)=(1+r \cdot t)p(0)$.

$1.00 at rate $r=1$ (like Israel and Brazil that once had 100% interest.) gives $2.00 at $t=1$ year.

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Trimester compounded interest gives $p(\frac{t}{3})=(1+r\cdot\frac{t}{3})p(0)$ at the $1/3^{rd}$-period $\frac{t}{3}$ or 1^{st} trimester and then use that to figure the 2^{nd} trimester and so on. Now $1.00 at rate $r=1$ earns $2.37.

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})p(2\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(0)=\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot 1=\tfrac{64}{27}=2.37$$

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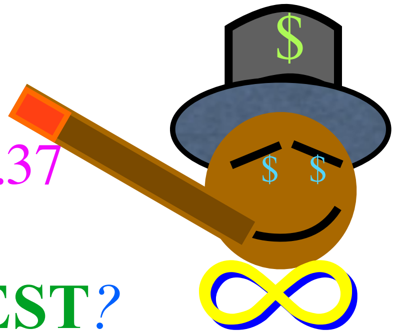

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})p(2\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(0)=\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot 1=\tfrac{64}{27}=2.37$$

So if you compound interest more and more frequently, do you approach **INFININTEREST***?*

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So if you compound interest more and more frequently, do you approach **INFININTEREST**?

NOT!!

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So if you compound interest more and more frequently, do you approach **INFININTEREST**?

NOT!!

$$p^{\frac{1}{1}}(t)=(1+r\cdot\tfrac{t}{1})^1 p(0)=\left(\tfrac{2}{1}\right)^1\cdot 1=\tfrac{2}{1}=2.00$$

+25¢

$$p^{\frac{1}{2}}(t)=(1+r\cdot\tfrac{t}{2})^2 p(0)=\left(\tfrac{3}{2}\right)^2\cdot 1=\tfrac{9}{4}=2.25$$

+12¢

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})^3 p(0)=\left(\tfrac{4}{3}\right)^3\cdot 1=\tfrac{64}{27}=2.37$$

+7¢

$$p^{\frac{1}{4}}(t)=(1+r\cdot\tfrac{t}{4})^4 p(0)=\left(\tfrac{5}{4}\right)^4\cdot 1=\tfrac{625}{256}=2.44$$

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$$p^{\frac{1}{2}}(t)=(1+r\cdot\tfrac{t}{2})p(\tfrac{t}{2})=(1+r\cdot\tfrac{t}{2})\cdot(1+r\cdot\tfrac{t}{2})p(0)=\tfrac{3}{2}\cdot\tfrac{3}{2}\cdot 1=\tfrac{9}{4}=2.25$$

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So if you compound interest more and more frequently, do you approach **INFININTEREST**?

NOT!!

$$p^{\frac{1}{1}}(t)=(1+r\cdot\tfrac{t}{1})^{1}p(0)=\left(\tfrac{2}{1}\right)^{1}\cdot 1=\tfrac{2}{1}=2.00$$

+25¢

$$p^{\frac{1}{2}}(t)=(1+r\cdot\tfrac{t}{2})^{2}p(0)=\left(\tfrac{3}{2}\right)^{2}\cdot 1=\tfrac{9}{4}=2.25$$

+12¢

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})^{3}p(0)=\left(\tfrac{4}{3}\right)^{3}\cdot 1=\tfrac{64}{27}=2.37$$

+7¢

$$p^{\frac{1}{4}}(t)=(1+r\cdot\tfrac{t}{4})^{4}p(0)=\left(\tfrac{5}{4}\right)^{4}\cdot 1=\tfrac{625}{256}=2.44$$

Monthly: $p^{\frac{1}{12}}(t)=(1+r\cdot\tfrac{t}{12})^{12}p(0)=\left(\tfrac{13}{12}\right)^{12}\cdot 1=2.613$

+17¢

+8¢

Weekly: $p^{\frac{1}{52}}(t)=(1+r\cdot\tfrac{t}{52})^{52}p(0)=\left(\tfrac{53}{52}\right)^{52}\cdot 1=2.693$

+2¢

Daily: $p^{\frac{1}{365}}(t)=(1+r\cdot\tfrac{t}{365})^{365}p(0)=\left(\tfrac{366}{365}\right)^{365}\cdot 1=\mathbf{2.714}5$

+zip¢

Hrly: $p^{\frac{1}{8760}}(t)=(1+r\cdot\tfrac{t}{8760})^{8760}p(0)=\left(\tfrac{8761}{8760}\right)^{8760}\cdot 1=\mathbf{2.718}1$

Interest product formula is *<u>really</u>* *inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...*

$$p^{1/m}(1) = (1+\tfrac{1}{m})^m \xrightarrow[m\to\infty]{} 2.718281828459.. = e$$

$$(1+\tfrac{1}{m})^{m\cdot r\cdot t} \xrightarrow[m\to\infty]{} e^{r\cdot t}$$

Let: $m{\cdot}r{\cdot}t=n$

or: $1/m= r{\cdot}t/n$

$$(1+\tfrac{r\cdot t}{n})^{n} \xrightarrow[n\to\infty]{} e^{r\cdot t}$$

$p^{1/m}(1)$ = **2.71**69239322 *for m = 1,000*

$p^{1/m}(1)$ = **2.718**1459268 *for m = 10,000*

$p^{1/m}(1)$ = **2.7182**682372 *for m = 100,000*

$p^{1/m}(1)$ = **2.71828**04693 *for m = 1,000,000*

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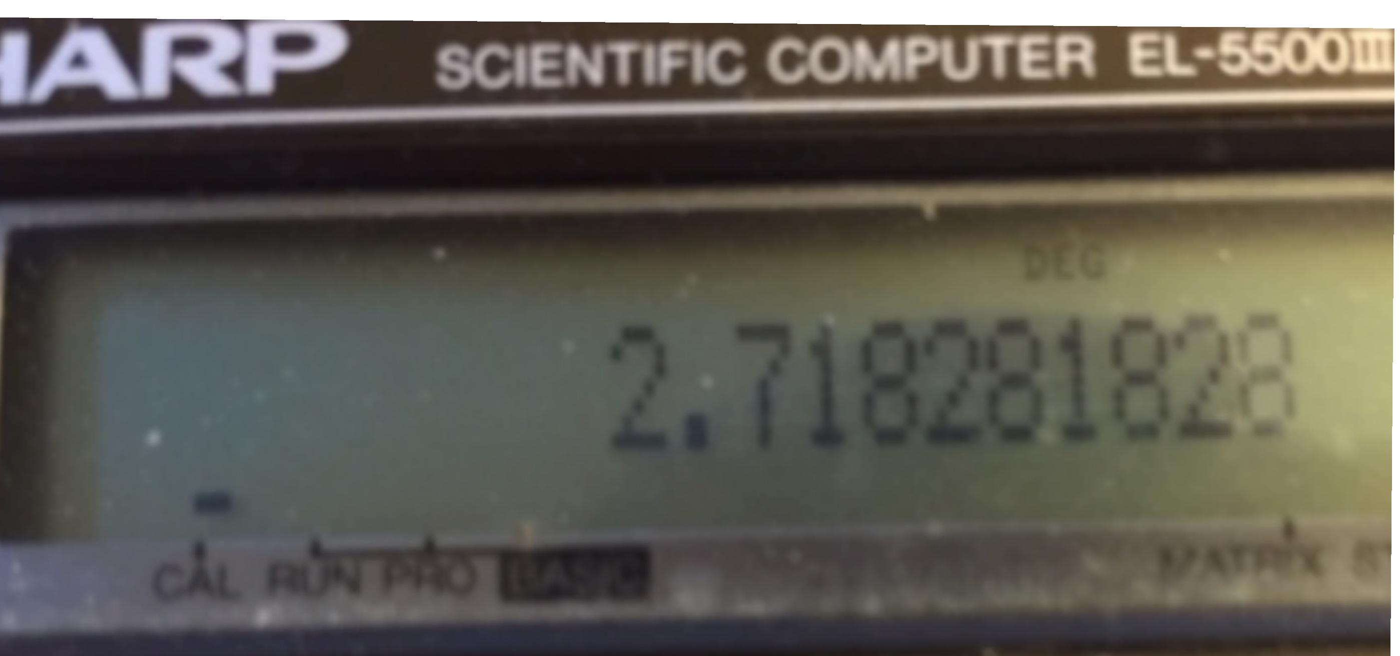

Interest product formula is <u>really</u> inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...

$$p^{1/m}(1) = \left(1+\frac{1}{m}\right)^m \xrightarrow[m\to\infty]{} \mathbf{2.718281828459..} = e$$

Let: $m\cdot r\cdot t = n$

or: $1/m = r\cdot t/n$

$$\left(1+\frac{1}{m}\right)^{m\cdot r\cdot t} \xrightarrow[m\to\infty]{} e^{r\cdot t}$$

$$\left(1+\frac{r\cdot t}{n}\right)^{n} \xrightarrow[n\to\infty]{} e^{r\cdot t}$$

$p^{1/m}(1) = \mathbf{2.71}69239322$ for $m = 1{,}000$

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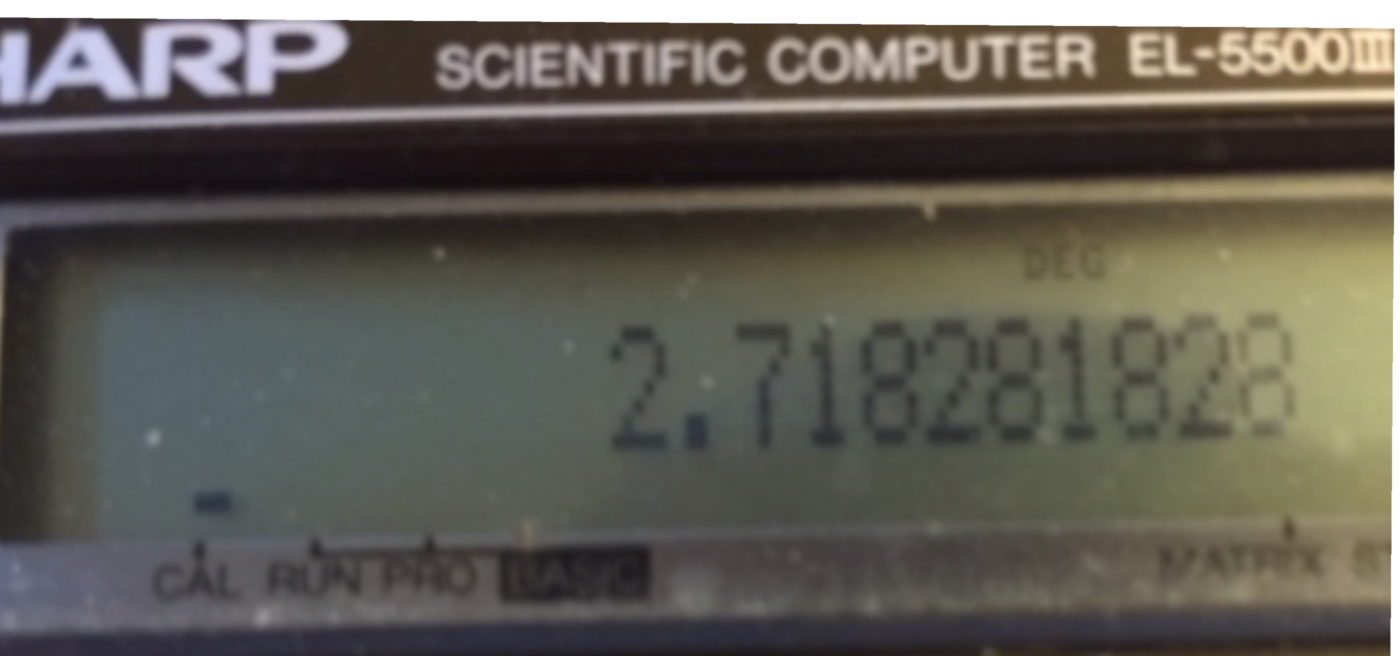
Can improve computational efficiency using binomial theorem:

$$(x+y)^n = x^n + n\cdot x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + n\cdot xy^{n-1} + y^n$$

$$\left(1+\frac{r\cdot t}{n}\right)^n = 1 + n\cdot\left(\frac{r\cdot t}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{r\cdot t}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{r\cdot t}{n}\right)^3 + \ldots$$

Define: *Factorials*(!):

$0! = 1 = 1!$, $2! = 1\cdot 2$, $3! = 1\cdot 2\cdot 3, \ldots$



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$p^{1/m}(1) = \mathbf{2.718280}4693$	*for m = 1,000,000*
$p^{1/m}(1) = \mathbf{2.7182816}925$	*for m = 10,000,000*
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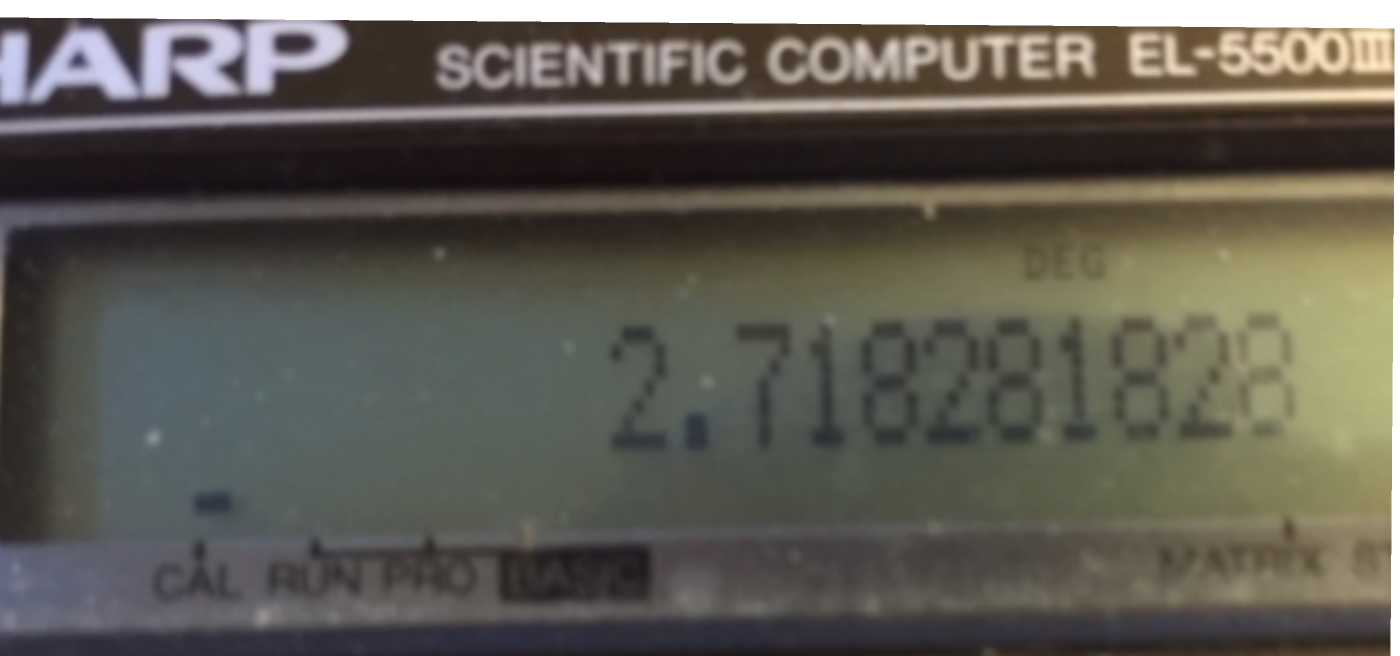
As $n \to \infty$ *let* :

$$n(n-1) \to n^2,$$

$$n(n-1)(n-2) \to n^3, \textit{ etc.}$$

$$\boxed{e^{r\cdot t} = 1 + r\cdot t + \frac{1}{2!}(r\cdot t)^2 + \frac{1}{3!}(r\cdot t)^3 + \ldots = \sum_{p=0}^{o} \frac{(r\cdot t)^p}{p!}}$$



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Precision order:

(o=1)-e-series = **2**.00000 =1+1

(o=2)-e-series = **2**.50000 =1+1+1/2

(o=3)-e-series = **2**.66667 =1+1+1/2+1/6

(o=4)-e-series = **2.7**0833 =1+1+1/2+1/6+1/24

(o=5)-e-series = **2.71**667 =1+1+1/2+1/6+1/24+1/120

(o=6)-e-series = **2.718**05 =1+1+1/2+1/6+1/24+1/120+1/720

(o=7)-e-series = **2.718**25

(o=8)-e-series = **2.71828**

About 12 summed quotients for 6-figure precision (A lot better!)

Power Series Good! Need general power series development

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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Rate of change of position $x(t)$ is *velocity $v(t)$*.

Set $t=0$ to get $c_1 = v(0)$.

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Gives Maclaurin (or Taylor) power series

$$\boxed{x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +}$$

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$$\boxed{x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2} + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +$$

Good old UP I formula!

Power Series Good! Need general power series development

Start with a general power series with constant coefficients c_0, c_1, *etc.* Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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Gives Maclaurin (or Taylor) power series

$$\boxed{\boxed{x(t) = x(0) + v(0)t + \tfrac{1}{2!} a(0)t^2} + \tfrac{1}{3!} j(0)t^3 + \tfrac{1}{4!} i(0)t^4 + \tfrac{1}{5!} r(0)t^5 + \ldots + \tfrac{1}{n!} x^{(n)} t^n +}$$

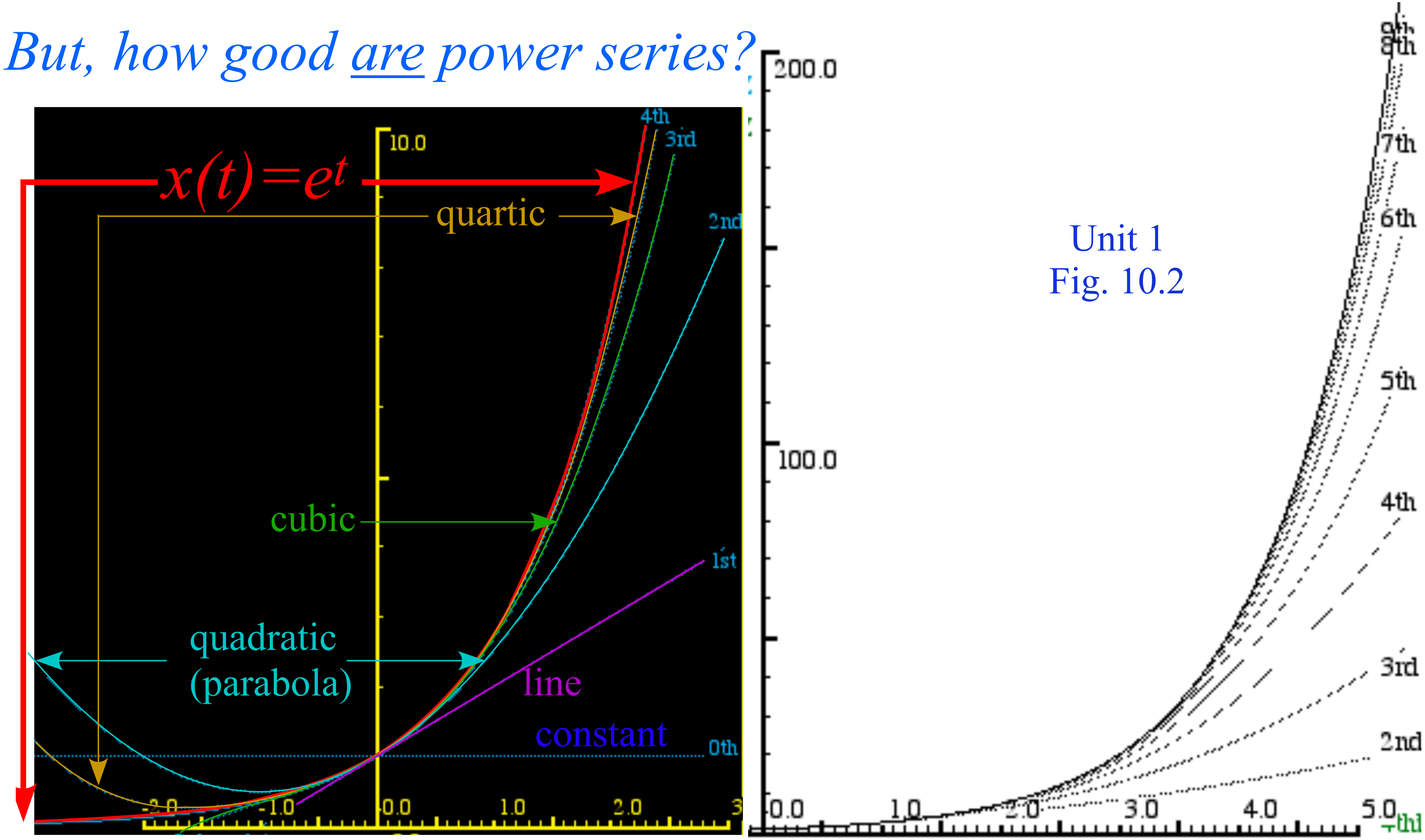
Good old UP I formula!

Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots$

gives *exponential:*

$$e^t = 1 + t + \tfrac{1}{2!} t^2 + \tfrac{1}{3!} t^3 + \tfrac{1}{4!} t^4 + \tfrac{1}{5!} t^5 + \ldots + \tfrac{1}{n!} t^n +$$

But, how good are power series?



Unit 1
Fig. 10.2

Gives Maclaurin (or Taylor) power series

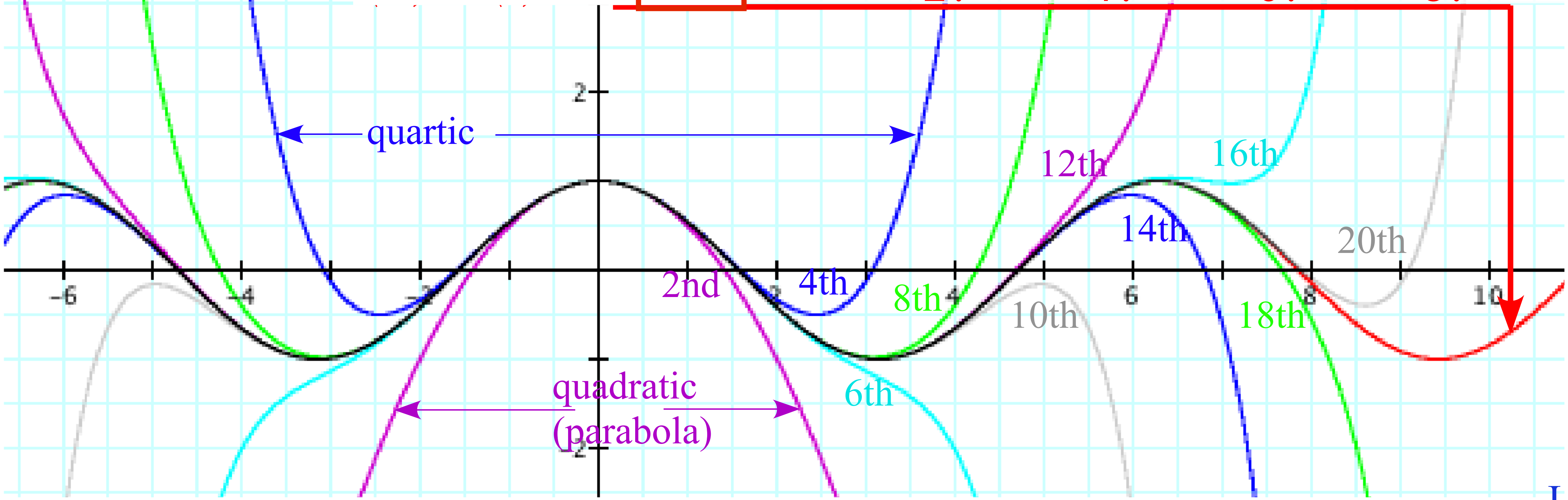
$$x(t) = x(0) + v(0)t + \tfrac{1}{2!}a(0)t^2 + \tfrac{1}{3!}j(0)t^3 + \tfrac{1}{4!}i(0)t^4 + \tfrac{1}{5!}r(0)t^5 + \ldots + \tfrac{1}{n!}x^{(n)}t^n +$$

Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots.$

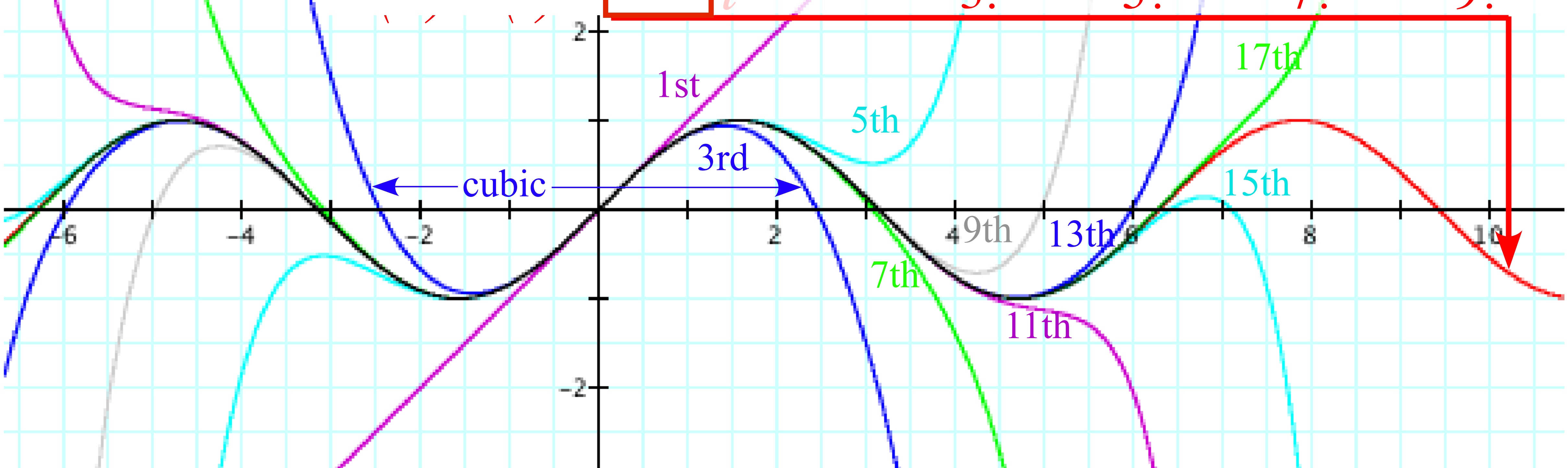
gives *exponential:* $$e^t = 1 + t + \tfrac{1}{2!}t^2 + \tfrac{1}{3!}t^3 + \tfrac{1}{4!}t^4 + \tfrac{1}{5!}t^5 + \ldots + \tfrac{1}{n!}t^n +$$

How good are power series? Depends...

$$x(t) = \boxed{\cos t} = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \ldots$$



$$x(t) = \boxed{\sin t} = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \ldots$$



Unit 1
Fig. 10.3

1. The Story of e (A Tale of Great \$Interest\$)

How good are those power series?

Taylor-Maclaurin series,

→ *imaginary interest, and complex exponentials*

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1, etc...$

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+... \qquad \text{(From exponential series)}$$

$$=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-... \qquad (i=\sqrt{-1}\text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,...)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-...\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-...\right)$$

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1, etc...$

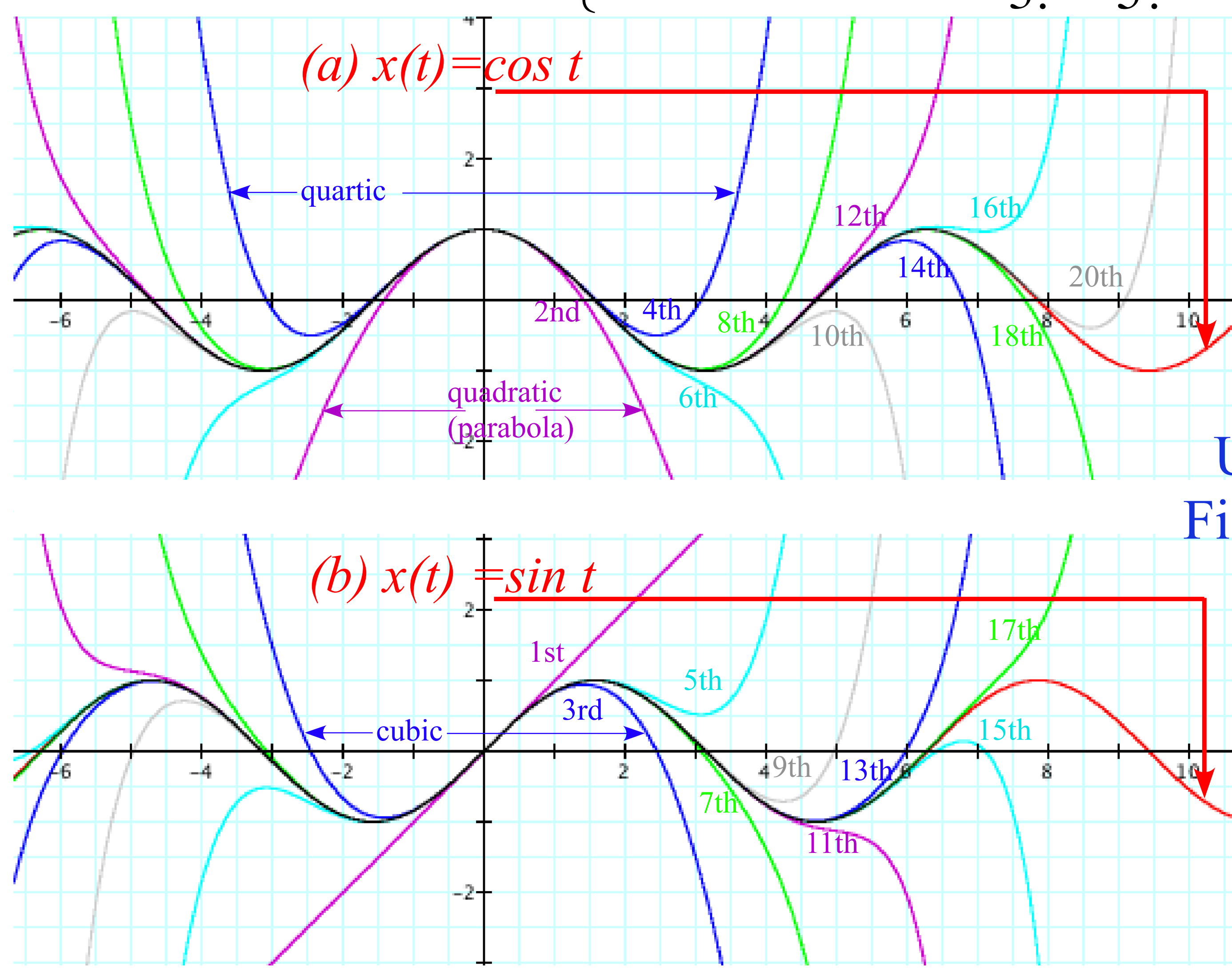
$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots \qquad \text{(From exponential series)}$$

$$=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\ldots \qquad (i=\sqrt{-1}\ \text{imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right) \quad \text{To match series for} \begin{cases} \textit{cosine}: \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \\ \textit{sine}: \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \end{cases}$$

$$e^{i\theta}=\cos\theta+i\sin\theta$$

Euler-DeMoivre Theorem



Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1$, *etc...*

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots \qquad \text{(From exponential series)}$$

$$=1+i\theta-\frac{\theta^2}{2!}\ -i\frac{\theta^3}{3!}\ +\frac{\theta^4}{4!}\ +i\frac{\theta^5}{5!}\ -\ldots \qquad (i=\sqrt{-1}\text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

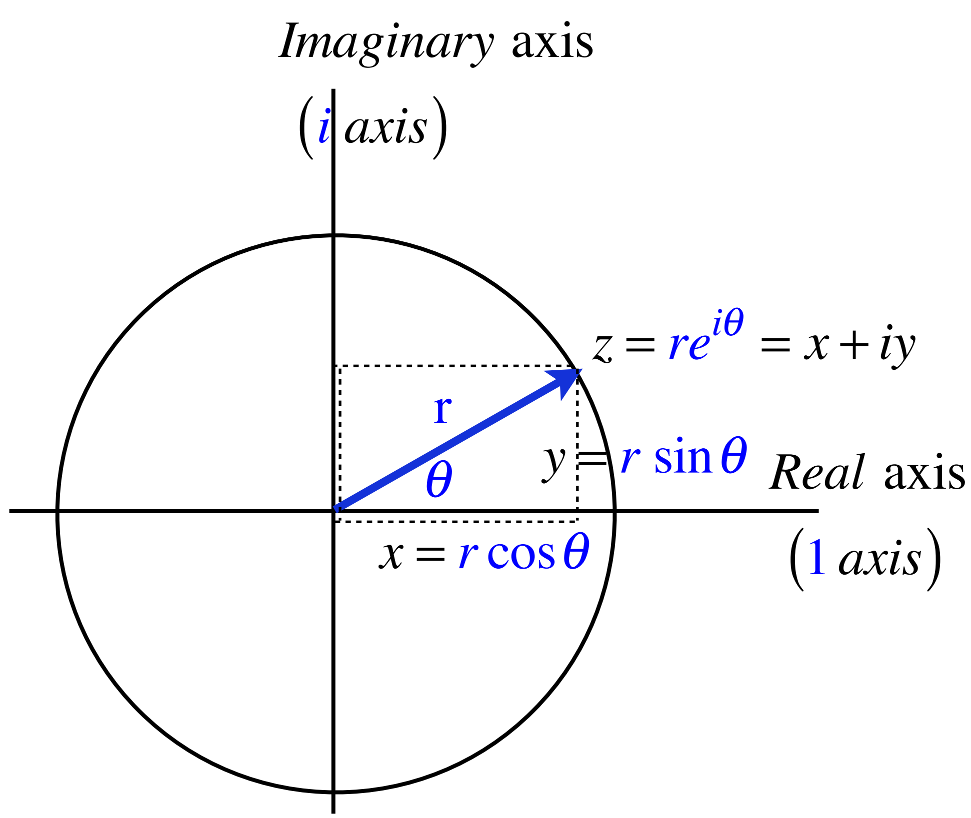
$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

To match series for

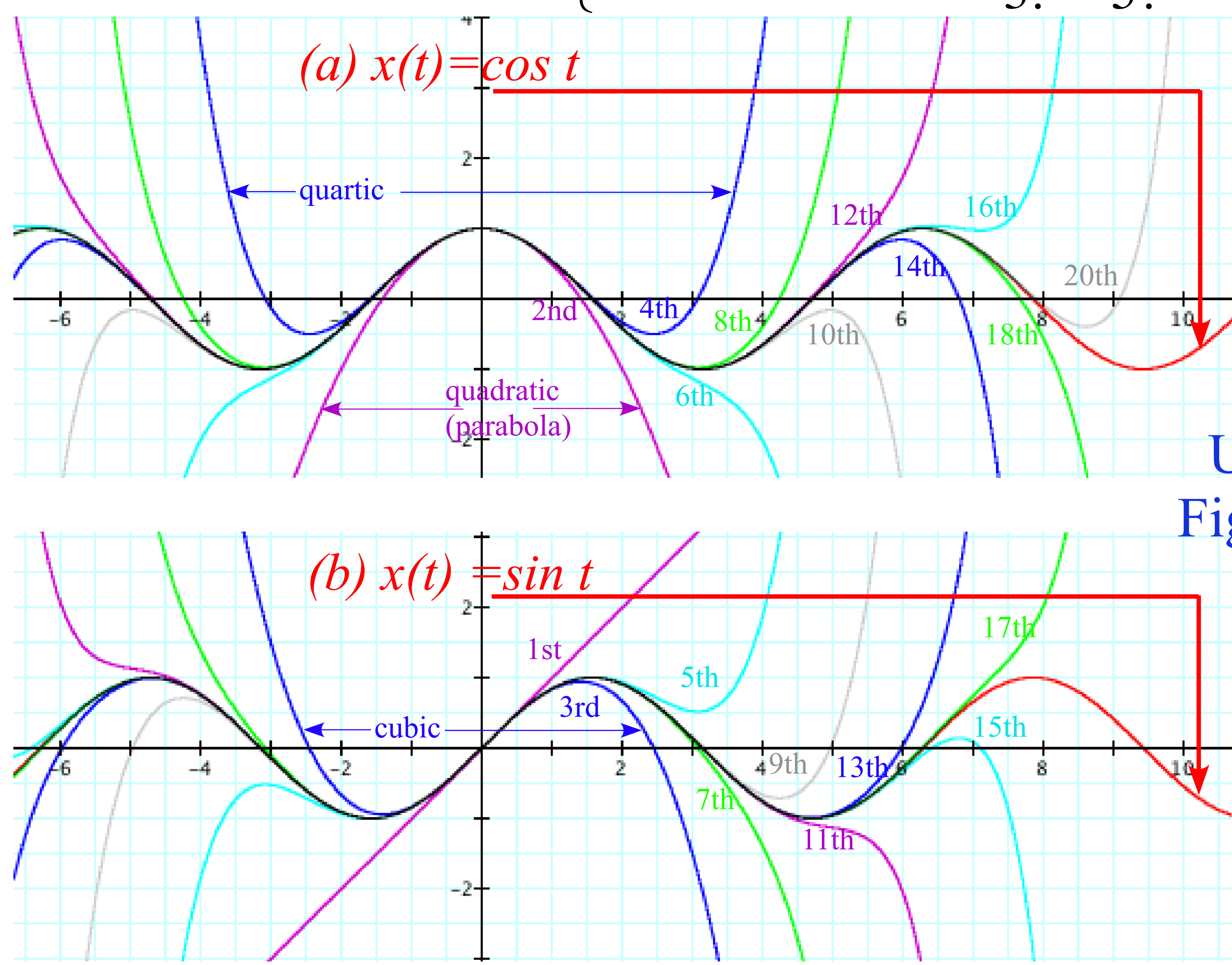
$$\begin{cases} \textit{cosine}: \cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots \\ \textit{sine}: \sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots \end{cases}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler-DeMoivre Theorem



$$re^{i\theta} = r\cos\theta + i\sin\theta$$



Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern: $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$, etc...*

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots \qquad \text{(From exponential series)}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots \qquad (i=\sqrt{-1} \text{ imples: } i^1{=}i,\ i^2{=}{-1},\ i^3{=}{-i},\ i^4{=}{+1},\ i^5{=}i,\ldots)$$

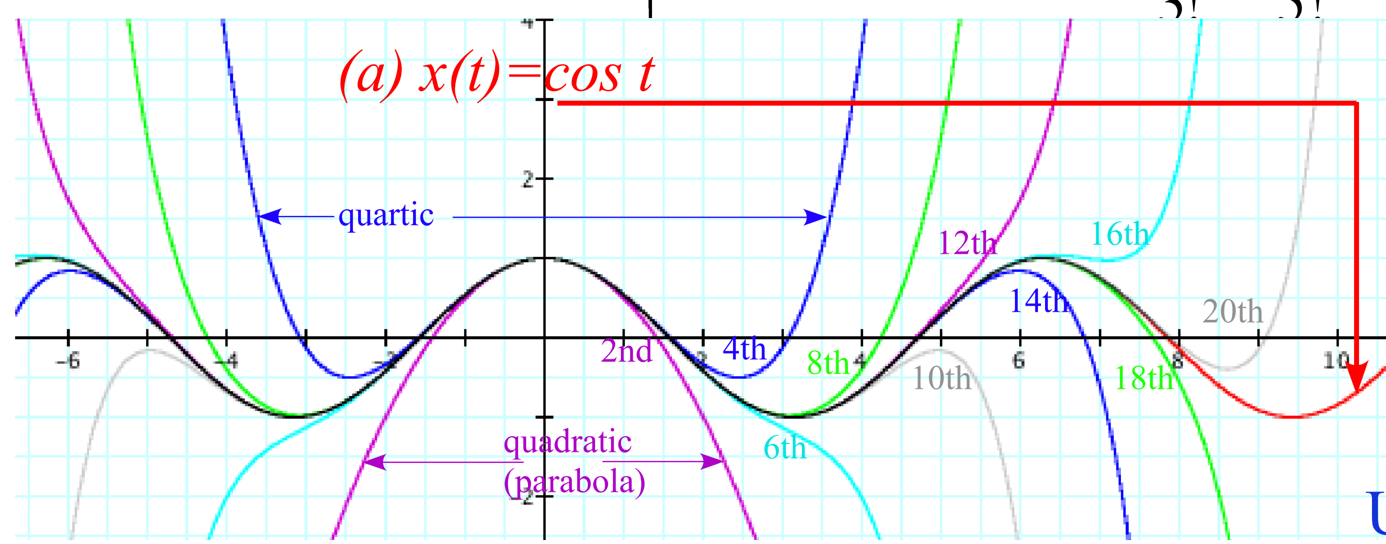
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \ldots\right)$$

To match series for

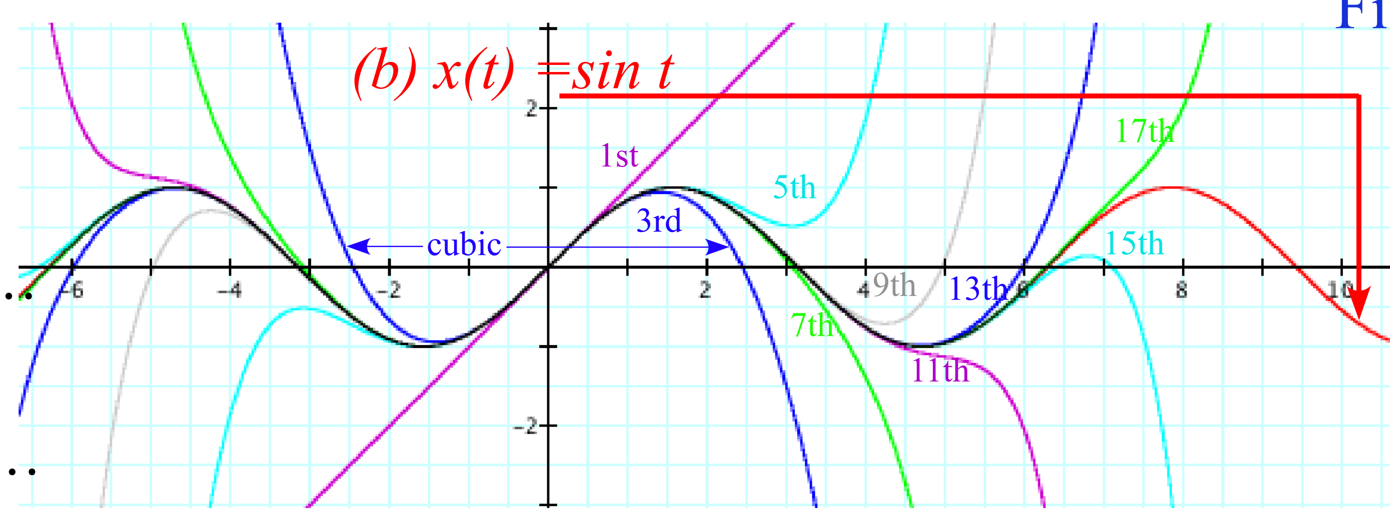
$$\begin{cases} \textit{cosine}: \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \\ \textit{sine}: \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \end{cases}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler-DeMoivre Theorem



Unit 1
Fig. 10.3

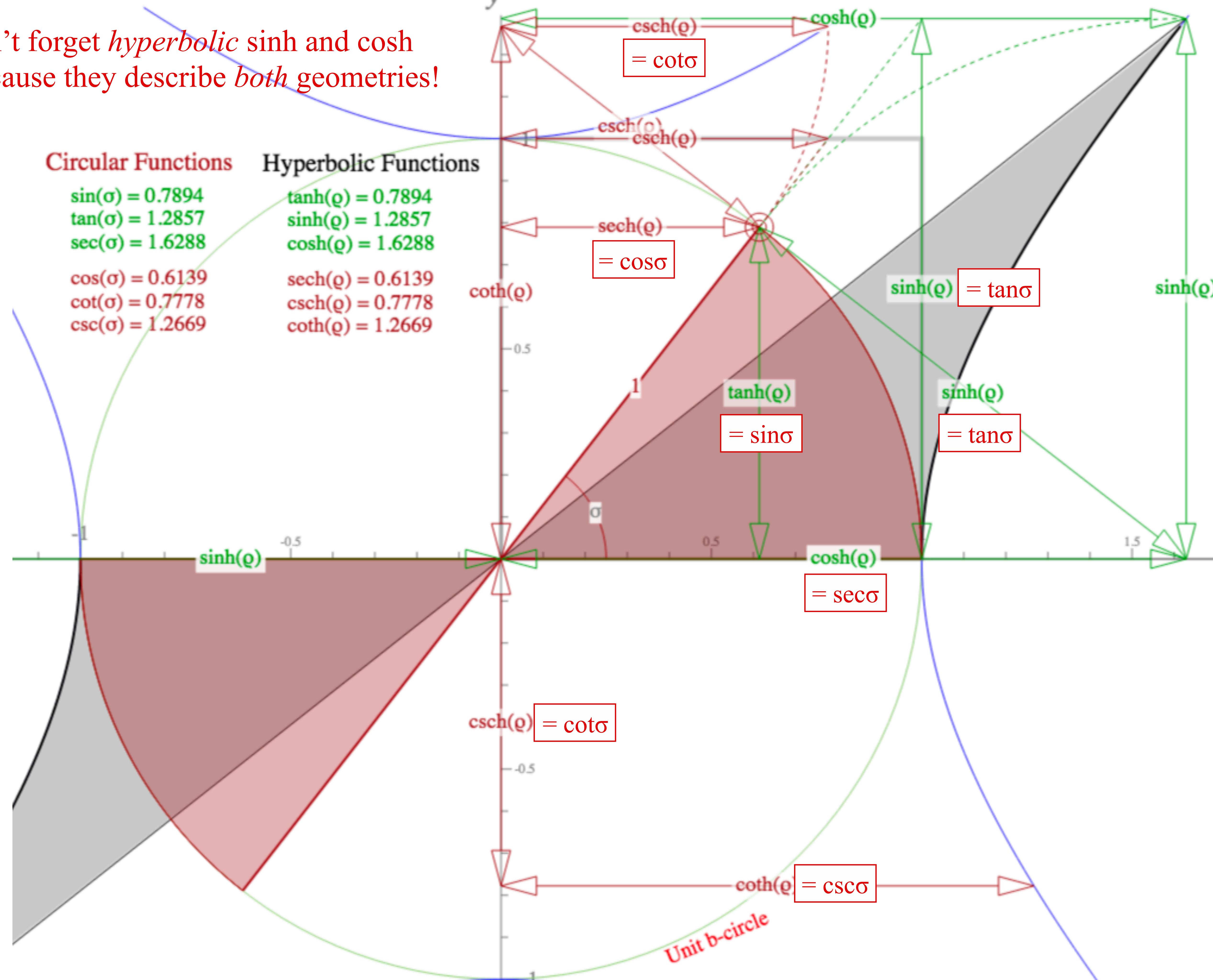


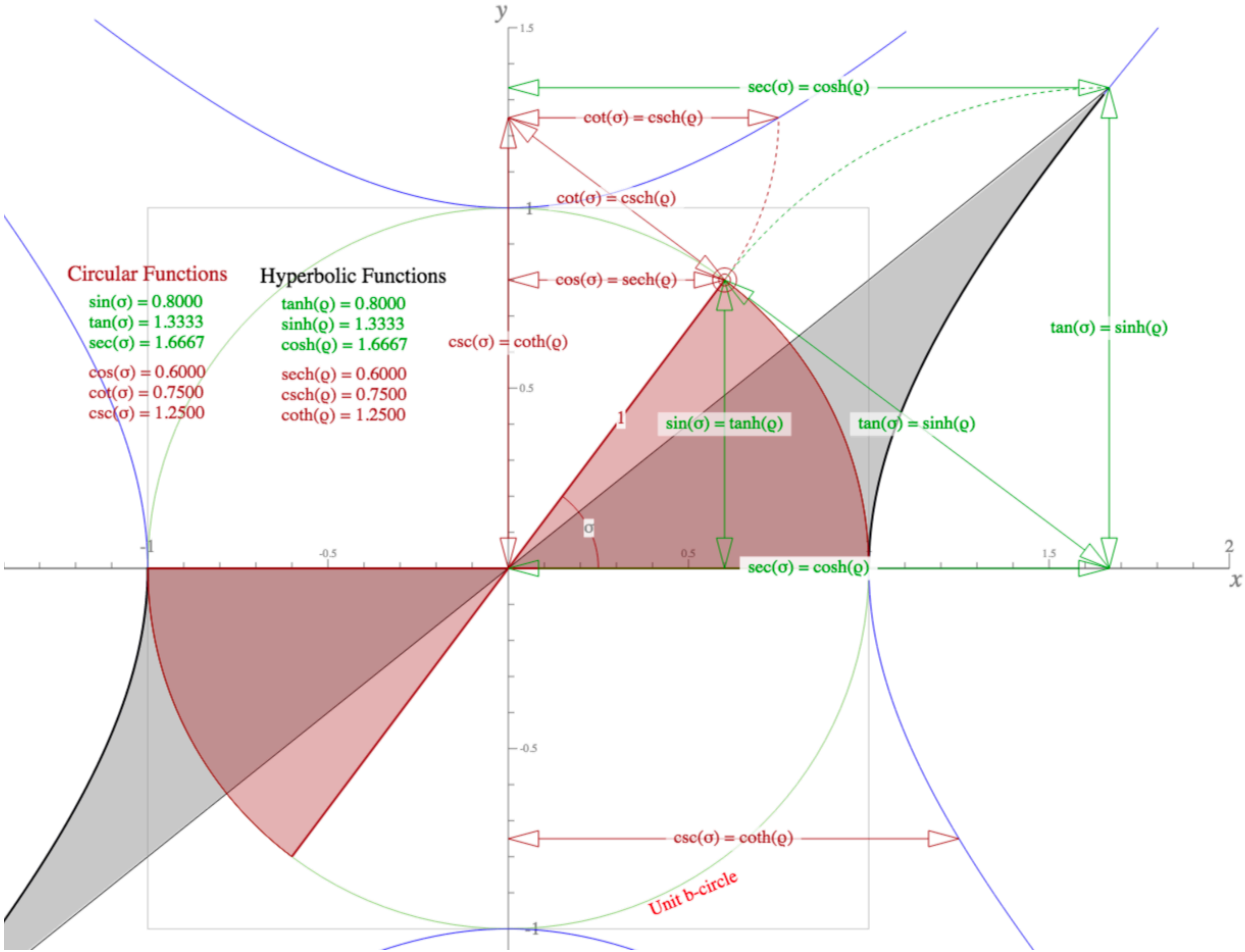
!!Don't forget *hyperbolic* sinh and cosh

$$e^{\rho} = \cosh\rho + \sinh\rho$$

$$\begin{cases} \textit{hypercosine}: \cosh\rho = 1 + \dfrac{\rho^2}{2!} + \dfrac{\rho^4}{4!} + \dfrac{\rho^6}{6!} + \cdots \\ \textit{hypersine}: \sinh\rho = \rho + \dfrac{\rho^3}{3!} + \dfrac{\rho^5}{5!} + \dfrac{\rho^7}{7!} + \cdots \end{cases}$$

!!Don't forget *hyperbolic* sinh and cosh
…because they describe *both* geometries!





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2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

Easy rotation and “dot” or “cross” products

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is cos$(a+b)$ *or* sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

$$e^{i(a+b)} = e^{ia} \qquad e^{ib}$$

$$\cos(a+b) + i\sin(a+b) = (\cos a + i \sin a)\,(\cos b + i \sin b)$$

$$\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} = \boxed{[\cos a \cos b - \sin a \sin b]} + i\,\boxed{[\sin a \cos b + \cos a \sin b]}$$

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

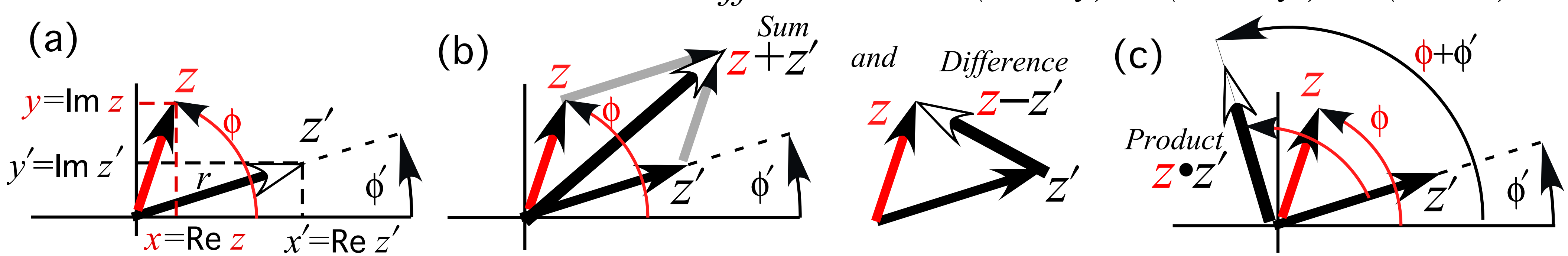
Can't remember is cos$(a+b)$ or sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

$$\begin{aligned}
e^{i(a+b)} &= e^{ia}\,e^{ib}\\
\cos(a+b) + i\sin(a+b) &= (\cos a + i\sin a)(\cos b + i\sin b)\\
\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} &= \boxed{[\cos a\cos b - \sin a\sin b]} + i\,\boxed{[\sin a\cos b + \cos a\sin b]}
\end{aligned}$$

2. Complex numbers add like vectors.

$$z_{sum} = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$$

$$z_{diff} = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y')$$



Unit 1
Fig. 10.6

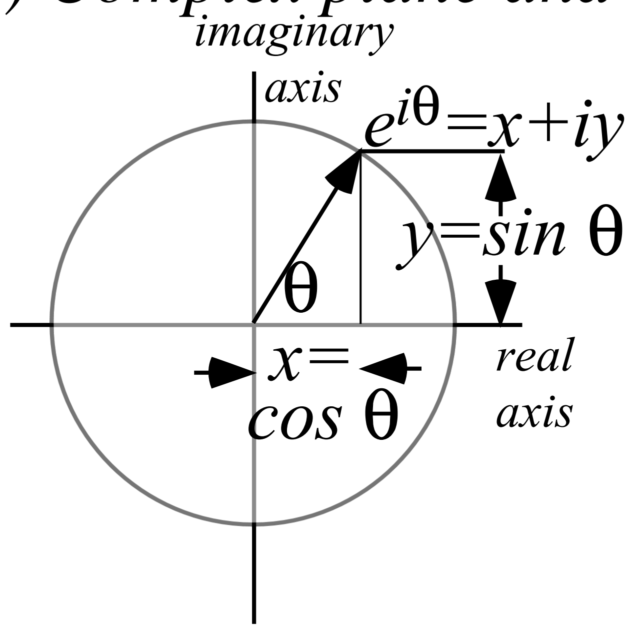
$$|z_{SUM}| = \sqrt{(z+z')^*(z+z')} = \sqrt{\left(re^{i\phi} + r'e^{i\phi'}\right)^*\left(re^{i\phi} + r'e^{i\phi'}\right)} = \sqrt{\left(re^{-i\phi} + r'e^{-i\phi'}\right)\left(re^{i\phi} + r'e^{i\phi'}\right)}$$

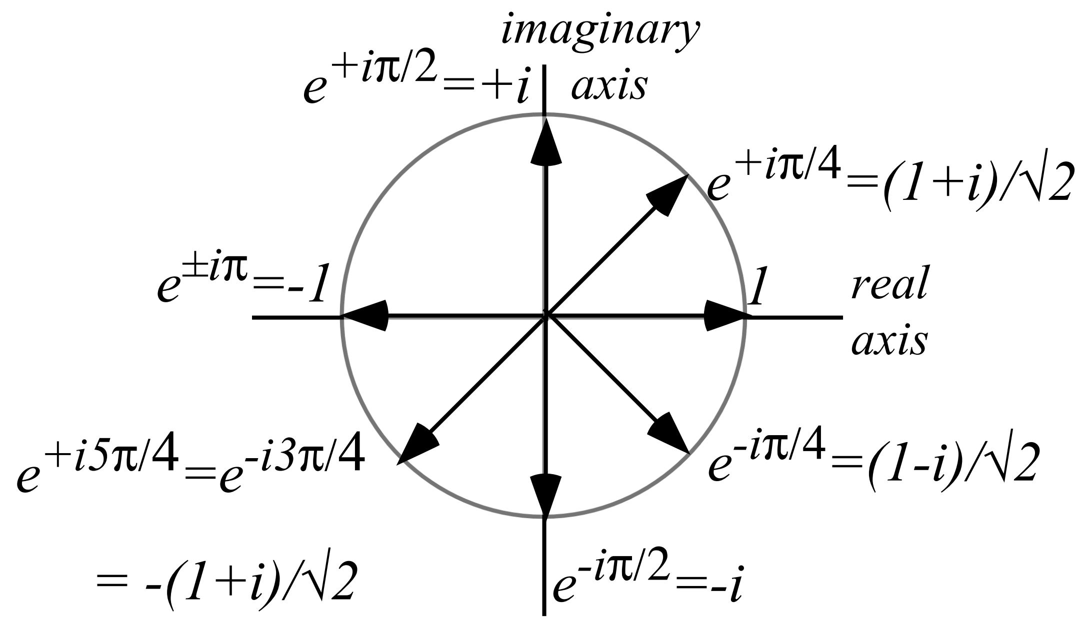
$$= \sqrt{r^2 + r'^2 + rr'\left(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right)} = \sqrt{r^2 + r'^2 + 2rr'\cos(\phi-\phi')}$$

(quick derivation of Cosine Law)

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

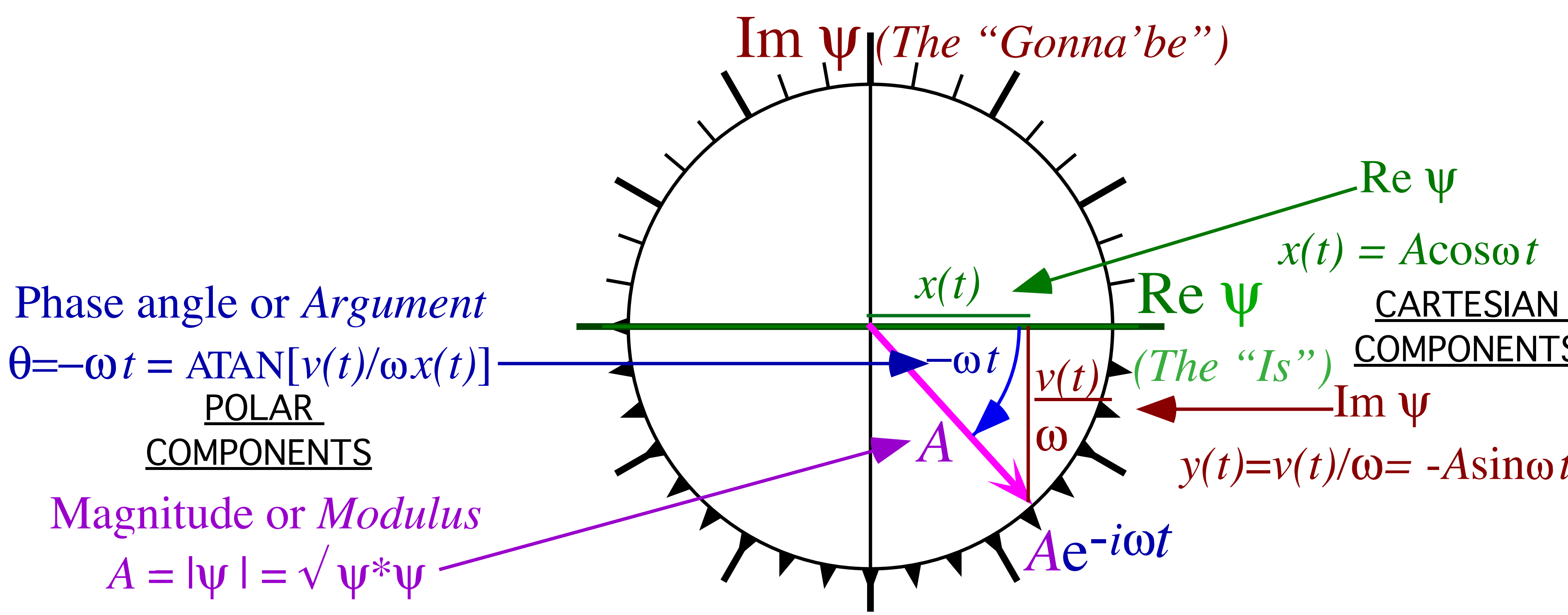
(a) Complex plane and unit vectors





(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$

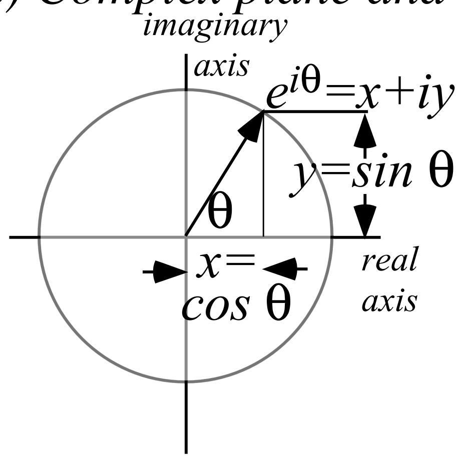
Unit 1
Fig. 10.5

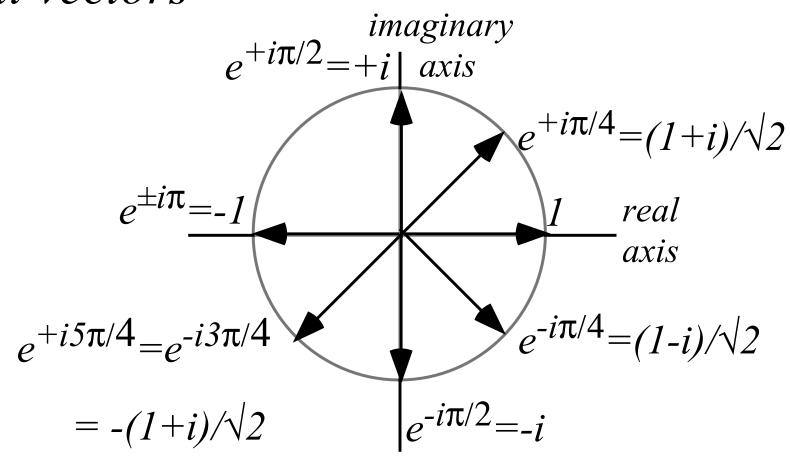


What Good Are Complex Exponentials? (contd.)

3. Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors





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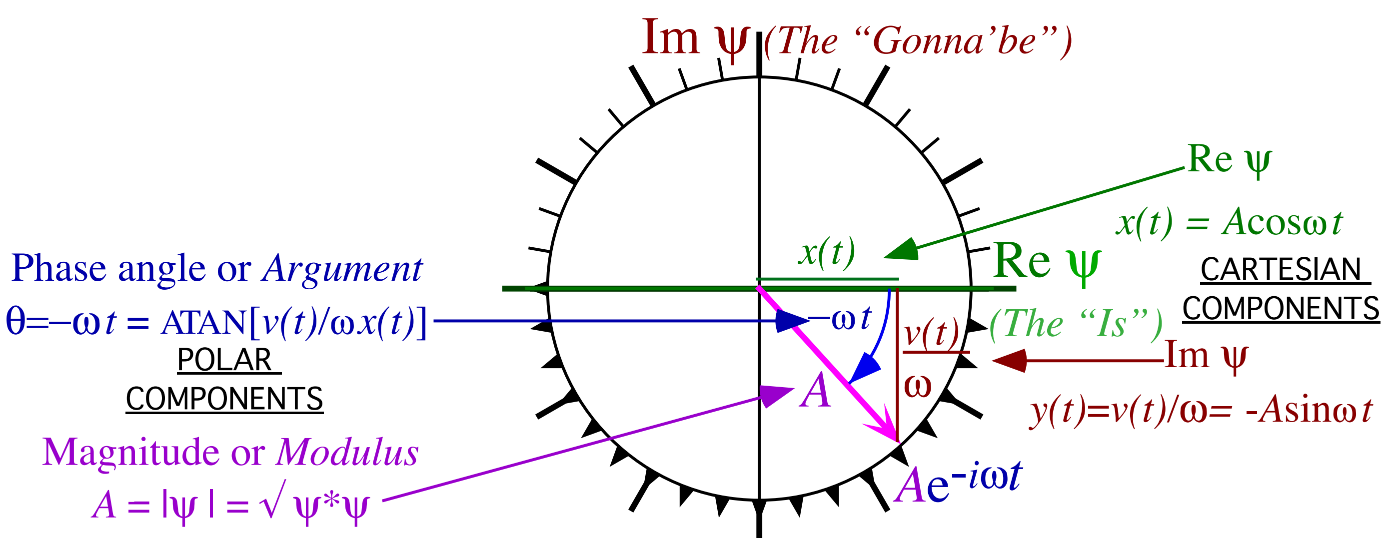

Unit 1
Fig. 10.5

Some Rect-vs-Polar *relations worth remembering*

Cartesian (x,y) form

$$\psi_x = \mathrm{Re}\,\psi(t) = x(t) = A\cos\omega t = \frac{\psi+\psi^*}{2}$$

$$\psi_y = \mathrm{Im}\,\psi(t) = \frac{v(t)}{\omega} = -A\sin\omega t = \frac{\psi-\psi^*}{2i}$$

$$\psi = re^{+i\theta} = re^{-i\omega t} = r(\cos\omega t - i\sin\omega t)$$

$$\psi^* = re^{-i\theta} = re^{+i\omega t} = r(\cos\omega t + i\sin\omega t)$$

Polar (r,θ) form

$$r = A = |\psi| = \sqrt{\psi_x^{\,2}+\psi_y^{\,2}} = \sqrt{\psi^*\psi}$$

$$\theta = -\omega t = \arctan(\psi_y/\psi_x)$$

$$\cos\theta = \tfrac{1}{2}(e^{+i\theta}+e^{-i\theta}) \qquad \mathrm{Re}\,\psi = \frac{\psi+\psi^*}{2}$$

$$\sin\theta = \tfrac{1}{2i}(e^{+i\theta}-e^{-i\theta}) \qquad \mathrm{Im}\,\psi = \frac{\psi-\psi^*}{2i}$$

2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

Easy rotation and "dot" or "cross" products

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x + iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x+iy) = x\cos\phi - y\sin\phi \quad + i\ (x\sin\phi + y\cos\phi)$$

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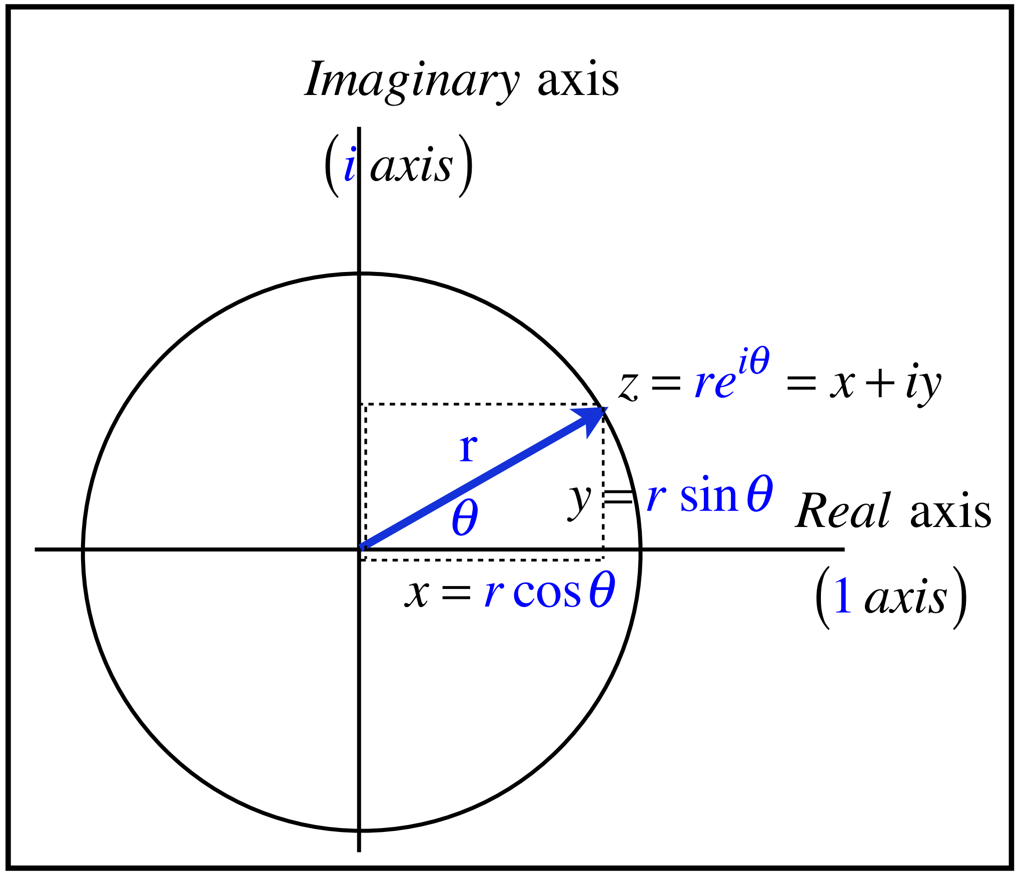
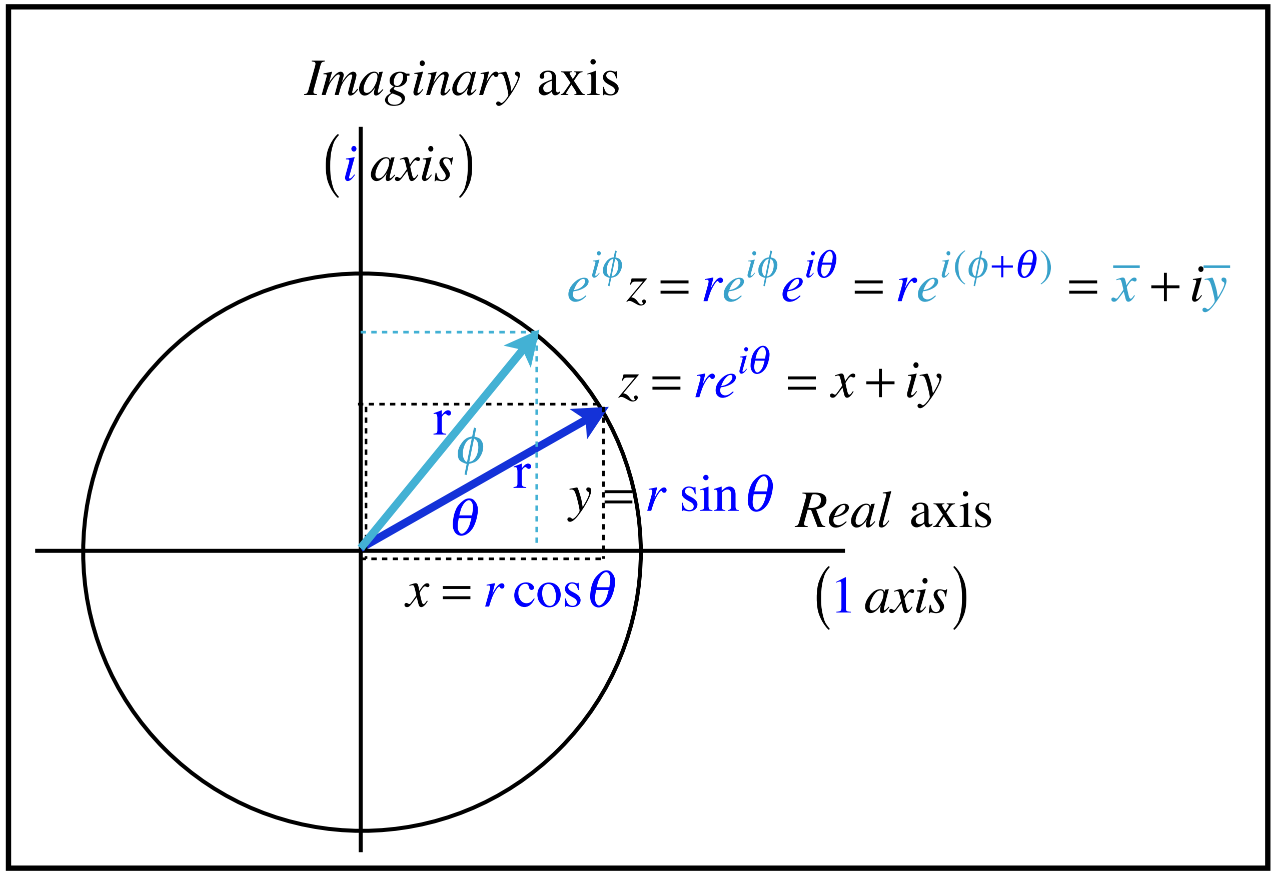
$e^{i\phi}$ acts on this: $z = re^{i\theta}$

to give this: $e^{i\phi}\ e^{i\phi}z = re^{i\phi}e^{i\theta}$





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5. Complex products provide 2D "dot"(•) and "cross"(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their "star" (*)-product A^*B.

$$A^*B = (A_x+iA_y)^*(B_x+iB_y) = (A_x-iA_y)(B_x+iB_y)$$

$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

Real part is scalar or "dot"(•) product **A•B**.

Imaginary part is vector or "cross"(×) product, but <u>just</u> the Z-component <u>*normal*</u> to xy-plane.

Rewrite A^*B in polar form.

$$A^*B = (|A|e^{i\theta_A})^*(|B|e^{i\theta_B}) = |A|e^{-i\theta_A}\,|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$

$$= |A||B|\cos(\theta_B-\theta_A) + i|A||B|\sin(\theta_B-\theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

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$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x+iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\cos\phi - y\sin\phi\\ x\sin\phi + y\cos\phi\end{pmatrix}$$

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$$\mathbf{A}\bullet\mathbf{B} = |A||B|\cos(\theta_B - \theta_A)$$
$$= |A|\cos\theta_A|B|\cos\theta_B + |A|\sin\theta_A|B|\sin\theta_B$$
$$= A_xB_x + A_yB_y$$

$$|\mathbf{A}\times\mathbf{B}| = |A||B|\sin(\theta_B - \theta_A)$$
$$= |A|\cos\theta_A|B|\sin\theta_B - |A|\sin\theta_A|B|\cos\theta_B$$
$$= A_xB_y - A_yB_x$$

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

6. Complex derivative contains "divergence"($\nabla \cdot \mathbf{F}$) and "curl"($\nabla \times \mathbf{F}$) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

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$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

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$$\boxed{\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\cdot\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{1}{2}\nabla\bullet\mathbf{f} + \frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

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$$\frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla \bullet \mathbf{f}$ and imaginary part has curl $\nabla \times \mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\overbrace{\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)} + \frac{i}{2}\overbrace{\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)} = \frac{1}{2}\nabla \bullet \mathbf{f} + \frac{i}{2}|\nabla \times \mathbf{f}|_{Z\perp(x,y)}$$

7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ and $\nabla \times \mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function $f(z)$, conjugate it (change all i's to $-i$) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz} = 0}$.

6. Complex derivative contains "divergence"($\nabla\bullet\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

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Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\underbrace{\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)}_{\nabla\bullet\mathbf{f}} + \frac{i}{2}\underbrace{\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)}_{\nabla\times\mathbf{f}} = \frac{1}{2}\nabla\bullet\mathbf{f} + \frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

7. Invent source-free 2D vector fields [$\nabla\bullet\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to *–i*) to give *f*(z*)* for which $\boxed{\frac{df^*}{dz} = 0}$.

For example: if *f(z)=a·z* then *f*(z*)=a·z*=a(x-iy)* is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ *has zero divergence*: $\nabla\bullet\mathbf{F}=0$ *and has zero curl*: $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ *and* $\nabla \times \mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to *–i*) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz}=0}$

For example: if $f(z)=a\cdot z$ then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

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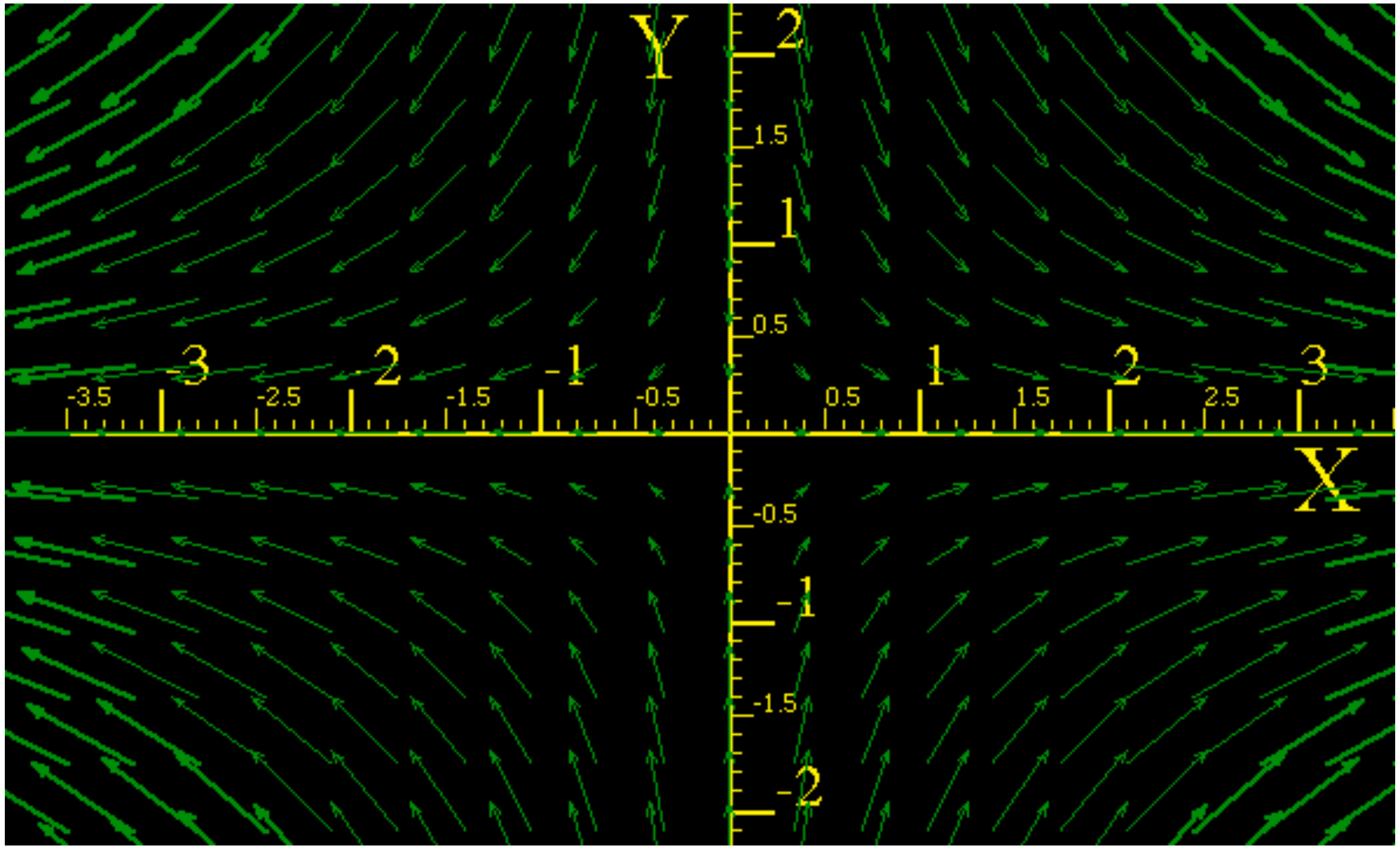

precursor to Unit 1 Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla x\mathbf{A}$) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* **Φ** or a curl of a *vector potential field* **A**.

$$\mathbf{F}=\nabla\mathbf{\Phi} \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\mathbf{\Phi}(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\mathbf{\Phi}(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

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To find $\phi=\Phi+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\,\phi = \Phi$) and imaginary ($\mathrm{Im}\,\phi = \mathbf{A}$) parts.

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$$f(z)=\frac{d\phi}{dz} \quad\Rightarrow\quad \phi = \quad \Phi \quad + i\ \mathbf{A} = \int f\cdot dz = \int az\cdot dz = \frac{1}{2}\,az^2$$

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\,\phi=\mathbf{\Phi}$) and imaginary ($\mathrm{Im}\,\phi=\mathbf{A}$) parts.

$$f(z)=\tfrac{d\phi}{dz} \quad\Rightarrow\quad \phi=\overbrace{\tfrac{1}{2}a(x^2-y^2)}^{\mathbf{\Phi}}+i\overbrace{axy}^{\mathbf{A}}=\int f\cdot dz=\int az\cdot dz=\tfrac{1}{2}az^2=\tfrac{1}{2}a(x+iy)^2$$

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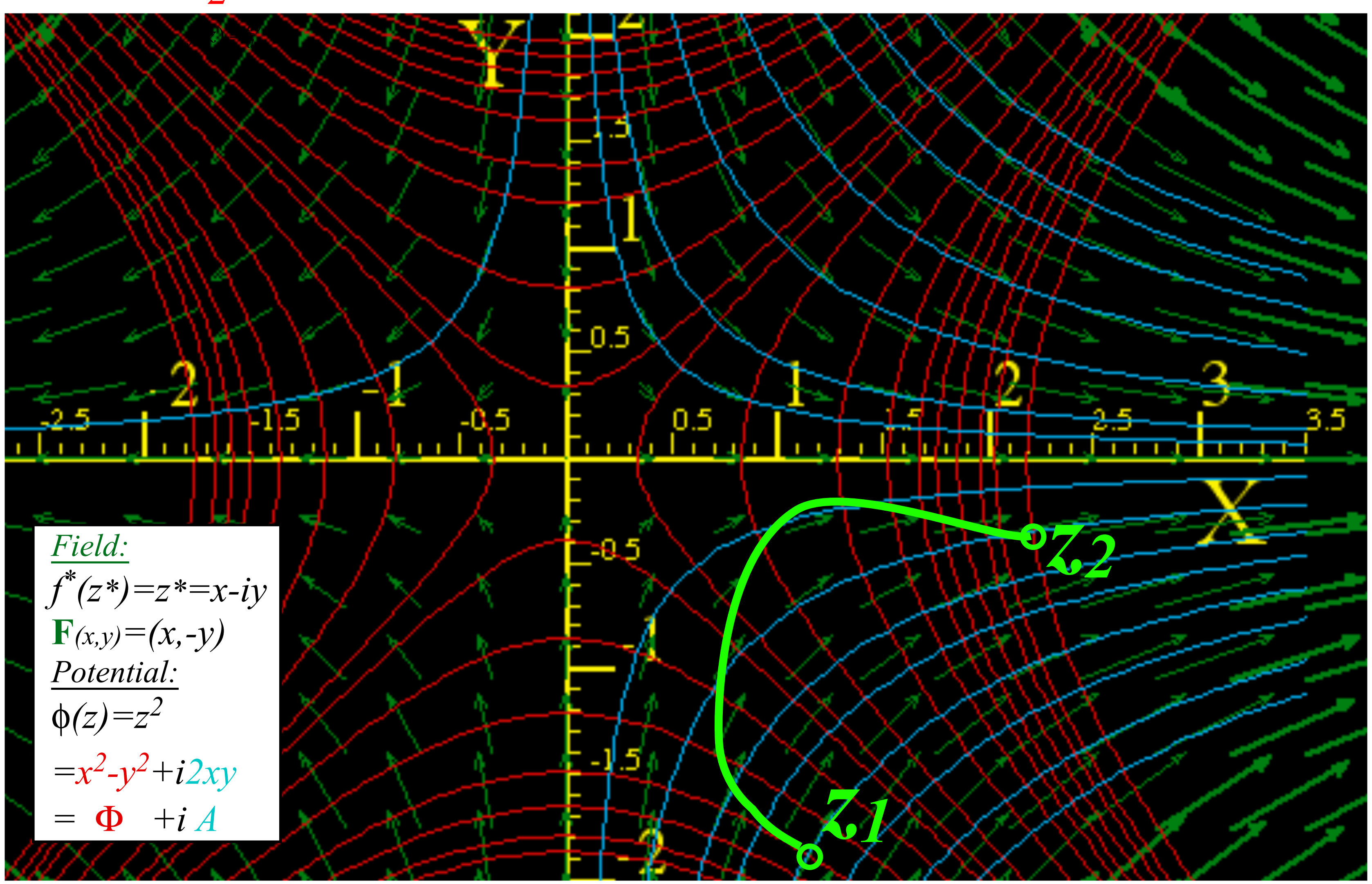

Unit 1
Fig. 10.7

What Good Are Complex Exponentials? (contd.)

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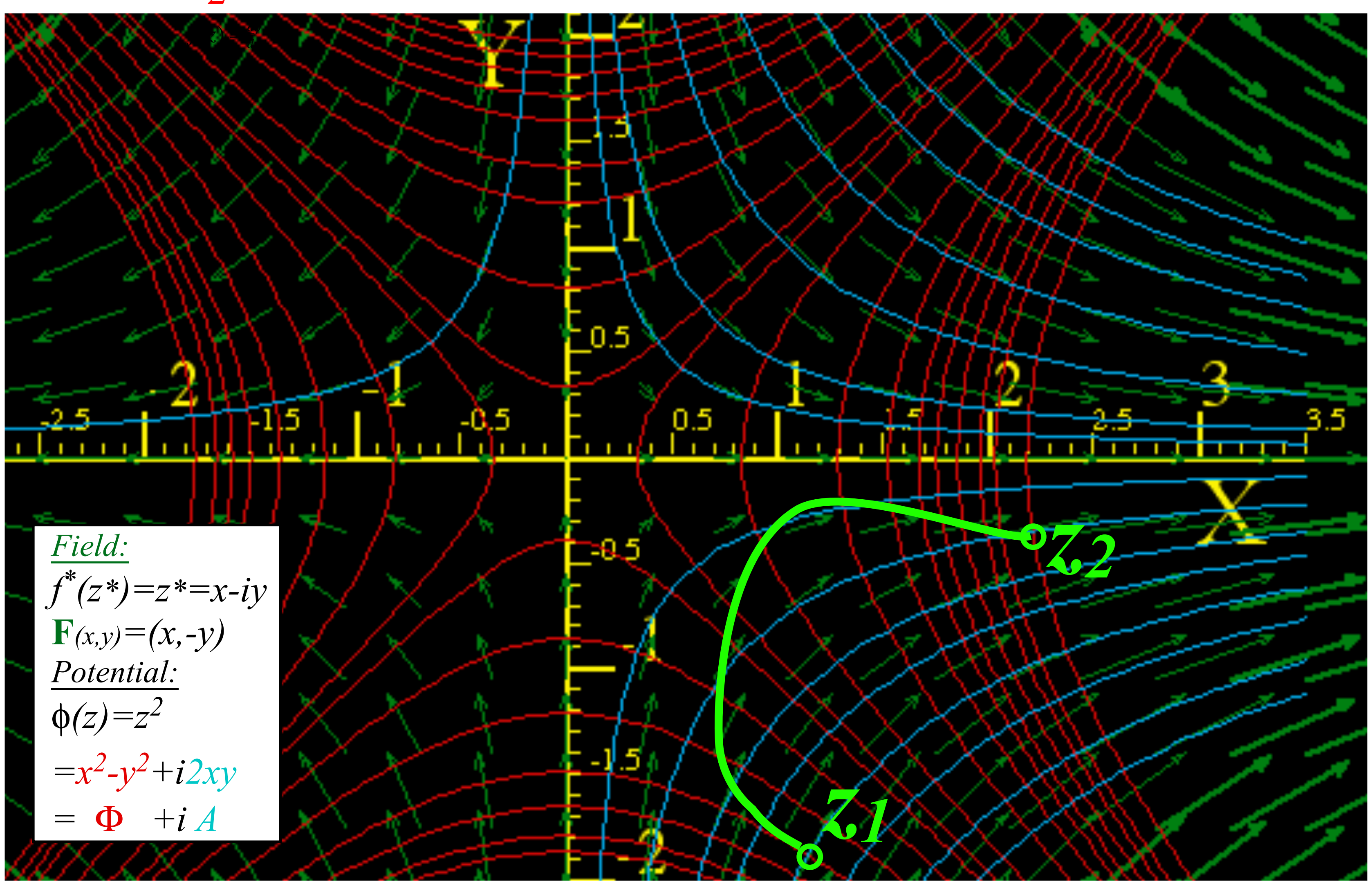
$$f(z)=\frac{d\phi}{dz} \quad\Rightarrow\quad \phi=\underbrace{\Phi}_{=\frac{1}{2}a(x^2-y^2)}+i\underbrace{\mathbf{A}}_{axy}=\int f\cdot dz=\int az\cdot dz=\frac{1}{2}az^2=\frac{1}{2}a(x+iy)^2$$

BONUS! Get a free coordinate system!



Unit 1 Fig. 10.7

The (**Φ**,**A**) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2=$ *const.*

$q^2=\mathbf{A}=(xy)=$ *const.*

*Actually it's OCC.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

The half-n'-half results: (Riemann-Cauchy Derivative Relations)

What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains "scalar"(**F=**$\nabla\boldsymbol{\Phi}$) and "vector"(**F=**∇x**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\boldsymbol{\Phi}=\begin{pmatrix}\frac{\partial\boldsymbol{\Phi}}{\partial x}\\ \frac{\partial\boldsymbol{\Phi}}{\partial y}\end{pmatrix}$ of scalar $\boldsymbol{\Phi}$ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}$ of vector **A** (*and they're <u>equal</u>!*)

$f(z)=\frac{d\phi}{dz} \quad \Rightarrow$

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\boldsymbol{\Phi}-i\mathbf{A})=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})(\boldsymbol{\Phi}-i\mathbf{A})=\frac{1}{2}\underbrace{(\frac{\partial\boldsymbol{\Phi}}{\partial x}+i\frac{\partial\boldsymbol{\Phi}}{\partial y})}+\frac{1}{2}\underbrace{(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x})}=\frac{1}{2}\nabla\boldsymbol{\Phi}+\frac{1}{2}\nabla\times\mathbf{A}$$

$$\frac{d}{dz}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz^*}=\frac{1}{2}\frac{\partial}{\partial x}+\frac{i}{2}\frac{\partial}{\partial y}$$

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

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$f(z)=\frac{d\phi}{dz}\quad\Rightarrow$

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\mathbf{\Phi}-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\mathbf{\Phi}-i\mathbf{A})=\frac{1}{2}\overbrace{\left(\frac{\partial\mathbf{\Phi}}{\partial x}+i\frac{\partial\mathbf{\Phi}}{\partial y}\right)}+\frac{1}{2}\overbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)}=\frac{1}{2}\nabla\mathbf{\Phi}+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\mathbf{\Phi}$ replaces usual physicist's definition $\mathbf{F}=-\nabla\mathbf{\Phi}$

What Good Are Complex Exponentials? (contd.)

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$f(z)=\frac{d\phi}{dz}\quad\Rightarrow$

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\boldsymbol{\Phi}-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\boldsymbol{\Phi}-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\boldsymbol{\Phi}}{\partial x}+i\frac{\partial\boldsymbol{\Phi}}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\boldsymbol{\Phi}+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+$\nabla\boldsymbol{\Phi}$ replaces usual physicist's definition **F**=-$\nabla\boldsymbol{\Phi}$

Given ϕ:
$$\boxed{\begin{aligned}\phi &= \boldsymbol{\Phi} + i\,\mathbf{A}\\ &=\tfrac{1}{2}a(x^2-y^2)+i\,axy\end{aligned}}$$

The half-n'-half result

find:

$$\nabla\boldsymbol{\Phi}=\begin{pmatrix}\frac{\partial\boldsymbol{\Phi}}{\partial x}\\ \frac{\partial\boldsymbol{\Phi}}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial \mathbf{A}}{\partial y}\\ -\frac{\partial \mathbf{A}}{\partial x}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

$f(z)=\frac{d\phi}{dz}\quad\Rightarrow$

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

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Given ϕ:
$$\boxed{\begin{aligned}\phi &= \Phi + i\,\mathbf{A}\\ &=\tfrac{1}{2}a(x^2-y^2)+i\,axy\end{aligned}}$$

The half-n'-half result

find:

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

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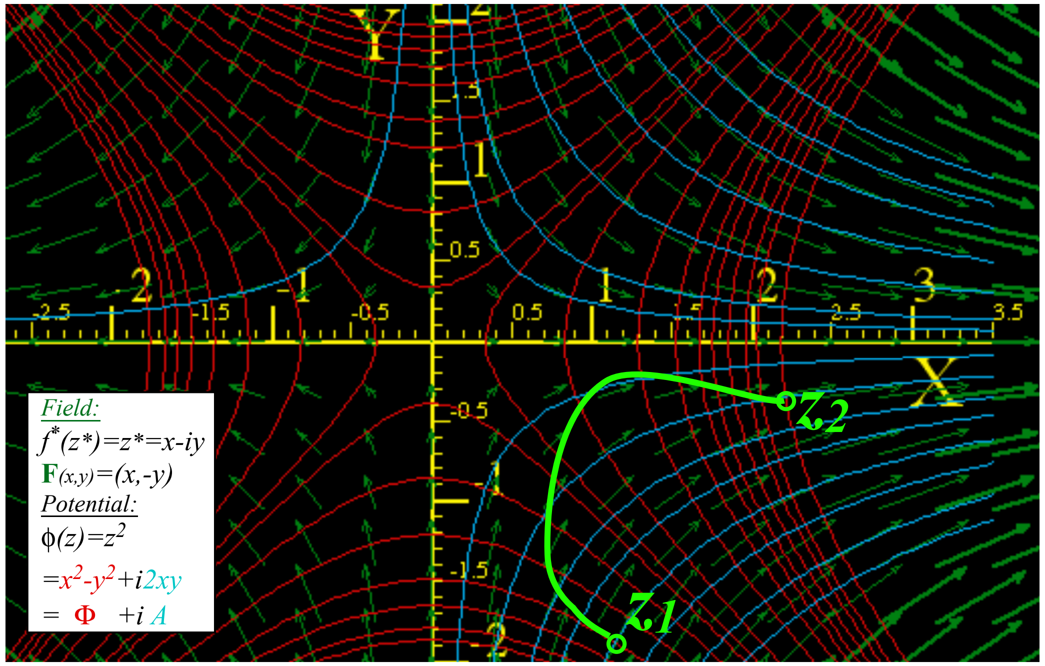
Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

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Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial \mathbf{A}}{\partial y}\\ -\frac{\partial \mathbf{A}}{\partial x}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\underbrace{\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)}+\frac{1}{2}\underbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)}=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

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The half-n'-half result

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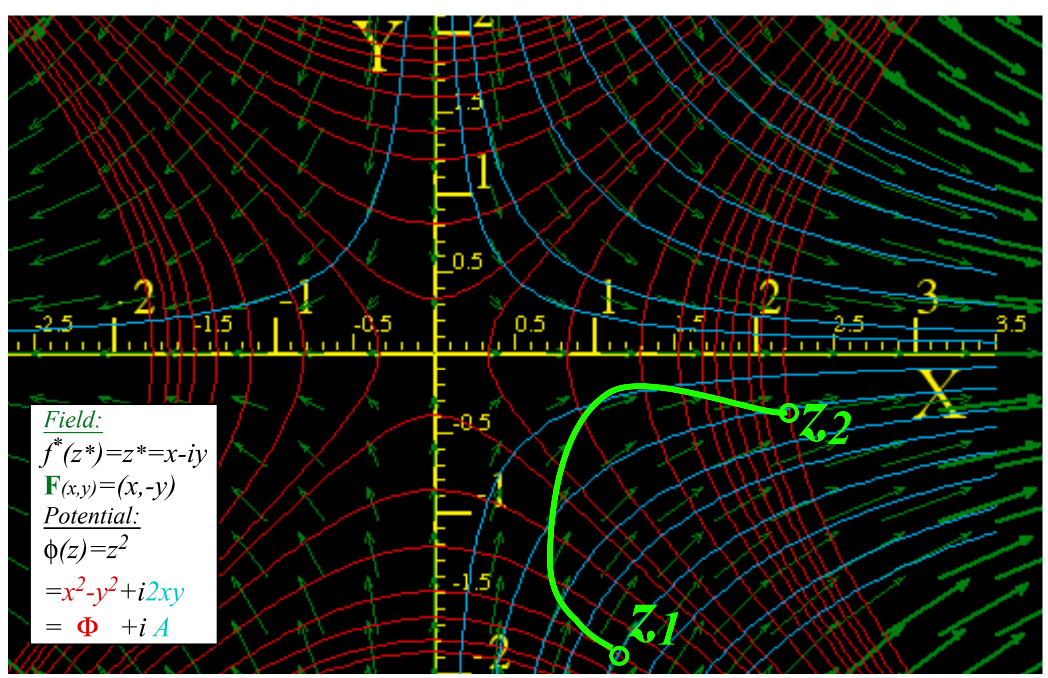
$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



The half-n'-half results are called *Riemann-Cauchy Derivative Relations*

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\mathbf{A}}{\partial y}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial x}=\frac{\partial\,\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial\mathbf{A}}{\partial x}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\,\mathrm{Im}f(z)}{\partial x}$$

→ *4. Riemann-Cauchy conditions* *What's analytic? (...and what's <u>not</u>?)*

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must <u>not</u> be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies : } \boxed{\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and : } \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

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Criteria for a field function $f = f_x(x,y) + i\,f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must <u>not</u> be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies: } \boxed{\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}}$$

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This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies: } \boxed{\frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}}$$

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What's analytic? (...and what's <u>not</u>?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z = x + iy$?

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

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$$
\begin{aligned}
f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\
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\end{aligned}
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A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

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Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: *NO!* *r(xy)=z*z is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

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$$
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f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\
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\end{aligned}
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A: *NO!* $r(xy)=z^*z$ *is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=x+iy$?

A: *YES!* $s(xy)=(x+iy)^2 =z^2$ *is analytic function of z.* *(Yay!)*

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

→ *Easy 2D circulation and flux integrals*

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F**•d**r**) and "flux"(∫**F**x d**r**)*

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

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In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

$$\int f(z)dz = \int \left(f^*(z^*)\right)^* dz = \int \left(f^*(z^*)\right)^* \left(dx + i\,dy\right) = \int \left(f_x^* + i\,f_y^*\right)^* \left(dx + i\,dy\right) = \int \left(f_x^* - i\,f_y^*\right)\left(dx + i\,dy\right)$$

$$= \int (f_x^*\,dx + f_y^*\,dy) + i \int (f_x^*\,dy - f_y^*\,dx)$$

$$= \int \mathbf{F} \bullet d\mathbf{r} + i \int \mathbf{F} \times d\mathbf{r} \bullet \hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F} \bullet d\mathbf{r} + i \int \mathbf{F} \bullet d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F} \bullet d\mathbf{r} + i \int \mathbf{F} \bullet d\mathbf{S} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

What Good Are Complex Exponentials? (contd.)

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫Fxdr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

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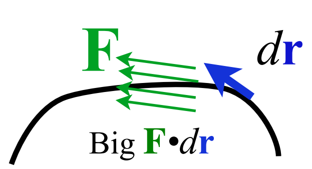
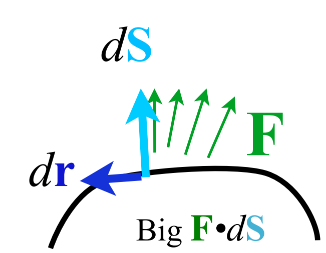


Real part $\int_1^2 \mathbf{F}\bullet d\mathbf{r} = \Delta\Phi$
sums **F** projections *along* path
$d\mathbf{r}$ that is, *circulation* on path
to get $\Delta\Phi$.

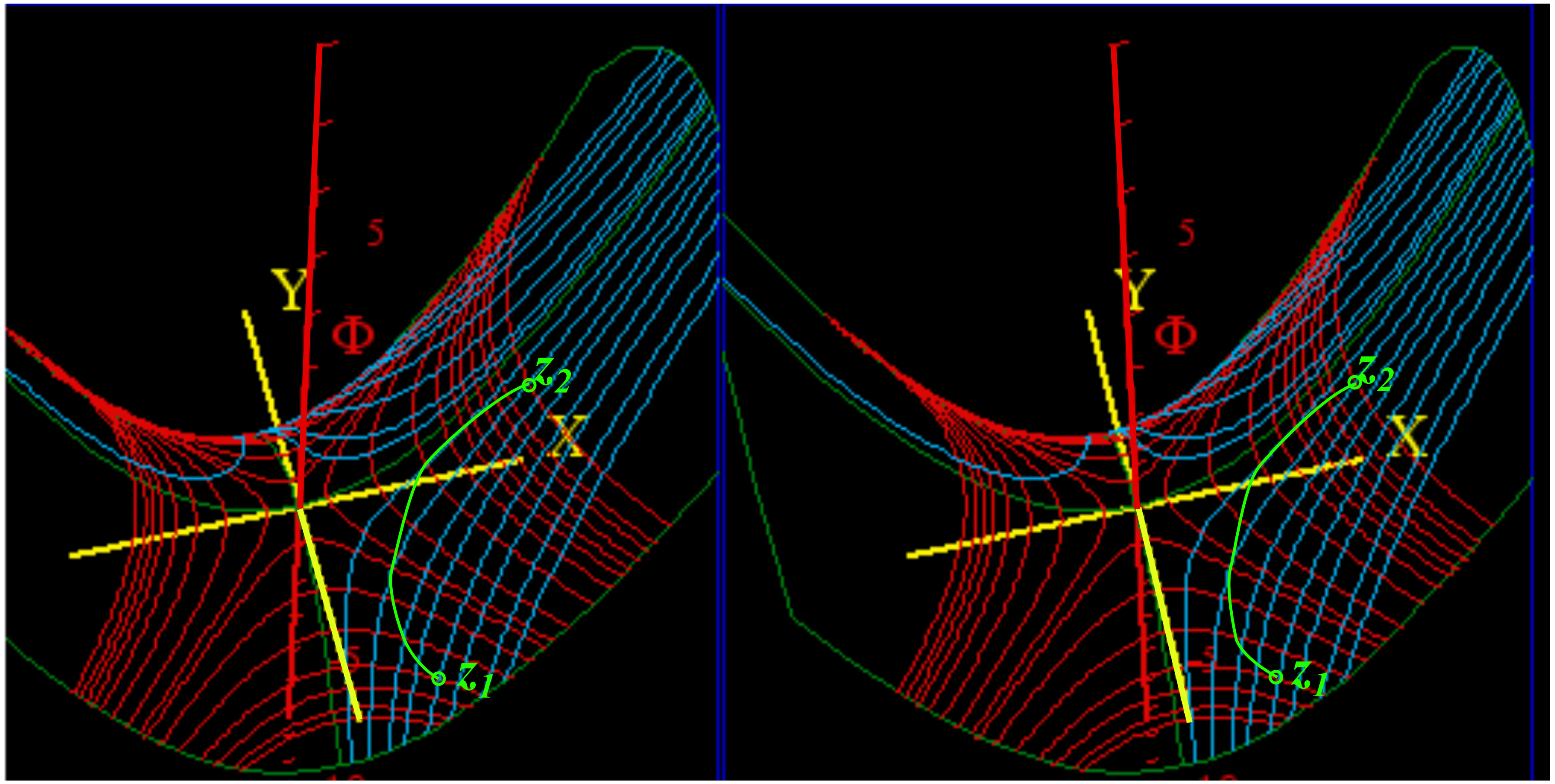


Imaginary part $\int_1^2 \mathbf{F}\bullet d\mathbf{S} = \Delta\mathbf{A}$
sums **F** projection *across* path $d\mathbf{r}$
that is, *flux* thru surface
elements $d\mathbf{S} = d\mathbf{r}\times\mathbf{e}_Z$ normal to $d\mathbf{r}$
to get $\Delta\mathbf{A}$.

Here the scalar potential $\Phi=(x^2-y^2)/2$ *is stereo-plotted vs.* (x,y)

The $\Phi=(x^2-y^2)/2=const.$ *curves are topography lines*

The $A=(xy)=const.$ *curves are streamlines normal to topography lines*



4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

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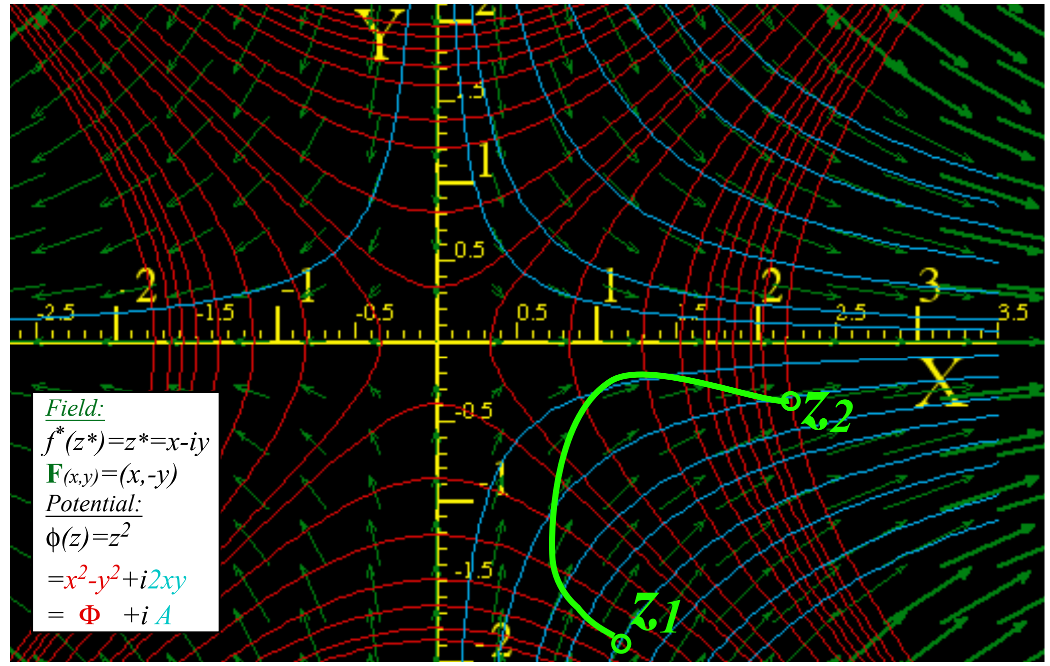
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{matrix}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{matrix} \uparrow & \quad \uparrow \\ \mathbf{E}_{\Phi} & \quad \mathbf{E}_{A} \end{matrix} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{matrix} \uparrow & \quad \uparrow \\ \mathbf{E}_{\Phi} & \quad \mathbf{E}_{A} \end{matrix} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} \\ \mathbf{E}_{A} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{A} \bullet \mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

What Good Are Complex Exponentials? (contd.)

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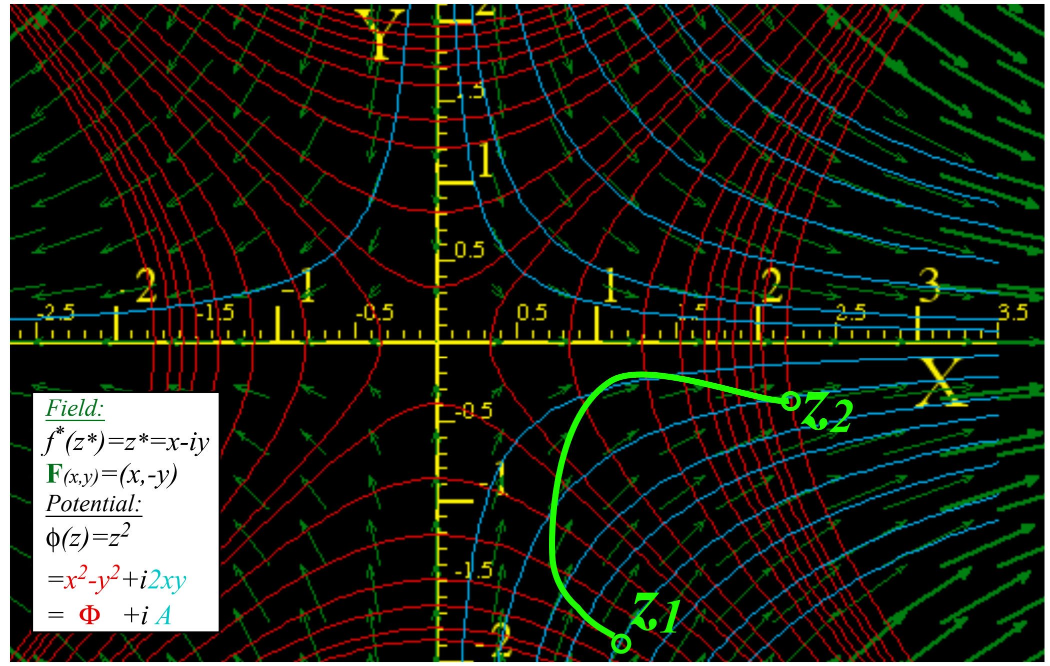

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Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y}$$

$$= -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

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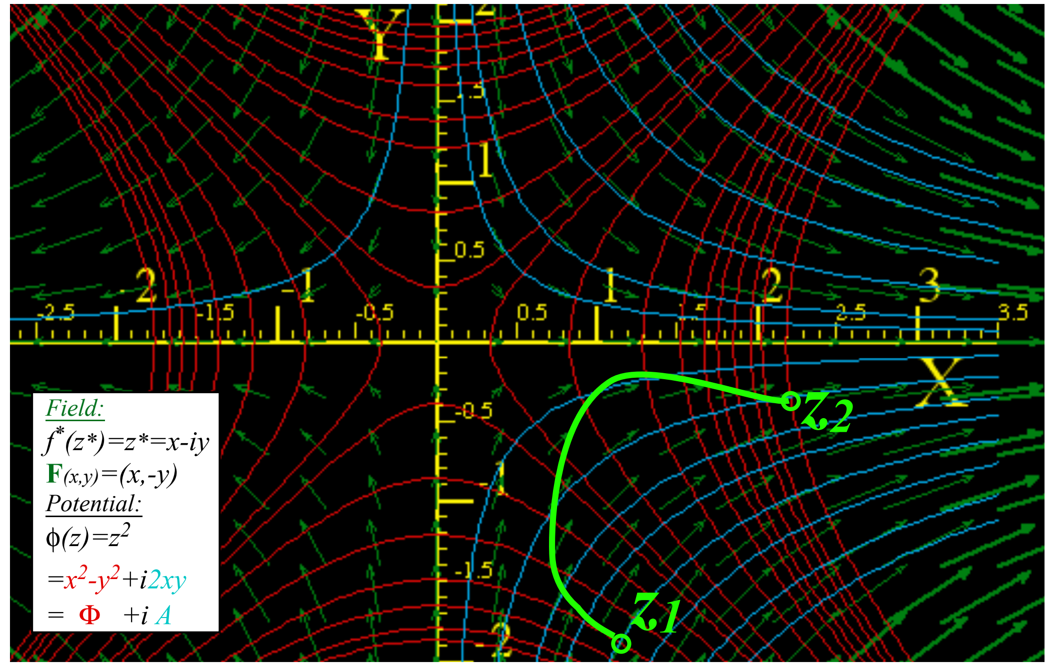

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$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} \\ \mathbf{E}_{A} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{A} \bullet \mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y} \frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

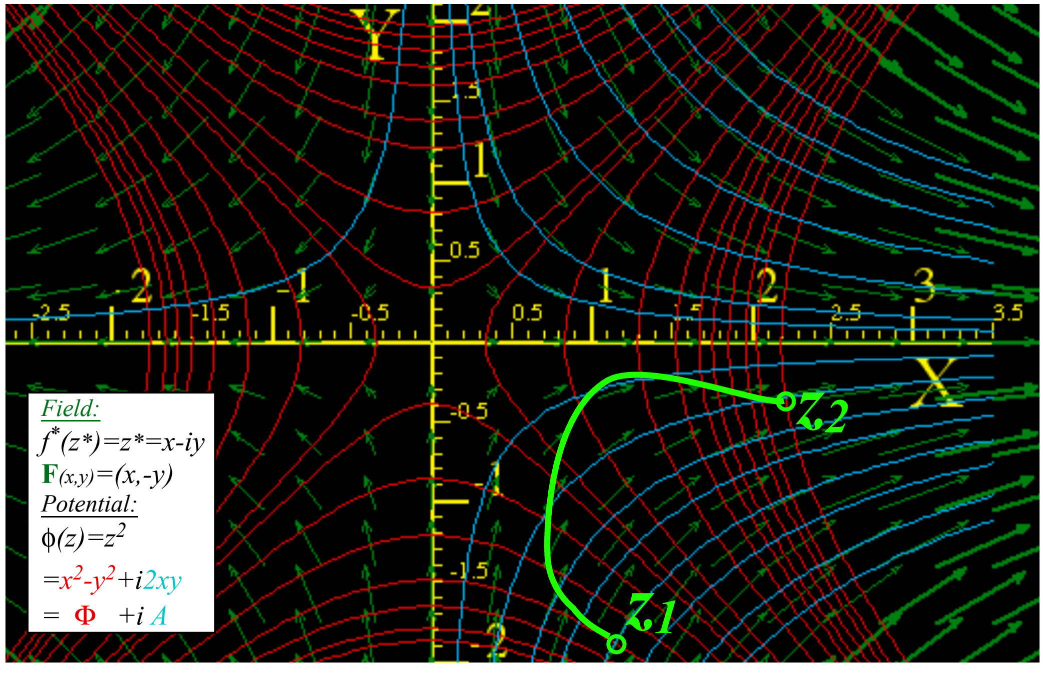
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \uparrow \quad \uparrow \\ \mathbf{E}_{\Phi} \quad \mathbf{E}_{A} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \uparrow \quad \uparrow \\ \mathbf{E}_{\Phi} \quad \mathbf{E}_{A} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} \\ \mathbf{E}_{A} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{A} \bullet \mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

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or Riemann-Cauchy

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

and so does A

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}=z^{-1}$ $\qquad$ $f(z)=\frac{a}{z}=az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

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$$\phi(z)=\quad \Phi \quad + \quad i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z)$$

What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials

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$$\phi(z) = \quad \Phi \quad + \quad i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= \underbrace{a\ln(r)} + i\,\underbrace{a\theta}$$

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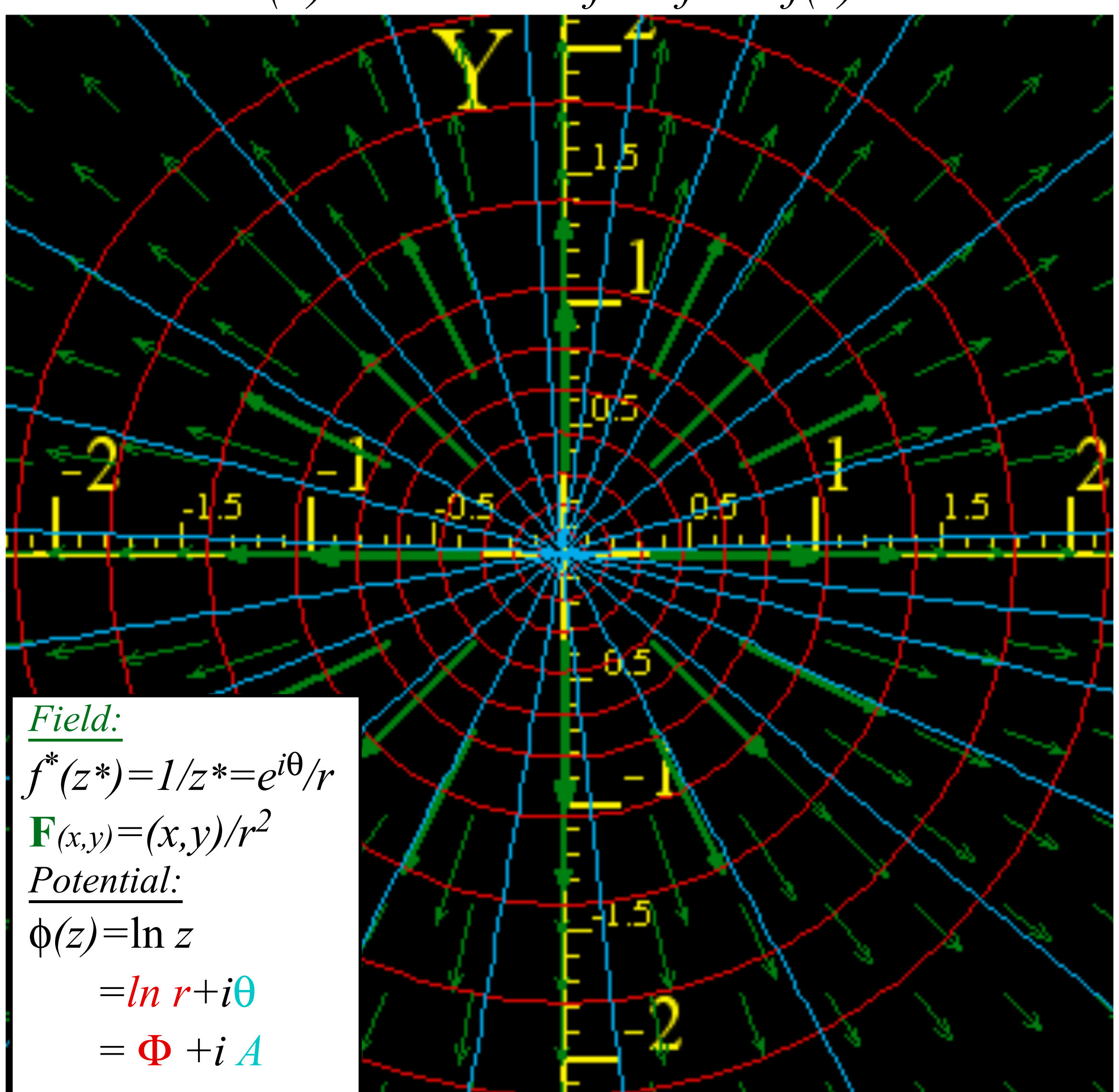
$$\phi(z)=\Phi+i\mathbf{A}=\int f(z)dz=\int \frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

$$=\overbrace{a\ln(r)}^{\Phi}+i\overbrace{a\theta}^{\mathbf{A}}$$

(a) Unit Z-line-flux field f(z)=1/z



Lecture 12 Mon. 10.01
May end here

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$$\phi(z)=\overbrace{\Phi}^{}+i\overbrace{\mathbf{A}}^{}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$
$$=a\ln(r)+i\,a\theta$$

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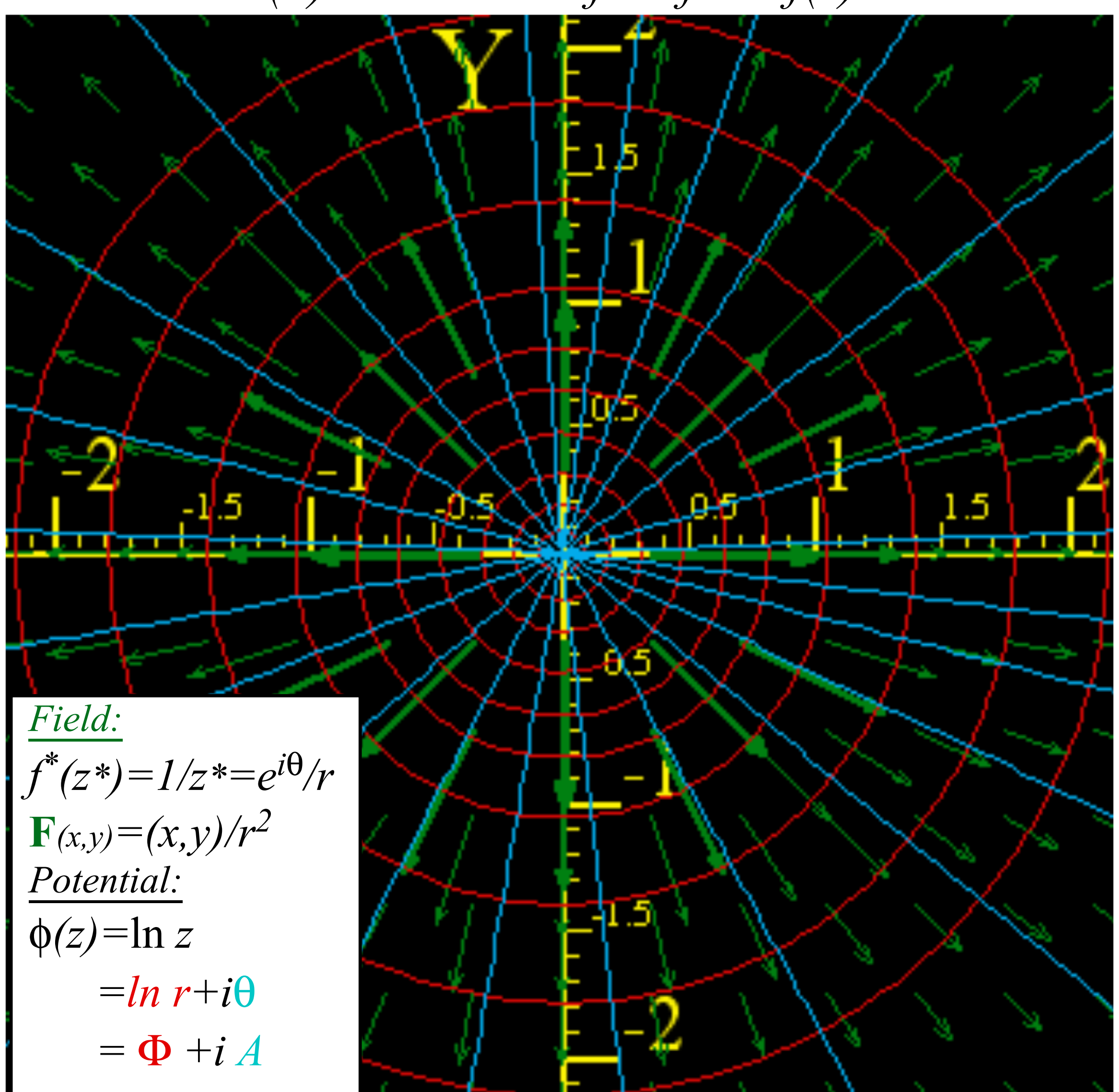
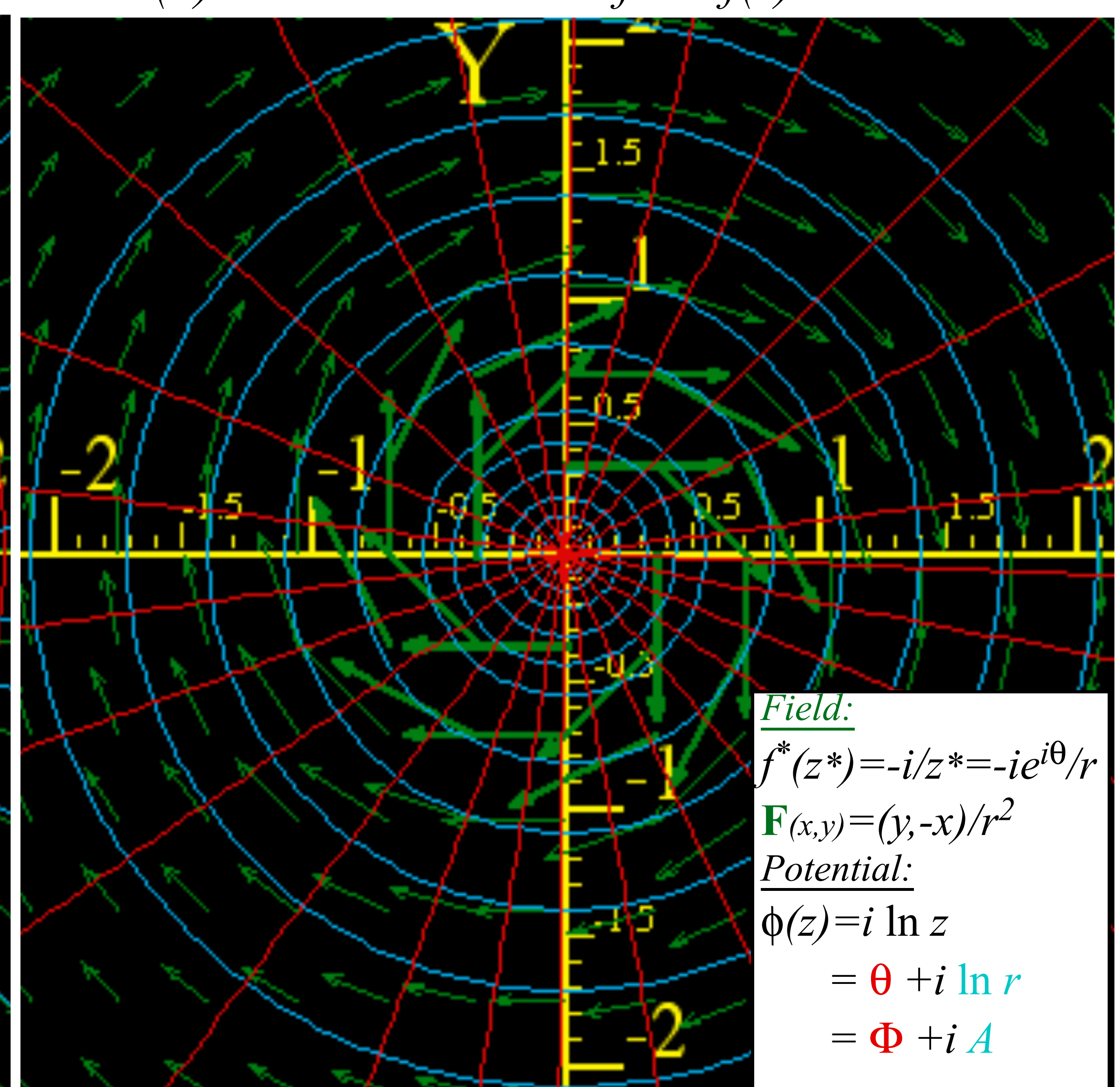


(b) Unit Z-line-vortex field f(z)=i/z



What Good Are Complex Exponentials? (contd.)

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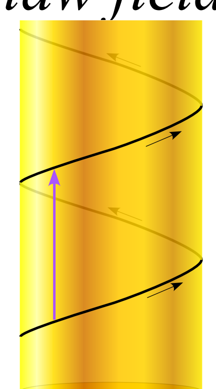
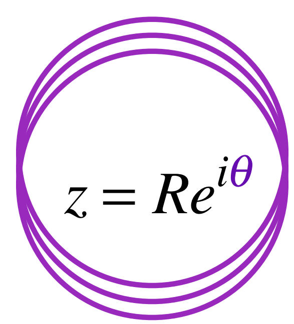
$$\phi(z) = \underbrace{\Phi}_{=a\ln(r)} + i\underbrace{A}_{a\theta} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= a\ln(r) + i\,a\theta$$

A monopole field is the only power-law field whose integral (potential) depends on path of integration.

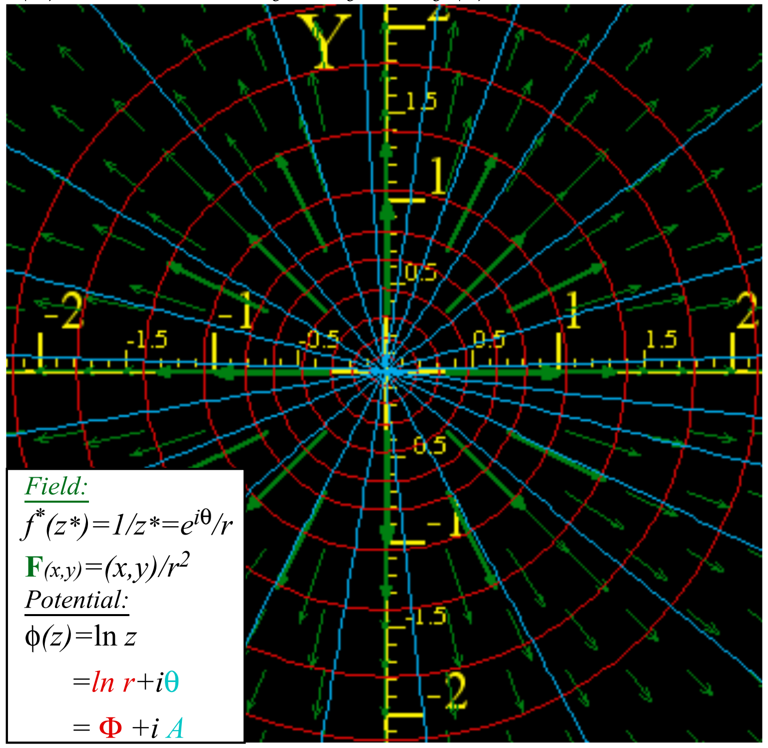




path that goes N times around origin (r=0) at constant r = R.

$$\Delta\phi = \oint f(z)dz = a\oint\frac{dz}{z} = a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}} = a\int_{\theta=0}^{\theta=2\pi N} id\theta = ai\,\theta\Big|_0^{2\pi N} = 2a\pi iN$$

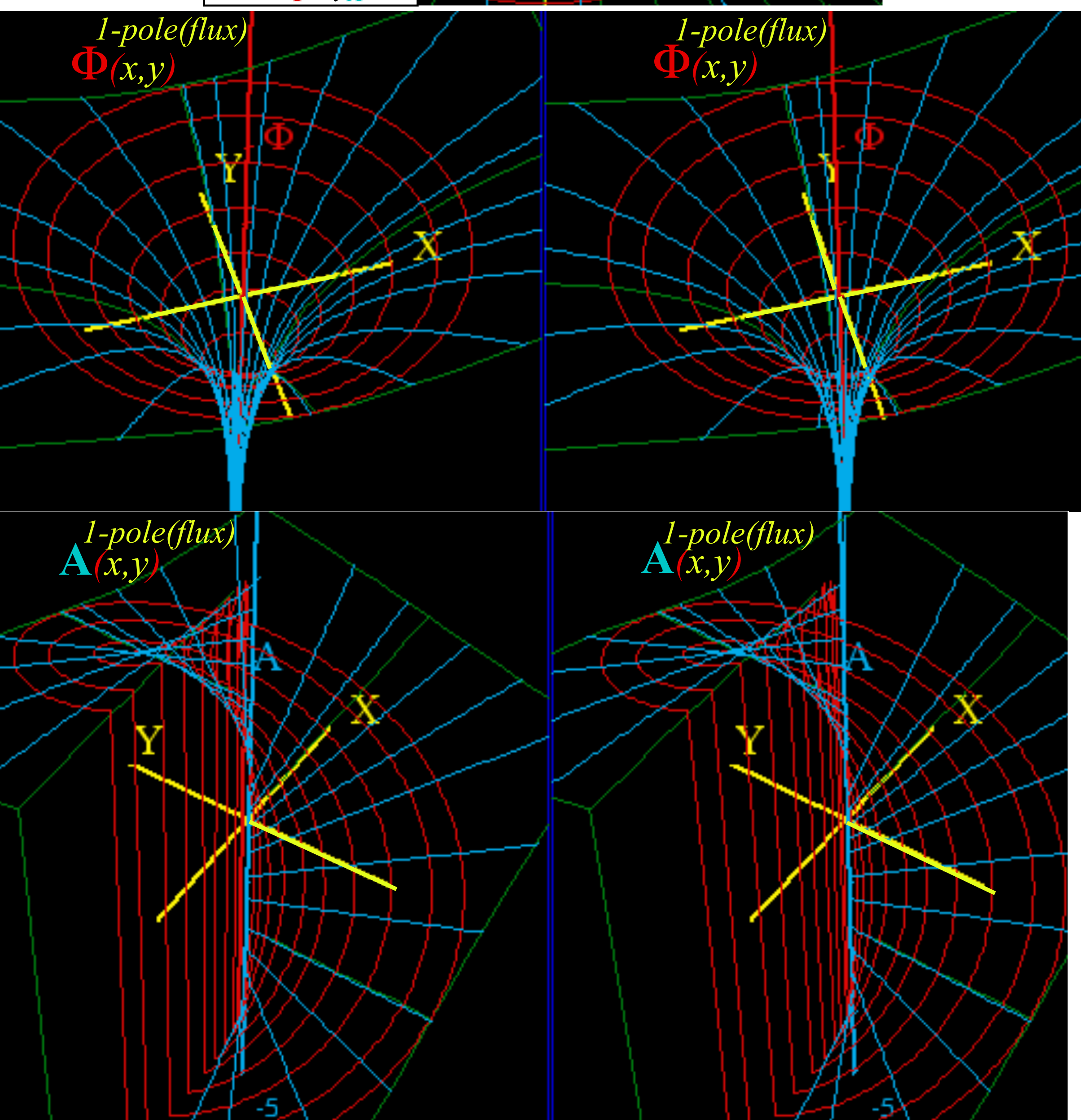
(a) Unit Z-line-flux field f(z)=1/z



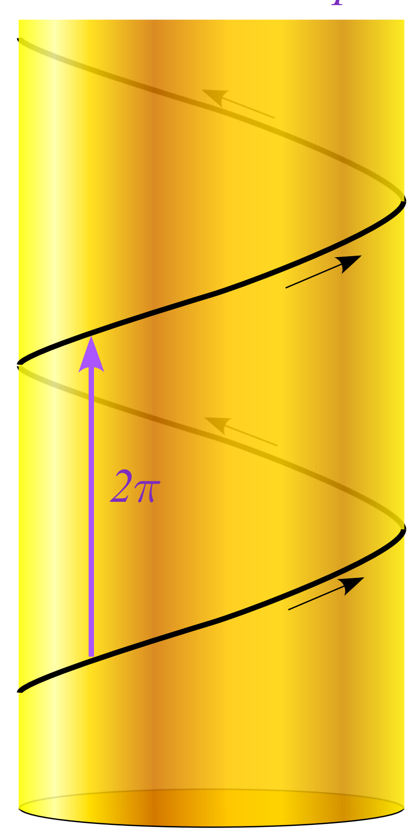
$$\phi(z)=\underbrace{\Phi}_{\ln(r)}+i\underbrace{\mathbf{A}}_{\theta}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(re^{i\theta})$$

$$=\ln(r)+i\theta$$

(For a=1)

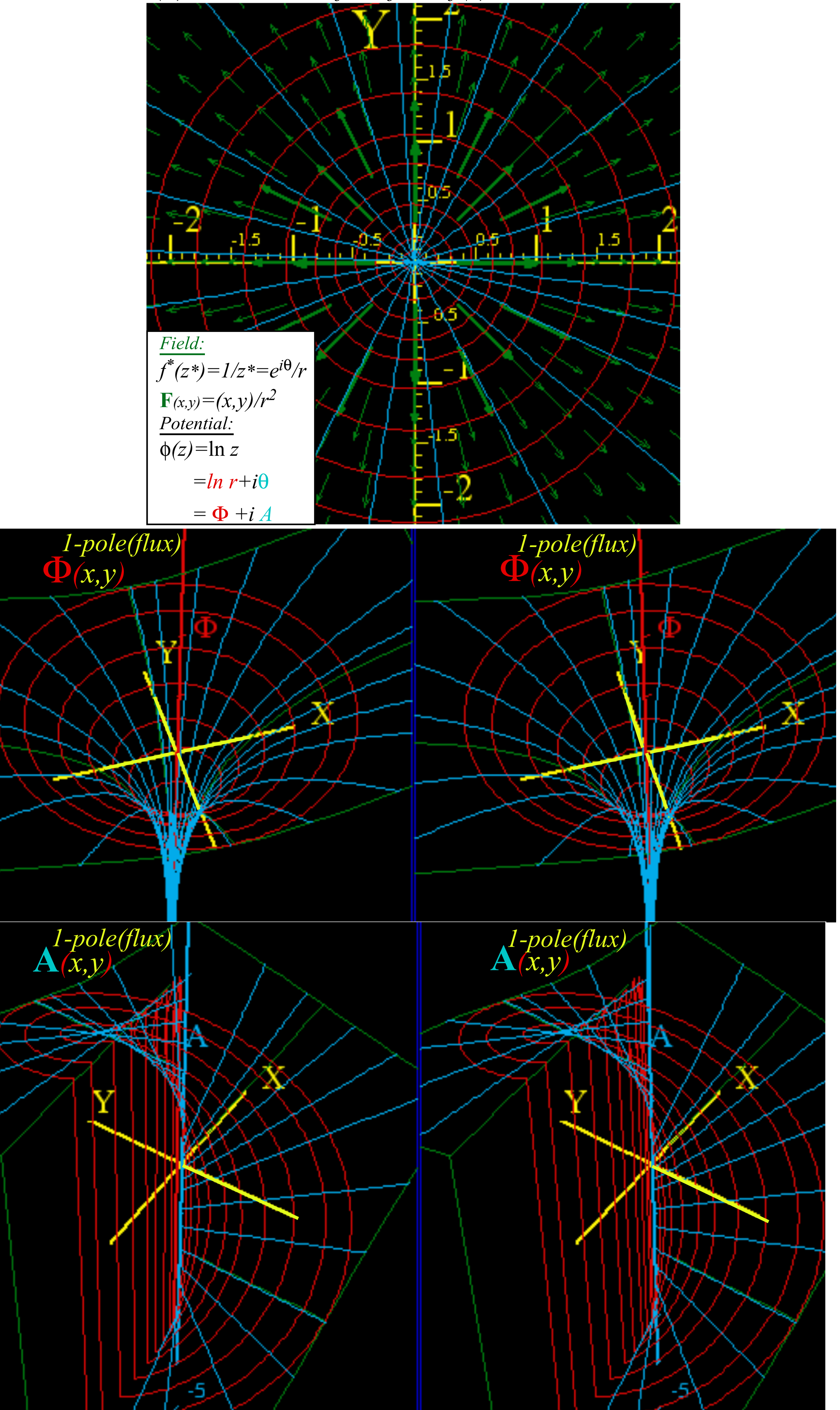


*Each turn around origin adds 2πi to vector potential i***A**

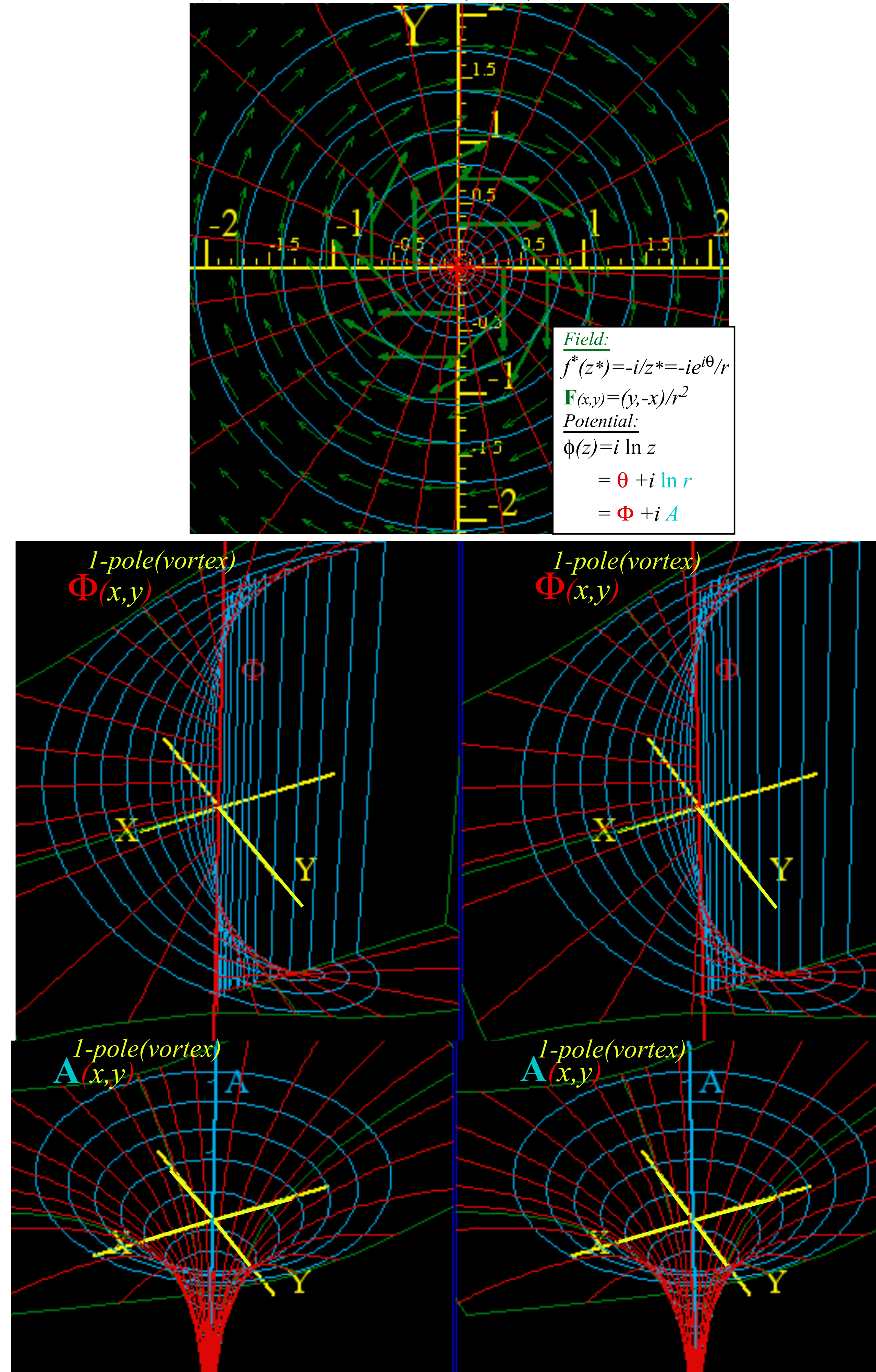


(For a=1)

(a) Unit Z-line-flux field f(z)=1/z



(b) Unit Z-line-vortex field f(z)=i/z

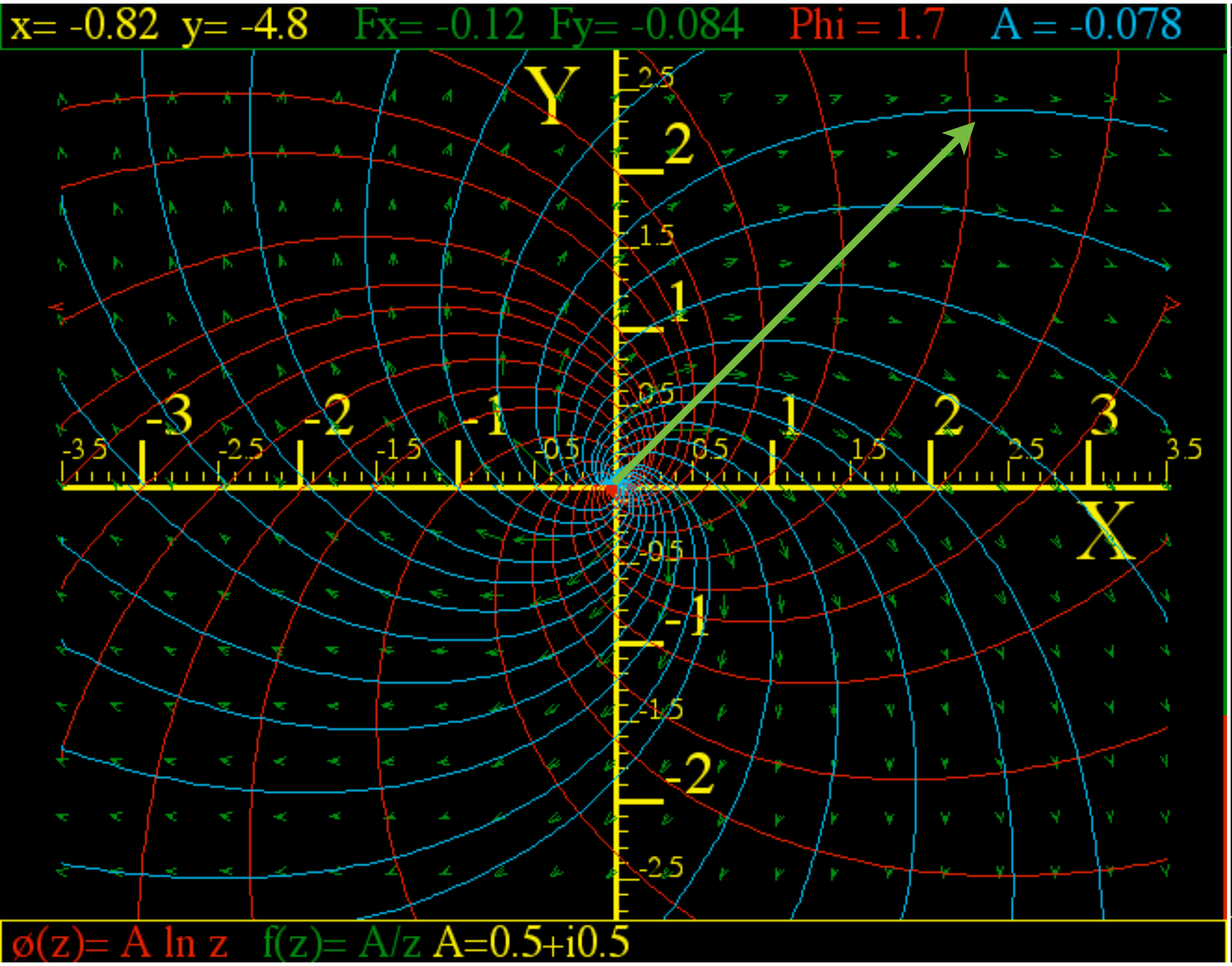


What Good Are Complex Exponentials? (contd.)

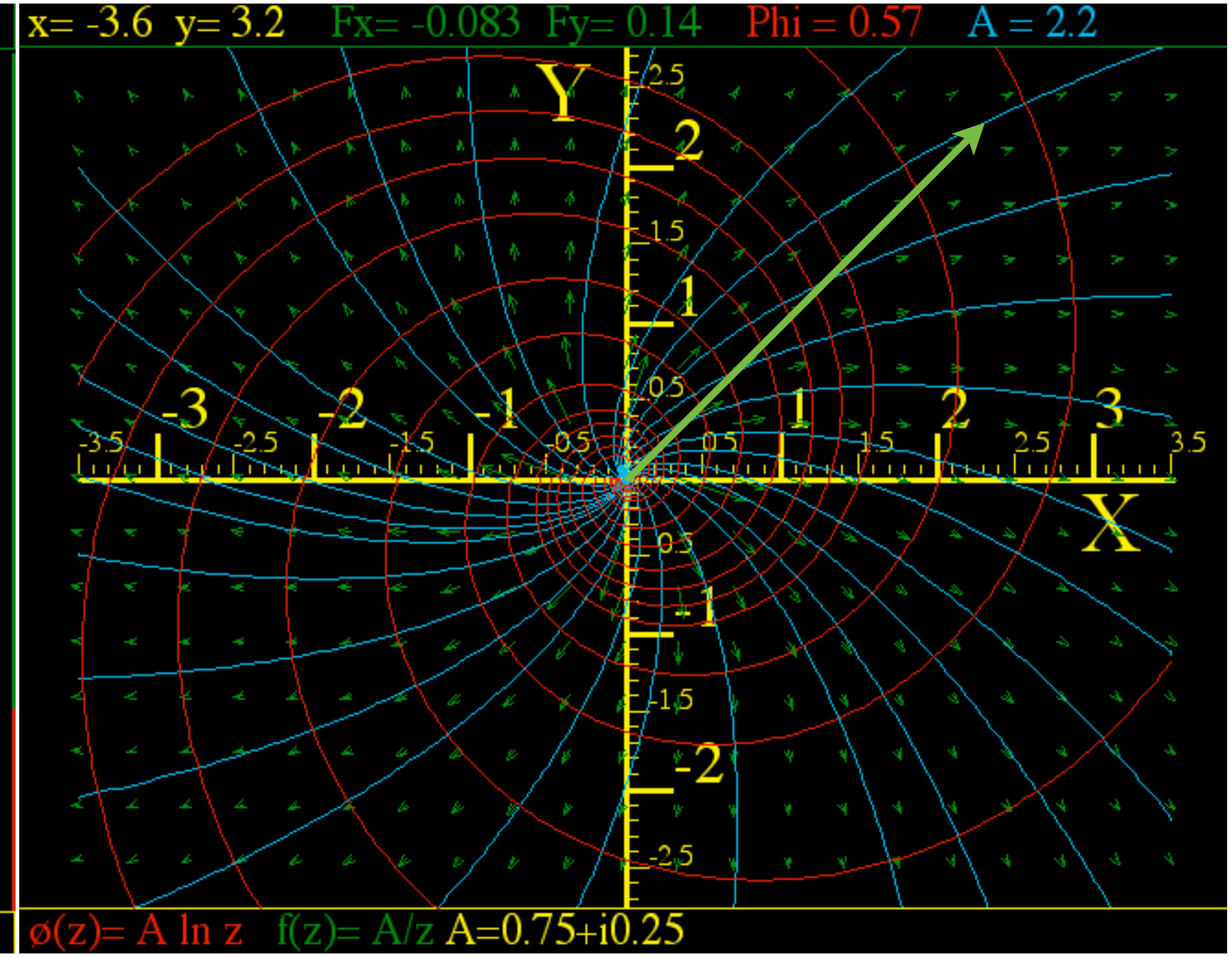
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

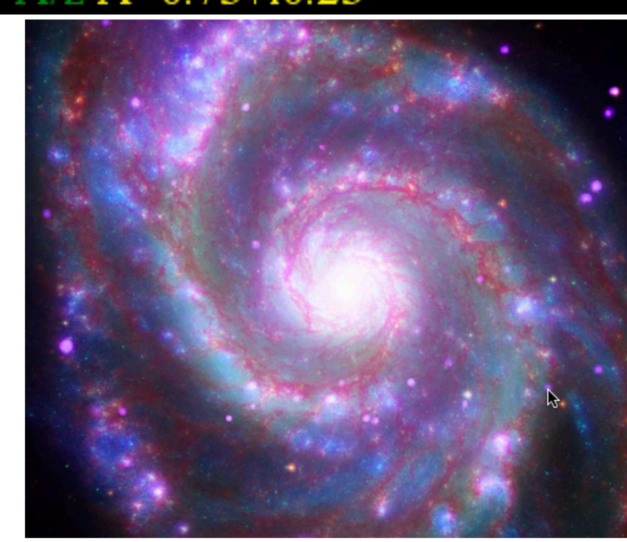
$f(z) = (0.75 + i0.25)/z = e^{i18^{\circ}}/z\sqrt{n}$

"Vortex"



"Hurricane"





4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

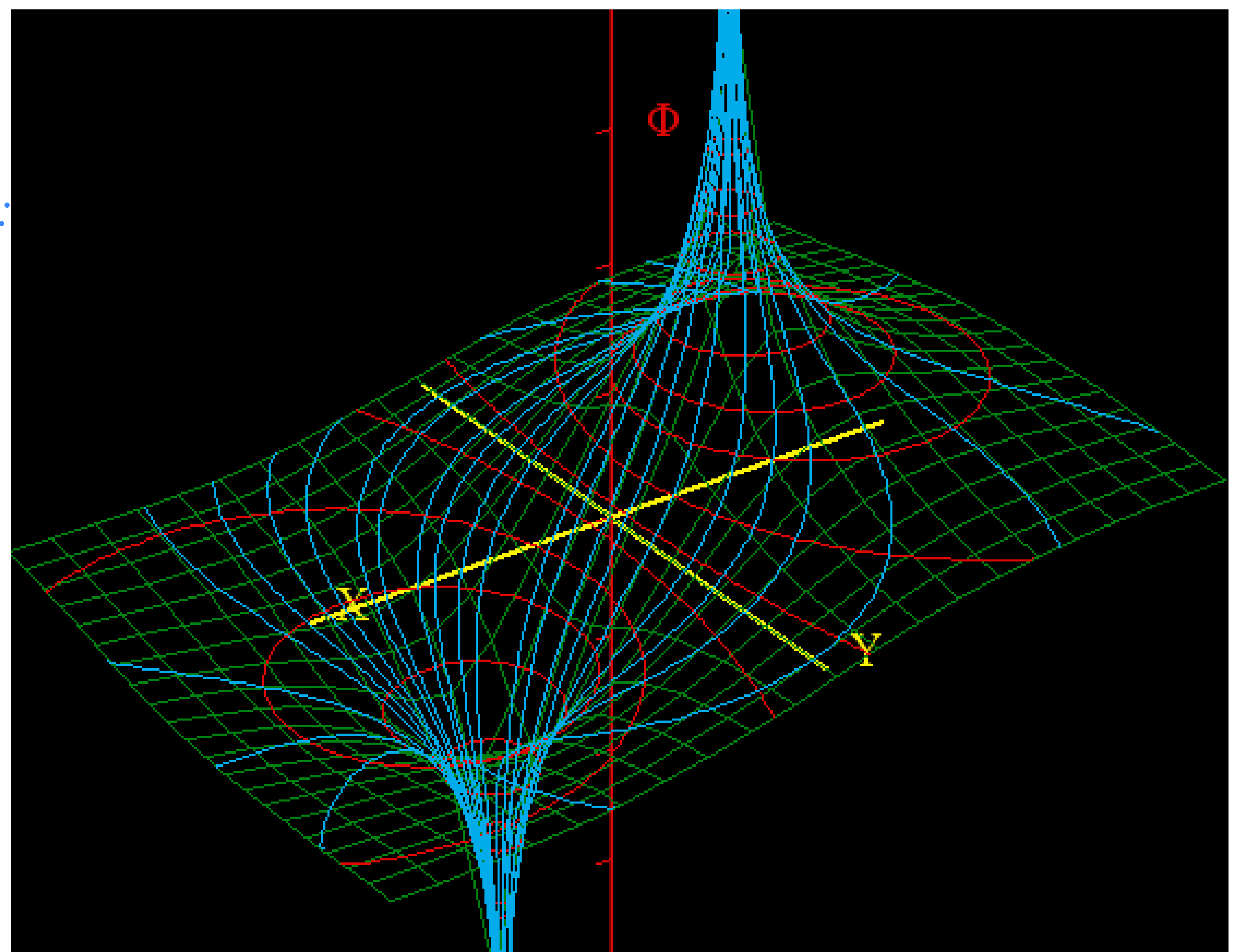
This is like the derivative definition:

$$\frac{df}{dz}=\frac{f(z+\Delta)-f(z)}{\Delta}$$

or:

$$\frac{df}{dz}=\frac{f(z+\frac{\Delta}{2})-f(z-\frac{\Delta}{2})}{\Delta}$$

if Δ is infinitesimal ($\Delta\to 0$)



So-called "physical dipole" has finite Δ (+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

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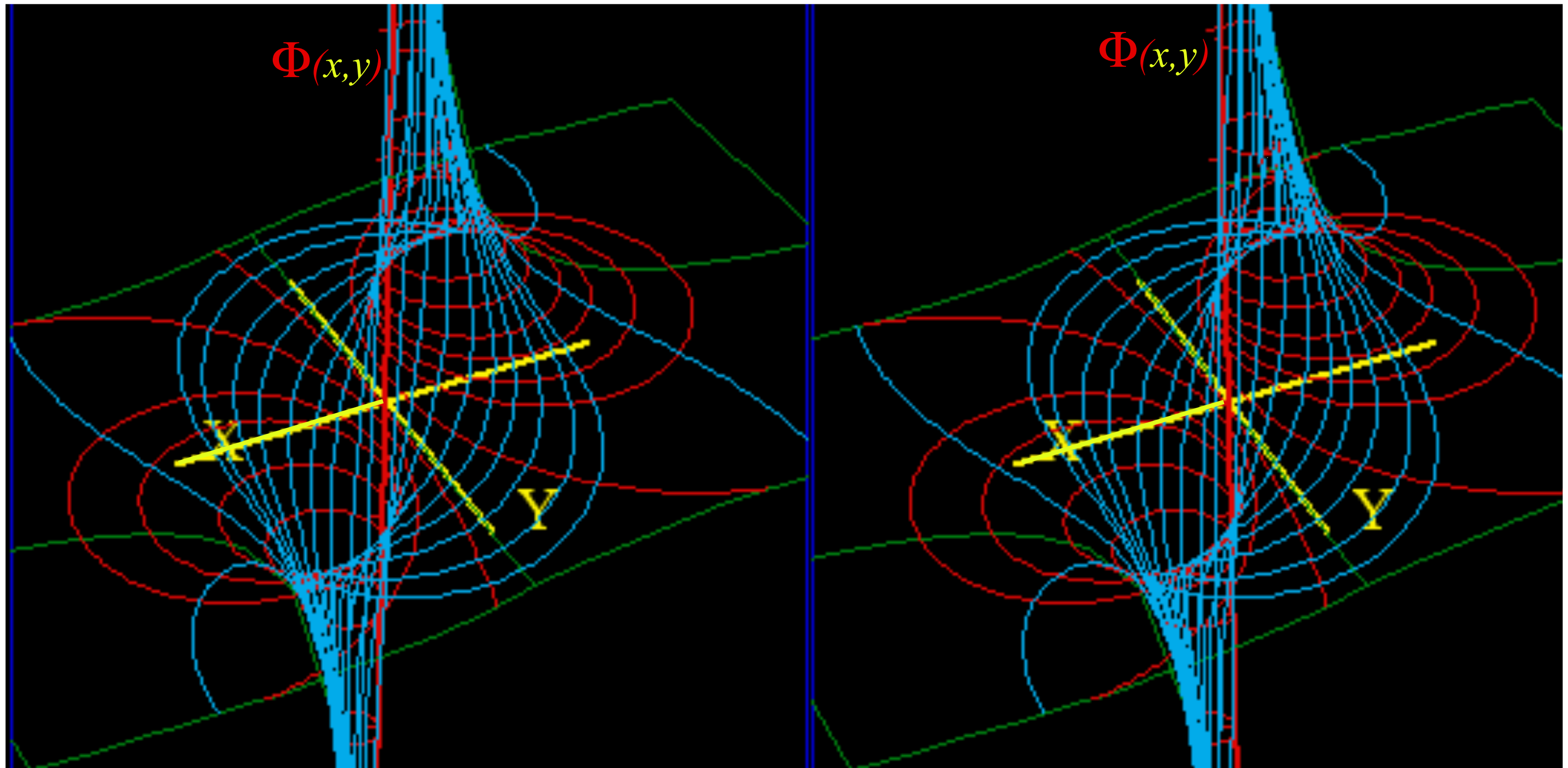

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A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

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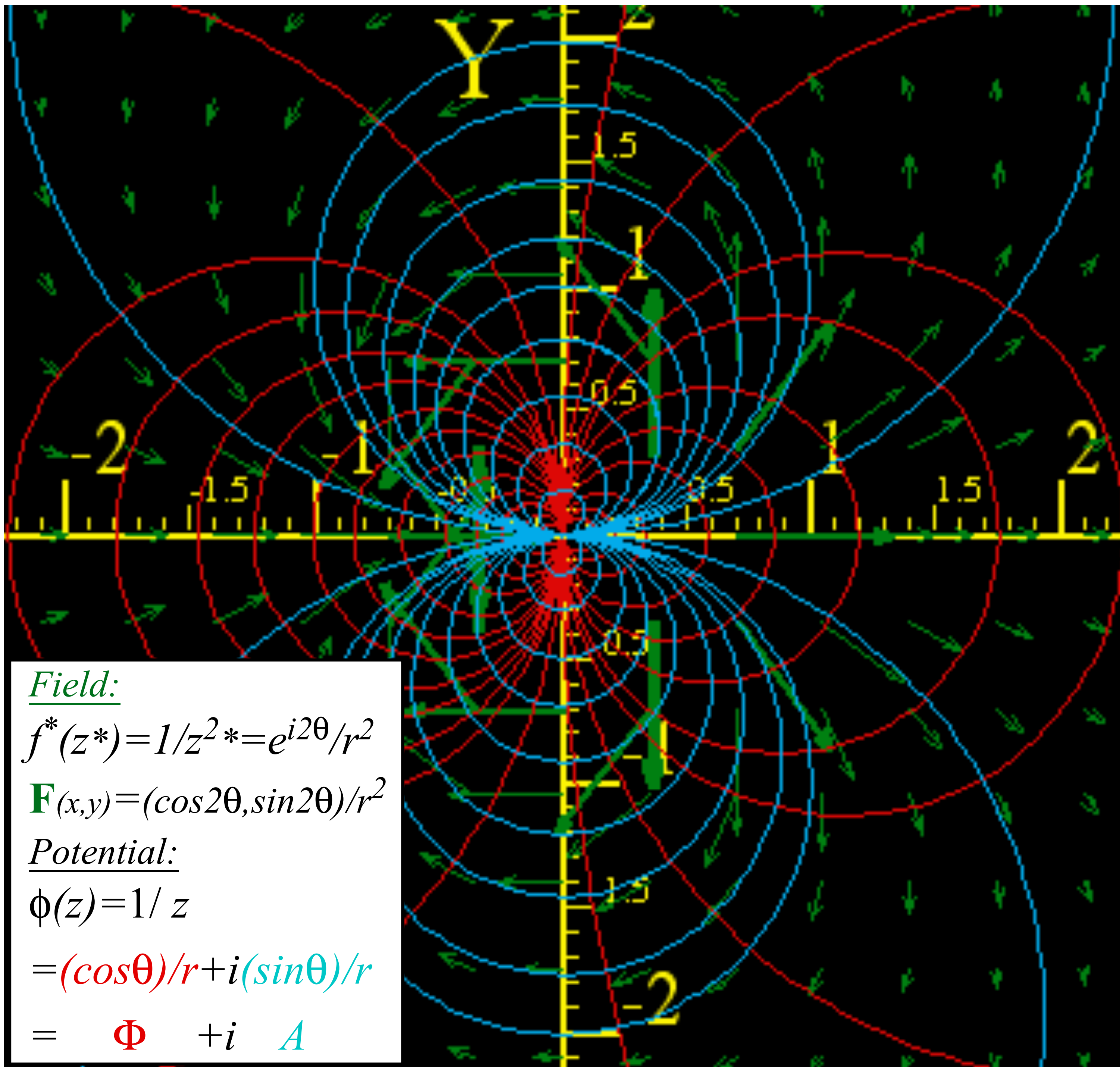
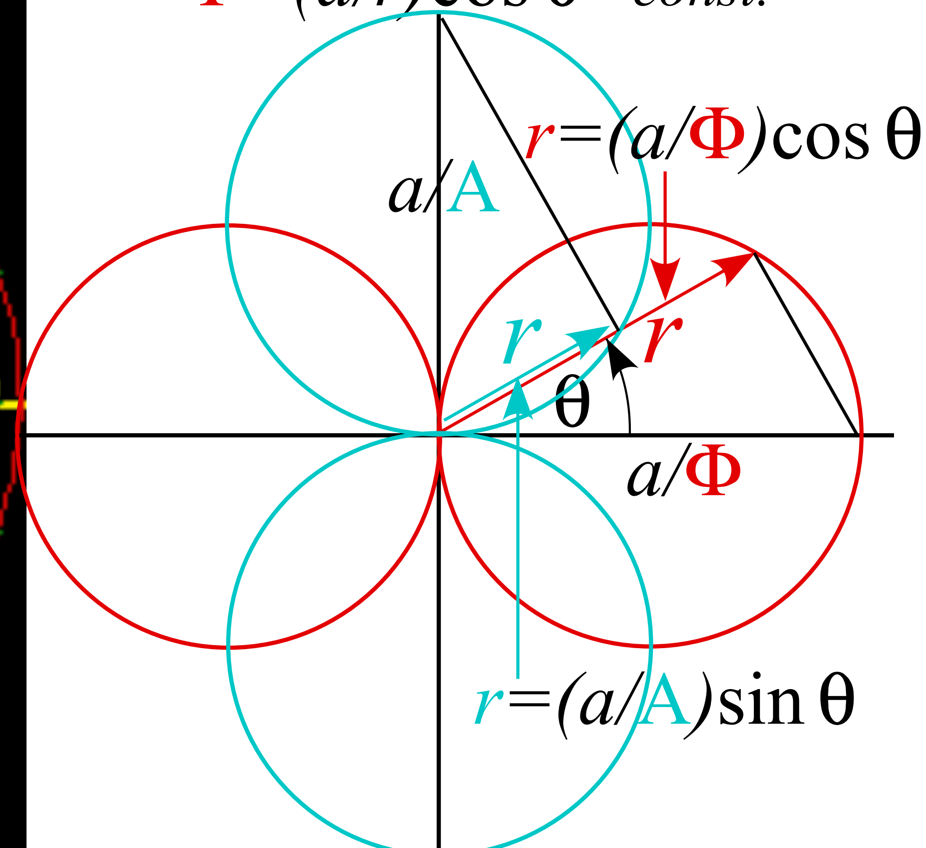


Scalar potentials

$\Phi=(a/r)\cos\theta=const.$



Vector potentials

$\mathrm{A}=(a/r)\sin\theta=const.$

2^n-*pole analysis* *(quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, ..., pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad\qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$

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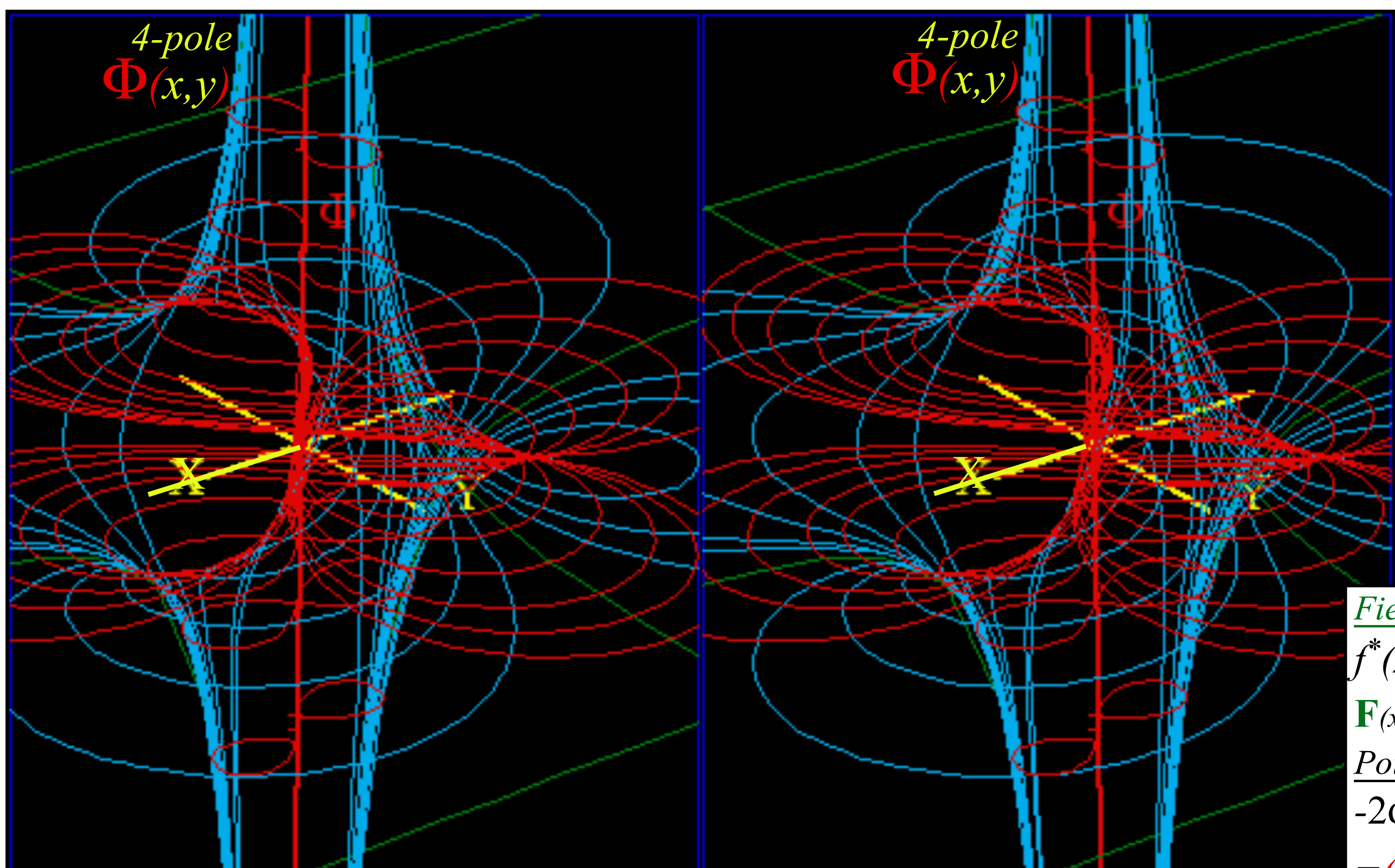
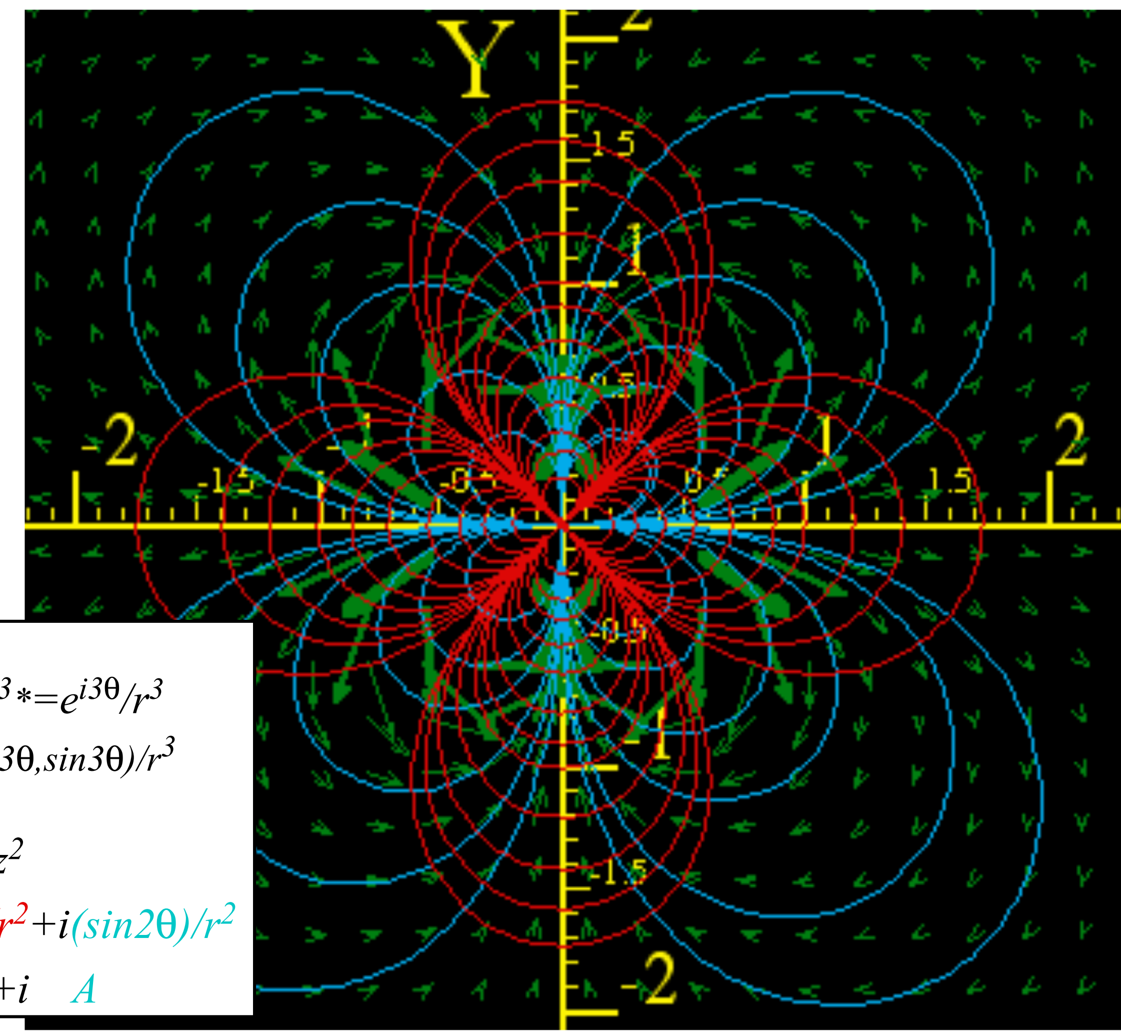
Each a *z*-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{\text{4-pole}} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{\text{2-pole}}}{dz} = \frac{d\phi^{\text{4-pole}}}{dz}$$

$$\phi^{\text{4-pole}} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{\text{2-pole}}}{dz}$$





Field:

$f^*(z^*)=1/z^3{}^*=e^{i3\theta}/r^3$

$\mathbf{F}_{(x,y)}=(\cos 3\theta, \sin 3\theta)/r^3$

Potential:

$-2\phi(z)=1/z^2$

$=(\cos 2\theta)/r^2+i(\sin 2\theta)/r^2$

$= \quad \Phi \quad +i \quad A$

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

$\cdots 2^2$-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz = \phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a 2^m*-pole*.

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

(octapole)$_0$	*(quadrupole)$_0$*	*(dipole)$_0$*	*(monopole)*	*(dipole)$_\infty$*	*(quadrupole)$_\infty$*	*(octapole)$_\infty$*

$$\phi(z)=\ldots\frac{a_{-4}}{-3}z^{-3} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \ldots$$

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at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz=$$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

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$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3}+\frac{a_{-3}}{-2}w^{-2}+\frac{a_{-2}}{-1}w^{-1}+a_{-1}\ln w+a_0w+\frac{a_1}{2}w^2+\frac{a_2}{3}w^3+\ldots$$

(*with* $z=w^{-1}$)

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(with $z\rightarrow w$)

$$=\ldots\frac{a_2}{3}z^{-3}+\frac{a_1}{2}z^{-2}+a_0z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^2+\frac{a_{-4}}{-3}z^3+\ldots$$

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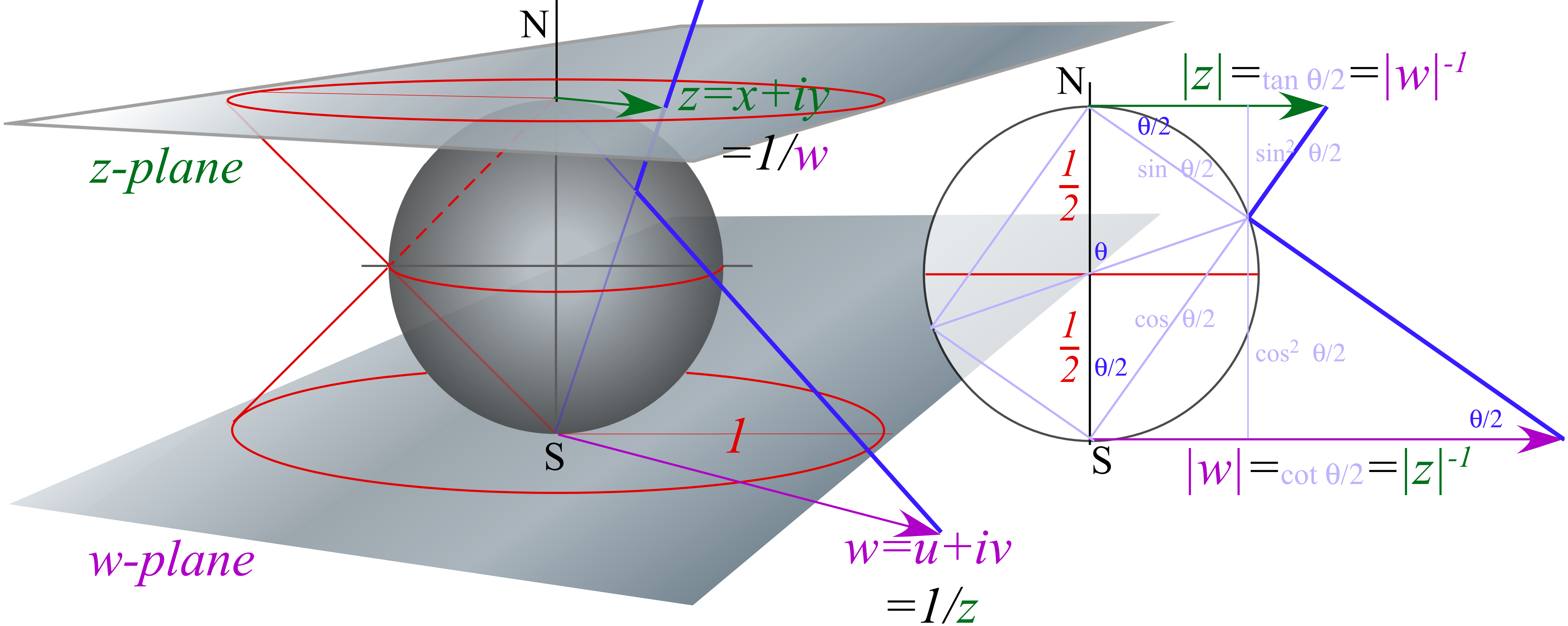

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(with $z\to w$)

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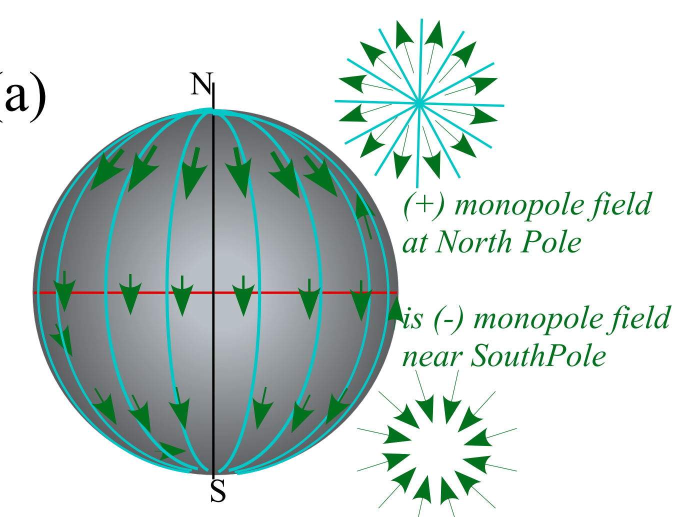
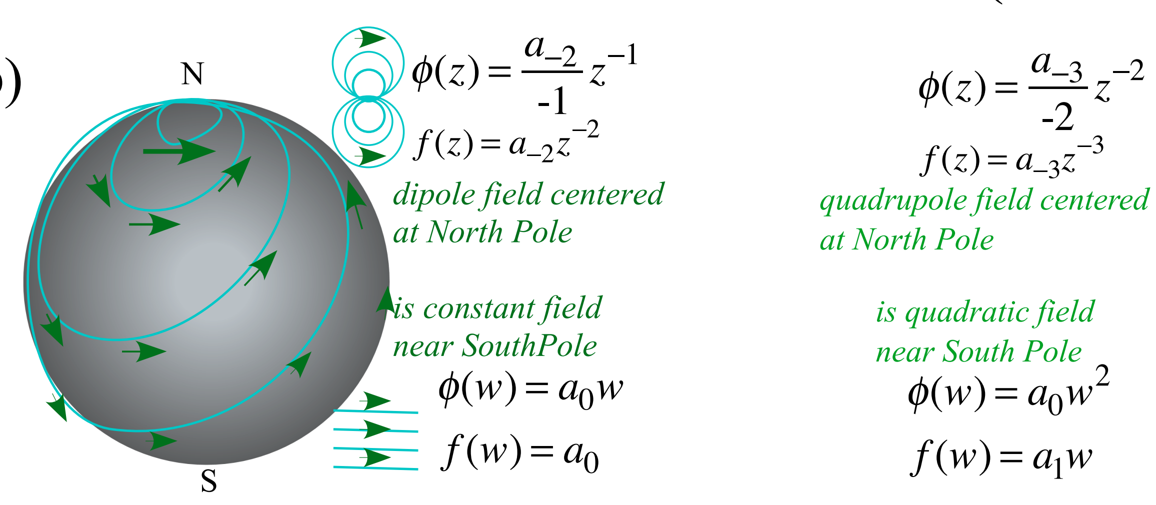
(a)



(b)



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Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

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(quadrupole)₀ *(dipole)₀* *(monopole)* *(dipole)∞* *(quadrupole)∞* *(octapole)∞* *(hexadecapole)∞* ...

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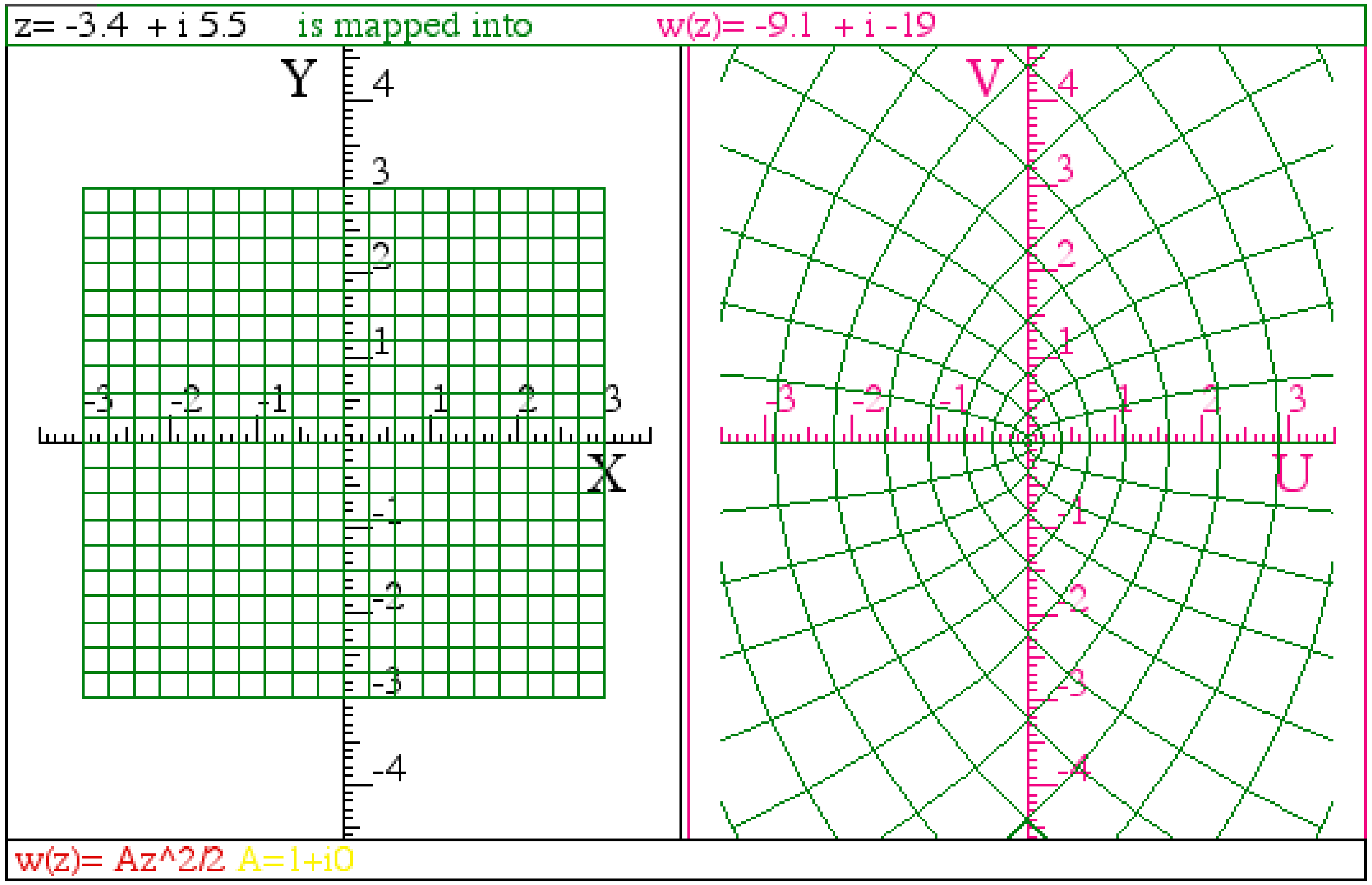
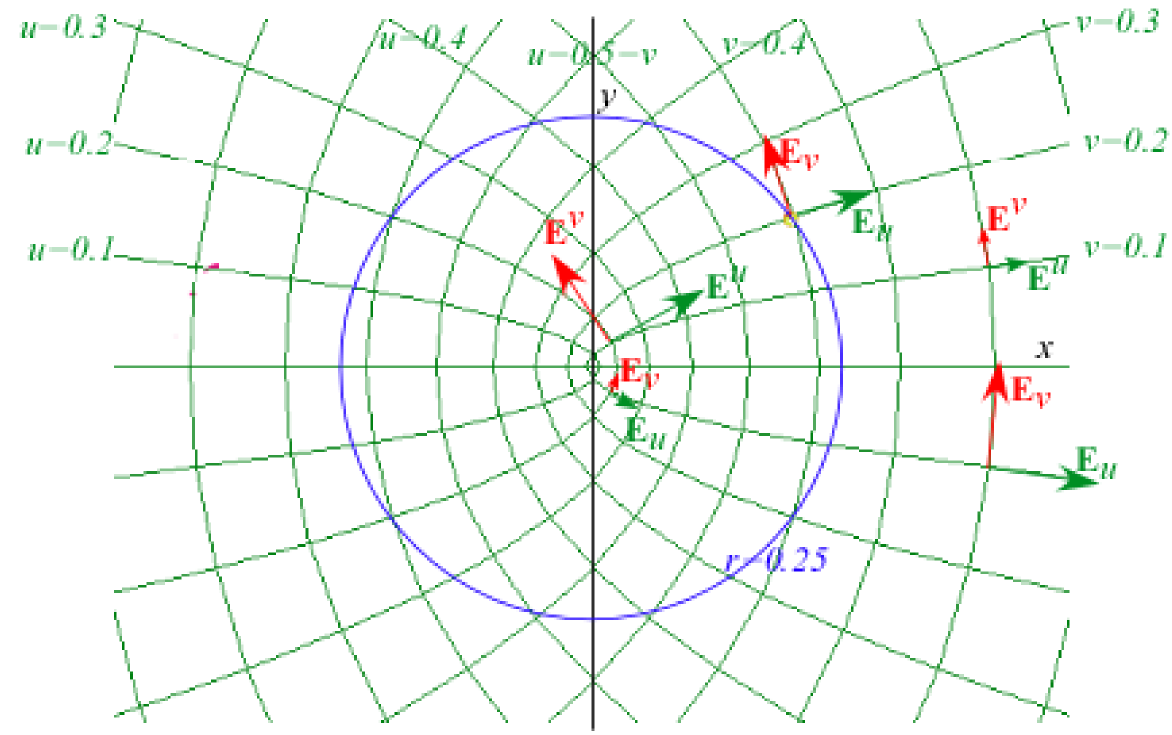
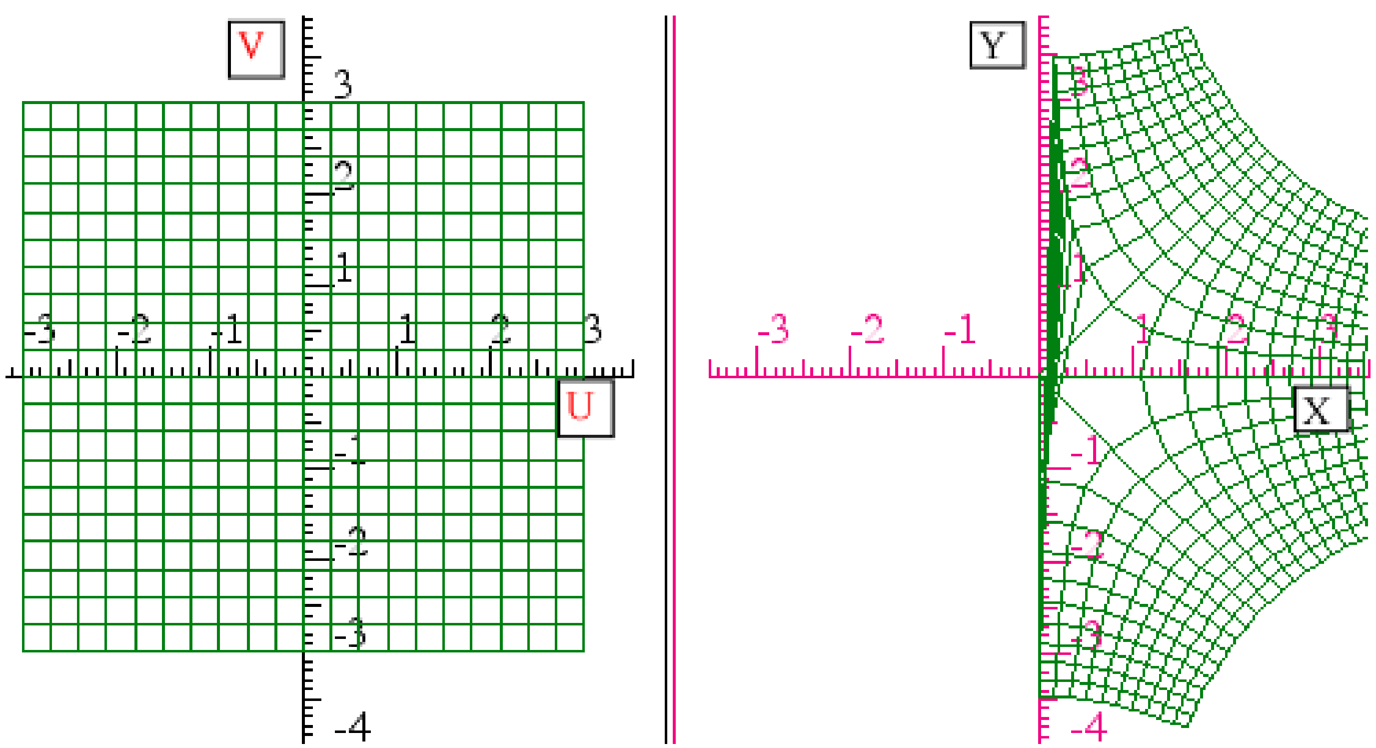
$w(z) = z^2$ gives parabolic OCC





Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



$w = (u + iv) = z^2 = (x + iy)^2$ is analytic function of z and w

Expansion: $u = x^2 - y^2$ and $v = 2xy$ may be solved using $|w| = |z^2| = |z|^2$

Expansion: $|w| = \sqrt{u^2 + v^2} = x^2 + y^2 = |z|^2$

Solution: $x^2 = \dfrac{u + \sqrt{u^2 + v^2}}{2}$ $\quad y^2 = \dfrac{-u + \sqrt{u^2 + v^2}}{2}$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}^u \\ \bar{\mathbf{E}}^v \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ +2y & 2x \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}_u & \bar{\mathbf{E}}_v \end{pmatrix} = \frac{\begin{pmatrix} 2x & +2y \\ -2y & 2x \end{pmatrix}}{4\left(x^2 + y^2\right)}$$