# Lecture 11
# Wed. 10.02.2019

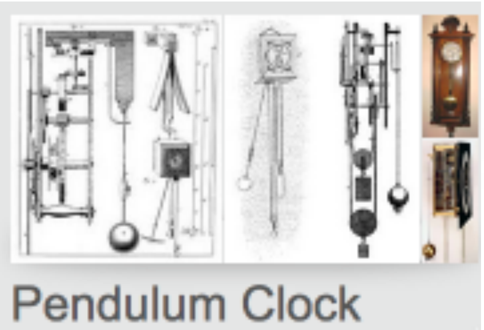
## *Poincare, Lagrange, Hamiltonian, and Jacobi mechanics*

*(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)*

*Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)*
- *Clues for future assignments (Web Simulation: CoulIt)*

*Examples of Hamiltonian mechanics in phase plots*
- *1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*
- *1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")*

*Exploring phase space and Lagrangian mechanics more deeply*
- *A weird "derivation" of Lagrange's equations*
- *Poincare identity and Action,* Jacobi-Hamilton *equations*
- *How Classicists might have "derived" quantum equations*
- *Huygen's contact transformations enforce minimum action*
- *How to do quantum mechanics if you only know classical mechanics*
- *("Color-Quantization" simulations:* Davis-Heller "Color-Quantization" or "Classical Chromodynamics"*)*




# *This Lecture's Reference Link Listing*



## *Lecture #11*

**Eric J Heller Gallery:**

*Main portal, Consonance and Dissonance II, Bessel 21, Chladni*

*The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981*
*Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981*

*Pendulum Web Simulation*
*Cycloidulum Web Simulation*

***CoulIt* Web Simulations:**

*Basic/Generic*
*Exploding Starlet*
*Volcanoes of Io (Color Quantized)*

***JerkIt* Web Simulations:**

*Basic/Generic*
*Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot*

**Select, exciting, and related Research & Articles of Interest:**

*An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019*
*An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019*
*Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_ Room_Temperature_Magnet_-_Belopolski-s-2019*
*"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019*
*Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019*
*What_Industry_Can_Teach_Academia_-_Mao-s-2019*

*Rovibrational_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)*
*A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019*

# *Running Reference Link Listing*

## ***Lectures #10 through #7***

*In reverse order*

**Links to previous lecture:** *Page=74*, *Page=75*, *Page=79*

*Pendulum Web Sim*
*Cycloidulum Web Sim*

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

***OscillatorPE* Web Simulation:**
*Coulomb-Newton-Inverse_Square*,
*Hooke-Isotropic Harmonic*,
*Pendulum-Circular_Constraint*

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

*Pirelli Site: Phasors animimation*
*CMwithBang Lecture #6, page=70 (9.10.18)*

***BoxIt Web Simulations:***
*Generic/Default*
*Most Basic A-Type*
*Basic A-Type w/reference lines*
*Basic A-Type A-Type with Potential energy*
*A-Type with Potential energy and Stokes Plot*
*A-Type w/3 time rates of change*
*A-Type w/3 time rates of change with Stokes Plot*
*B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)*

***RelaWavity Web Elliptical Motion Simulations:***
*Orbits with b/a=0.125*
*Orbits with b/a=0.5*
*Orbits with b/a=0.7*
*Exegesis with b/a=0.125*
*Exegesis with b/a=0.5*
*Exegesis with b/a=0.7*
*Contact Ellipsometry*

**Select, exciting, and related Research & Articles of Interest:**
*Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019*
*Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019*
*Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019*

*A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019*
*Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018*

*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
*Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,*
*Nuclear_Collisions - Mishustin-PhysRevC-2007*, *APS Link & Abstract*
*Hadronic Molecules - Guo-x-2017*
*Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016*

# Running Reference Link Listing

## Lectures #6 through #1

*In reverse order*

*RelaWavity Web Simulation: Contact Ellipsometry*
*BoxIt Web Simulation: Elliptical Motion (A-Type)*
*CMwBang Course: Site Title Page*
*Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors*
UAF Physics UTube channel

*Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971*
*MIT OpenCourseWare: High School/Physics/Impulse and Momentum*
*Hubble Site: Supernova - SN 1987A*

***BounceItIt Web Animation - Scenarios:***
*49:1 y vs t*, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*, *1:500:1 - 1D Gas (Warm)*, *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
*Farey Sequence - Wolfram*
*Fractions - Ford-AMM-1938*

***Monstermash BounceItIt Animations:***
*1000:1 - V2 vs V1*, *1000:1 with t vs x - Minkowski Plot*
*Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013*
*Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015*
*Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)*
*Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971*

***WaveIt Web Animation - Scenarios:***
*Quantum_Carpet*, *Quantum_Carpet_wMBars*,
*Quantum_Carpet_BCar*, *Quantum_Carpet_BCar_wMBars*
*Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001*
*Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)*

***BounceIt Web Animation - Scenarios:***
*Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)*
*1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4*
*7:1 - V vs V Plot: Power=1*
*3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4*
*3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1*
*3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps*
*4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4*
*4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps*
*6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot*
*5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps*
*1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps*

***BounceIt Dual plots***
***$m_1$:$m_2$ = 3:1***
*$v_2$ vs $v_1$ and $V_2$ vs $V_1$*, *($v_1$, $v_2$)=(1, 0.1)*, *($v_1$, $v_2$)=(1, 0)*
*$y_2$ vs $y_1$ plots:* *($v_1$, $v_2$)=(1, 0.1)*, *($v_1$, $v_2$)=(1, 0)*, *($v_1$, $v_2$)=(1, -1)*
Estrangian plot *$V_2$ vs $V_1$:* *($v_1$, $v_2$)=(0, 1)*, *($v_1$, $v_2$)=(1, -1)*
***$m_1$:$m_2$ = 4:1***
*v2 vs v1*, *y2 vs y1*
***$m_1$:$m_2$ = 100:1,*** *($v_1$, $v_2$)=(1, 0): V2 vs V1 Estrangian plot*, *y2 vs y1 plot*

*With g=0 and 70:10 mass ratio*
*With non zero g, velocity dependent damping and mass ratio of 70:35*
*$M_1$=49, $M_2$=1 with Newtonian time plot*
*$M_1$=49, $M_2$=1 with $V_2$ vs $V_1$ plot*
*Example with friction*
*Low force constant with drag displaying a Pass-thru*, *Fall-Thru*, *Bounce-Off*
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

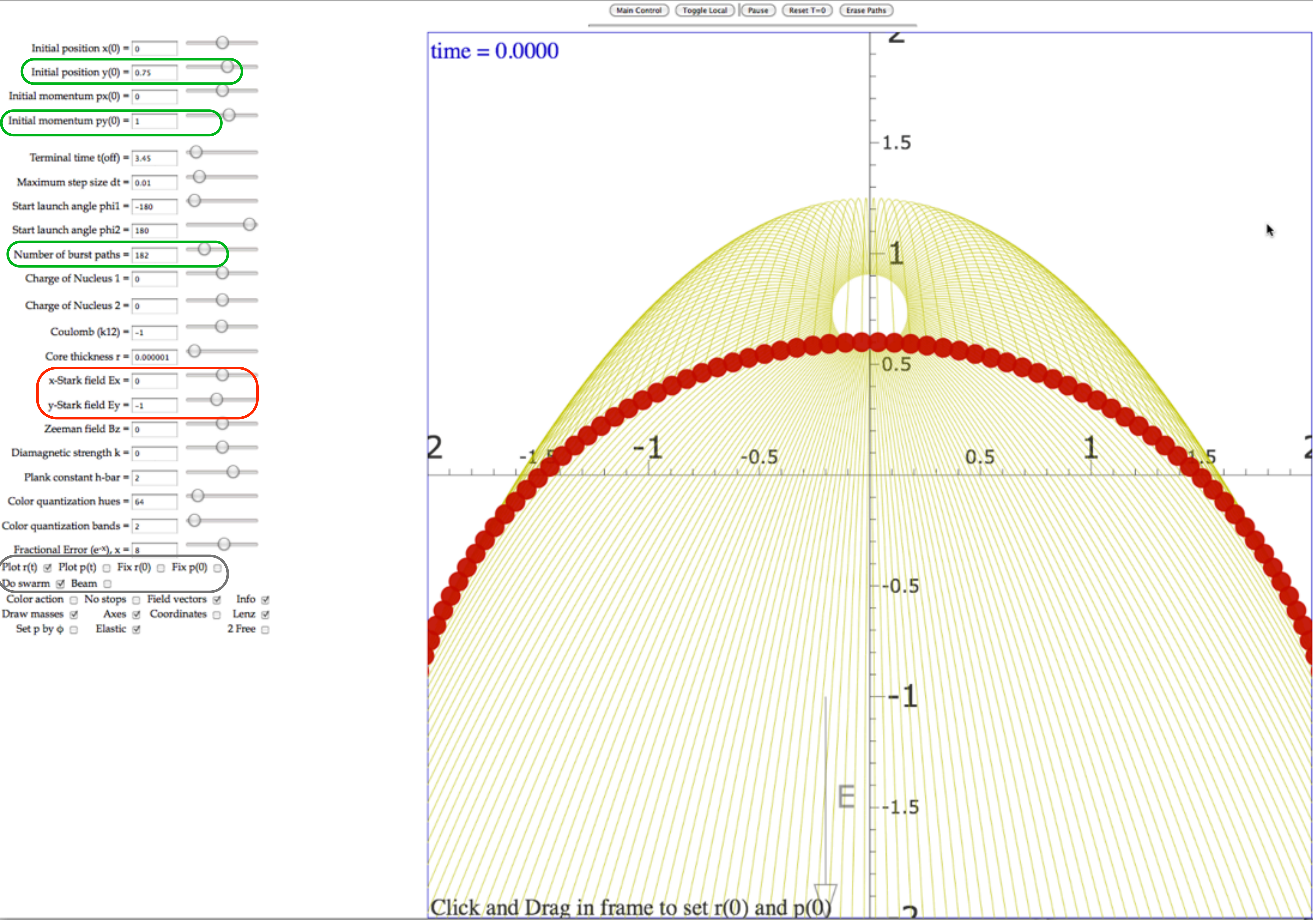
*AJP article on superball dynamics*
*AAPT Summer Reading List*
*Scitation.org - AIP publications*
*HarterSoft Youtube Channel*

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
*Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html*
*Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007*
*BounceIt web simulation with g=0 and 70:10 mass ratio*
*With non zero g, velocity dependent damping and mass ratio of 70:35*
*Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)*
*Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)*
*Matrix Collision Simulator: $M_1$=49, $M_2$=1 $V_2$ vs $V_1$ plot* <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
*(Now in Development)*

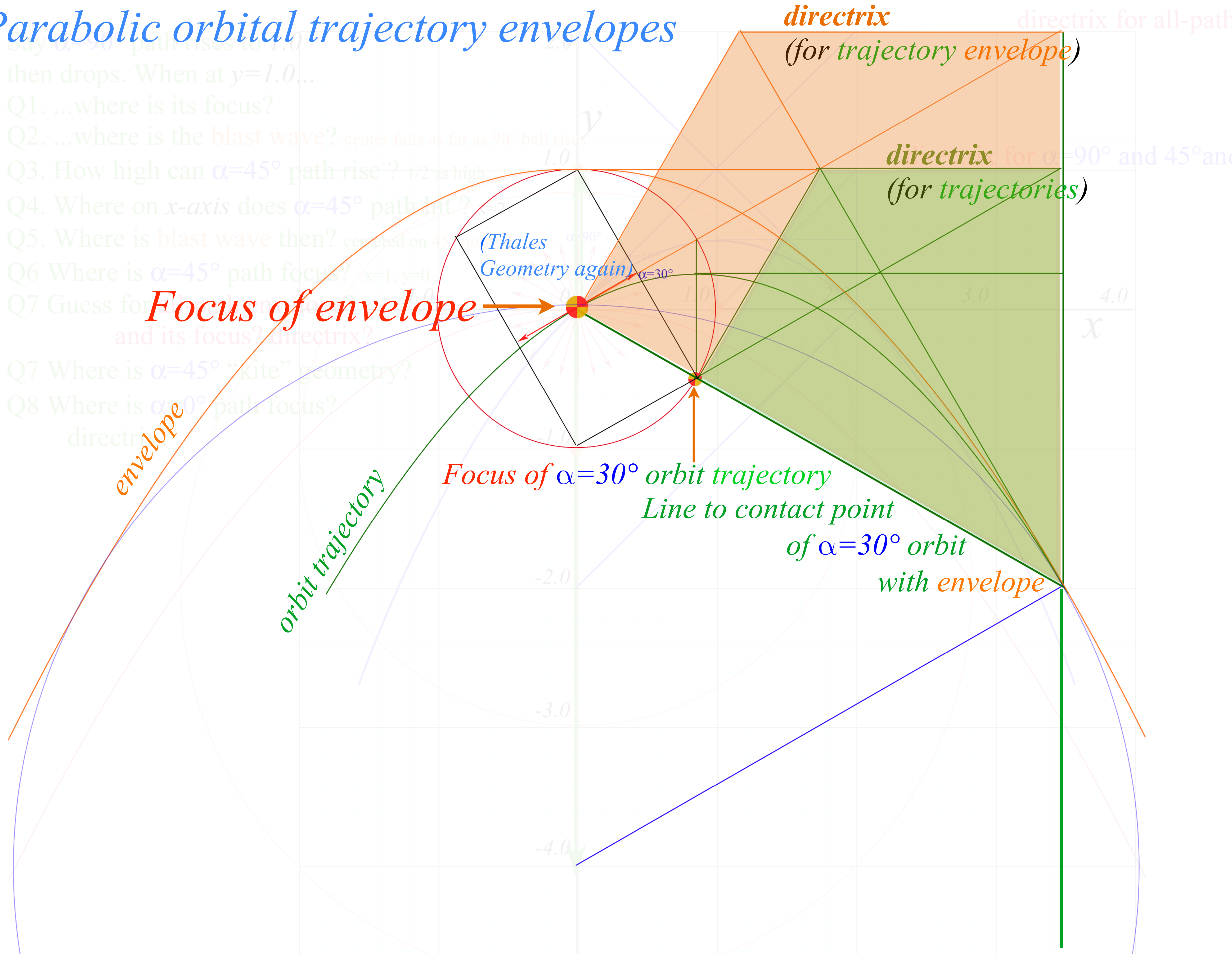
*Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)*

*Some clues for future assignments* *(Web Simulation: CoulIt)*




[Link ⇒ CoulIt - Simulation of the *Volcanoes of Io*]

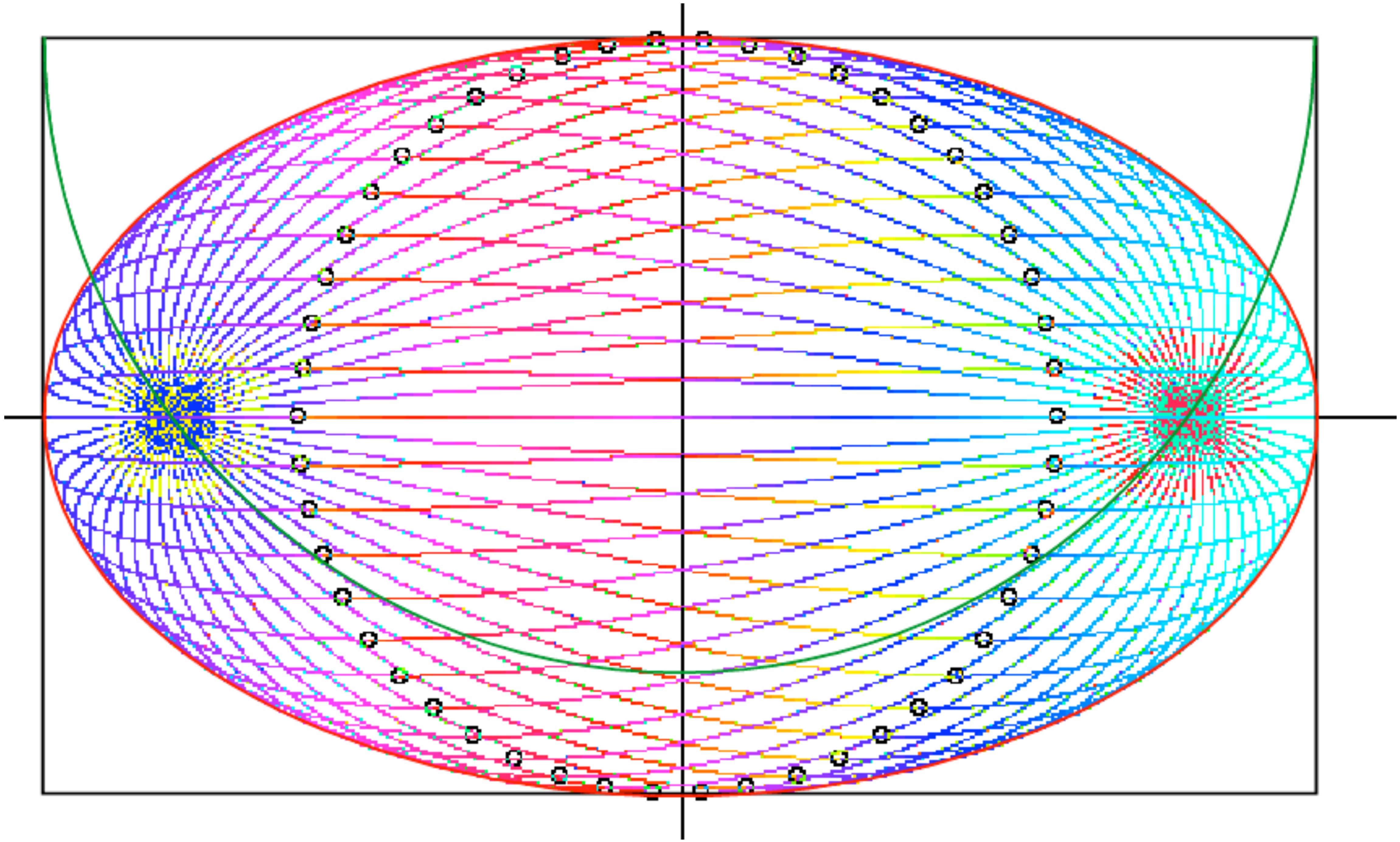
# *Parabolic orbital trajectory envelopes*



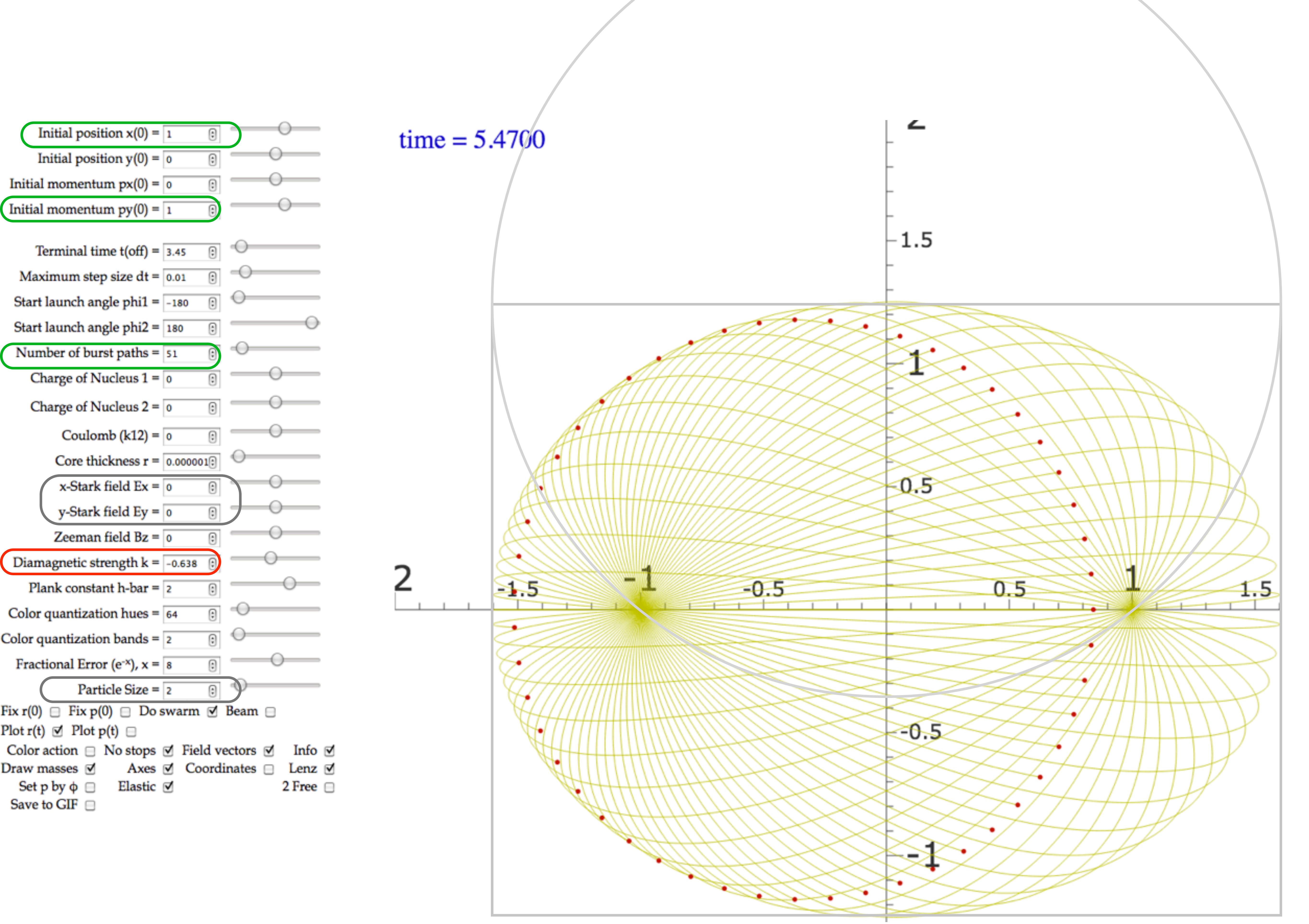
[redacted] 2D-IHO orbital envelopes ([redacted] a generalization.)

Some clues for future assignments (Web Simulation: CouIIt)

# *Exploding-starlet elliptical envelope and contacting elliptical trajectories*



*(Web Simulation: CouIIt - Exploding*Starlet {IHO Potential})*




*(Web Simulation: CoulIt - Exploding*Starlet {IHO Potential})*

## *Examples of Hamiltonian mechanics in phase plots*

*1D Pendulum and phase plot* *(Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*

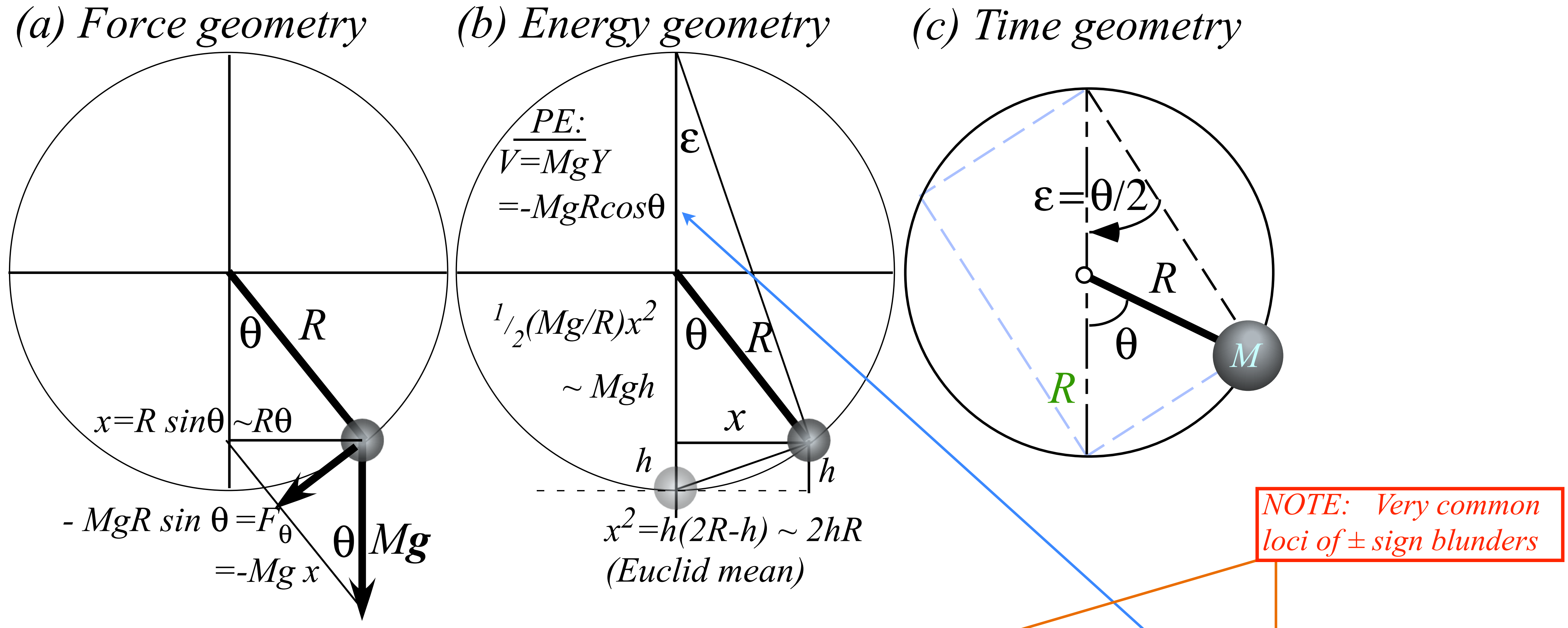
*Circular pendulum dynamics and elliptic functions*

*Cycloid pendulum dynamics and "sawtooth" functions*

*1D-HO phase-space control* *(Old Mac OS & Web Simulations of "Catcher in the Eye")*

# 1D Pendulum and phase plot

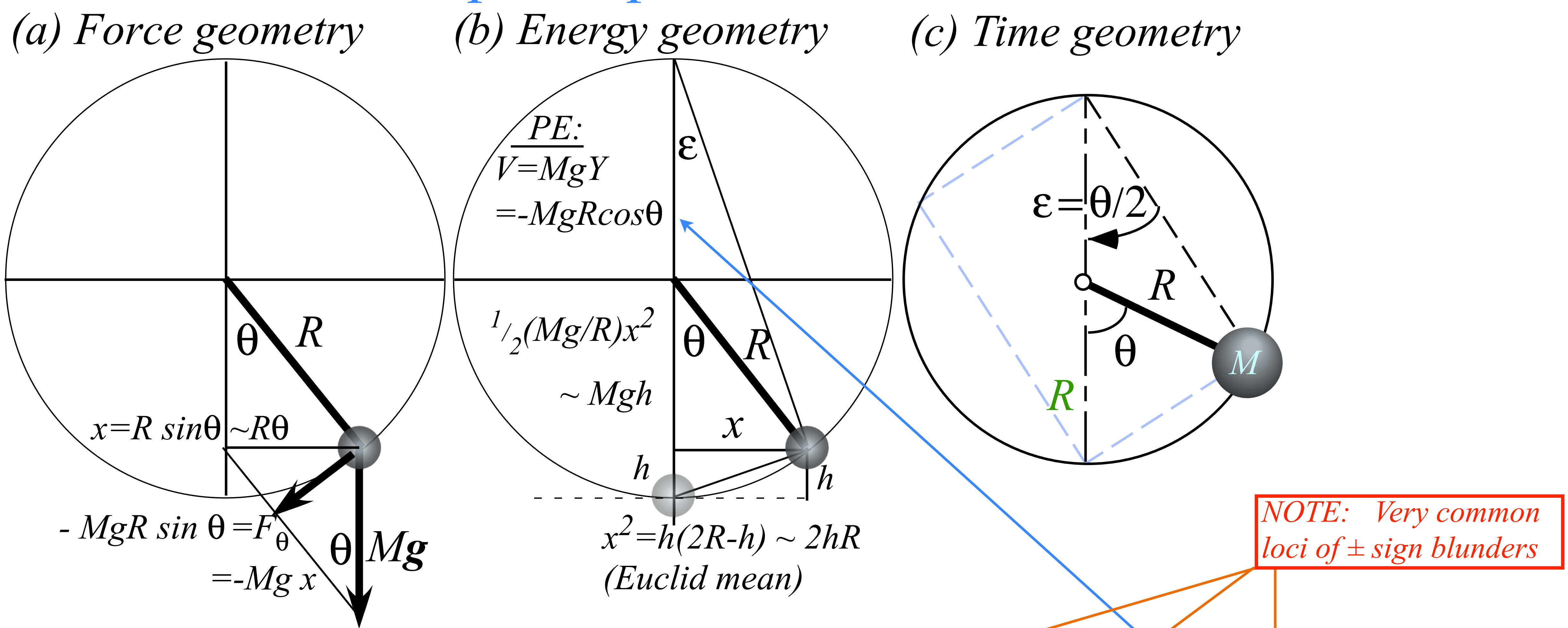
(Unit 2 Chapter 7 Fig. 2)




Lagrangian function L= KE - PE = T - U where potential energy is $U(\theta) = -MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

# 1D Pendulum and phase plot



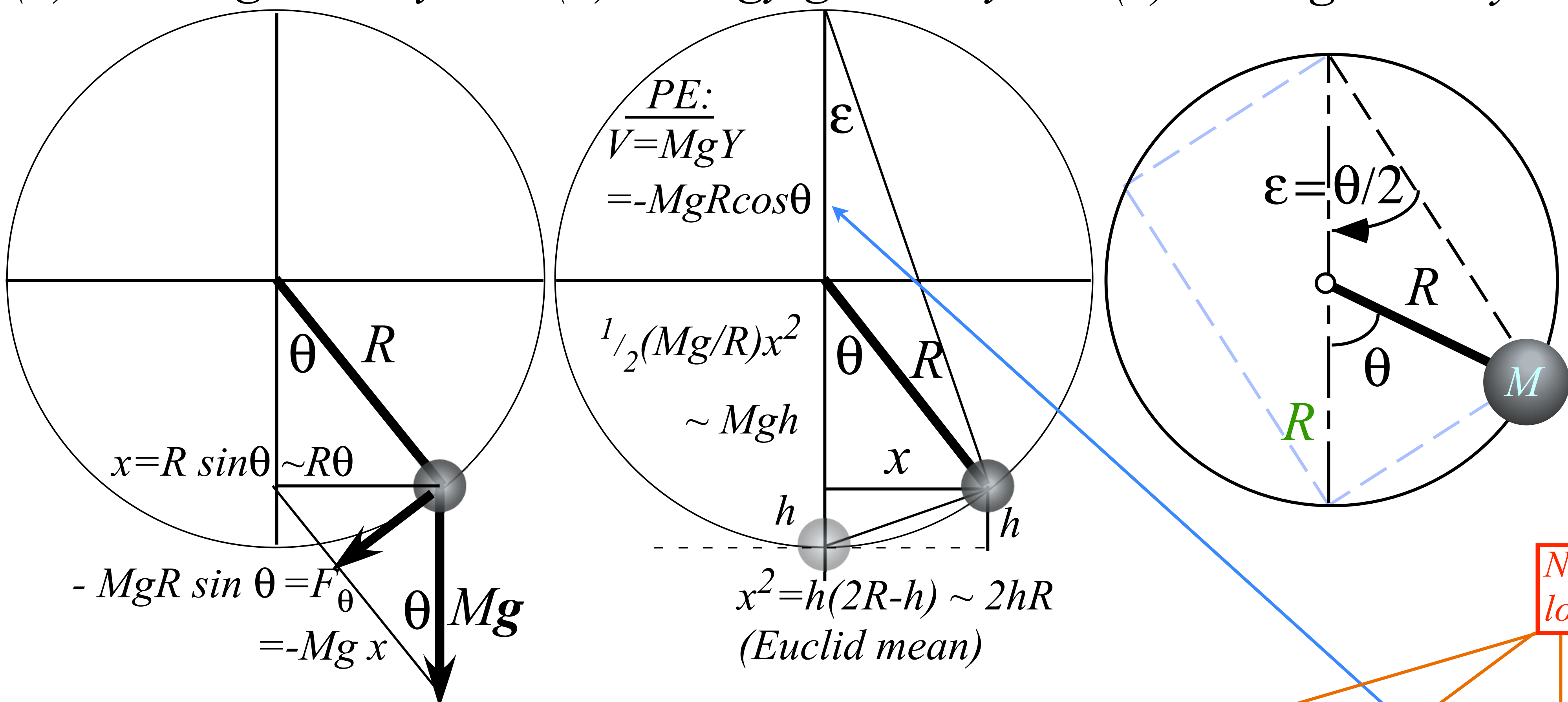

*Lagrangian function L= KE - PE = T - U where potential energy is U(θ) =* $-MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

*Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) =* $-MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;= E \quad = const.$$

# 1D Pendulum and phase plot




NOTE: Very common loci of ± sign blunders

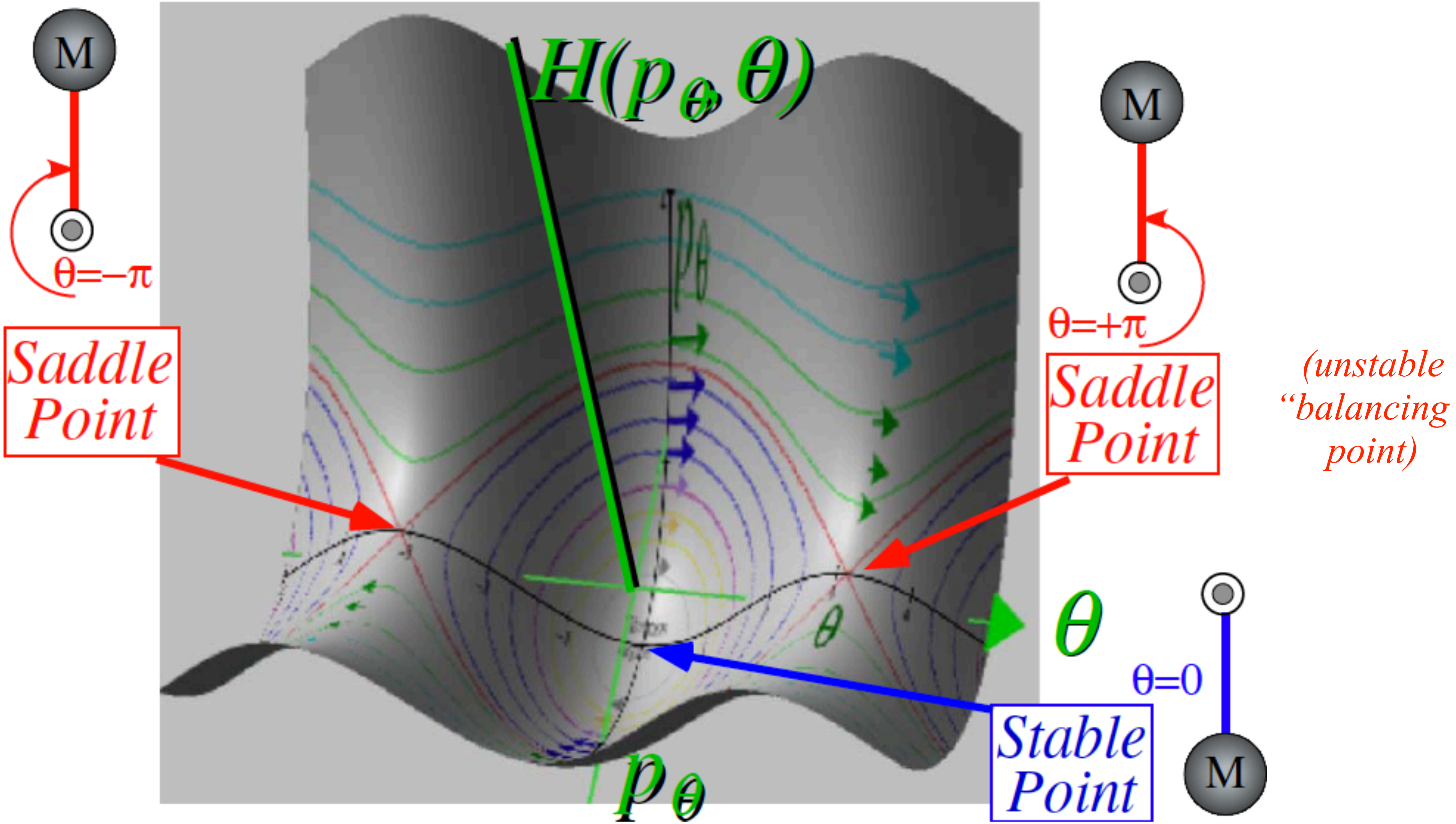
Lagrangian function L= KE - PE = T - U where potential energy is $U(\theta) = -MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

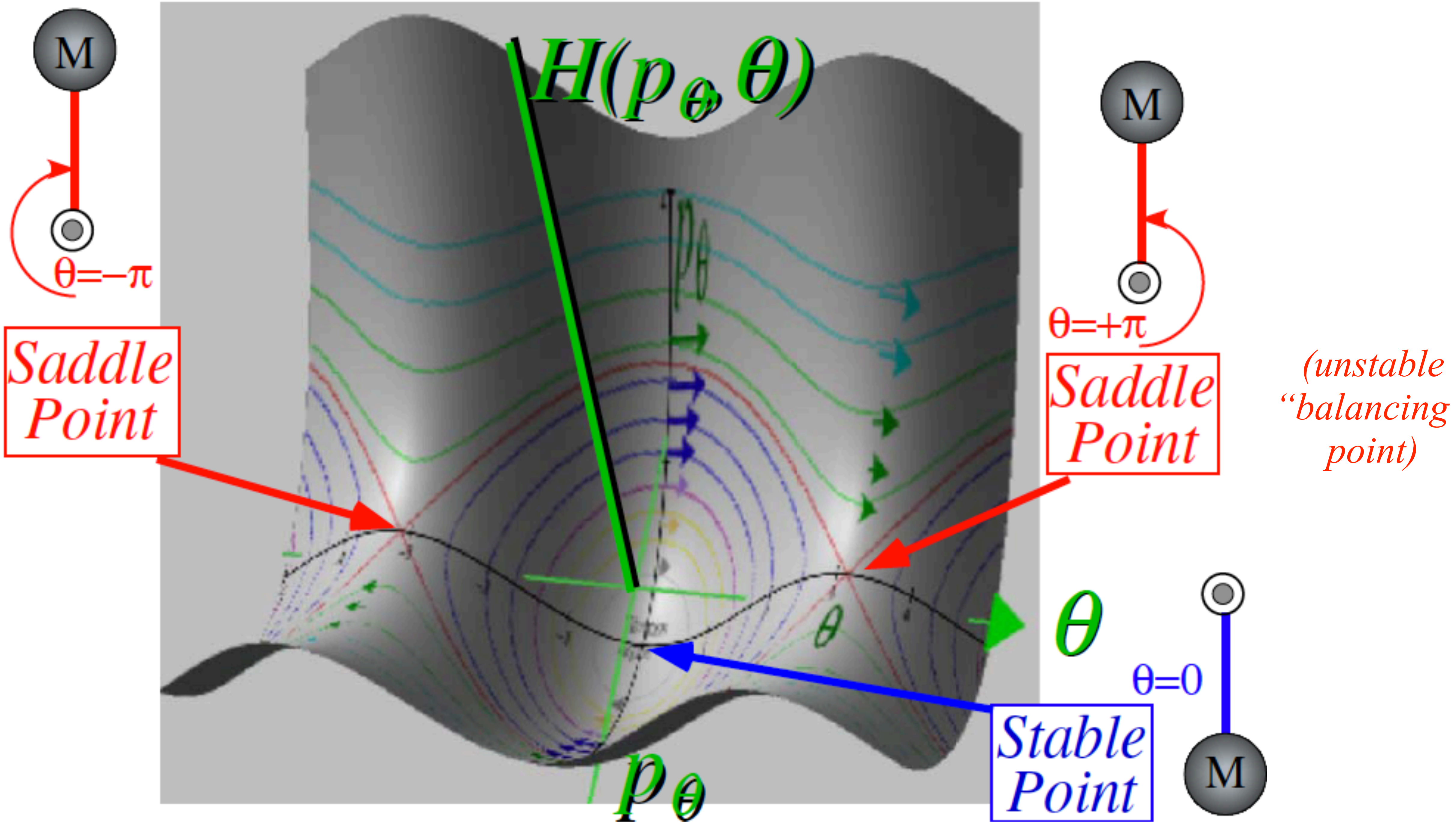
$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;=E \quad =\text{const.}$$

implies: $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$




Example of plot of Hamilton for 1D-solid pendulum in its Phase Space $(\theta, p_\theta)$

$$H(p_\theta,\theta) = E = \frac{1}{2I} p_\theta{}^2 - MgR\cos\theta \ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$




*Example of plot of Hamilton for 1D-solid pendulum in its Phase Space ($\theta$,$p_\theta$)*

$$H(p_\theta,\theta) = E = \frac{1}{2I}p_\theta^{\ 2} - MgR\cos\theta\ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$

*Funny way to look at Hamilton's equations:*

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix} = \mathbf{e_H}\times\left(-\nabla H\right) = \overrightarrow{(\text{H-axis})}\times\overrightarrow{(\text{fall line})}, \text{ where:}\begin{cases}\overrightarrow{(\text{H-axis})} = \mathbf{e_H} = \mathbf{e_q}\times\mathbf{e_p}\\ \overrightarrow{(\text{fall line})} = -\nabla H\end{cases}$$

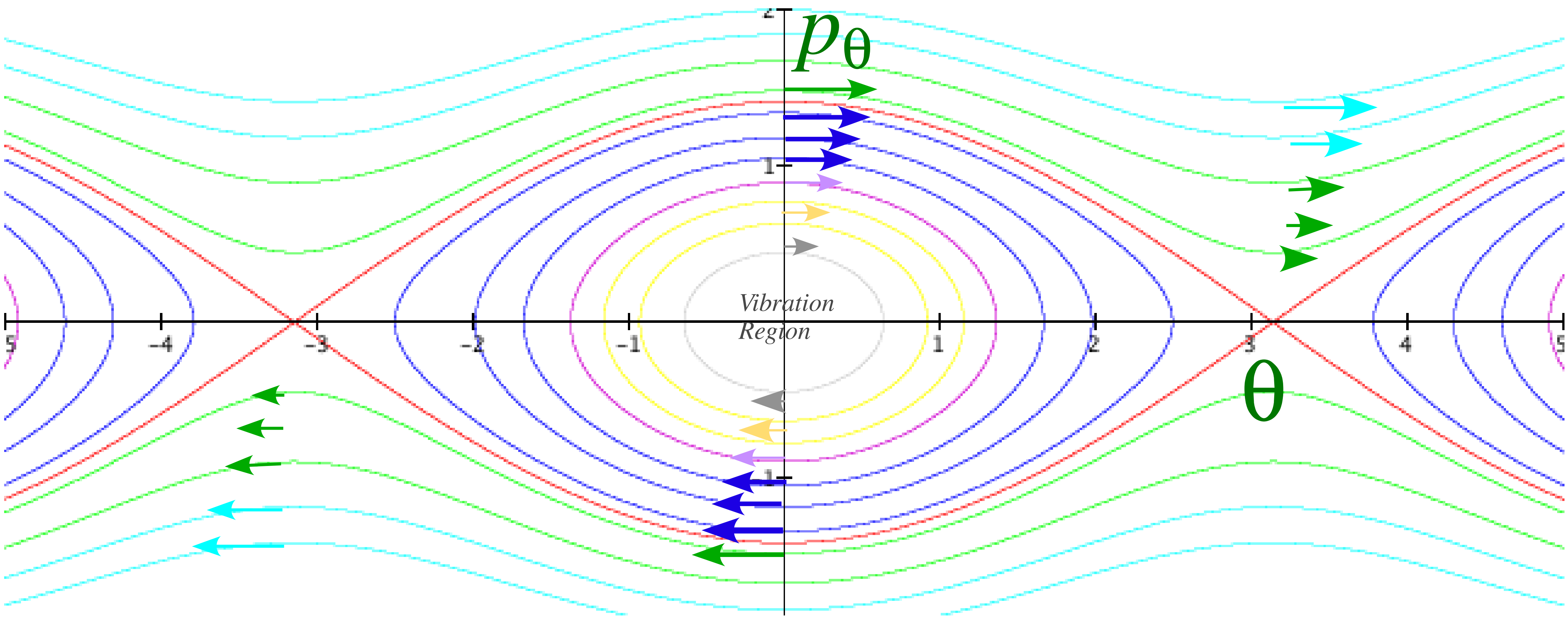



*Fig. 2.7.2 Phase portrait or topography map for simple pendulum*

*(Unit 2 Chapter 7 Fig. 2)*

## *Examples of Hamiltonian mechanics in phase plots*

*1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*

→ *Circular pendulum dynamics and elliptic functions*

*Cycloid pendulum dynamics and "sawtooth" functions*

*1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")*

# Circular pendulum dynamics and elliptic functions

*Hamiltonian function* $H = KE + PE = T + U$ *where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

*implies:* $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$

*Let* $E = MgY = -MgR\cos\theta_0$ *be potential energy where* $KE = 0$ *or* $p_\theta = 0$

# Circular pendulum dynamics and elliptic functions

*Hamiltonian function H= KE + PE = T +U where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

*implies:* $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$

*Let* $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I\left(E + MgR\cos\theta\right)} / I \quad \text{where: } I = MR^2$$

# *Circular pendulum dynamics and elliptic functions*

*Hamiltonian function H= KE + PE = T +U where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

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$E=MgY=-MgR\cos\theta_0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I\left(E + MgR\cos\theta\right)} / I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2\left(E + MgR\cos\theta\right)/I}}$$

# *Circular pendulum dynamics and elliptic functions*

*Hamiltonian function H= KE + PE = T +U where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

*implies:* $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

*Let* $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR\cos\theta)} / I \quad \text{where: } I = MR^2 \quad \text{or} : dt = \frac{d\theta}{\sqrt{2(E + MgR\cos\theta)/I}}$$

($E=MgY=-MgR\cos\theta_0$)

*Quadrature integral gives quarter-period* $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

# Circular pendulum dynamics and elliptic functions

*Hamiltonian function* $H = KE + PE = T + U$ *where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;= E \quad = const.$$

*implies:* $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

*Let* $E = MgY = -MgR\cos\theta_0$ *be potential energy where* $KE = 0$ *or* $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR\cos\theta)}/I \quad \text{where: } I = MR^2 \quad \text{or}: dt = \frac{d\theta}{\sqrt{2(E + MgR\cos\theta)/I}}$$

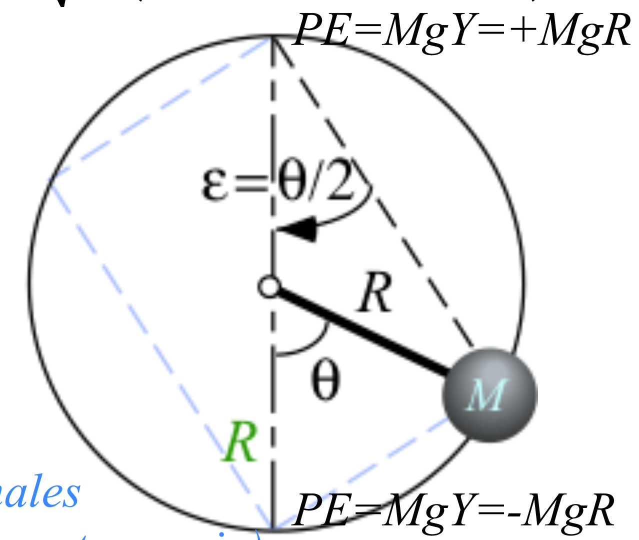
$E = MgY = -MgR\cos\theta_0$

*Quadrature integral gives quarter-period* $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

*Uses a half-angle coordinate* $\varepsilon = \theta/2$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\varepsilon,$$




*(Thales Geometry again)*

# *Circular pendulum dynamics and elliptic functions*

*Hamiltonian function* $H = KE + PE = T + U$ *where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

*implies:* $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

*Let* $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E+MgR\cos\theta)}/I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2(E+MgR\cos\theta)/I}}$$

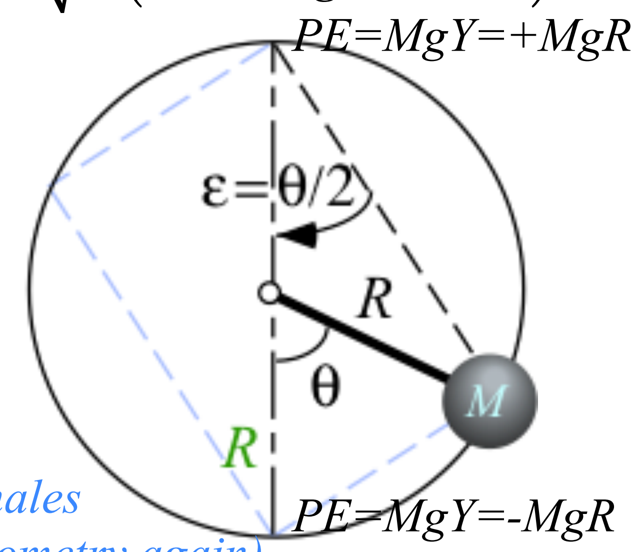
$E=MgY=-MgR\cos\theta_0$

*Quadrature integral gives quarter-period* $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

*Uses a half-angle coordinate* $\varepsilon=\theta/2$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1-2\sin^2\varepsilon, \qquad \cos\theta - \cos\theta_0 = 2\sin^2\varepsilon_0 - 2\sin^2\varepsilon$$



*(Thales Geometry again)*

# Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}{p_\theta}^2 + U(\theta) = \frac{1}{2I}{p_\theta}^2 - MgR\cos\theta = E \quad = const.$$

implies: $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

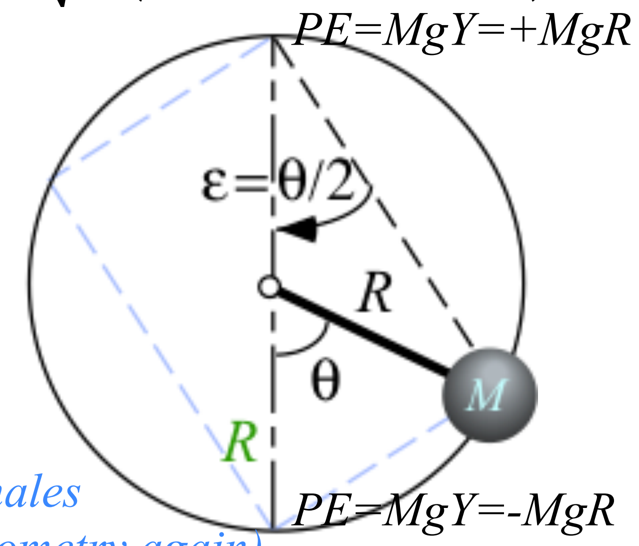
Let $E=MgY=-MgR\cos\theta_0$ be potential energy where $KE=0$ or $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E+MgR\cos\theta)} / I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2(E+MgR\cos\theta)/I}}$$

$E=MgY=-MgR\cos\theta_0$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$



(Thales Geometry again)

Uses a half-angle coordinate $\varepsilon=\theta/2$

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$$\tau_{1/4} = \sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}$$

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E=MgY=-MgRcosθ₀

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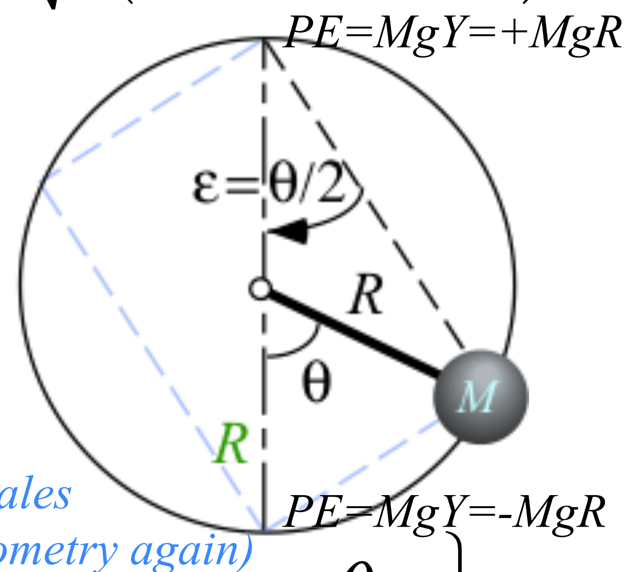


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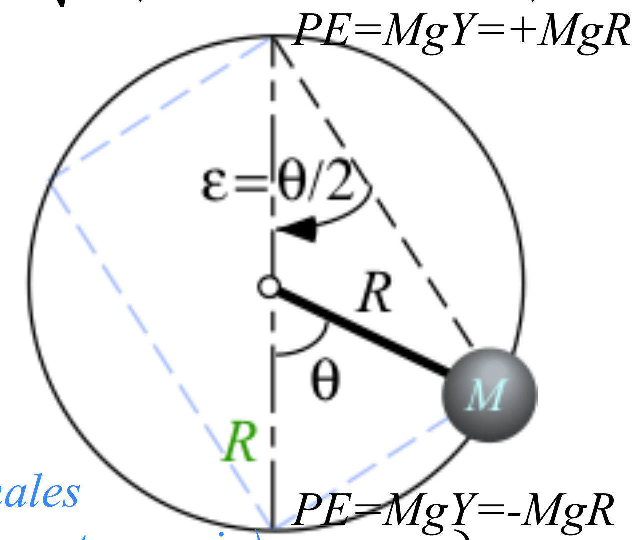

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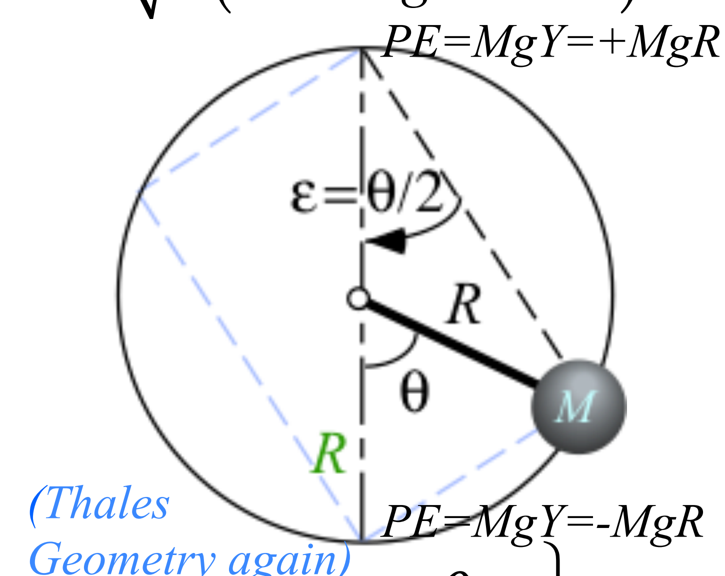


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For low amplitude $\varepsilon \ll 1$: $\sin\varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

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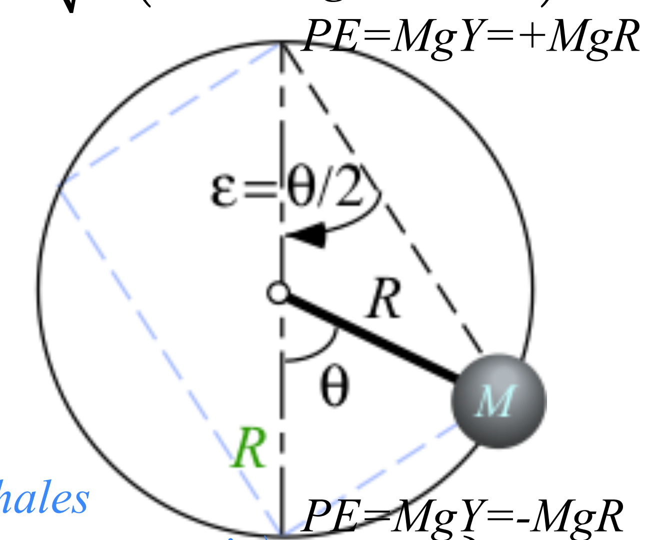


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low $\varepsilon \ll 1$: $t = \sqrt{\frac{R}{g}}\int_0^{\varepsilon(t)}\frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon}{\varepsilon_0}\Bigg|_0^{\varepsilon(t)} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon(t)}{\varepsilon_0}$

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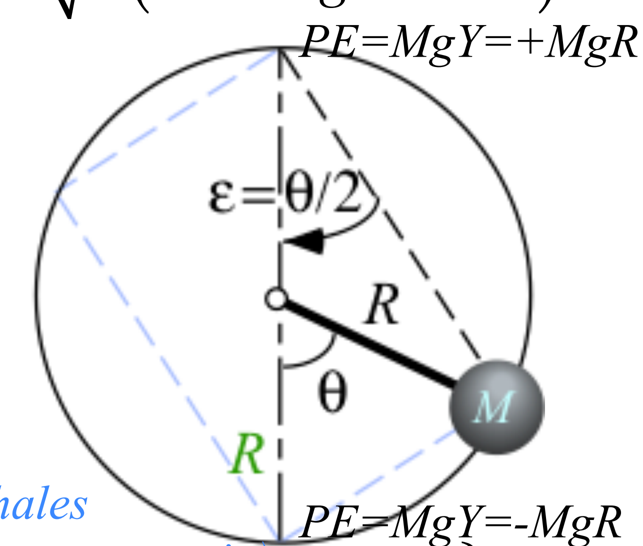


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Uses a half-angle coordinate $\varepsilon = \theta/2$

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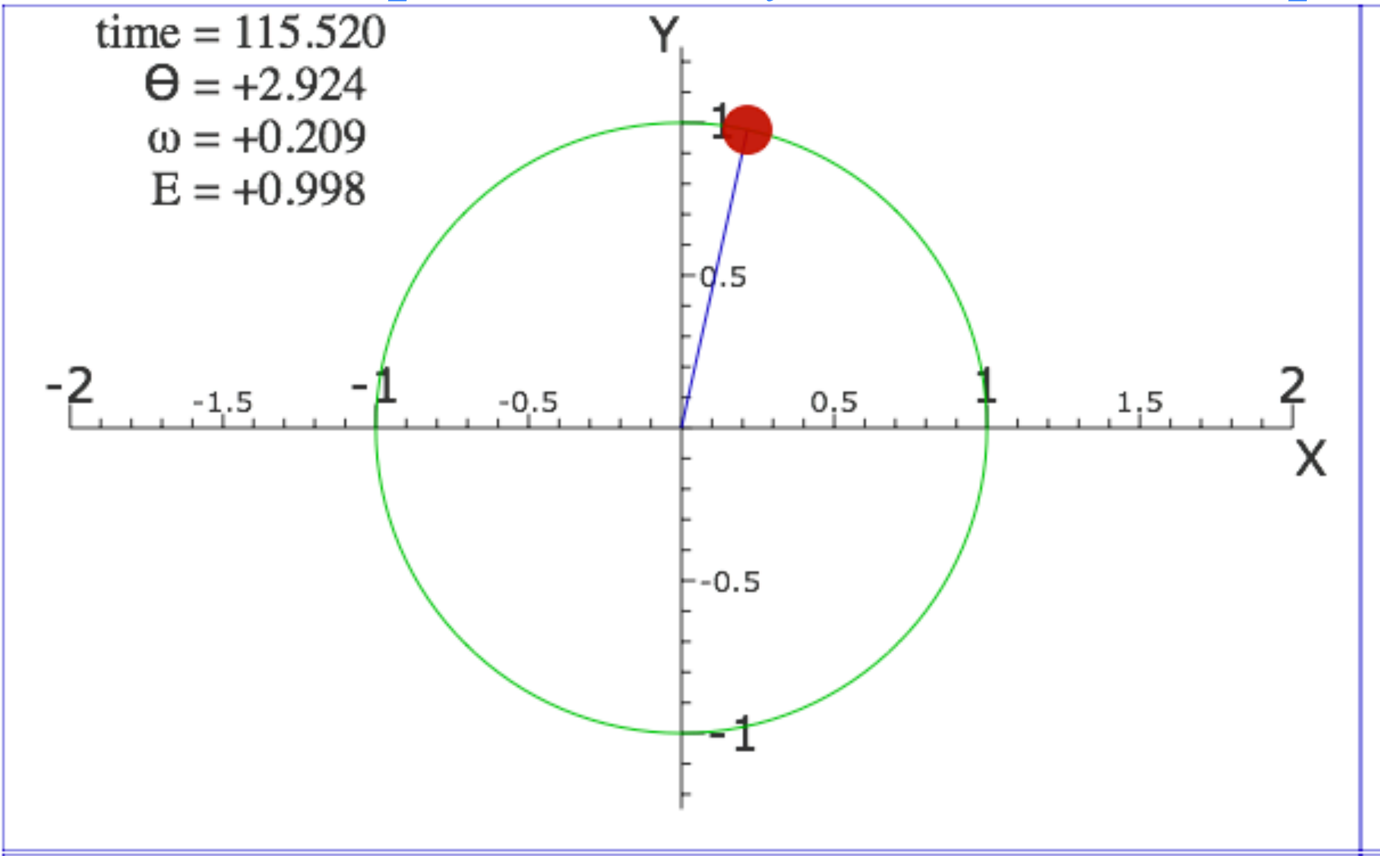
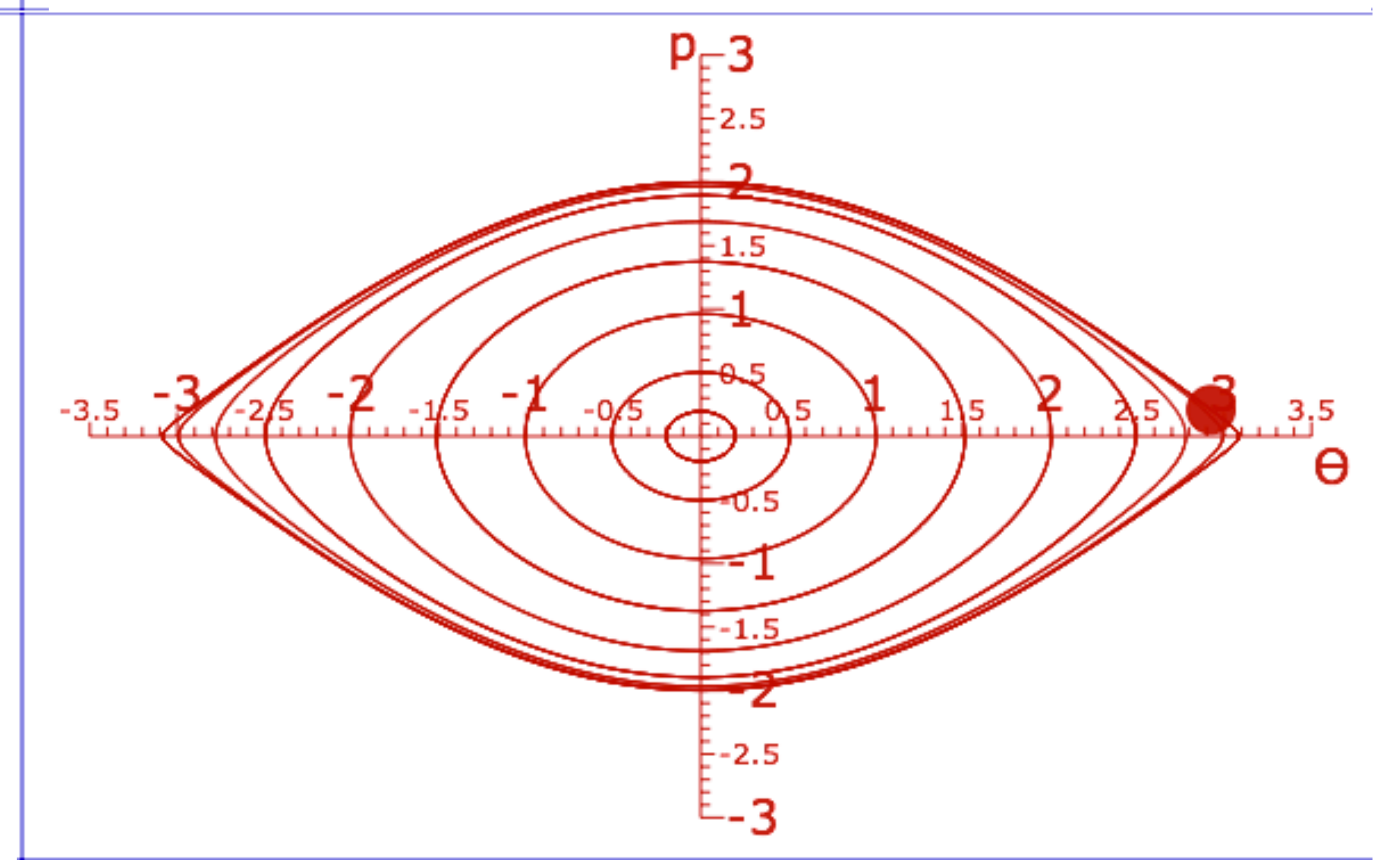
..reduces to sine...

$$\varepsilon(t) = \varepsilon_0\sin\sqrt{\frac{g}{R}}t = \varepsilon_0\sin\omega t, \quad \text{where: } \omega = \sqrt{\frac{g}{R}}$$

For low amplitude $\varepsilon \ll 1$: $\sin\varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

# Circular pendulum dynamics and elliptic functions

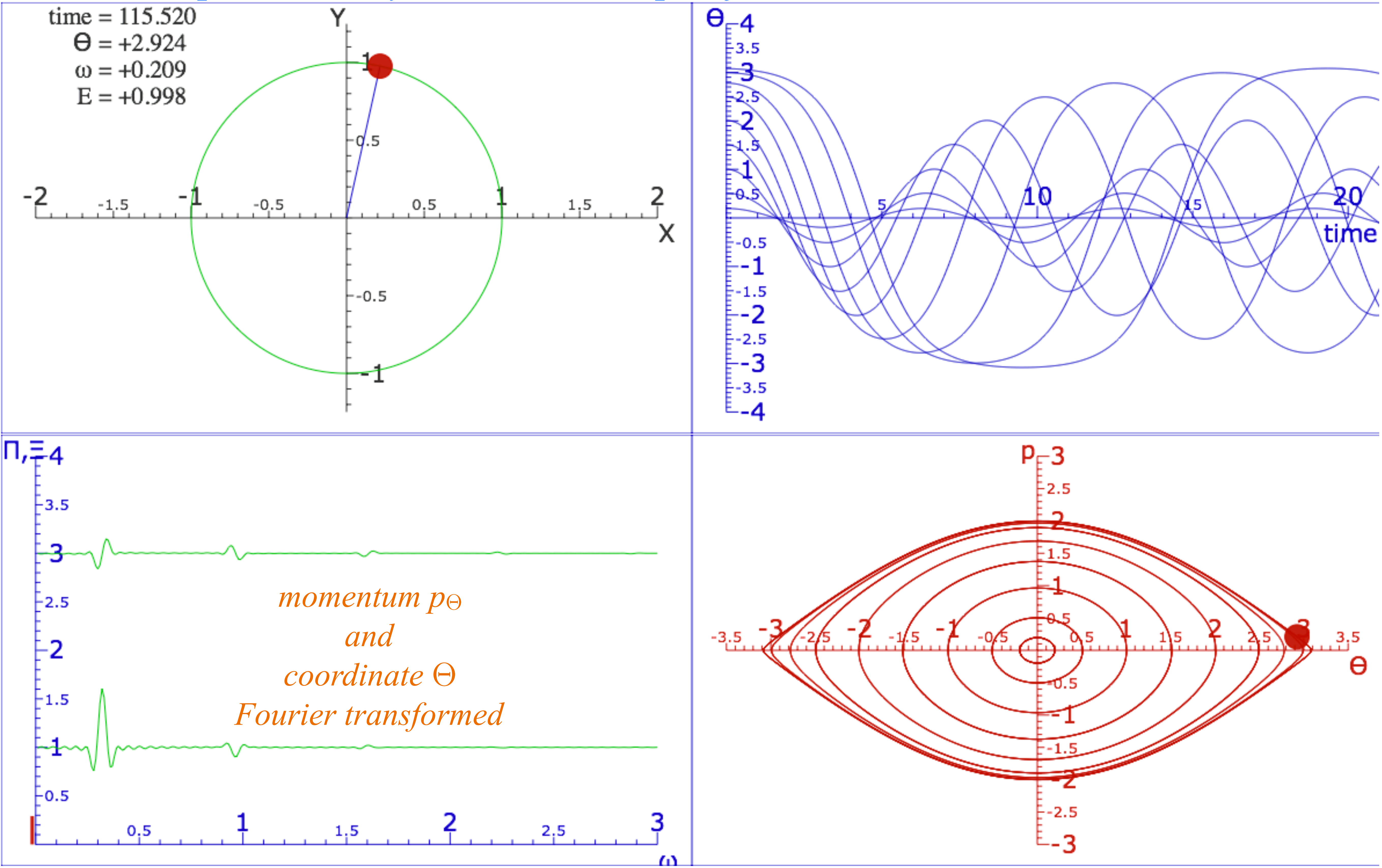



(Simulations of pendulum)

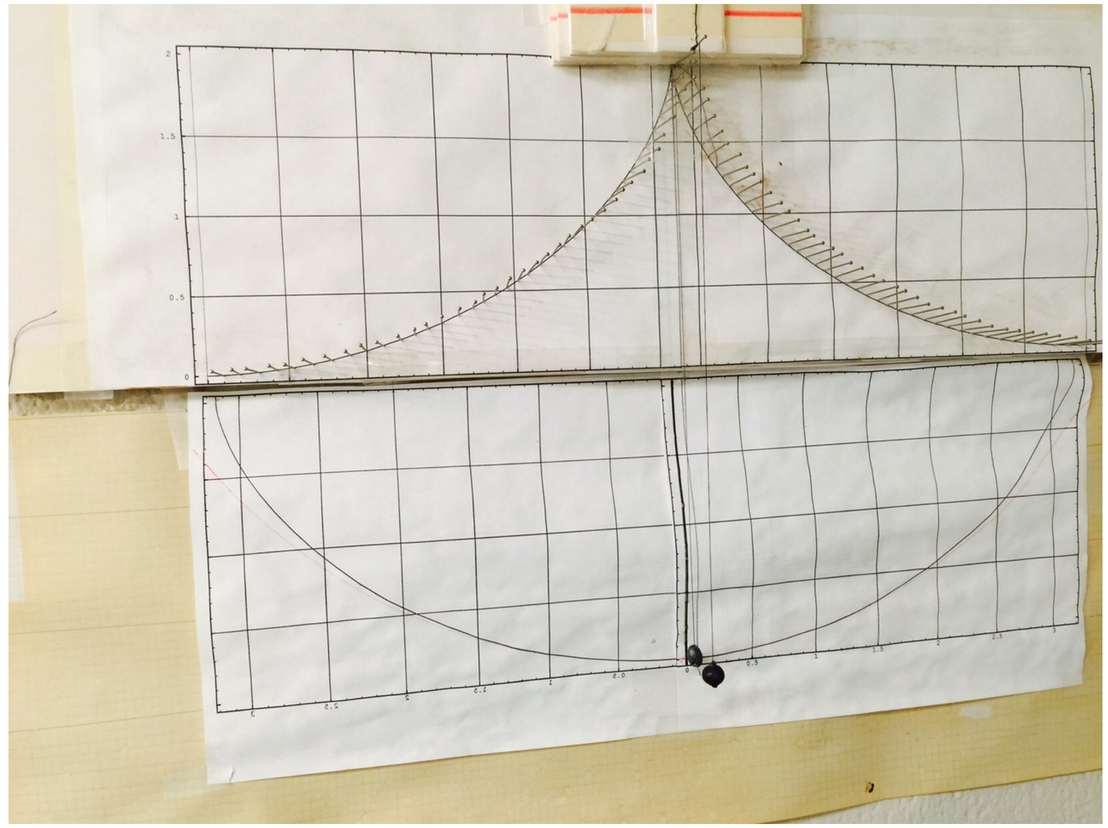
(See also: Simulation of cycloidally constrained pendulum)

# *Circular pendulum dynamics and elliptic functions*




*(Simulations of pendulum)*

*(See also: Simulation of cycloidally constrained pendulum)*



*U of A* (PHYS 241)
*Cycloid pendulum*

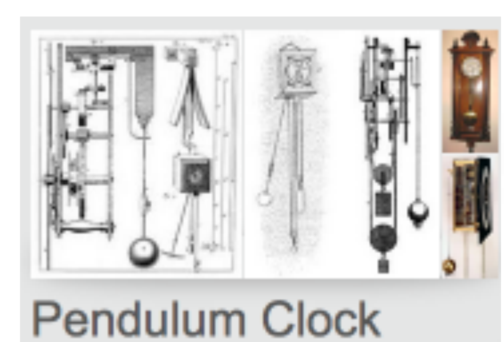
# *Examples of Hamiltonian mechanics in phase plots*

*1D Pendulum and phase plot* *(Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*

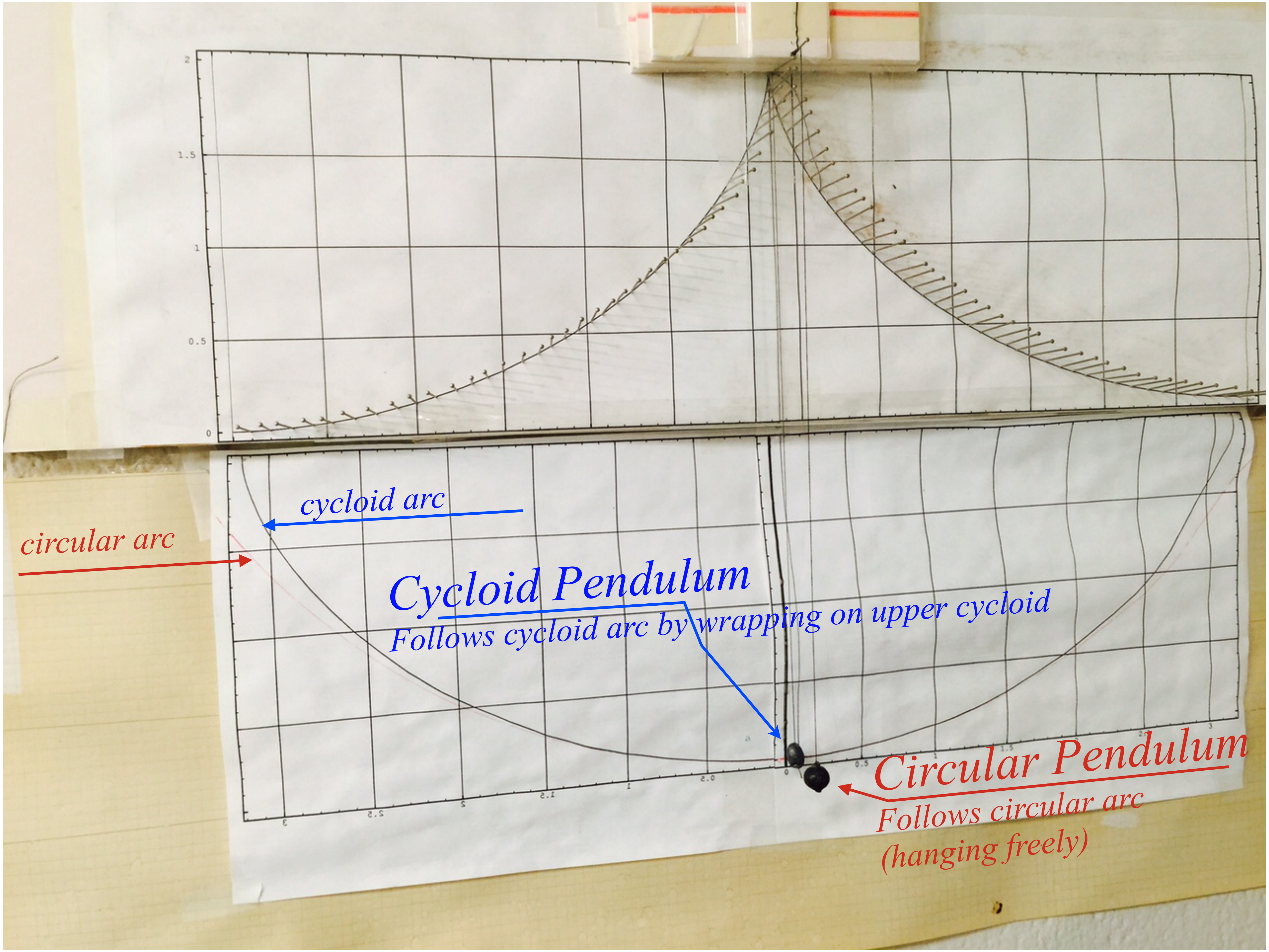
*Circular pendulum dynamics and elliptic functions*

→ *Cycloid pendulum dynamics and "sawtooth" functions*

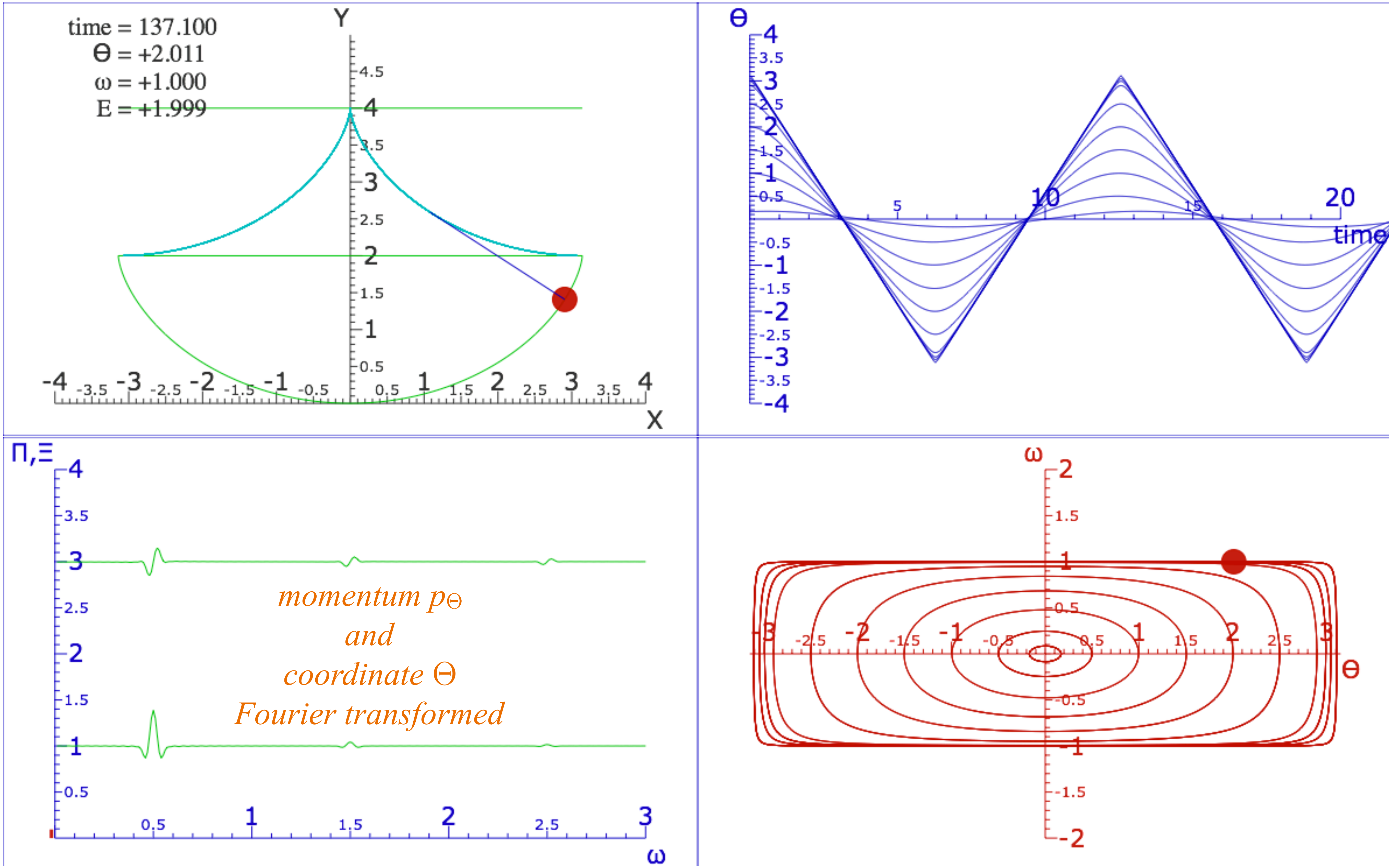
*1D-HO phase-space control* *(Old Mac OS & Web Simulations of "Catcher in the Eye")*




Christaan Huygens
(1629-1695)

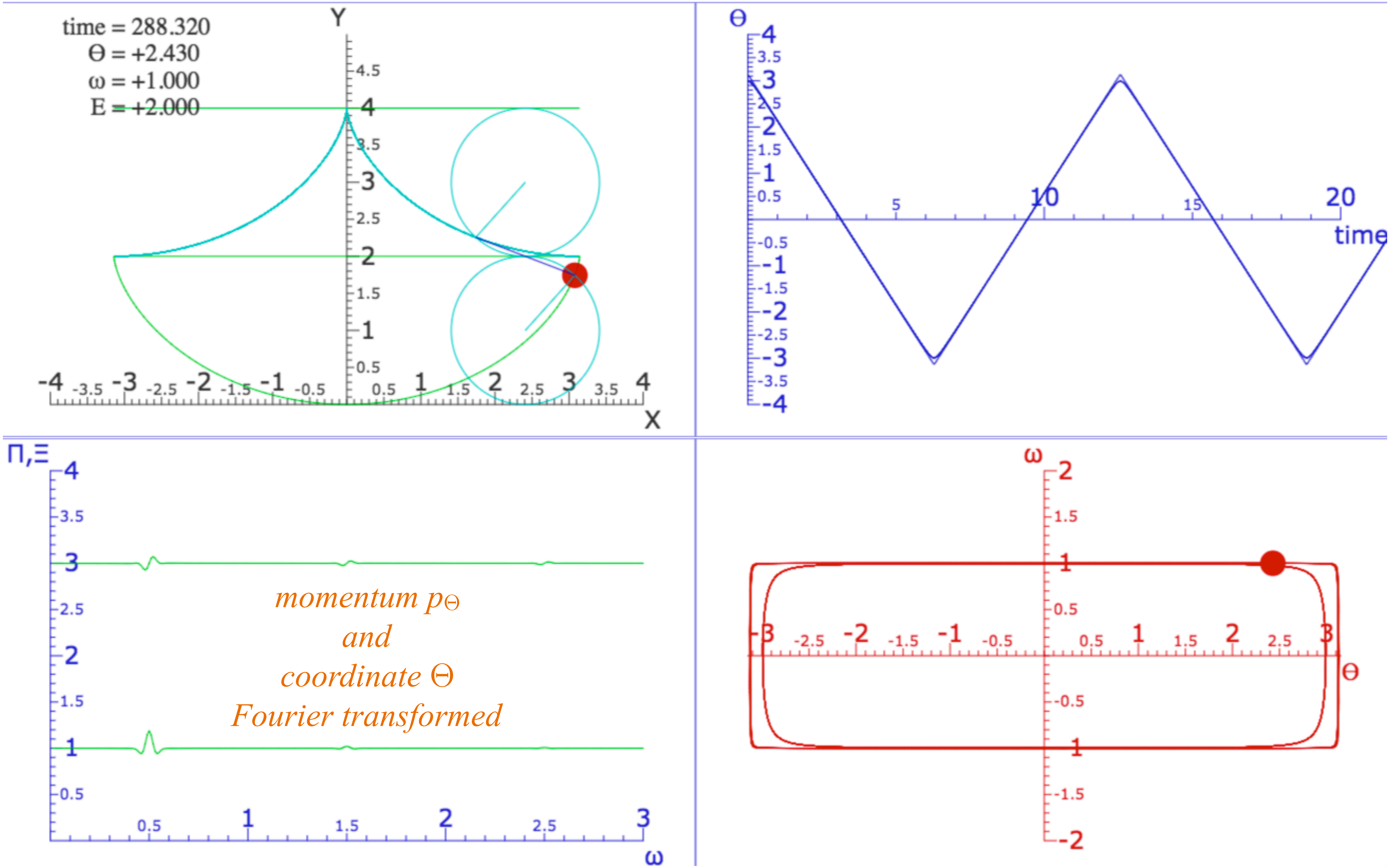

cycloid arc
circular arc
Cycloid Pendulum
Follows cycloid arc by wrapping on upper cycloid
Circular Pendulum
Follows circular arc
(hanging freely)

# *Cycloid pendulum dynamics and "sawtooth" functions*



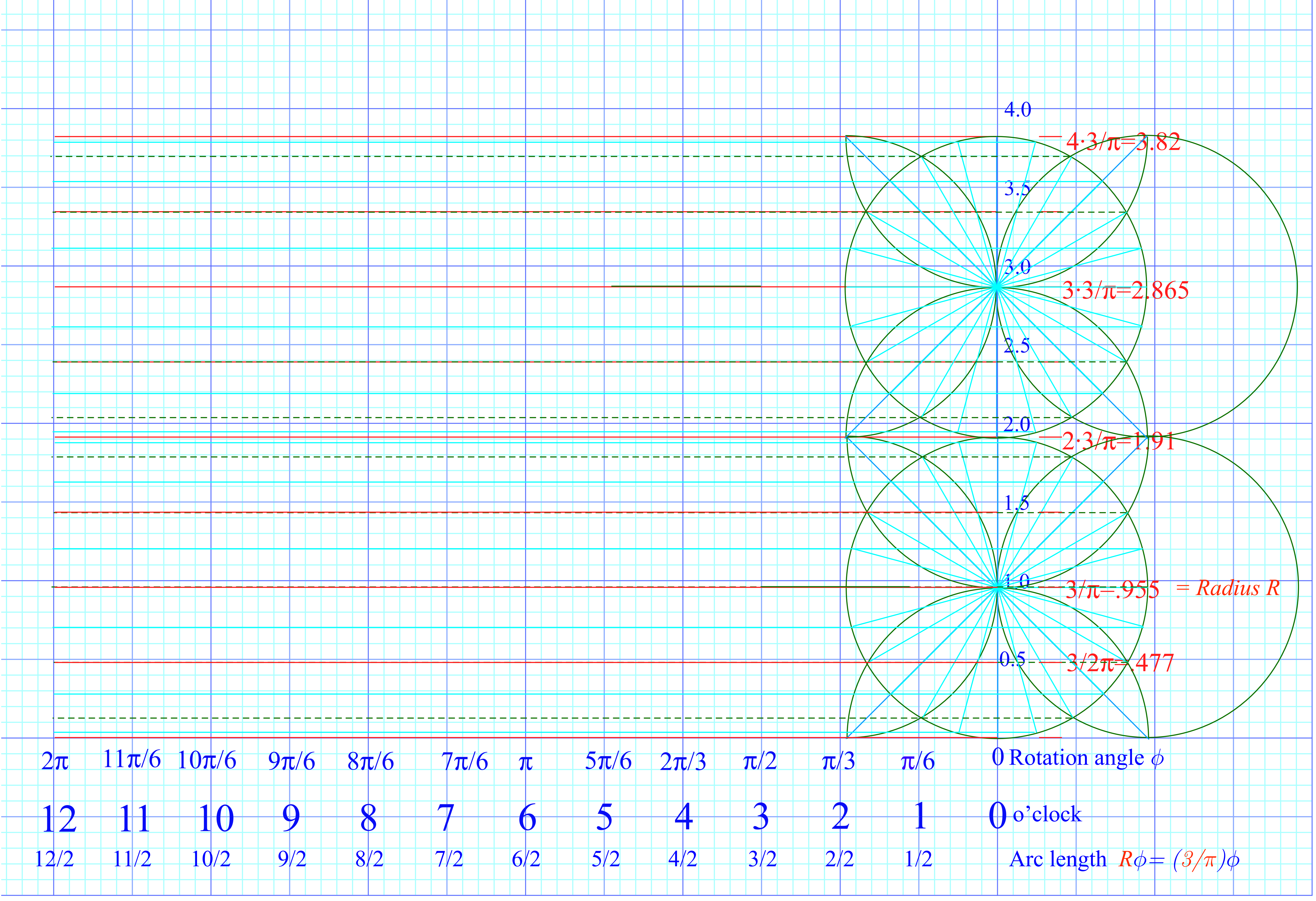

*(Simulations of cycloidally constrained pendulum)*

# *Cycloid pendulum dynamics and "sawtooth" functions*




*(Simulations of cycloidally constrained pendulum)*

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled -out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$



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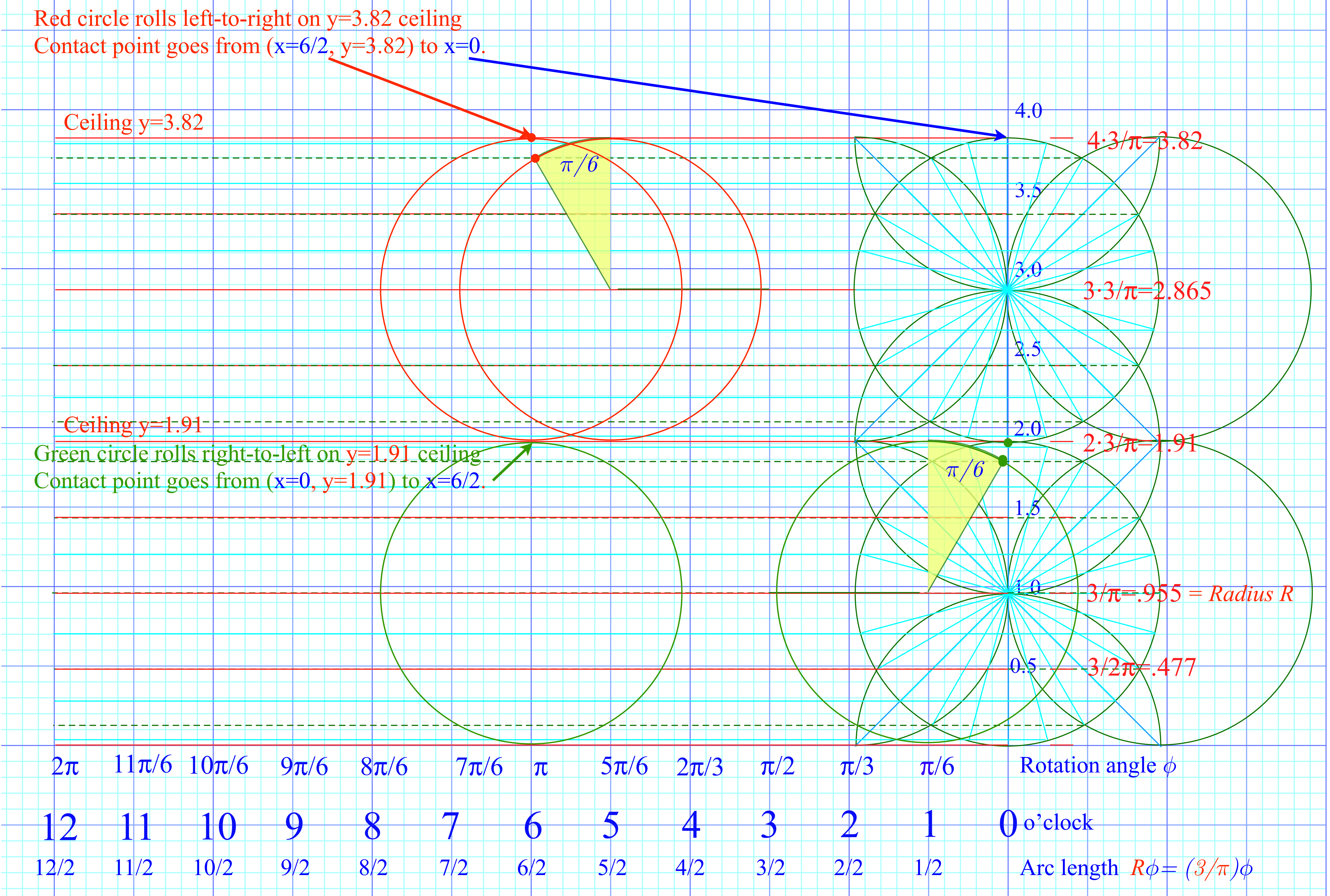

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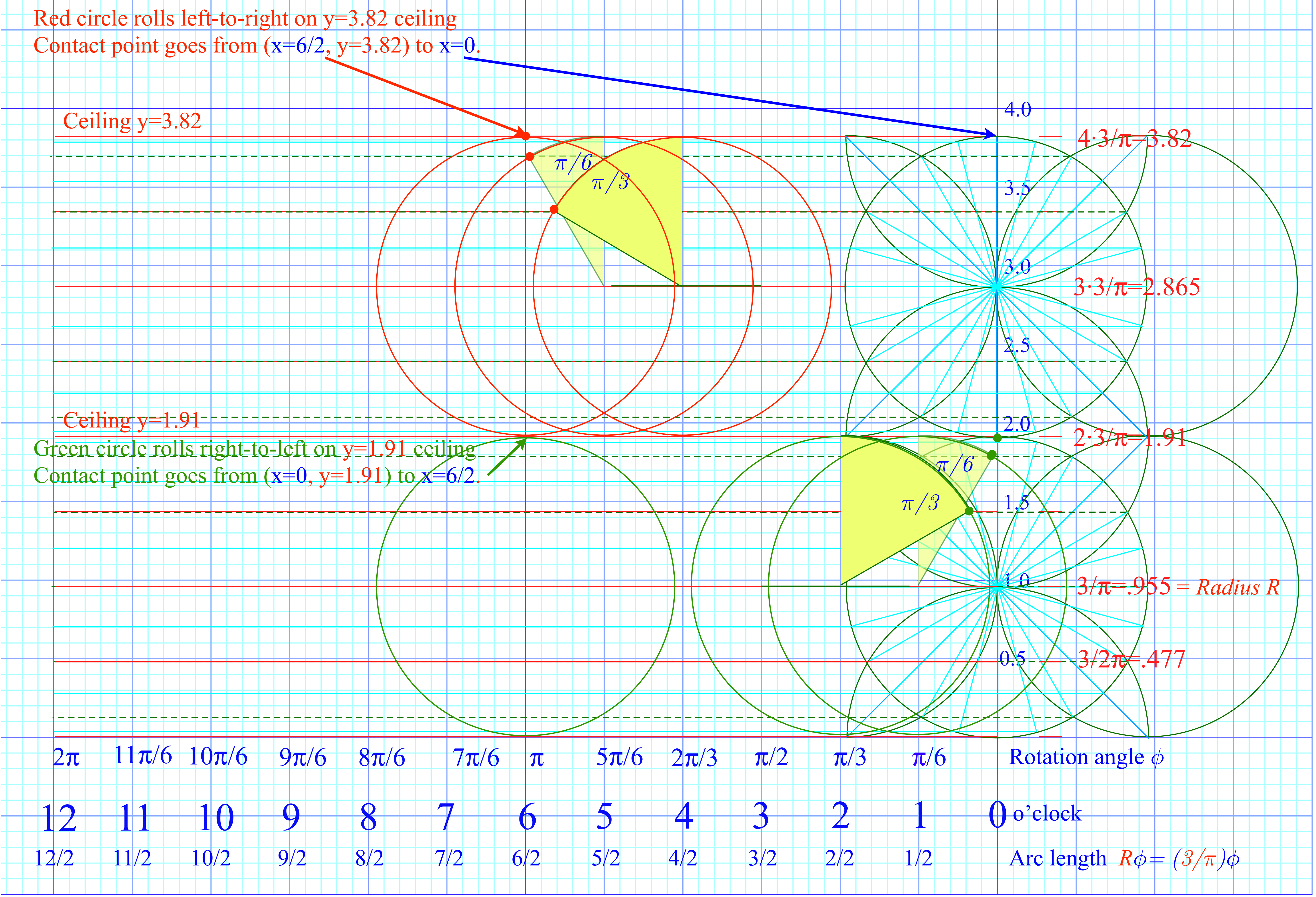

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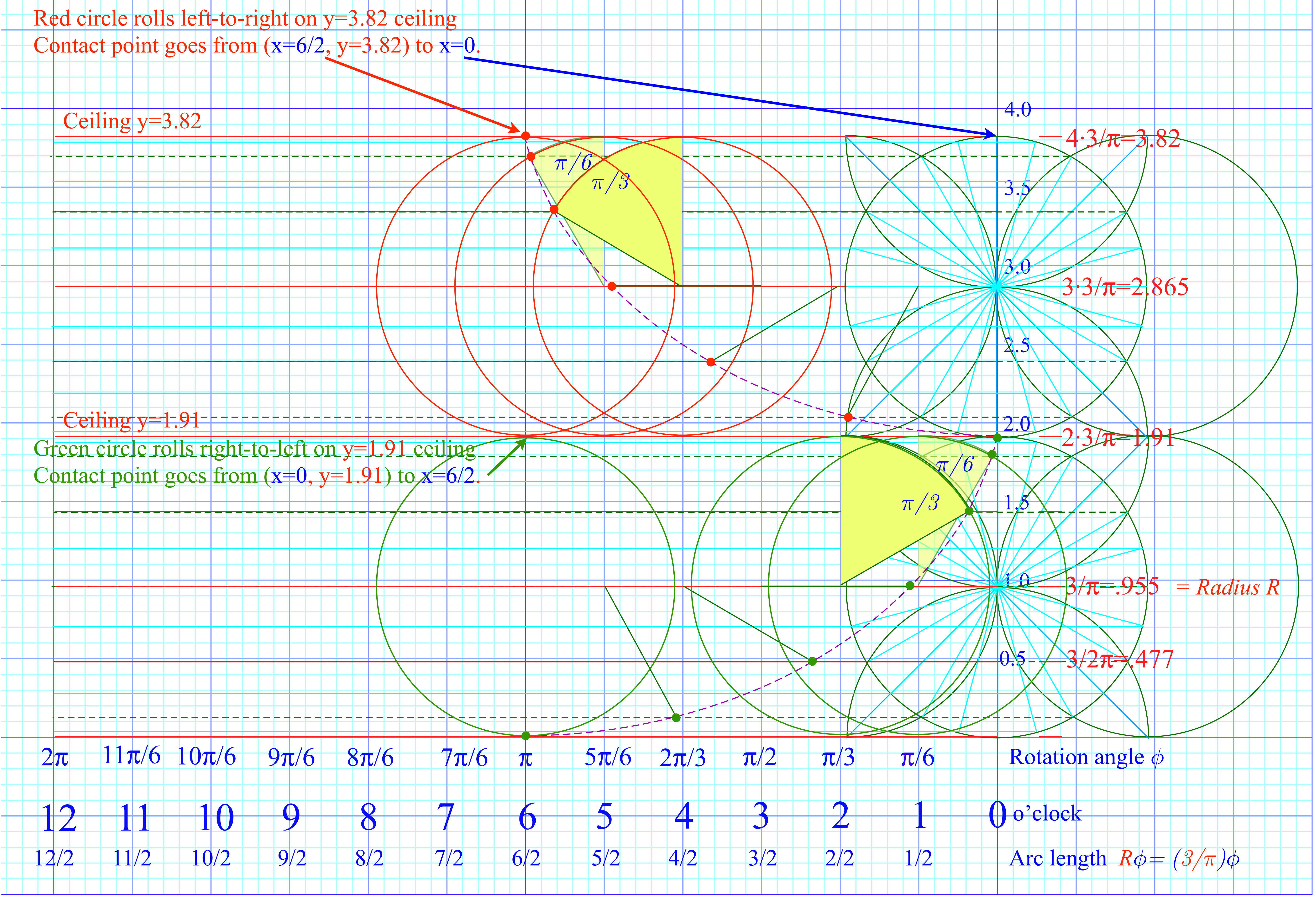
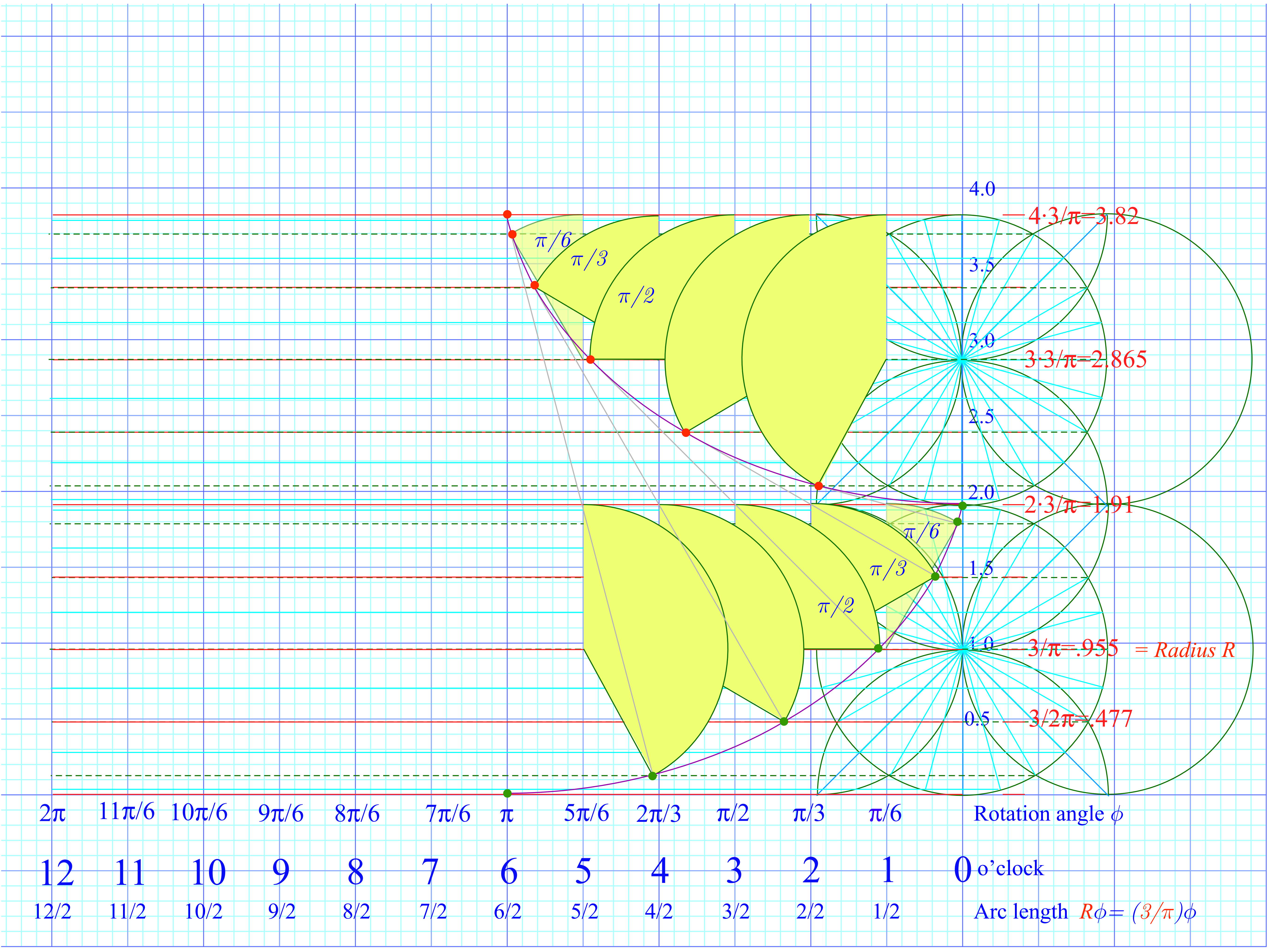
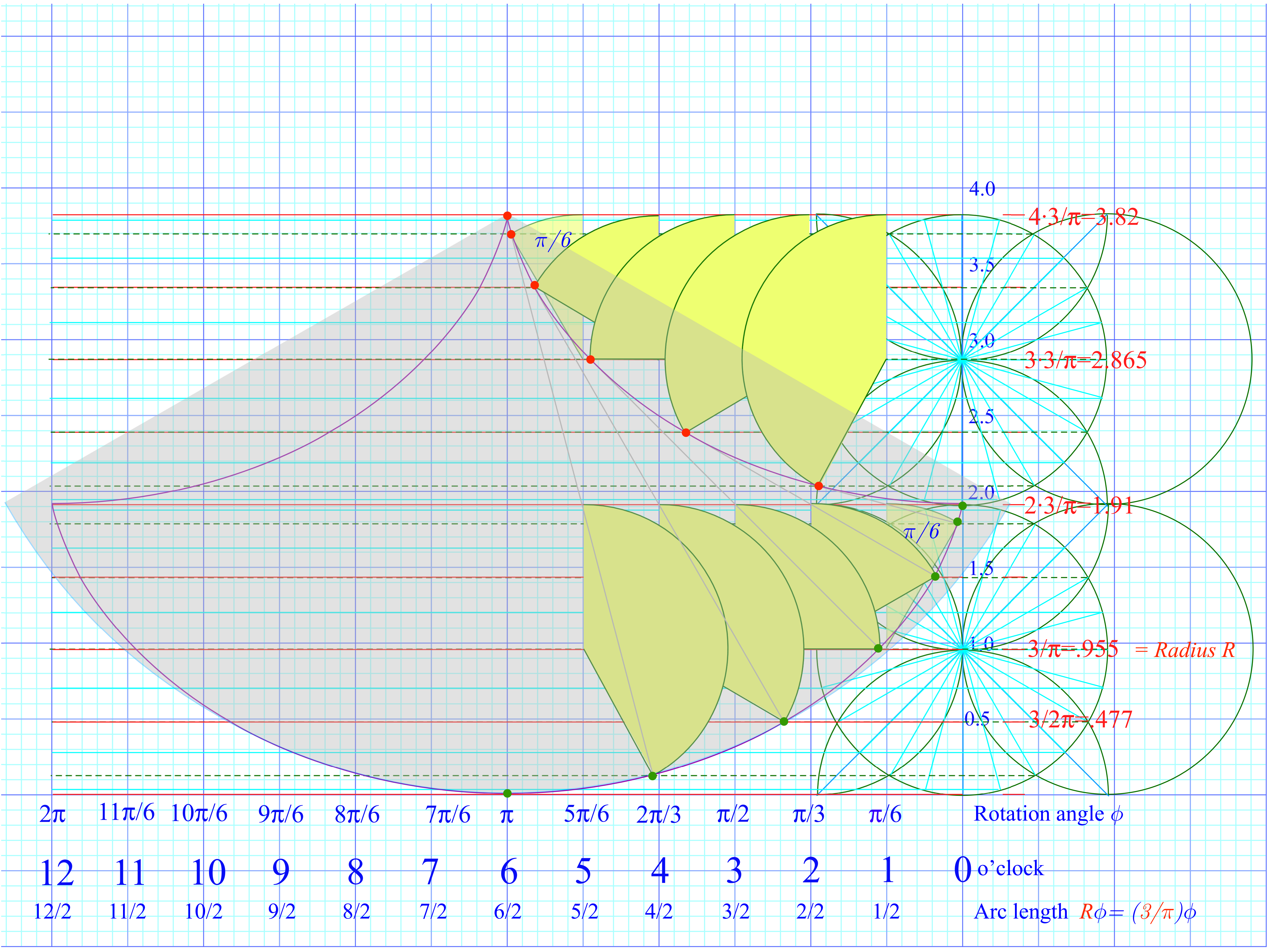



4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/2
π/6
π/3
π/2
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6 Rotation angle φ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rφ= (3/π)φ



4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/6
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle ϕ
12
11
10
9
8
7
6
5
4
3
2
1
0 o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rϕ= (3/π)ϕ

## *Examples of Hamiltonian mechanics in phase plots*

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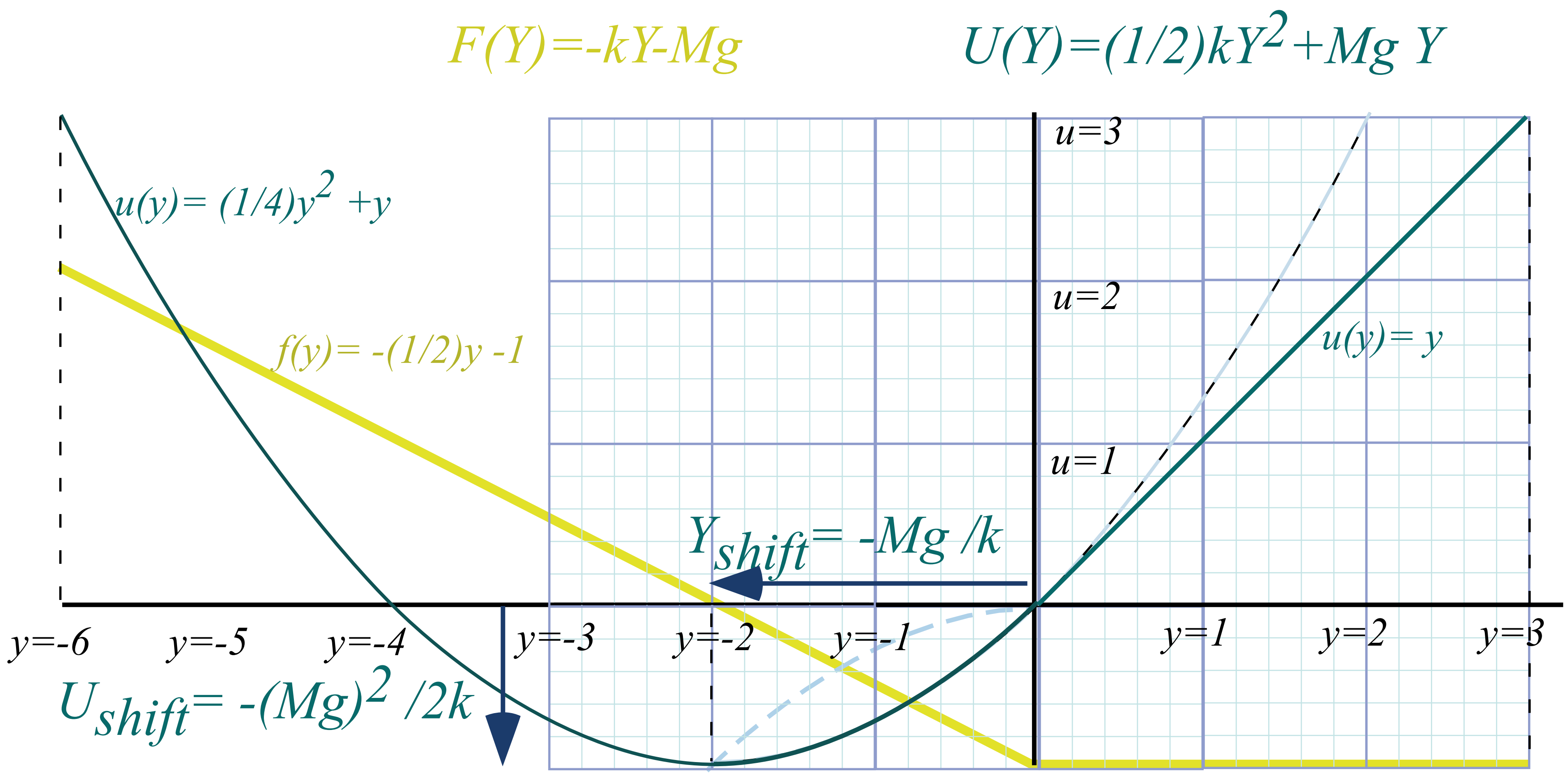
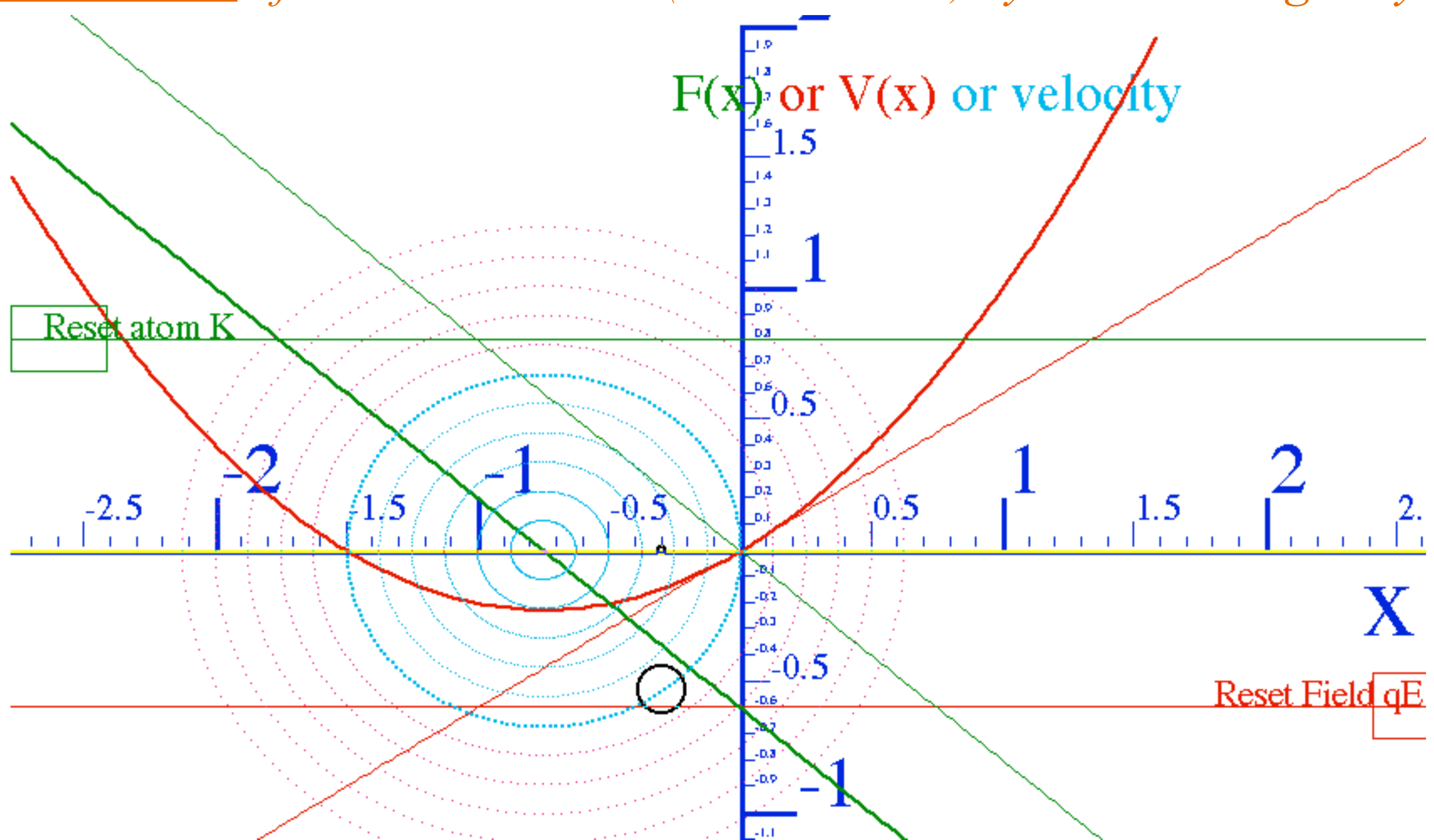



Unit 1
Fig. 7.4

*Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control*



## *Exploring phase space and Lagrangian mechanics more deeply*

→ ***A weird "derivation" of Lagrange's equations***

*Poincare identity and Action,* Jacobi-Hamilton *equations*

*How Classicists might have "derived" quantum equations*

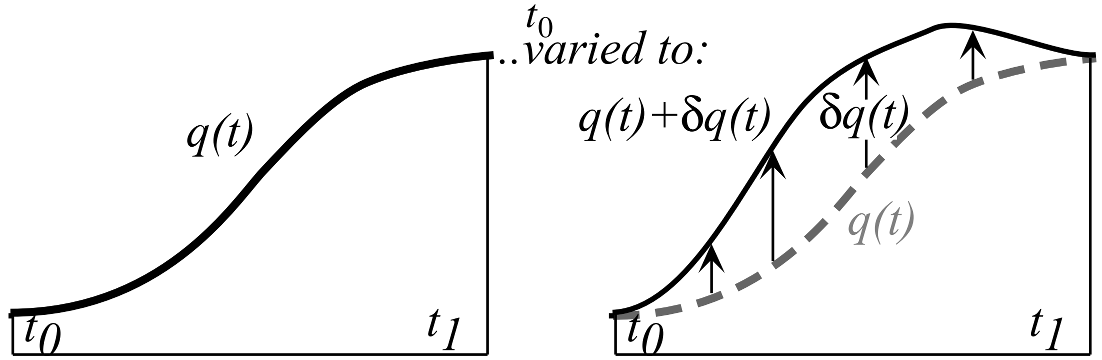
*Huygen's contact transformations enforce minimum action*

*How to do quantum mechanics if you only know classical mechanics*

*A strange "derivation" of Lagrange's equations by Calculus of Variation*

Variational calculus finds extreme (minimum or maximum) values to entire integrals

*Minimize (or maximize):* $S(q) = \int_{t_0}^{t_1} dt\, L(q(t), \dot{q}(t), t).$




An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all.(1)

$$\delta q(t_0) = 0 = \delta q(t_1) \qquad (1)$$

*1st order L(q+δq) approximate:*

$$S(q+\delta q) = \int_{t_0}^{t_1} dt \left[ L(q,\dot{q},t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt}\delta q$$

*A weird "derivation" of Lagrange's equations*

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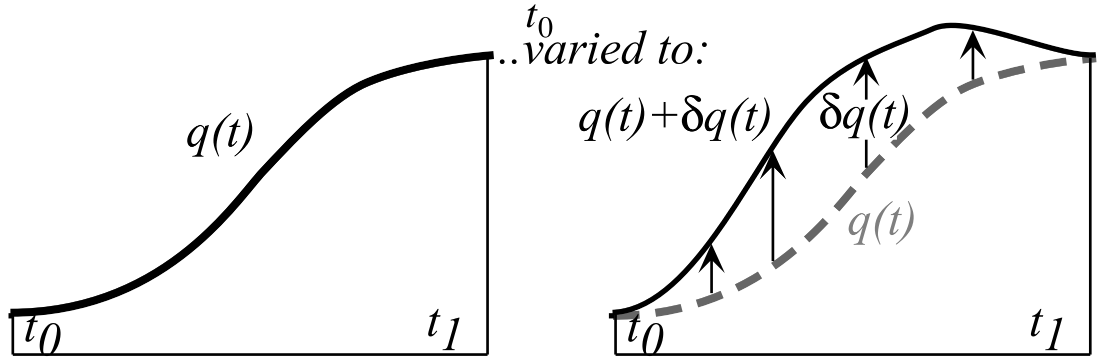


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Replace $\frac{\partial L}{\partial \dot{q}}\delta\dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

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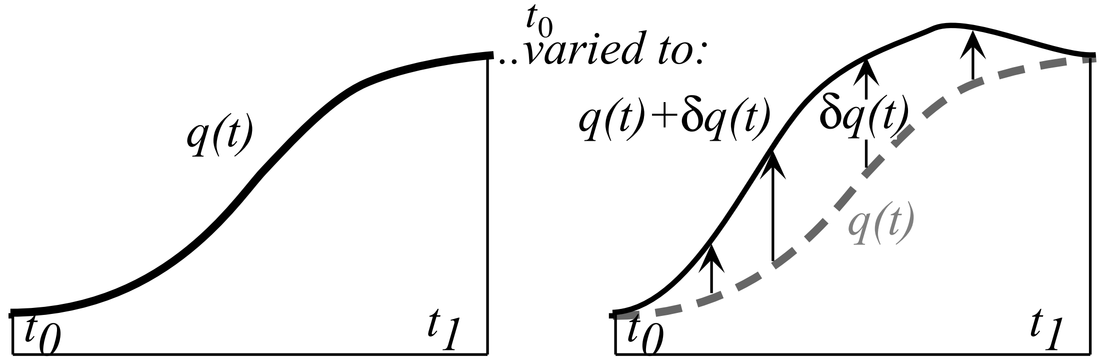


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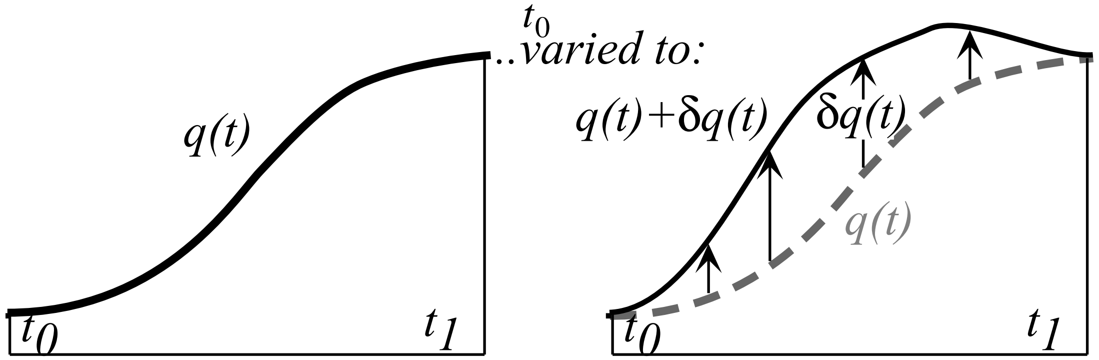


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$$=\int_{t_0}^{t_1} dt\ L(q,\dot{q},t)+\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q+\left.\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\right|_{t_0}^{t_1}$$

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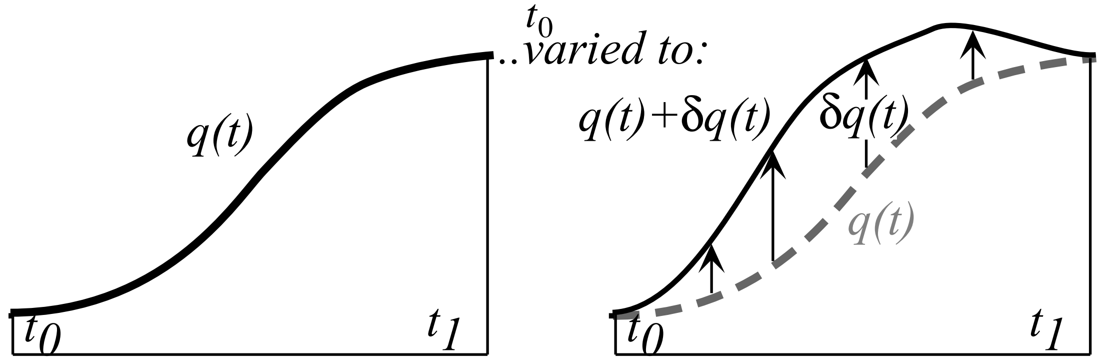


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due to requiring (1)

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q+\delta q) - S(q) = \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q$

Extreme value (actually *minimum* value) of *S(q)* occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

*Euler-Lagrange equation(s)*

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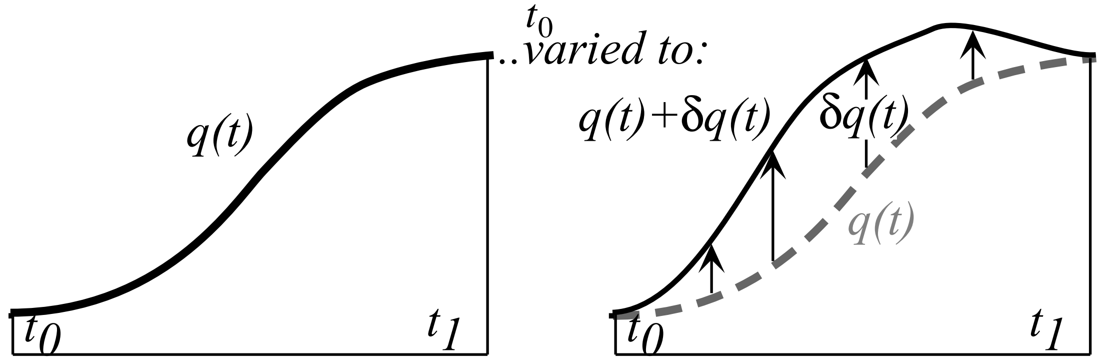


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*Euler-Lagrange equation(s)*

*But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian L = T - U???*

# *Exploring phase space and Lagrangian mechanics more deeply*

*A weird "derivation" of Lagrange's equations*

→ ***Poincare identity and Action,*** Jacobi-Hamilton *equations*

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# *Legendre-Poincare identity and Action*

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if $dt$ is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \left( \mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ implies: } \mathbf{v}\cdot dt = d\mathbf{r} \right)$$

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This is the time differential $dS$ of *action* $S = \int L \cdot dt$ whose time derivative is rate $L$ of *quantum phase*.

$$dS = L \cdot dt = \mathbf{p} \bullet d\mathbf{r} - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

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Unit 8 shows *DeBroglie law* $\mathbf{p}=\hbar\mathbf{k}$ and *Planck law* $H=\hbar\omega$ make *quantum plane wave phase* $\Phi$*:*

$$\Phi = S/\hbar = \int L\cdot dt/\hbar$$

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Q:When is the *Action*-differential $dS$ integrable?

A: A differential $dW=f_x(x,y)dx+f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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*Similar to conditions for integrating work differential* $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$. *That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* ∂-symmetry *of W:*

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y\partial x} = \frac{\partial^2 W}{\partial x\partial y} = \frac{\partial f_y}{\partial x}$$

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*These conditions are known as* Jacobi-Hamilton *equations*

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# *How* Jacobi-Hamilton *could have "derived" Schrodinger equations*

*(Given "quantum wave")*

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

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*Try 1st* **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}} e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial \mathbf{r}} e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = (i/\hbar)\mathbf{p}\psi(\mathbf{r},t) \quad \text{or:} \quad \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\psi(\mathbf{r},t)$$

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*Momentum Operator*
*or* **p**-*op in* **r**-*basis*

$$\mathbf{p} \Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}$$

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*Momentum Operator or* **p***-op in* **r***-basis*

$$\mathbf{p} \Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}$$

*Try 1st t-derivative of wave* ψ

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{\partial}{\partial t} e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t} e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial t}\psi(\mathbf{r},t)$$

$$= (i/\hbar)(-H)\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)}$$

# How Jacobi-Hamilton could have "derived" Schrodinger equations

(Given "quantum wave")

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

$dS$ is integrable if: $\dfrac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\dfrac{\partial S}{\partial t} = -H$

*These conditions are known as Jacobi-Hamilton equations*

*Try 1st* **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}} e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial \mathbf{r}} e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = (i/\hbar)\mathbf{p}\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\psi(\mathbf{r},t)}$$

*Momentum Operator or* **p**-*op in* **r**-*basis*

$$\mathbf{p} \Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}$$

*Try 1st* $t$-*derivative of wave* ψ

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{\partial}{\partial t} e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t} e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial t}\psi(\mathbf{r},t)$$

$$= (i/\hbar)(-H)\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)}$$

*Schrodinger time equation*

$$\mathrm{i}\hbar\dot{\psi}(\mathbf{r},t) = H\psi(\mathbf{r},t)$$

*Exploring phase space and Lagrangian mechanics more deeply*

*A weird "derivation" of Lagrange's equations*

*Poincare identity and Action,* Jacobi-Hamilton *equations*

*How Classicists might have "derived" quantum equations*

***Huygen's contact transformations enforce minimum action***

*How to do quantum mechanics if you only know classical mechanics*

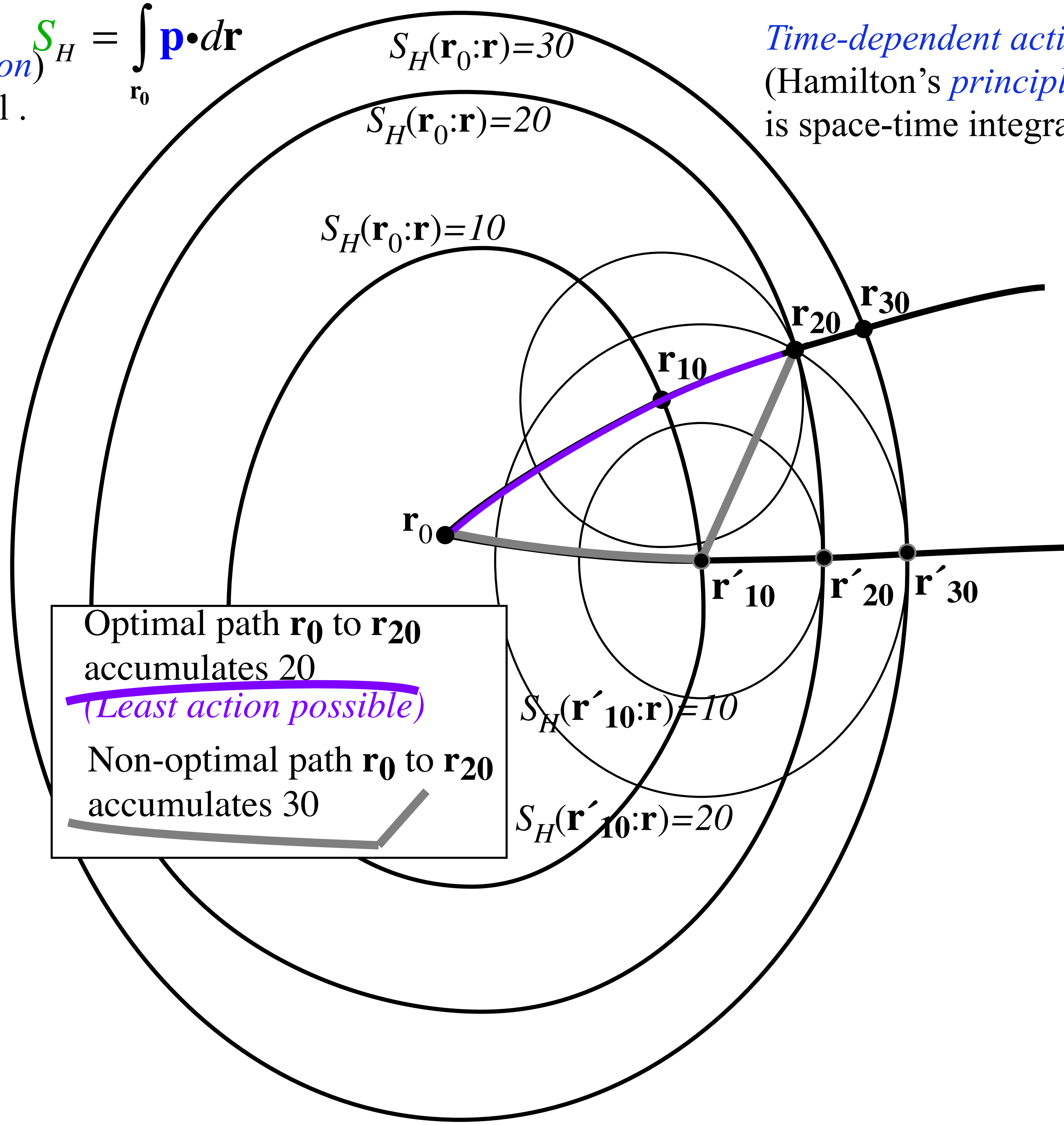
Christaan Huygens
(1629-1695)

# *Huygen's contact transformations enforce minimum action*

*Each point* $\mathbf{r}_k$ *on a wavefront "broadcasts" in all directions.*
*Only* ***minimum action*** *path interferes constructively*

*Time-independent action* $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \bullet d\mathbf{r}$
(Hamilton's *reduced action*)
is a purely spatial integral .

*Time-dependent action* $S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p} \bullet d\mathbf{r} - H \cdot dt)$
(Hamilton's *principle action*)
is space-time integral .




Unit 1
Fig. 12.12

# *Huygen's contact transformations enforce minimum action*

*Each point* $\mathbf{r}_k$ *on a wavefront "broadcasts" in all directions.*
*Only* ***minimum action*** *path interferes constructively*

*Time-independent action* (Hamilton's *reduced action*) is a purely spatial integral .

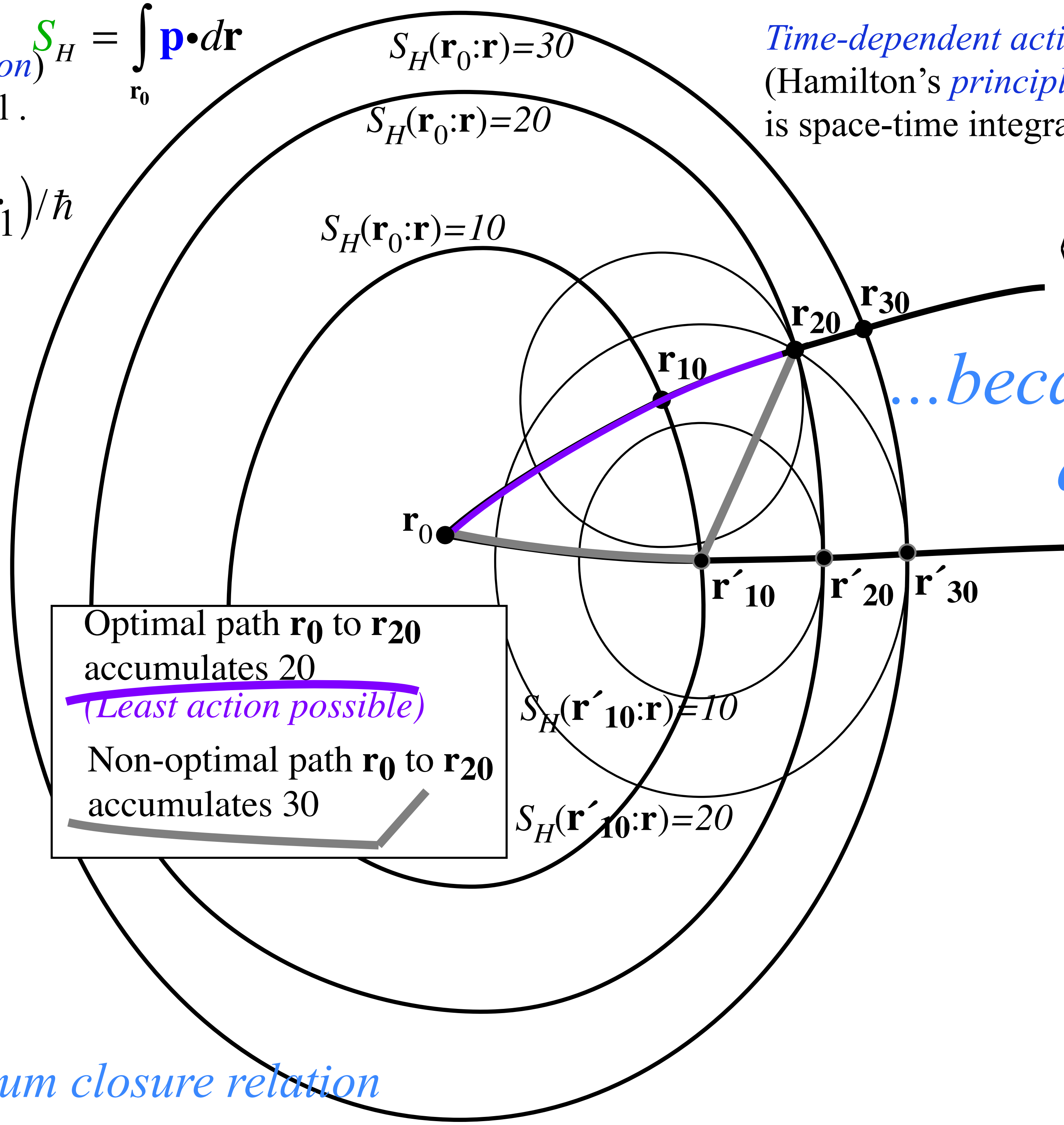
$$S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p}\bullet d\mathbf{r}$$

$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1)/\hbar}$$

*Time-dependent action* (Hamilton's *principle action*) is space-time integral .

$$S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p}\bullet d\mathbf{r} - H \cdot dt)$$

$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1)/\hbar}$$




*...because action is quantum wave phase*

Unit 1
Fig. 12.12

*Feynman's path-sum closure relation*

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i(S_H(\mathbf{r}_0:\mathbf{r}') + S_H(\mathbf{r}':\mathbf{r}_1))/\hbar} = e^{i S_H(\mathbf{r}_0:\mathbf{r}_1)/\hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

## *Exploring phase space and Lagrangian mechanics more deeply*

*A weird "derivation" of Lagrange's equations*

*Poincare identity and Action,* Jacobi-Hamilton *equations*

*How Classicists might have "derived" quantum equations*

*Huygen's contact transformations enforce minimum action*

→ ***How to do quantum mechanics if you only know classical mechanics***

Davis-Heller "Color-Quantization" or "Classical Chromodynamics"

# How to do quantum mechanics if you only know classical mechanics

*Bohr quantization* requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $S_H(\mathbf{r}_0 : \mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \bullet d\mathbf{r}$ . The integer $n$ ($n = 0, 1, 2, \ldots$) is a *quantum number*.

$$1 = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi\hbar n = hn$$

Numerically integrate Hamilton's equations and Lagrangian $L$. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L\, dt + Ht .$$

# *How to do quantum mechanics if you only know classical mechanics*

*Bohr quantization* requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple *n* of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0)=\int_{r_0}^{r_0}\mathbf{p}\bullet d\mathbf{r}$ . The integer *n* (*n = 0, 1, 2,...*) is a *quantum number*.

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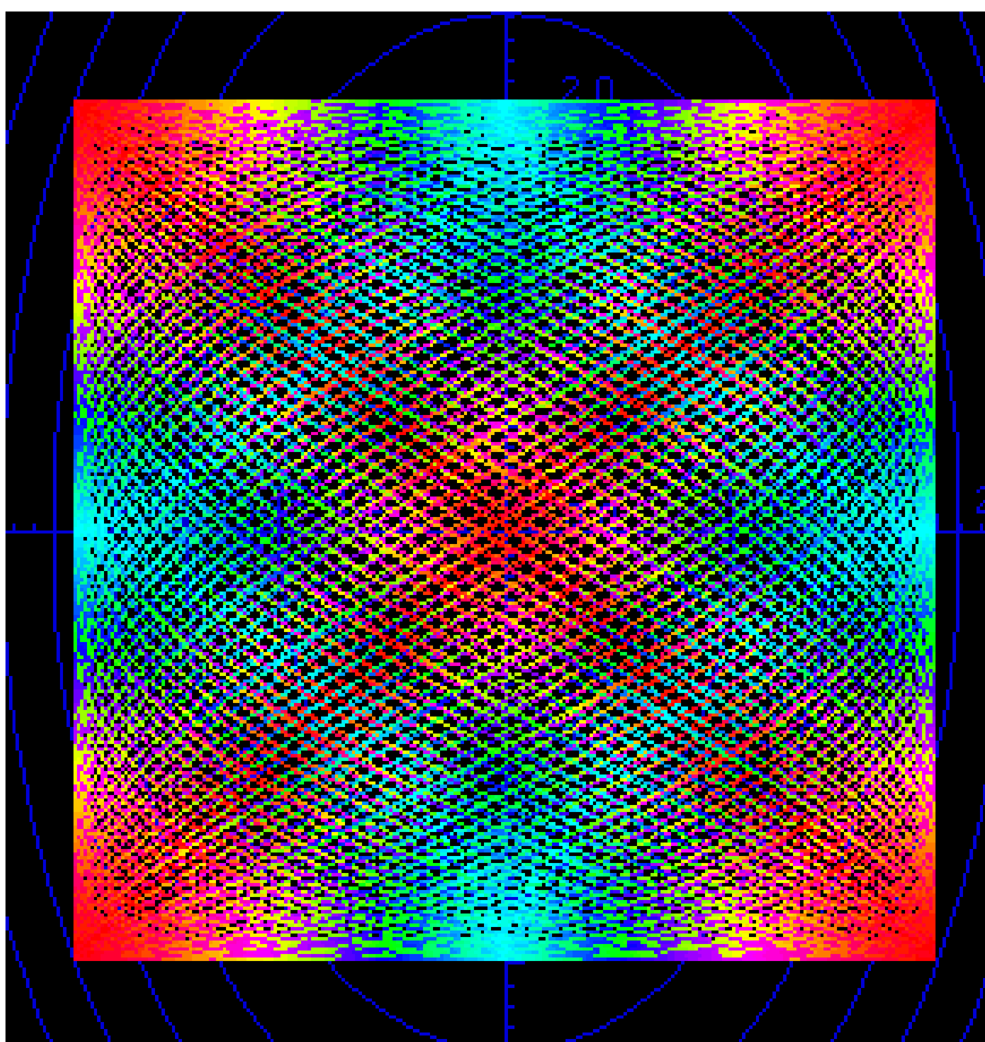
Numerically integrate Hamilton's equations and Lagrangian *L*. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0}:\mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0}:\mathbf{r}) = S_p(\mathbf{0},0:\mathbf{r},t) + Ht = \int_0^t L\,dt + Ht .$$

The hue should represent the phase angle $S_H(\mathbf{0}:\mathbf{r})/\hbar$ *modulo* $2\pi$ as, for example,
*0=red, π/4=orange, π/2=yellow, 3π/4=green, π=cyan* (opposite of red), *5π/4=indigo, 3π/2=blue, 7π/4=purple, and 2π=red* (full color circle).
Interpolating action on a palette of *32* colors is enough precision for low quanta.



*simulation by "Color U(2)"*

Unit 1 Fig. 12.13

*closed system has quantized E. Standing wave has only two phases(±) *cyan* and *red*

Quantum dynamical tunneling in bound states - Wavepacket and Color-quantization - M. J. Davis and E. J. Heller, J. Chem. Phys. 75, 246 (1981)

The Semiclassical Way to Molecular Spectroscopy: Eric J. Heller, Acc. Chem. Res. 1981, 14, 368-375

# How to do quantum mechanics if you only know classical mechanics

*Bohr quantization* requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0)=\int_{r_0}^{r_0}\mathbf{p}\bullet d\mathbf{r}$ . The integer $n$ ($n = 0, 1, 2,...$) is a *quantum number*.

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Numerically integrate Hamilton's equations and Lagrangian $L$. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0}:\mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0}:\mathbf{r})=S_p(\mathbf{0},0:\mathbf{r},t)+Ht=\int_0^t L\,dt+Ht\,.$$

The hue should represent the phase angle $S_H(\mathbf{0}:\mathbf{r})/\hbar$ *modulo* $2\pi$ as, for example,
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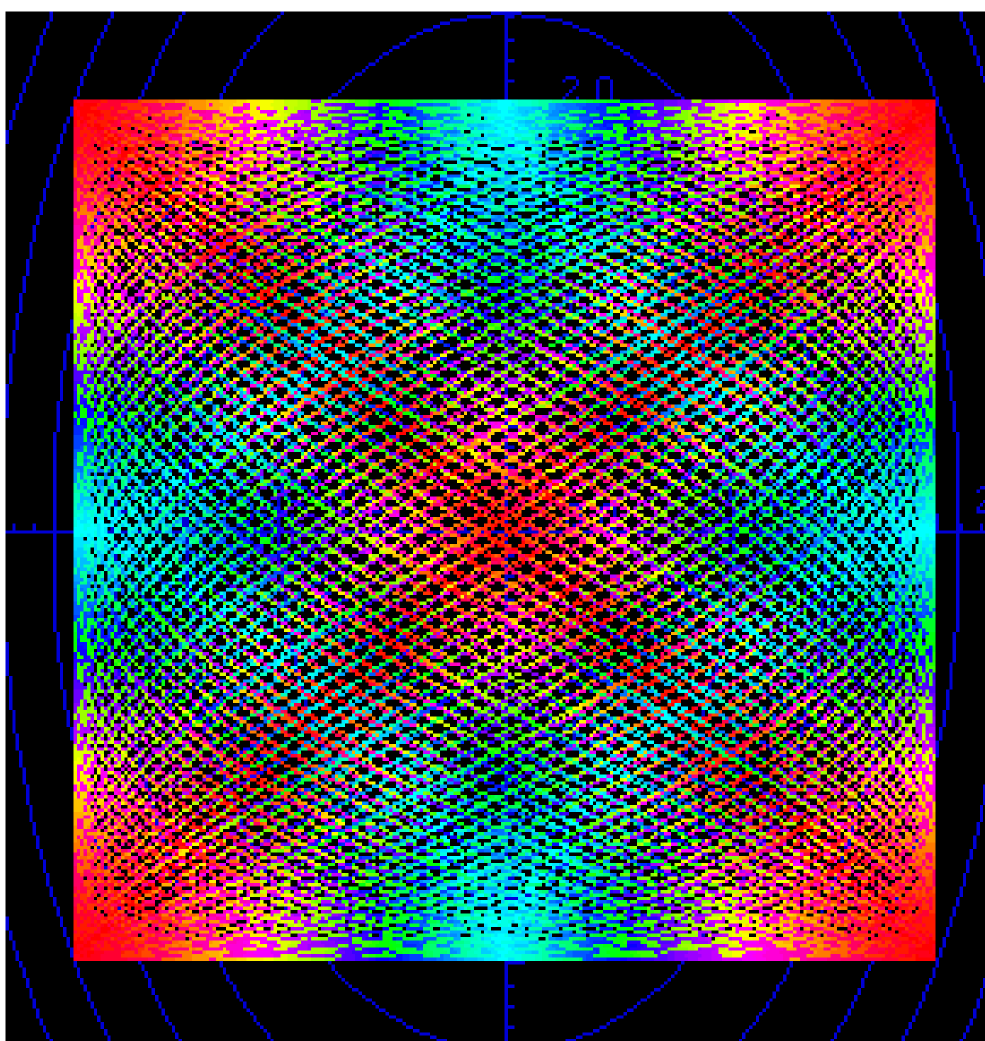
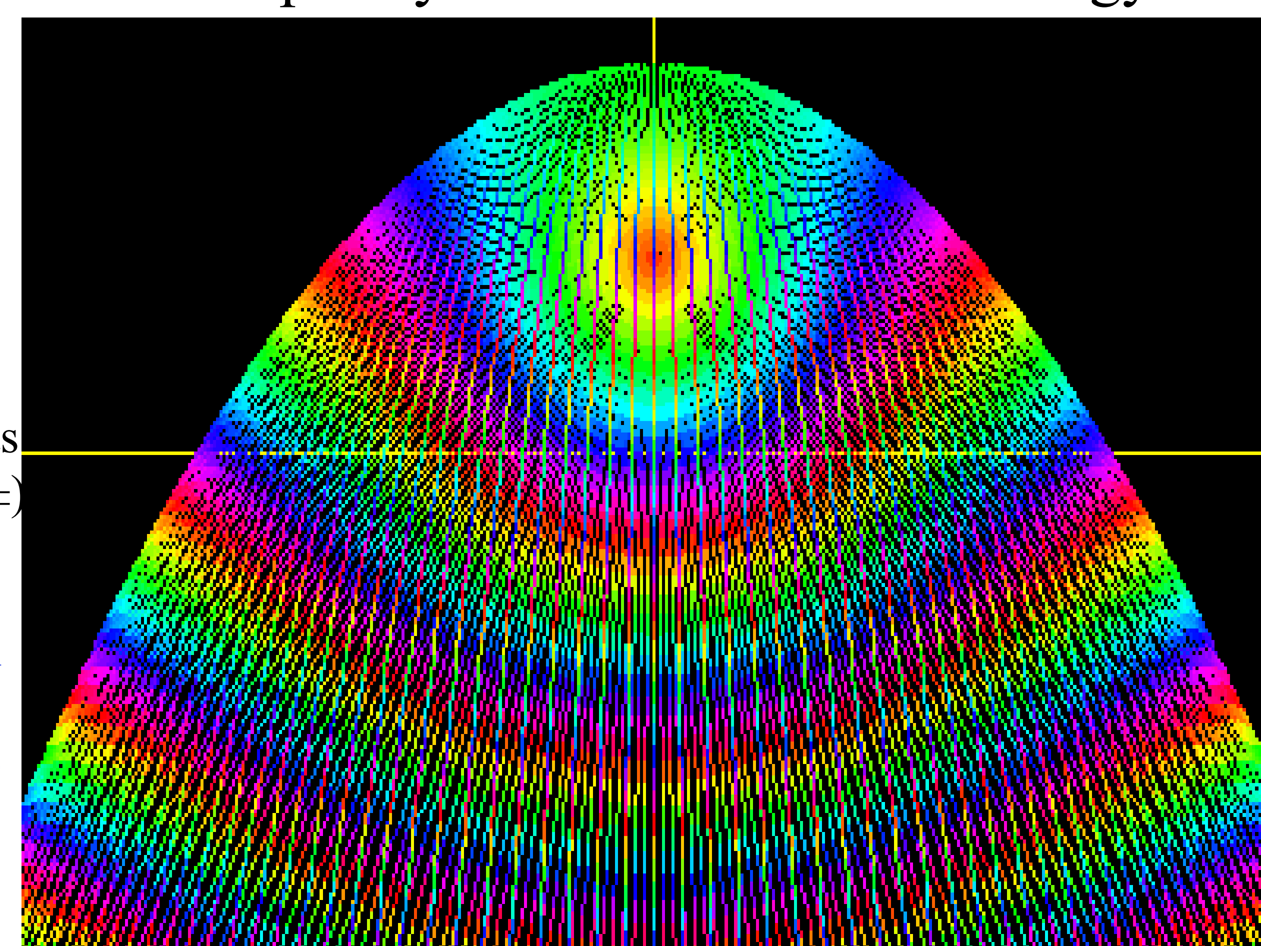


*Simulation by "Color U(2)"*

Unit 1 Fig. 12.13

*closed system has quantized E
Standing wave has only two phases(±) *cyan* and *red*

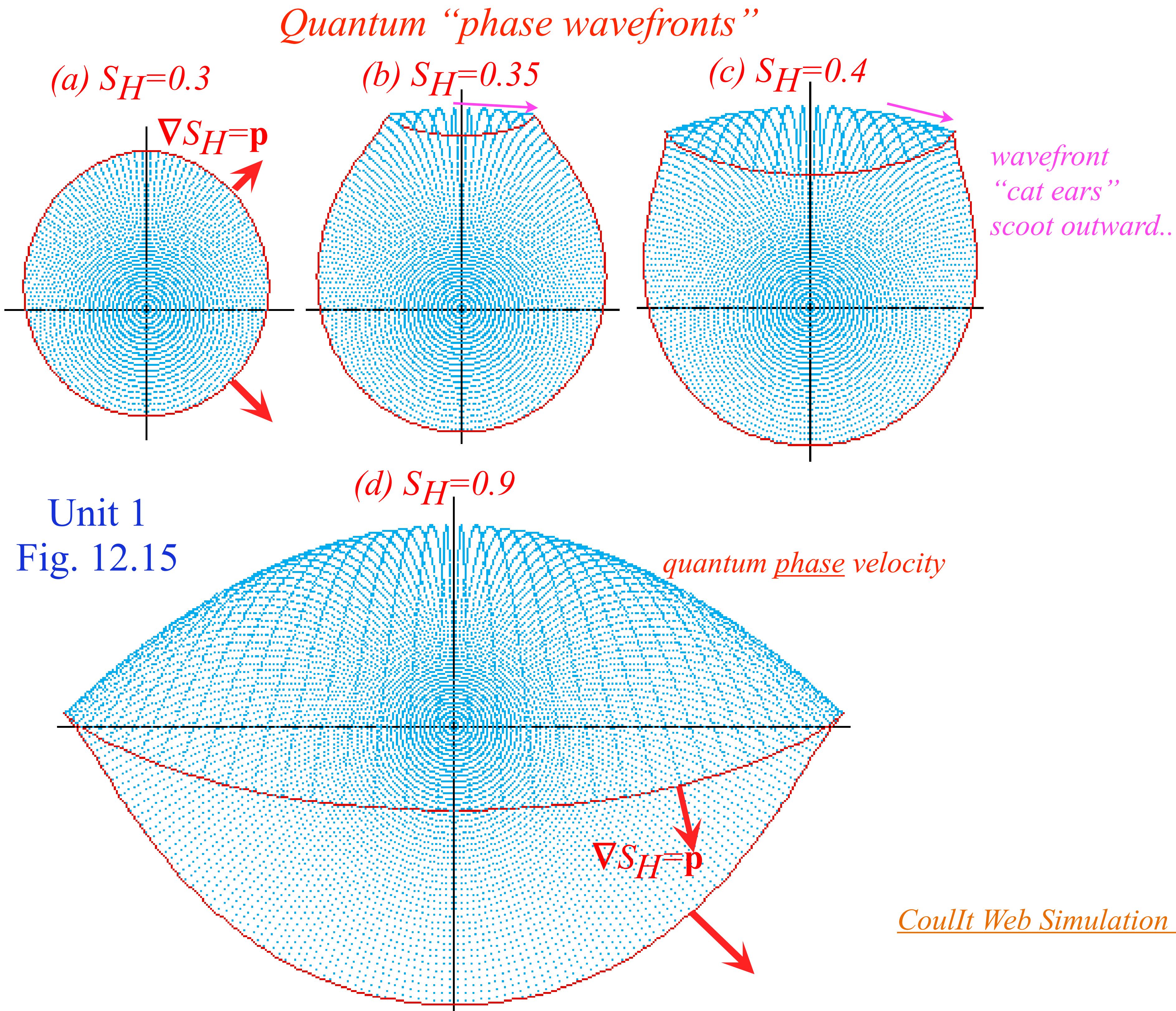
*open system has continuous energy



Unit 1 Fig. 12.14

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$ .

$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$




Unit 1
Fig. 12.15

CoulIt Web Simulation with "Quantum phase front"

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$ .

$$V_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

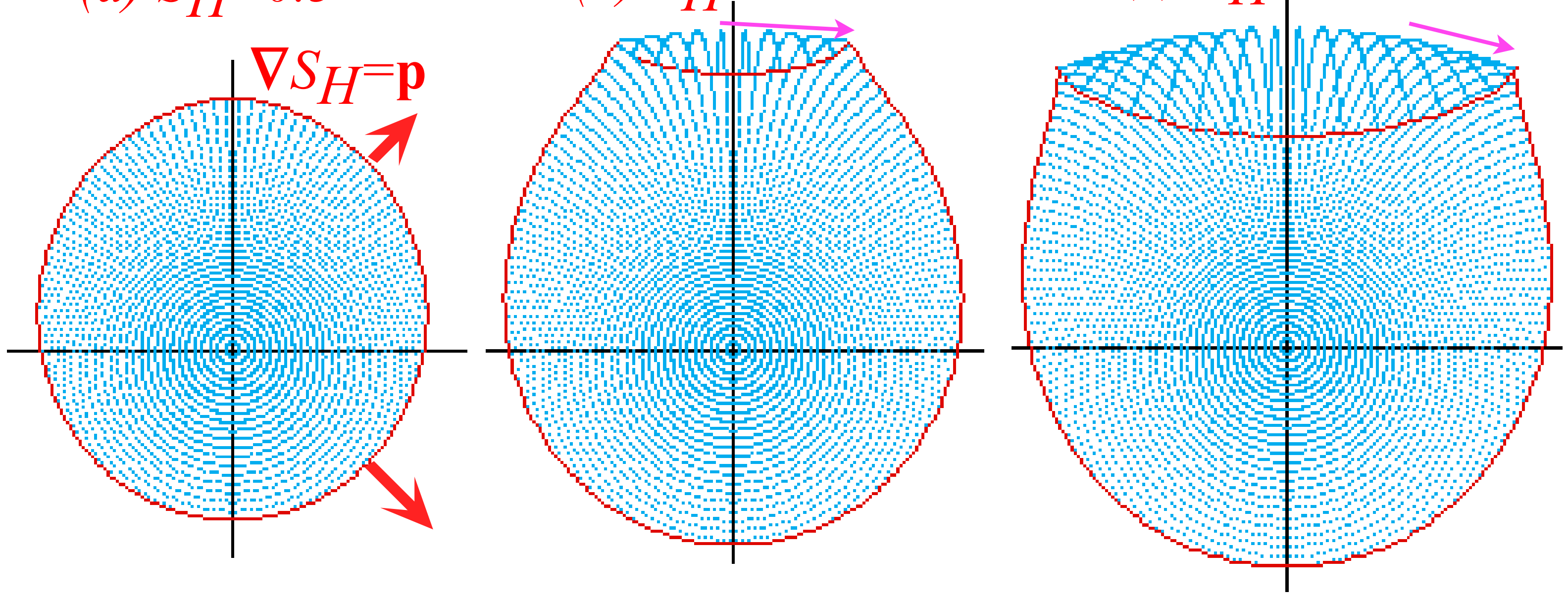
This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$V_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

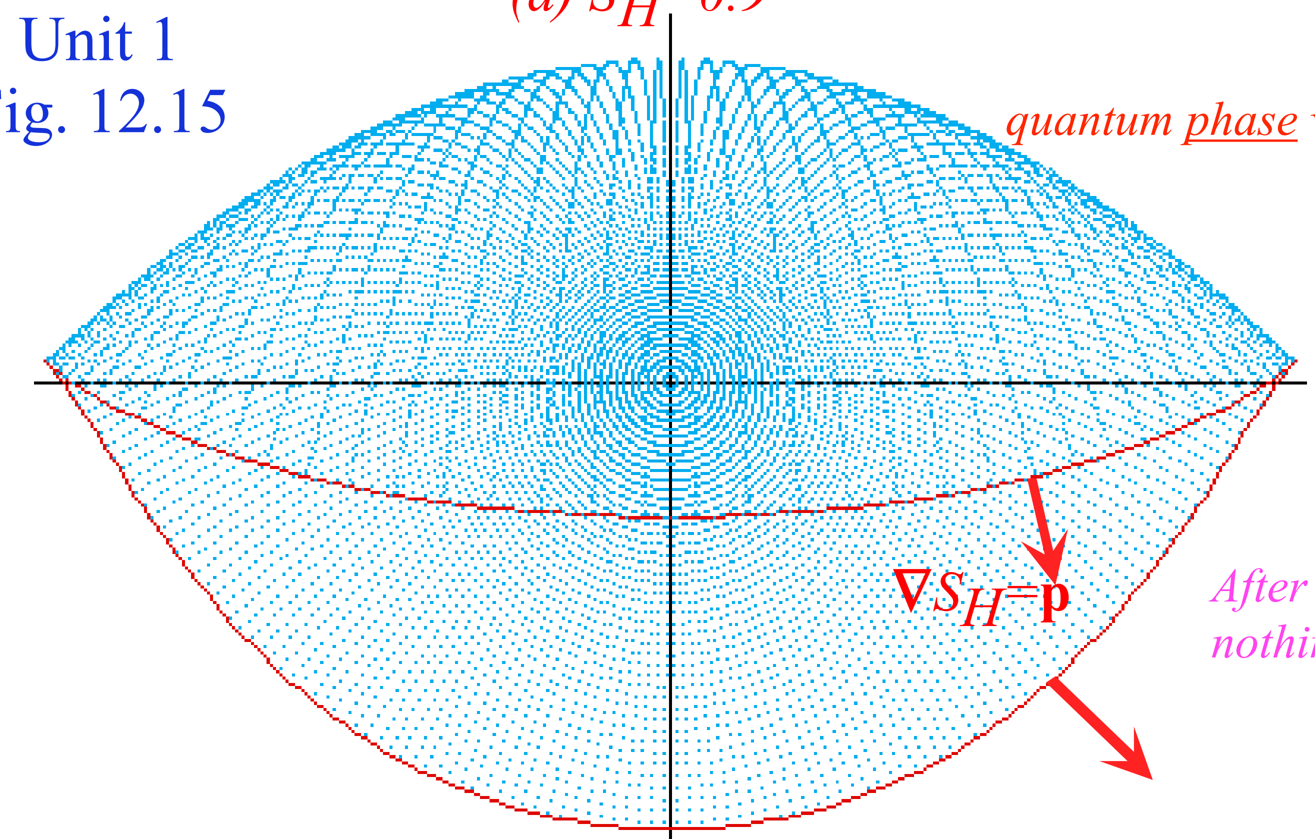
*Quantum "phase wavefronts"*

(a) $S_H=0.3$ (b) $S_H=0.35$ (c) $S_H=0.4$




Unit 1
Fig. 12.15

(d) $S_H=0.9$




CoulIt Web Simulation with "Quantum phase front"

A moving wave has a *quantum phase velocity* found by setting $S$=*const.* or $dS(0,0{:}r,t)=0=\mathbf{p}\bullet d\mathbf{r}-Hdt$ .

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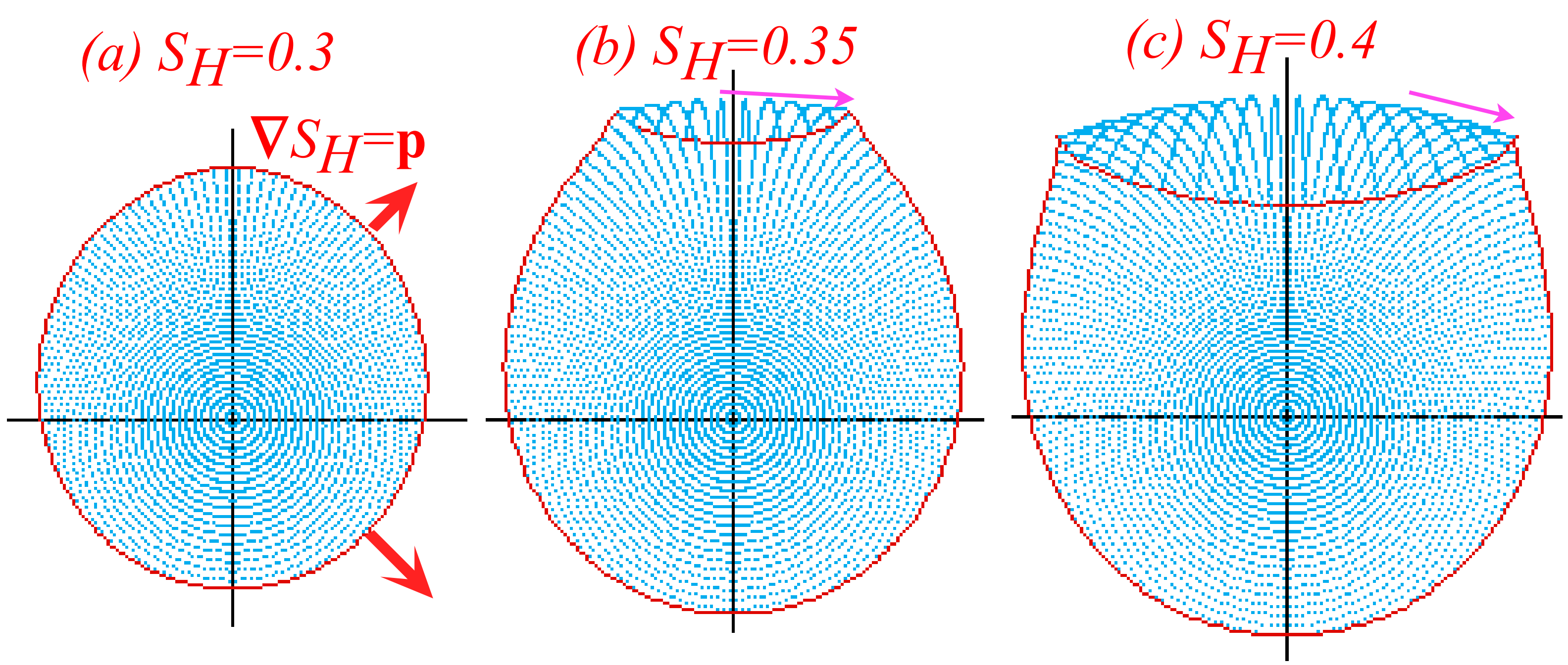
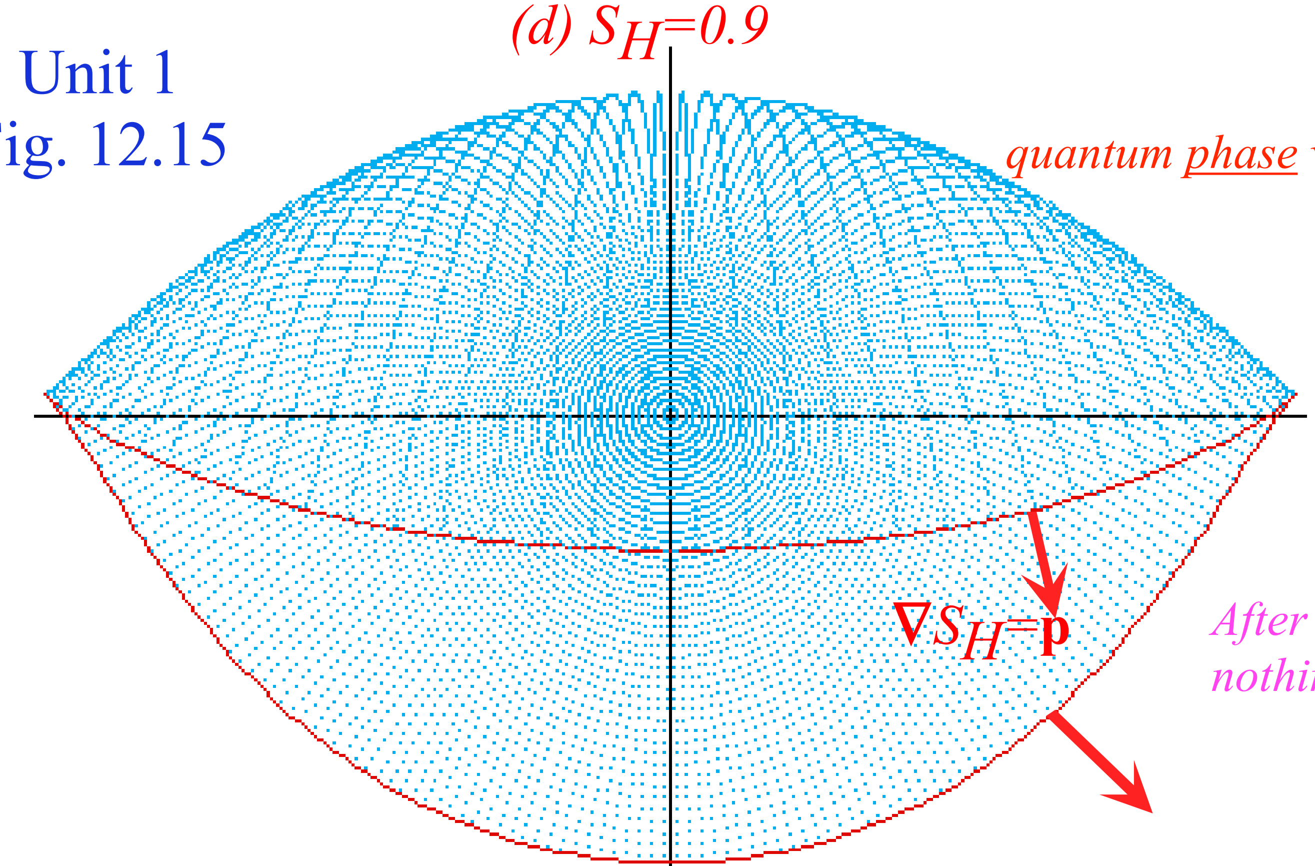
*Quantum "phase wavefronts"*




*wavefront "cat ears" scoot outward..*

Unit 1
Fig. 12.15




*After a while ...*
*nothing left but a smile!*

16th Century carving on St. Wifred's in Grappenhall

...on St. Nicolas

From *Alice's Adventures in Wonderland* by Lewis Carrol (1865)

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\bullet d\mathbf{r}-Hdt$ .

$$\mathbf{V}_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

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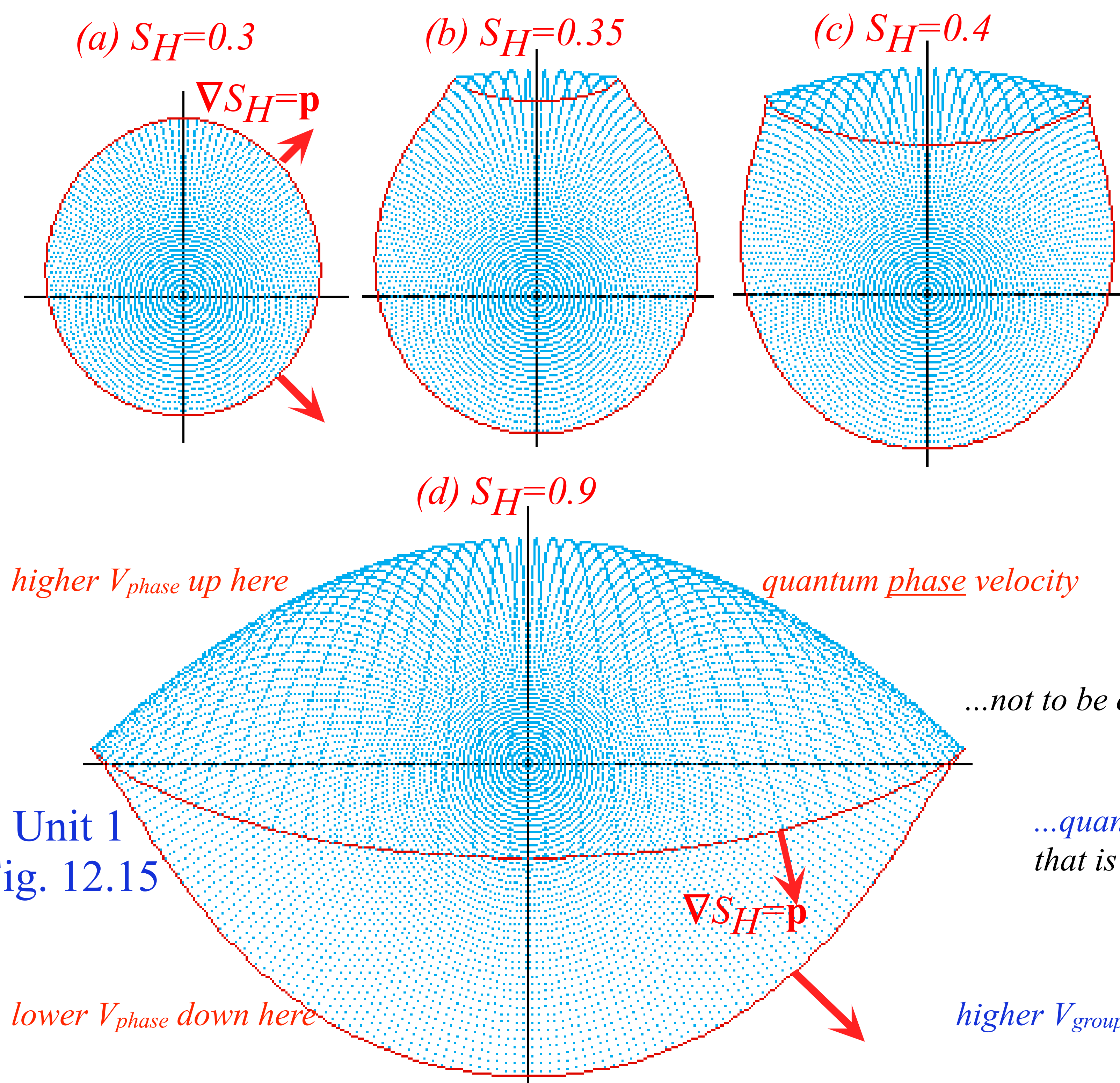
Note: This is Hamilton's 1st Equation

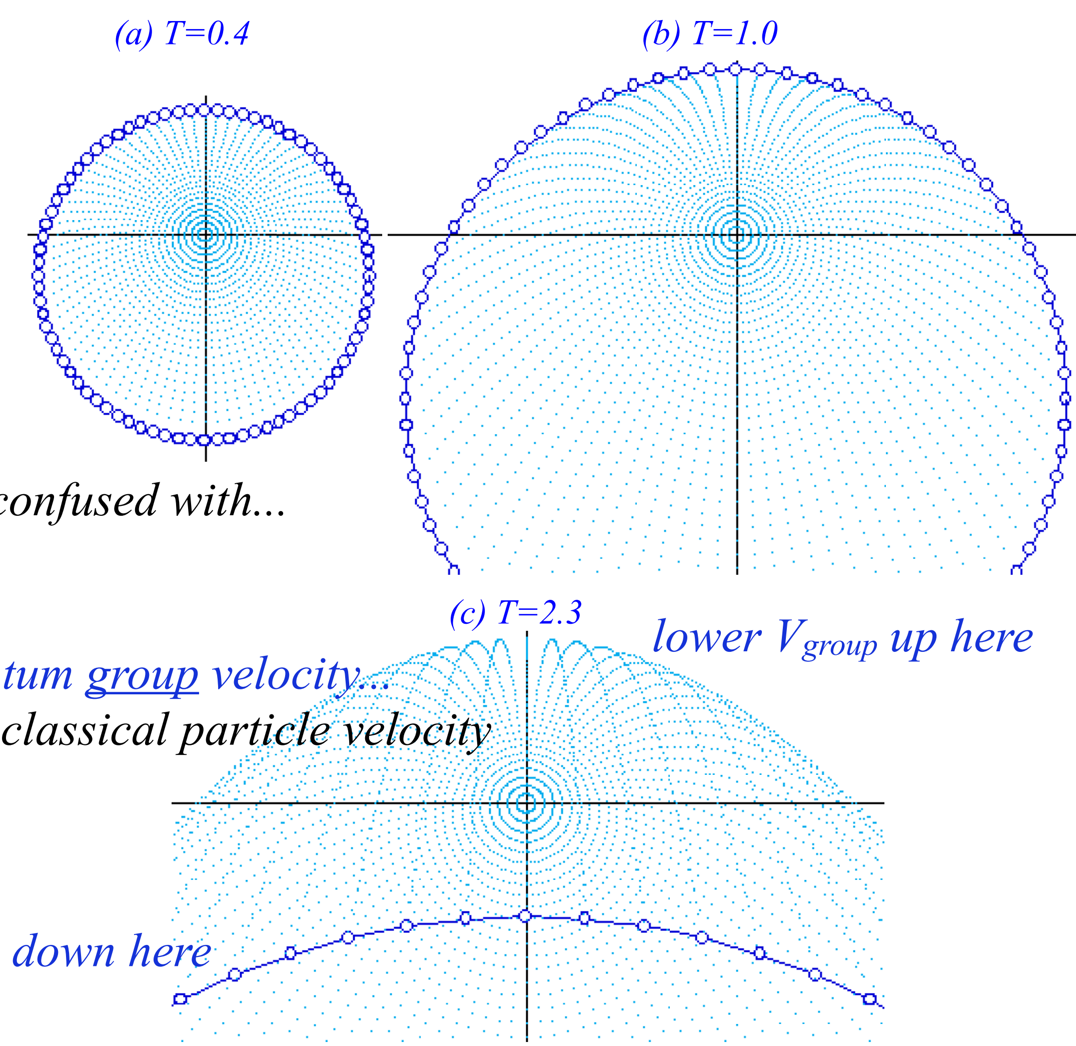
*Quantum "phase wavefronts"*




Unit 1
Fig. 12.15

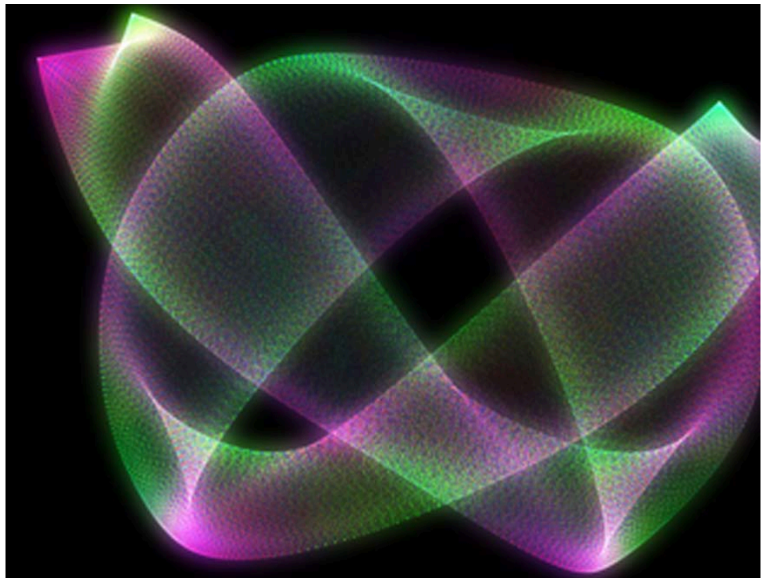
*Classical "blast wavefronts"*

*...not to be confused with...*




*...quantum group velocity...*
*that is classical particle velocity*

Check out the Heller Galleries http://jalbum.net/en/browse/user/album/1696720



# Resonance Fine Art

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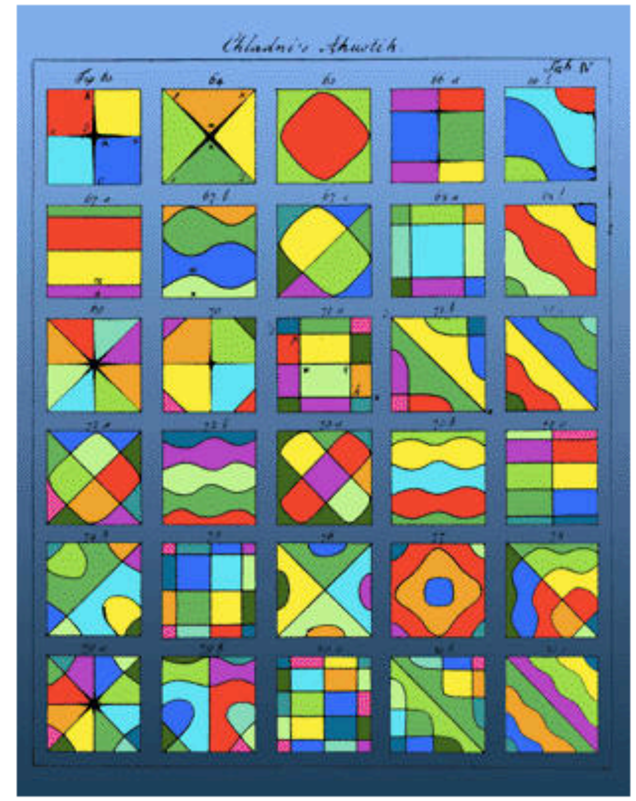
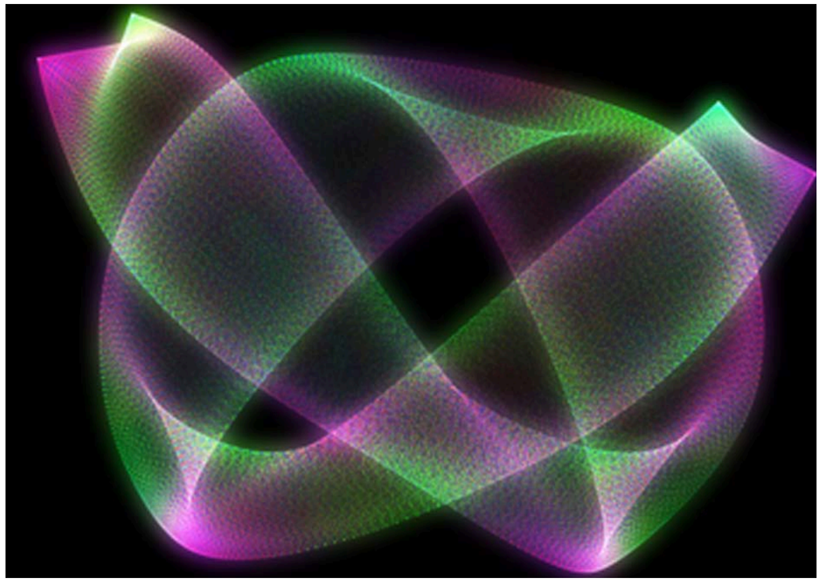
## Chladni



The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took he prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chaldni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.

Check out the Heller Galleries http://jalbum.net/en/browse/user/album/1696720



**National Science Foundation (NSF)**
**Arlington, VA**

September-November 2002

Selected images.

http://search.nsf.gov/search?ie=&site=nsf&output=xml_no_dtd&proxyreload=1&client=nsf&lr=&proxystylesheet=http%3A%2F%2Fwww.nsf.gov%2Fsearch%2Fnsf_new.xslt&oe=&btnG.x=0&btnG.y=0&q=eric+heller

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**University Museum, University of Arkansas, Fayetteville, AK**

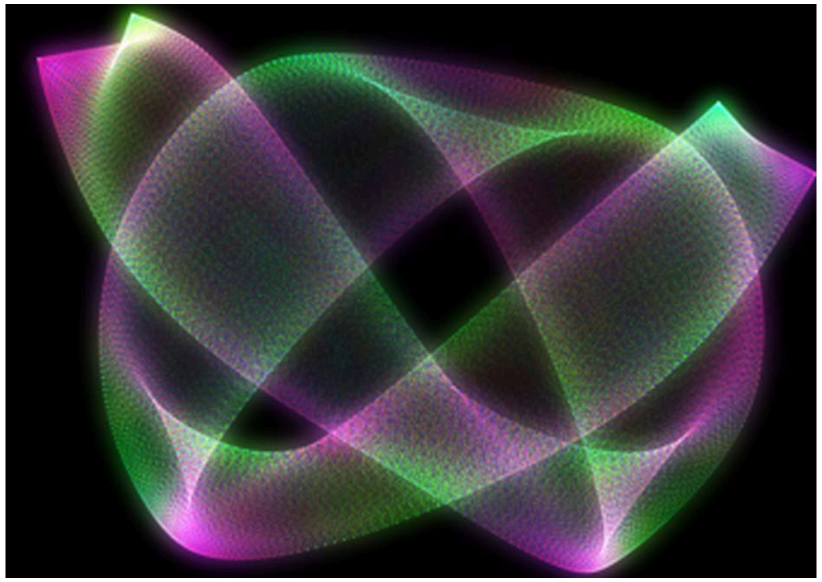
October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

Bessel 21

Check out the Heller Galleries http://jalbum.net/en/browse/user/album/1696720



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**University Museum,*** **University of Arkansas, Fayetteville, AK**

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Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

*UAF *Museum closed after this exhibit*

Bessel 21

Lecture 11 ends here
Wed. 10.02.2019