Lecture 10
Mon. 9.30.2019

Hamiltonian vs. Lagrangian mechanics in Generalized Curvilinear Coordinates (GCC)

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lectures 8-9 procedures:

Lagrange prefers Co*variant g_{mn} with* Contra*variant velocity* $\dot{q}^m$

Hamilton prefers Contra*variant g^{mn} with* Co*variant momentum p_m*

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

Hamilton's equations in Runga-Kutta (computer solution) form

Examples of Hamiltonian mechanics in effective potentials

*I*sotropic *H*armonic *O*scillator *in polar coordinates and effective potential* (Web Simulation: OscillatorPE - IHO)

Coulomb orbits in polar coordinates and effective potential (Web Simulation: OscillatorPE - Coulomb)

Examples of Hamiltonian mechanics in phase plots (Mostly for next Lecture 11)

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum,)

This Lecture's Reference Link Listing

Web Resources - front page | Quantum Theory for the Computer Age | 2017 Group Theory for QM
UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2018 Adv CM
Classical Mechanics with a Bang! | 2018 AMOP
Modern Physics and its Classical Foundations | **2019 Advanced Mechanics**

Lecture #10

Links to previous lecture: *Page=74*, *Page=75*, *Page=79*

Pendulum Web Sim
Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

Select, exciting, and related Research & Articles of Interest:

An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_ Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019

Rovibrational_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

From last lecture listing:

Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-Daily KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019

A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018

Running Reference Link Listing

Lectures #9 through #7

In reverse order

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,
Nuclear_Collisions - Mishustin-PhysRevC-2007, *APS Link & Abstract*
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, *(v_1, v_2)=(1, 0.1)*, *(v_1, v_2)=(1, 0)*
y_2 vs y_1 plots: *(v_1, v_2)=(1, 0.1)*, *(v_1, v_2)=(1, 0)*, *(v_1, v_2)=(1, -1)*
Estrangian plot *V_2 vs V_1:* *(v_1, v_2)=(0, 1)*, *(v_1, v_2)=(1, -1)*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *(v_1, v_2)=(1, 0):* *V2 vs V1 Estrangian plot*, *y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, *Fall-Thru*, *Bounce-Off*
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot *<<Under Construction>>*

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

Quick Review of Lagrange Relations in Lectures 8-9

0^{th} and 1^{st} equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian*** *have no explicit dependence on* ***momentum*** $\mathbf{p}$

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian*** *have no explicit dependence on* ***velocity*** $\mathbf{v}$

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian*** *have no explicit dependence on* ***speedinum*** $\mathbf{V}$

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2}$$
$$= \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1^{st} equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2}$$
$$= \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

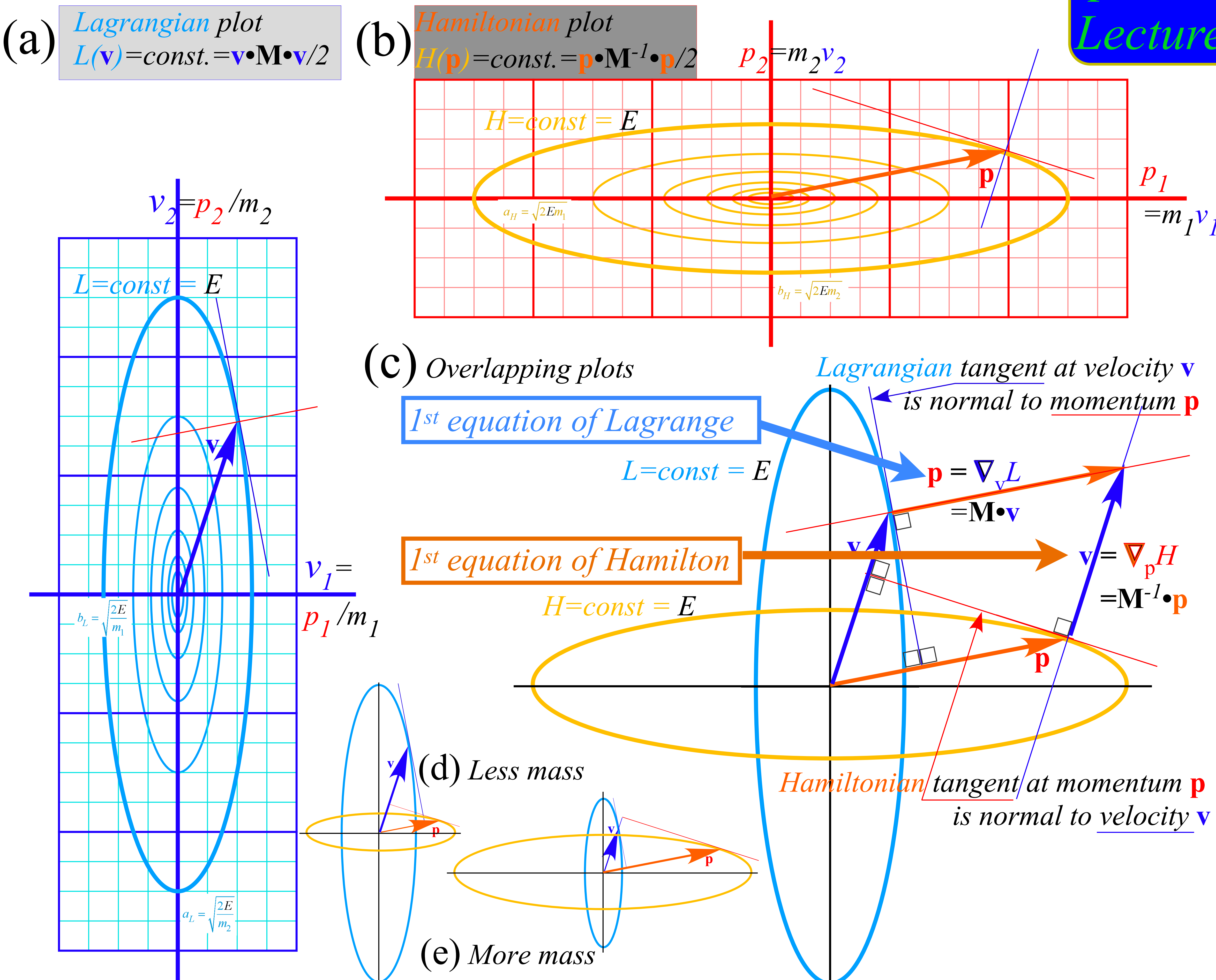
Hamilton's 1^{st} equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

Estrangian is neglected for now.
(It is related to dual ellipse geometry in Lecture 8 p. 71-79 and 99-101)

†non-dependency due to stationary-value effects as shown on p. 28-31

Unit 1
Fig. 12.2



Review of Lagrange Equations in Lecture 9

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

GCC Lagrangian definition

GCC "canonical" momentum p_m definition

→ *GCC "canonical" force F_m definition*

Coriolis "fictitious" forces (... and weather effects)

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 9)**

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\,g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising;
radial momentum p_r has the usual linear $M\cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi = Mr^2\omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$

$$= M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Rewriting GCC Lagrange equations :

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r}$$

$$= M\,r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

$$= 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Conventional forms

radial force: $M\ddot{r} = M\,r\dot{\phi}^2 - \frac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

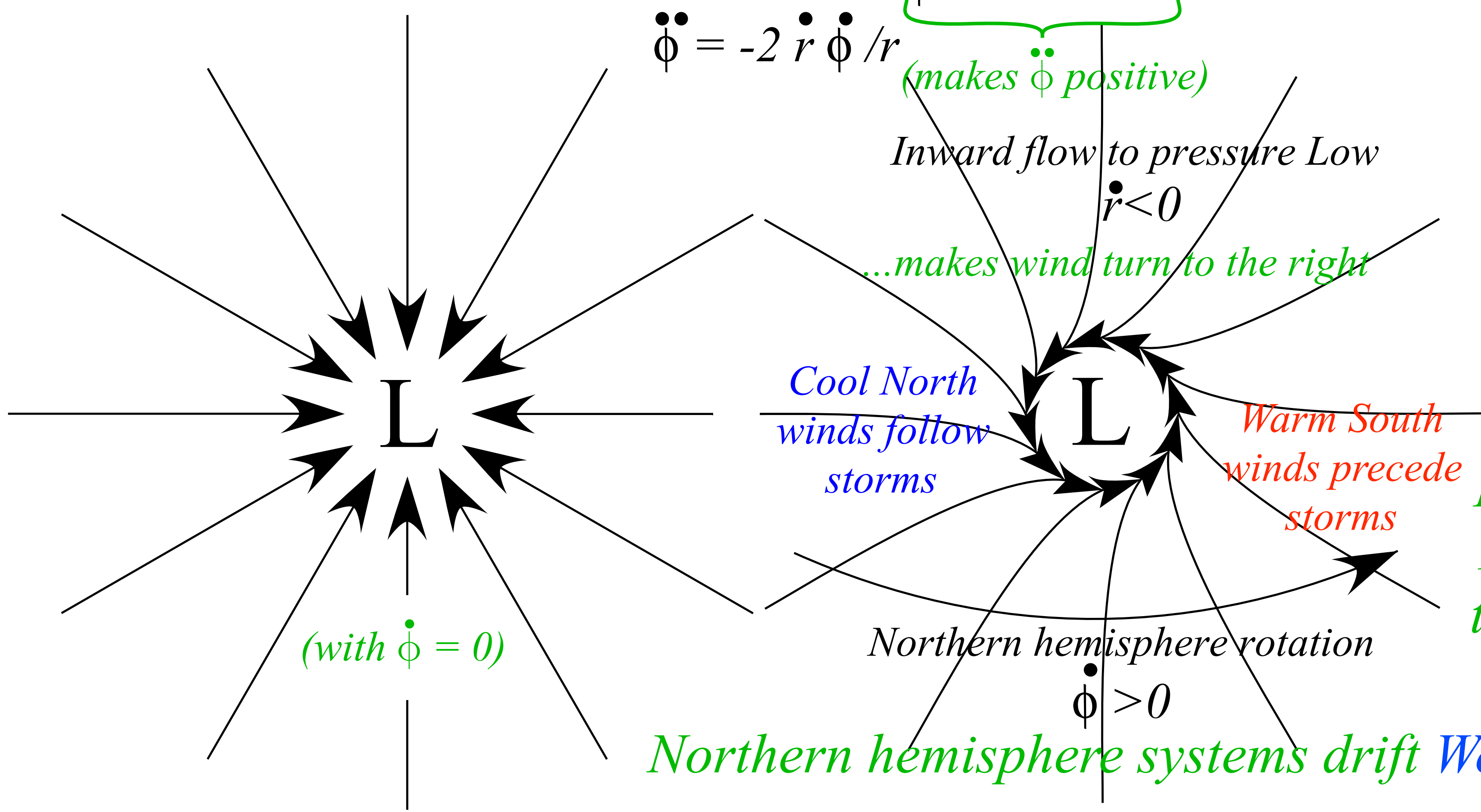
Field-free (U=0)

radial acceleration: $\ddot{r} = r\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\dot{\phi}>0$ and $\dot{r}<0$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}/r$



Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Deep quantum rule: Flow tries to mimic the external rotation (least relative v)

Northern hemisphere systems drift West to East

Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

GCC velocity: $\dot{q}^m=\frac{dq^m}{dt}$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

GCC velocity: $\dot{q}^m = \frac{dq^m}{dt}$

...of coordinates and velocity and time, too. *(You can safely drop last chain-rule factor [1=dt/dt])*

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}\frac{dt}{dt}$$

Deriving Hamilton's equations from Lagrangian theory

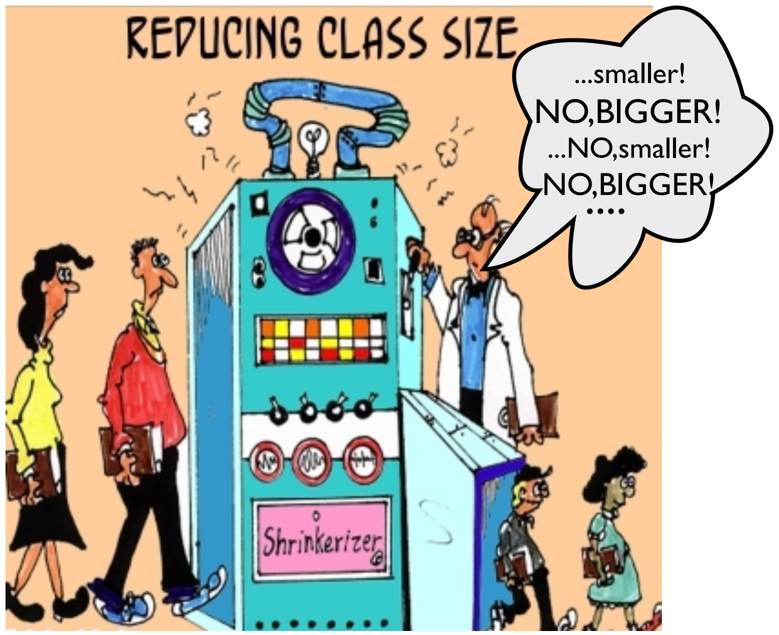
Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

GCC velocity: $\dot{q}^m = \dfrac{dq^m}{dt}$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U(t)-dial.)

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$



Cartoonish way to imagine explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

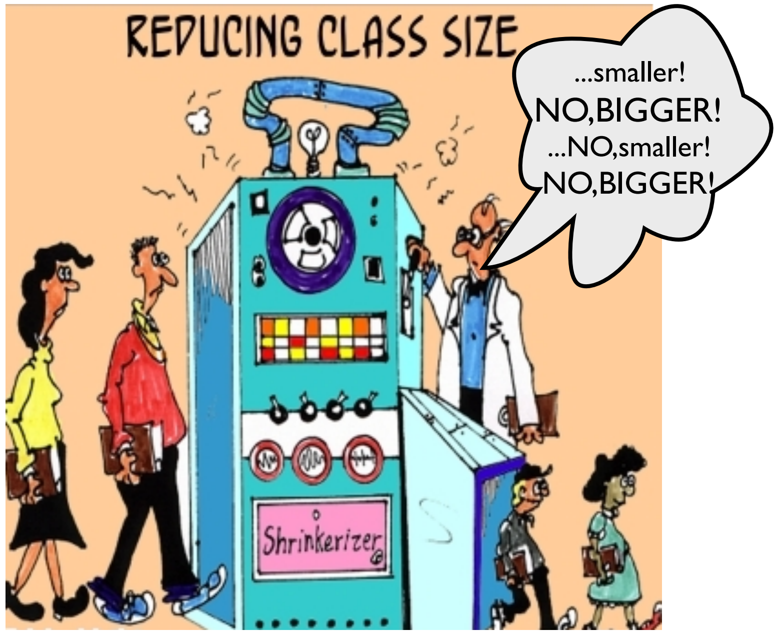
GCC velocity: $\dot{q}^m = \frac{dq^m}{dt}$

...of coordinates and velocity and <u>*time*</u>*, too. (Imagine Mad Scientist turning U-dial.)*

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Recall Lagrange equations: $\boxed{\dot{p}_m = \frac{\partial L}{\partial q^m}}$ $\boxed{p_m = \frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \dot{p}_m\frac{dq^m}{dt} + p_m\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$



Cartoonish way to imagine explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ *...*

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

GCC velocity: $\dot{q}^m = \dfrac{dq^m}{dt}$

...of coordinates and velocity and <u>*time*</u>*, too. (Imagine Mad Scientist turning U-dial.)*

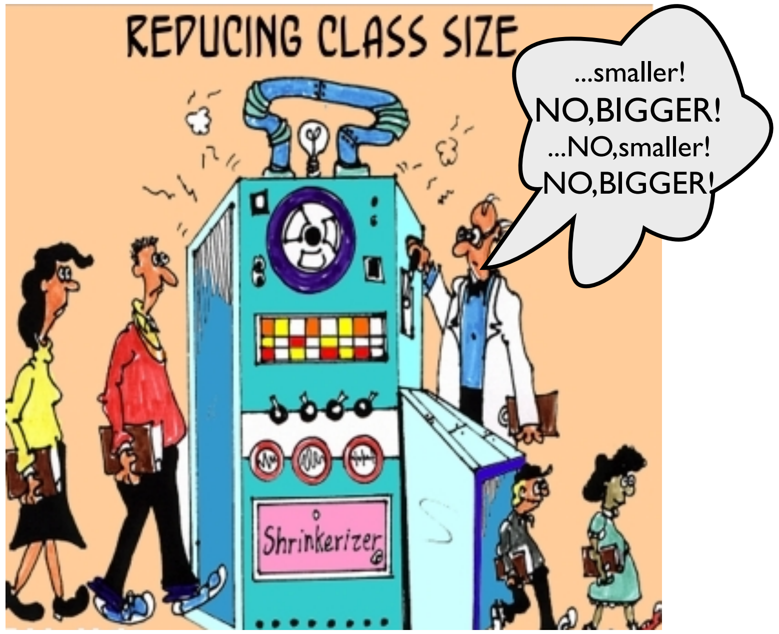
$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Recall Lagrange equations: $\boxed{\dot{p}_m = \dfrac{\partial L}{\partial q^m}}$ $\boxed{p_m = \dfrac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \dot{p}_m\frac{dq^m}{dt} + p_m\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Use product rule:
$\dot{u}\dfrac{dv}{dt} + u\dfrac{d\dot{v}}{dt} = \dfrac{d}{dt}(u\dot{v})$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m\dot{q}^m\right) + \frac{\partial L}{\partial t}$$



Cartoonish way to imagine explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

GCC velocity: $\dot{q}^m = \frac{dq^m}{dt}$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

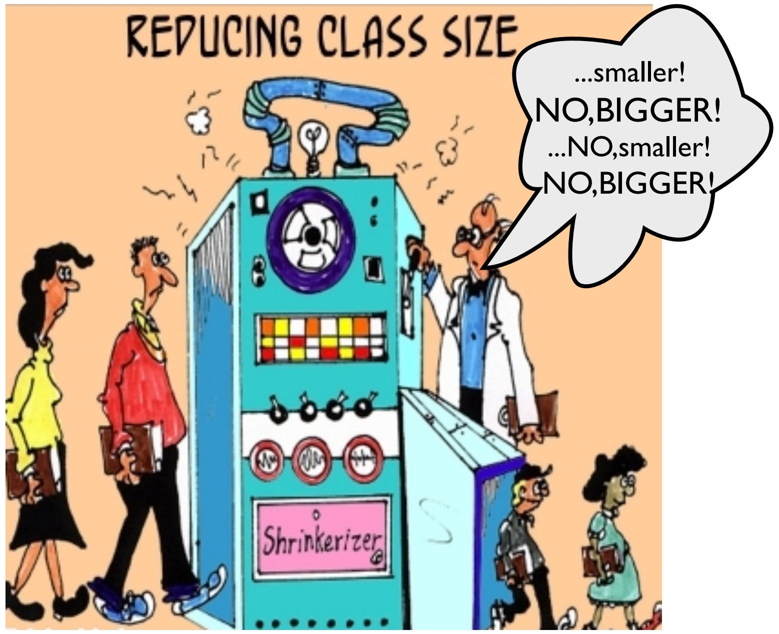
$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Recall Lagrange equations: $\boxed{\dot{p}_m = \frac{\partial L}{\partial q^m}}$ $\boxed{p_m = \frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \dot{p}_m\frac{dq^m}{dt} + p_m\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Use product rule: $\dot{u}\frac{dv}{dt} + u\frac{d\dot{v}}{dt} = \frac{d}{dt}(u\dot{v})$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m\dot{q}^m\right) + \frac{\partial L}{\partial t}$$



Cartoonish way to imagine explicit time dependence

and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p}\bullet\mathbf{v} - L(\mathbf{v})$

$$\frac{d}{dt}\left(p_m\dot{q}^m - L\right) = -\frac{\partial L}{\partial t} \qquad where: H \equiv p_m\dot{q}^m - L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

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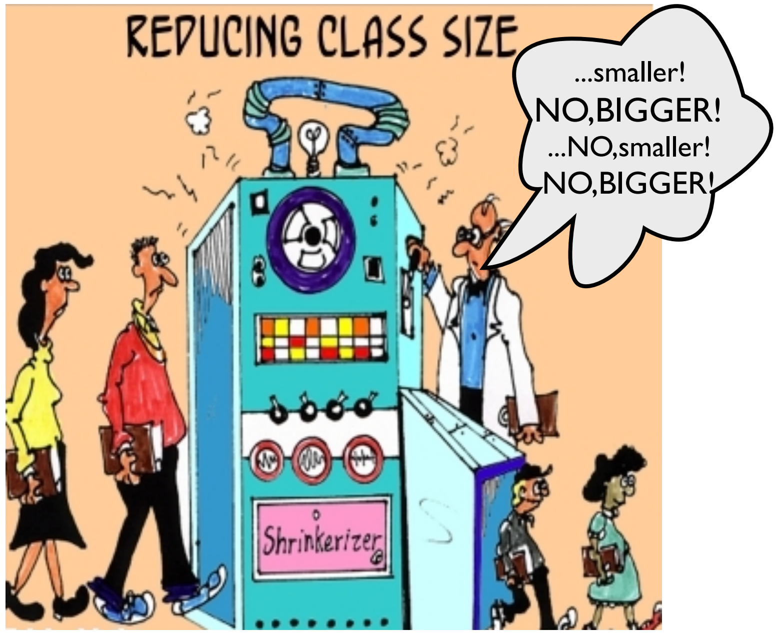

Cartoonish way to imagine explicit time dependence

and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

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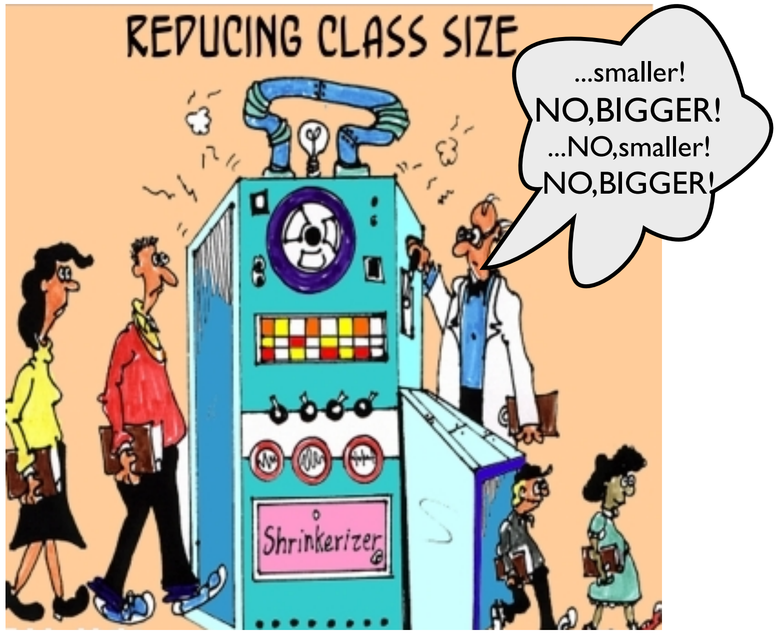

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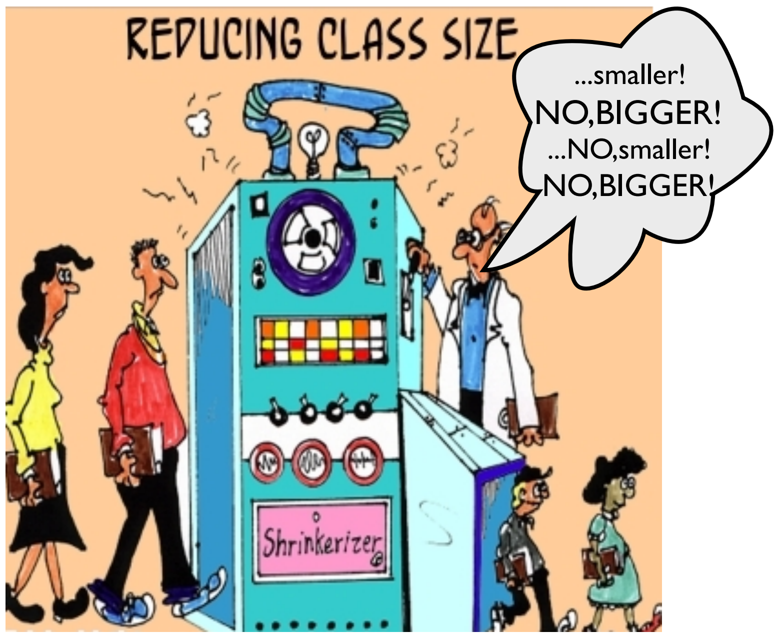

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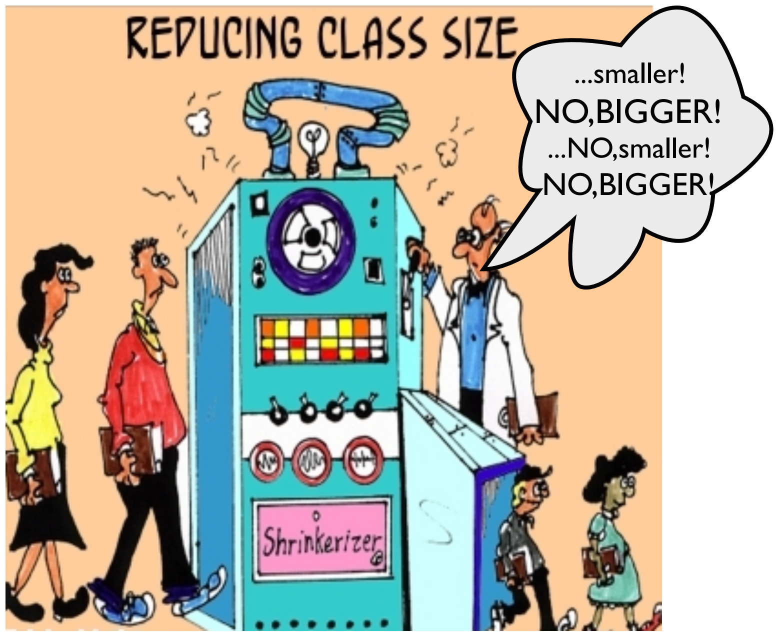

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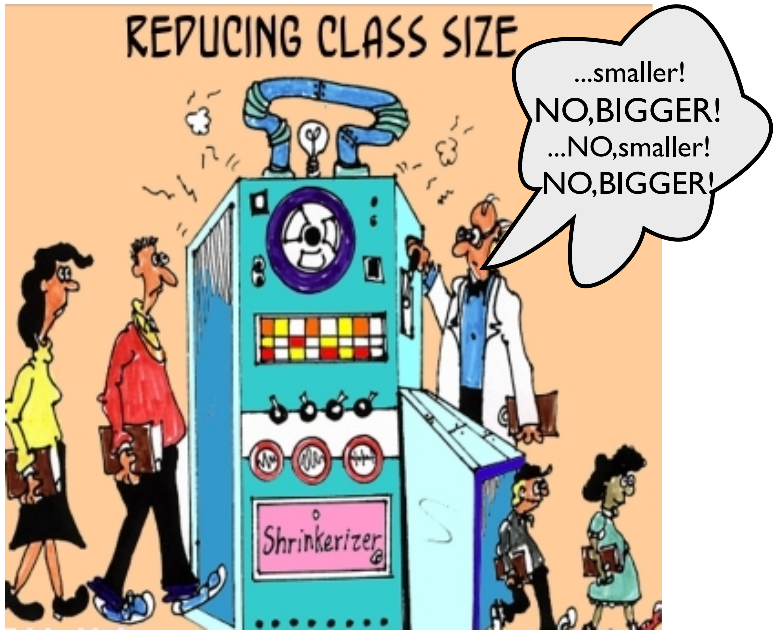

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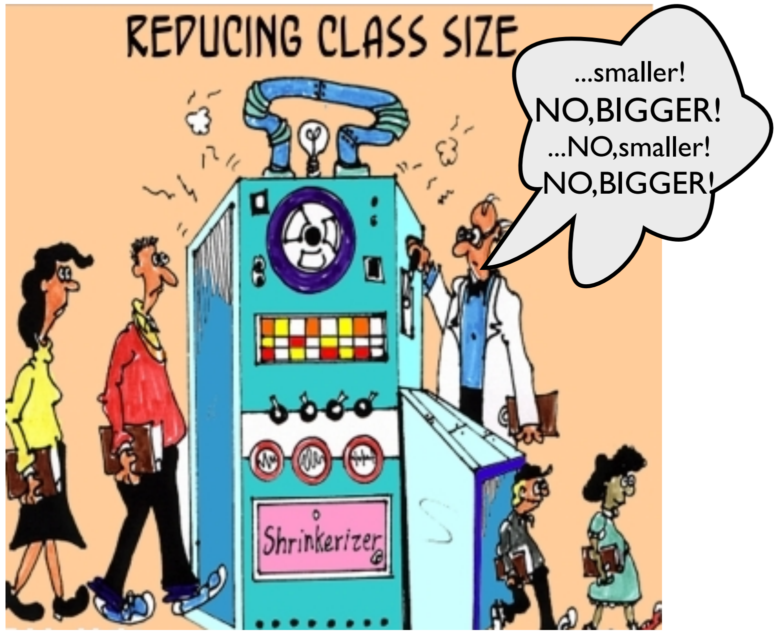

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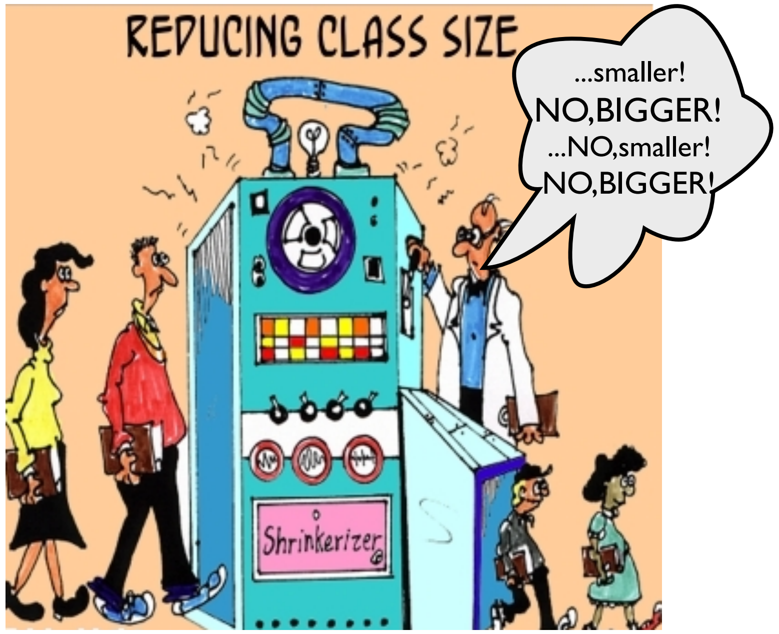

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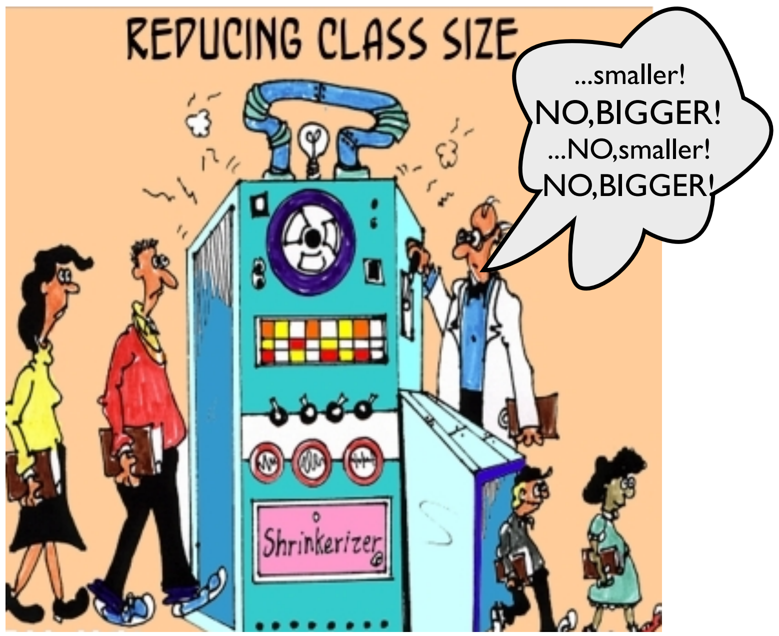

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Hamilton's 2nd GCC equation

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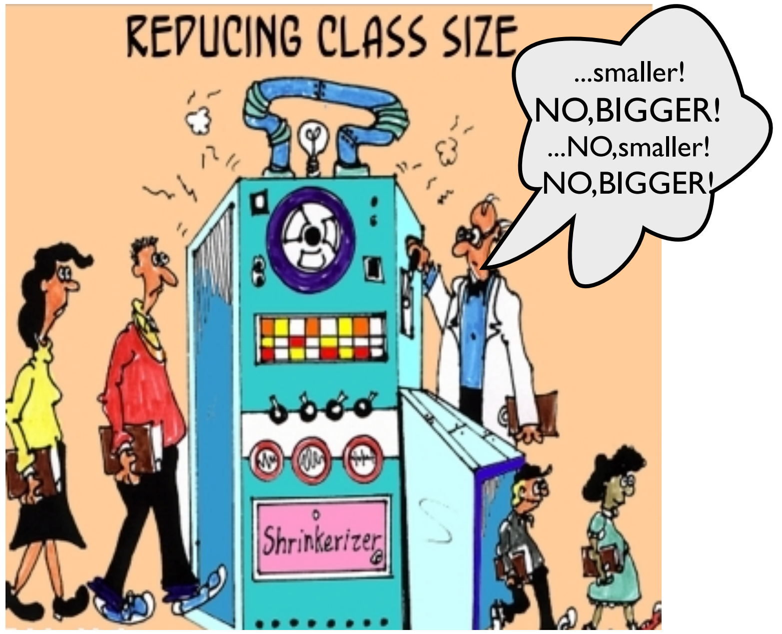

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a most peculiar relation involving partial vs total

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Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

Hamilton's equations in Runga-Kutta (computer solution) form

Hamilton prefers Contra*variant* g^{mn} *with* Co*variant momentum* p_m

Using Legendre transform of Lagrangian $L=T-U$ *with covariant metric definitions of* L *and* p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \dfrac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$*.)*

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

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An inverse metric relation $\dot{q}^m = \frac{1}{M} g^{mn} p_n$ *gives correct form that depends on momentum p_m.*

$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

details on next pages

(Formally **and** Numerically correct)

Details of metric tensor algebra:

Given: $H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$

Let: $\dot{q}^m = \frac{1}{M} g^{mn'} p_{n'}$

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Metric tensor symmetry:

$$g_{mn} = g_{nm}$$

$$g^{mn'} = g^{n'm}$$

(Always applies)

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$$H = \frac{1}{2} M g_{mn} \frac{1}{M} g^{mn'} p_{n'} \dot{q}^n + U$$

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$$= \frac{1}{2} \delta_n^{n'} p_{n'} \dot{q}^n + U$$ where: $$\dot{q}^n = \frac{1}{M} g^{m'n} p_{m'}$$

Metric tensor symmetry:

$$g_{mn} = g_{nm}$$

$$g^{mn'} = g^{n'm}$$

(Always applies)

Metric inversion symmetry:

$$g_{mn} g^{mn'} = \delta_n^{n'}$$

Details of metric tensor algebra:

Given: $H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$

Let: $\dot{q}^m = \frac{1}{M} g^{mn'} p_{n'}$

$$H = \frac{1}{2} M g_{mn} \frac{1}{M} g^{mn'} p_{n'} \dot{q}^n + U$$

$$= \frac{1}{2} g_{mn} g^{mn'} p_{n'} \dot{q}^n + U$$

$$= \frac{1}{2} \delta^{n'}_{n} p_{n'} \dot{q}^n + U \quad \text{where:} \quad \dot{q}^n = \frac{1}{M} g^{m'n} p_{m'}$$

$$= \frac{1}{2} p_n \dot{q}^n + U = \frac{1}{2} p_n \frac{1}{M} g^{m'n} p_{m'} + U$$

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$$= \frac{1}{2} p_n \dot{q}^n + U = \frac{1}{2} p_n \frac{1}{M} g^{m'n} p_{m'} + U$$

$$= \frac{1}{2M} g^{mn} p_m p_n + U$$

Metric tensor symmetry:

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Hamilton prefers Contravariant g^{mn} *with* Covariant *momentum* p_m

Using Legendre transform of Lagrangian $L=T-U$ *with covariant metric definitions of* L *and* p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:

$$H = p_m \dot{q}^m - L = \left(M g_{mn} \dot{q}^n \right) \dot{q}^m - \left(\frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U \right)$$

$$= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$$

This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$*.)*

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

An inverse metric relation $\dot{q}^m = \frac{1}{M} g^{mn} p_n$ *gives correct form that depends on momentum* p_m*.*

$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M \left(g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right) - U(r, \phi)$$

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Polar coordinate Lagrangian was given as: *See covariant polar metric $g_{\mu\upsilon}$ on next page (p35)*

$$L(\dot{r}, \dot{\phi}, r, \phi) = \tfrac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

Covariant polar metric $g_{\mu\upsilon}$ [from p53 of Lecture 9] *Contravariant polar metric* $g^{\mu\upsilon}$

Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor
g^{mn}

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad\qquad \uparrow \mathbf{E}_r \qquad\qquad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{matrix}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$.)

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$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as: *See covariant polar metric $g_{\mu\upsilon}$ on p39*

$$L(\dot{r},\dot{\phi},r,\phi) = \tfrac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

Polar coordinate Hamiltonian is given here:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r{}^2 + g^{\phi\phi} p_\phi{}^2) + U(r,\phi)$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

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(Numerically correct ONLY!)

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$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \frac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

Polar coordinate Hamiltonian is given here: *Contravariant polar metric $g^{\mu\upsilon}$ on p35*

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r,\phi) = \frac{1}{2M} \left(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2\right) + U(r,\phi)$$

Hamilton prefers Contra*variant* g^{mn} *with* Co*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

Hamilton's equations in Runga-Kutta (computer solution) form

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r,\phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$ *in 2D-polar coordinates satisfies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m} = \dot{q}^m$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r,\phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$$

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$$\frac{\partial H}{\partial p_r} = \dot{r}$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

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$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

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$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r,\phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$$

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$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{M r^2}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial r} = -\cancel{2}\frac{p_\phi^2}{\cancel{2}Mr^3} + \frac{\partial U(r,\phi)}{\partial r}$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$p_\phi = Mr^2\dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial r} = -\cancel{2}\frac{p_\phi^2}{\cancel{2}Mr^3} + \frac{\partial U(r,\phi)}{\partial r}$$

$$\dot{p}_r = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi} = \frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial p_r}=\dot{r} \qquad \frac{\partial H}{\partial p_r}=\frac{p_r}{M} \qquad p_r=M\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi} \qquad \frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2} \qquad p_\phi=Mr^2\dot{\phi}$$

$$\frac{\partial H}{\partial r}=-\dot{p}_r \qquad \frac{\partial H}{\partial r}=-\cancel{2}\frac{p_\phi^2}{\cancel{2}Mr^3}+\frac{\partial U(r,\phi)}{\partial r} \qquad \dot{p}_r=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi \qquad \frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi} \qquad \dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

Hamiltonian $H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$ *in 2D-polar coordinates satisfies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

$$p_r=M\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

$$\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2}$$

$$p_\phi=Mr^2\dot{\phi}$$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial r}=-\cancel{2}\frac{p_\phi^2}{\cancel{2}Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

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$$\dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

$$p_\phi=Mr^2\dot{\phi}$$

$$\dot{p}_\phi=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}=-\partial_\phi U(r,\phi)$$

Compare these Hamilton's equations to Lagrange's on next page...

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 9)**

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\,g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising; radial momentum p_r has the usual linear $M\cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi = Mr^2\omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$

$$= M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

→ *Hamilton's equations in Runga-Kutta (computer solution) form*

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$= Mr\dot{\phi}^2 - \partial_r U(r,\phi)$$

$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_2 = \dot{x}_2(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_3 = \dot{x}_3(x_1, x_2, x_3, \ldots)$$

$$\vdots$$

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

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$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi)$$

Hamiltonian eqs. in Runga-Kutta form:

$$\dot{r} = \dot{r}(r, p_r, \phi, p_\phi) \quad = \frac{p_r}{M}$$

$$\dot{p}_r = \dot{p}_r(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi^2}{Mr^3} - \partial_r U(r,\phi)$$

$$\dot{\phi} = \dot{\phi}(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi}{Mr^2}$$

$$\dot{p}_\phi = \dot{p}_\phi(r, p_r, \phi, p_\phi) \quad = -\partial_\phi U(r,\phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \ldots)$$

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$$\dot{x}_3 = \dot{x}_3(x_1, x_2, x_3, \ldots)$$

$$\vdots$$

Examples of Hamiltonian mechanics in effective potentials

Isotropic Harmonic Oscillator in polar coordinates and effective potential (*Web Simulation: OscillatorPE - IHO*)

Coulomb orbits in polar coordinates and effective potential (*Web Simulation: OscillatorPE - Coulomb*)

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r{}^2 + g^{\phi\phi} p_\phi{}^2) + k \cdot r^2/2 = \frac{1}{2M}(p_r{}^2 + \frac{1}{r^2} \cdot p_\phi{}^2) + \frac{k \cdot r^2}{2} = E = const.$$

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H is not explicit function of ϕ , and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

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Same applies to any radial potential U(r)

$$E = \frac{p_r{}^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} + \overbrace{U(r)}^{\text{"real" PE}}}^{\text{"effective" PE}}$$

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Solving for momentum: $p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2$

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$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2} = \sqrt{2M}\sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2}$$

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Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2} \cdot r^2$

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}$$

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$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2} = \sqrt{2M}\sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2}$$

Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2$

Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}$$

Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}}$

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H is not explicit function of ϕ, and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

$$\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k\cdot r^2}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k\cdot r^2}{2} = E = const.$$

Same applies to any radial potential U(r)

$$E = \frac{p_r{}^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} + \overbrace{U(r)}^{\text{"real" PE}}}^{\text{"effective" PE}}$$

Solving for momentum: $p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2$

$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2} = \sqrt{2M}\sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2}$$

Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}\cdot r^2$

Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}$$

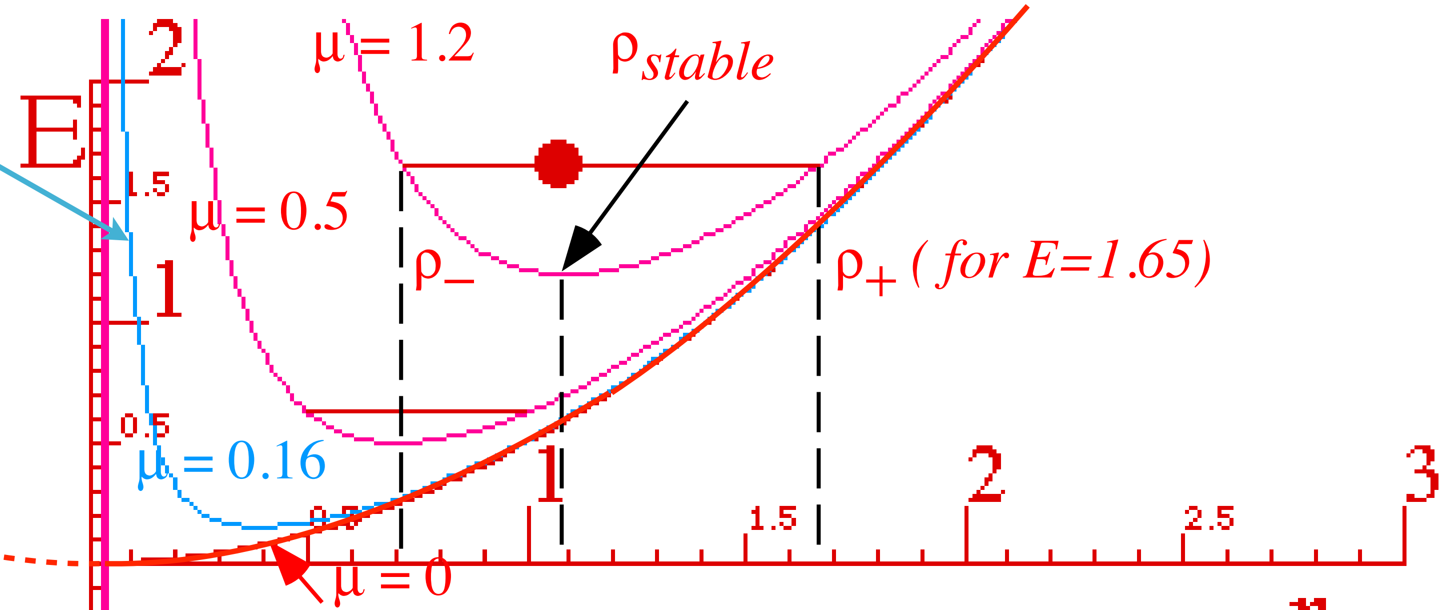
Called the "quadrature" or 1/4-cycle solution if $r_< = 0$ and $r_> =$ max amplitude

Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}}$

Time vs r for any radial U(r):

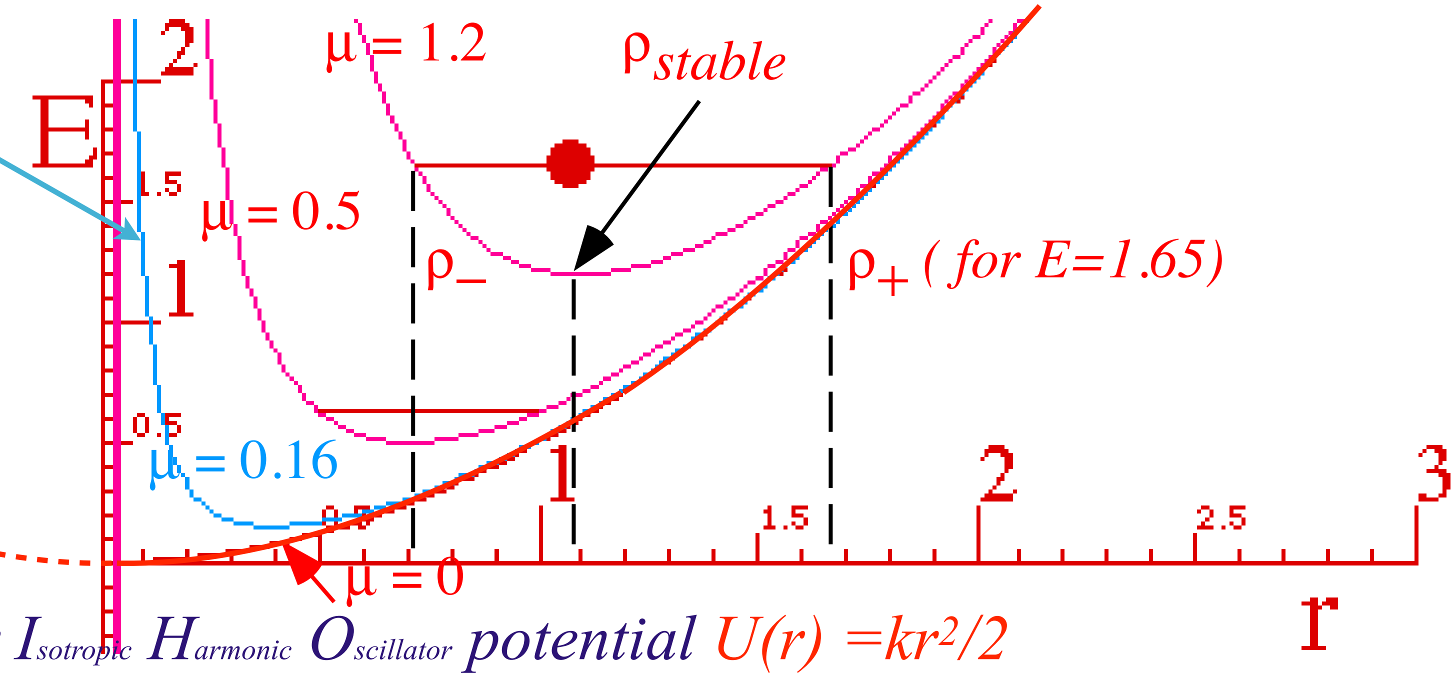
$$t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{2U(r)}{M}}}$$

$$E=\frac{p_r^{\;2}}{2M}+\overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}}+\overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential U(r) =kr²/2

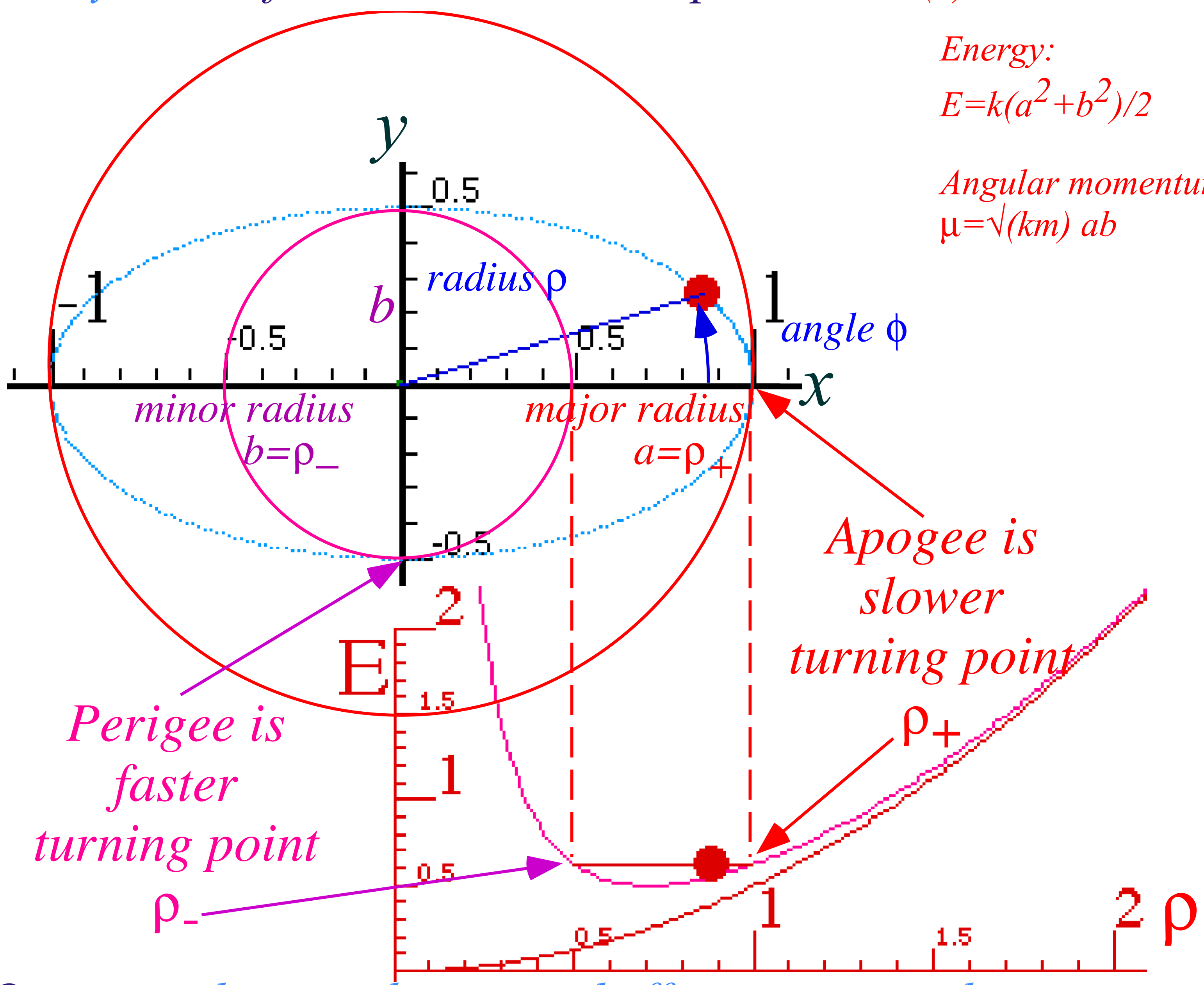
$$E=\frac{p_r^{\,2}}{2M}+\overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}}+\overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$

Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$



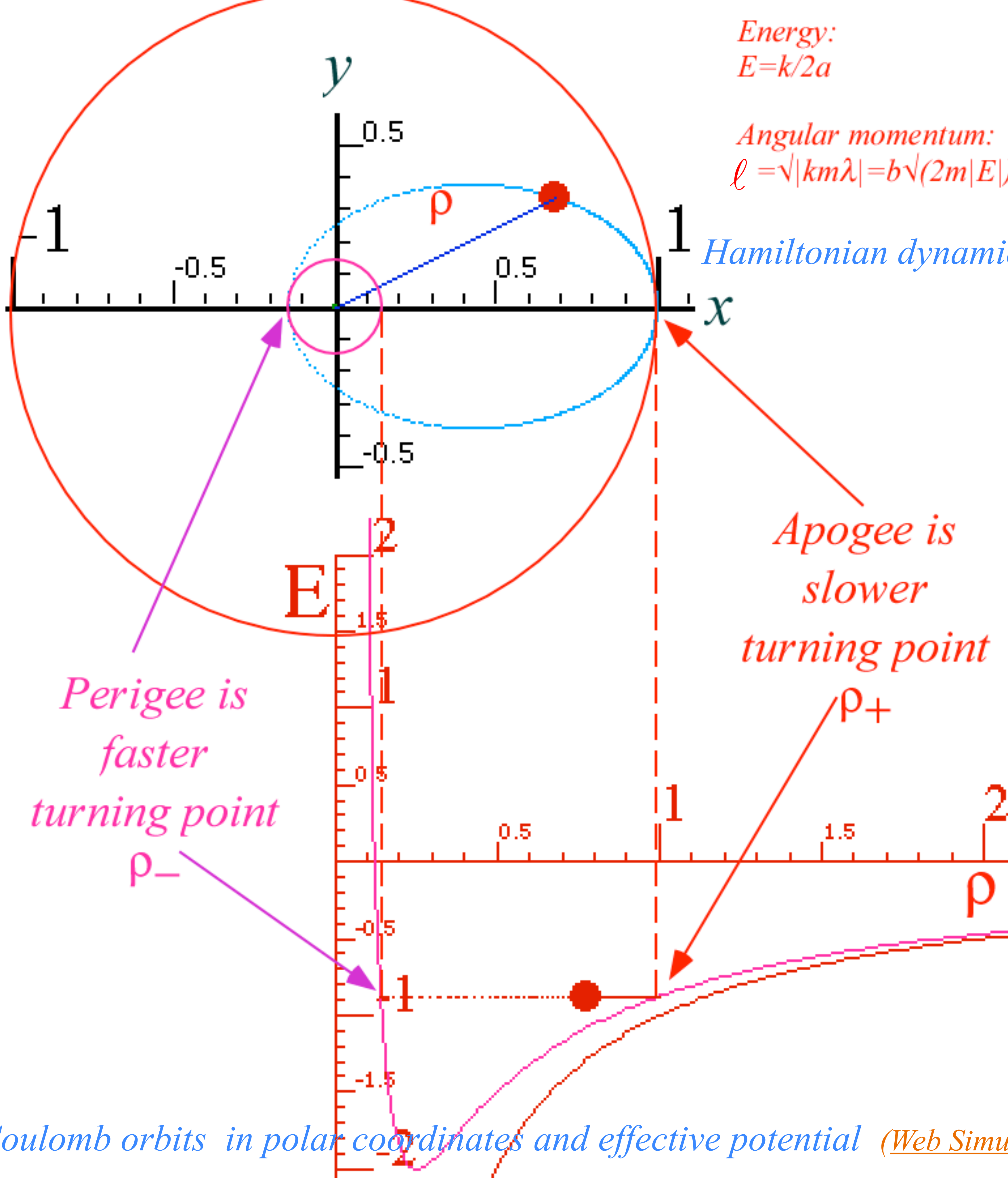
$I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ in polar coordinates and effective potential (Web Simulation: OscillatorPE - IHO)

Examples of Hamiltonian mechanics in effective potentials

Isotropic Harmonic Oscillator in polar coordinates and effective potential *(Web Simulation: OscillatorPE - IHO)*

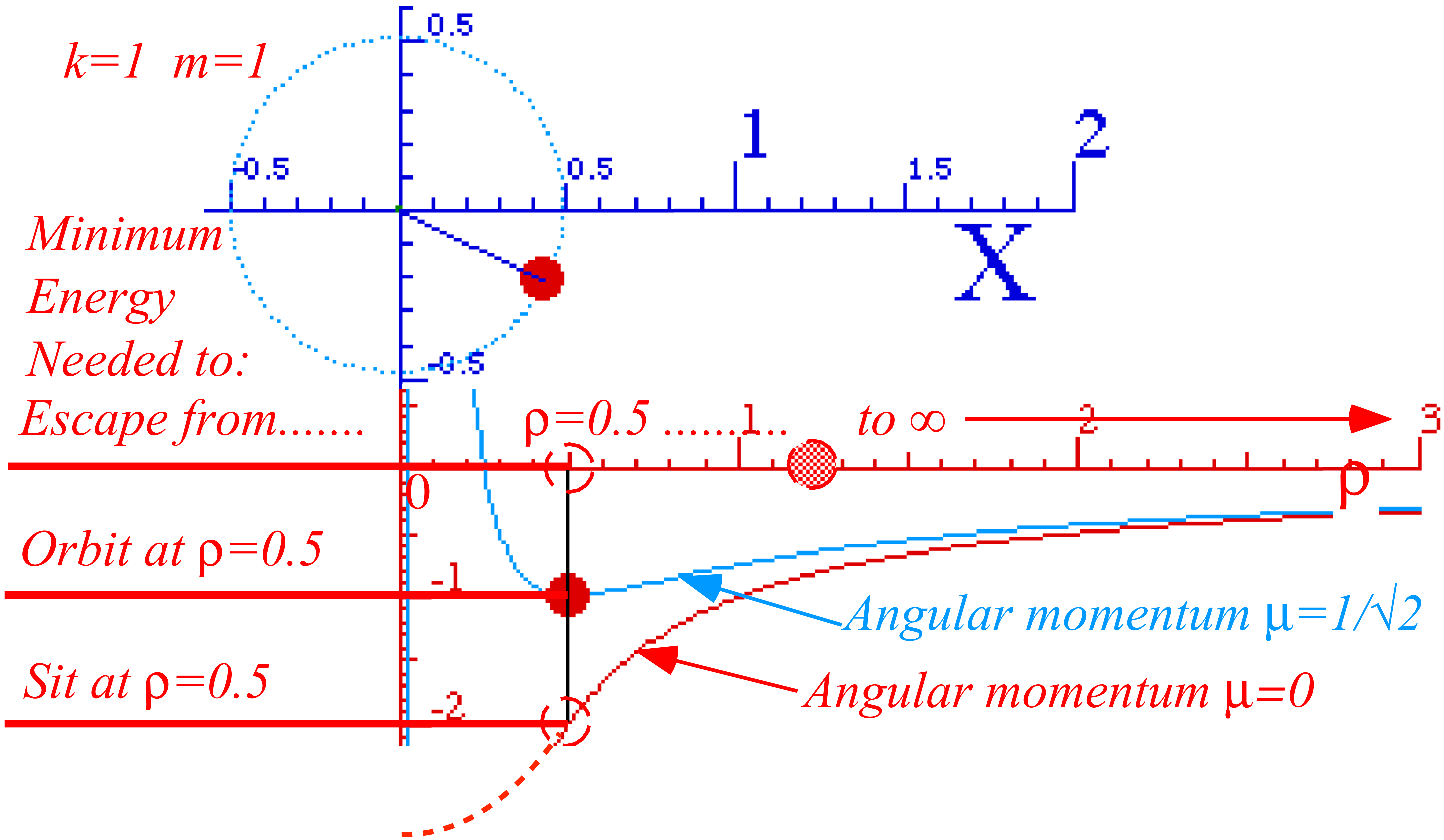
Coulomb orbits in polar coordinates and effective potential *(Web Simulation: OscillatorPE - Coulomb)*

Hamiltonian dynamics for Coulomb potential $U(r) = -k/r$

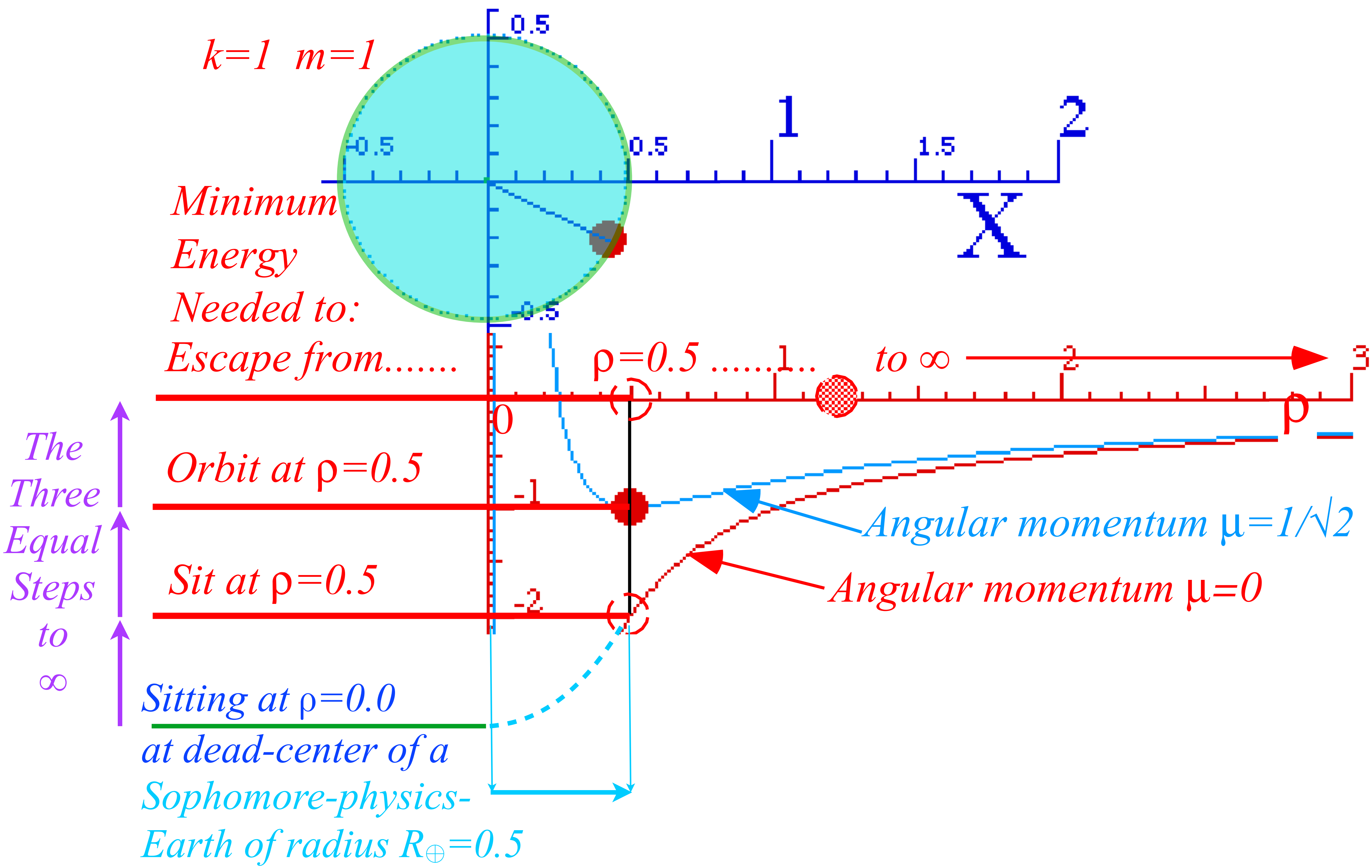


$$E = \frac{p_r^2}{2M} + \underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} - \overbrace{\frac{k}{r}}^{\text{"real" PE}}$$

Coulomb orbits in polar coordinates and effective potential (Web Simulation: OscillatorPE - Coulomb)



k=1 m=1
Minimum
Energy
Needed to:
Escape from.......
ρ=0.5 ......1...
to ∞
Orbit at ρ=0.5
Sit at ρ=0.5
Angular momentum μ=1/√2
Angular momentum μ=0
X
ρ



From p. 74 Lect. 6, on next page

Sophomore-physics-Earth inside and out: "3-steps out of (or into) Hell" ...and surface orbit at $r=R_\oplus$

From p. 75 Lect. 6

surface gravity: $g=-G\frac{M_\oplus}{R_\oplus^2}$

Centifugal force = surface gravity:

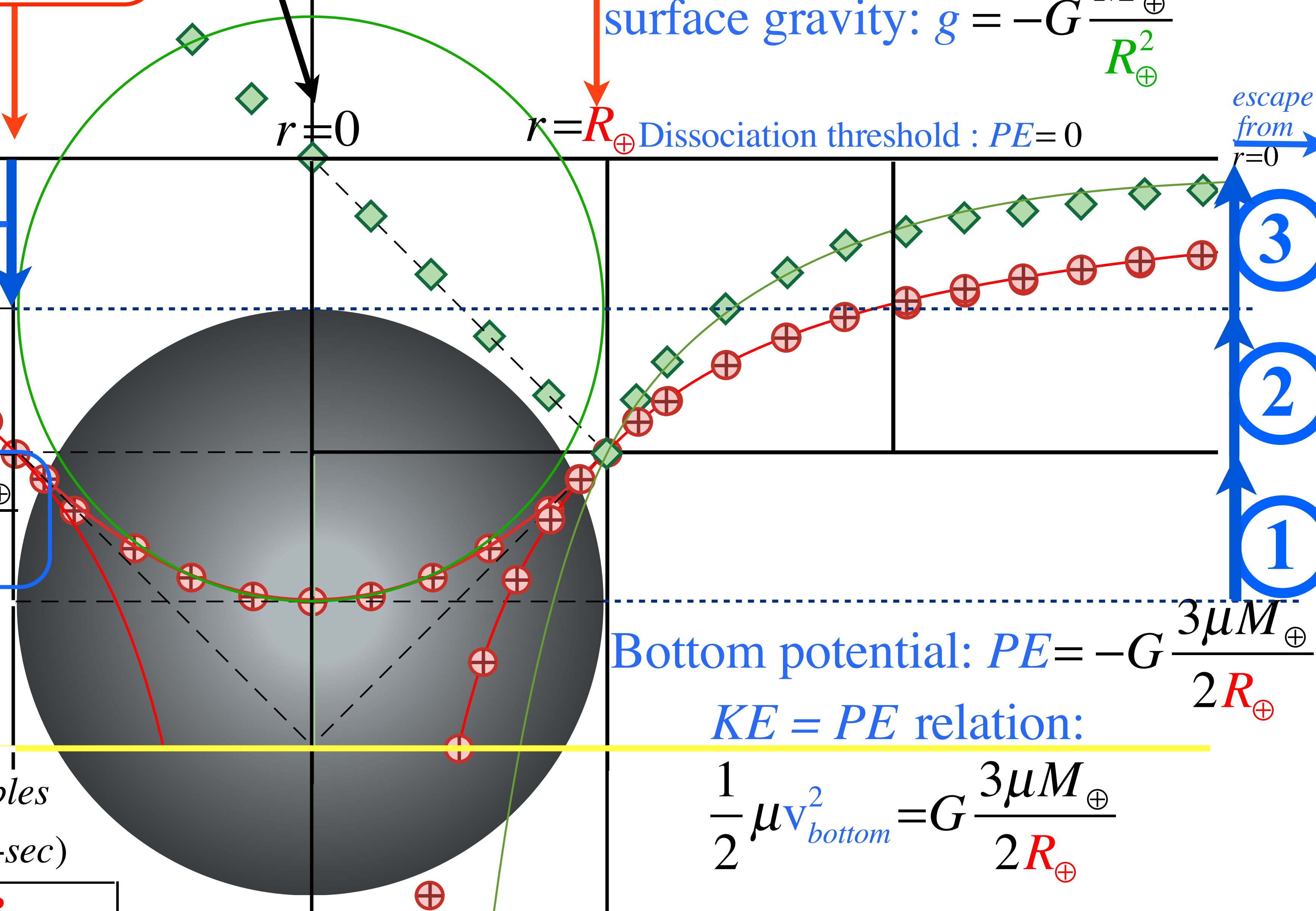
$$\frac{\mu v_\odot^2}{R_\oplus}=\mu g=G\frac{\mu M_\oplus}{R_\oplus^2}$$

Orbit KE $=\frac{1}{2}\mu v_\odot^2=G\frac{\mu M_\oplus}{2R_\oplus}$

Orbit $E_\odot^{Total}=\frac{1}{2}\mu v_\odot^2-G\frac{\mu M_\oplus}{R_\oplus}=\boxed{-G\frac{\mu M_\oplus}{2R_\oplus}}$

$(r=R_\oplus)$-orbit angular frequency:

$$\omega_\odot^2 R_\oplus=G\frac{M_\oplus}{R_\oplus^2}\Rightarrow\omega_\odot=\sqrt{G\frac{M_\oplus}{R_\oplus^3}}$$



Bottom potential: $PE=-G\frac{3\mu M_\oplus}{2R_\oplus}$

$KE=PE$ relation:

$$\frac{1}{2}\mu v_{bottom}^2=G\frac{3\mu M_\oplus}{2R_\oplus}$$

Geometric (x,y) (Dimensionless)	Scaling relations	mks variables (meter-kg-sec)		
space coord.: x	$r=R_\oplus x$	$x=r/R_\oplus$		
PE for $\lvert x\rvert\geq 1$: $y^{PE}=\frac{-1}{x}$	$PE^{mks}(r)=\frac{GM\mu}{R_\oplus}y^{PE}$	$PE^{mks}(r)=-\frac{GM\mu}{r}=-\frac{GM\mu}{R_\oplus}\frac{1}{x}$	PE for $\lvert x\rvert<1$: $y^{PE}=\frac{x^2}{2}-\frac{3}{2}$	$PE^{mks}(r)=\frac{GM\mu}{R_\oplus}\left(\frac{r^2}{2R_\oplus^2}-\frac{3}{2}\right)$
$Force$ for $\lvert x\rvert\geq 1$: $y^{Force}=\frac{-1}{x^2}$	$F^{mks}(r)=\frac{GM\mu}{R_\oplus^2}y^{Force}$	$F^{mks}(r)=-\frac{GM\mu}{r^2}=-\frac{GM\mu}{R_\oplus^2}\frac{1}{x^2}$	$Force$ for $\lvert x\rvert<1$: $y^{Force}=-x$	$F^{mks}(r)=-\frac{GM\mu}{R_\oplus^3}r$

$(r=0)$-escape-velocity

$$v_{bottom}=\sqrt{3G\frac{M_\oplus}{R_\oplus}}$$

$(r=R_\oplus)$-escape velocity:

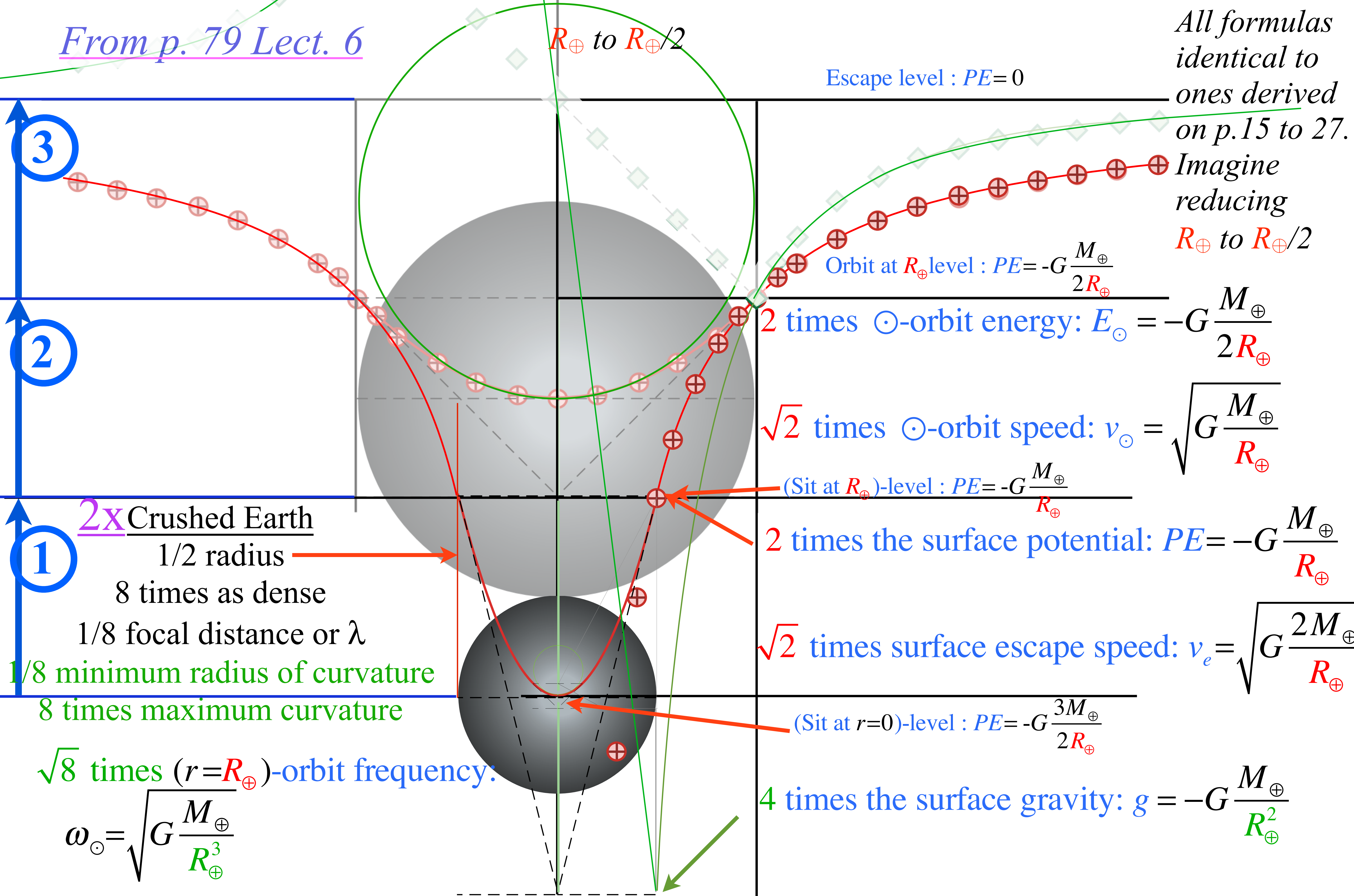
$$v_{escape}=\sqrt{2G\frac{M_\oplus}{R_\oplus}}$$

Sophomore-physics-Earth inside and out: "3-steps to Hell"

Suppose Earth radius crushed to 1/2: ($R_\oplus=6.4\cdot 10^6 m$ crushed to $R_\oplus/2=3.2\cdot 10^6 m$)

From p. 79 Lect. 6

All formulas identical to ones derived on p.15 to 27. Imagine reducing $R_\oplus$ to $R_\oplus/2$



Next Hamiltonian Lecture 11…

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vertically Driven Pendulum))

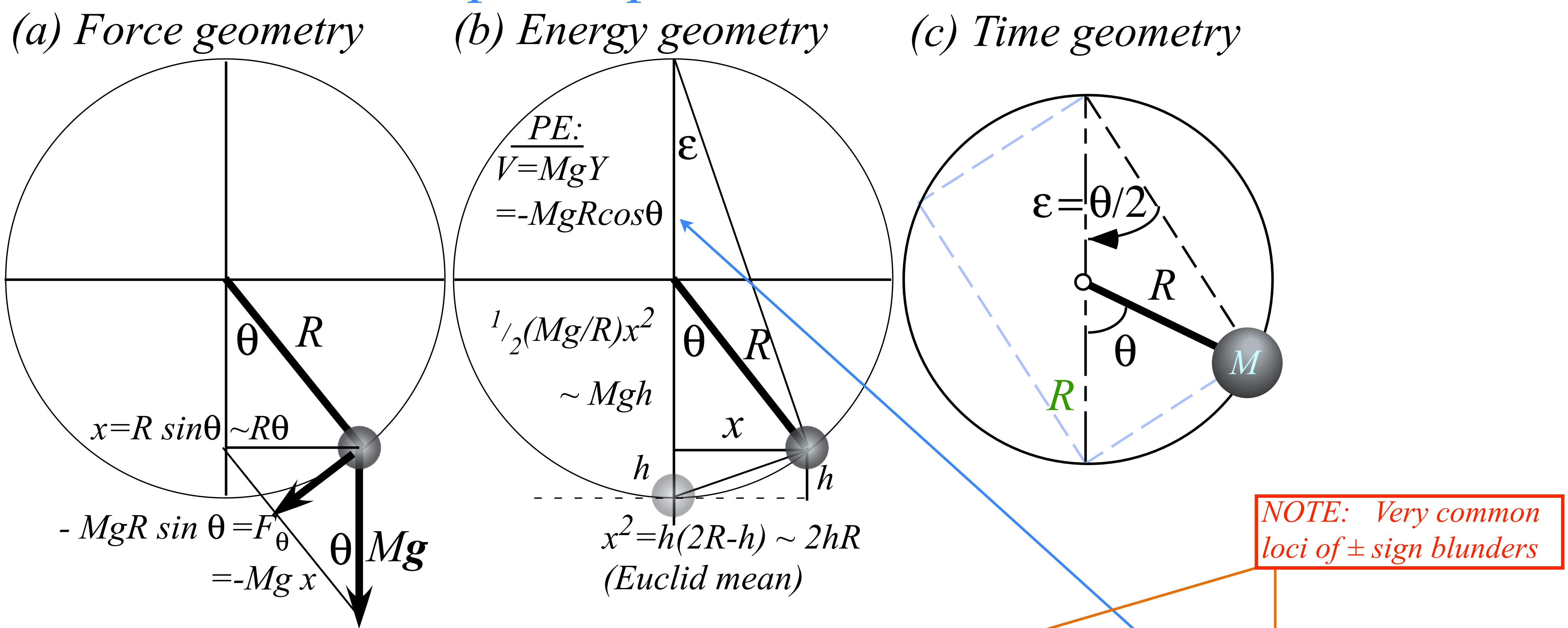
1D-HO phase-space control (Classic Simulation of "Catcher in the Eye", Web Simulation:JerkIt)

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control

Normally we'd stop here

Lecture 10 ends here
Mon. 9.30.2019

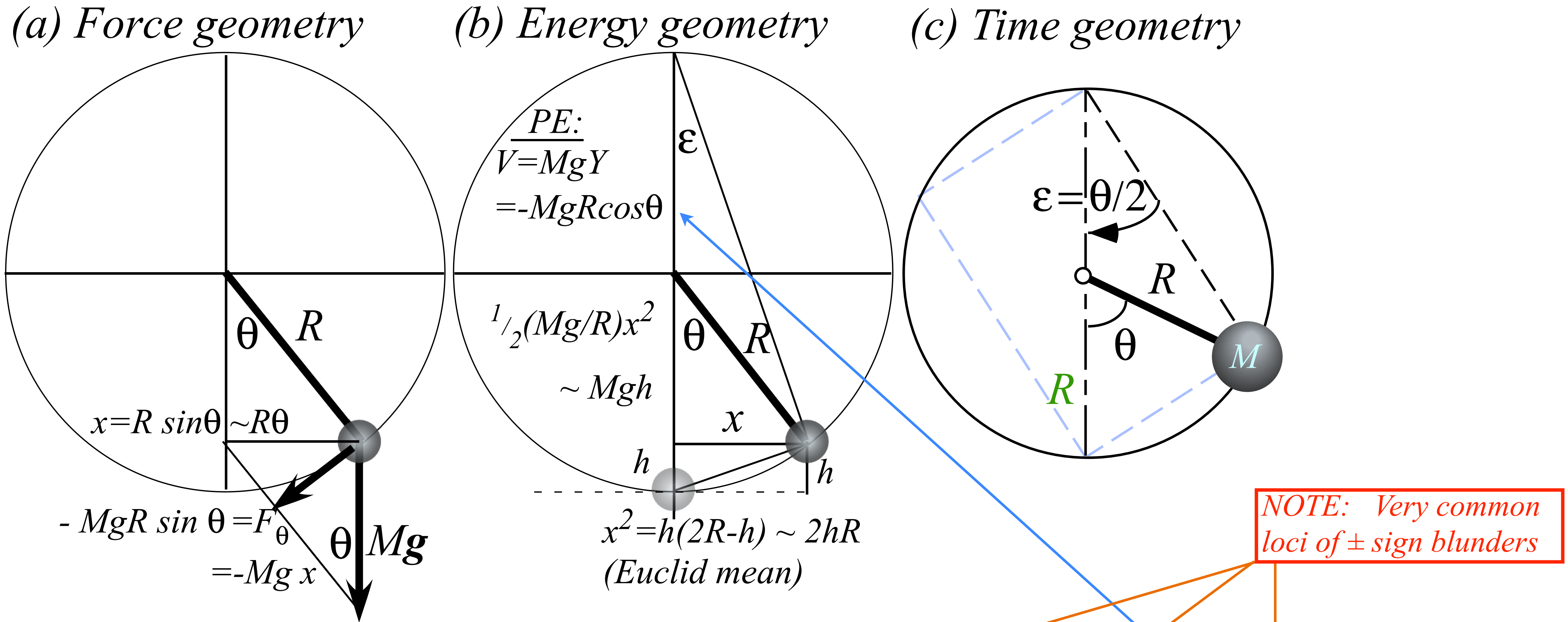
1D Pendulum and phase plot



Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

1D Pendulum and phase plot



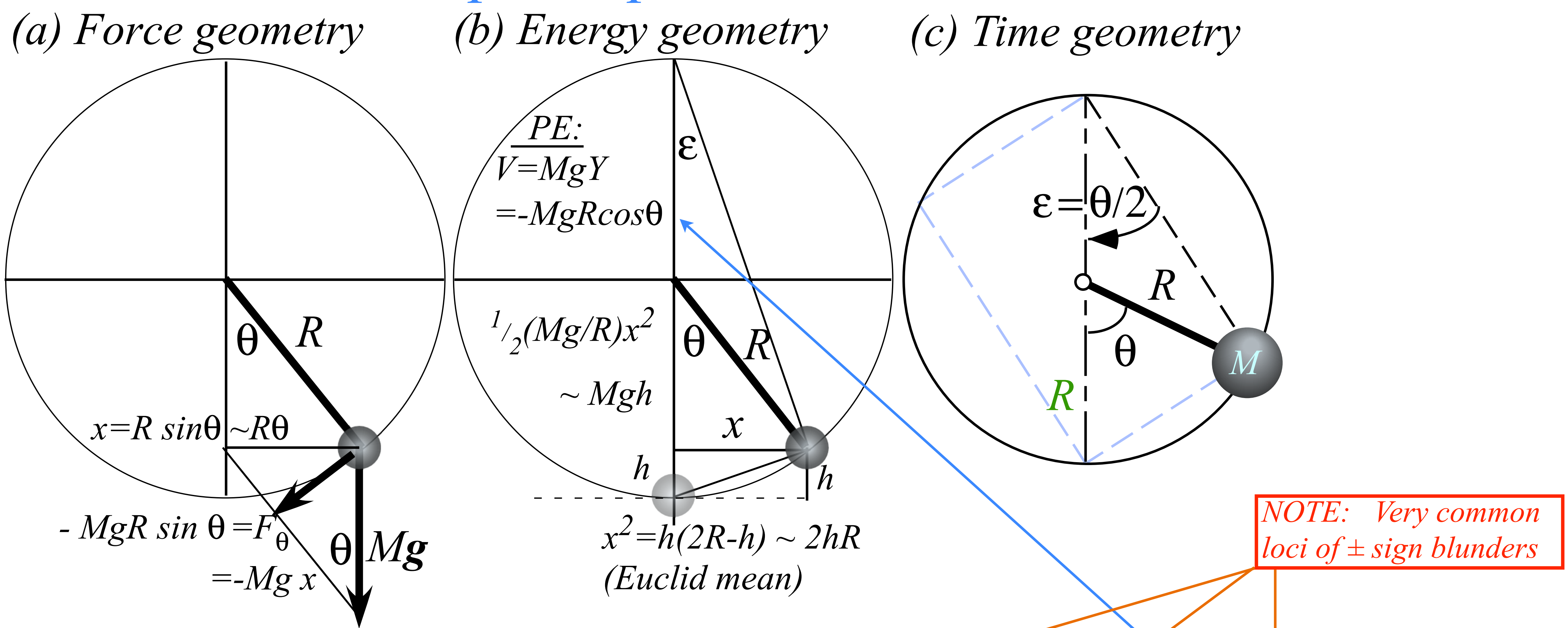
Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \ = E \quad \text{=const.}$$

1D Pendulum and phase plot



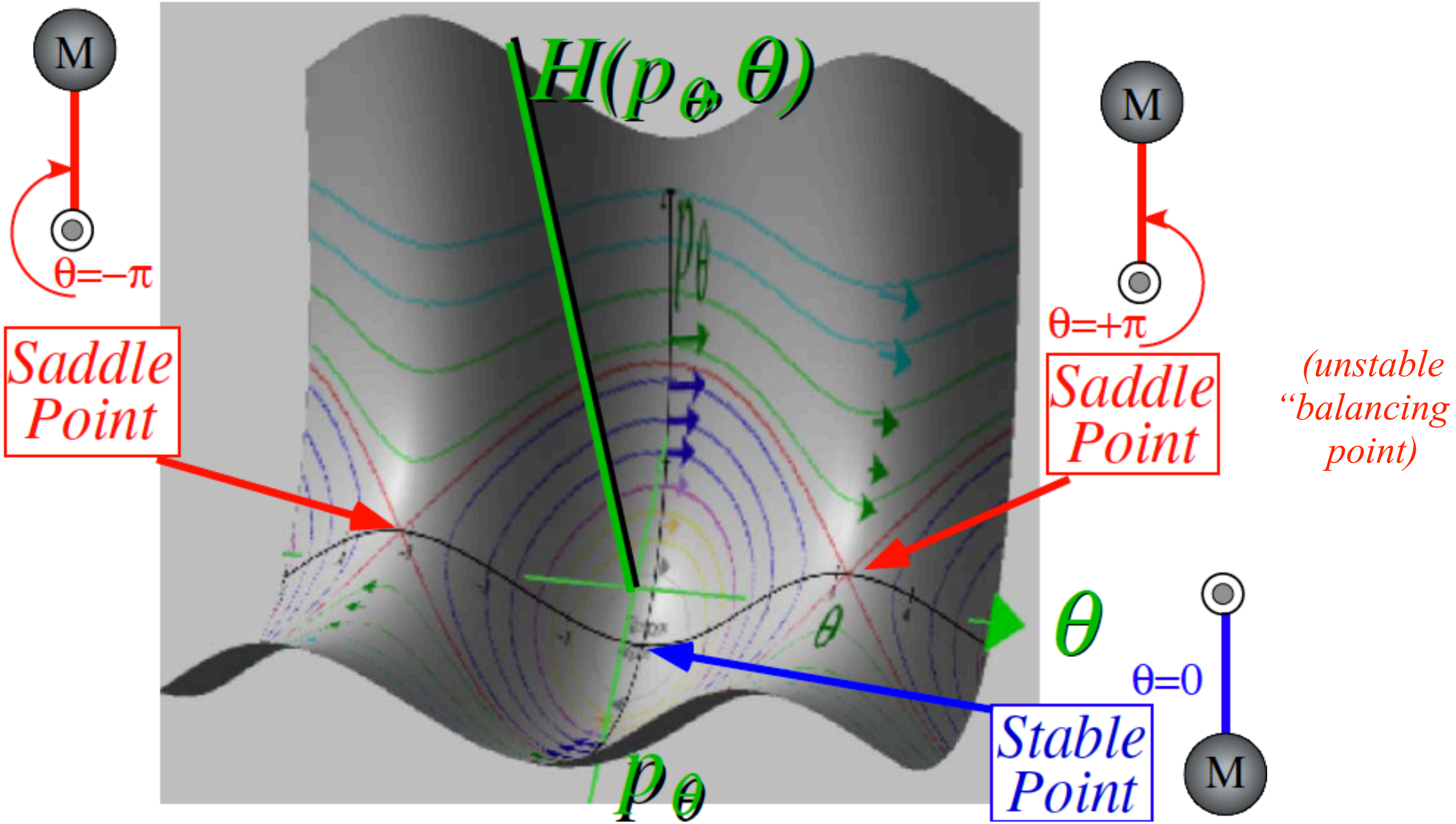
Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot\theta,\theta)=\frac{1}{2}I\dot\theta^2-U(\theta)=\frac{1}{2}I\dot\theta^2+MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

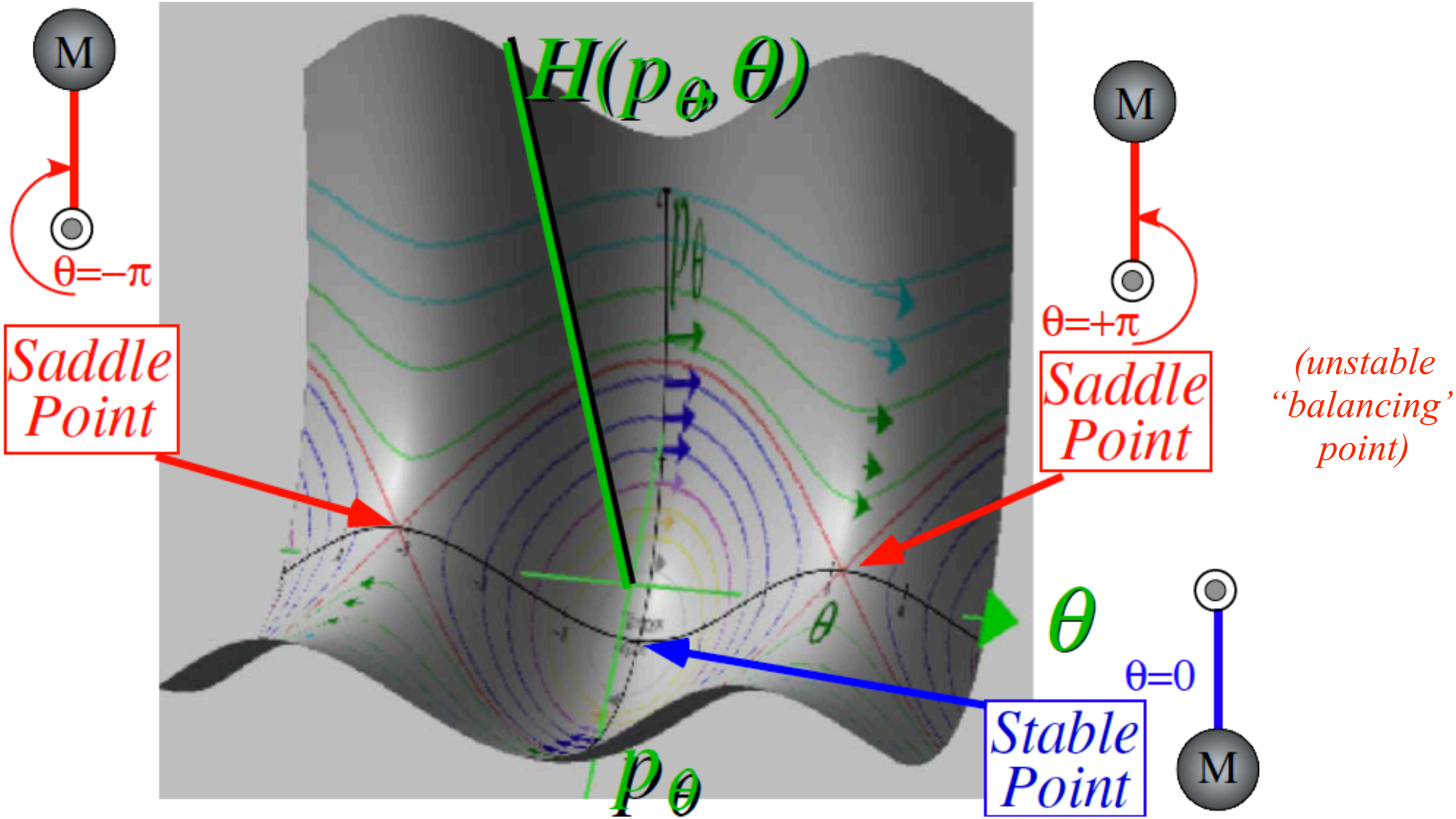
$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta \ = E \quad =const.$$

implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta{}^2 - MgR\cos\theta \ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta) = E = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix} = \mathbf{e_H}\times\left(-\nabla H\right) = (\overrightarrow{\text{H-axis}})\times(\overrightarrow{\text{fall line}}),\ \text{where:}\begin{cases}(\overrightarrow{\text{H-axis}}) = \mathbf{e_H} = \mathbf{e_q}\times\mathbf{e_p}\\ (\overrightarrow{\text{fall line}}) = -\nabla H\end{cases}$$

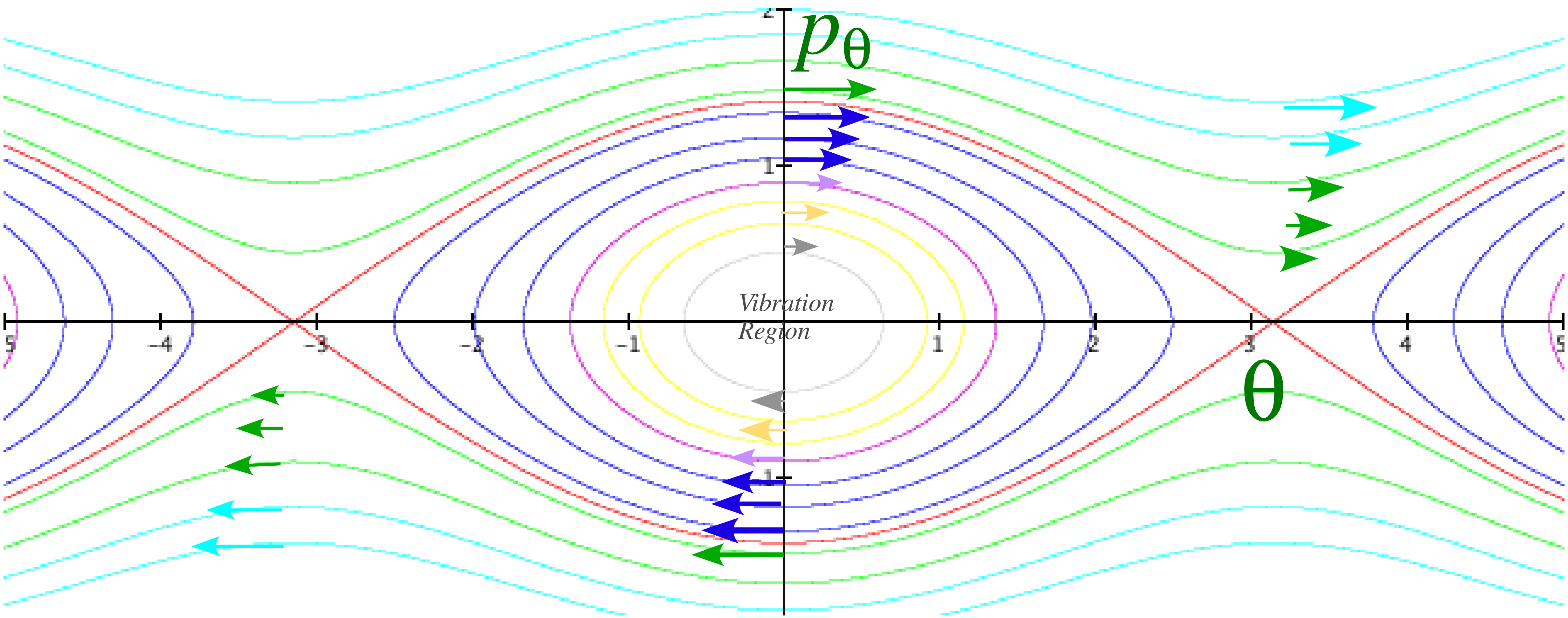


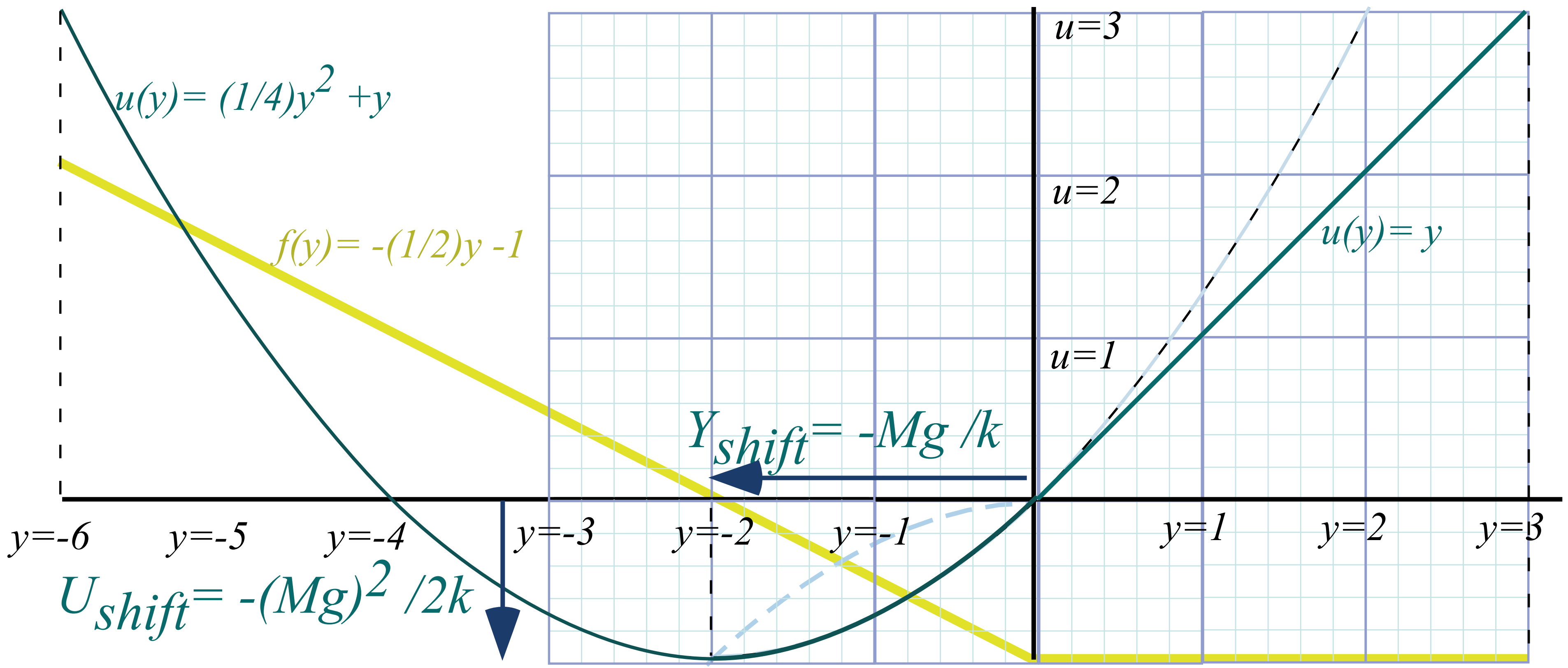
Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

F(Y)=-kY-Mg

U(Y)=(1/2)kY²+Mg Y

Unit 1
Fig. 7.4



Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control