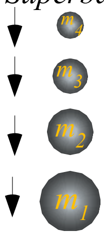
9/17/19 Assignment Set 4 - Read *Unit 1 Ch. 3* thru *Ch.8 Lect. 4-5* Due Wed. *9/25/19* Name______________

Some lesser known properties of parabolic PE functions

1.(a) Mechanics problems of atomic oscillators affected by electric fields is basic to spectroscopy. A useful model is potential $V^{atom}(x)=k\,x^2/2$ function of center x of charge Q with polarizability spring constant k. A uniform electric field E applies force $F=Q\cdot E$ to charge by adding potential $V^E(x)$ to $V^{atom}(x)$. (Give $V^E(x)=$______ and $F^E(x)=$_____) Consider the resulting potential $V^{total}(x)$ for an atom for unit constants $k=1$ and $Q=1$. Derive and plot the new values for equilibrium position $x^{equil}(E)$, energy $V^{equil}(E)$, dipole moment $p^{equil}(E)=Q\cdot x^{equil}$. Plot $V^{total}(x)$ for field values of $E=-3,-2,-1, 0, 1, 2, and\ 3$. Does frequency $\omega^{equil}(E)$ vary with field E? What curve do $x^{equil}(E)$ points form?

(b) Follow the steps to construct to external and internal potential energy $V(r)$ and Force $F(r)$ plots of the Sophomore-Physics Earth model. (Lect, 6 p.39-41 and p,62-65.) Describe the 3 equally spaced energy levels.

Superball tower IBM model constructions (With initial $V_k=-1$) See Fig. 8.1(b) p.103 of *Text* Unit 1 or Lect. 5 p.60



The 100% energy transfer limit (IBM values are $v_1^{IN}=1$ and $-1=v_2^{IN}=v_3^{IN}=v_4^{IN}=\ldots$ *after 1st floor bang.)*

2. Suppose each m_k has just the right mass ratio m_k/m_{k+1} with the m_{k+1} above it to pass on all its energy to m_{k+1} so the top ball-N, a *1gm* pellet, goes off with the total energy. Construct velocity-velocity diagrams, indicate velocity at each stage, and derive the required intermediate mass values for (a) $N=2$, (b) $N=3$, (c) $N=4$.

(d) Give algebraic formula for this *Maximum Amplified Velocity* factor in terms of N *(MAV(N)=___________?).*

(e) Give algebraic formula neighbor-mass ratios $R=M_{N-1}/M_N$ in terms of N *(R(N)=_________?).*

N-Ball tower ∞-limits

3. Suppose each m_k is very much larger than m_{k+1} above it so that final v_{k+1} approaches its upper limit. Then top m_N goes off with nearly the highest velocity v_N attainable. Construct the velocity-velocity diagrams. Indicate each intermediate velocity limit value at each stage and the limiting top value for (a) $N=2$, (b) $N=3$, (c) $N=4$.

(d) Give algebraic formula for *Absolute Maximum Amplified Velocity* factor in terms of N *(AMAV(N)=_______?).*

The optimal idler (An algebra/calculus vs. *geometry problem)*

4.(a) To get highest final v_3 of mass m_3 find optimum mass m_2 in terms of masses m_1 and m_3 that will do that.

(b) Consider this problem in Galileo-shifted frame with: $v_1^{IN}=2$ and $0=v_2^{IN}=v_3^{IN}$ (Algebra simplifies for this.)

(c) Do V-V plots for case m_1=4 and m_3=1 (where m_2=____?) …for non-optimal case m_1=4, m_2=3, m_3=1.

(d) Give formula for optimal top mass final velocity in terms of m_1, m_2, and m_3 and compare to result of 4(a). Plot that final velocity versus the idler mass $x=m_2$=0 to 4. How sensitive is the optimal final v_3 to x?

The backsides of exponentials

5. *Some lesser known properties of exponentials and logarithms*

(a) Do plots of exponential $y=e^x$ and $y=\log_e x$ functions on the same graph and draw any tangent-triangle whose hypotenuse is tangent to one of the curves and intercepts the x or y axis at integers -2, -1, 0, 1, 2,..

(b) As a roller-coaster car moves down a track $y=e^x$ it shines one laser beam along the track and another beam vertically down so both makes spots on baseline $y=0$. Find the distance between spots as function of x.

Solutions to Assignment 4

Properties of all-important parabolic PE functions

Ex. 1 A most important mechanics problems is that of atomic oscillators affected by electric fields since it is basic to all spectroscopy. A useful approximate model is potential $V^{atom}(x)=k\,x^2/2$ function of center x of charge Q where k is a spring constant of atomic polarizability. A uniform electric field E is assumed to apply a force $F=Q\cdot E$ to the charge by adding a potential $V^E(x)$ to $V^{atom}(x)$. (Give $V^E(x)=$______ and $F^E(x)=$______)

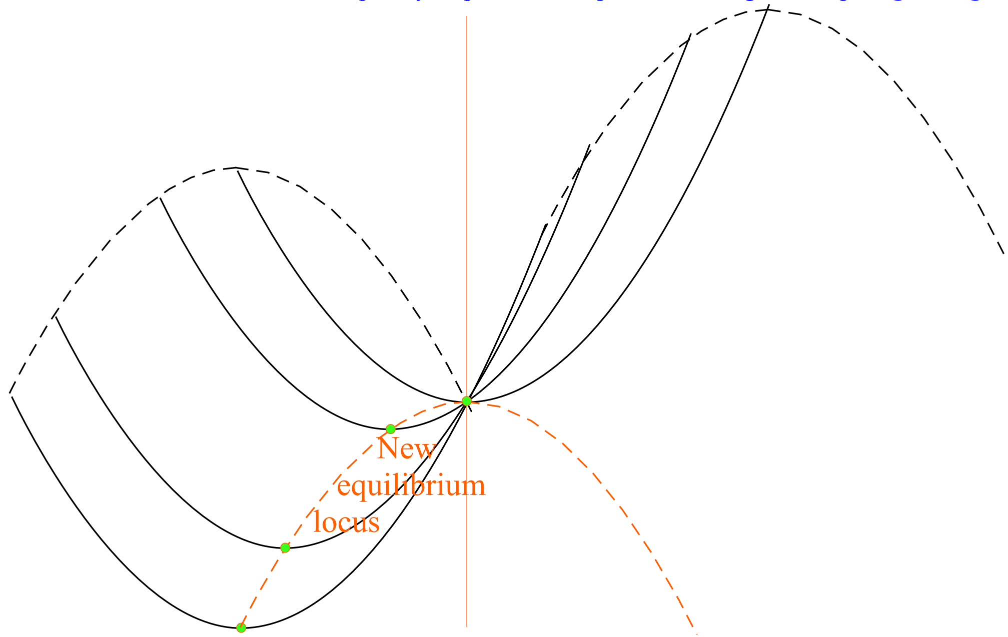
Consider the resulting potential $V^{total}(x)$ for an atom for unit constants $k=1$ and $Q=1$. Derive and plot the new values for equilibrium position $x^{equil}(E)$, energy $V^{equil}(E)$, dipole moment $p^{equil}(E)=Q\cdot x^{equil}$. Plot $V^{total}(x)$ for field values of $E=-3,-2,-1, 0, 1, 2, and\ 3$. Does oscillation frequency $\omega^{equil}(E)$ vary with field E? If so, how?

Ex.1 Parabolic potential changes due to uniform field F=*-const.* that slides $V(x)$ equilibrium point to the side by Δ and down by $-B\Delta^2$ in an Energy-*vs.*-x plot. The parabola rigidly follows an inverted copy of the original zero-Field potential $B\,x^2=(k/2)\,x^2$.

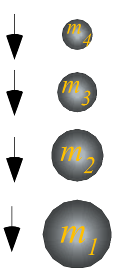
Adding $E{\cdot}x$ to $B\,x^2$ gives $V(x)=B\,x^2+E{\cdot}x$ that may be rewritten $V(x)=B(x+E/2B)^2-E^2/4B=B(x-\Delta)^2-B\Delta^2$.

That is just the same zero-Field parabola shape but it's x-shifted by $\Delta=-E/2B=-E/k$ and drops down by $-B\Delta^2=-(k/2)\Delta^2$.

Being the same parabola means it has the same frequency. Equilibrium dipole moment grows to $p=Q{\cdot}\Delta=-Q{\cdot}E/2B=-Q{\cdot}E/k$.



Superball tower IBM model constructions (Independent Bang Model with initial V_k=-1)



The 100% energy transfer limit

Ex.2 Suppose each m_k has just the right mass ratio m_k/m_{k+1} with the m_{k+1} above it to pass on all its energy to m_{k+1} so the top ball-N, a *1gm* pellet, goes off with the total energy. Construct velocity-velocity diagrams, indicate velocity at each stage, and derive the required intermediate mass values for (a) $N=2$, (b) $N=3$, (c) $N=4$.

(d) Give algebraic formula for this *Maximum Amplified Velocity* factor in terms of N ($MAV(N)=$____________?).

(e) Give algebraic formula neighbor-mass ratios $R=M_{N-1}/M_N$ in terms of N ($R(N)=$__________?).

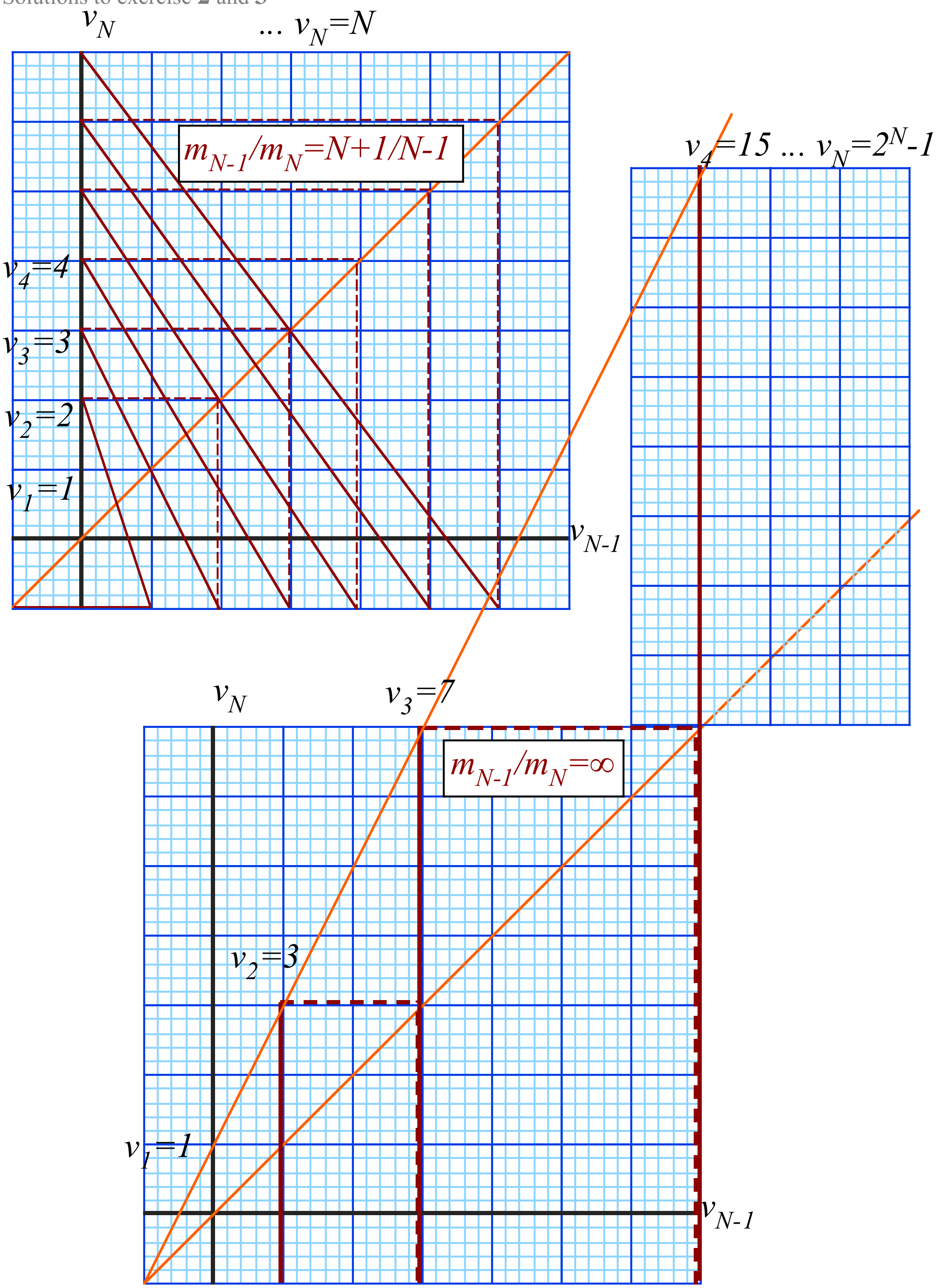
The towering limit

Ex.3 Suppose each m_k is very much larger than m_{k+1} above it so that final v_{k+1} approaches its upper limit. Then top m_N goes off with nearly the highest velocity v_N attainable. Construct the velocity-velocity diagrams. Indicate each intermediate velocity limit value at each stage and the limiting top value for (a) $N=2$, (b) $N=3$, (c) $N=4$.

(d) Give algebraic formula for *Absolute Maximum Amplified Velocity* factor in terms of N ($AMAV(N)=$____?).

Exercise Set 4

Solutions to exercise **2** and **3**



1st case shows *linear* series of final velocity. 2nd case shows *geometric* or *exponential* series of velocity.

(Solutions to Assignment 4) The optimum idler:

4(a) Find optimum mass m_2 to get highest final v_3 of mass m_3 in terms of masses m_1 and m_3.

Let $m_1 = M$, $m_2 = x$ and $m_3 = m$. Then use (5.1b): $\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in stages. 1st stage gives: $v_x^{FIN} = \frac{3M-x}{M+x}$

$$\begin{pmatrix} v_M^{FIN} \\ v_x^{FIN} \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-x & 2x \\ 2M & x-M \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-3x \\ 3M-x \end{pmatrix}$$

. The 2nd stage: $\begin{pmatrix} v_x^{FIN} \\ v_m^{FIN} \end{pmatrix} = \frac{1}{x+m}\begin{pmatrix} x-m & 2m \\ 2x & m-x \end{pmatrix}\begin{pmatrix} \frac{3M-x}{M+x} \\ -1 \end{pmatrix}$

The velocity v_m is to be maximized. (A quicker approach involving a frame-change in part (b) has less algebra.)

$$v_m^{FIN} = \frac{2x\frac{3M-x}{M+x} - (m-x)}{x+m} = \frac{6Mx - 2x^2 + (x-m)(M+x)}{(M+x)(x+m)} = \frac{-x^2 + (7M-m)x - mM}{x^2 + (M+m)x + mM} = \frac{N(x)}{D(x)}$$

Derivative $\frac{1}{D(x)}\frac{dN}{dx} - N(x)\frac{d}{dx}\frac{1}{D(x)} = \frac{D(x)\frac{dN(x)}{dx} - N(x)\frac{dD(x)}{dx}}{D(x)^2}$ is set to zero.

$$\left(x^2 + (M+m)x + mM\right)\left(-2x + (7M-m)\right) \qquad -\left(-x^2 + (7M-m)x - mM\right)\left(2x + (M+m)\right) = 0$$

	x^2	$+(M+m)x$	mM
$-2x$	$-2x^3$	$-2(M+m)x^2$	$-2mMx$
$(7M-m)$	$(7M-m)x^2$	$(7M-m)(M+m)x$	$(7M-m)mM$

	x^2	$-(7M-m)x$	mM
$2x$	$-2x^3$	$-2(7M-m)x^2$	$2mMx$
$(M+m)$	$(M+m)x^2$	$-(M+m)(7M-m)x$	$(M+m)mM$

Cancellations simplify it.

$$\left(x^2 + (M+m)x + mM\right)\left(-2x + (7M-m)\right) \qquad -\left(-x^2 + (7M-m)x - mM\right)\left(2x + (M+m)\right) = 0$$

	x^2	$+(M+m)x$	mM
$-2x$		$-2(M)x^2$	
$(7M-m)$	$(7M)x^2$		$(7M)mM$

	x^2	$-(7M-m)x$	mM
$2x$		$-2(7M)x^2$	
$(M+m)$	$(M)x^2$		$(M)mM$

Result is quadratic and not cubic equation: $-8Mx^2+8M^2m=0$ or $-x^2+Mm=0$.

The result is geometric mean! $x=\sqrt{(Mm)}$ or: $m_2=\sqrt{(m_1 m_3)}$. The resulting final velocity *(Not assigned)* is as follows:

$$v_m^{FIN} = \frac{-\sqrt{mM}^2 + (7M-m)\sqrt{mM} - mM}{\sqrt{mM}^2 + (M+m)\sqrt{mM} + mM} = \frac{-mM + (7M-m)\sqrt{mM} - mM}{mM + (M+m)\sqrt{mM} + mM} = \frac{(7M-m)\sqrt{mM} - 2mM}{(M+m)\sqrt{mM} + 2mM}$$

Xtra-Credit (Not assigned)

Now try more difficult problem for next stage where lowest mass is coming up with higher speed S but top one is still falling at speed -1.

Let $m_1 = M$, $m_2 = x$ and $m_3 = m$. Use (5.1b): $\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in stages. 1st stage gives: $v_x^{FIN} = \frac{3M-x}{M+x}$

$$\begin{pmatrix} v_M^{FIN} \\ v_x^{FIN} \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-x & 2x \\ 2M & x-M \end{pmatrix}\begin{pmatrix} S \\ -1 \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} SM-(S+2)x \\ (2S+1)M-x \end{pmatrix}$$

. 2nd stage: $\begin{pmatrix} v_x^{FIN} \\ v_m^{FIN} \end{pmatrix} = \frac{1}{x+m}\begin{pmatrix} x-m & 2m \\ 2x & m-x \end{pmatrix}\begin{pmatrix} \frac{3M-x}{M+x} \\ -1 \end{pmatrix}$

Again, velocity v_m could be maximized but algebra is complicated. Consider instead simpler algebra of Solution to 4(b) that follows.

(Solutions to Assignment 4 contd) The optimum idler:

4(b) Find optimum mass m_2 to get highest final v_3 of mass m_3 in terms of masses m_1 and m_3 assuming a frame that has zero initial velocity $0=v_2^{IN}=v_3^{IN}=v_4^{IN}=\ldots$ for all "falling" balls m_2, m_3, m_4,…except m_1 that has $v_1^{IN}=v_0=2$.

Let $m_1=M$, $m_2=x$ and $m_3=m$. Then use (5.1b): $\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$ in stages. 1st stage gives:

$$\begin{pmatrix} v_M^{FIN} \\ v_x^{FIN} \end{pmatrix} = \frac{1}{M+x}\begin{pmatrix} M-x & 2x \\ 2M & x-M \end{pmatrix}\begin{pmatrix} v_0=2 \\ 0 \end{pmatrix} = \frac{v_0}{M+x}\begin{pmatrix} M-x \\ 2M \end{pmatrix}$$

. The 2nd stage:

$$\begin{pmatrix} v_x^{FIN} \\ v_m^{FIN} \end{pmatrix} = \frac{1}{x+m}\begin{pmatrix} x-m & 2m \\ 2x & m-x \end{pmatrix}\begin{pmatrix} \frac{2Mv_0}{M+x} \\ 0 \end{pmatrix}$$

The velocity v_m^{FIN} is to be maximized by setting x-derivative to zero

$$v_m^{FIN} = \frac{2x\frac{2Mv_0}{M+x}+0}{x+m} = \frac{4Mv_0 x}{(M+x)(x+m)} = \frac{4Mv_0 x}{x^2+(M+m)x+Mm} \text{ has max } v_m^{FIN} \text{ if: } 0 = \frac{d}{dx}\frac{x}{x^2+(M+m)x+Mm}$$

$$\text{solving: } 0 = \frac{d}{dx}\frac{x}{x^2+(M+m)x+Mm} = \frac{1}{x^2+(M+m)x+Mm} - \frac{x(2x+M+m)}{(x^2+(M+m)x+Mm)^2}$$

$$\text{reducing: } 0 = x^2+(M+m)x+Mm - x(2x+M+m) = -x^2+Mm$$

This gives optimal middle mass m_2 to be *geometric mean* of m_1 and m_3: $x=m_2=\sqrt{Mm}=\sqrt{m_1 m_3}=\sqrt{4\cdot 1}=2$

4(c) VV graphs shown on following page

It should be noted that the momentum line slopes for the optimal pair of IBM collisions are equal.

$$\text{slope(1:2)} \quad \frac{m_1}{m_2} = \frac{m_1}{\sqrt{m_1 m_3}} = \frac{\sqrt{m_1}}{\sqrt{m_3}} \quad \text{equals the slope(2:3)} \quad \frac{m_2}{m_3} = \frac{\sqrt{m_1 m_3}}{m_3} = \frac{\sqrt{m_1}}{\sqrt{m_3}}$$

4(d) The resulting maximum velocity of the top mass $m_3=m$ is found by substitution of x value.

$$v_m^{FIN} = \frac{4v_0 Mx}{x^2+(M+m)x+Mm} = \frac{4v_0 M\sqrt{Mm}}{Mm+(M+m)\sqrt{Mm}+Mm} = \frac{4v_0 M\sqrt{Mm}}{2Mm+(M+m)\sqrt{Mm}} = \frac{8m_1\sqrt{m_1 m_3}}{2m_1 m_3+(m_1+m_3)\sqrt{m_1 m_3}} = \frac{8m_1 m_2}{2m_1 m_3+m_1 m_2+m_2 m_3}$$

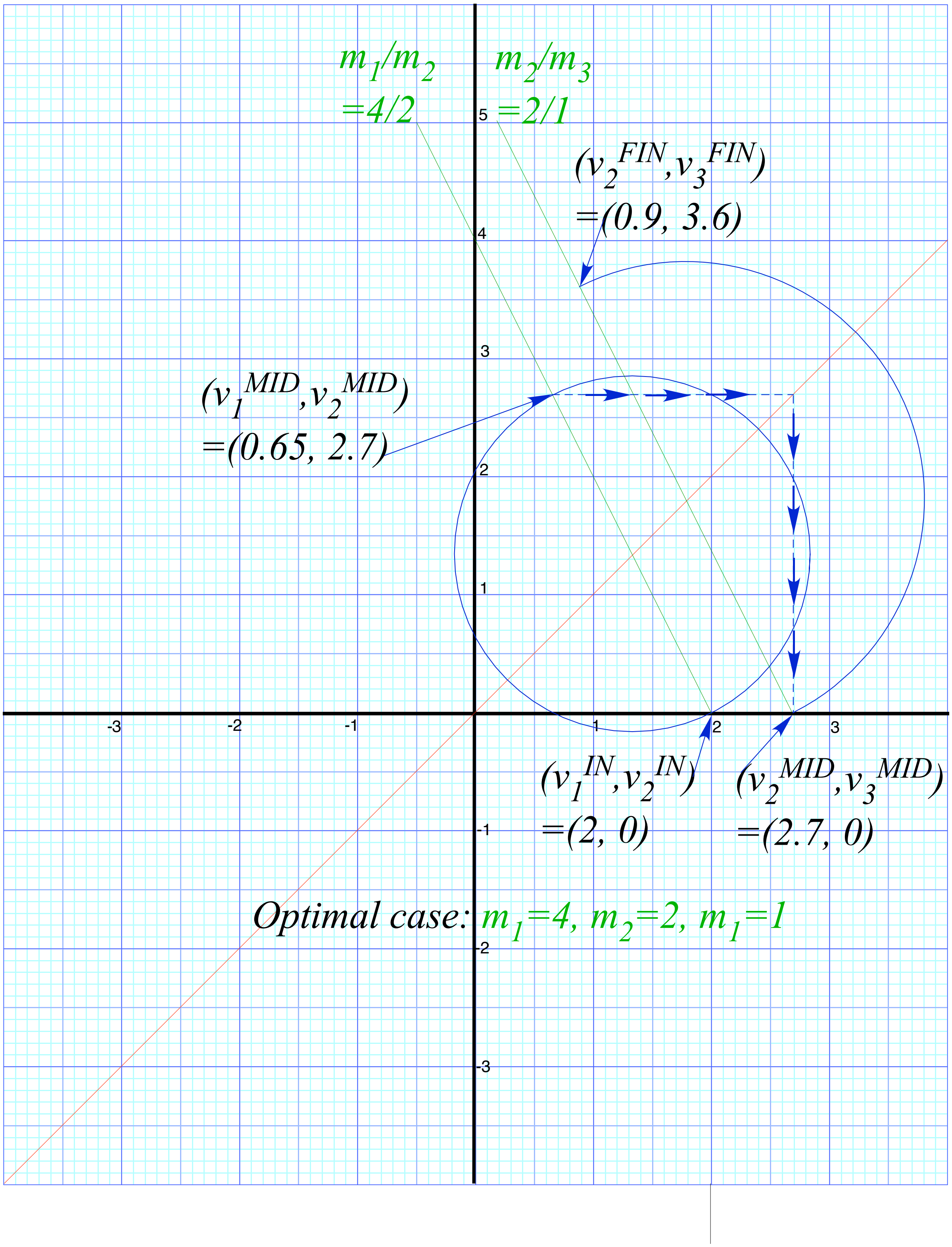
For case $m_1=4$ and $m_3=1$ we get $m_2=\sqrt{4\cdot 1}=2$ with optimal speed $v_m^{FIN}=64/18=3.56$ consistent with figure below.

Note: This is in frame with $v_1^{IN}=2$ and $0=v_2^{IN}=v_3^{IN}=v_4^{IN}=\ldots$. In IBM lab frame: $v_1^{IN}=1$ and $-1=v_2^{IN}=v_3^{IN}=v_4^{IN}=\ldots$

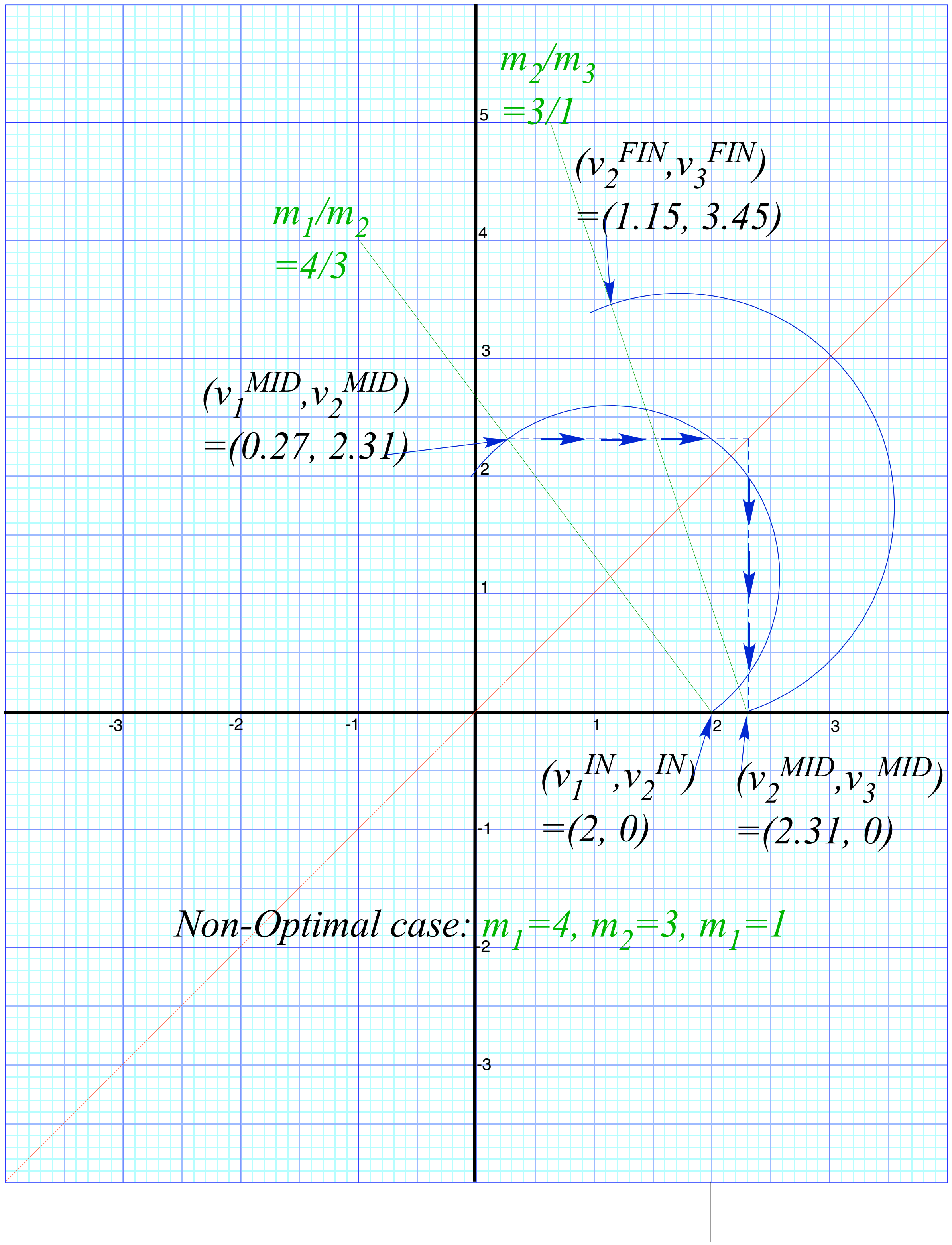
Thus we need to subtract 1.0 to get $v_m^{FIN(Lab)}=2.56$ and slide the graph down 45° line by 1.0 unit. Then the result matches the formula given by **4**(a).

The optimal sequence may be continued to a 5-mass tower by choosing m_5 arbitrarily and making $m_4=\sqrt{m_3 m_5}$. However, having uniform slopes appears to be the optimal overall strategy. Picking that value is an open problem for ball towers of 4 or greater since the IBM approximation degrades potential details become important..

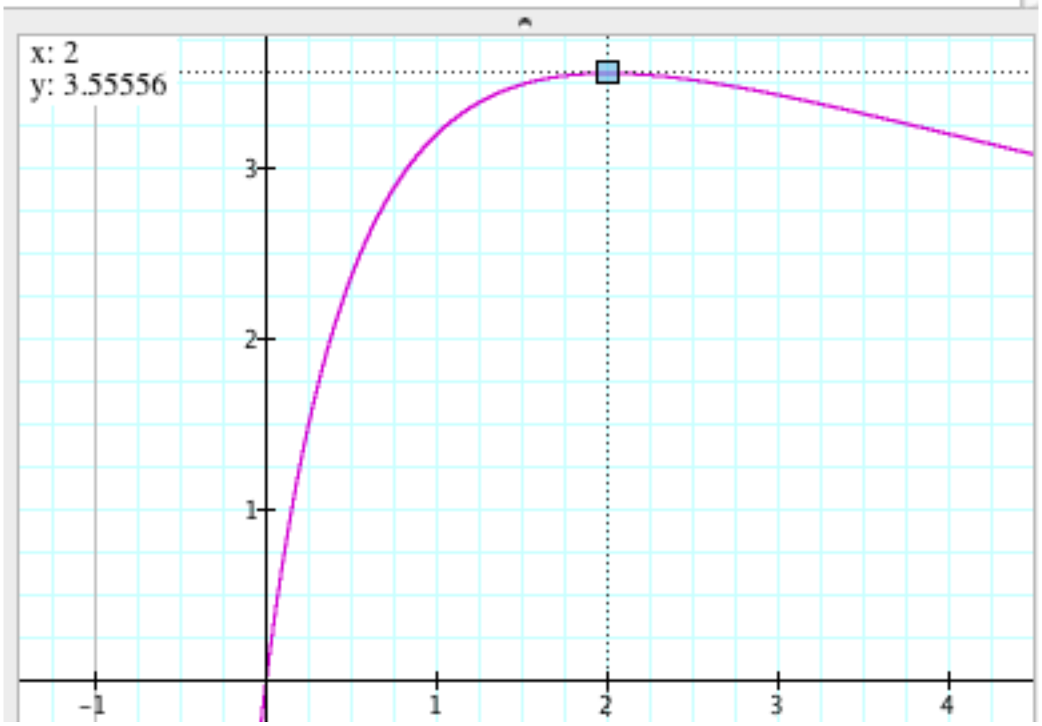
Ex.4(c)Optimal case VV-graph shows final velocity V_3=3.65:



Ex.4(c) Non-Optimal case VV-graph shows smaller final velocity V_3=3.45 than Optimal 3.65.:



$$y = 32\frac{x}{(x+4)\,(x+1)}$$



(Solutions to Assignment 4 contd) The backsides of exponentials

(b) Plot exponential $y=e^x$ and $y=\log_e x$ functions on same graph and draw tangent-triangle whose hypotenuse is tangent to a curves and intercepts x or y axes at -2, -1, 0, 1, 2,.. Give the base and altitude coordinates of the tangent point in each case.

Note $y=e^x$ tangents at x=integer-N intercept x-axis at integer-(N-1). Distance between vertical spot and tangent spot is always 1.0.