

Lecture 28

Mon. 11.26.2018

Multi-particle orbits and rotating body dynamics

(Ch. 2-7 of Unit 6 12.07.17)

2-Particle orbits

Ptolemetric or LAB view and reduced mass

Copernican or COM view and reduced coupling

2-Particle orbits and scattering: LAB-vs.-COM frame views

Ruler & compass construction (or not)

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity ω -ellipsoid vs. angular momentum \mathbf{L} -ellipsoid

Lagrangian ω -equations vs. Hamiltonian momentum \mathbf{L} -equation

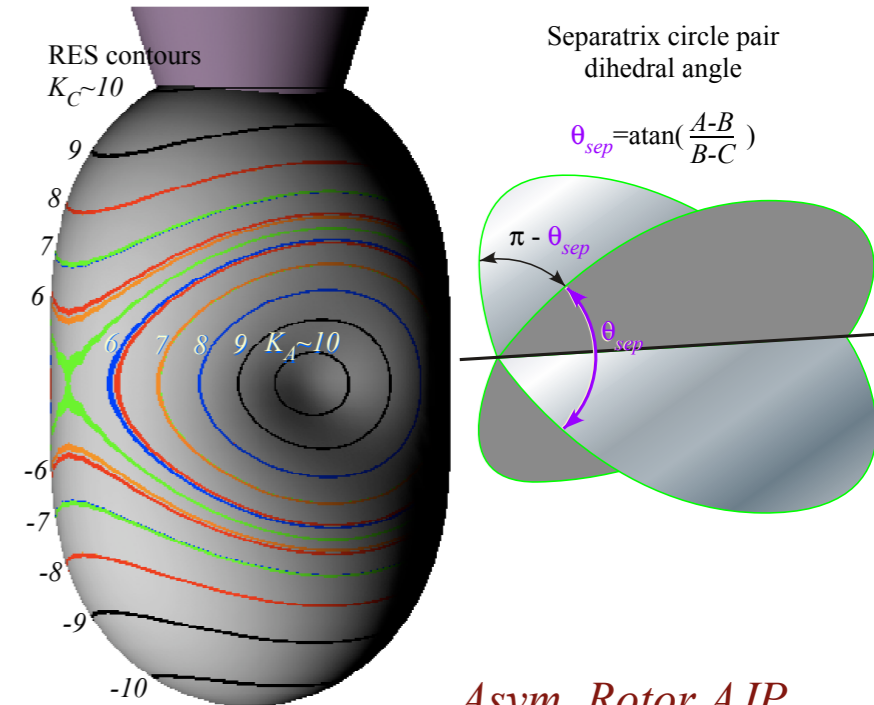
Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)

Symmetric, asymmetric, and spherical-top dynamics (Constant \mathbf{L})

BOD-frame cone rolling on LAB frame cone

Deformable spherical rotor RES and semi-classical rotational states and spectra

Cycloidal geometry of flying levers and Practical poolhall application



*Asym. Rotor AJP
44,11 1976*



A running collection of links to course-relevant sites and articles

Physics Web Resources

[Comprehensive Harter-Soft Resource Listing](#)

[UAF Physics YouTube channel](#)

[LearnIt Physics Web Applications](#)

Neat external material to start the class:

[AIP publications](#)

[AJP article on superball dynamics](#)

[AAPT summer reading](#)

These are hot off the presses:

[Sorting ultracold atoms in a 3D optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018](#)

[Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018](#)

Slightly Older ones:

[Wave-particle duality of C60 molecules](#)

[Optical vortex knots – One Photon at a Time](#)

Older Links from Lectures 14-20

<http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html>

<http://www.sussexcountyonline.com/news/photos/punkinchunkin.html>

[Shooting-range-for-medieval-siege-weapons-Anybody-knows](#)

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge>

<https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth>

[The trebuchet, Chevedden, Sci Am 1995](#)

'Simple' Pendulum Sim: <https://modphys.hosted.uark.edu/markup/PendulumWeb.html>

'Cycloid' Pendulum: <https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html>

Google search on: "[Satelite view of Patricia](#)" (Images)

[Physics Girl Channel - Fun with Vortex Rings in the Pool](#)

[iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDe](#)

<https://modphys.hosted.uark.edu/markup/CoulltWeb.html?scenario=SynchrotronMotion>

<https://modphys.hosted.uark.edu/markup/CoulltWeb.html?scenario=SynchrotronMotion2>

Mechanical Analog to EM Motion (YouTube video) - <https://youtu.be/hTd5FTJ-vRk>

Coullt Web Simulation: [Bound-state motion in parabolic coordinates](#)

Coullt Web Simulation: [Bound-state motion in hyperbolic coordinates](#)

Oscillt Web App: Simulations of various types of resonance: [18](#), [27](#), [31](#), [35](#), [38](#), [39](#)

[Smith Chart](#)

<http://nobelprize.org/>

Analyt Web Application, posted 10/22/2018 in our *testing area*:

<https://modphys.hosted.uark.edu/testing/markup/AnalytBJS.html>

"Texts"

[Classical Mechanics with a Bang!](#)

[Quantum Theory for the Computer Age](#)

[Principles of Symmetry, Dynamics, and Spectroscopy](#)

[Modern Physics and its Classical Foundations](#)

[AMOP Ch 0 Space-Time Symmetry - 2019](#)

[Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018](#)

[Springer AMO Handbook - Ch 32 - Harter-Reimer-2019](#)

"Relawavity" and quantum basis of *Lagrangian & Hamiltonian* mechanics:

[2-CW laser wave - Bohrlt Web App](#)

[Lagrangian vs Hamiltonian - RelaWavity Web App](#)

Classes

[2014 AMOP](#)

[2017 Group Theory for QM](#)

[2018 AMOP](#)

[2018 Adv Mechanics](#)

Older Links from Lectures 21-23

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: [5](#), [61](#)

Boxlt Web Simulations

[Pure A-Type w/Cosine](#)

[Pure B-Type w/Cosine](#)

[Pure B-Type w/Freq ratios](#)

[Mixed AB-Type 2:1 Freq ratio](#)

[Pure A-Type A=4.9, B=0, C=0, & D=4.0](#)

[Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0](#)

[Pure C-Type A,D=4.055, B=0, C=0.1](#)

[Mixed AB-Type w/Cosine](#)

[Mixed AB Type A=4.0, BU2=0.866..., CU2=0, & D=1.0 w/Stokes & Freq rats](#)

[Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot](#)

[Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot](#)

[Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot](#)

Classical Mechanics with a Bang! 2018

Lectures [8](#), [9](#), [23](#) page [93](#)

Text [Unit 6](#), page=[27](#)

ColorU2 for the Web - in development

Group Theory for Quantum Mechanics - 2017 Lectures: [6](#), [7](#), [8](#), and the combined [9-10](#)

Quantum Theory for the Computer Age [Unit 3 Ch.7-10](#), page=[90](#)

Web based 3D & XR ($x \in \{A, M, V\}$, R=Reality) <https://www.babylonjs.com/>

Web based 3D graphics [WebGL API](#) (Graphics Layer modeled after OpenGL)

[Wiki on Pafnuty Chebyshev](#)

continued ↘

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“Texts”

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[Quantum Theory for the Computer Age](#)

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[Modern Physics and its Classical Foundations](#)

Classes

[2014 AMOP](#)

[2017 Group Theory for QM](#)

[2018 AMOP](#)

[2018 Adv Mechanics](#)


Repeated from previous page

Older Links from Lectures 24-27

[JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap\(1\)](#)

[MolVibes Web App: C3vN3](#)

[Wavelt Web App:](#)

[Dim = 3 w/Wave Components;](#)

[Static Char Table: 6, 12, 12\(b\), 16, 36, 256](#)

[Quantum Carpet with N=20: Gaussian, Boxcar](#)

[Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015](#)

[QTCA Unit 5 Ch14 2013](#)

[Lester. R. Ford, Am. Math. Monthly 45,586\(1938\)](#)

[John Farey, Phil. Mag.\(1816\) Wolfram](#)

[Harter, J. Mol. Spec. 210, 166-182 \(2001\)](#)

[Harter, Li IMSS \(2013\)](#)

[Li, Harter, Chem.Phys.Letters \(2015\)](#)

[OscillatorPE Web App: IHO Scenario 2, Coulomb Scenario 3](#)

[RelaWavity Web App/Simulator/Calculator: Elliptical - IHO orbits](#)

[Coullt Web App Simulations: p19, p32, p72, p73, p92, R=-0.375, R=+0.5, Rutherford](#)

Links to supplement Lecture 28

[CMwBang Text 2012 Unit 6 page=5](#)

[Bouncelt Web App/Scenarios:](#)

[5002, 5003](#)

[Coullt Web App/Scenarios:](#)

[TwoParticleCollision_LToR, TwoParticleCollision_LToR_CM, TwoParticleOrbit_Coulomb,](#)

[TwoParticleOrbit_Coulomb_CM, TwoParticleOrbit_Hooke, TwoParticleOrbit_Hooke_CM](#)

[Singular Motion of Asymmetric Rotators AJP 44, 11 p1080 Harter-Kim-1976](#)

[Molecular Eigensolution Symmetry Analysis and Fine Structure - Int.J.MolSci1.4.13 Harter-Mitchell-IJMS-2013](#)

[Lenz Vector and Orbital Analog Computers - AJP 44 p348 1976](#)

[Some Geometric Aspects of Classical Coulomb Scattering AJP 40 4 p1852 1972](#)

[How Molecules do Self-NMR - Harter-Mitchell-Columbus-2009](#)

[Classical Mechanics with a Bang! - Asymmetric Top Demo](#)

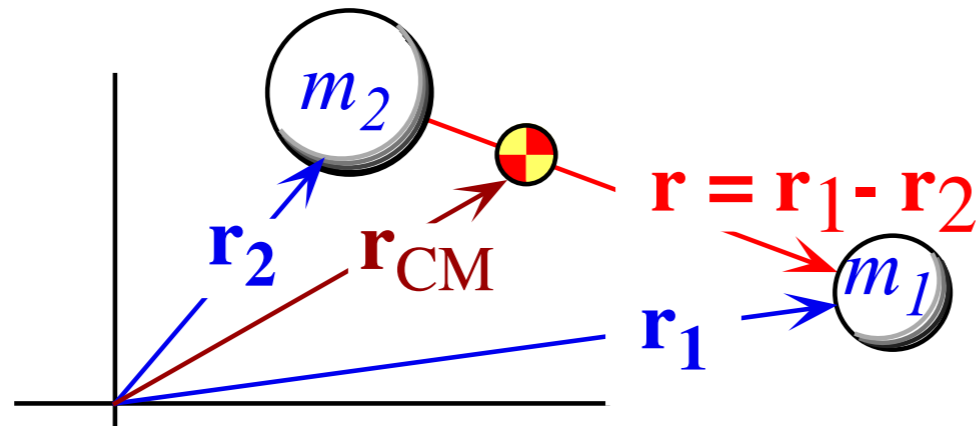
[Allbookstores.com - Compare for Heller's SemiClassical Way - 0691163731](#)

["My Bomerang Won't Come Back" \(YouTube: Playlist\)](#)

[Rotating Solid Bodies in Microgravity \(YouTube\)](#)

[Dancing T-handle in zero-g \(YouTube\)](#)

2-Particle orbits and center-of-mass (CM) coordinate frame



$$\mathbf{r}_{\text{CM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

Defining *relative coordinate vector*

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and *mass-weighted-average* or *center-of-mass coordinate vector* \mathbf{r}_{CM}

$$\bar{\mathbf{r}} = \mathbf{r}_{\text{CM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

The inverse coordinate transformation.

$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2 \mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1 \mathbf{r}}{m_1 + m_2}$$

2-Particle orbits

➔ *Ptolemaic or LAB view and reduced mass*
Copernican or COM view and reduced coupling

Reduced mass: Ptolemaic views

Radial inter-particle force \mathbf{F}_{12} is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = F(r)\mathbf{e}_r = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

\mathbf{F}_{12} acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

Re-scaled force: A Copernican view

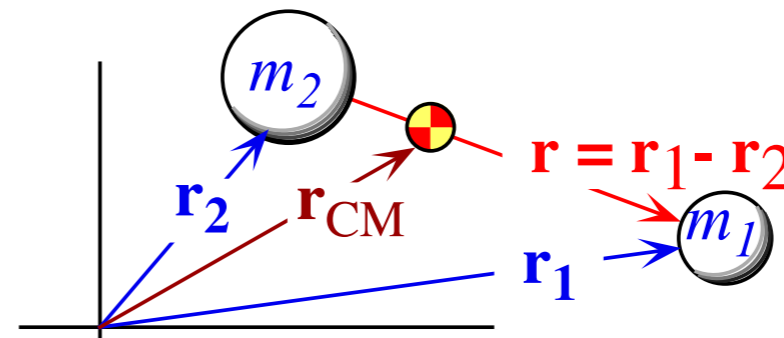
relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}$$

$$\mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$



$$\mathbf{r}_{\text{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Reduced mass: Ptolemaic views

Radial inter-particle force \mathbf{F}_{12} is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

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Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

Sum $\mathbf{F}_{12} + \mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\text{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Re-scaled force: A Copernican view

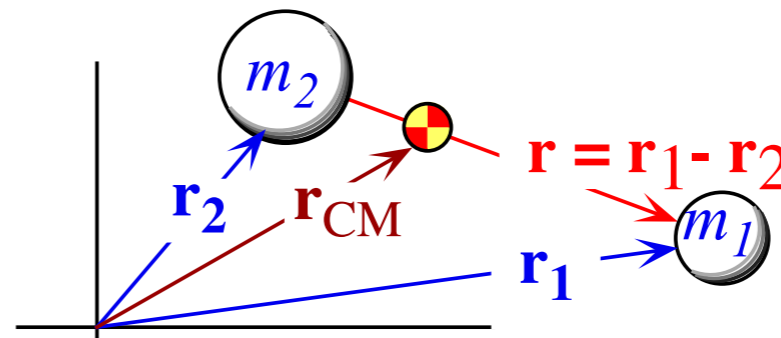
relative radius vector

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Reduced mass: Ptolemaic views

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Sum $\mathbf{F}_{12} + \mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\text{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12} - \mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using **reduced mass:** $\mu = \frac{m_2 m_1}{m_1 + m_2}$ $\ddot{\mathbf{r}}_{\text{CM}} = \mathbf{0}$

Re-scaled force: A Copernican view

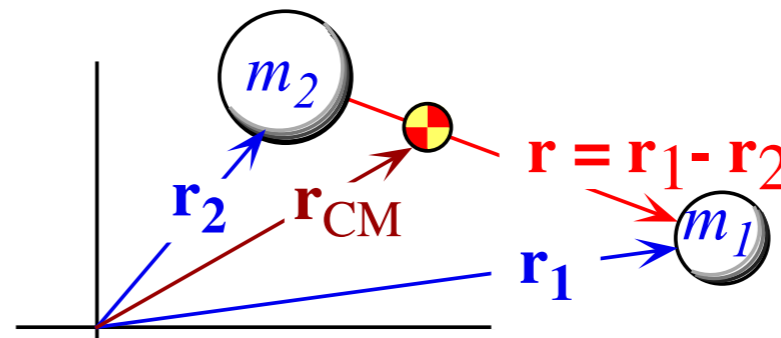
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$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$



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$$[m_1\ddot{\mathbf{r}}_1] - [m_2\ddot{\mathbf{r}}_2] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

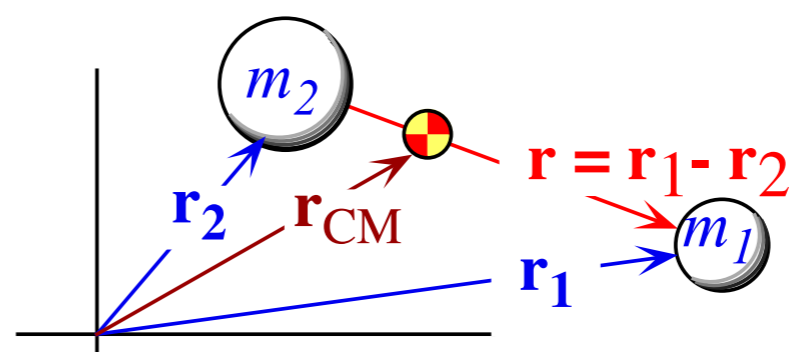
Re-scaled force: A Copernican view

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \quad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$



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$$(m_1 + m_2)\ddot{\mathbf{r}}_{\text{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

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$$[m_1\ddot{\mathbf{r}}_1] - [m_2\ddot{\mathbf{r}}_2] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\left[m_1\ddot{\mathbf{r}}_{\text{CM}} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2} \right] - \left[m_2\ddot{\mathbf{r}}_{\text{CM}} - \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2} \right] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e}_r = \mathbf{F}(r)$

Re-scaled force: A Copernican view

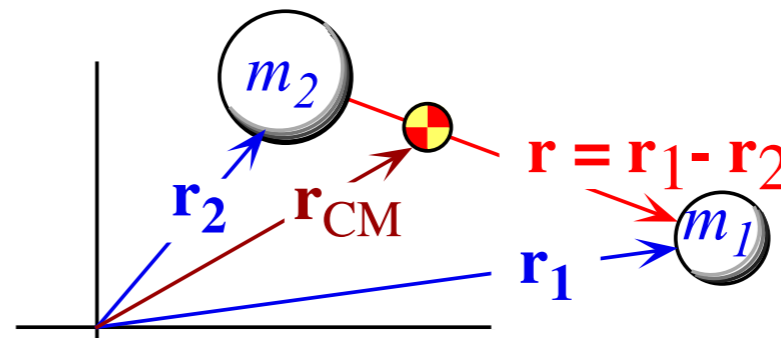
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$$\mathbf{r}_2 = \frac{-m_1 \mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2} \mathbf{r}$$

$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2 \mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1 \mathbf{r}}{m_1 + m_2}$$



$$\mathbf{r}_{\text{CM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

Reduced mass: Ptolemaic views

Radial inter-particle force \mathbf{F}_{12} is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

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Sum $\mathbf{F}_{12} + \mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

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Difference $\mathbf{F}_{12} - \mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = F(r)$ using **reduced mass:** $\mu = \frac{m_2 m_1}{m_1 + m_2}$ and: $\ddot{\mathbf{r}}_{\text{CM}} = \mathbf{0}$

$$[m_1\ddot{\mathbf{r}}_1] - [m_2\ddot{\mathbf{r}}_2] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\left[\cancel{m_1\ddot{\mathbf{r}}_{\text{CM}}} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2} \right] - \left[\cancel{m_2\ddot{\mathbf{r}}_{\text{CM}}} - \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2} \right] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\frac{2m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2} = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e}_r = \mathbf{F}(r)$

Re-scaled force: A Copernican view

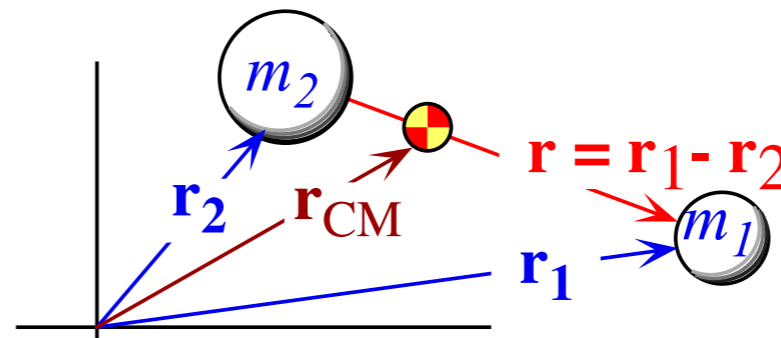
relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \frac{m_2 \mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1} \mathbf{r}$$

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$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2 \mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1 \mathbf{r}}{m_1 + m_2}$$



$$\mathbf{r}_{\text{CM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

Reduced mass: Ptolemaic views

Radial inter-particle force \mathbf{F}_{12} is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = F(r)\mathbf{e}_r = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

\mathbf{F}_{12} acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

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Sum $\mathbf{F}_{12} + \mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\text{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12} - \mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = F(r)$ using **reduced mass: $\mu = \frac{m_2 m_1}{m_1 + m_2}$** and: $\ddot{\mathbf{r}}_{\text{CM}} = \mathbf{0}$

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(Why it's reduced)

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Re-scaled force: A Copernican view

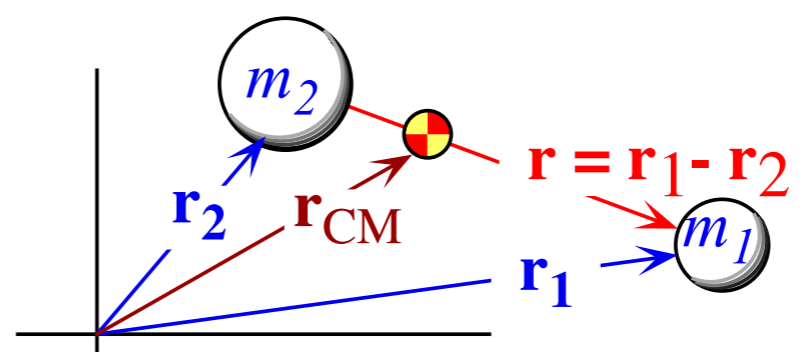
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2-Particle orbits

Ptolemaic view and reduced mass

➔ *Copernican view and reduced coupling*

Reduced mass: Ptolemaic views

Radial inter-particle force \mathbf{F}_{12} is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

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\mathbf{F}_{12} acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ $\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$

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(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1 \left(1 - \frac{m_1}{m_2} \dots \right) \quad (m_2 \gg m_1)$$

Re-scaled force: A Copernican view

relative radius vector

$$\frac{m_1}{\mu} \mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu} \mathbf{r}_2$$

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(Here we get "reduced" coupling constants)

each particle keeps its original mass m_1 or m_2 , but feels

coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

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Re-scaled force: A Copernican view

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$$F(r) = \frac{k}{r^2} \text{ becomes: } F\left(\frac{m_1}{\mu} r_1\right) = \frac{\mu^2}{m_1^2} \frac{k}{r_1^2}$$

(Coulomb)

$$k \rightarrow k_1 = k \mu^2 / m_1^2, \quad k \rightarrow k_2 = k \mu^2 / m_2^2$$

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Re-scaled force: A Copernican view

relative radius vector

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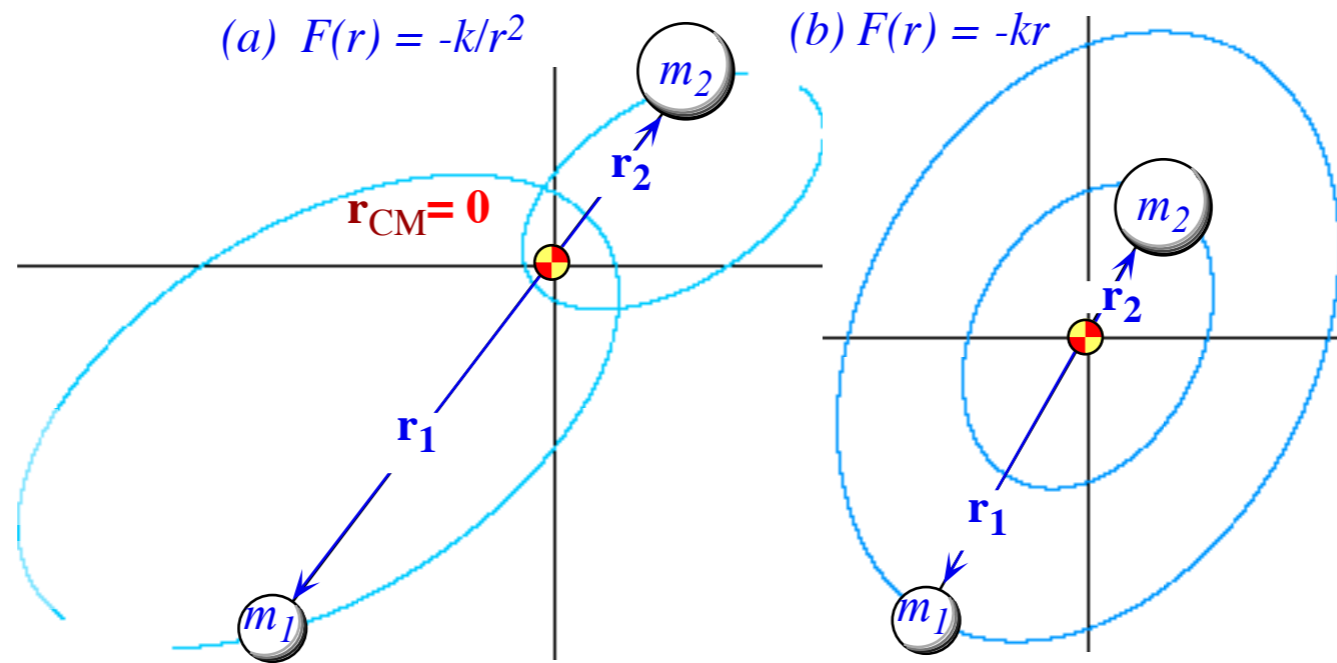
$$F(r) = -kr \text{ becomes: } F\left(\frac{m_1}{\mu} r_1\right) = -\frac{m_1}{\mu} k r_1$$

(Harmonic Oscillator)

$$k \rightarrow k_1 = k m_1 / \mu, \quad k \rightarrow k_2 = k m_2 / \mu$$

2-Particle orbits and scattering: LAB-vs.-COM frame views
Ruler & compass construction (or not)

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



[CoulIt Web Simulations](#)
[Coulombic Orbit \(CM Frame\)](#)
[Coulombic Orbit \(Lab Frame\)](#)

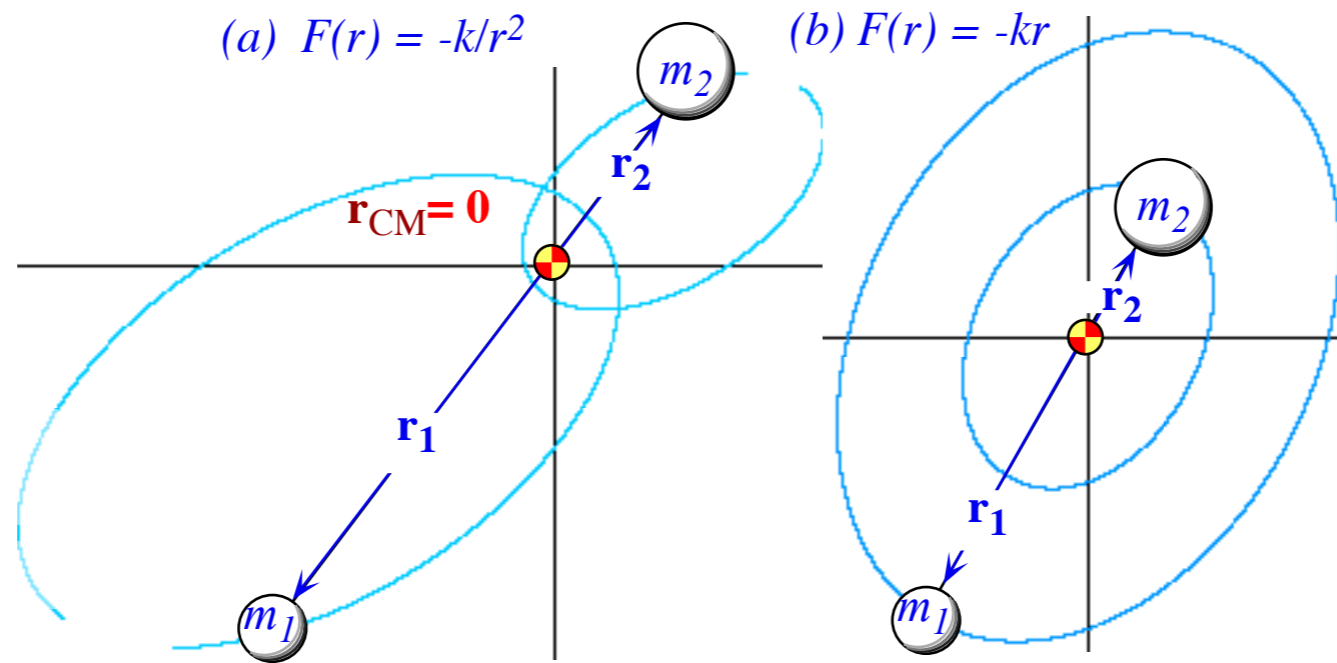
[CoulIt Web Simulations](#)
[Hooke Orbit \(CM Frame\)](#)
[Hooke Orbit \(Lab Frame\)](#)

Two particles are in synchronous motion around fixed CM origin.

Orbit periods are identical to each other.

Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.

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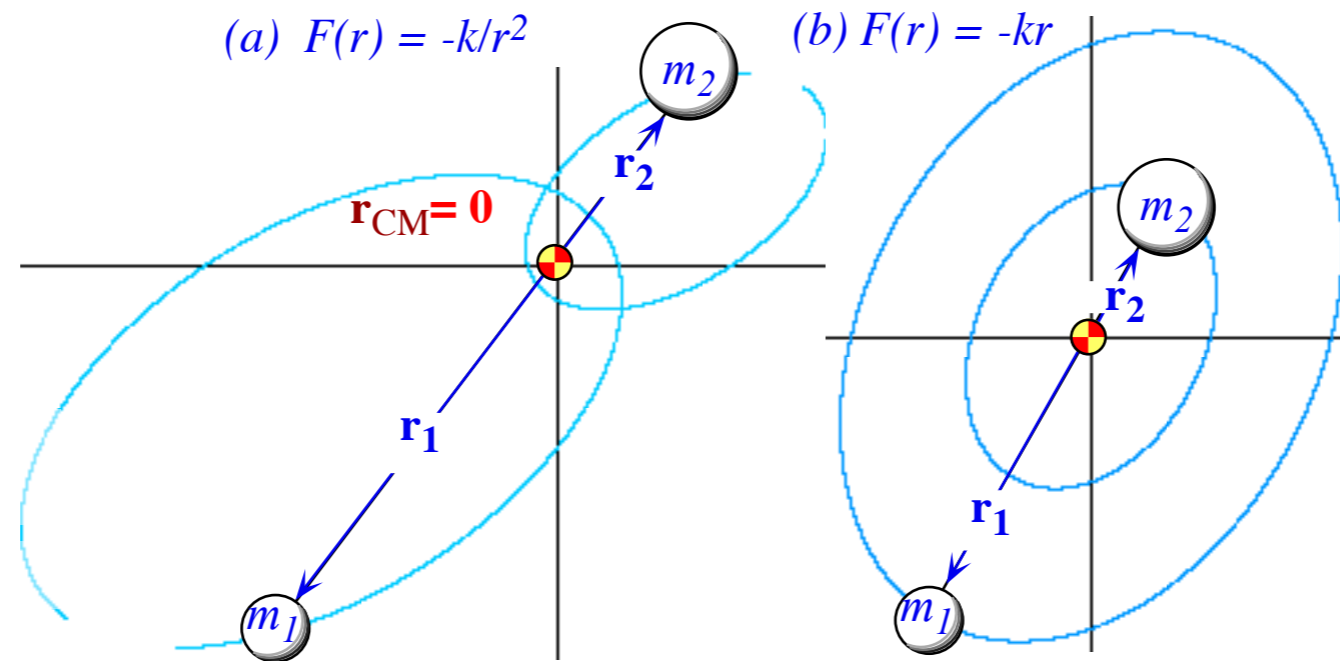
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Orbits differ in size of axes (a_1, b_1) and (a_2, b_2)

Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).

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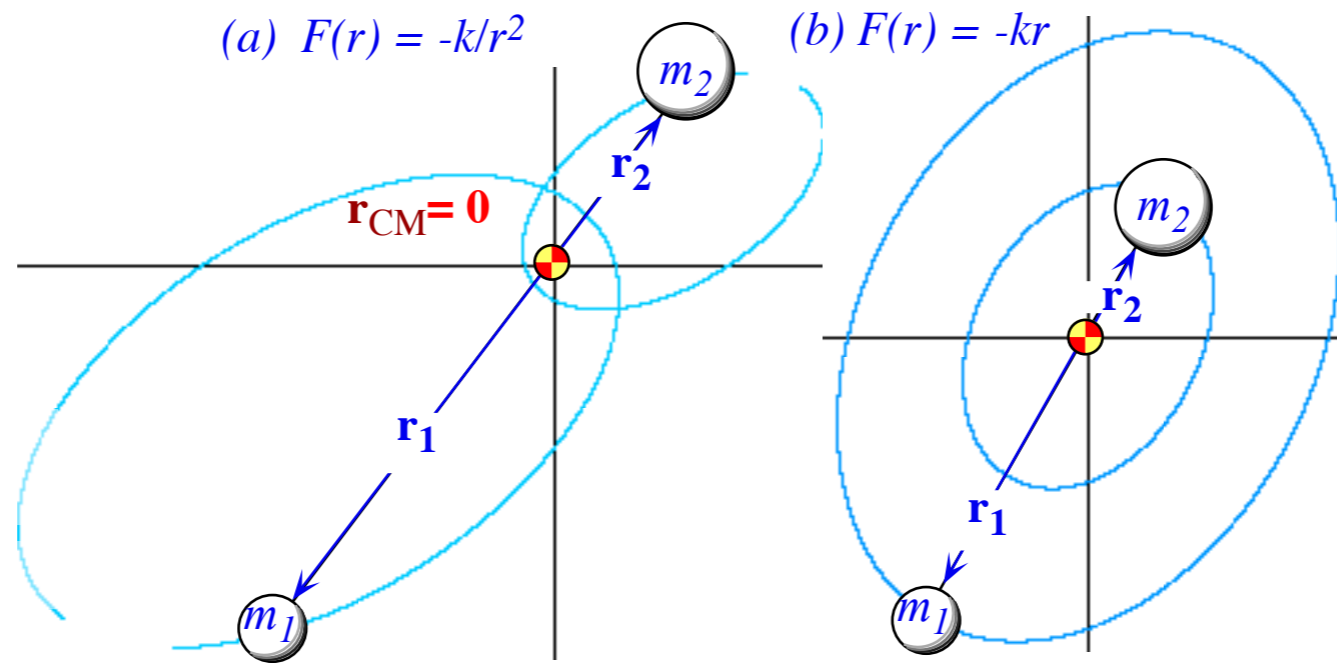
Orbit axial dimensions (a_k, b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a \mu,$$

$$b_1 m_1 = b_2 m_2 = b \mu$$

$$\lambda_1 m_1 = \lambda_2 m_2 = \lambda \mu$$

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Harmonic oscillator periods

and Coulomb orbit periods

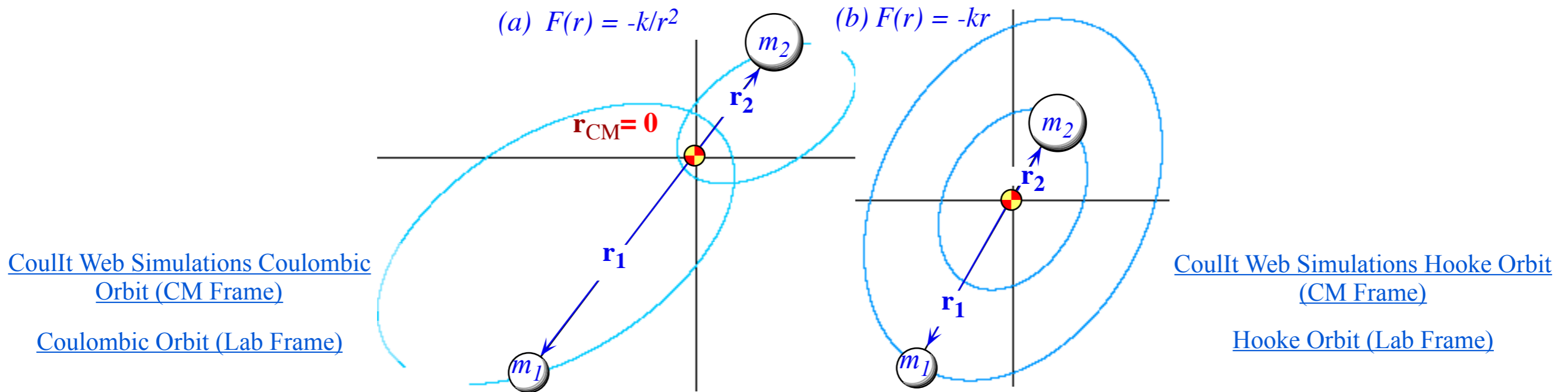
and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}}$$

$$T_{Coul} = 2\pi\sqrt{\frac{\mu a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1^3}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2^3}{k_2}}$$

$$\epsilon_1 = \epsilon_2 = \epsilon$$

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



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$$a_1 m_1 = a_2 m_2 = a \mu, \quad b_1 m_1 = b_2 m_2 = b \mu, \quad \lambda_1 m_1 = \lambda_2 m_2 = \lambda \mu$$

Harmonic oscillator periods and Coulomb orbit periods and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}} \quad T_{Coul} = 2\pi\sqrt{\frac{\mu a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1^3}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2^3}{k_2}} \quad \epsilon_1 = \epsilon_2 = \epsilon$$

Three Coulomb orbit energy values satisfy the same proportion relation as their axes

$$E_1 m_1 = E_2 m_2 = E \mu, \quad \text{where: } |E_1| = \frac{|k_1|}{2a_1}, \quad |E_2| = \frac{|k_2|}{2a_2}, \quad |E| = \frac{|k|}{2a}.$$

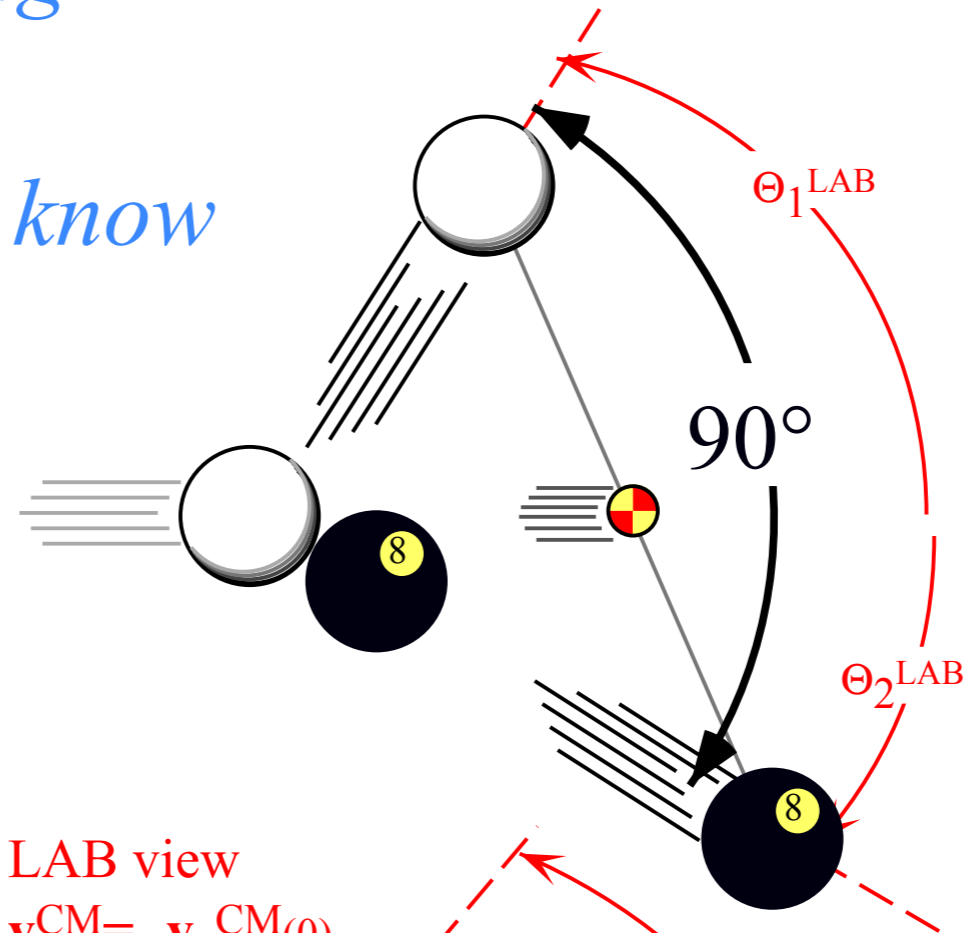
Energy values and axes satisfy similar sum relations

$$E_1 + E_2 = \frac{m_1}{\mu} E + \frac{m_2}{\mu} E = E, \quad \text{and: } a_1 + a_2 = \frac{m_1}{\mu} a + \frac{m_2}{\mu} a = a$$

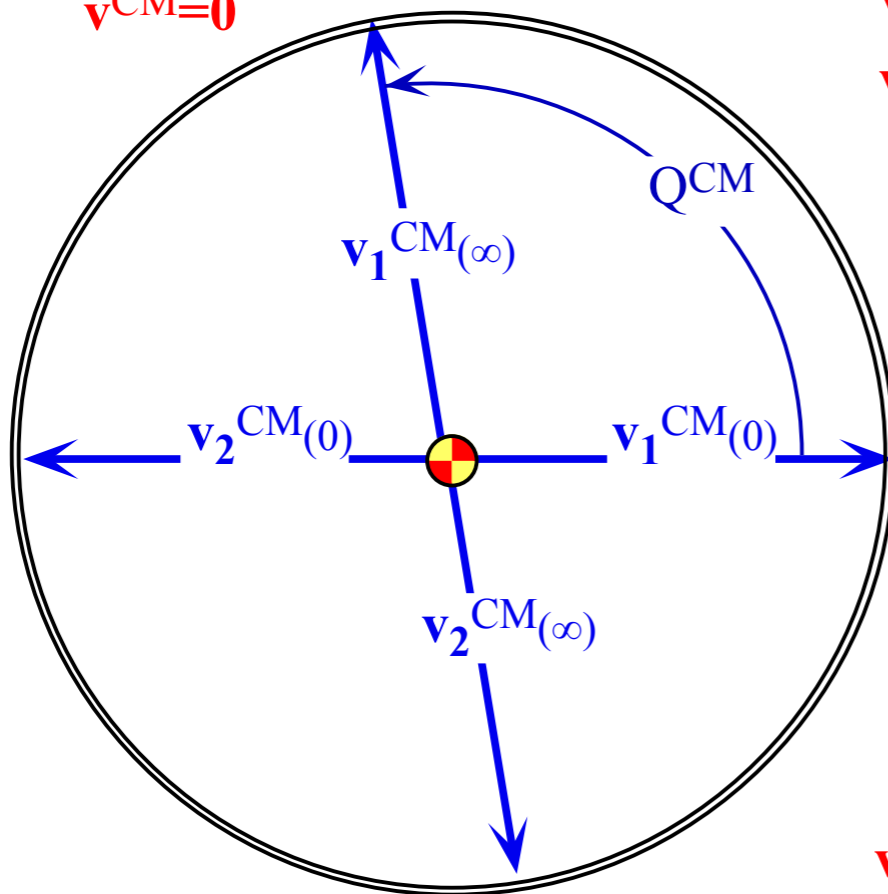
A common type of scattering

$(m_1=m_2)$

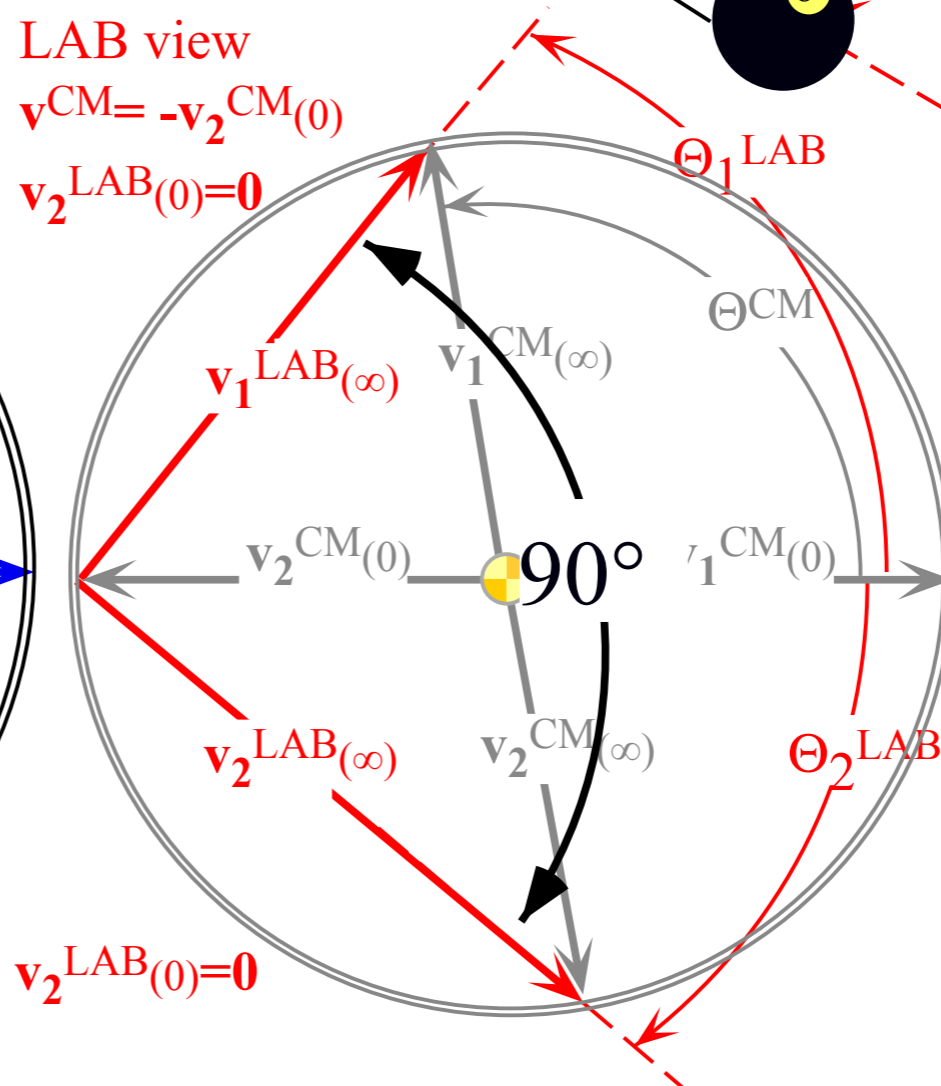
...that every pool shark should know



CM view
 $v^{CM}=0$



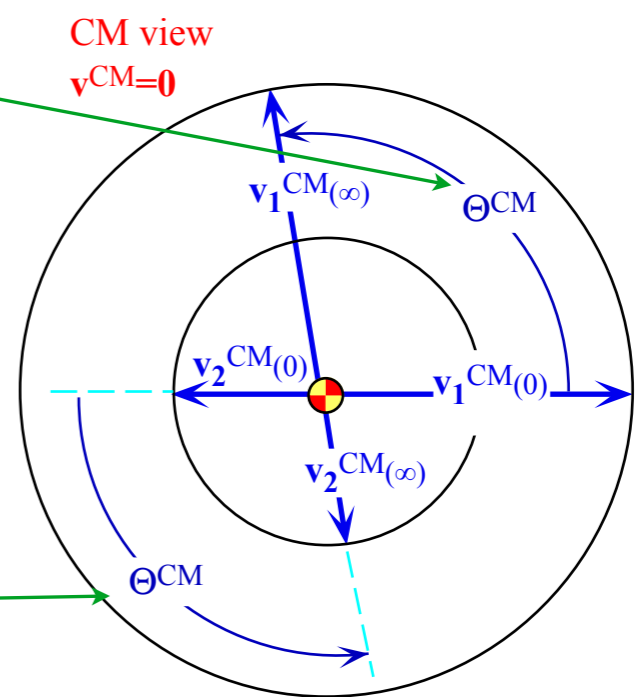
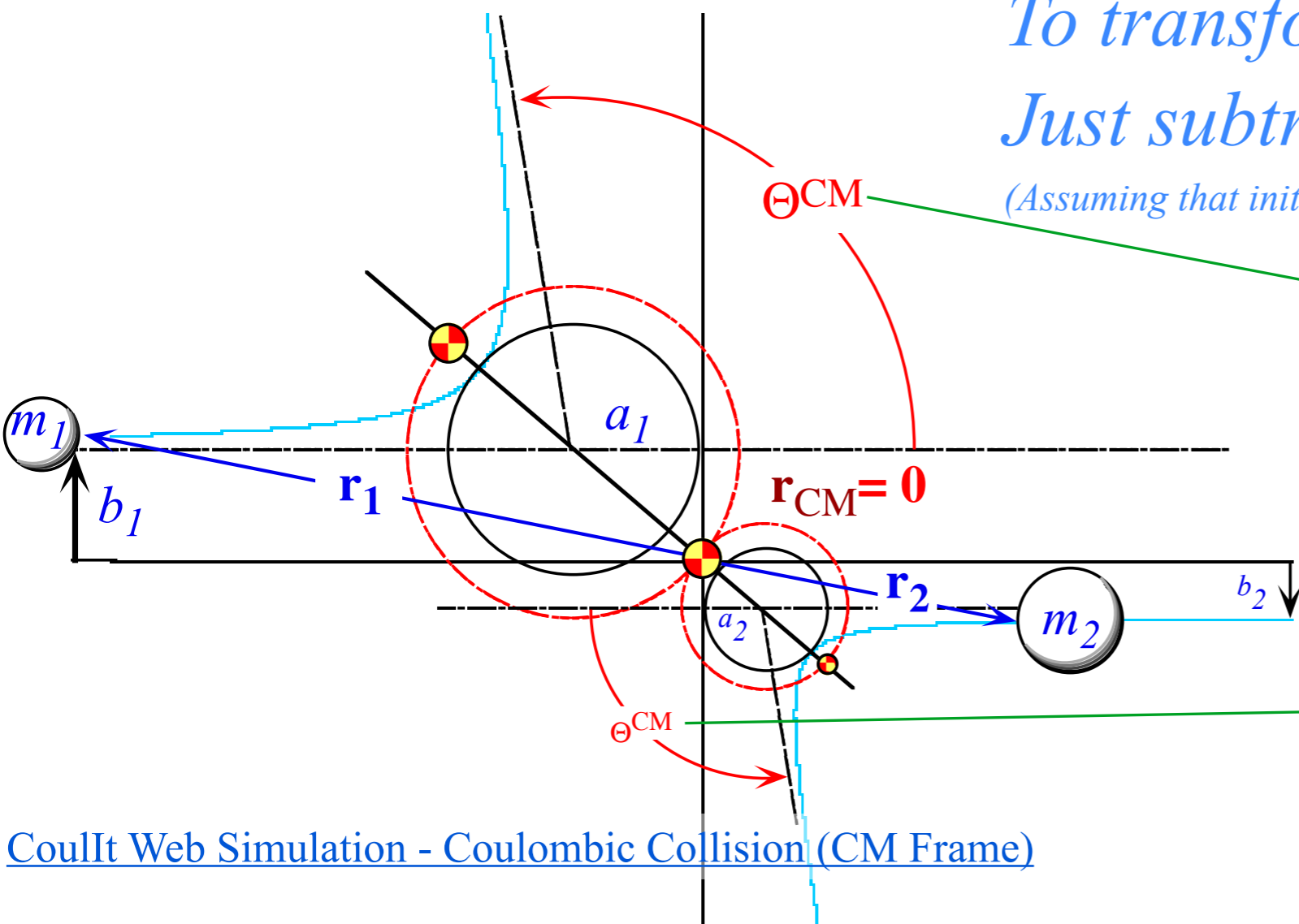
LAB view
 $v^{CM} = -v_2^{CM(0)}$
 $v_2^{LAB(0)} = 0$



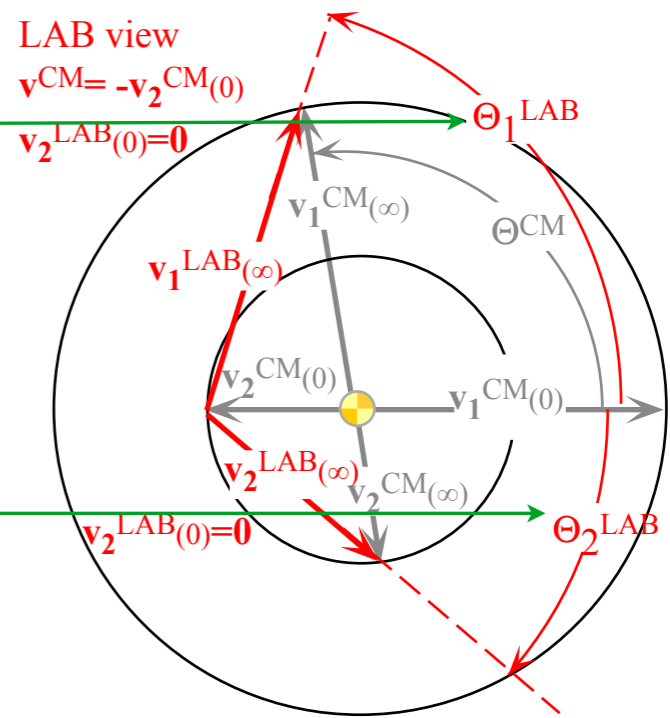
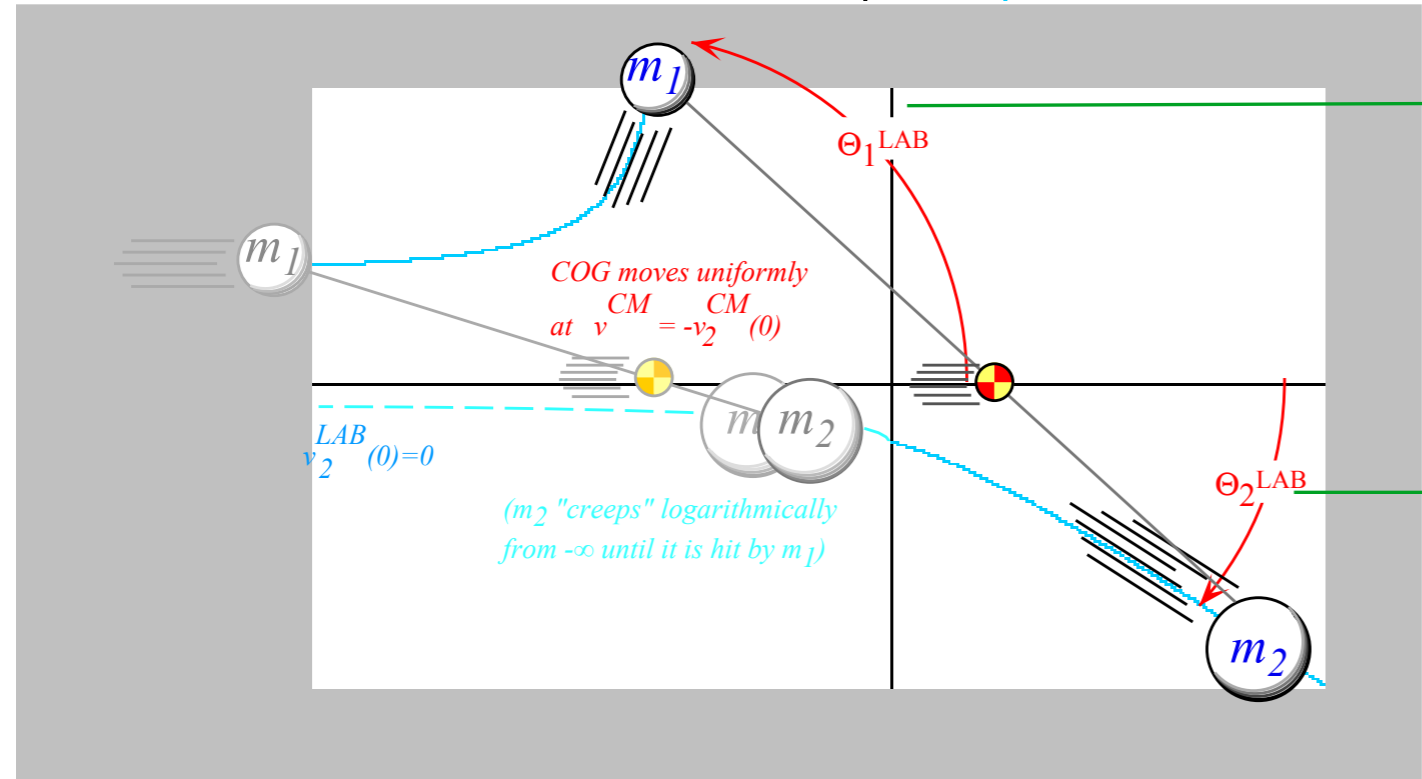
To transform CM to LAB frame
 Just subtract $v_2^{CM}(0)$ from all v

(Said: Galileo)

(Assuming that initial $v_2^{LAB}(0)$ is zero so $v_2^{CM}(0)$ is CM velocity in LAB)



[CoulIt Web Simulation - Coulombic Collision \(CM Frame\)](#)



[CoulIt Web Simulation - Coulombic Collision \(LAB Frame\)](#)

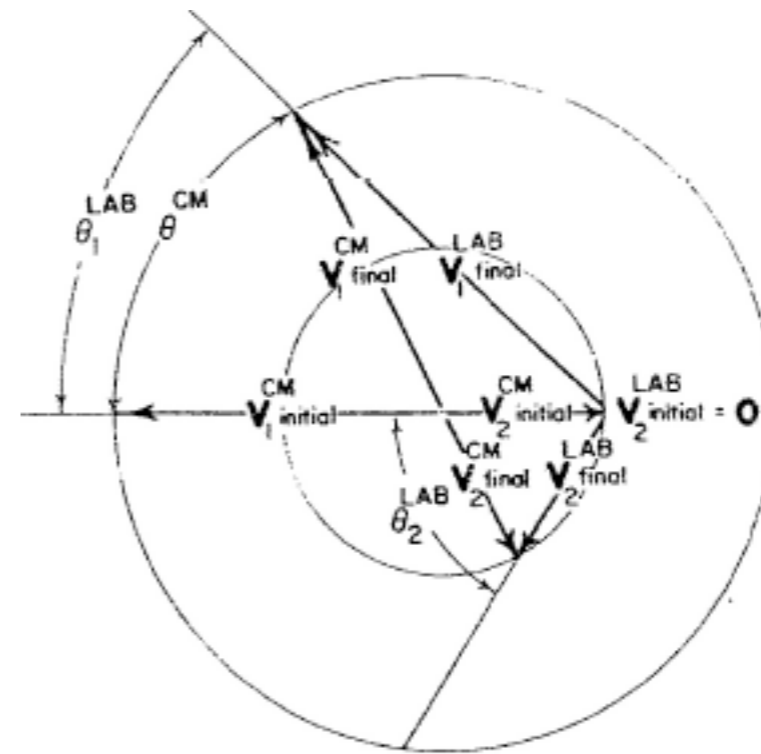


FIG. 4. Given the center of mass scattering angle θ^{CM} (from Fig. 3) and the mass ratio (2:1 in this case) a vector addition construction produces angles θ_1^{LAB} and θ_2^{LAB} shown here.

From: [Geometric aspects of classical Coulomb scattering](#)
American Journal of Physics 40,1852-1856 (1972)
 Class project when I taught Jr. CM at Georgia Tech
 (Just 5 students)

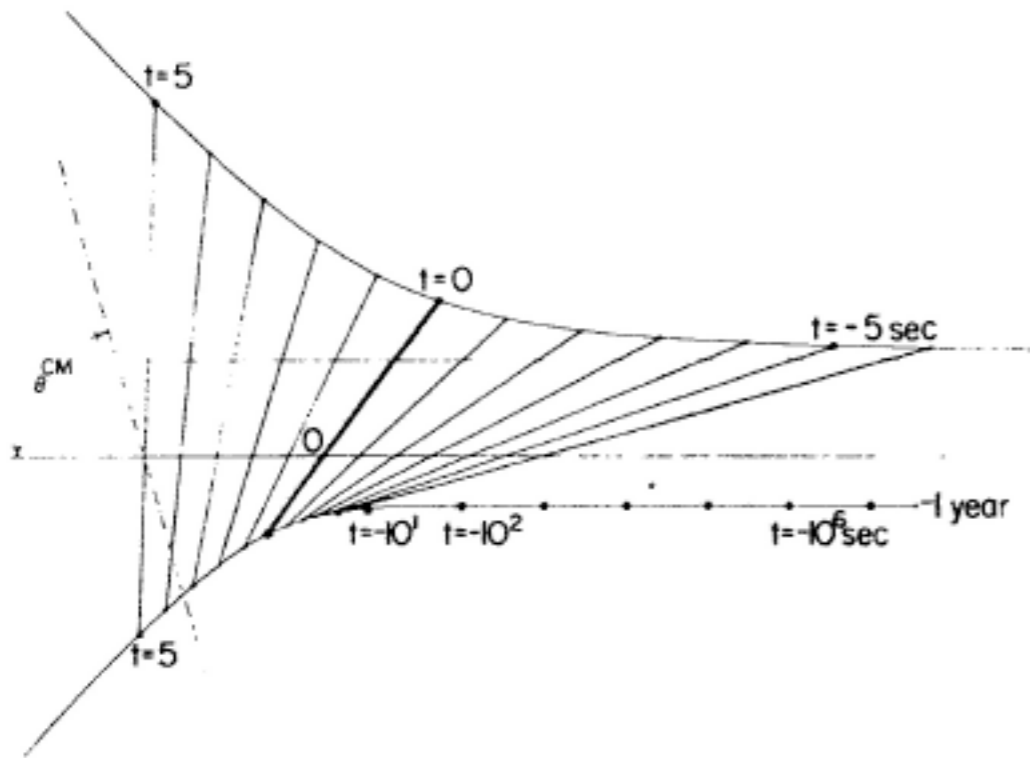
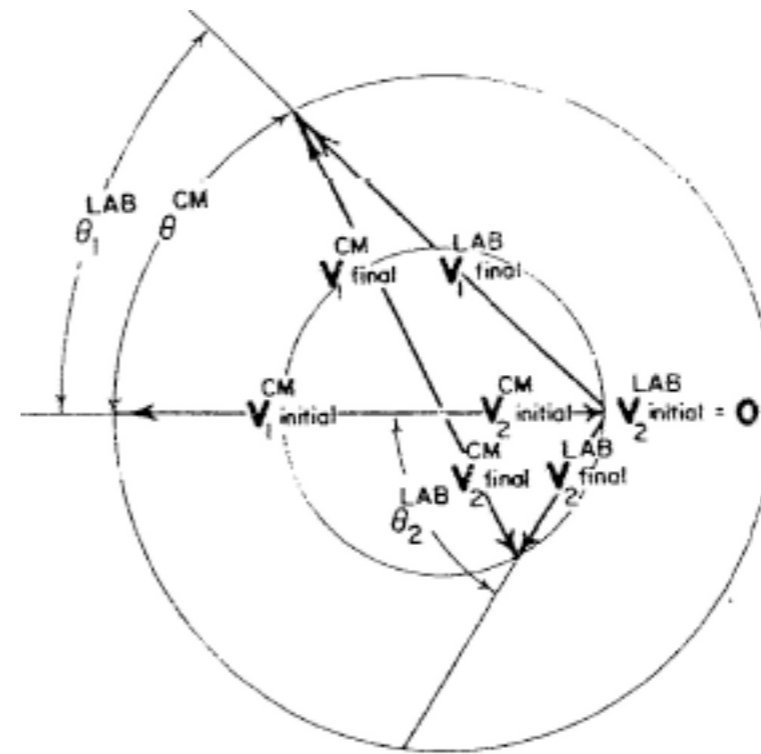


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

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The trouble with the Coulomb field is...

$$\int t^{-1} dt = \ln t + C$$

$$\begin{aligned} v_2^{\text{LAB}}(t) &= \int (|F|/m_2) dt \\ &\cong \int k dt / m_2 [v_1^{\text{CM}}(\text{initial})t]^2 \\ &\cong [-k/m_2 v_1^{\text{CM}}(\text{initial})^2] t^{-1} \end{aligned}$$

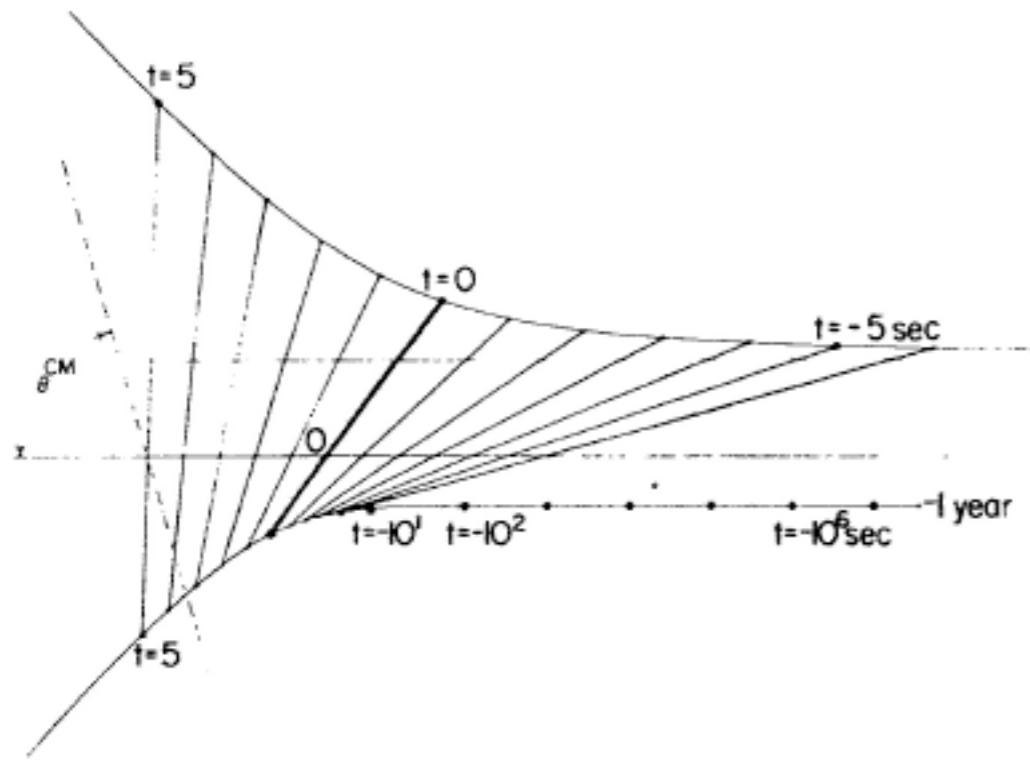


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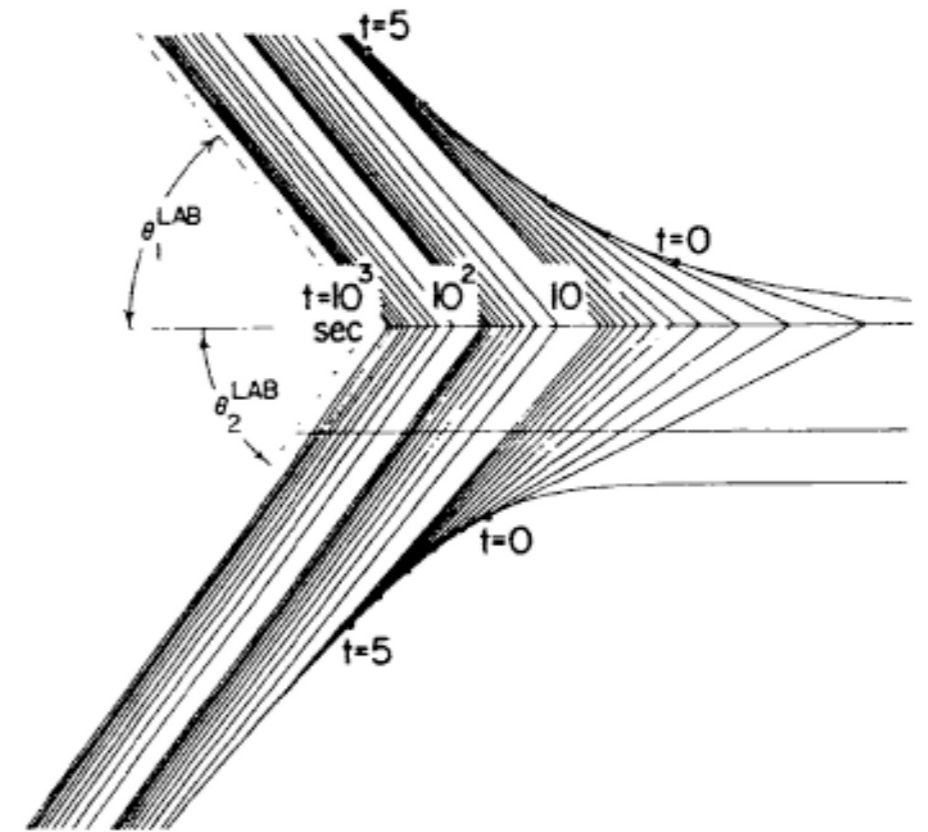


FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t = 10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04° , respectively.

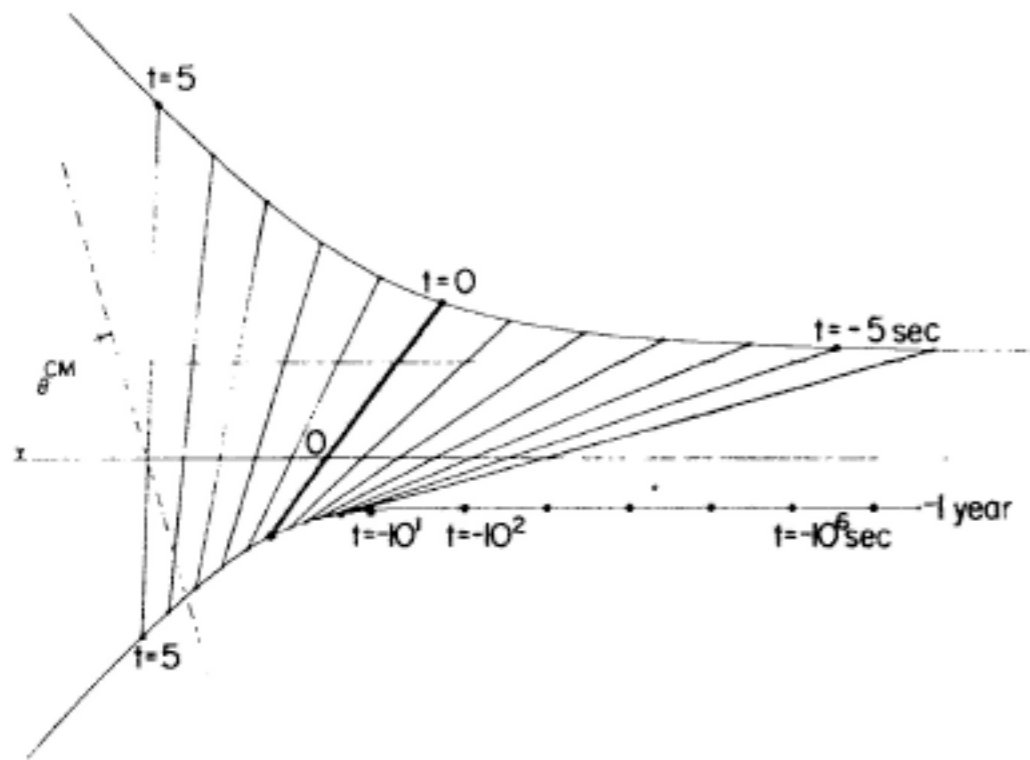


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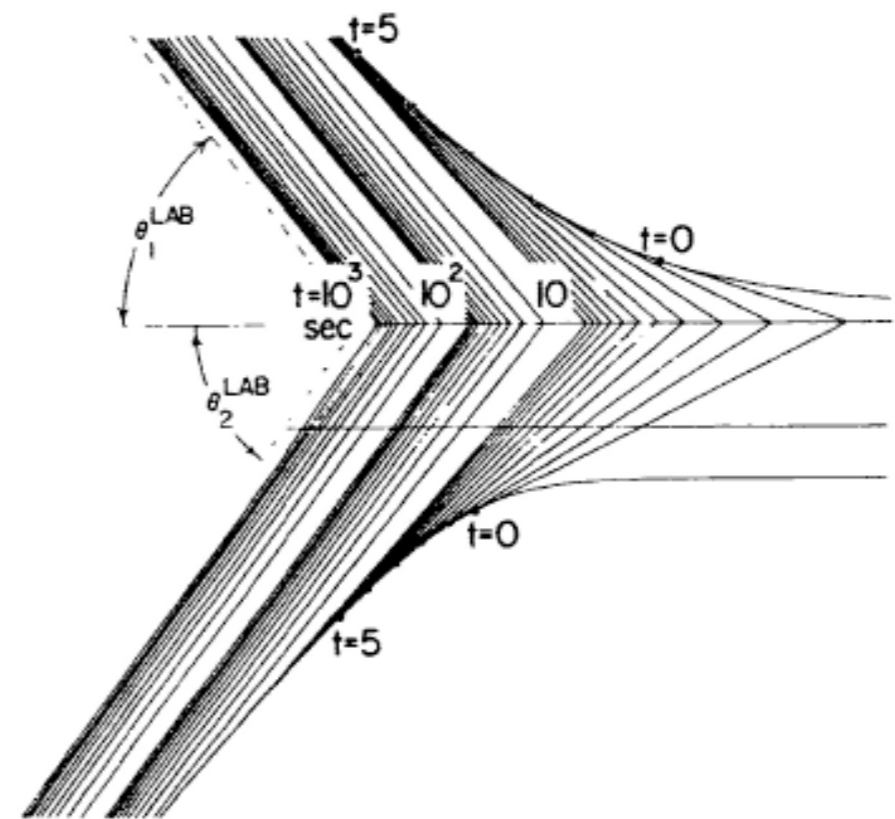


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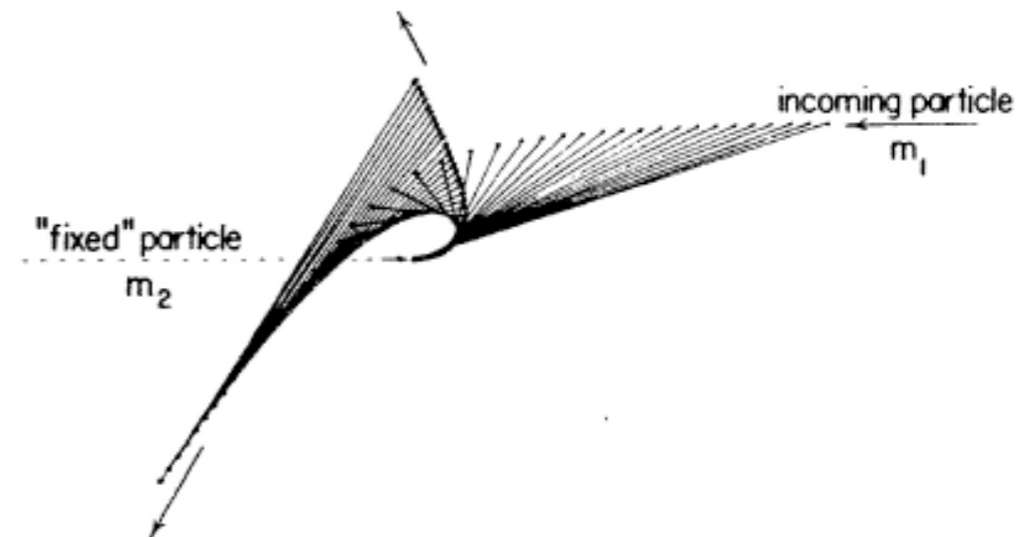


FIG. 7. Attractive Coulomb scattering in laboratory system. This has the same "anomalies" as the repulsive case.

➔ *Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$*
How to make my boomerang come back
The gyrocompass and mechanical spin analogy

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

Angular momentum vector \mathbf{L}_j of a mass m_j is its linear momentum \mathbf{p}_j times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

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The sum-total angular momentum is

$$\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^3 \mathbf{L}_j = \sum_{j=1}^3 \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$$

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$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_{j=1}^3 \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{total} \\ &= \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^3 \mathbf{r}_j \times \left(\sum_{k=1(k \neq j)}^3 \mathbf{F}_{jk}^{constraint} \right) \end{aligned}$$

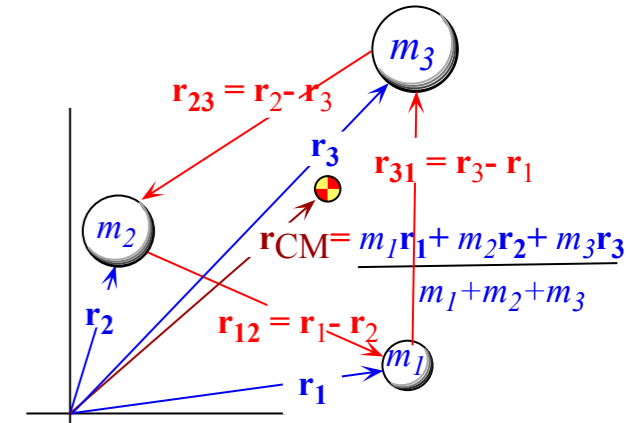


Fig. 6.4.1 Three-particle coordinate vectors

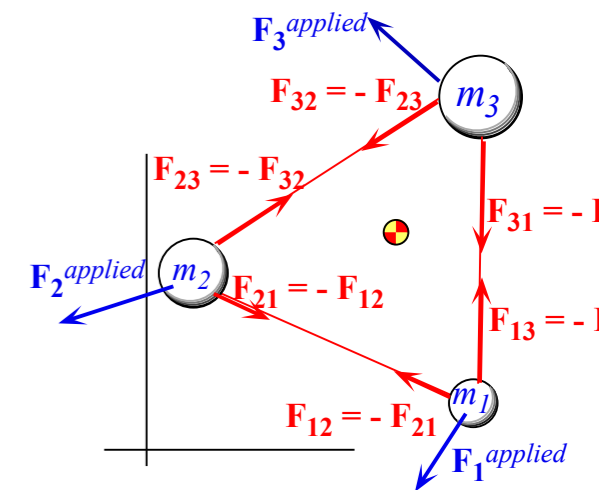


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

Angular momentum vector \mathbf{L}_j of a mass m_j is its linear momentum \mathbf{p}_j times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

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Internal constraint or coupling force terms appear at first to be a nuisance.

$$\begin{aligned} \sum_{j=1}^3 \sum_{k=1(k \neq j)}^3 \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} &= \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}) + \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}) + \mathbf{r}_3 \times (\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}) \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12}^{constraint} + (\mathbf{r}_1 - \mathbf{r}_3) \times \mathbf{F}_{13}^{constraint} + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_{23}^{constraint} = \mathbf{0} \end{aligned}$$

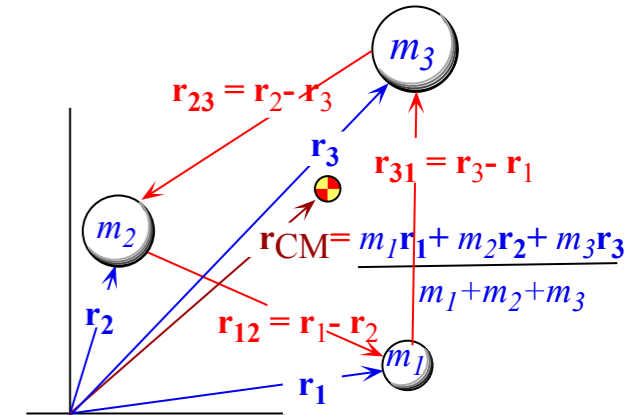


Fig. 6.4.1 Three-particle coordinate vectors

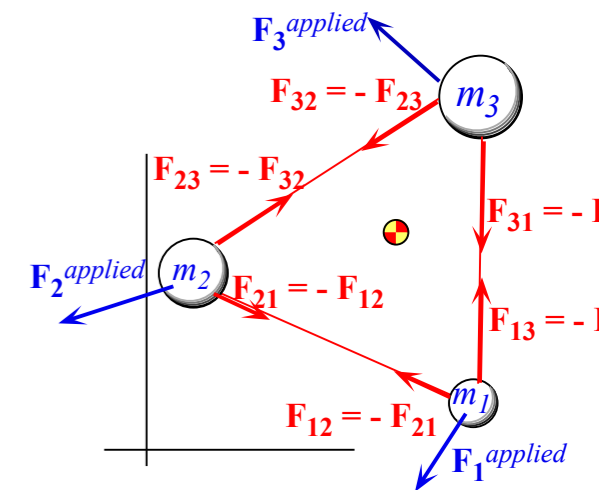


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$

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However, they vanish if coupling forces act along lines connecting the masses.

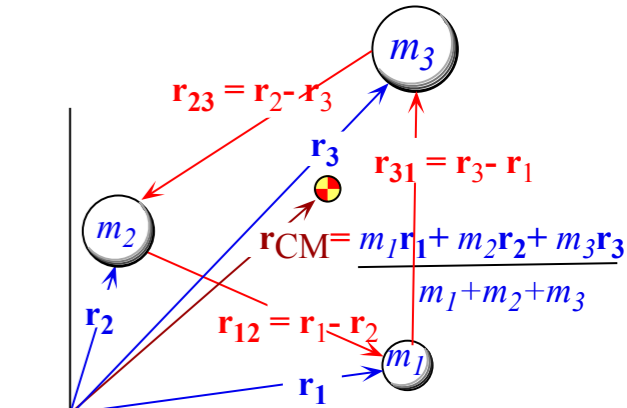


Fig. 6.4.1 Three-particle coordinate vectors

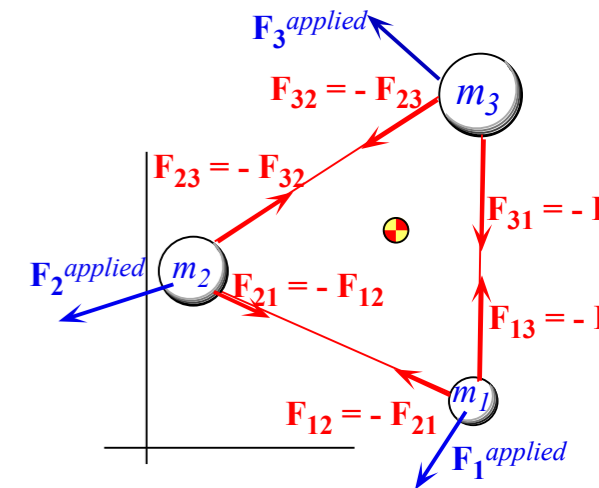


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=\mathbf{dp}/dt$ equations: $\mathbf{N}=\mathbf{dL}/dt$

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The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}, \text{ where: } \mathbf{N} = \sum_{j=1}^3 \mathbf{N}_j \text{ and: } \mathbf{N}_j = \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{applied}$$

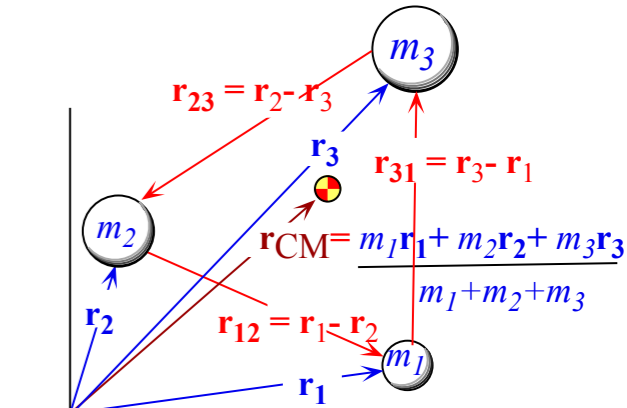


Fig. 6.4.1 Three-particle coordinate vectors

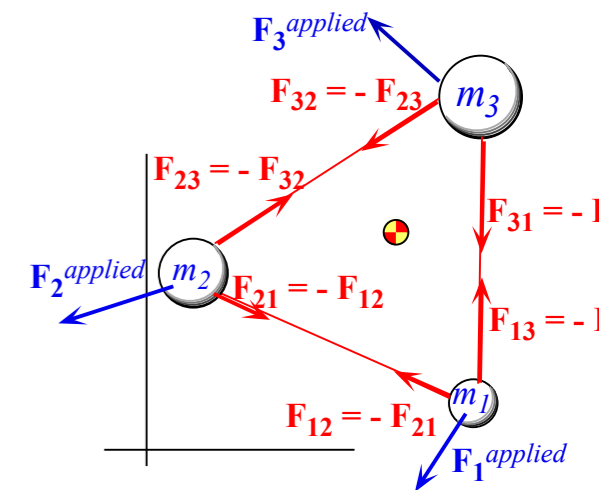


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=\mathbf{dp}/dt$ equations: $\mathbf{N}=\mathbf{dL}/dt$

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However, they vanish if coupling forces act along lines connecting the masses.

The results are the *rotational Newton's equation*.

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Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}, \text{ where: } \mathbf{F} = \sum_{j=1}^3 \mathbf{F}_j^{applied}$$

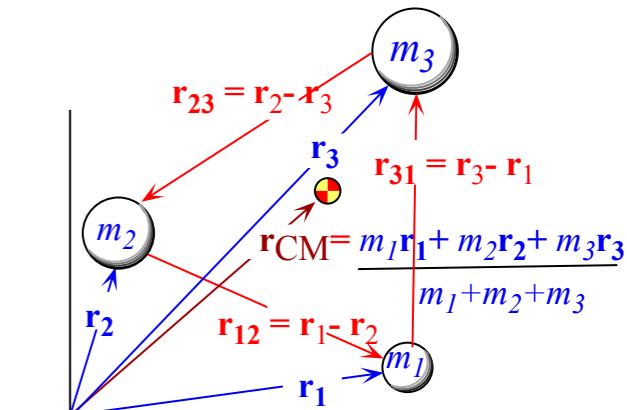


Fig. 6.4.1 Three-particle coordinate vectors

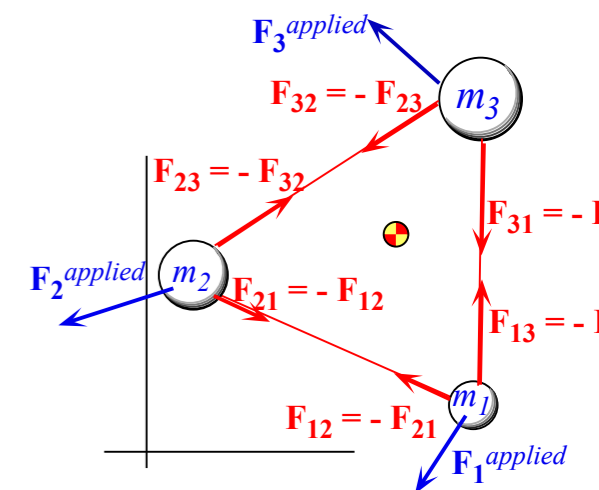


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

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However, they vanish if coupling forces act along lines connecting the masses.

The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}, \text{ where: } \mathbf{N} = \sum_{j=1}^3 \mathbf{N}_j \text{ and: } \mathbf{N}_j = \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{applied}$$

Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}, \text{ where: } \mathbf{F} = \sum_{j=1}^3 \mathbf{F}_j^{applied}$$

Remaining $3N-6$ equations consist of normal mode or GCC equations of some kind.

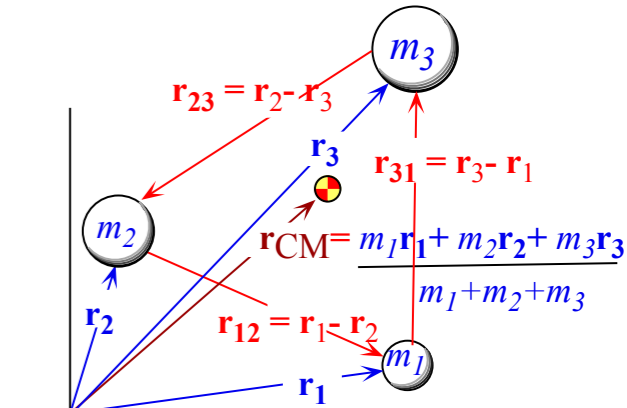


Fig. 6.4.1 Three-particle coordinate vectors

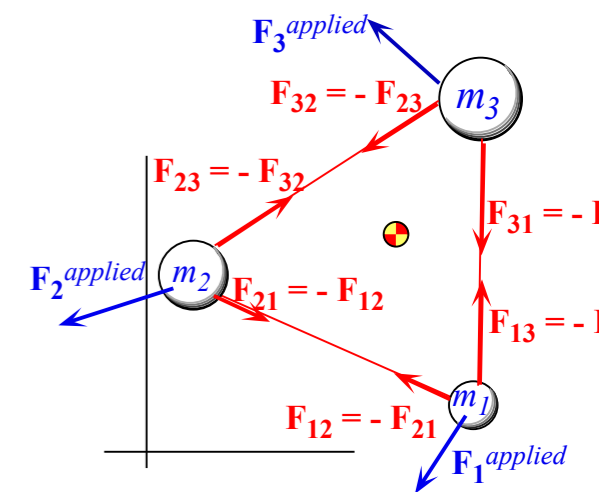


Fig. 6.4.2 Three-particle force vectors

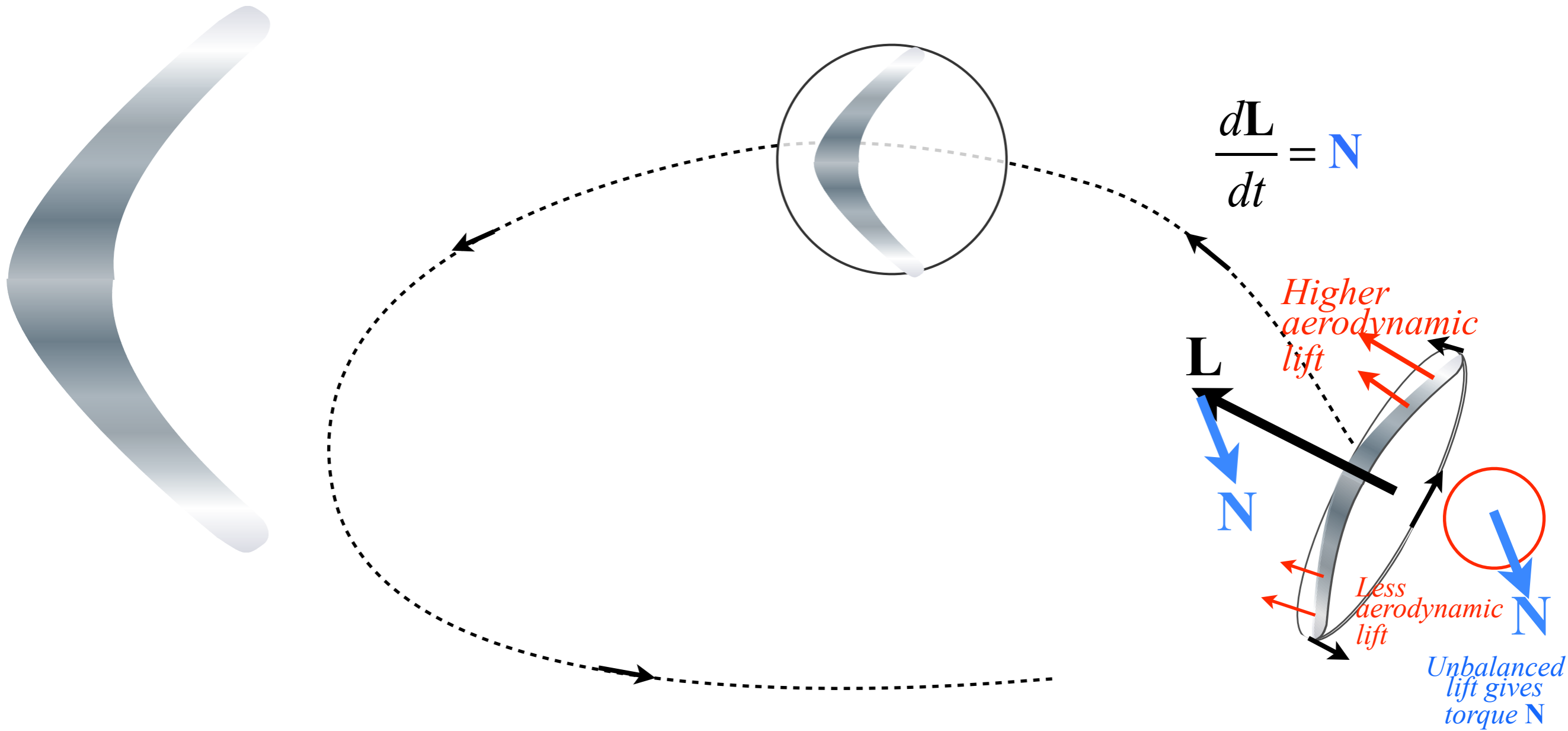
Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$



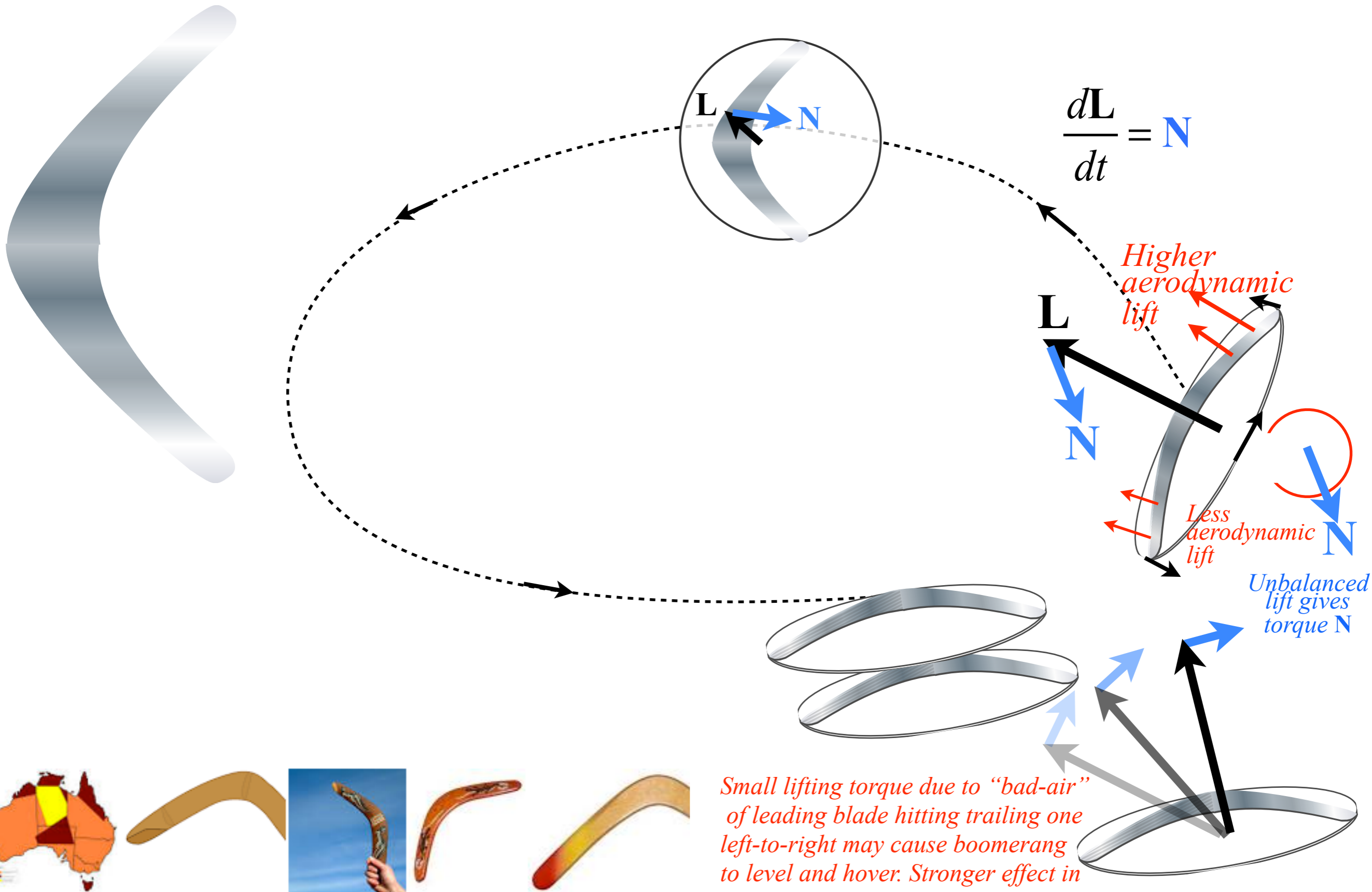
How to make my boomerang come back

The gyrocompass and mechanical spin analogy

The Australian Boomerang (that comes back!)



The Australian Boomerang (that comes back and hovers down!)



Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.

The Australian Boomerang (that comes back and hovers down!)

Charlie Drake's famous 1961 song:

My boomerang won't come back!

My boomerang won't come back!

My boomerang won't come back!

I've waved the thing all over the place

Practiced til' I was blue in the face*

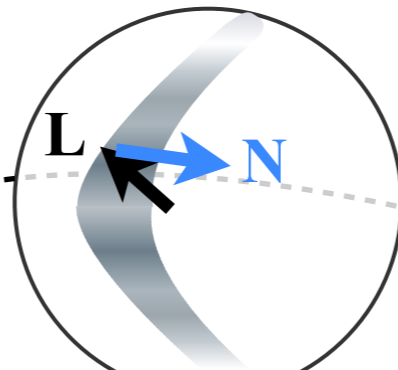
I'm a big disgrace

to the Aborigine Race

My boomerang won't come back!



*blue later replaced black



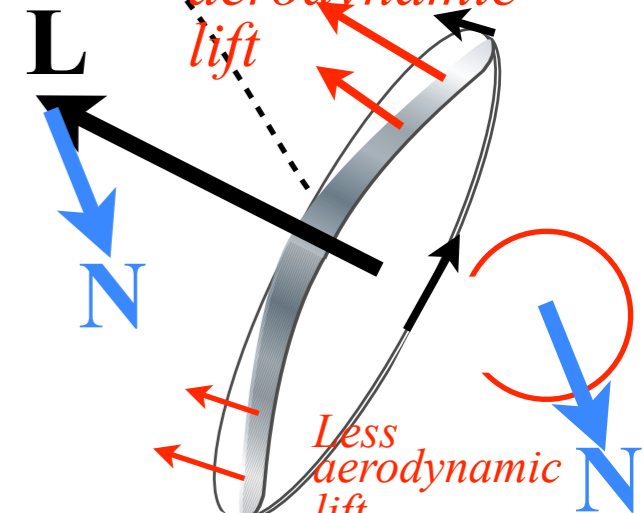
$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$



Aluminum boomerang I made in 1965.

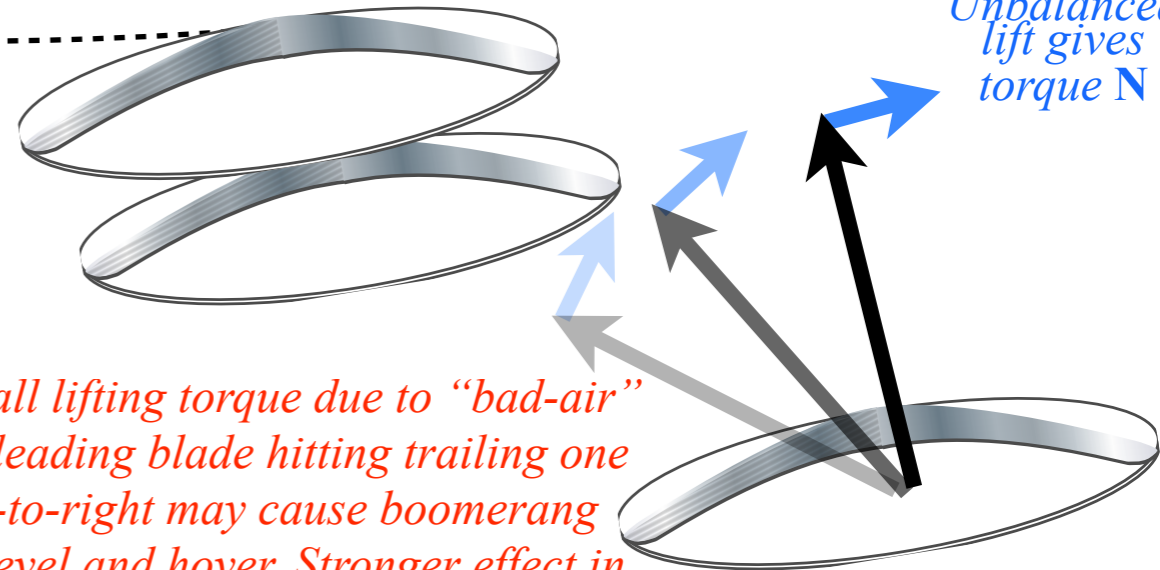
It once flew over 18 seconds with hover-return!

Higher aerodynamic lift



Less aerodynamic lift

Unbalanced lift gives torque N



Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.



Popular Charlie Drake & My Boomerang Won't Come Back videos

32 videos • 842 views • Last updated on Oct 27, 2015



<https://www.youtube.com/playlist?list=PLGwmGldCxzLxbPIFVG8Z89WZIBuT4m0Ii>



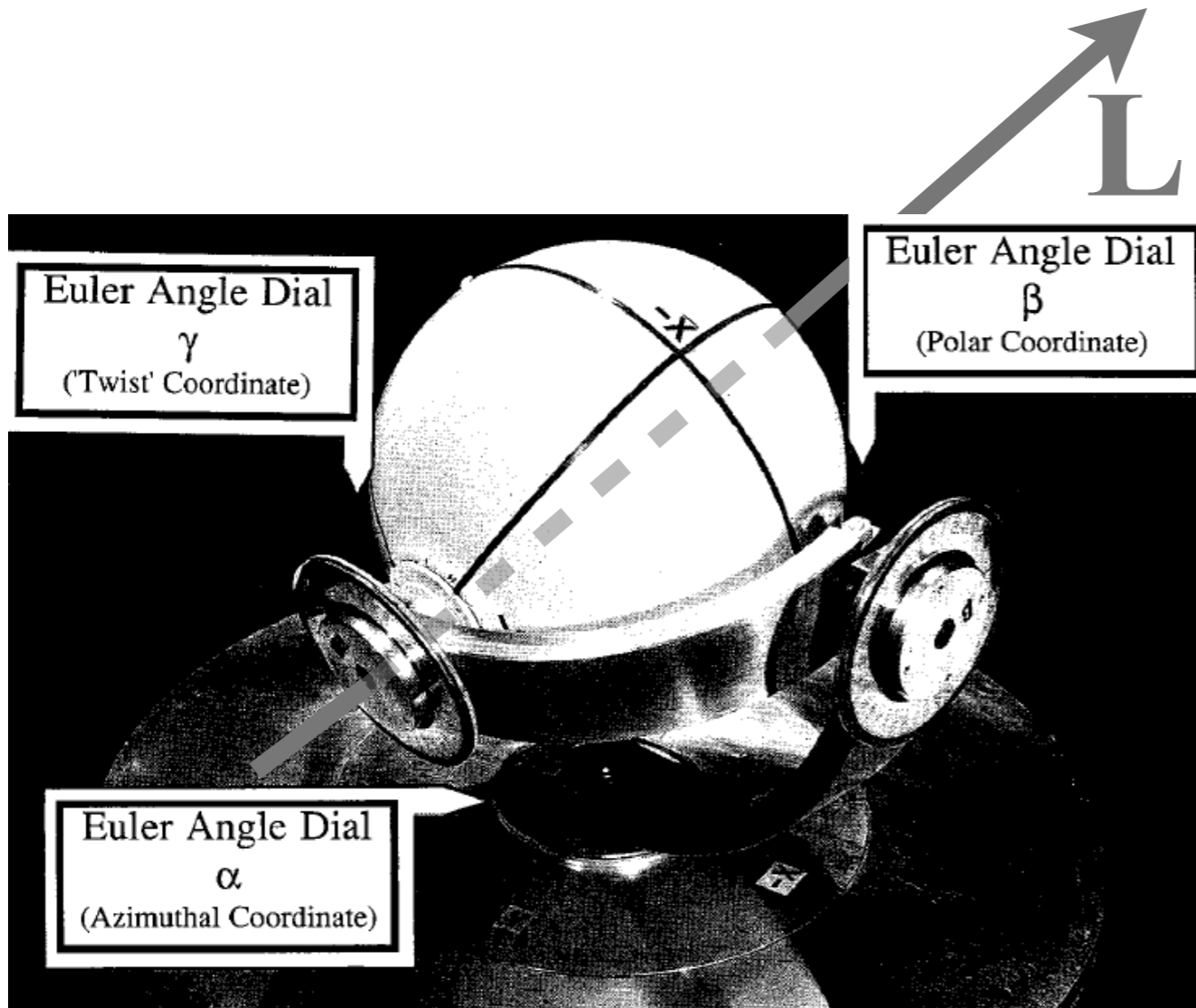
Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

How to make my boomerang come back

 *The gyrocompass and mechanical spin analogy*

The gyrocompass and mechanical spin analogy

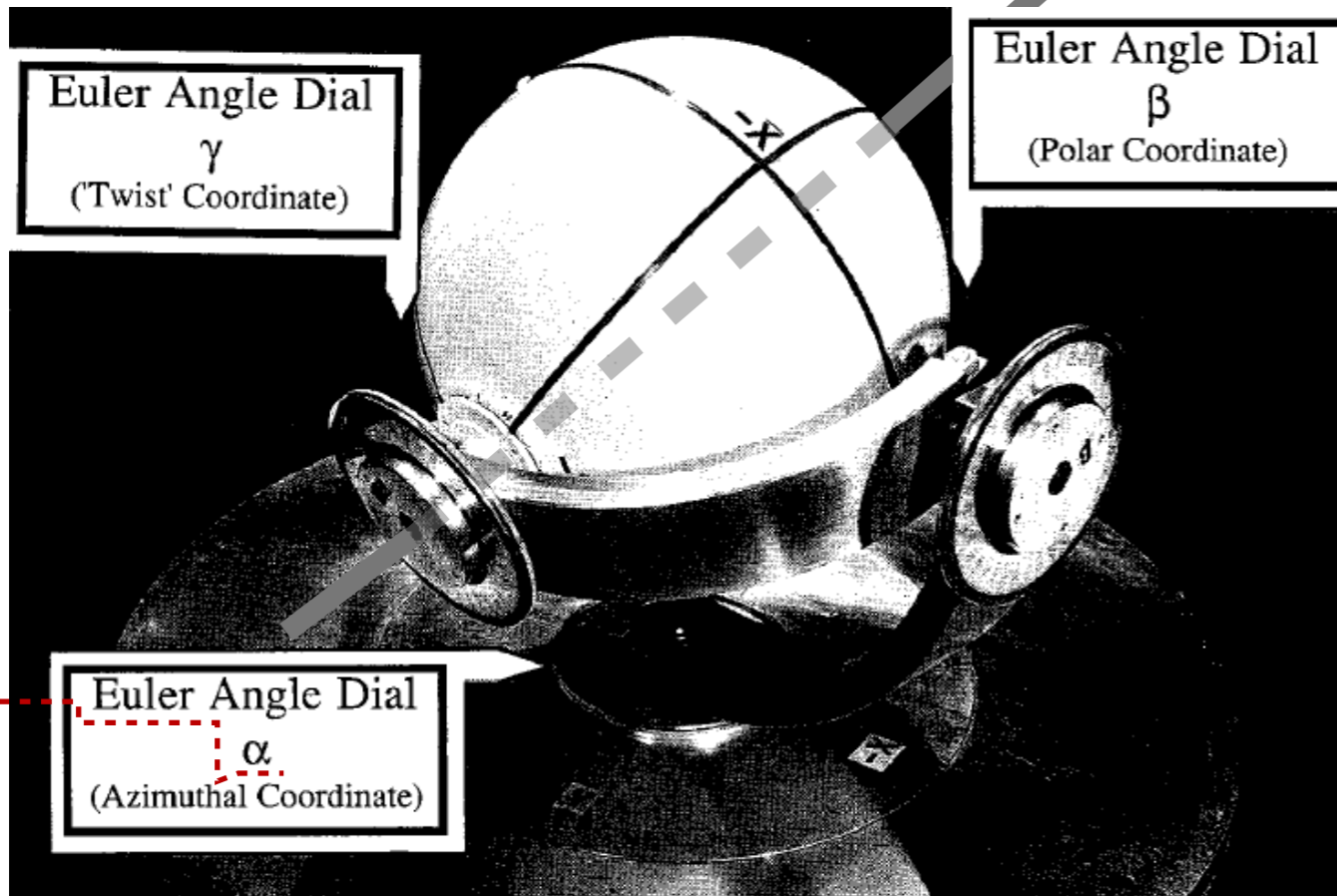
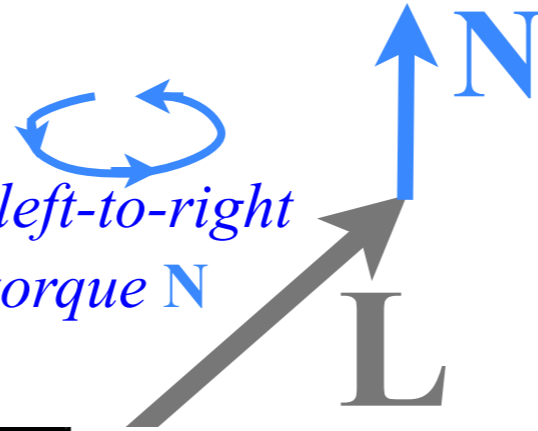
Suppose Euler ball has right-hand rotation with angular momentum \mathbf{L}



The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum \mathbf{L}

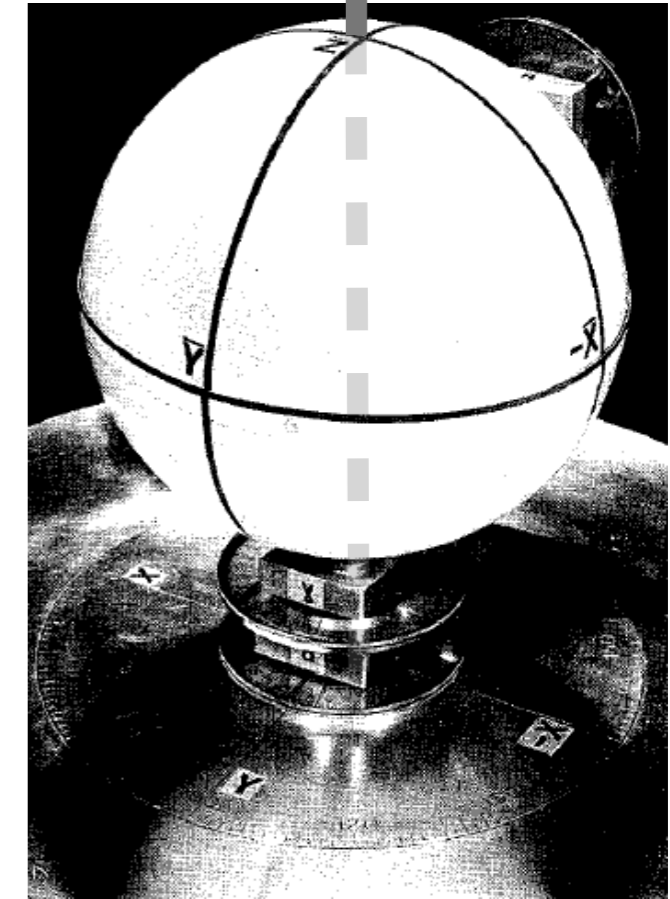
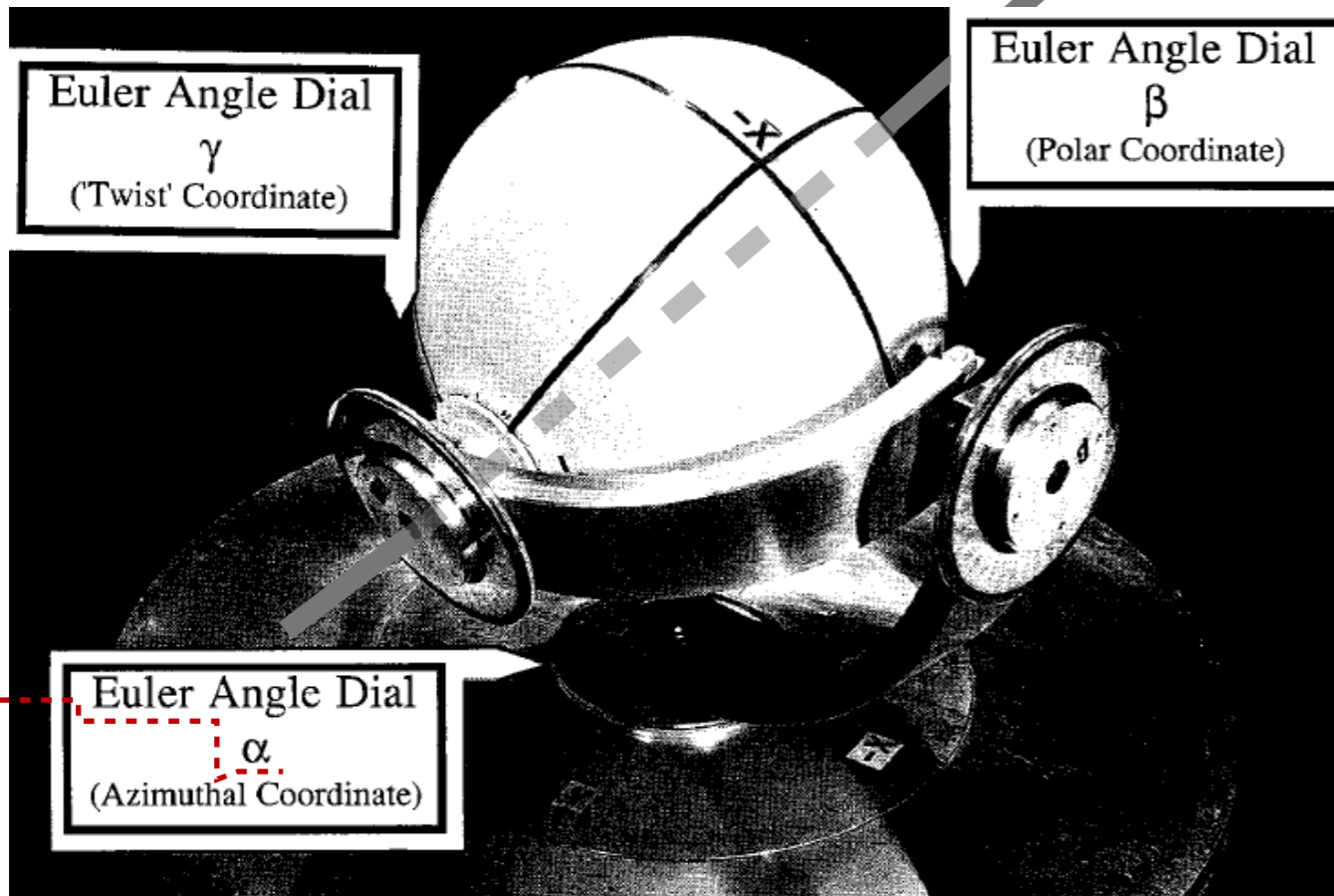
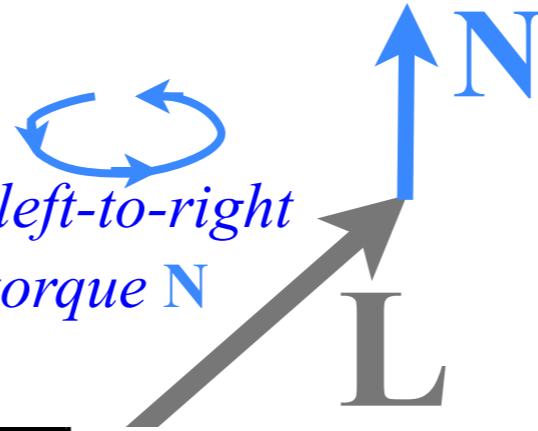
If the α -dial for z -rotation is turning left-to-right this applies righthand "thumbs-up" torque \mathbf{N}



The gyrocompass and mechanical spin analogy

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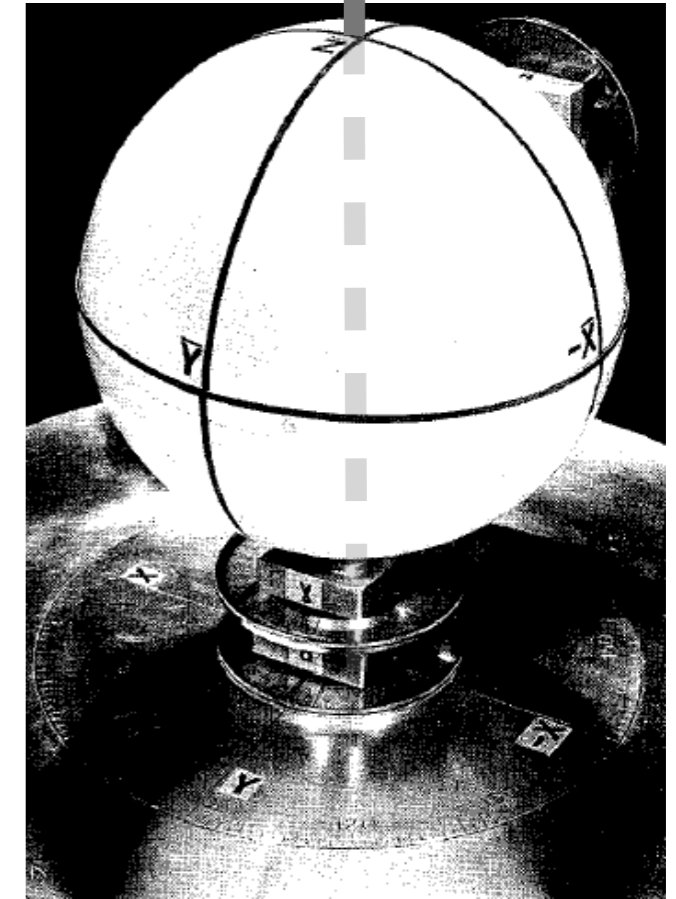
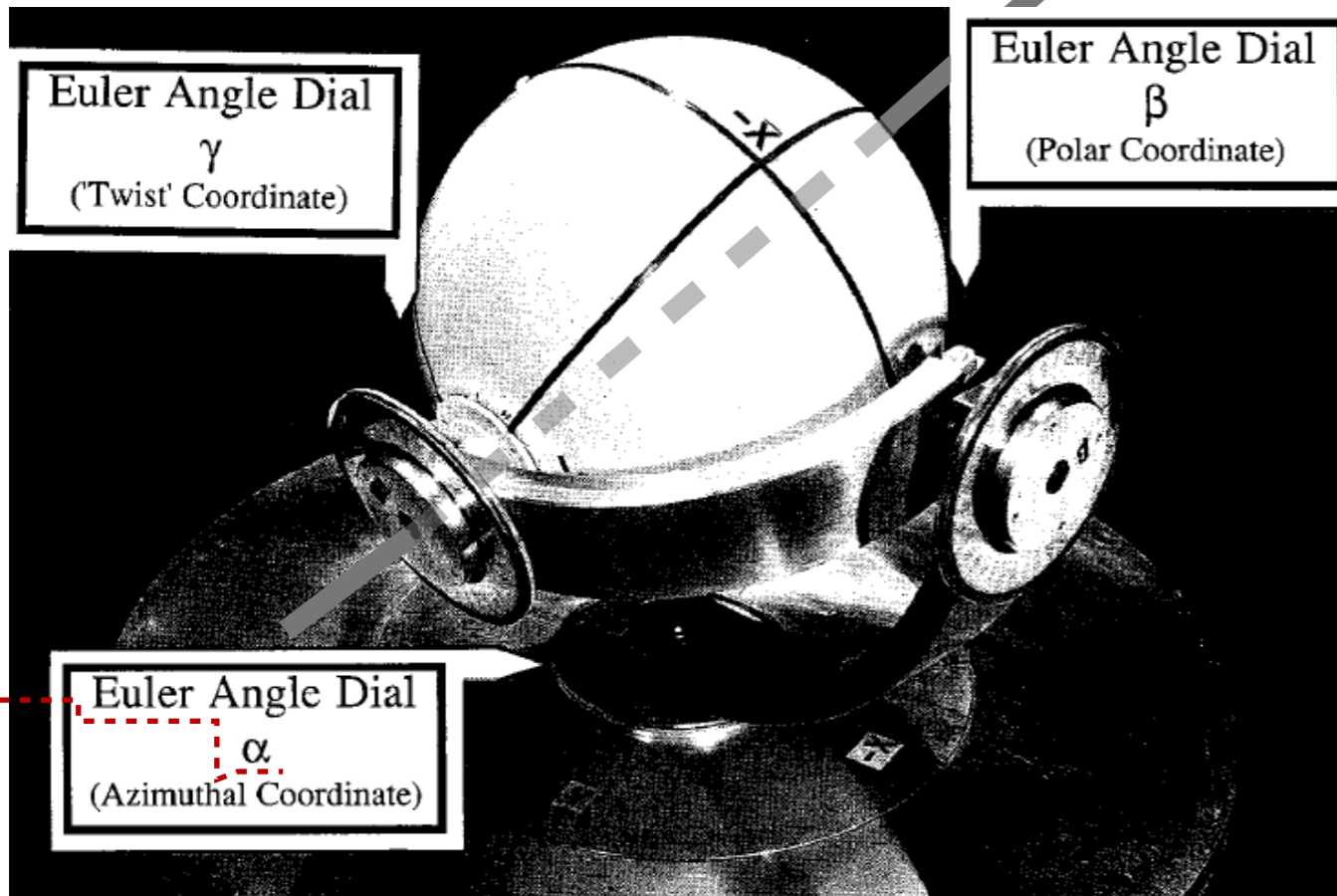
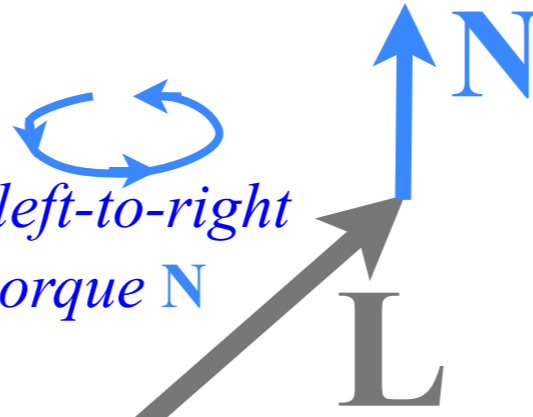


Then the ball tends to line-up with z -axis (and may go past z , then come back, etc. in a precessional or “hunting” motion)

The gyrocompass and mechanical spin analogy

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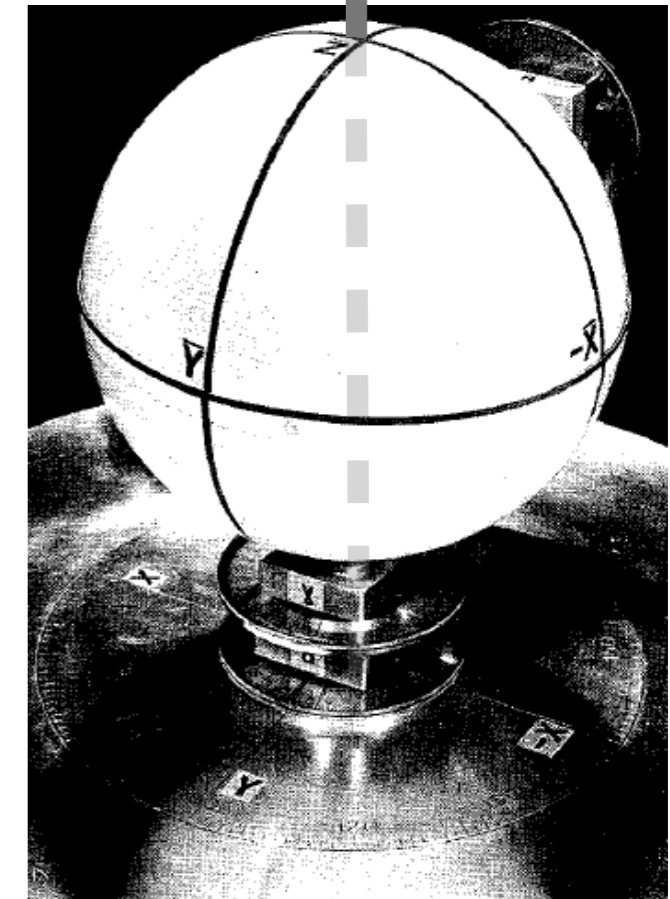
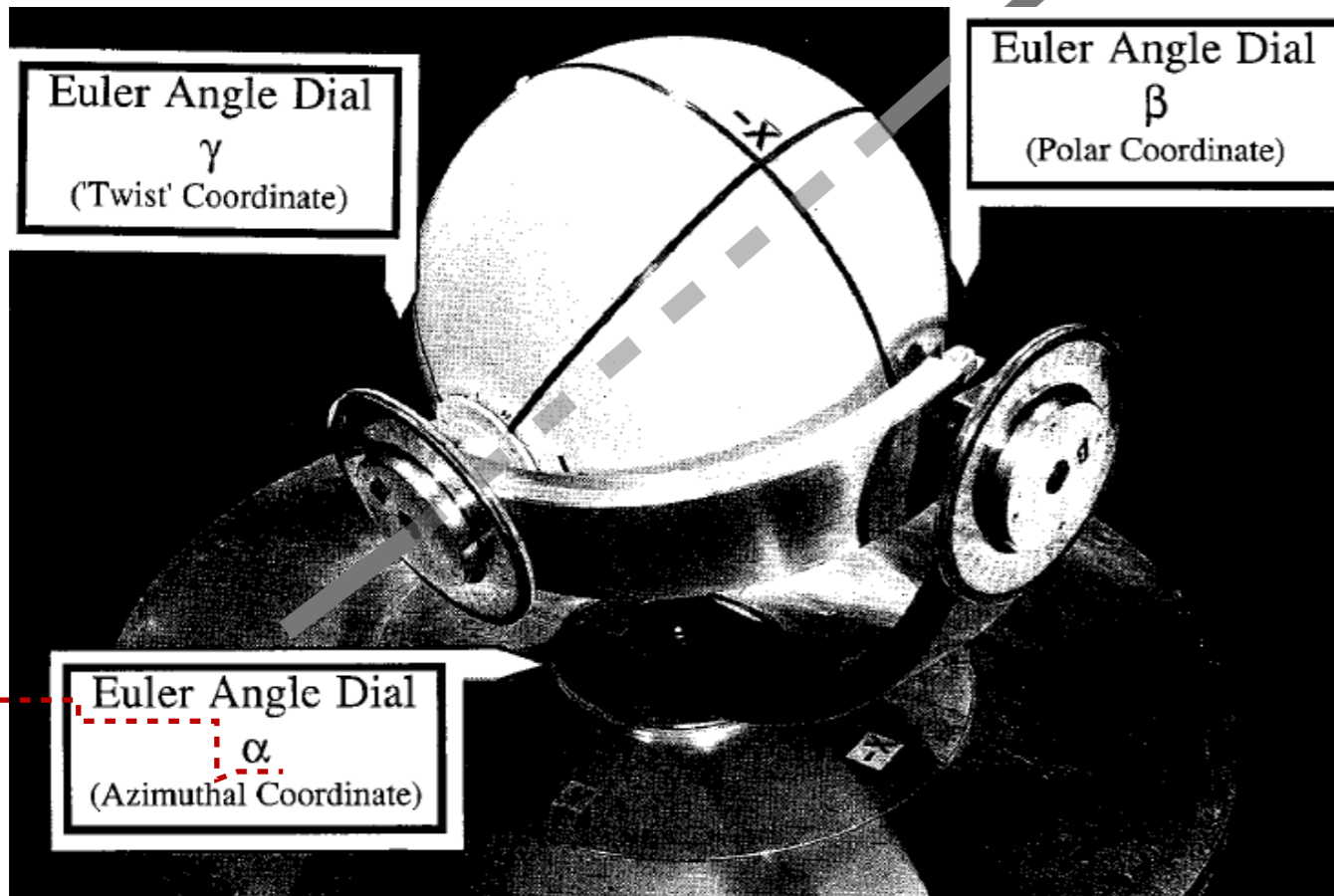
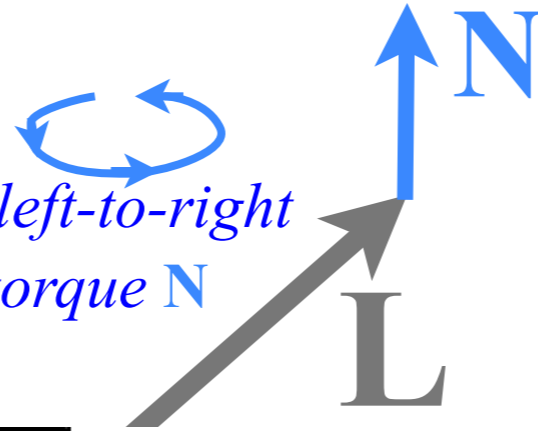
A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.

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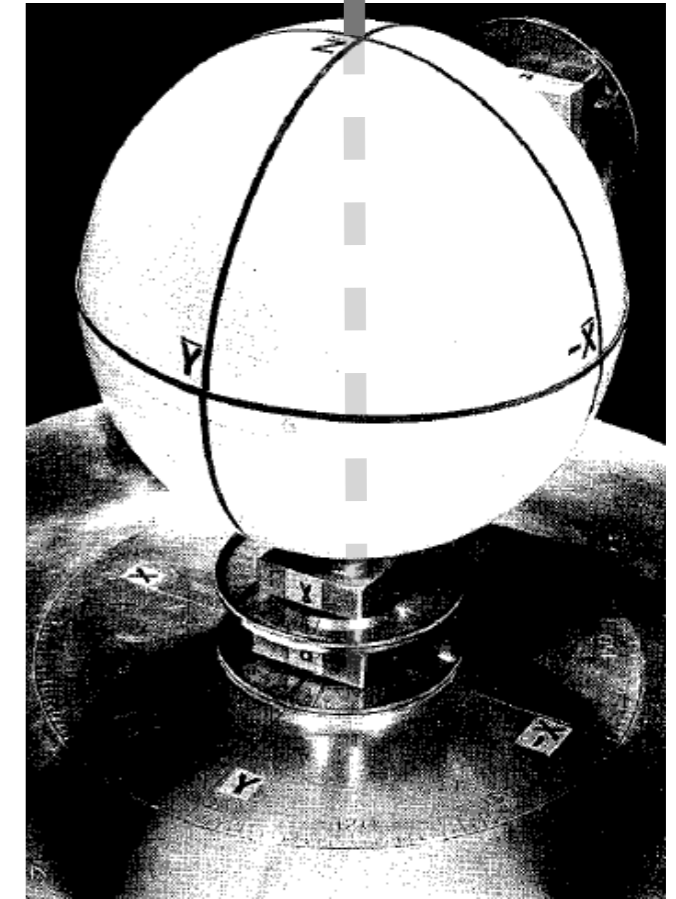
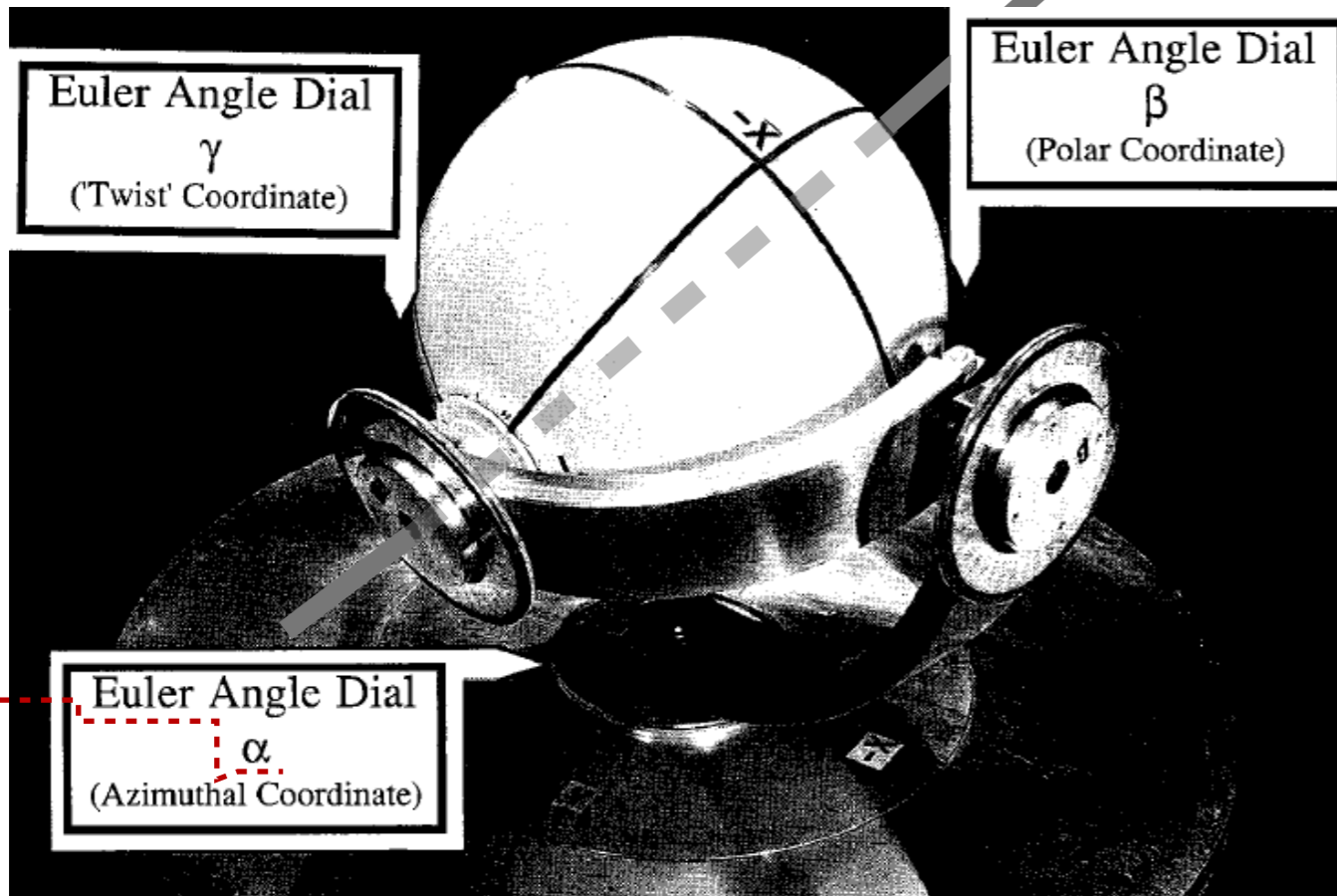
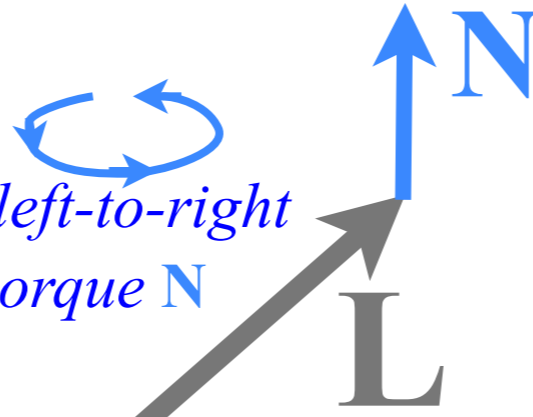
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This is analogous to the tendency for spin magnetic moments to align (or precess about) the B -direction of a magnetic field
Recall S-precession discussion in CMwB Unit 4 Ch.4 and Lect.26

The gyrocompass and mechanical spin analogy

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A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.

General Rule: Gyros tend to “line-up” so they are rotating with whatever is most closely coupled to them.

Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or “hunting” motion)

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Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity ω -ellipsoid vs. angular momentum \mathbf{L} -ellipsoid

Lagrangian ω -equations vs. Hamiltonian momentum \mathbf{L} -equation

Inertia tensors

Consider N -body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* \mathbf{L} relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^N \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^N m_j \mathbf{r}_j \times (\boldsymbol{\omega} \times \mathbf{r}_j) \quad \text{with} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:

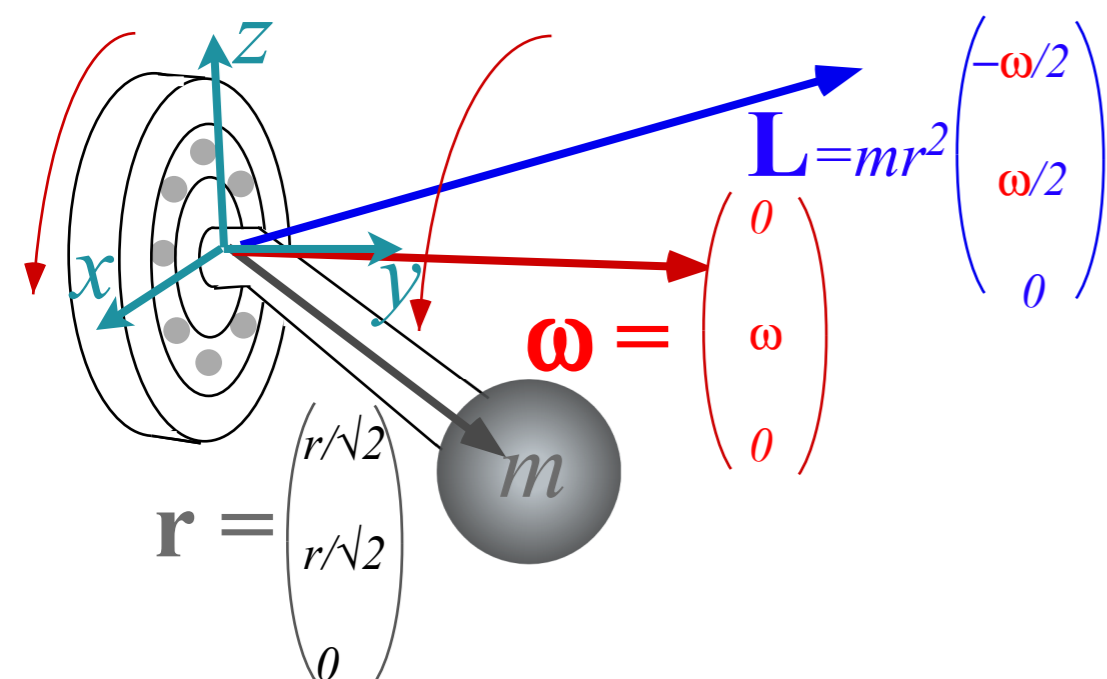


Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

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This produces the *rotational inertia tensor* \mathbf{I} :
$$\vec{\mathbf{I}} = \sum_{j=1}^N \vec{\mathbf{I}}_j = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right]$$

in the $\boldsymbol{\omega}$ -to- \mathbf{L} relation:
$$\mathbf{L} = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega}) \mathbf{r}_j \right] = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right] \cdot \boldsymbol{\omega} = \vec{\mathbf{I}} \cdot \boldsymbol{\omega}$$

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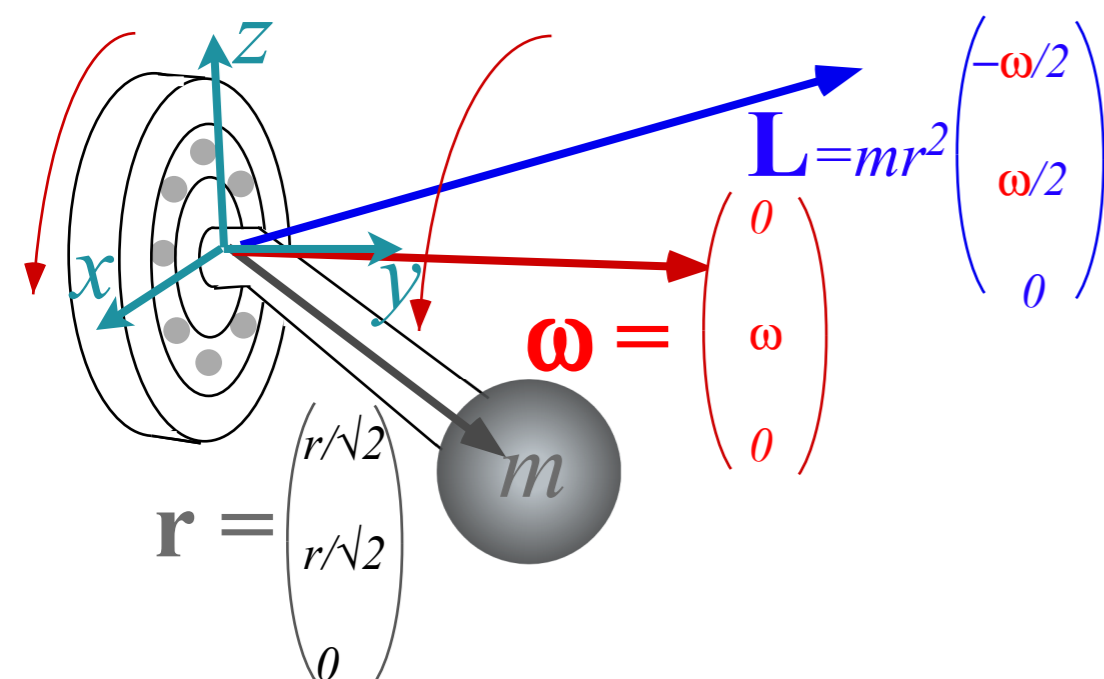


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Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation

using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \langle \vec{\mathbf{I}} \rangle = \sum_{j=1}^N \langle \vec{\mathbf{I}}_j \rangle = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix}$$

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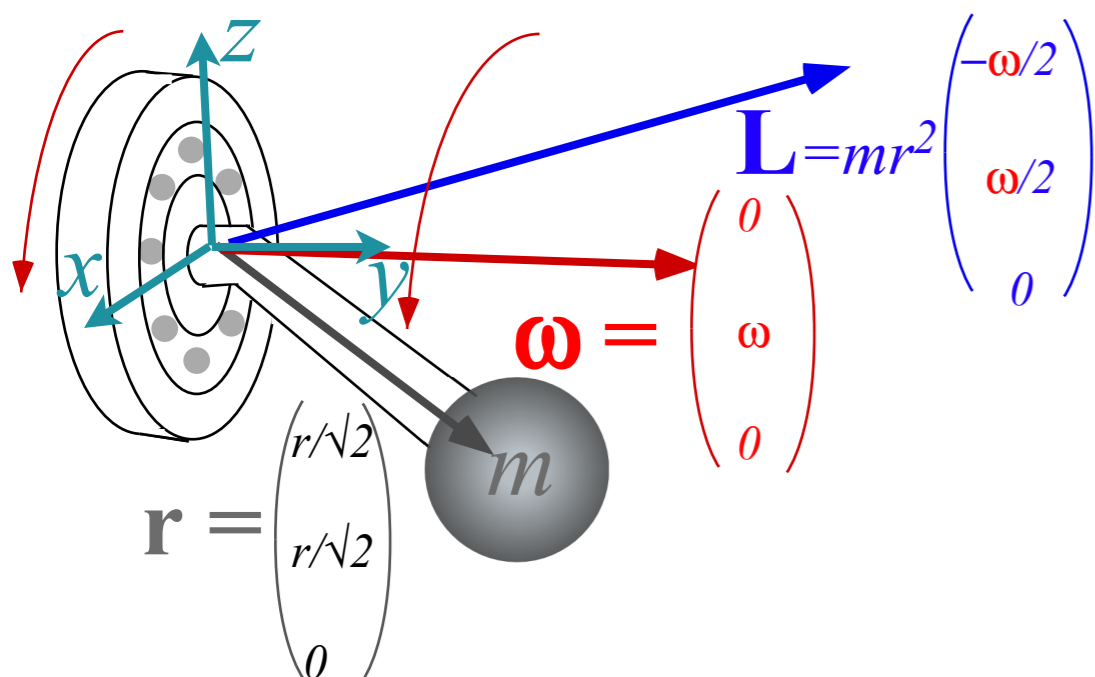


Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N -body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* \mathbf{L} relations with Levi-Civita analysis

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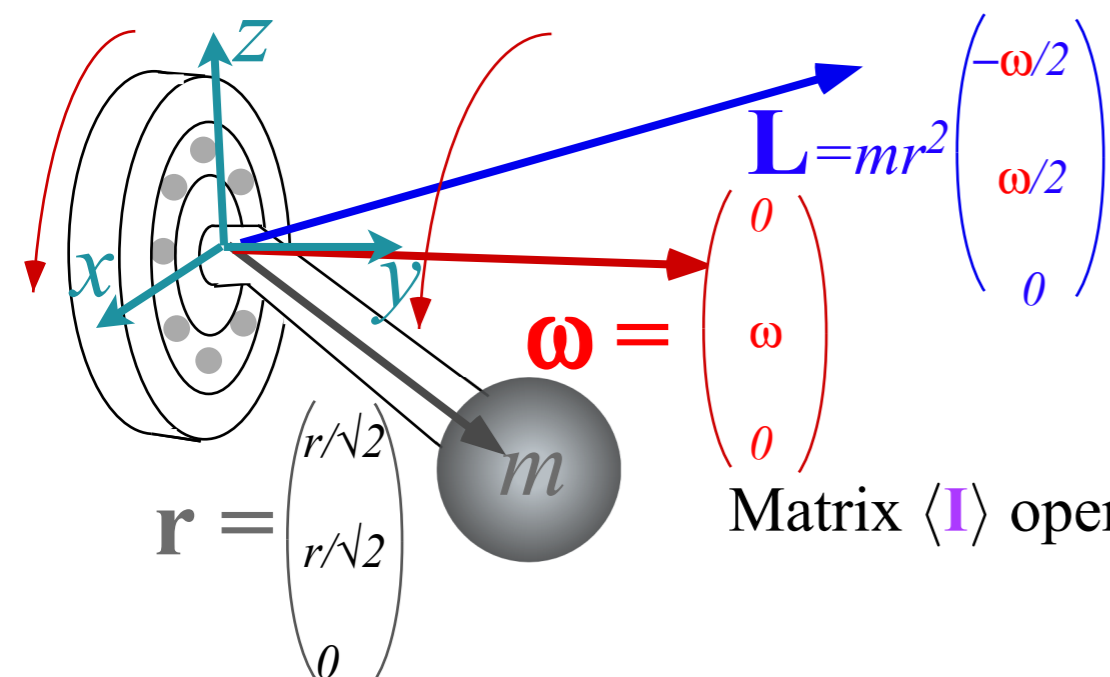
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Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \langle \tilde{\mathbf{I}} \rangle = \sum_{j=1}^N \langle \tilde{\mathbf{I}}_j \rangle = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix}$$

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Instantaneous matrix $\langle \mathbf{I} \rangle$ of inertia is:

$$\langle \tilde{\mathbf{I}} \rangle = mr^2 \begin{pmatrix} (1/\sqrt{2})^2 + 0 & -(1/\sqrt{2})(1/\sqrt{2}) & -(1/\sqrt{2})0 \\ -(1/\sqrt{2})(1/\sqrt{2}) & (1/\sqrt{2})^2 + 0 & -(1/\sqrt{2})0 \\ -0(1/\sqrt{2}) & -0(1/\sqrt{2}) & (1/\sqrt{2})^2 + (1/\sqrt{2})^2 \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle \mathbf{I} \rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum \mathbf{L}

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N -body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* \mathbf{L} relations with Levi-Civita analysis

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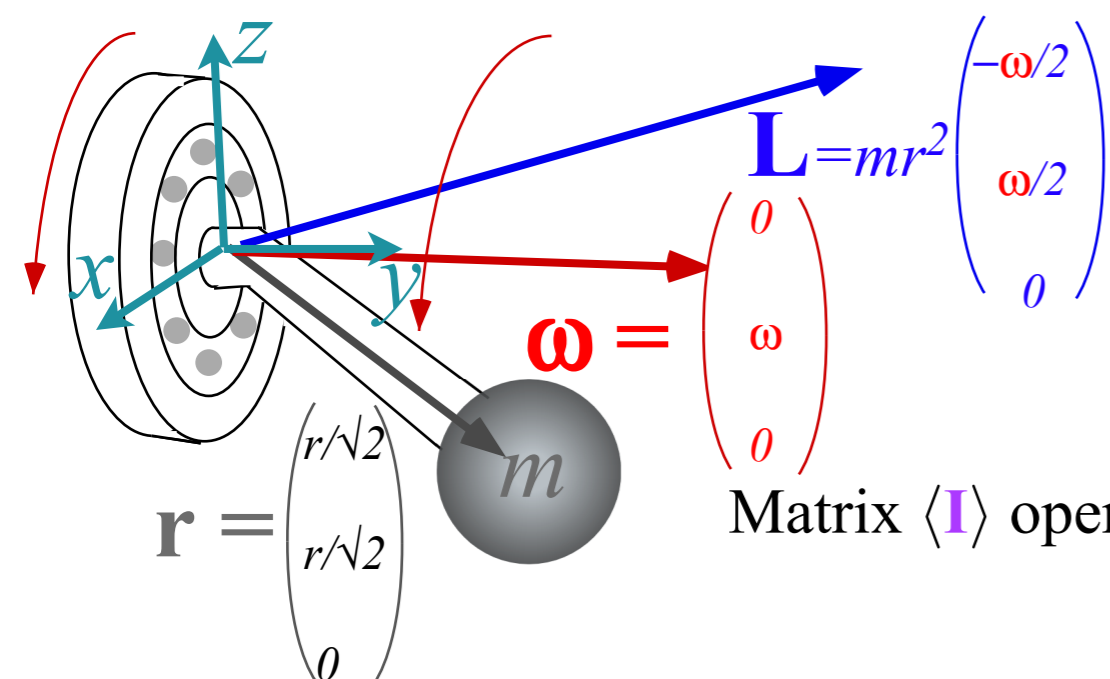
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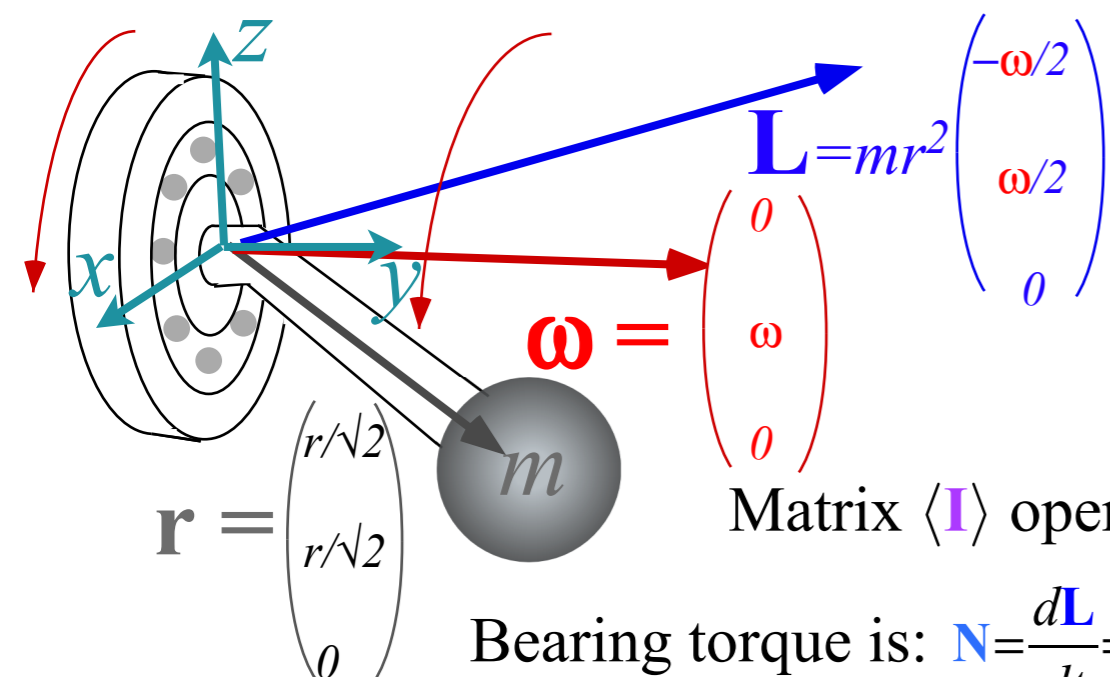
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Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \langle \tilde{\mathbf{I}} \rangle = \sum_{j=1}^N \langle \tilde{\mathbf{I}}_j \rangle = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:



Instantaneous matrix $\langle \mathbf{I} \rangle$ of inertia is:

$$\langle \tilde{\mathbf{I}} \rangle = mr^2 \begin{pmatrix} (1/\sqrt{2})^2 + 0 & -(1/\sqrt{2})(1/\sqrt{2}) & -(1/\sqrt{2})0 \\ -(1/\sqrt{2})(1/\sqrt{2}) & (1/\sqrt{2})^2 + 0 & -(1/\sqrt{2})0 \\ -0(1/\sqrt{2}) & -0(1/\sqrt{2}) & (1/\sqrt{2})^2 + (1/\sqrt{2})^2 \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle \mathbf{I} \rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum \mathbf{L}

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2 \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Kinetic energy in terms of velocity ω and rotational Lagrangian

Kinetic energy T of a rotating rigid body can be expressed in terms of the inertia matrix \mathbf{I}

$$T = \frac{1}{2} \sum_{j=1}^3 m_j \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j = \frac{1}{2} \sum_{j=1}^3 m_j (\boldsymbol{\omega} \times \mathbf{r}_j) \cdot (\boldsymbol{\omega} \times \mathbf{r}_j)$$

Levi-Civita identity
 $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$

$$T = \frac{1}{2} \sum_{j=1}^3 m_j [(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_j \cdot \mathbf{r}_j) - (\boldsymbol{\omega} \cdot \mathbf{r}_j)(\mathbf{r}_j \cdot \boldsymbol{\omega})]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \sum_{j=1}^3 m_j [(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - (\mathbf{r}_j)(\mathbf{r}_j)] \cdot \boldsymbol{\omega}$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \vec{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Kinetic energy is a *quadratic form*

$$T = \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \langle \omega | x \rangle & \langle \omega | y \rangle & \langle \omega | z \rangle \end{pmatrix} \begin{pmatrix} \langle x | \mathbf{I} | x \rangle & \langle x | \mathbf{I} | y \rangle & \langle x | \mathbf{I} | z \rangle \\ \langle y | \mathbf{I} | x \rangle & \langle y | \mathbf{I} | y \rangle & \langle y | \mathbf{I} | z \rangle \\ \langle z | \mathbf{I} | x \rangle & \langle z | \mathbf{I} | y \rangle & \langle z | \mathbf{I} | z \rangle \end{pmatrix} \begin{pmatrix} \langle x | \omega \rangle \\ \langle y | \omega \rangle \\ \langle z | \omega \rangle \end{pmatrix} \quad (\text{Dirac notation})$$

$$= \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \sum_{j=1}^3 m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Simplifies in *principle inertial axes* $\{X, Y, Z\}$ or *body eigen-axes*

$$T = \frac{1}{2} \begin{pmatrix} \omega_X & \omega_Y & \omega_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix} \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \omega_X & \omega_Y & \omega_Z \end{pmatrix} \begin{pmatrix} I_{XX} & 0 & 0 \\ 0 & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{pmatrix} \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix} = \frac{I_{XX} \omega_X^2}{2} + \frac{I_{YY} \omega_Y^2}{2} + \frac{I_{ZZ} \omega_Z^2}{2}$$

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

$$\mathbf{L} = \vec{\mathbf{I}} \cdot \boldsymbol{\omega}, \quad \text{generally implies:} \quad \boldsymbol{\omega} = \vec{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

Express kinetic energy T in terms of angular velocity $\boldsymbol{\omega}$, momentum \mathbf{L} , or both at once. once

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \vec{\mathbf{I}} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} = \frac{1}{2} \mathbf{L} \cdot \vec{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

$$\begin{aligned} T &= \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}} \end{aligned}$$

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

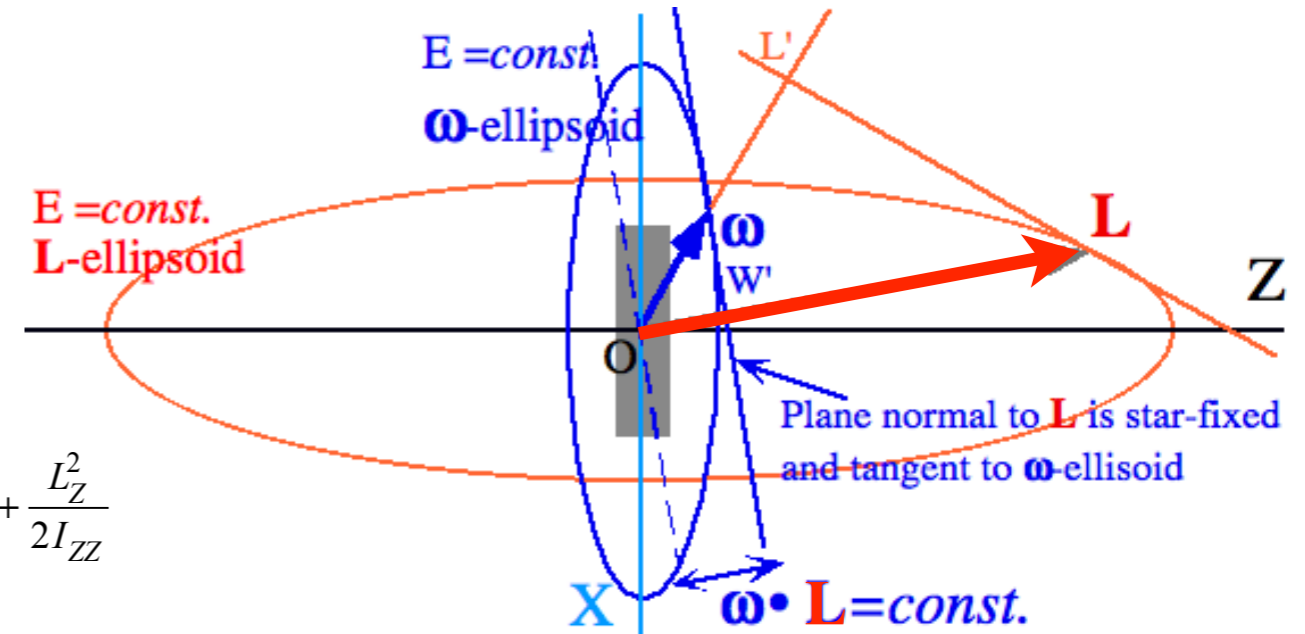
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Hamiltonian form is the equation of the *angular momentum or L-ellipsoid*
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Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

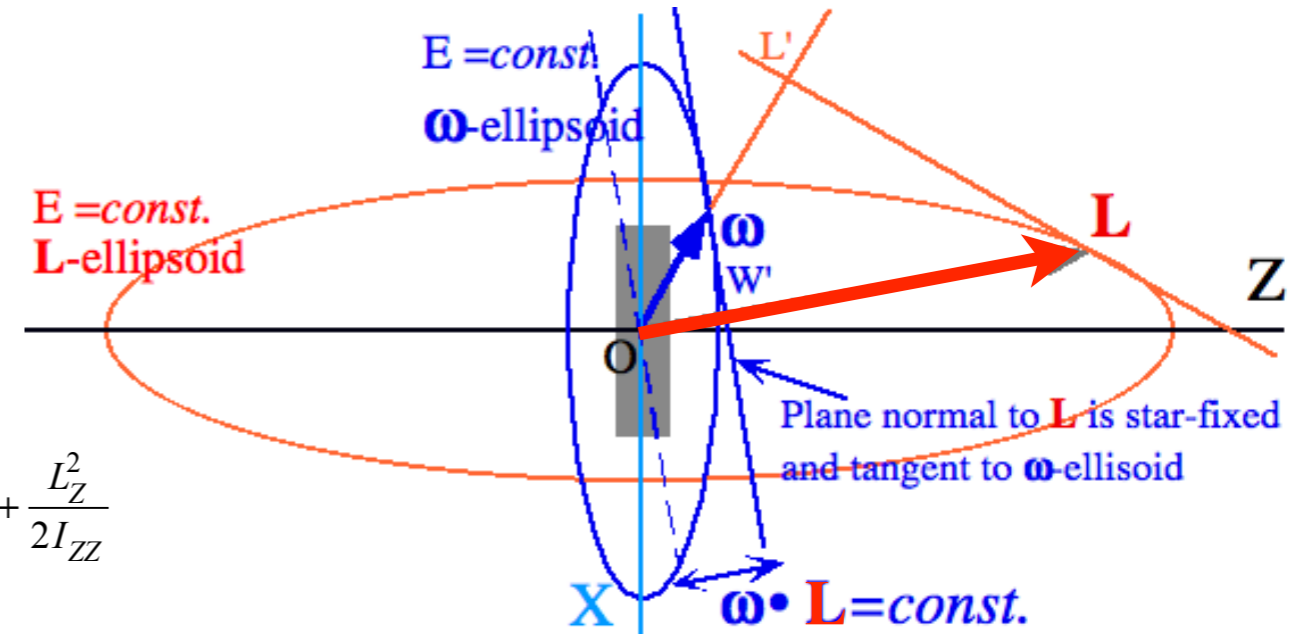
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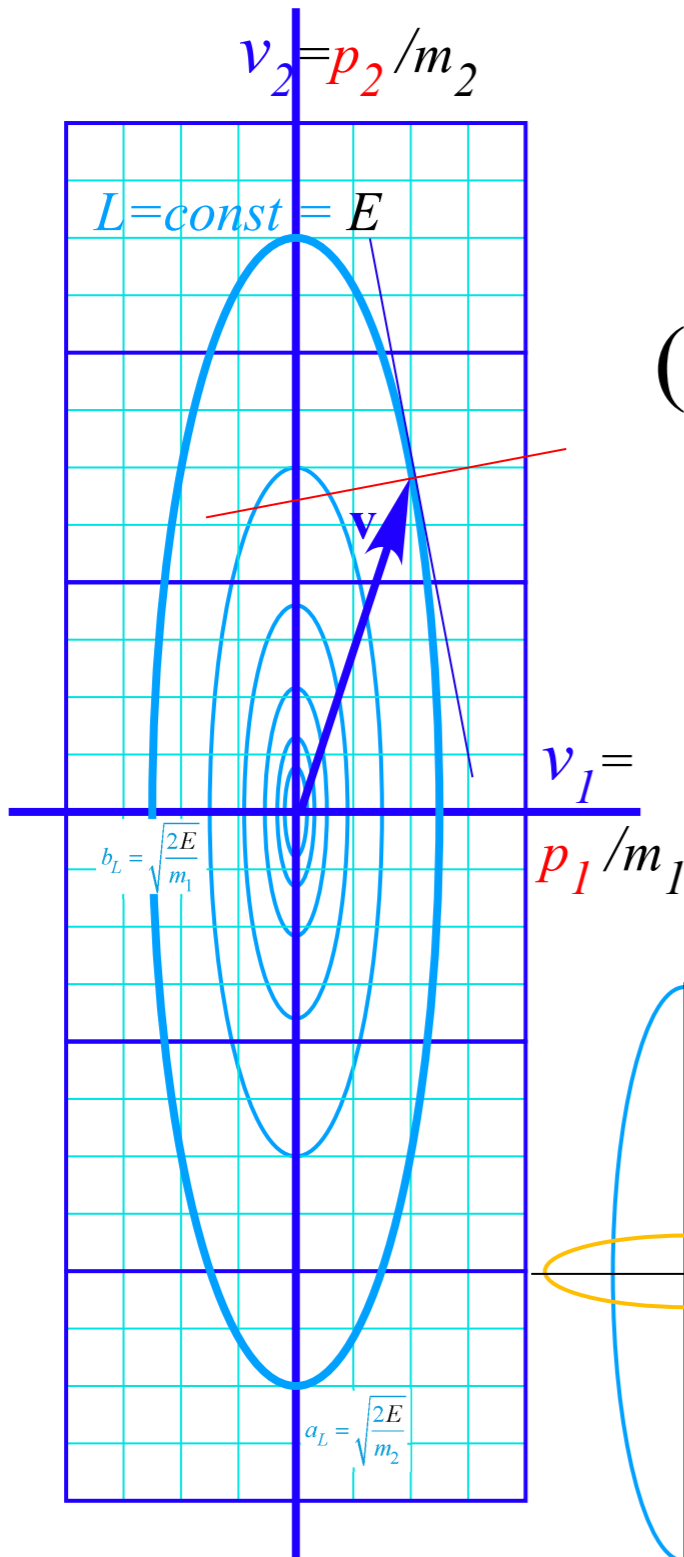
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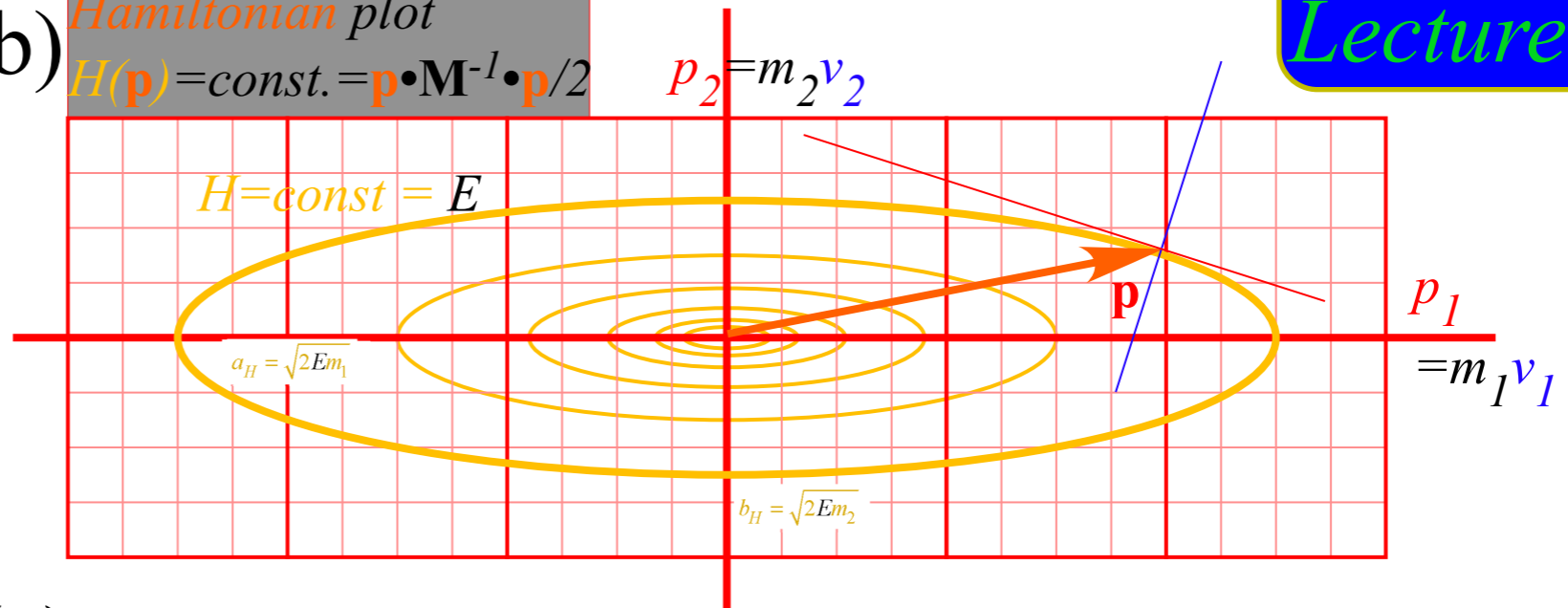
Hamiltonian form is the equation of the *angular momentum or L-ellipsoid*
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Recall quadratic forms for Lagrangian and Hamiltonian in Lecture 10 unit 1?

(a) *Lagrangian plot*
 $L(\mathbf{v}) = \text{const.} = \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} / 2$



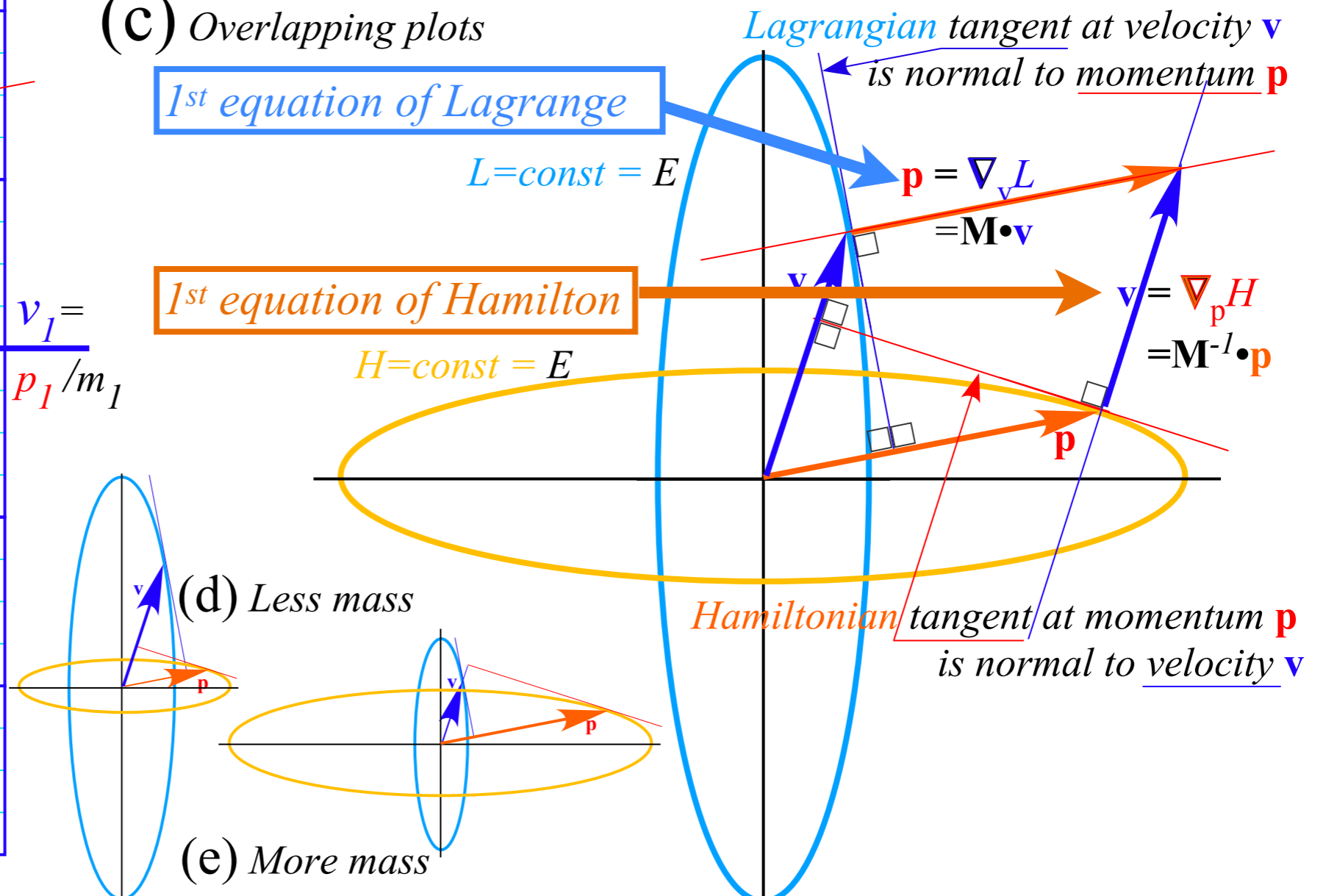
(b) *Hamiltonian plot*
 $H(\mathbf{p}) = \text{const.} = \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} / 2$



(c) *Overlapping plots*

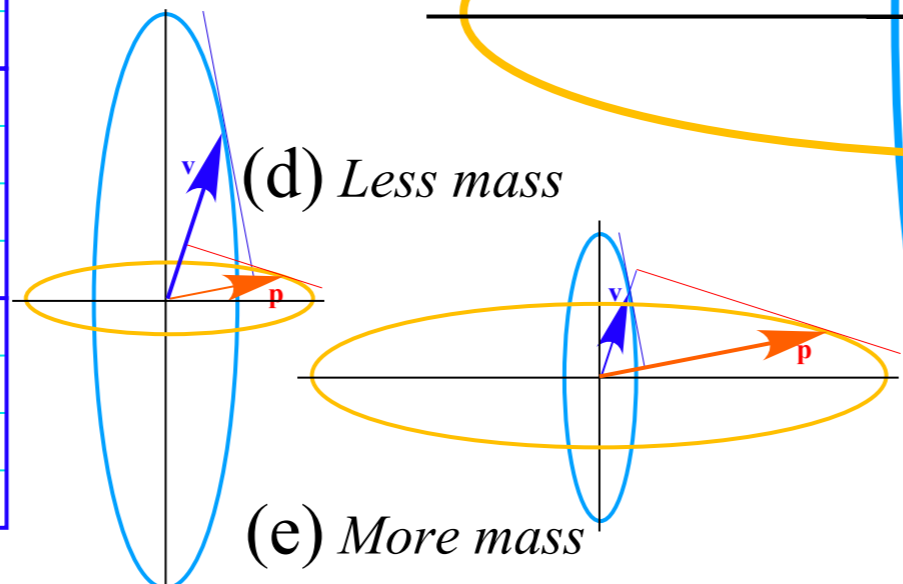
1st equation of Lagrange

1st equation of Hamilton



(d) *Less mass*

(e) *More mass*



Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

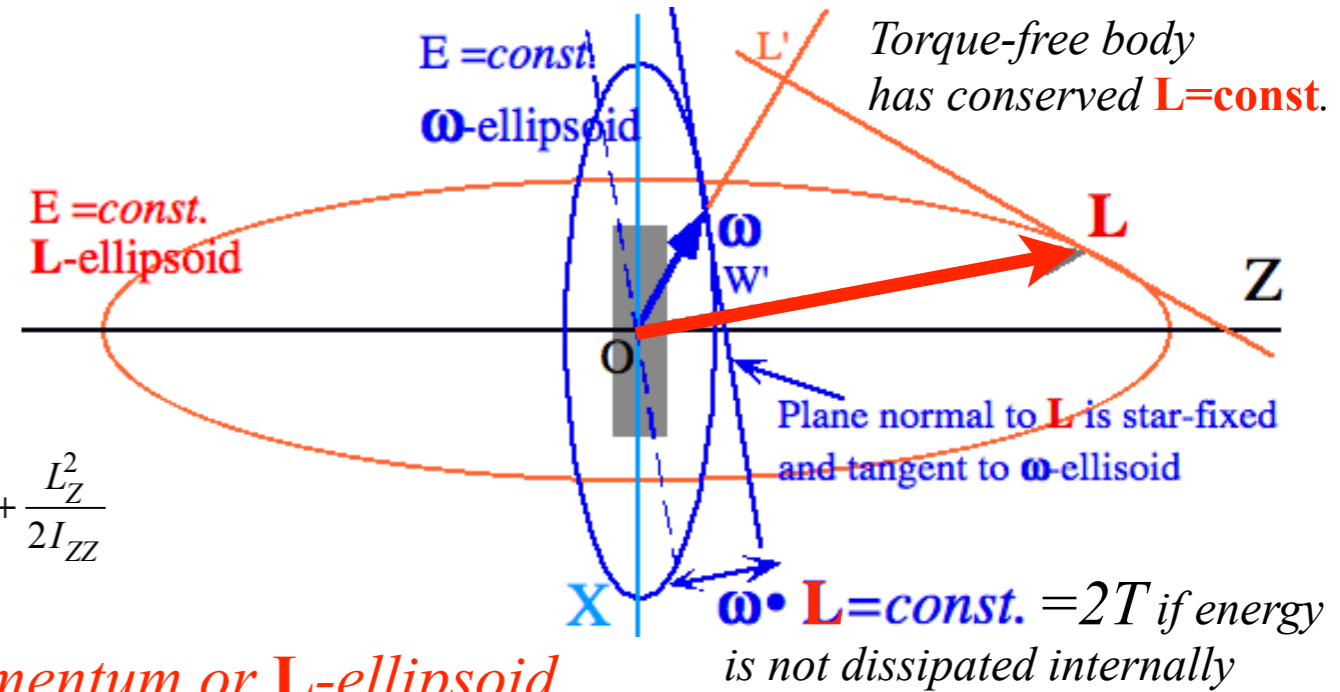
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Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

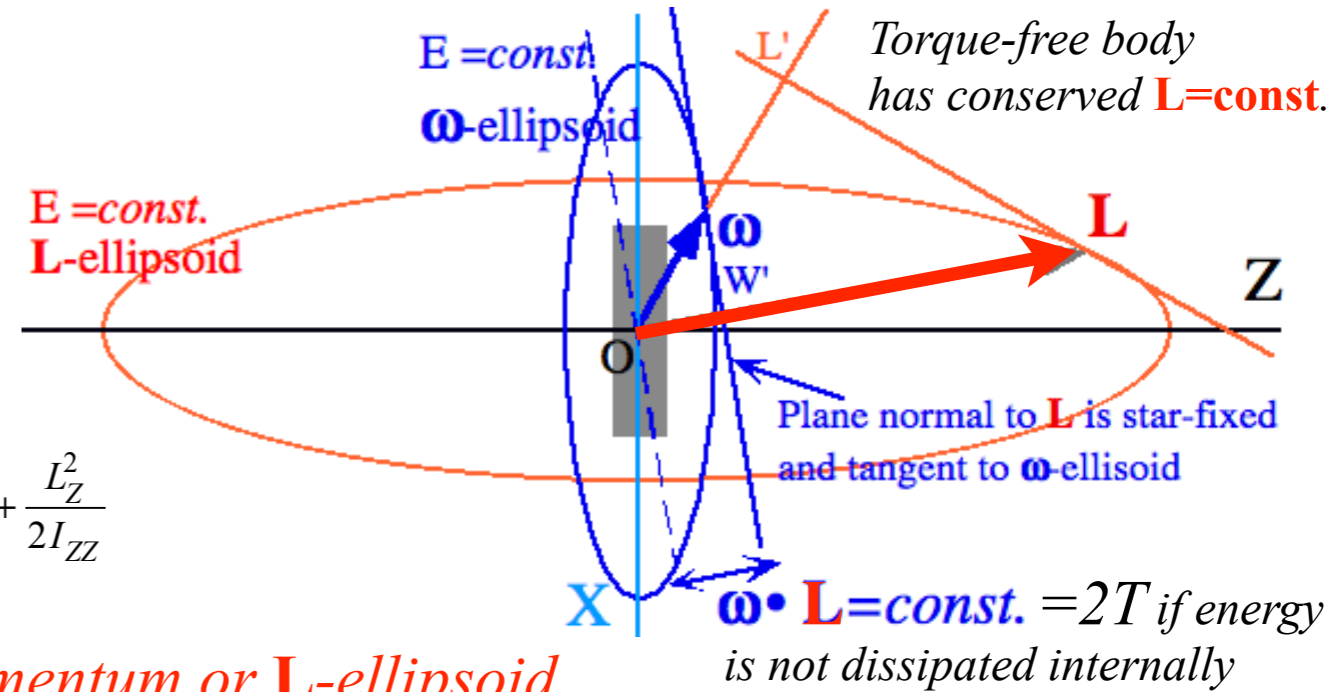
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Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

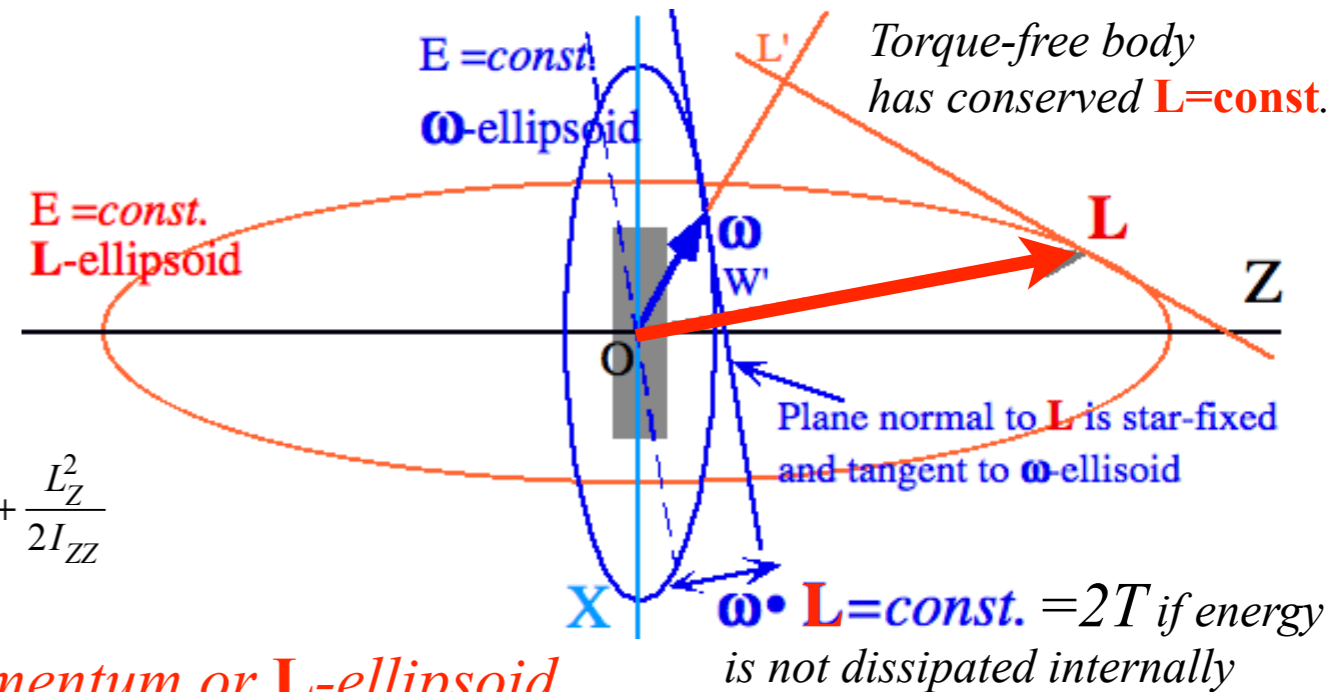
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Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

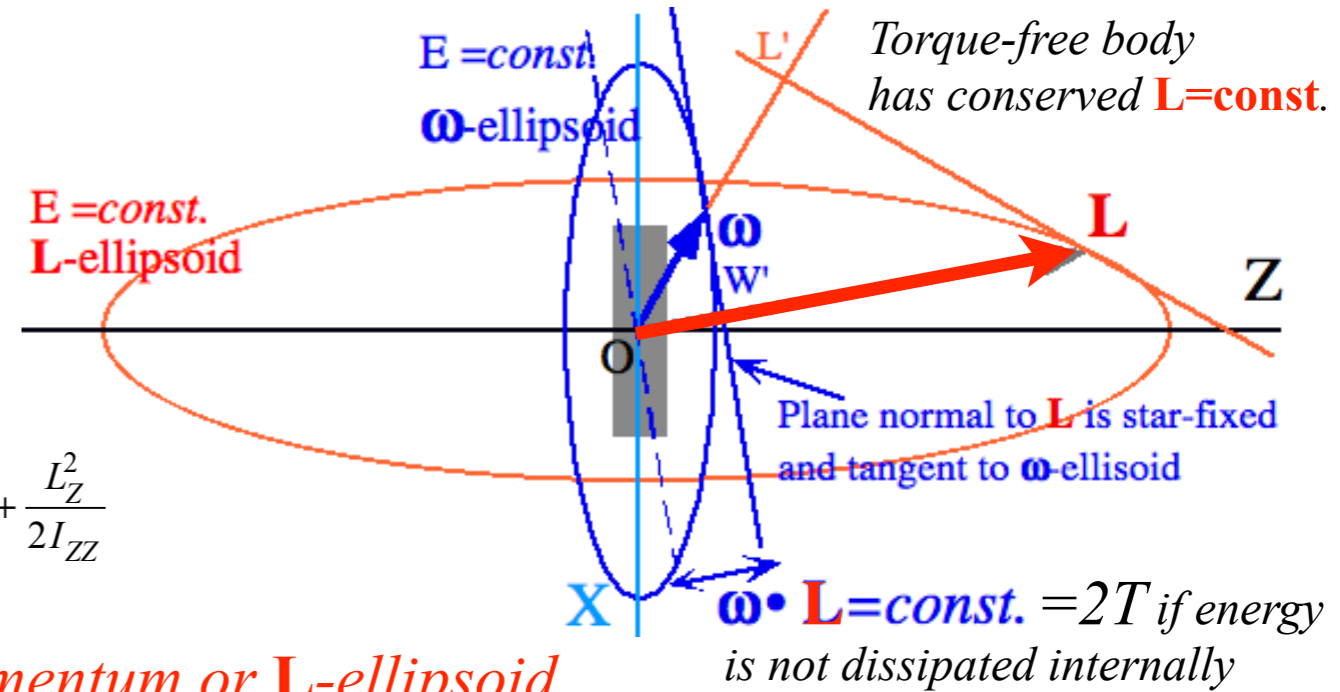
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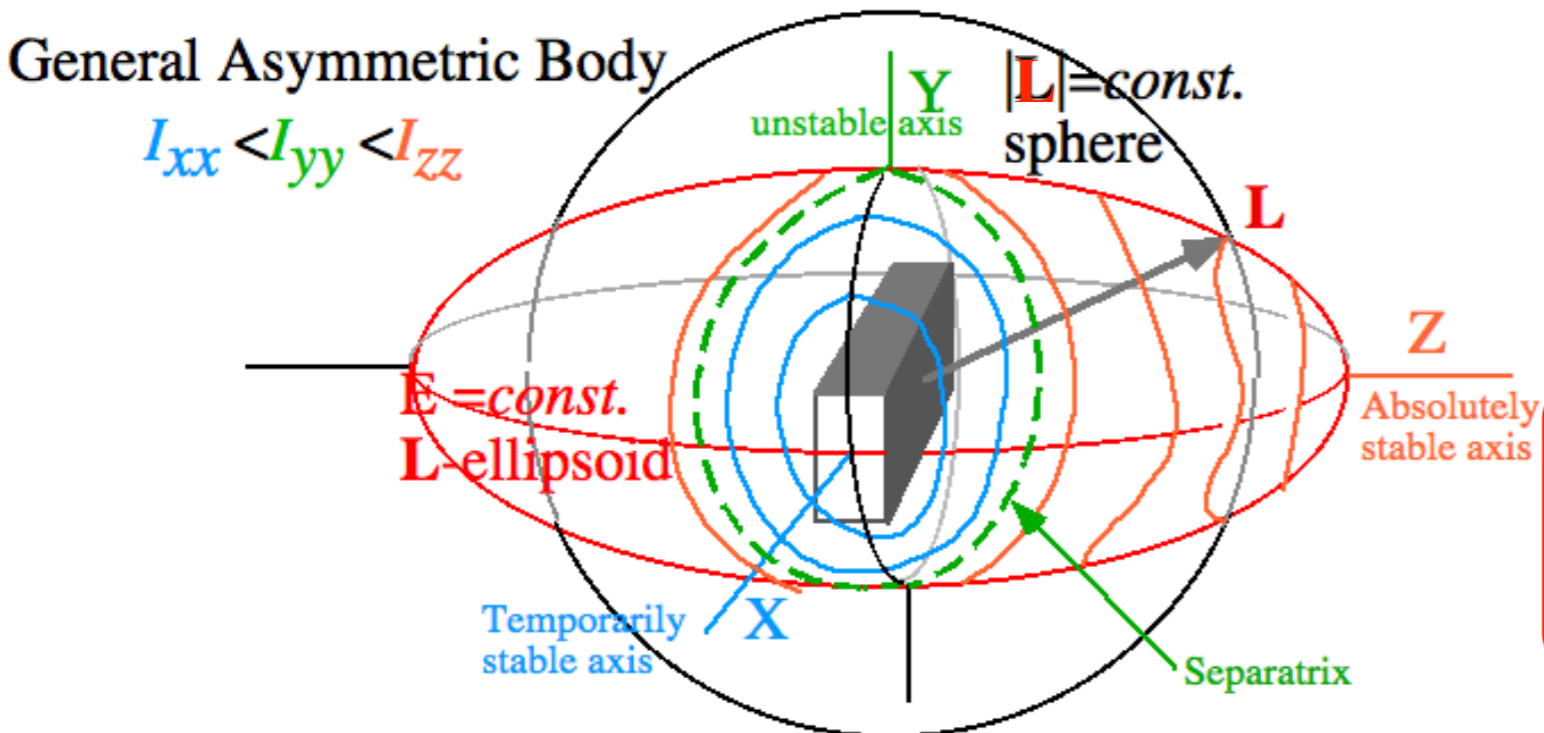
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Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

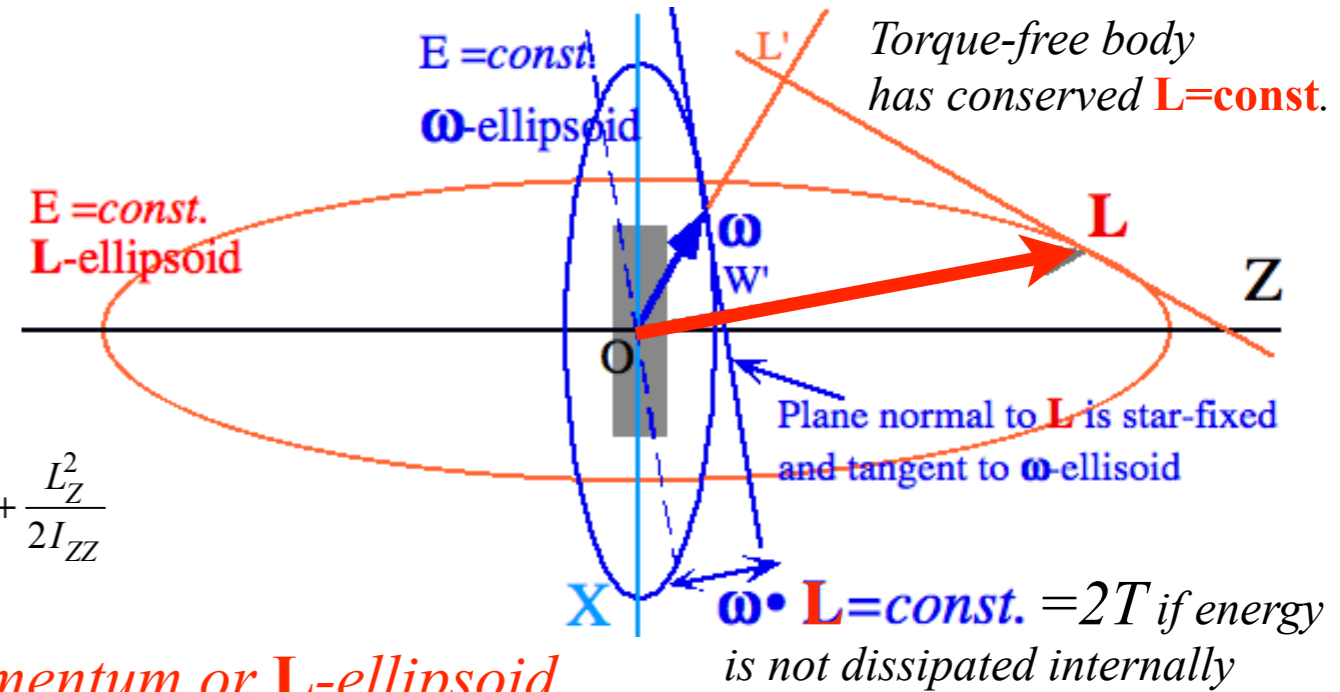
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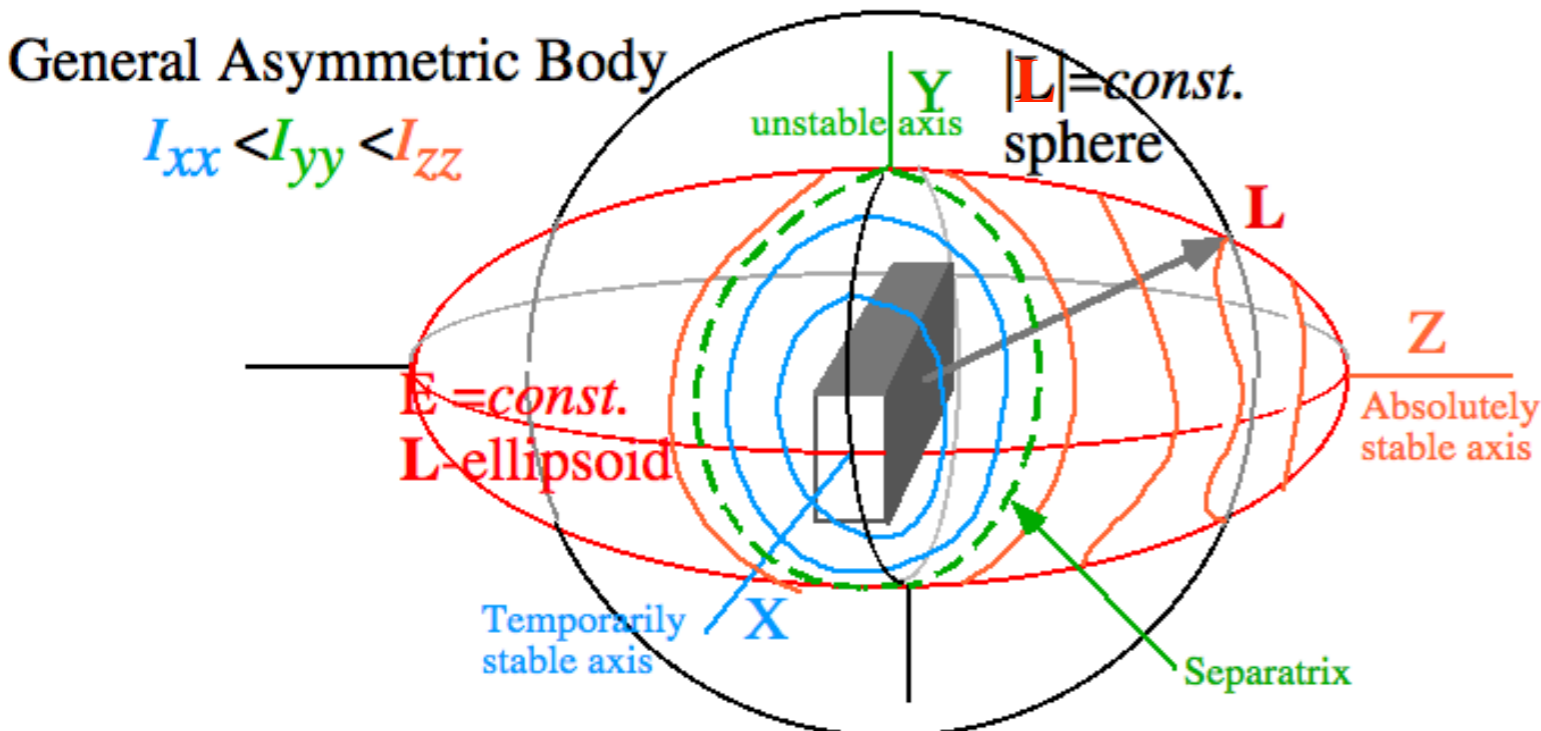
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$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \cdot \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}}{2} = \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

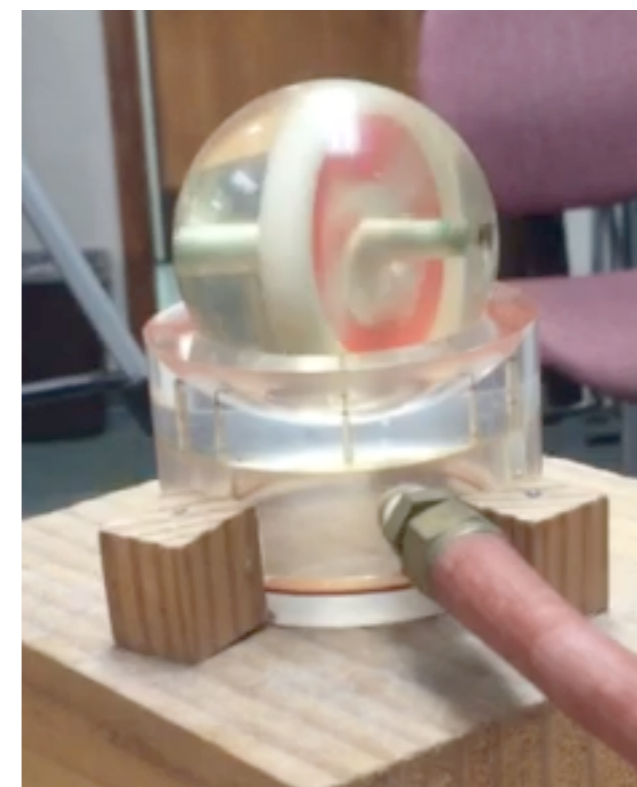
In body frame momentum \mathbf{L} moves along intersection of *L-ellipsoid* and *L-sphere* (Length $|\mathbf{L}|$ is constant in any classical frame.)



[Asymmetric Top
Demo video](#)

Rotational Energy Surfaces (RES)

➔ *Symmetric, asymmetric, and spherical-top dynamics (Constant L)
BOD-frame cone rolling on **LAB** frame cone*



[Singular Motion of
Asymmetric Rotators](#)
AJP 44, 11 p1080 1976

Asymmetric-top dynamics (Constant L)

1. NASA Space station video



<https://youtu.be/1n-HMSCDYtM>

For those physicist who are brave of heart, make note the video's comments

2. UAF lab air-supported asymmetric top video



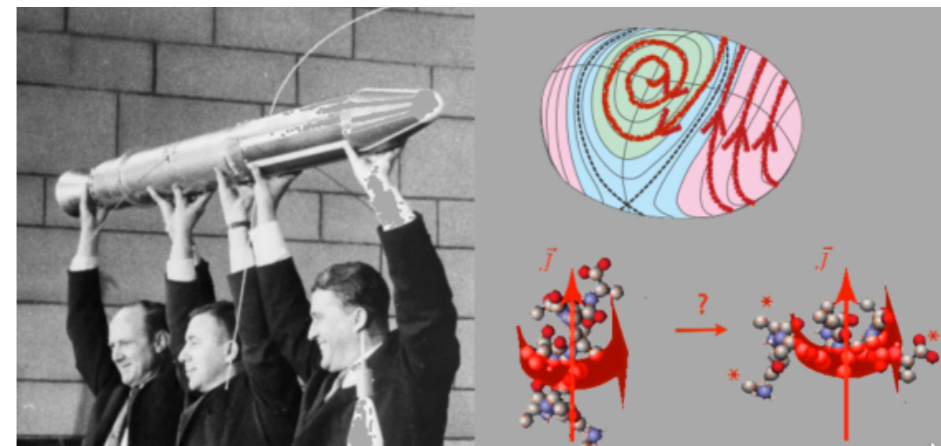
<https://youtu.be/HWjGvCaqx5g>

3. NASA-Rotating Solid Bodies in Microgravity (2008)



<https://www.youtube.com/watch?v=BPMjcN-sBJ4>

4. Early NASA-JPL satellite blunder (1958)



To be Continued \Rightarrow several pages ahead

Comments following Space Lab video of asymmetric rotation show that it is not a widely understood phenomenon



Bagnon DuJour • 3 months ago

As the handles spins out it dips down a bit before becoming detached and that linear momentum travels through the angular momentum until the equilibrium requires the flip to maintain the path of least resistance. If they could spin it perfectly without the dip, it would not turn like that.

^ | v • Reply • Share ›



Bill Aldridge → Bagnon DuJour • 3 months ago

So you are saying, when they put their hands on the tip, i dip, you dip, we dip.

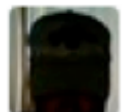
^ | v • Reply • Share ›



EVERYONE is born an atheist → Bagnon DuJour • 3 months ago

Exactly. Not sure why this was even posted. Maybe it was just going to b used as a basic physics example for schools.

^ | v • Reply • Share ›

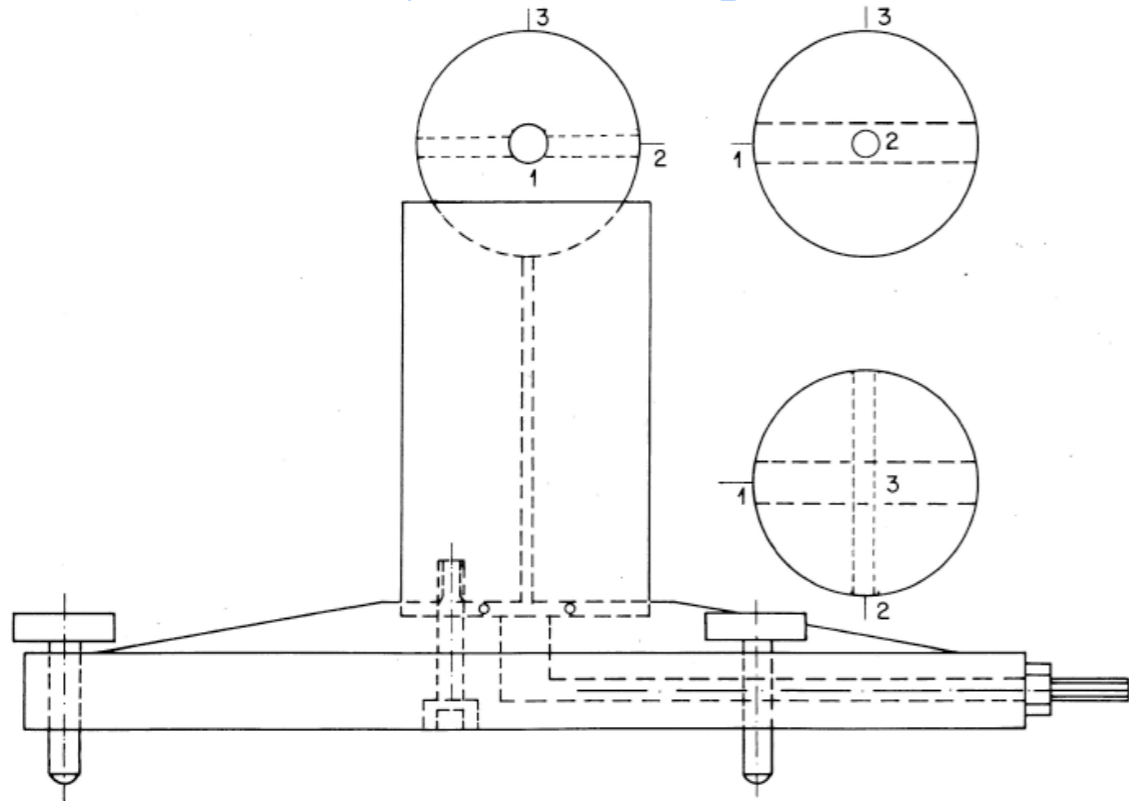


Tim Johnson → Bagnon DuJour • 3 months ago

It sounds like you have a handle on what's going on here.

1 ^ | v • Reply • Share ›





$$\begin{aligned}
 I_1 &= (2M/5 + m_2/3)R^2 + m_1r_1^2/2 + m_2r_2^2/4, \\
 I_2 &= (2M/5 + m_1/3)R^2 + m_2r_2^2/2 + m_1r_1^2/4, \quad (8) \\
 I_3 &= (2M/5 + m_1/3 + m_2/3)R^2 + m_1r_1^2/4 + m_2r_2^2/4,
 \end{aligned}$$

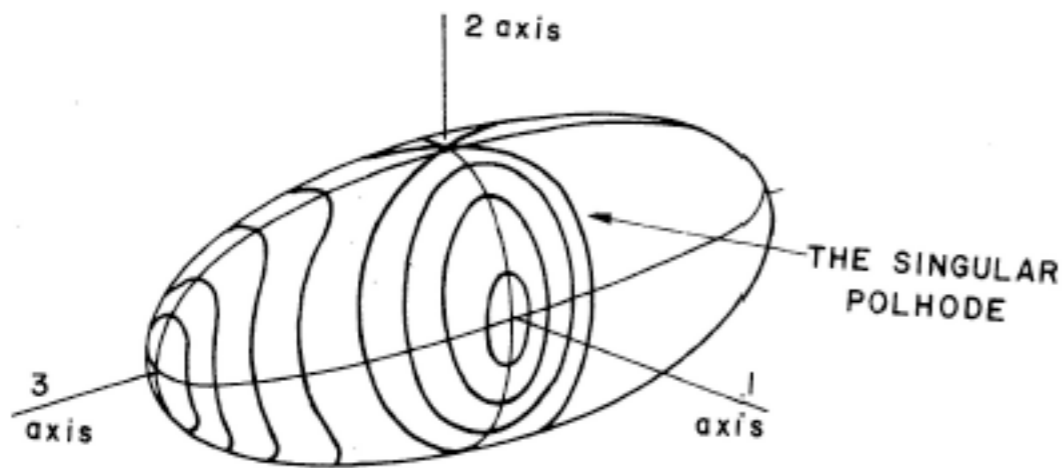


Fig. 3. Polhodes. A family of constraint curves for the vector ω in the body system, or "polhodes," are separated into two distinct groups by a curve called the singular polhode.

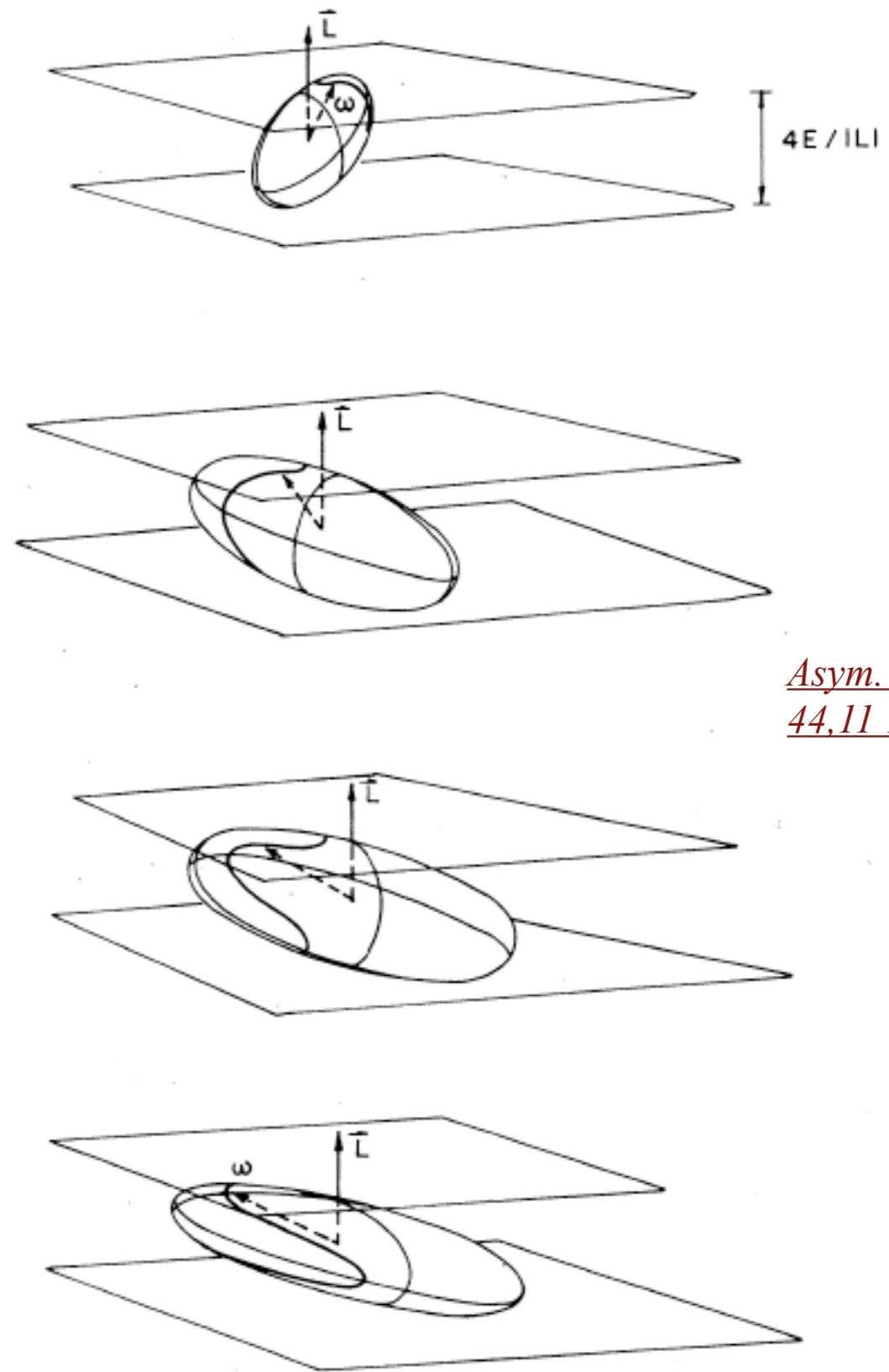


Fig. 4. Model of rotational motion near the singular polhode.

*Asym. Rotor AJP
44,11 1976*

Bocce-Ball Asymmetric Top Motion solved by Euler's equation and elliptic integrals

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L}, \quad (9)$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) / I_2 = 0. \quad (10)$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2} (c - d\omega^2)^{1/2} / I_2 (I_1 I_3)^{1/2}, \quad (11)$$

where the constants $a-d$ [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2),$$

$$c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1),$$

$$a = I_2(I_3 - I_2)W^2 \cos^2 \epsilon,$$

$$c = [I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon]W^2, \quad (12)$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W \cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \quad (13)$$

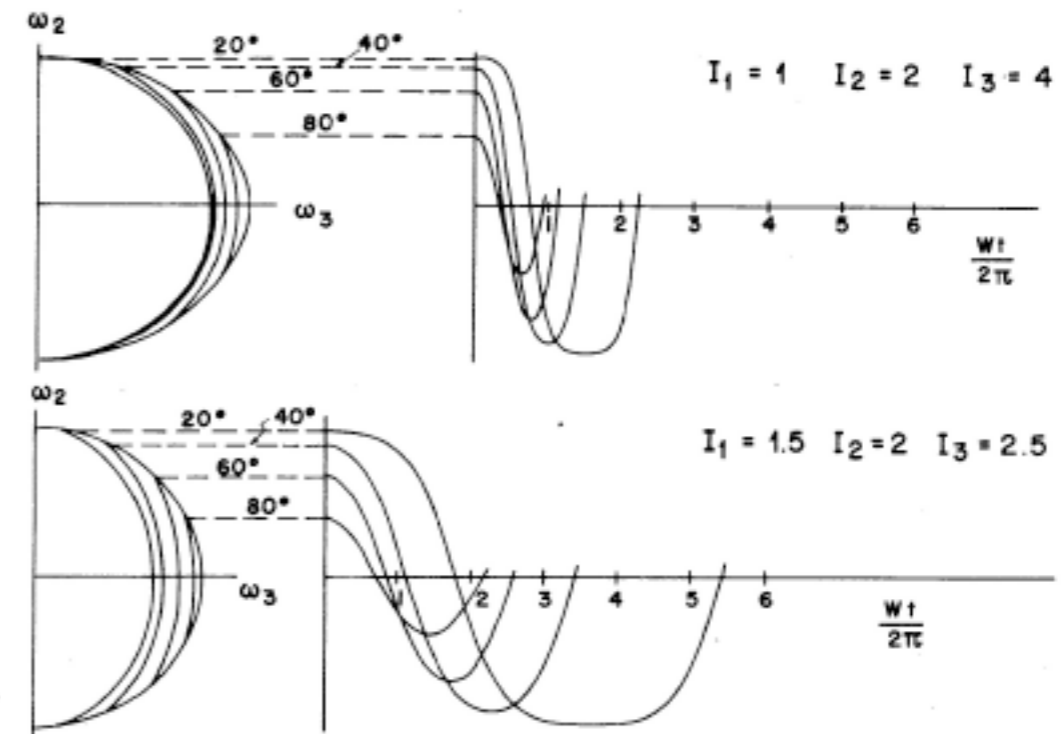


Fig. 6. Exact solutions. The motion of the $\boldsymbol{\omega}$ vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

Bocce-Ball Asymmetric Top Motion solved by Euler's equation and elliptic integrals

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$$c = [I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon]W^2, \quad (12)$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W \cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \quad (13)$$

$$t = \left(\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2EI_1)} \right)^{1/2} \times \int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2} (1 - k^2 \Omega^2)^{1/2}}, \quad (14)$$

where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2} \Omega = \Omega W \cos \epsilon. \quad (15)$$

A further substitution $\Omega = \sin \phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2} (1 - k^2 \Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \equiv \text{sn}^{-1}(\phi', k). \quad (16)$$

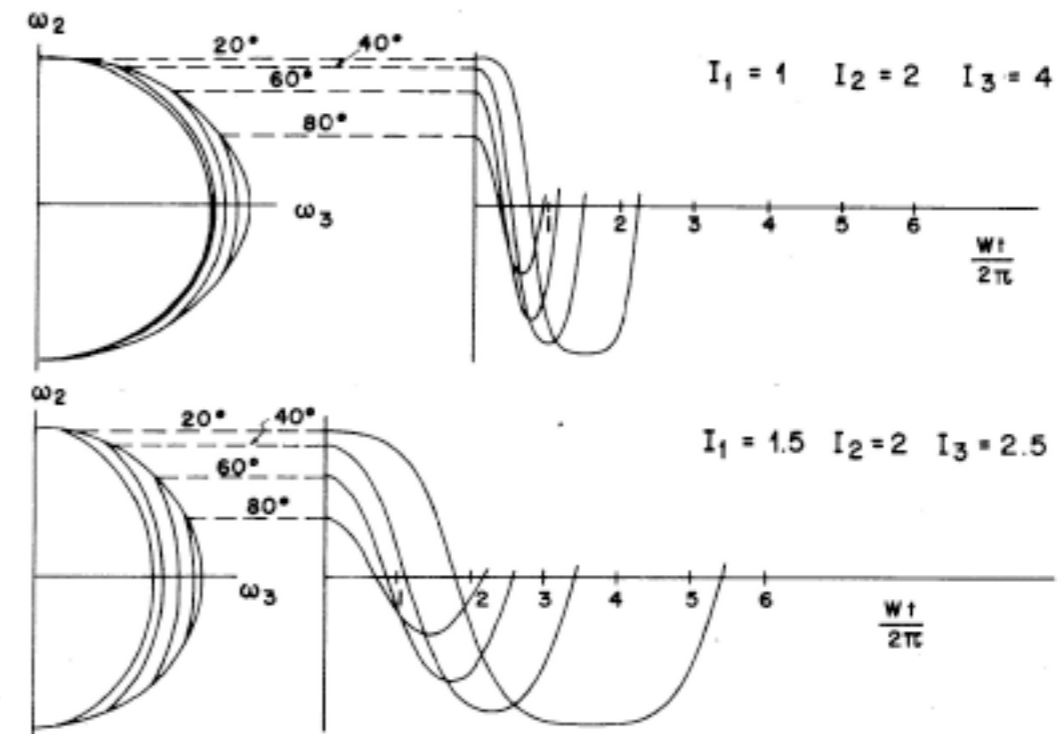


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$$\dot{\omega} = (a - b\omega^2)^{1/2}(c - d\omega^2)^{1/2}/I_2(I_1I_3)^{1/2}, \quad (11)$$

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where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W \cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \quad (13)$$

$$t = \left(\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2EI_1)} \right)^{1/2} \times \int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2 \Omega^2)^{1/2}}, \quad (14)$$

where the following substitutions were made:

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A further substitution $\Omega = \sin \phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2 \Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \equiv \text{sn}^{-1}(\phi', k). \quad (16)$$

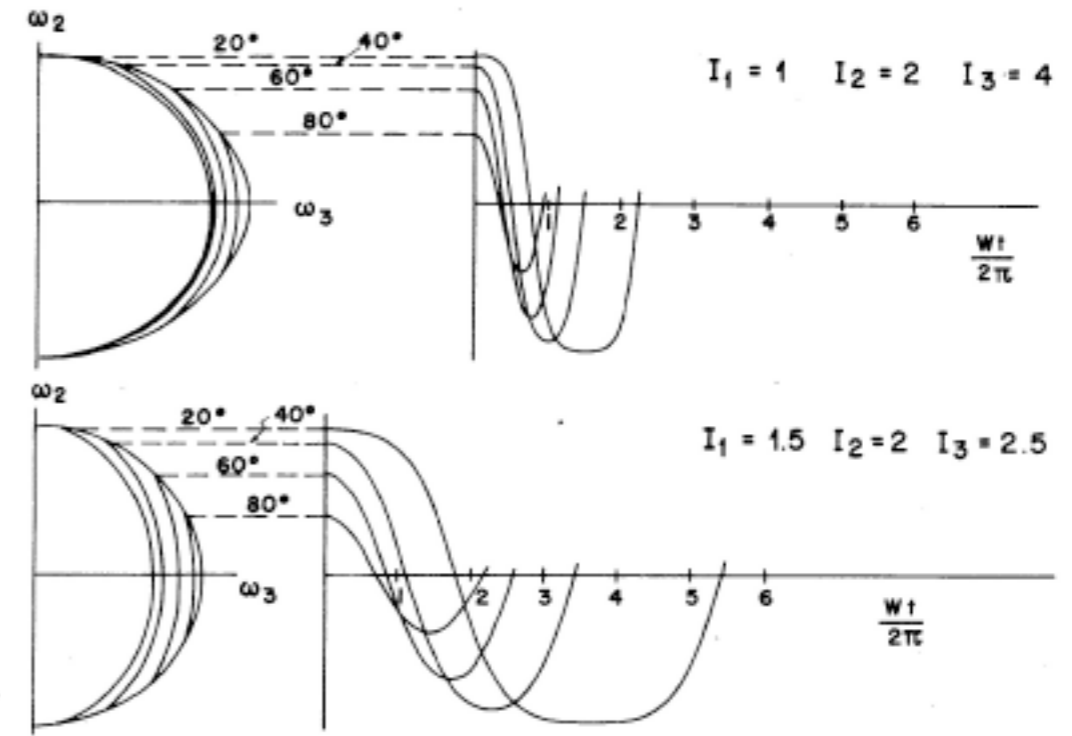


Fig. 6. Exact solutions. The motion of the $\boldsymbol{\omega}$ vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

$$t = \frac{2}{W} \times \left(\frac{I_1 I_2 I_3}{(I_3 - I_2)[I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon]} \right)^{1/2} \times \text{sn}^{-1} \left(\frac{\pi}{2}, k \right), \quad (17a)$$

$$t \rightarrow \frac{2}{W} \left(\frac{I_1 I_2}{(I_3 - I_2)(I_2 - I_1)} \right)^{1/2} \text{sn}^{-1} \left(\frac{\pi}{2}, k \right), \quad (17b)$$

where

$$k = \left(\frac{I_2(I_2 - I_1)}{I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon} \right)^{1/2} \cos \epsilon, \quad (18a)$$

$$k \rightarrow 1 - (I_3/I_2)[(I_3 - I_1)/(I_2 - I_1)](\epsilon^2/2). \quad (18b)$$

Bocce-Ball Asymmetric Top Motion solved by Euler's equation and elliptic integrals

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L}, \quad (9)$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) / I_2 = 0. \quad (10)$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2} (c - d\omega^2)^{1/2} / I_2 (I_1 I_3)^{1/2}, \quad (11)$$

where the constants $a-d$ [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2),$$

$$c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1),$$

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$$c = [I_2(I_2 - I_1) \cos^2 \epsilon + I_3(I_3 - I_1) \sin^2 \epsilon]W^2, \quad (12)$$

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where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2} \Omega = \Omega W \cos \epsilon. \quad (15)$$

A further substitution $\Omega = \sin \phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2} (1 - k^2 \Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \equiv \text{sn}^{-1}(\phi', k). \quad (16)$$

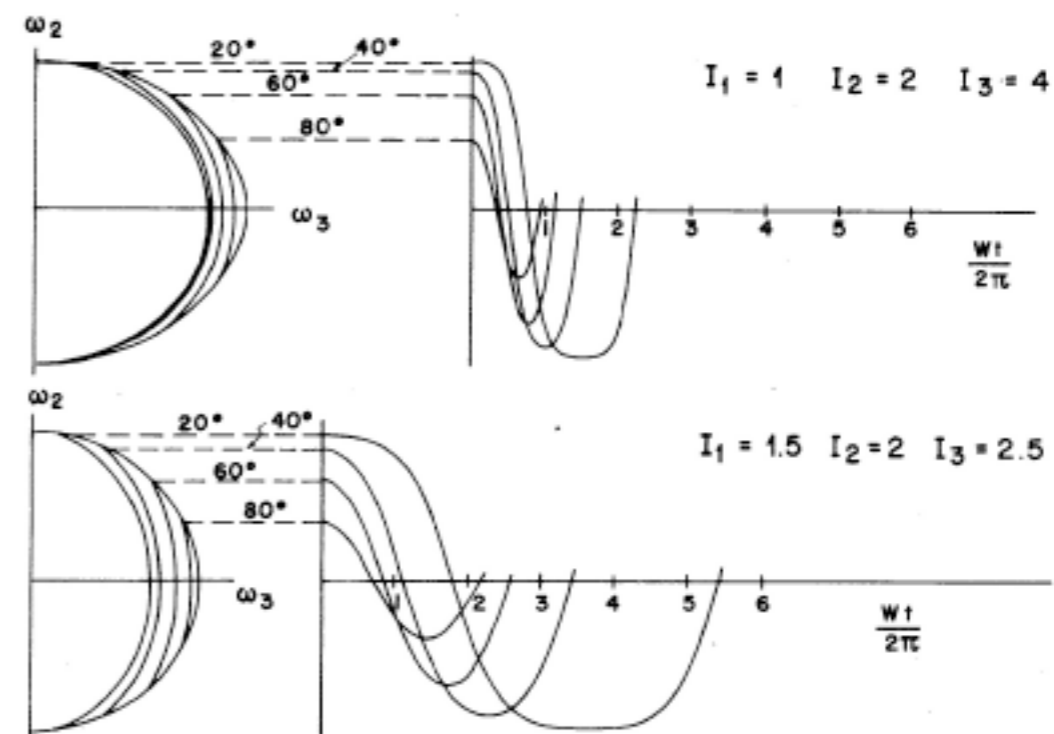
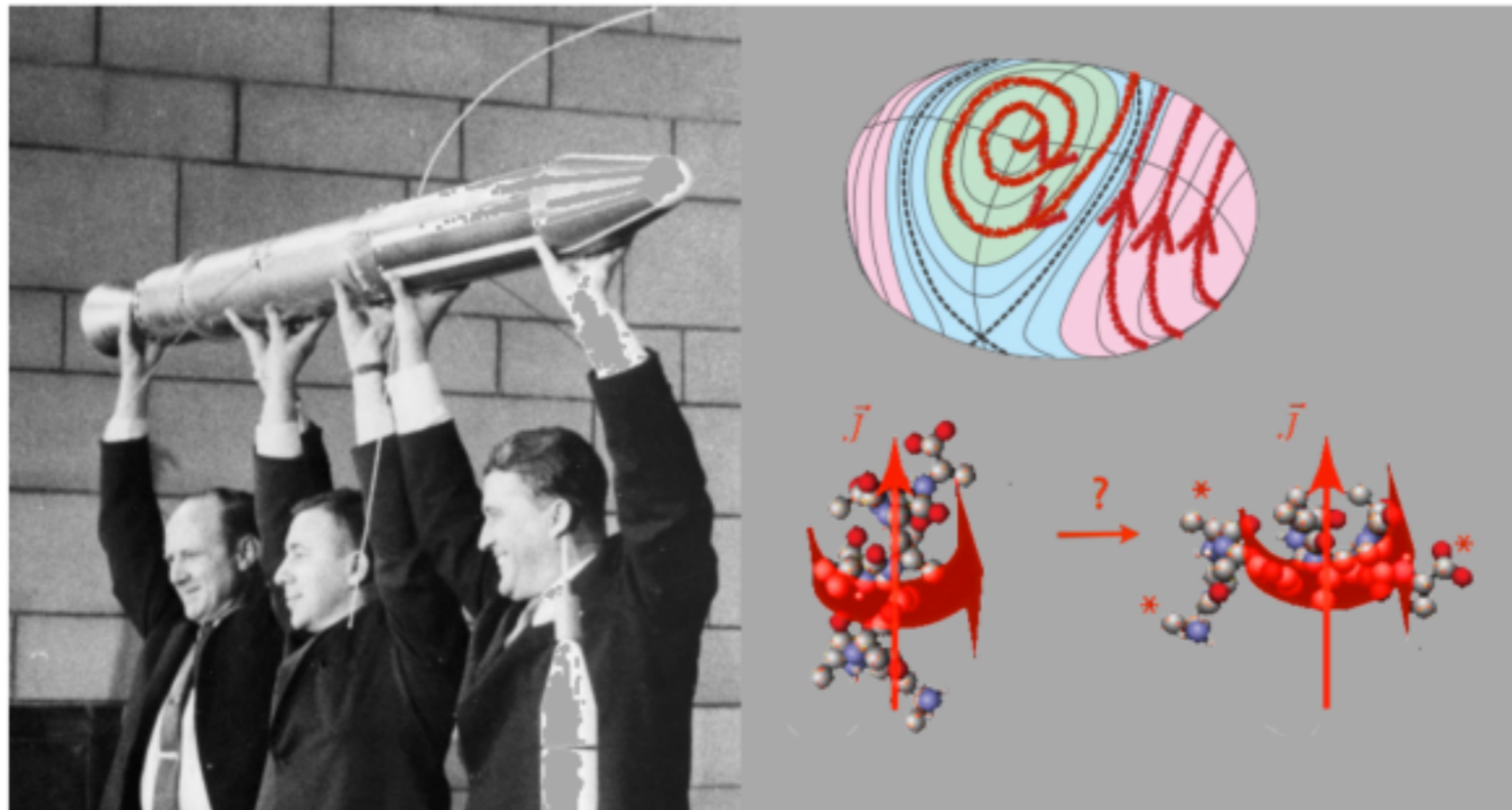


Fig. 6. Exact solutions. The motion of the $\boldsymbol{\omega}$ vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

The limiting forms [Eqs. (17) and (18b)] become good approximations for $\epsilon < 10^\circ$. The approximate number of revolutions accomplished by a body before it overturns is given by the product of $W/2\pi$, the number of revolutions per second, and the right-hand side of Eq. (17b). Exact solutions for various I_j and ϵ are displayed in Fig. 6.

If one desires to increase the reversal time, it should be done through the first factor in Eq. (17b). The integral in the second factor is usually only as large as 7 or 8 in our experiments ($\epsilon = 10^\circ$ gives 3.1, $\epsilon = 1^\circ$ gives 5.4, and $\epsilon = 0^\circ$ gives 9.5). This is a good demonstration of the behavior of an elliptic function near its singularity.

4. Early NASA-JPL satellite blunder (1958)



*From text*in preparation
by Rick Heller on semi-
classical dynamics of
polyatomic molecules*

**Now available:*

Eric Heller,

[The Semiclassical Way](#)

Princeton University Press (2018)

Figure 10.3: NASA-JPL early blunder. Rockets are not rigid bodies, especially with floppy whip antennas attached. The Explorer 1 satellite was the first one launched successfully by the United States. Seen in the left panel are James van Allen (center), William Pickering (left), and Werner von Braun, with a full-size model of the satellite, just after it was successfully orbited in 1958. As this press conference took place, the satellite was busily tumbling out of control. Van Allen soon realized that the intermittent signal from the satellite was due to tumbling. Fortunately, enough antennas were bristling from the satellite that it still gave much useful data, resulting in discovery of the van Allen radiation belts. The tumbling took place because friction due to slight wobbling is converted to heat, lowering the rotational energy, but not changing the angular momentum. The only way for this to happen is for the satellite to start rotating around a lower energy axis, until it bottoms out in end and over and tumbling at the lowest possible rotational energy for the given angular momentum. The author thanks Prof. William Harter for pointing out the existence and the physics of this story.

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES) replace Lagrange Poinsot $\frac{1}{2}\omega \cdot I \cdot \omega$

Rotational Energy Surface (RES) is quadratic multipole function plotted radially

$$E = \frac{\mathbf{J}_x^2}{2I_x} + \frac{\mathbf{J}_y^2}{2I_y} + \frac{\mathbf{J}_z^2}{2I_z} \text{ with } J = \text{const.}$$

$$= J^2 \left(\frac{\sin^2\theta \cos^2\phi}{2I_x} + \frac{\sin^2\theta \sin^2\phi}{2I_y} + \frac{\cos^2\theta}{2I_z} \right)$$

Constant Energy Surface (CES) is asymmetric ellipsoid of constant E

$$E = \frac{\mathbf{J}_x^2}{2I_x} + \frac{\mathbf{J}_y^2}{2I_y} + \frac{\mathbf{J}_z^2}{2I_z} = \text{const.}$$

or : $\frac{\mathbf{J}_x^2}{2EI_x} + \frac{\mathbf{J}_y^2}{2EI_y} + \frac{\mathbf{J}_z^2}{2EI_z} = 1$

Here notation L or \mathbf{L} for angular momentum is replaced by J or \mathbf{J}

(a) RE surface

(b) CE surface

(c) RES intersecting CES

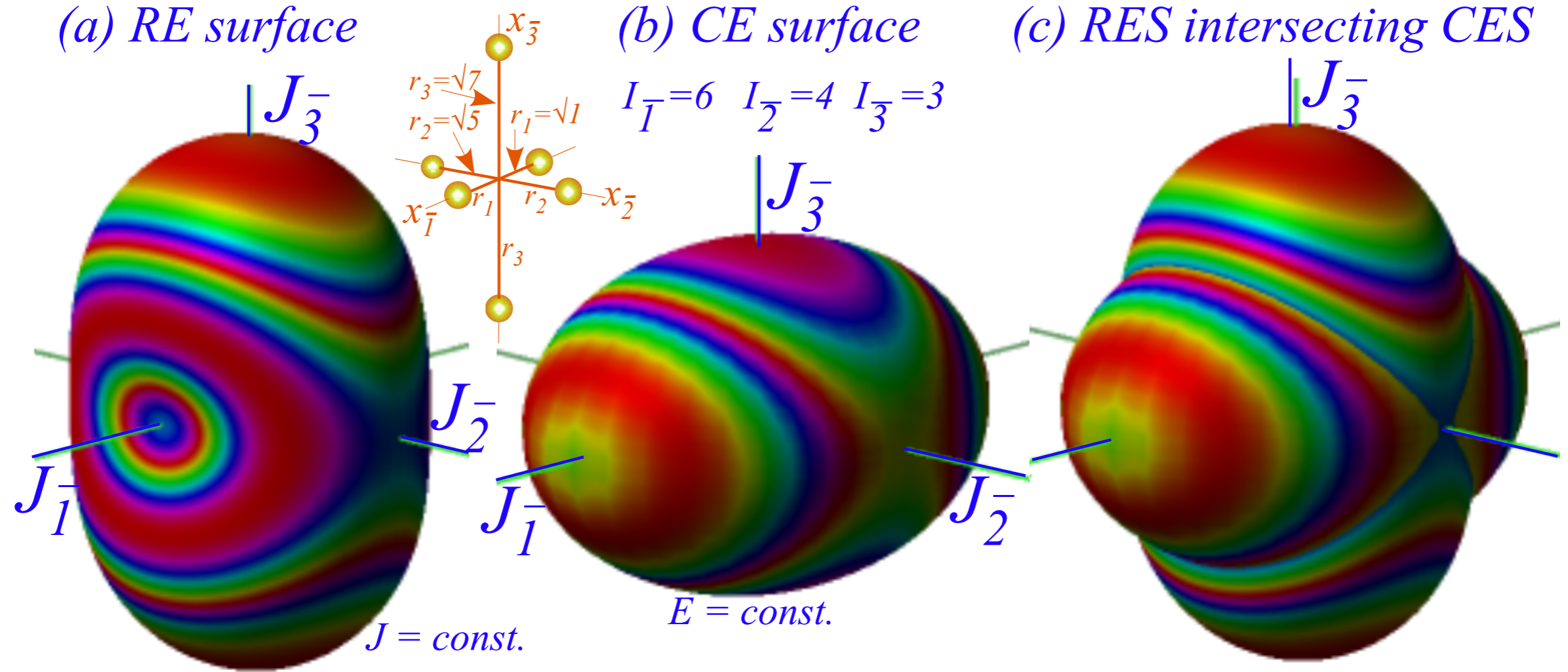


Fig. 6.8.1 Rigid rotor surfaces (a) RES polynomial, (b) CES ellipsoid, and (c) RES and CES intersected.

RES and CES for nearly-symmetric prolate rotors and nearly-symmetric oblate rotors

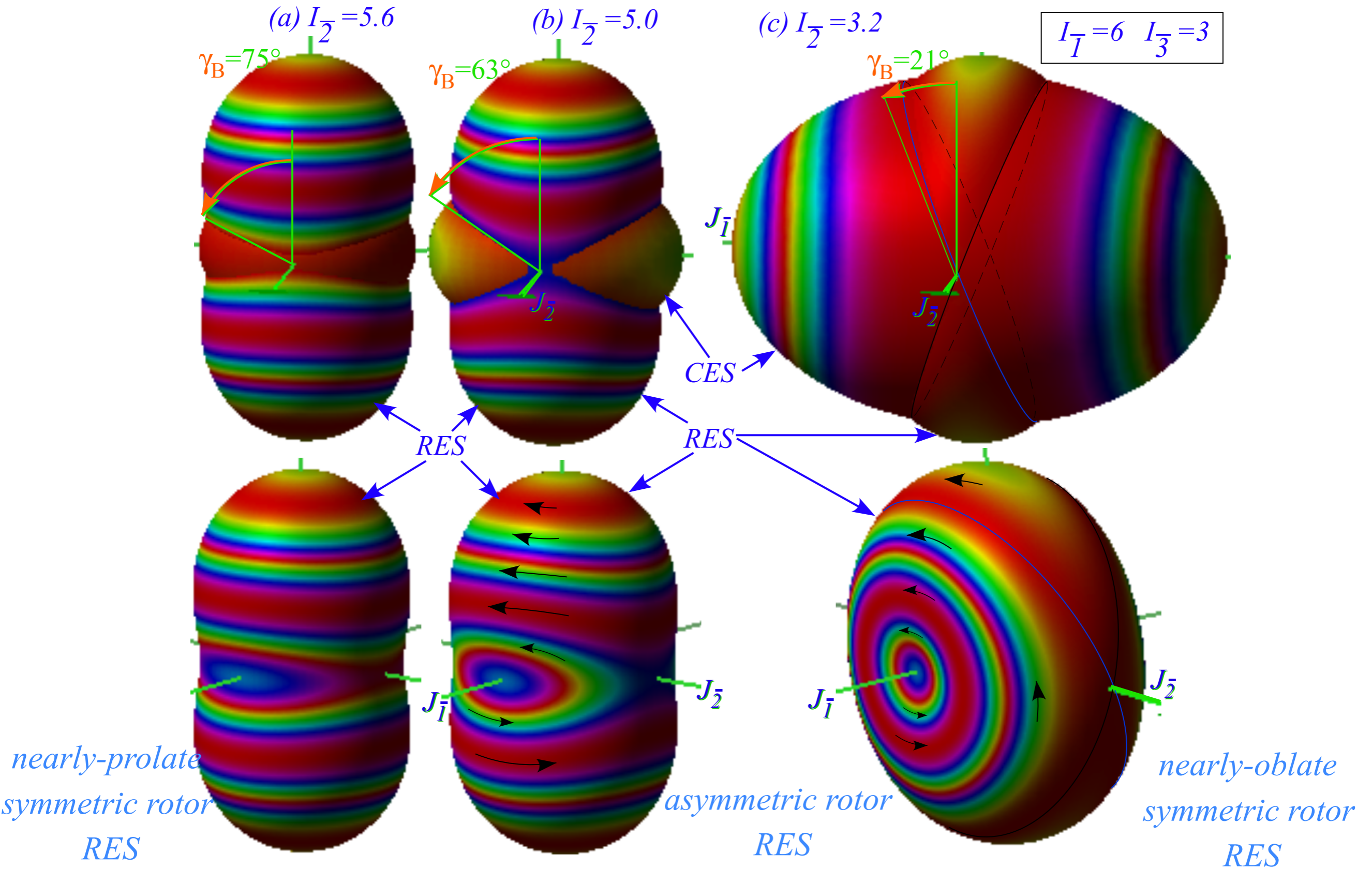
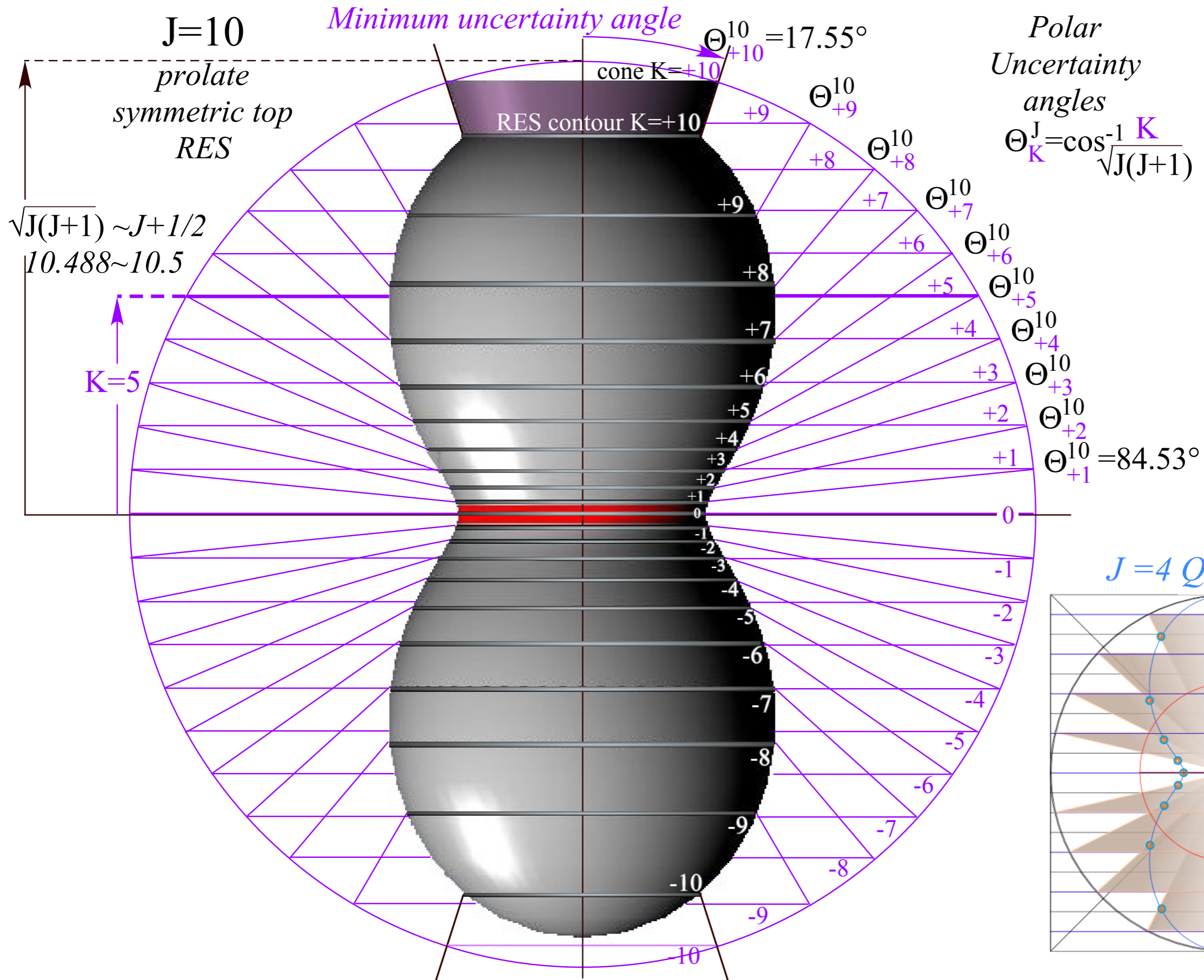
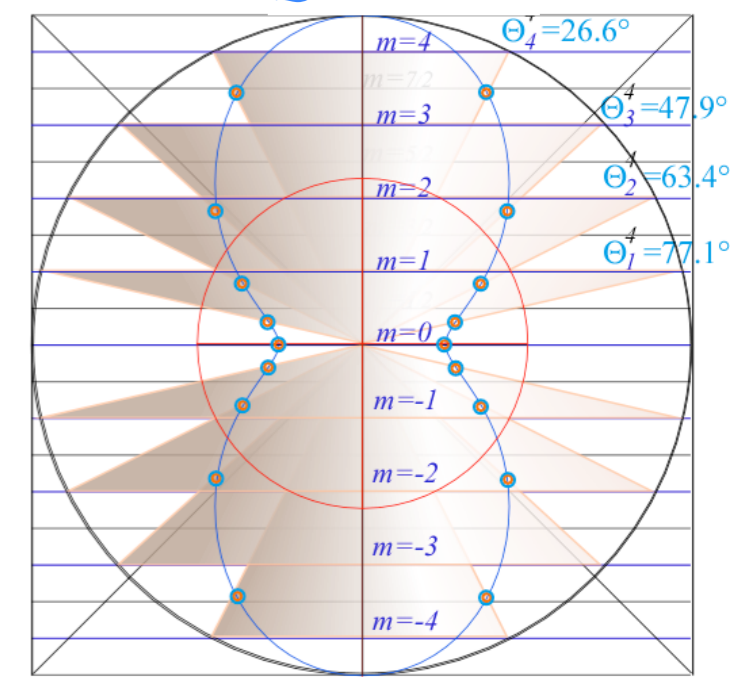


Fig. 6.8.2 Fixed- J - RES with CES at separatrix $E = J^2 / 2I_2$ as I_2 varies. (a) $I_2 = 5.6$ and $\gamma_B = 75.5^\circ$ (Nearly prolate low- E CES), (b) $I_2 = 5.0$ and $\gamma_B = 63.4^\circ$, (c) $I_2 = 3.2$ and $\gamma_B = 20.7^\circ$ (Nearly oblate high- E CES).

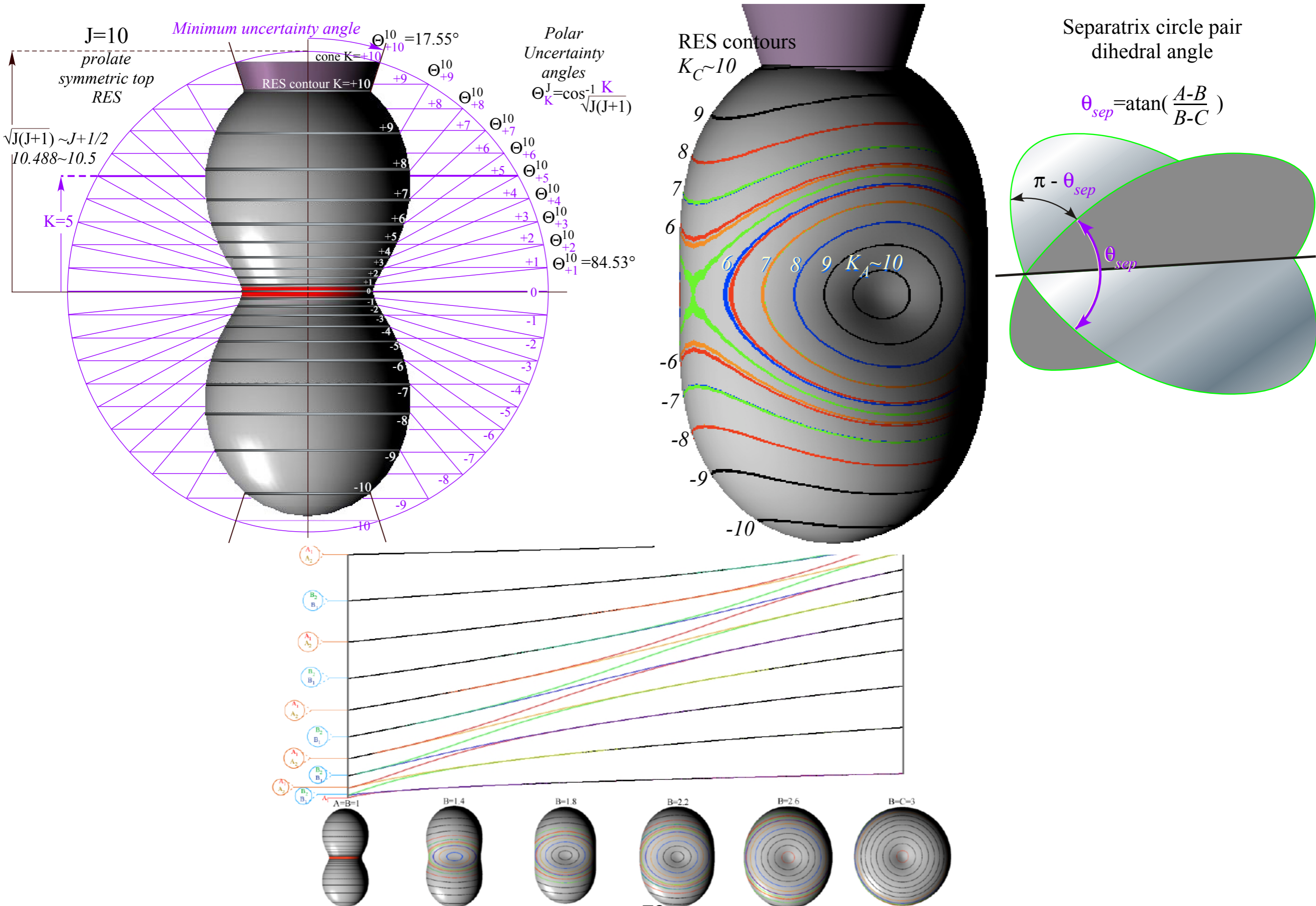
RES for symmetric prolate rotor locates $J = 10$ quantum ($-J < K < J$) levels (at RES-quantum cone intersections)



J = 4 Quantum cones



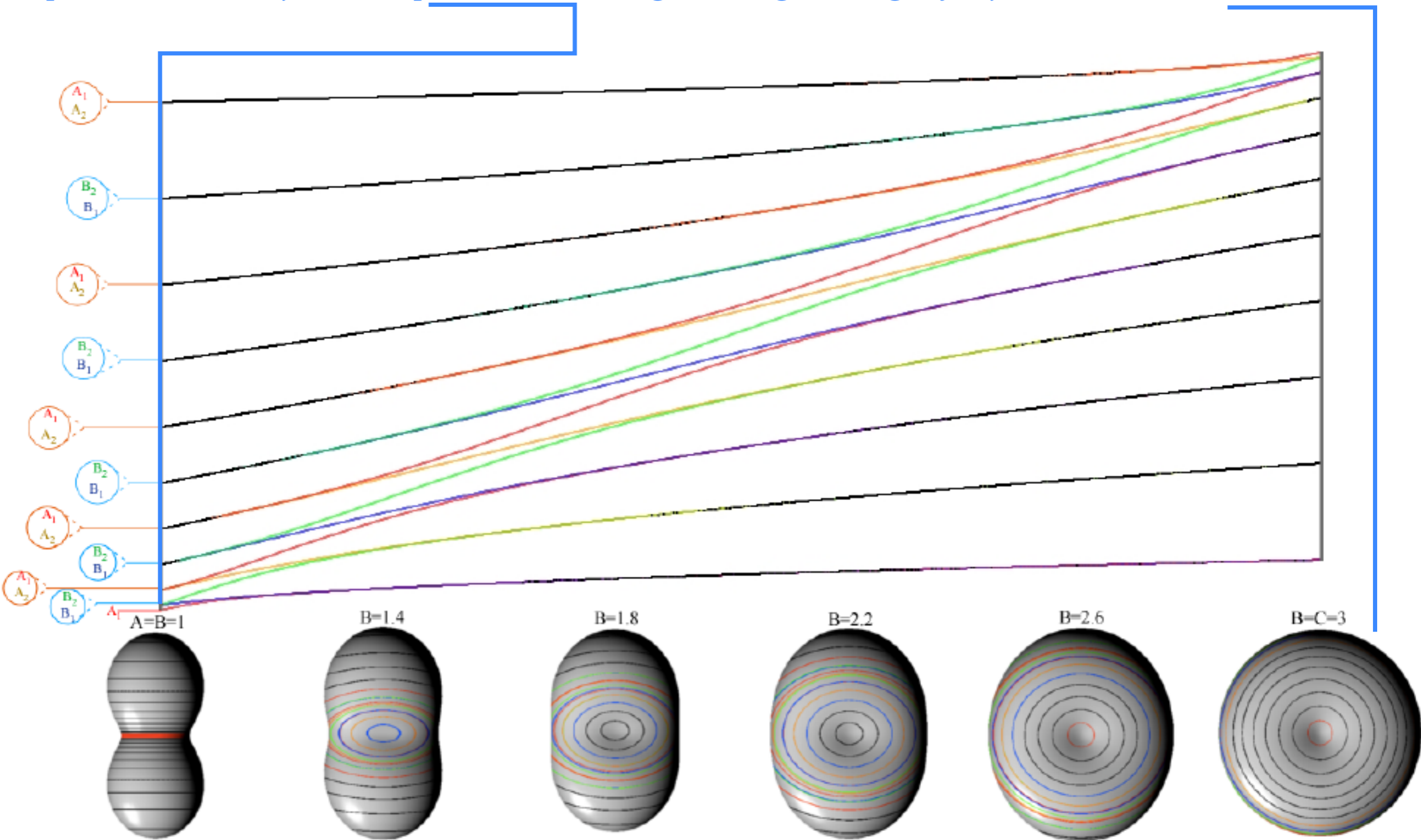
RES for symmetric and asymmetric rotor approximates $J=10$ ($-J < K < J$) levels (near RES-quantum cone levels)

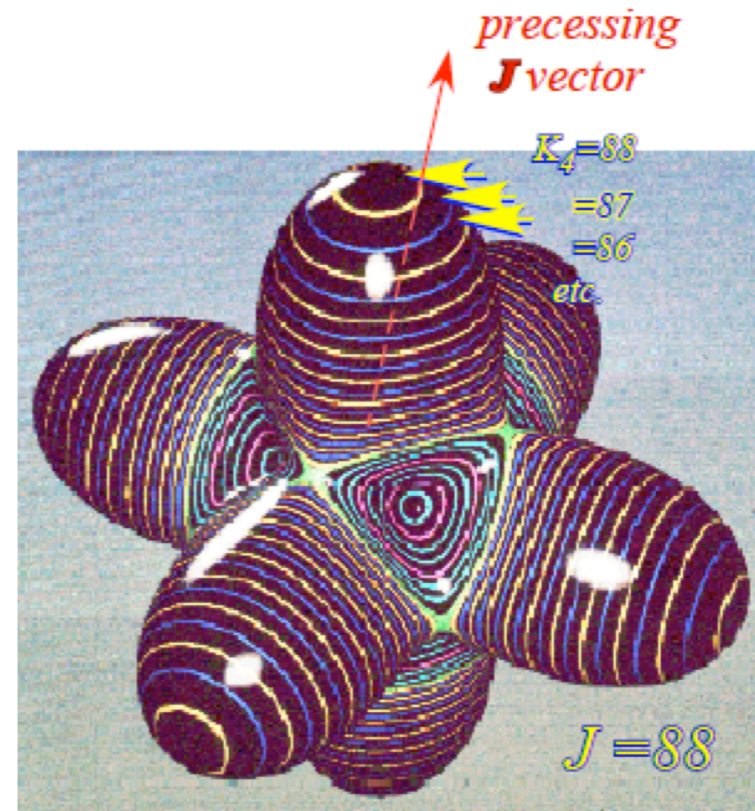


RES for symmetric prolate rotor locates $J = 10$ quantum ($-J < K < J$) levels (at RES-quantum cone intersections)

$$E = AJ_x^2 + BJ_y^2 + CJ_z^2 \quad \text{with } J = \text{const.}$$

Spectra varies as symmetric prolate RES changes through a range of asymmetric RES to oblate RES





Rotational Energy Surfaces (RES)

→ *Symmetric, asymmetric, and spherical-top dynamics (Constant **J**)*
BOD-frame cone rolling on LAB frame cone

RES for spherical rotor approximates $J=88$ ($-J < K < J$) levels of SF_6

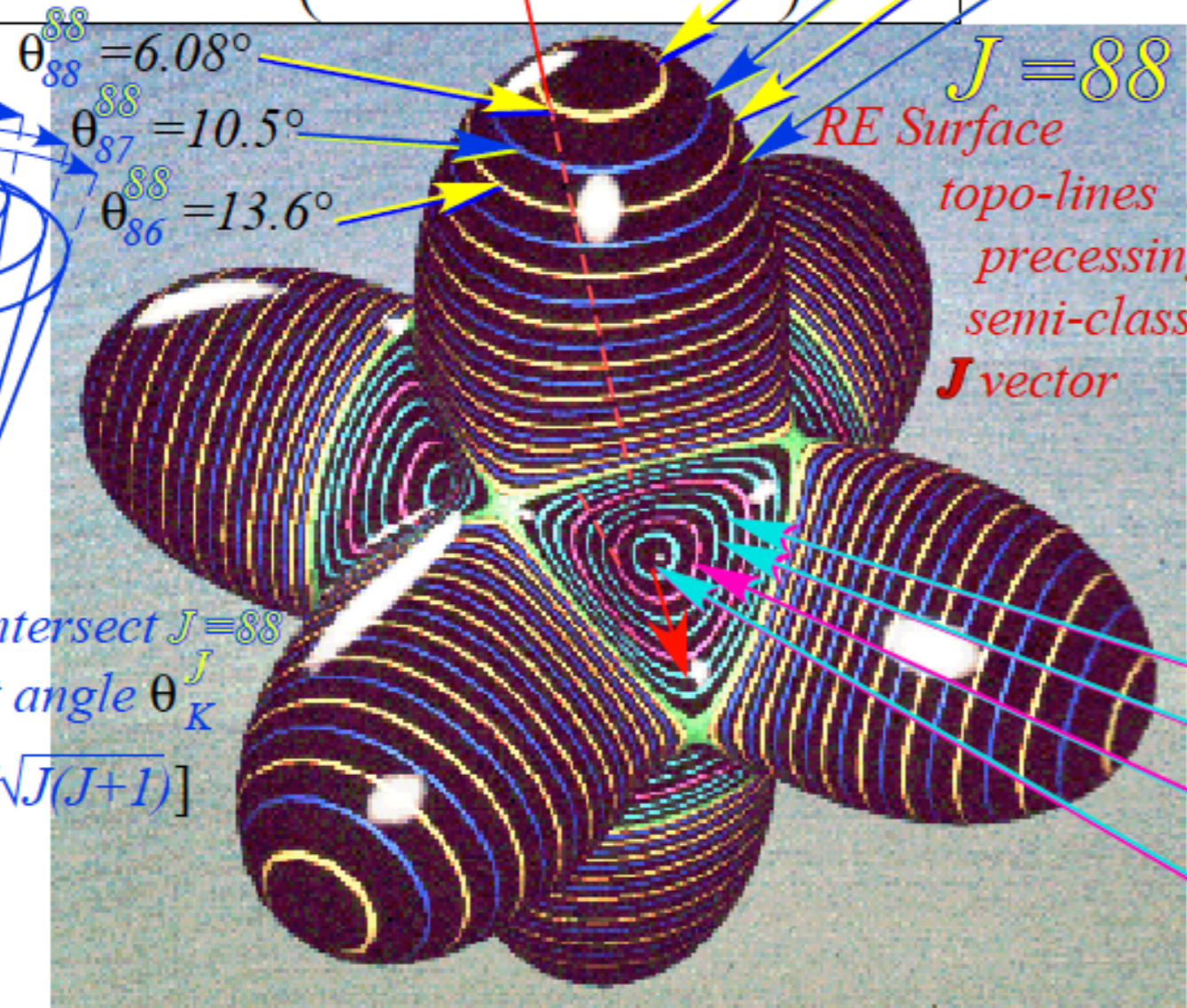
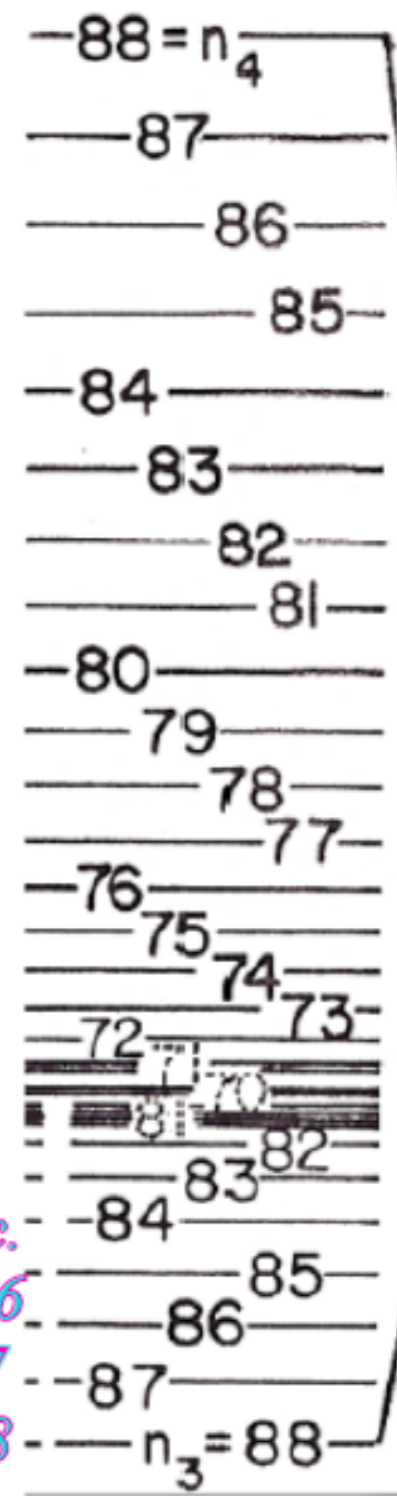
$$\langle H \rangle \sim v_{\text{vib}} + BJ(J+1) + \langle H^{\text{Scalar Coriolis}} \rangle + \langle H^{\text{Tensor Centrifugal}} \rangle + \langle H^{\text{Tensor Coriolis}} \rangle + \langle H^{\text{Nuclear Spin}} \rangle + \dots$$

O_h or T_d Spherical Top: (Hecht CH_4 Hamiltonian 1960)

$$H = B(J_x^2 + J_y^2 + J_z^2) + t_{440} \left(J_x^4 + J_y^4 + J_z^4 - \frac{3}{5} J^4 \right) + \dots$$

$$= BJ^2 + t_{440} \left(T_0^4 + \sqrt{\frac{5}{14}} [T_4^4 + T_{-4}^4] \right) + \dots$$

$K_4 = 88$
 $= 87$
 $= 87$
 etc.



$J=88$
 RE Surface
 topo-lines track precessing semi-classical J vector

$\theta_{88}^{88} = 6.08^\circ$
 $\theta_{87}^{88} = 10.5^\circ$
 $\theta_{86}^{88} = 13.6^\circ$
 (J,K) cones intersect $J=88$ RE surface at angle θ_K^J
 $\theta_K^J = \text{acos}[K/\sqrt{J(J+1)}]$

etc.
 $= 86$
 $= 87$
 $K_3 = 88$

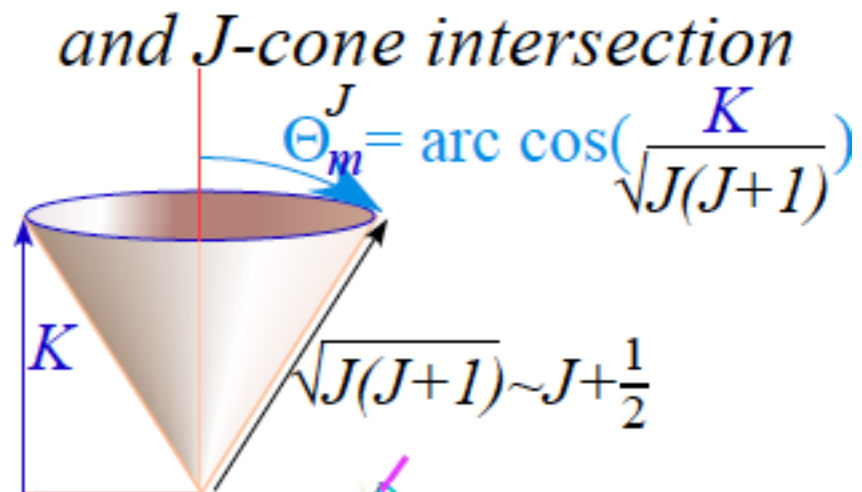
(next page shows slice)



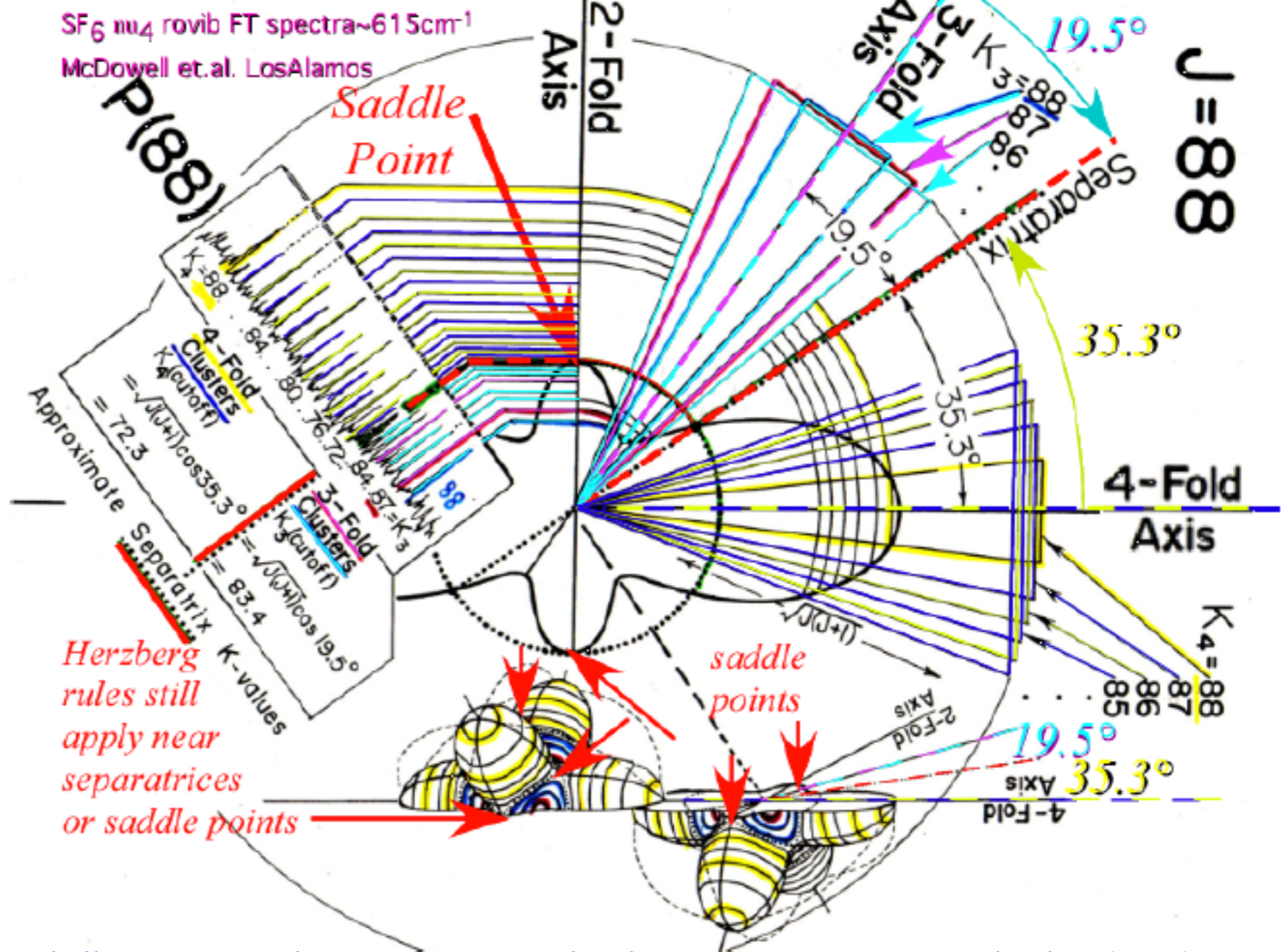
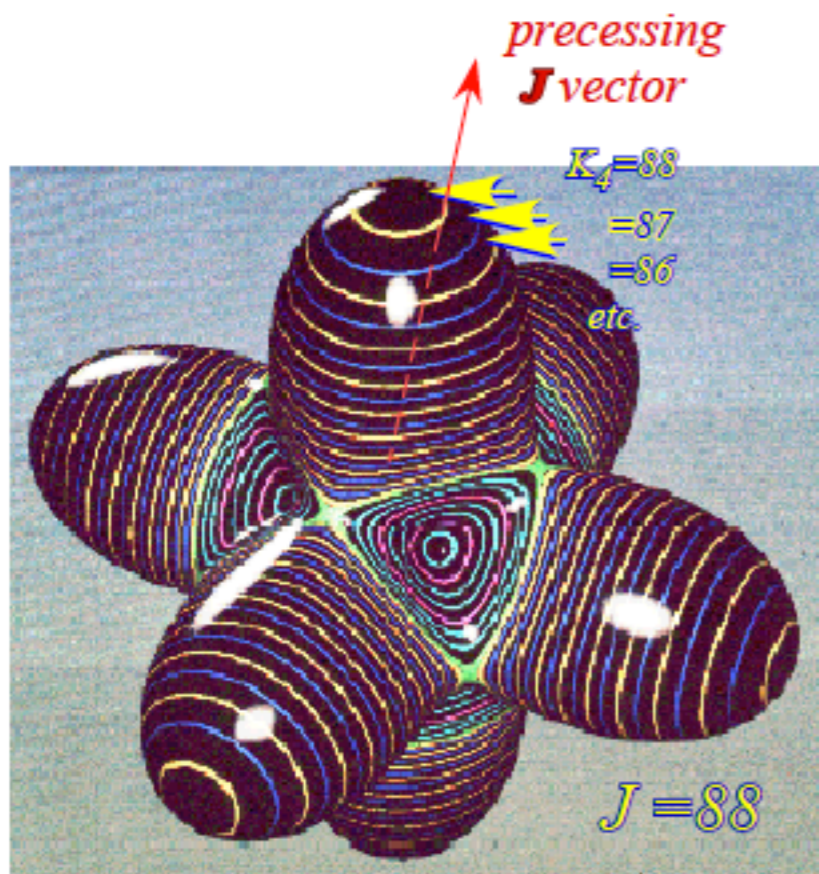
SF₆ Spectra of O_h Ro-vibronic Hamiltonian described by RE Tensor Topography and J-cone intersection

$$\mathbf{H} = B(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2) + t_{440} \left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5} J^4 \right) + \dots$$

$$= BJ^2 + t_{440} \left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}} [\mathbf{T}_4^4 + \mathbf{T}_{-4}^4] \right) + \dots$$



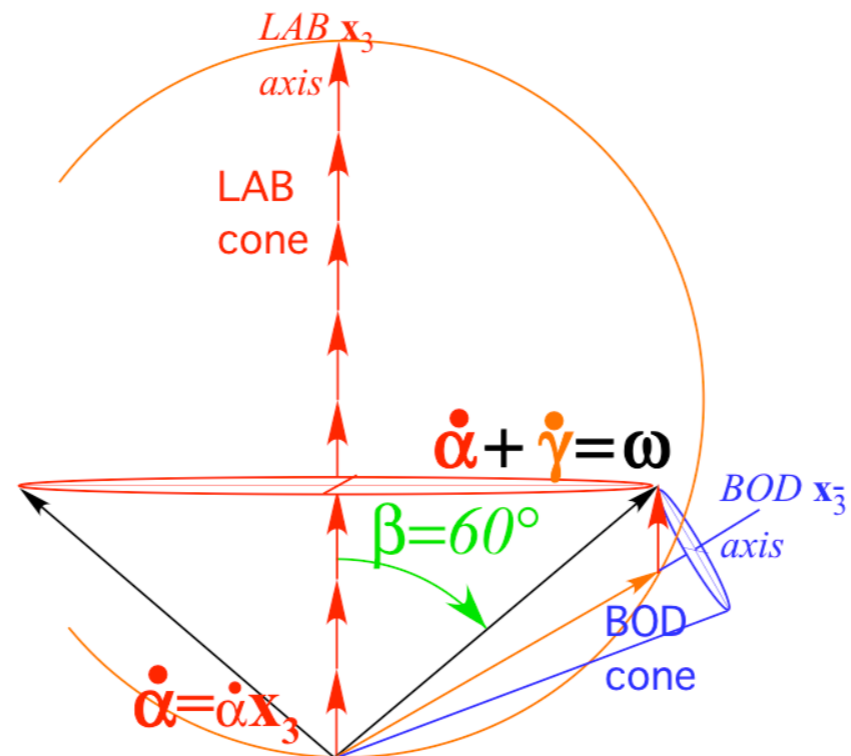
Rovibronic Energy (RE) Tensor Surface



Rotational Energy Surfaces (RES)

Symmetric, asymmetric, and spherical-top dynamics (Constant **J**)

➔ *BOD-frame cone rolling on LAB frame cone*



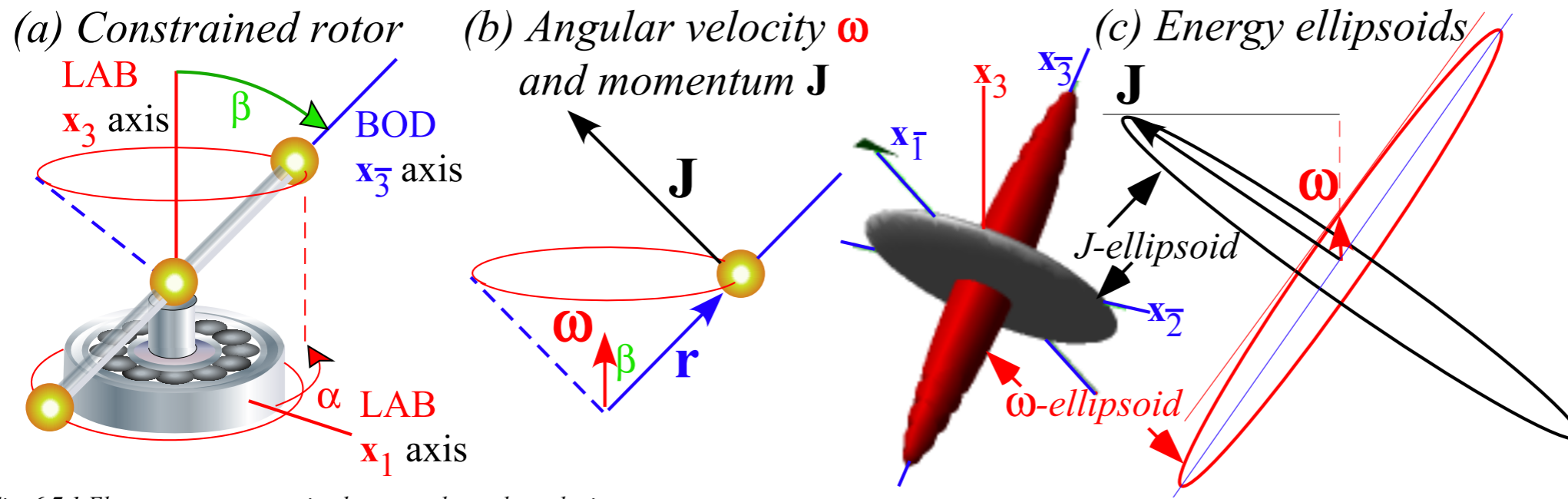


Fig. 6.7.1 Elementary ω -constrained rotor and angular velocity-momentum geometry.

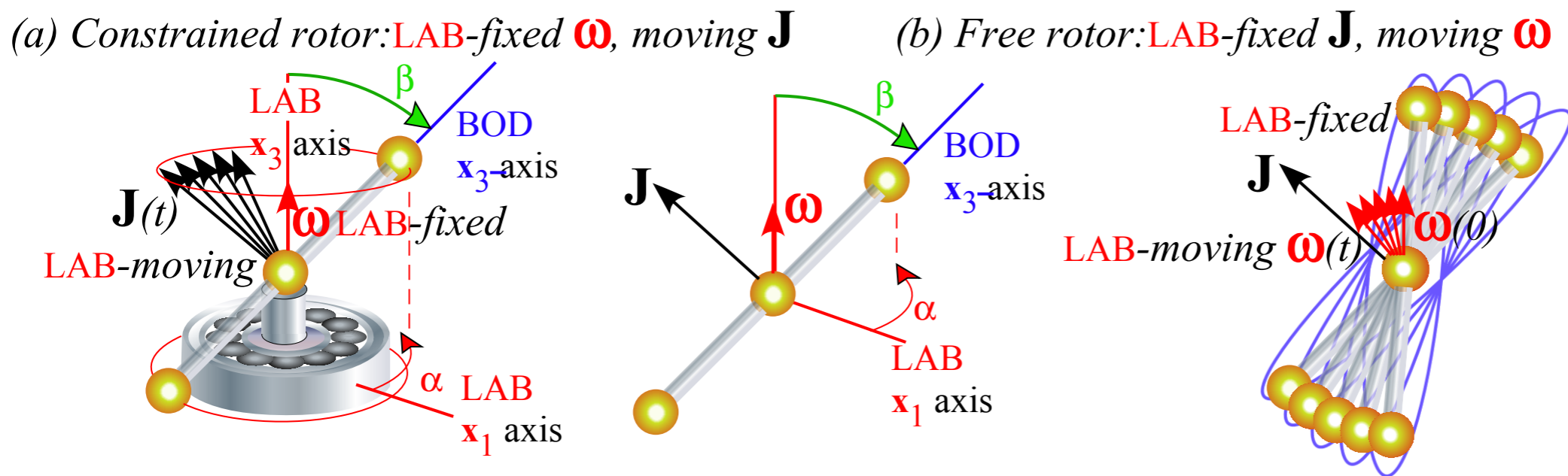


Fig. 6.7.2 Free rotor cut loose from LAB-constraining ω -axis changes dynamics accordingly.

..this was the kind of dynamics that started me dropping superballs...

Prolate tops: (a) $I_{II}=4I_3$

$$\dot{\gamma} = 3\dot{\alpha} \cos\beta$$

$$\dot{\gamma} = (3/4)\omega_{\bar{3}}$$

(b) $I_{II}=2I_3$

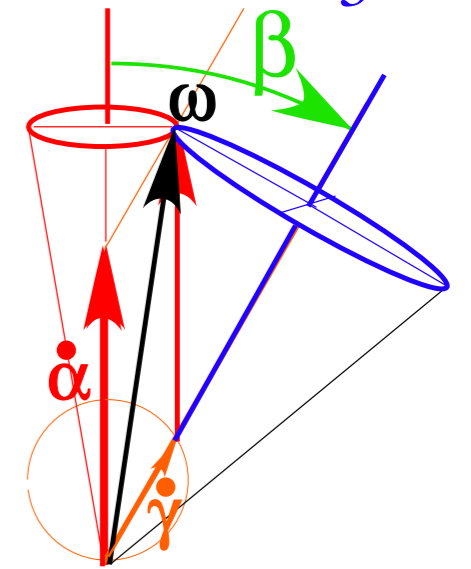
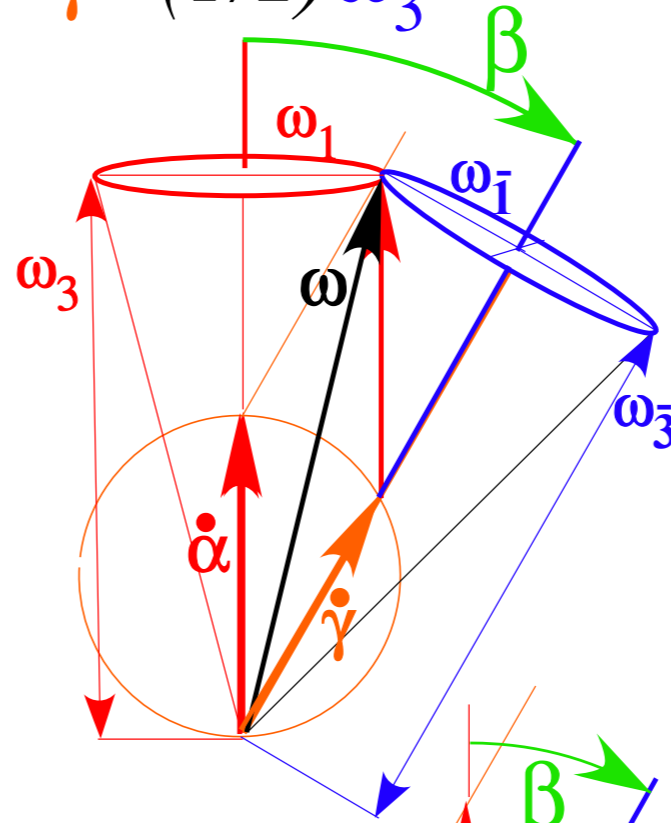
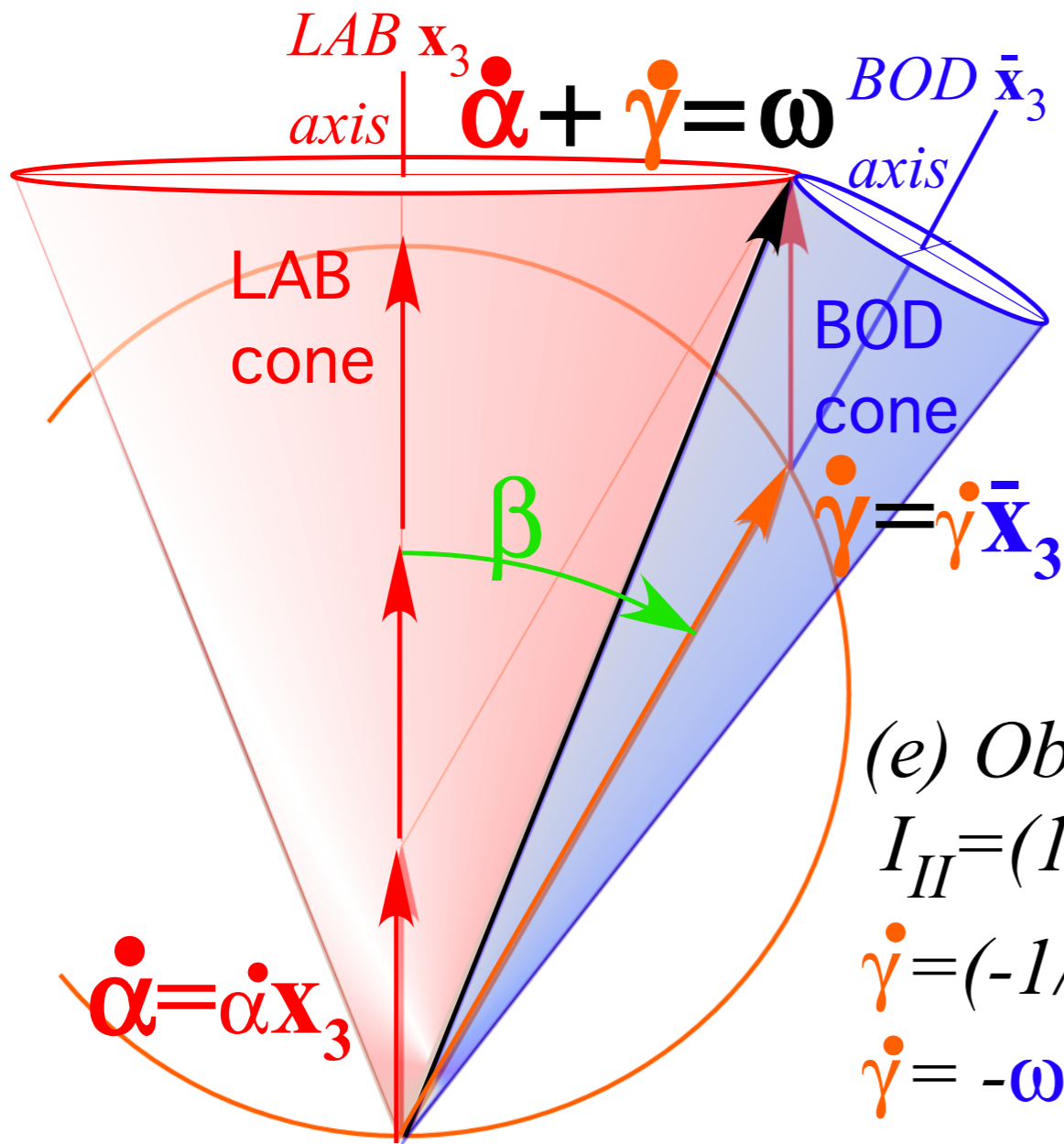
$$\dot{\gamma} = \dot{\alpha} \cos\beta$$

$$\dot{\gamma} = (1/2)\omega_{\bar{3}}$$

(c) $I_{II}=(3/2)I_3$

$$\dot{\gamma} = (1/2)\dot{\alpha} \cos\beta$$

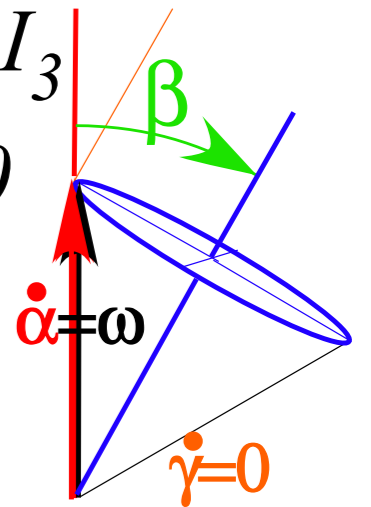
$$\dot{\gamma} = (1/3)\omega_{\bar{3}}$$



(d) Spherical top:

$$I_{II} = I_3$$

$$\dot{\gamma} = 0$$

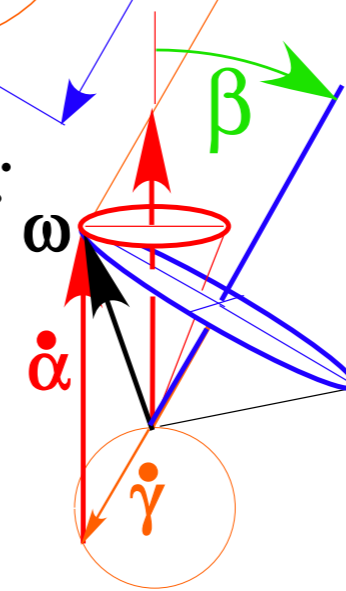


(e) Oblate limit:

$$I_{II} = (1/2)I_3$$

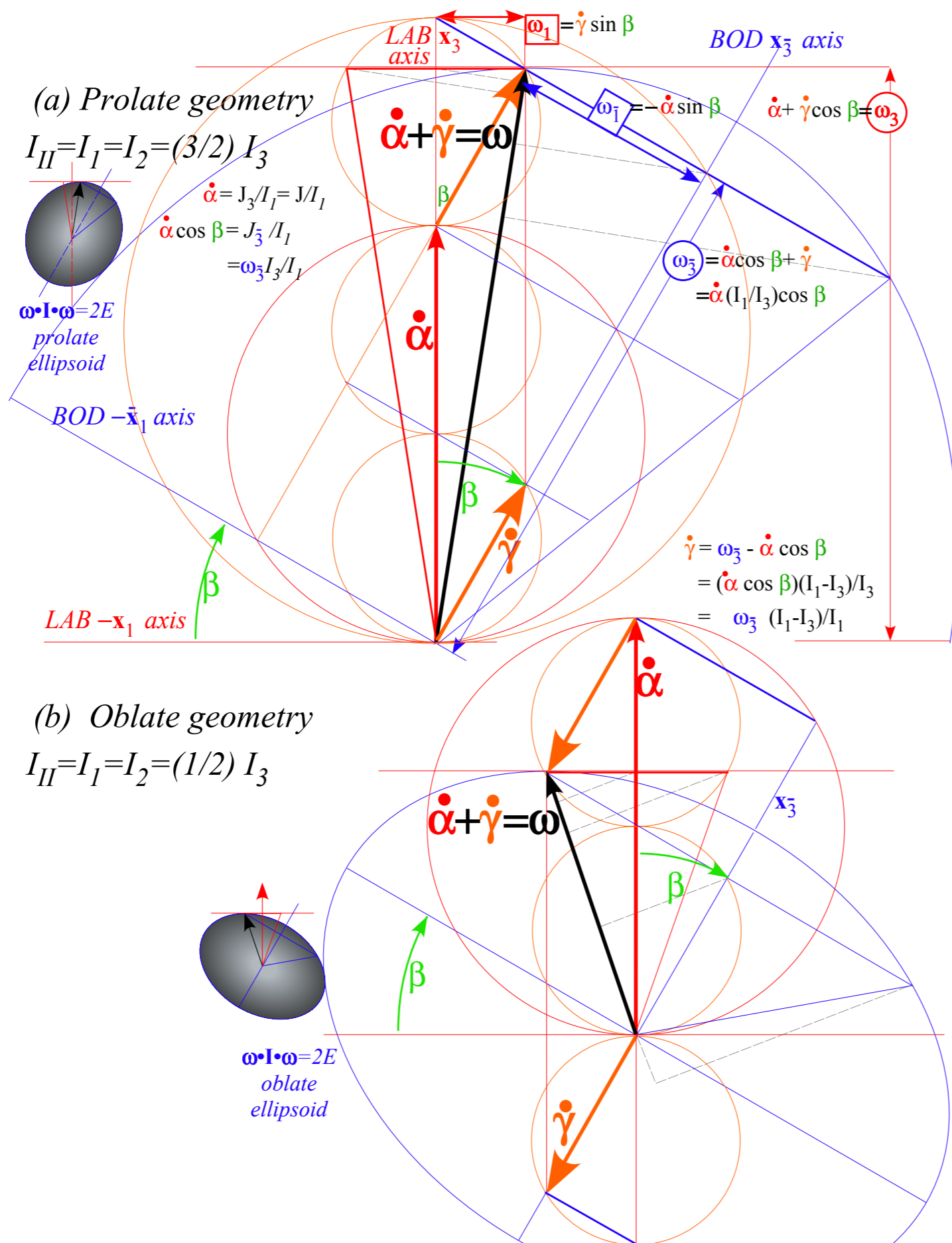
$$\dot{\gamma} = (-1/2)\dot{\alpha} \cos\beta$$

$$\dot{\gamma} = -\omega_{\bar{3}}$$



Blue BOD-frame cones roll (around ω -sticking axis) without slipping on red LAB-frame cone

Fig. 6.7.3 Symmetric top ω -cones for $\beta=30^\circ$ and inertial ratios: (a) $\frac{I_{II}-I_3}{I_3} = 3$, (b) 1, (c) $\frac{1}{2}$, (d) 0, (e) $-\frac{1}{2}$.



Blue BOD-frame cones roll without slipping on red LAB-frame cone

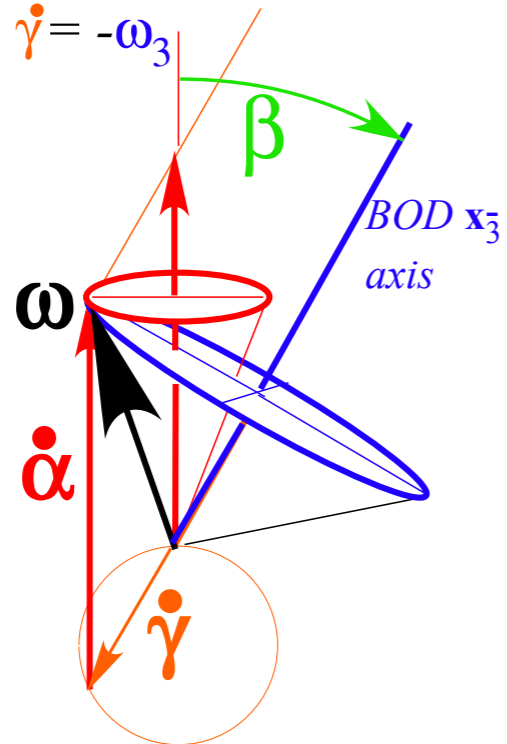
Fig. 6.7.4 Detailed geometry of symmetric top kinetics. (a) Prolate case. (b) Most-oblate case

Oblate limit:

$$I_{II} = (1/2) I_3$$

$$\dot{\gamma} = (-1/2) \dot{\alpha} \cos \beta$$

$$\dot{\gamma} = -\omega_3$$

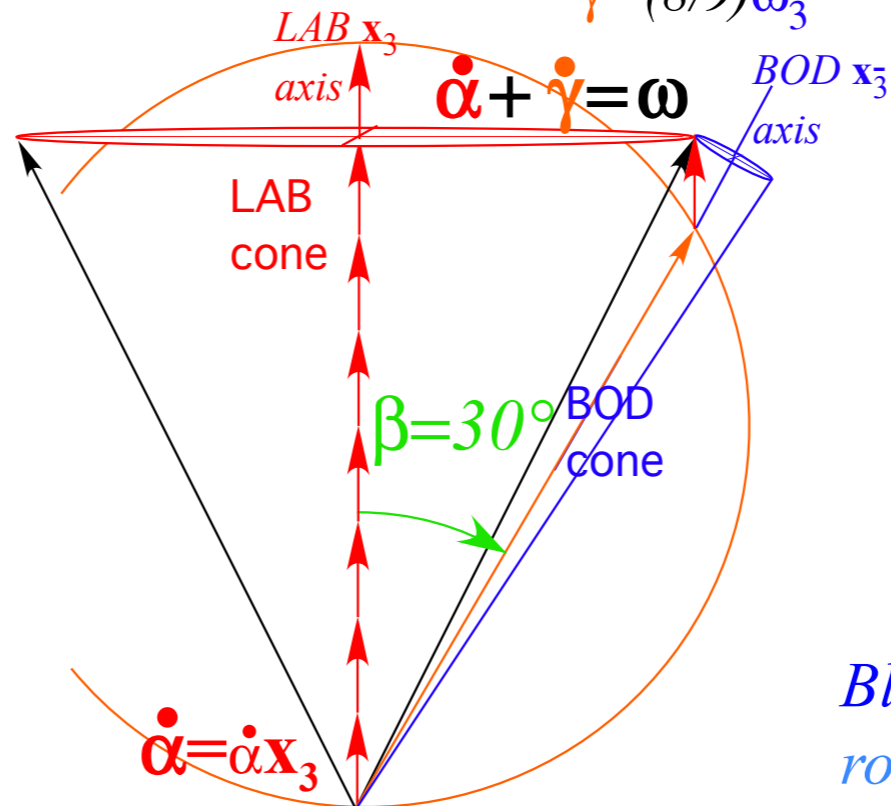


$$\begin{aligned} \dot{\gamma} &= \omega_3 - \dot{\alpha} \cos \beta \\ &= (\dot{\alpha} \cos \beta)(I_1 - I_3)/I_3 \\ &= \omega_3 (I_1 - I_3)/I_1 \end{aligned}$$

Very prolate top: $I_{II} = 9I_3$

$$\dot{\gamma} = 8\dot{\alpha} \cos \beta$$

$$\dot{\gamma} = (8/9)\omega_3$$



Blue BOD-frame cones roll without slipping on red LAB-frame cone

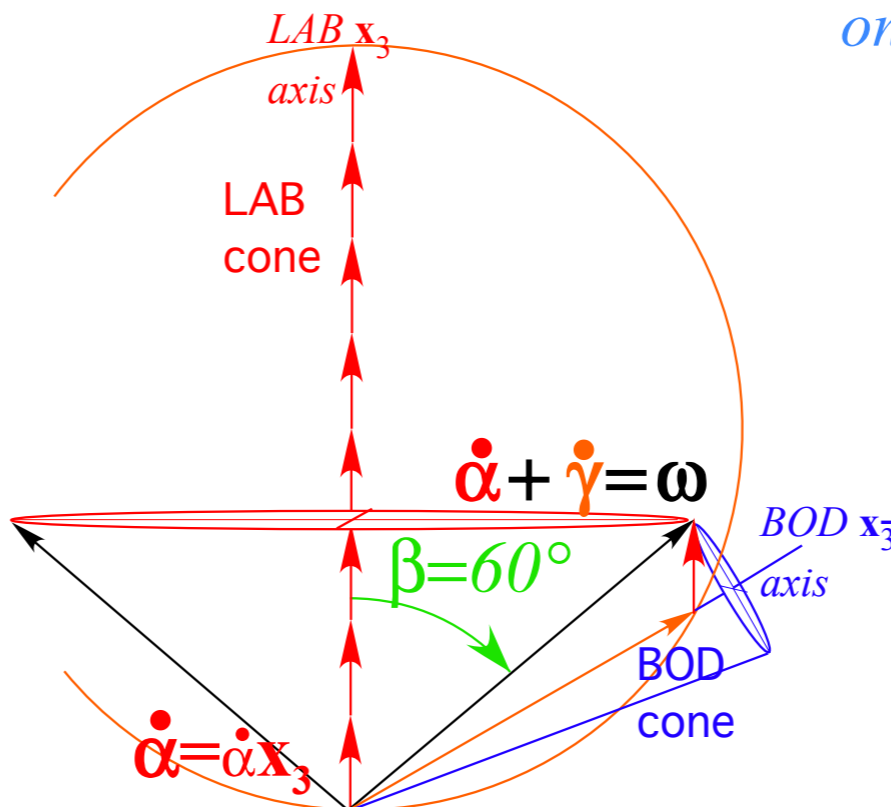
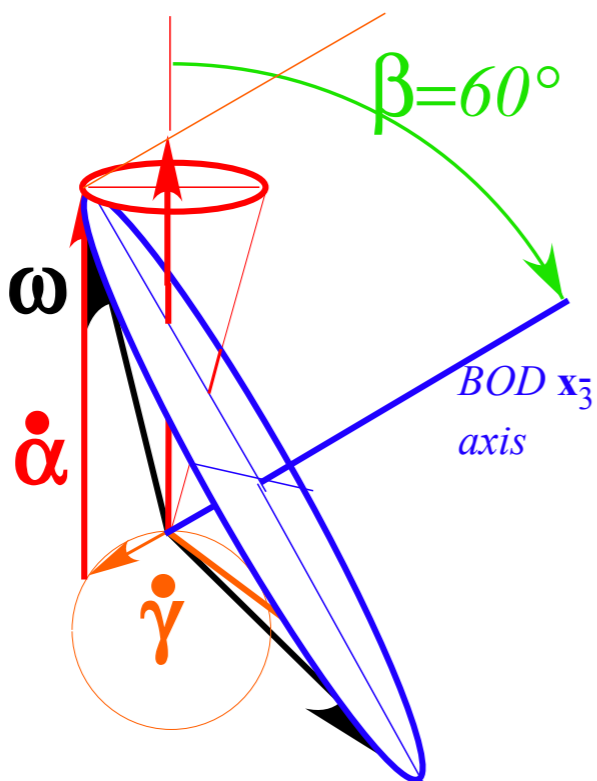


Fig. 6.7.5 Extreme cases (Oblate vs. Prolate) of symmetric-top geometry.

 *Cycloidal geometry of flying levers*
Practical poolhall application

If you hammer a stick at a point h meters from its center
 you give it some linear momentum Π
 and some angular momentum $\Lambda = h \cdot \Pi$

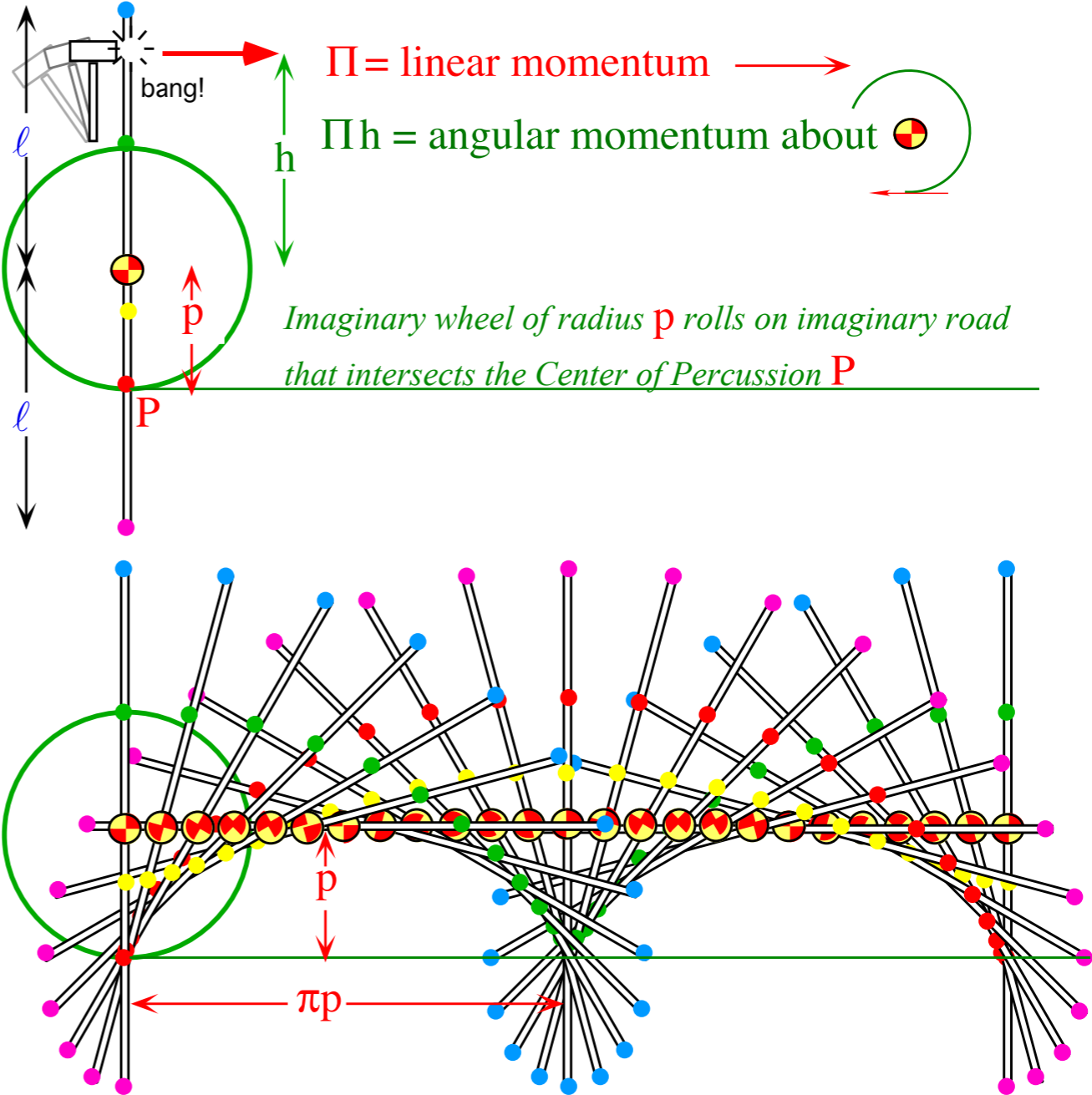


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
 you give it some linear momentum Π
 and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
 is angular momentum Λ divided by
 moment of inertia $I = M \ell^2/3$ of the stick.

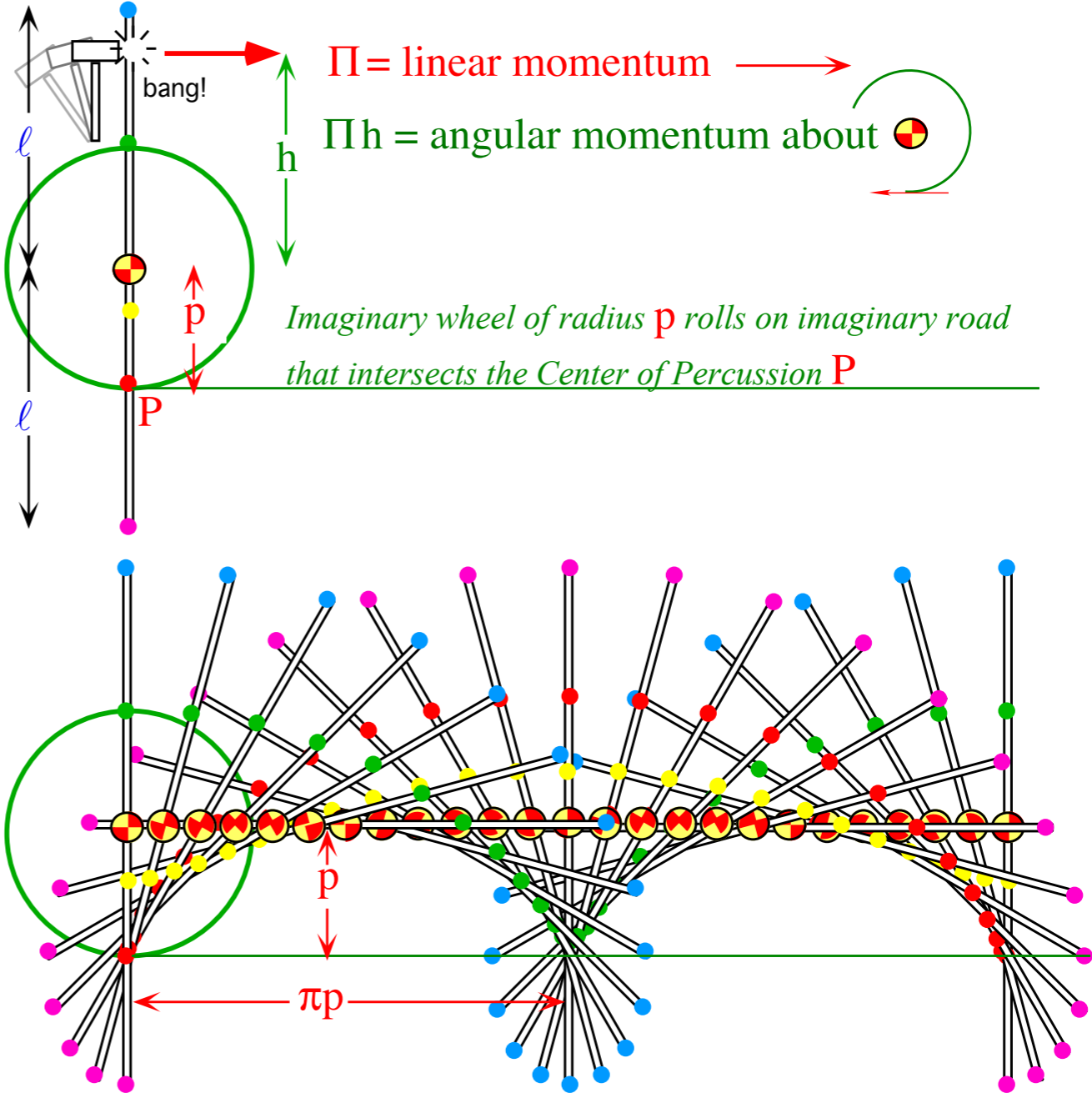


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 moment of inertia $I = M \ell^2/3$ of the stick.

$$\omega = \Lambda / I \quad (=3\Lambda / (M \ell^2) \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi / (M \ell^2) \text{ for stick})$$

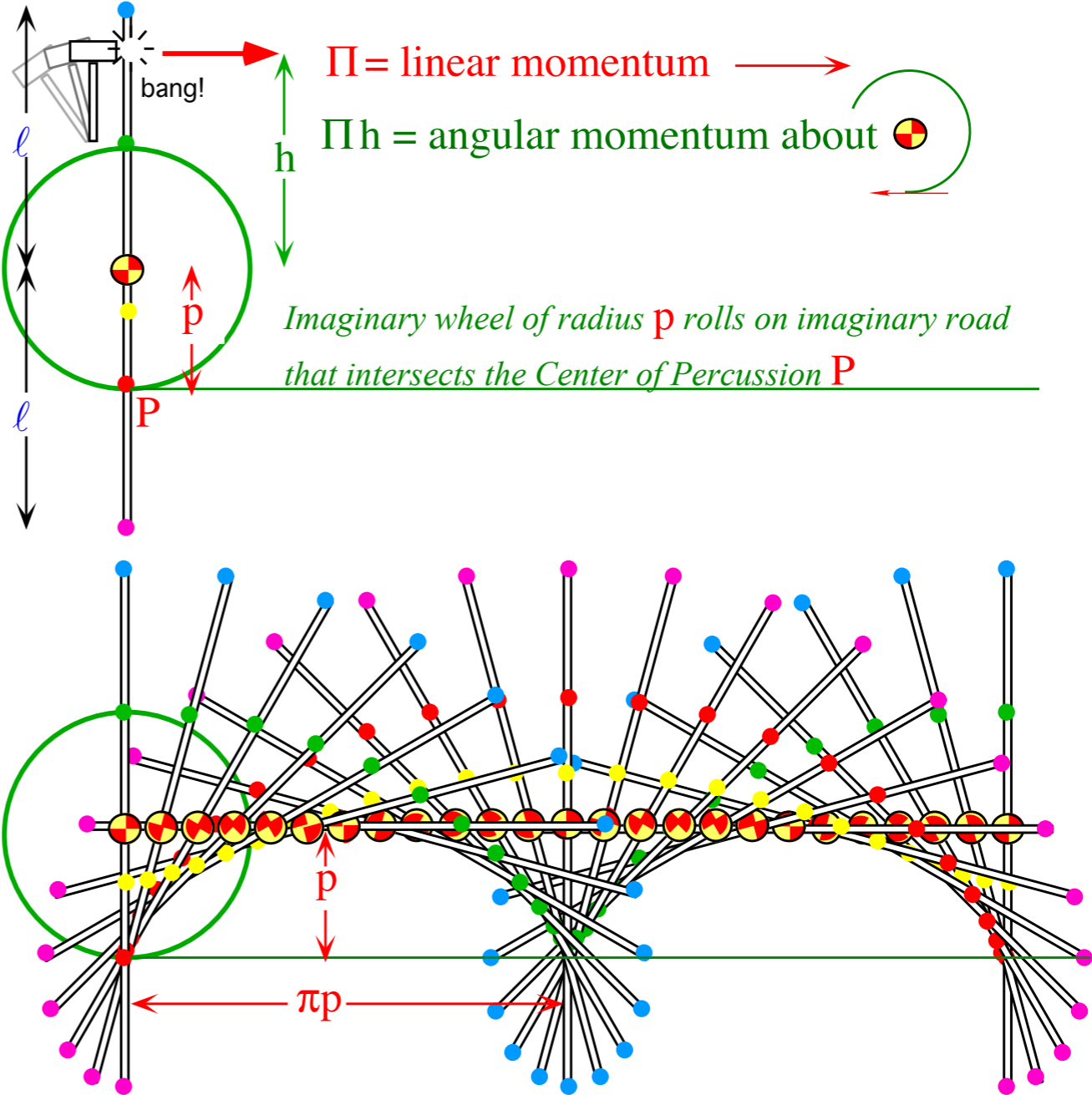


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One point **P**, or *center of percussion (CoP)*, is
 on the wheel where speed $p\omega$ due to rotation
 just cancels translational speed V_{Center} of stick.

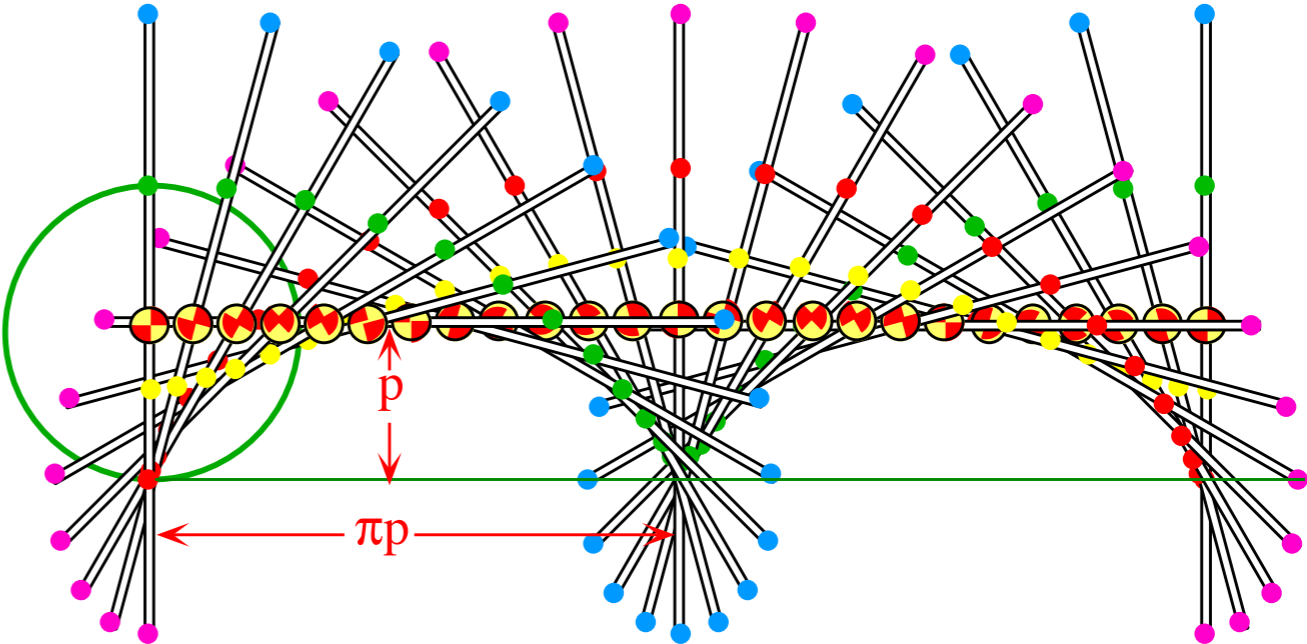
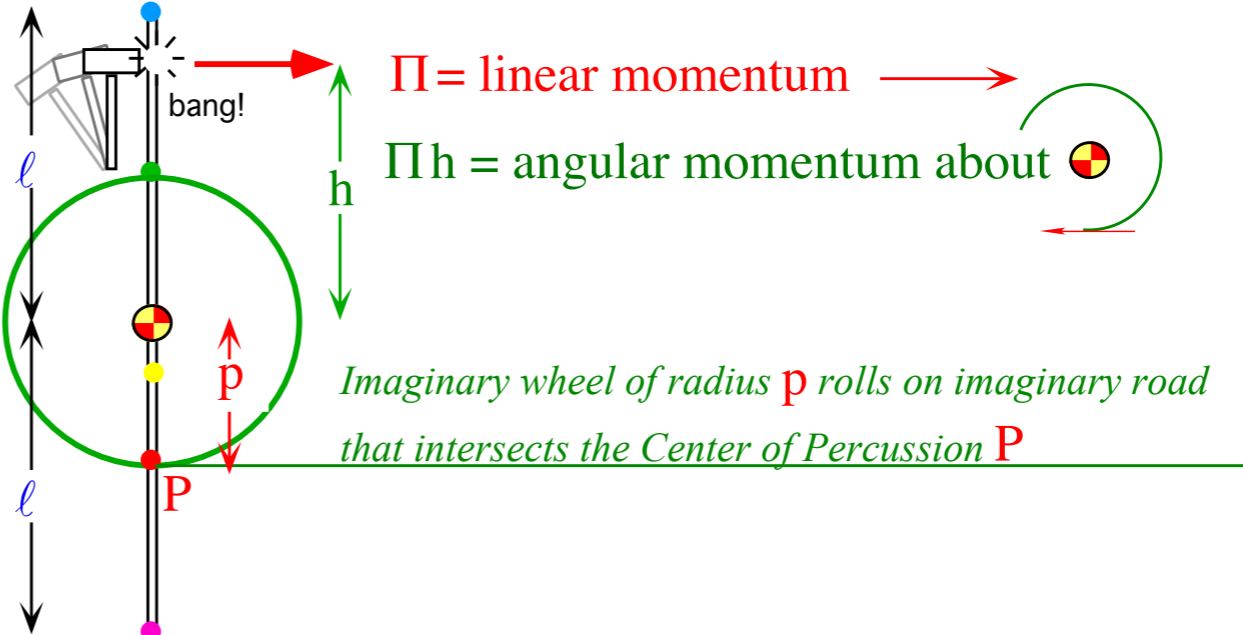


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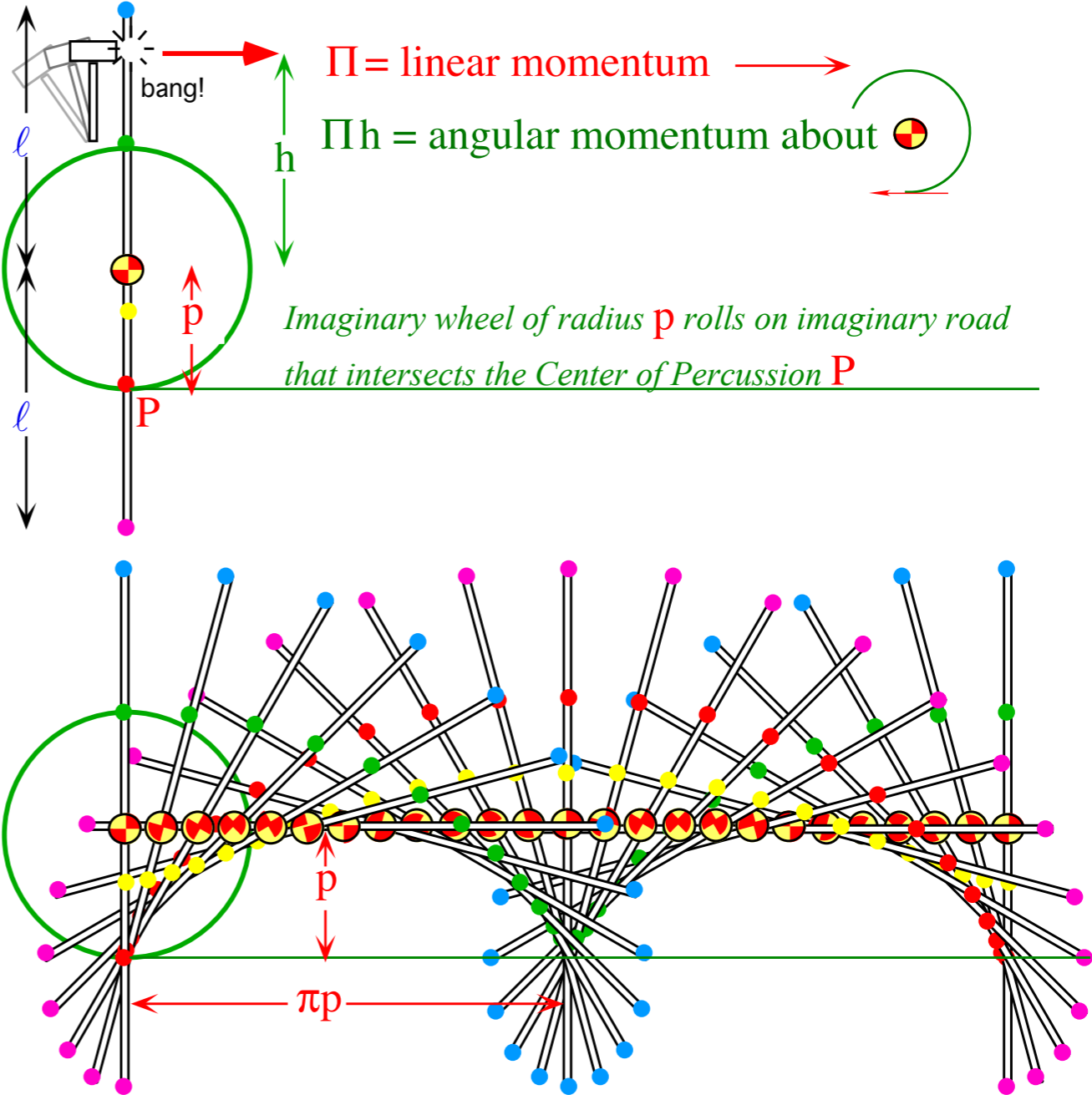


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$$I / M = \quad = \quad = p \cdot h$$

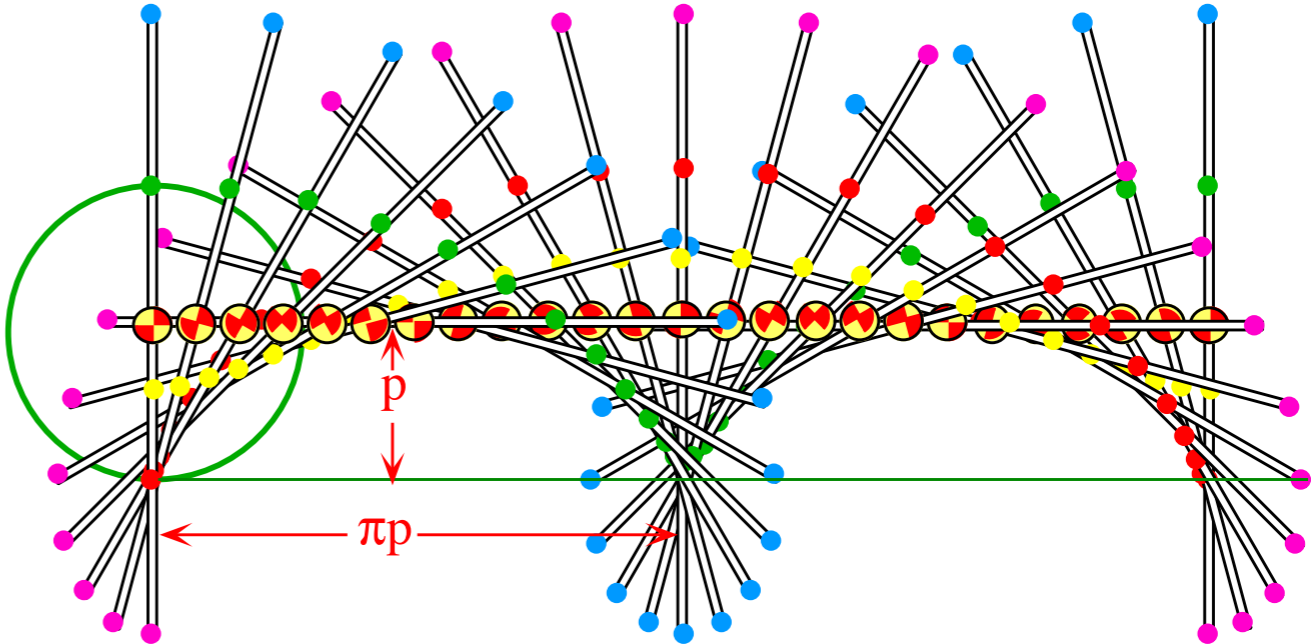
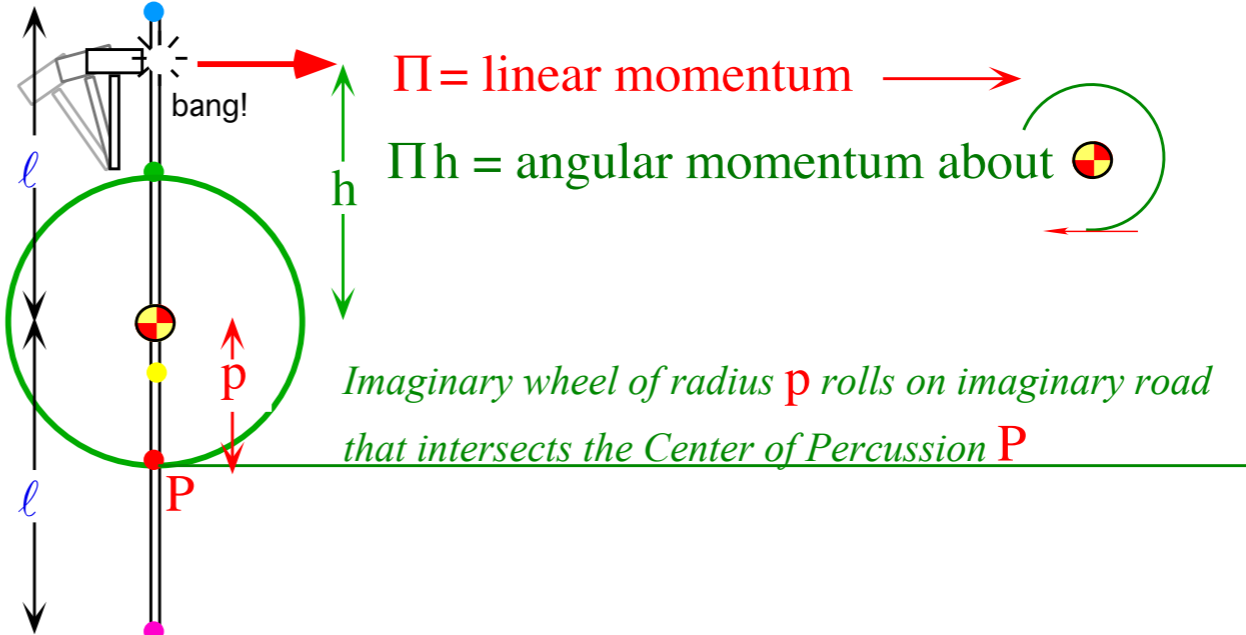


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$$\Pi / M = V_{Center} = |p\omega| = p \cdot h\Pi / I$$

$$I / M = \quad = \quad = p \cdot h \quad \text{or: } p = I / (Mh)$$

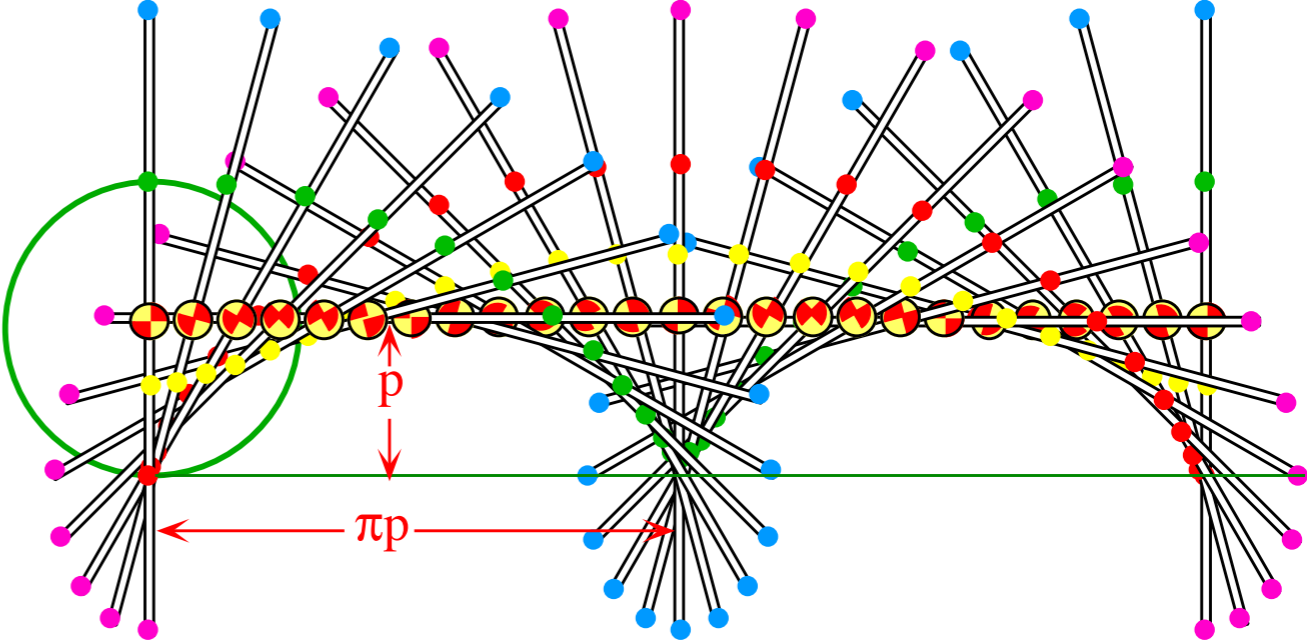
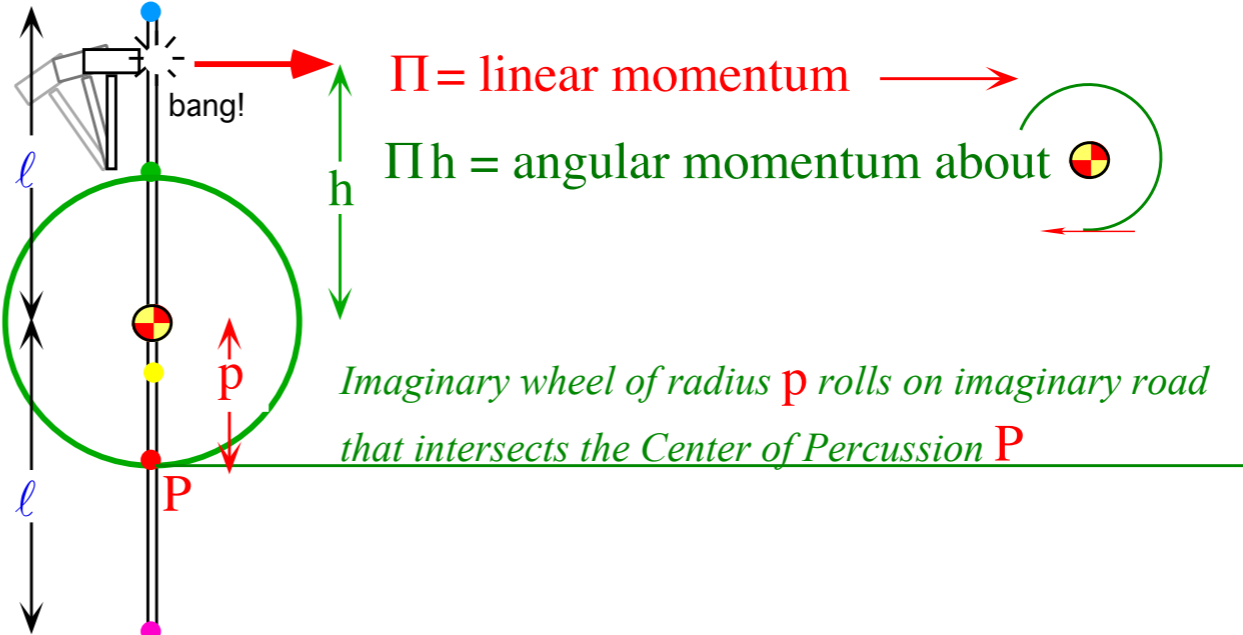


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$$I / M = \quad = \quad = p \cdot h \quad \text{or: } p = I / (Mh)$$

P follows a normal cycloid made by a circle
 of radius $p = I / (Mh)$ rolling on an imaginary road
 thru point P in direction of Π .

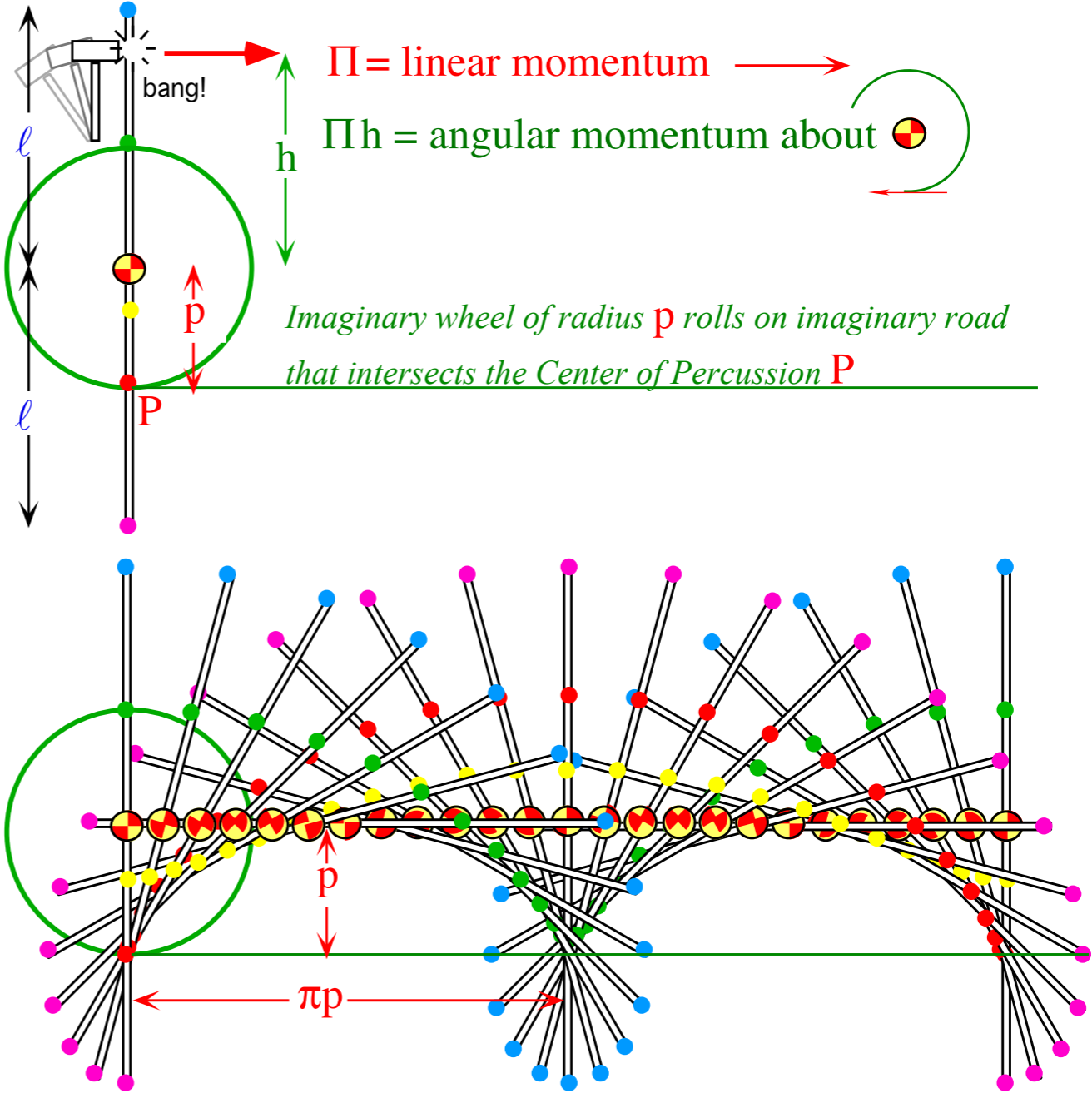


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

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P follows a normal cycloid made by a circle of radius $p = I / (Mh)$ rolling on an imaginary road thru point P in direction of Π .

The *percussion radius* $p = \ell^2/3h$ is of the **CoP** point that has no velocity just after hammer hits at h .

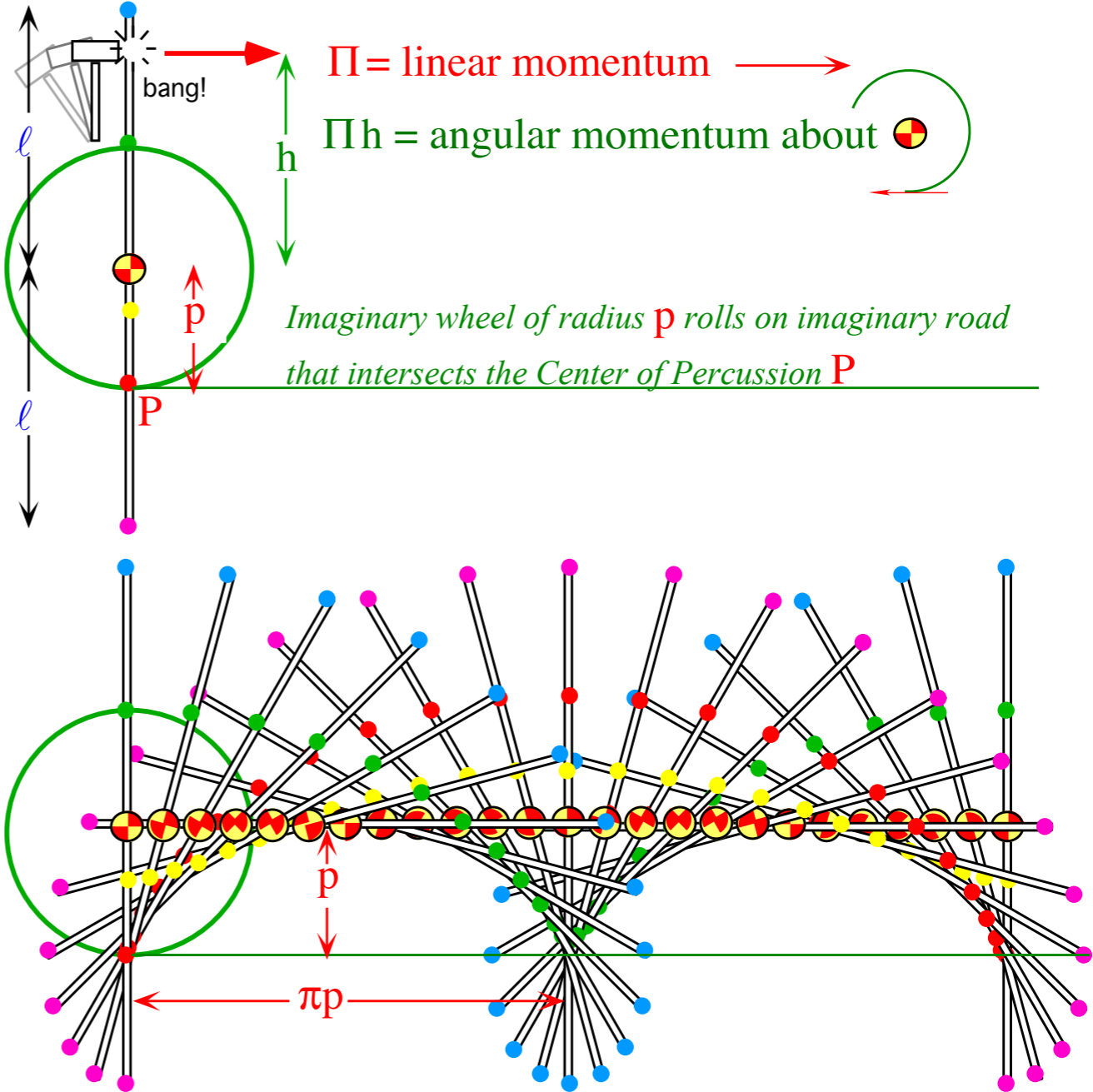
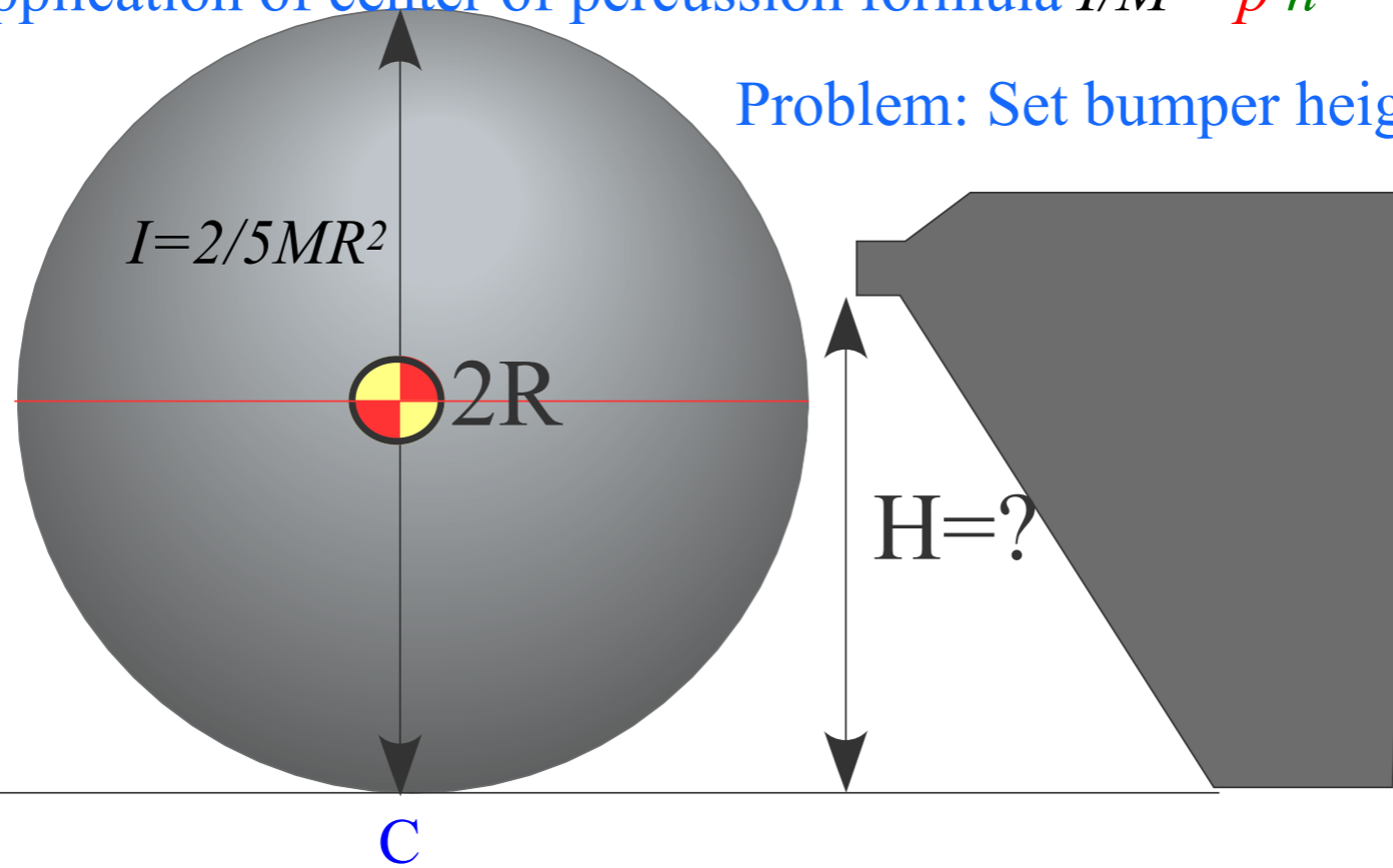


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

Cycloidal geometry of flying levers
→ *Practical poolhall application*

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

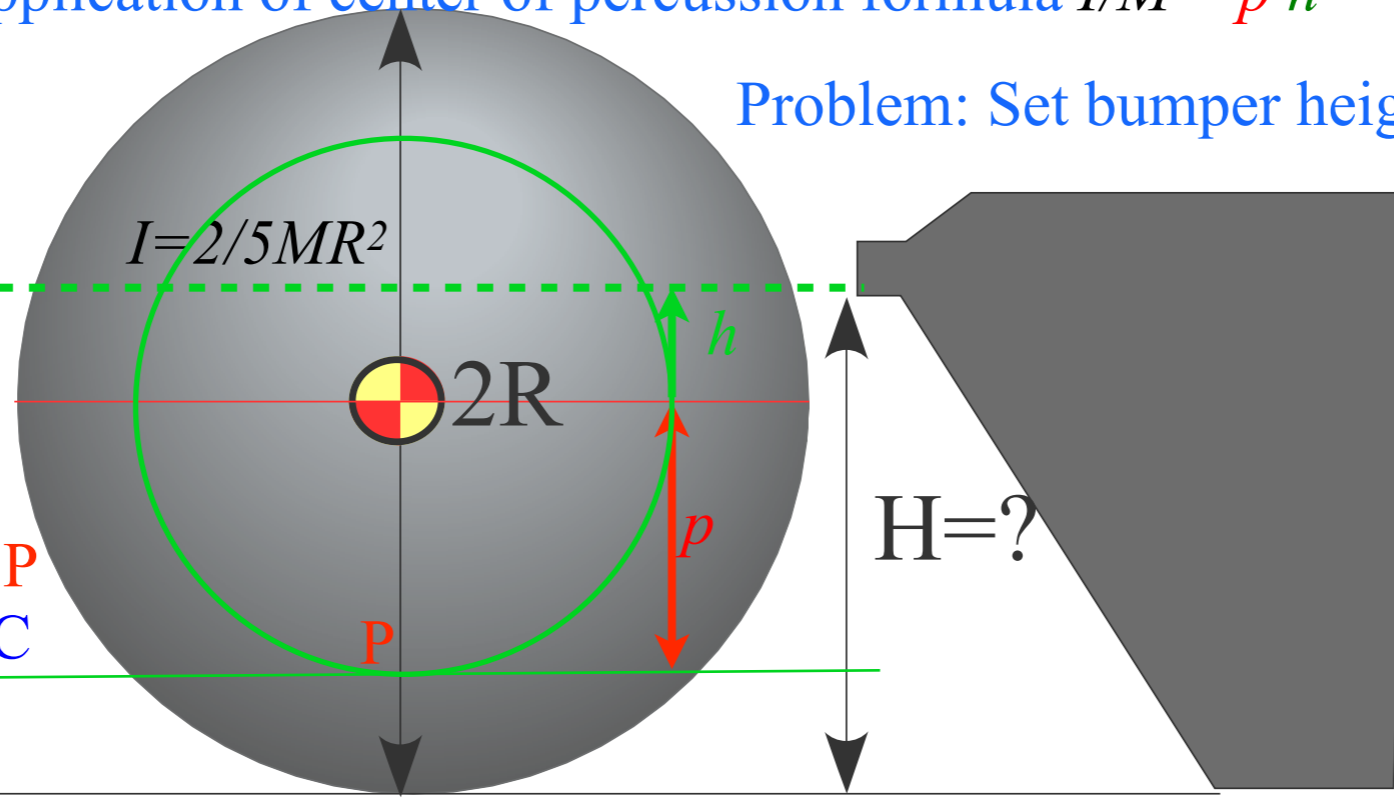
Problem: Set bumper height H so ball does not skid.



Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

center of percussion P
above contact point C

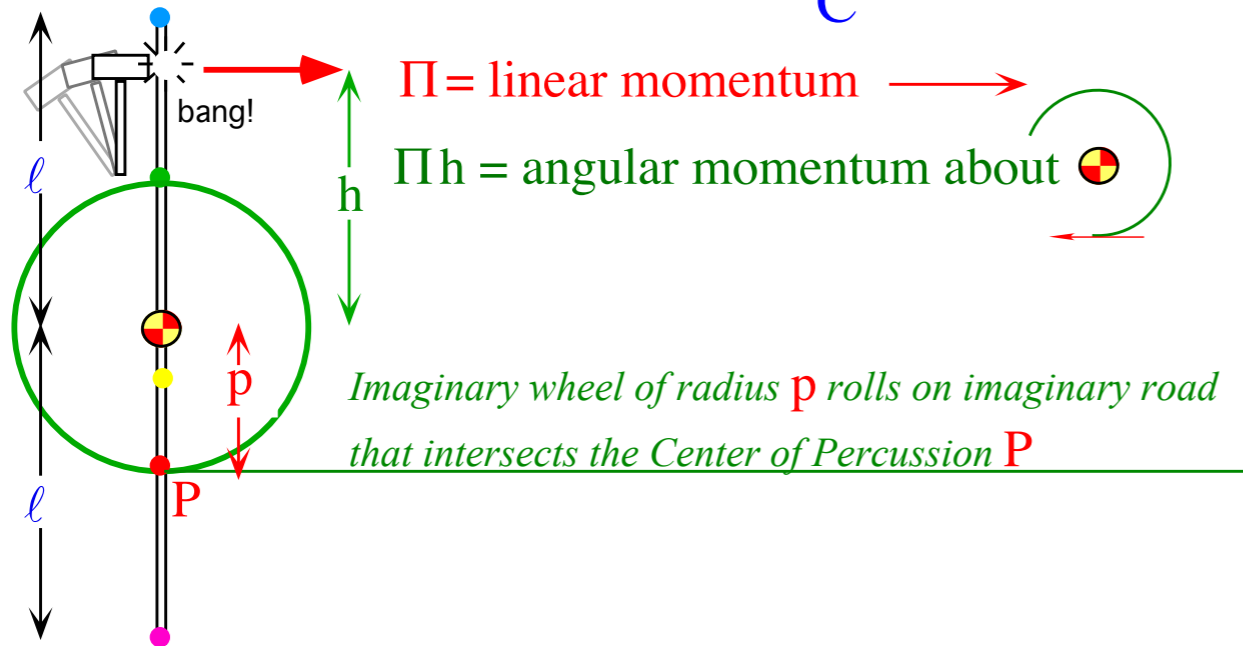


Where should bumper height H be set to make ball contact point C at the center of percussion P ?

$H = ?$

C

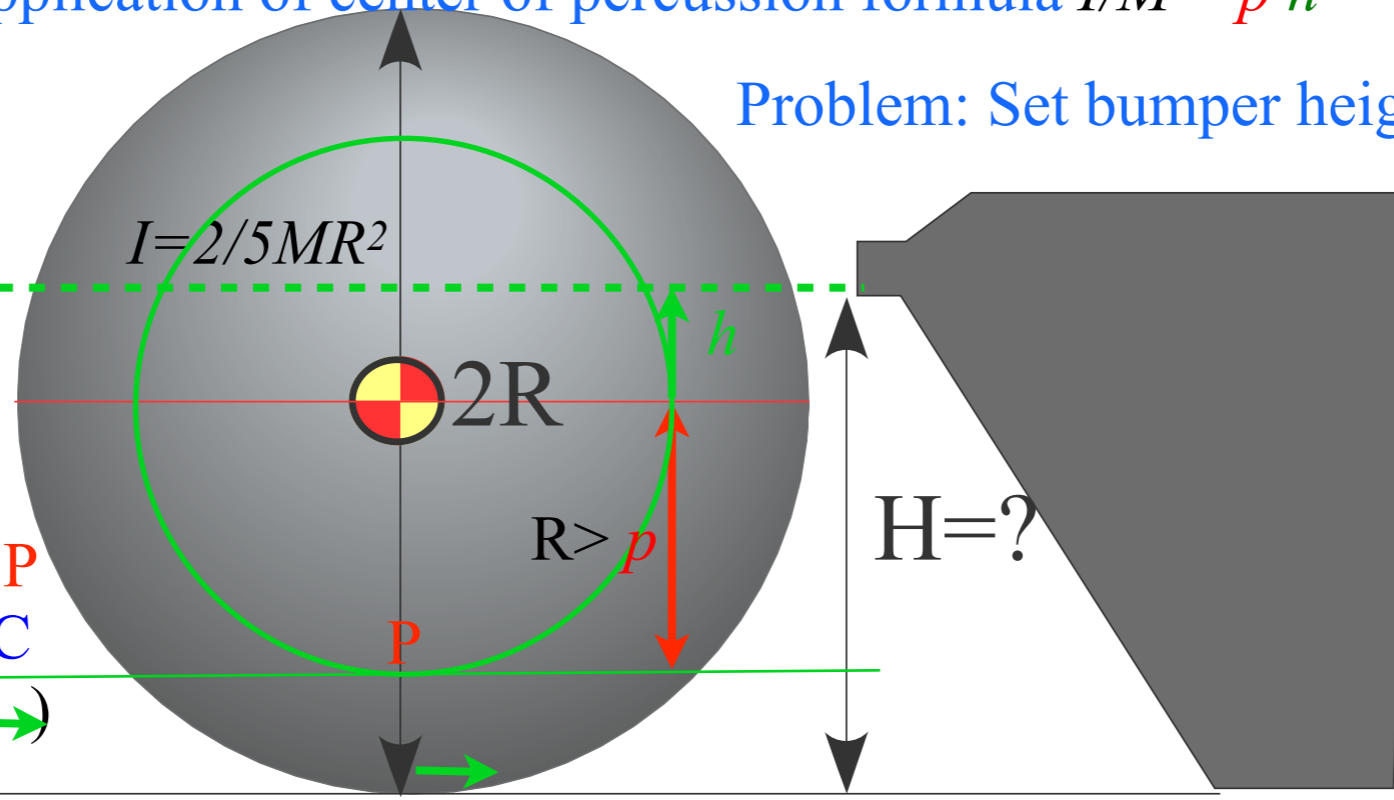
$$I/M = p \cdot h$$



Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

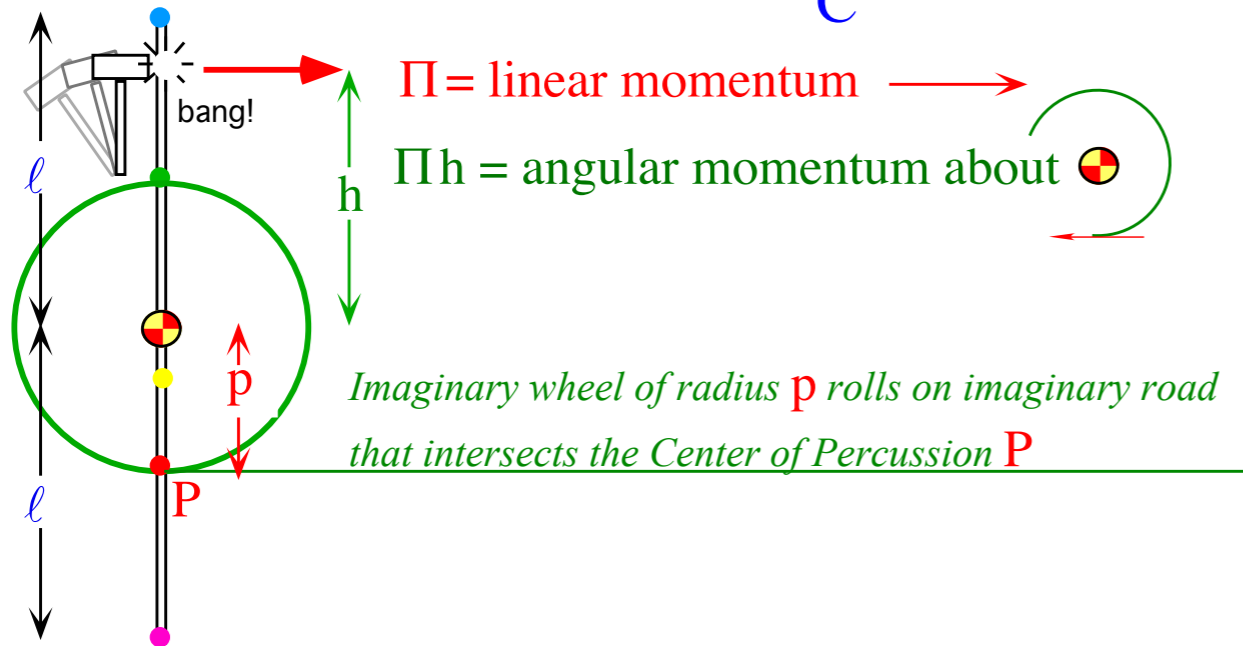
center of percussion P
above contact point C
(Ball skids to right \rightarrow)



Where should bumper height H be set to make ball contact point C at the center of percussion P ?

$H = ?$

C



$\Pi =$ linear momentum \rightarrow
 $\Pi h =$ angular momentum about

$$I/M = p \cdot h$$

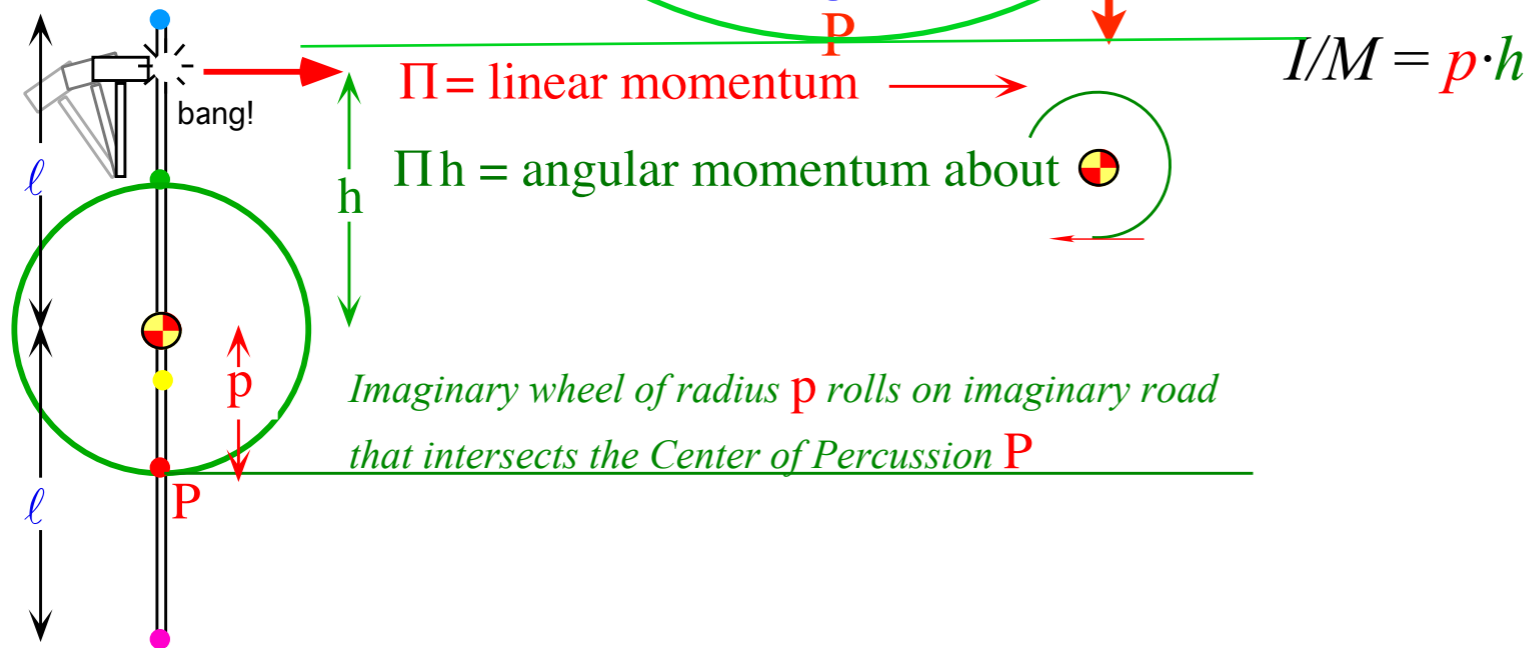
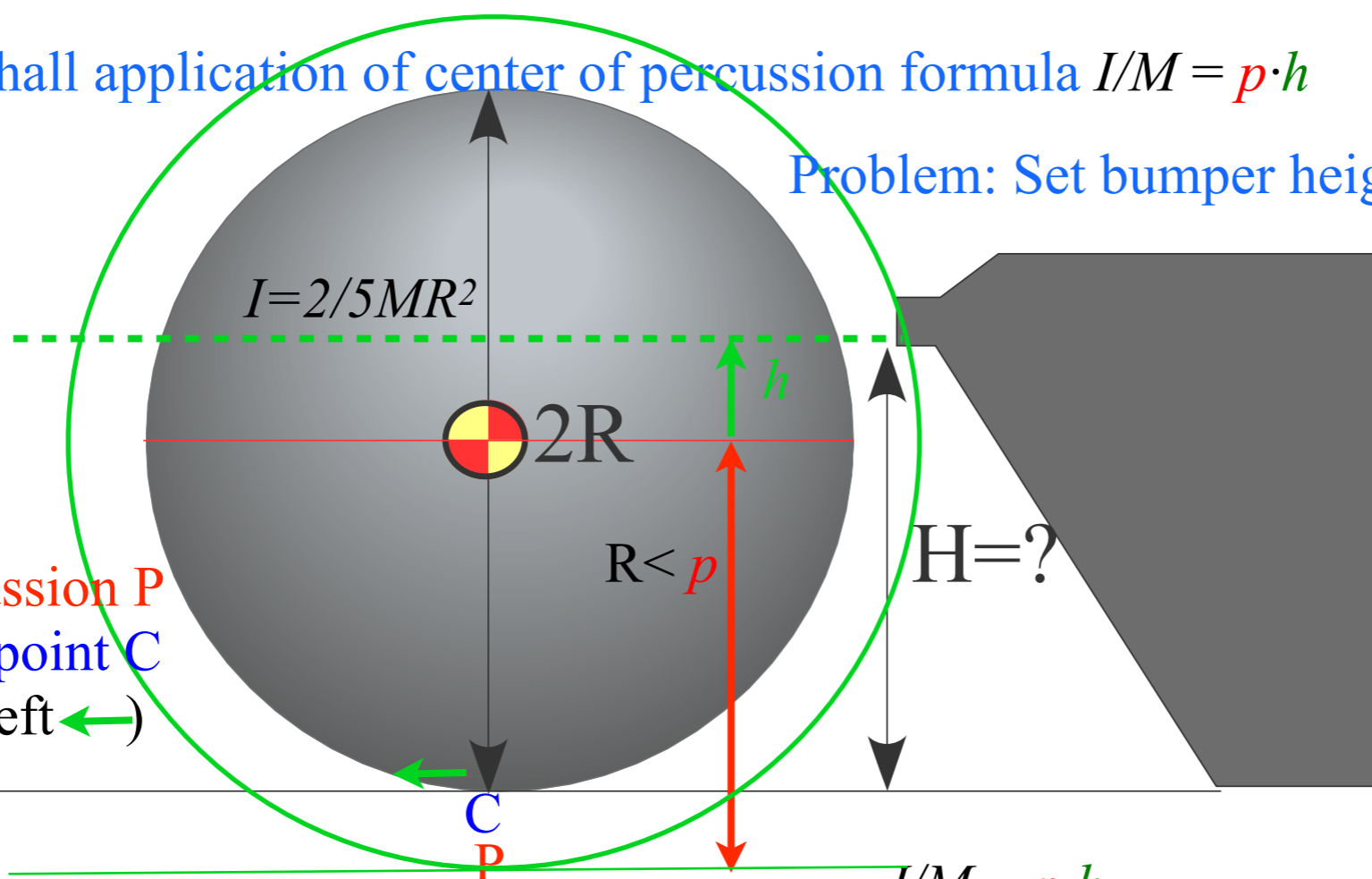
Imaginary wheel of radius p rolls on imaginary road that intersects the Center of Percussion P

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

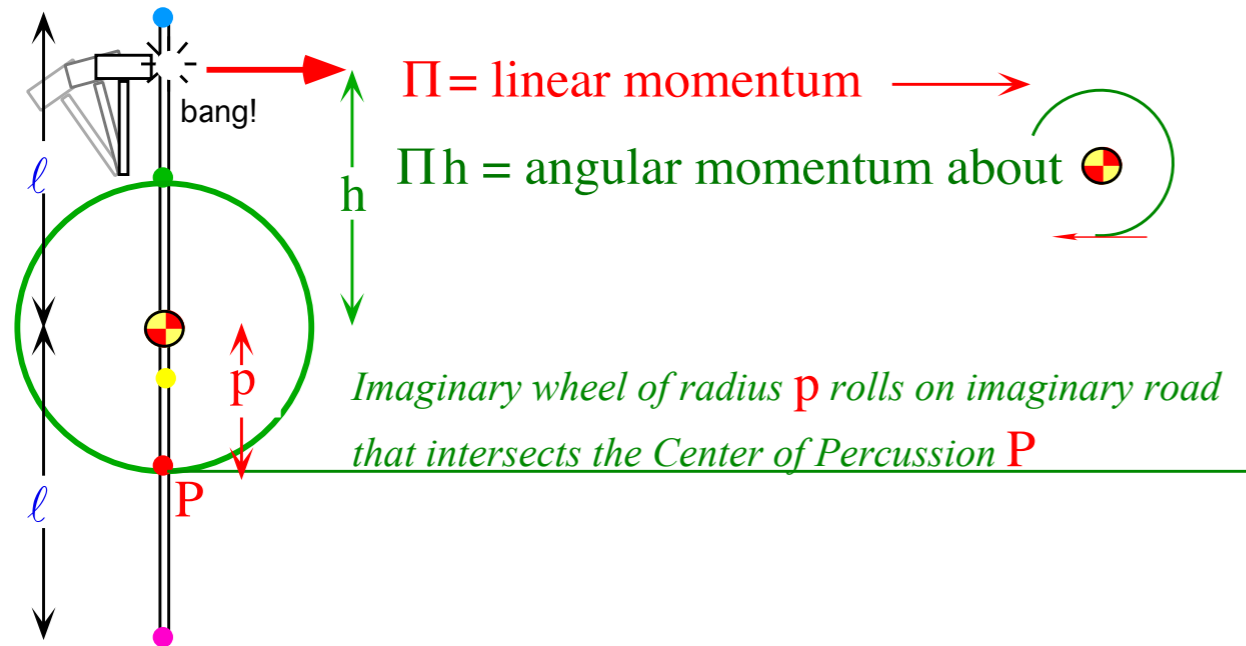
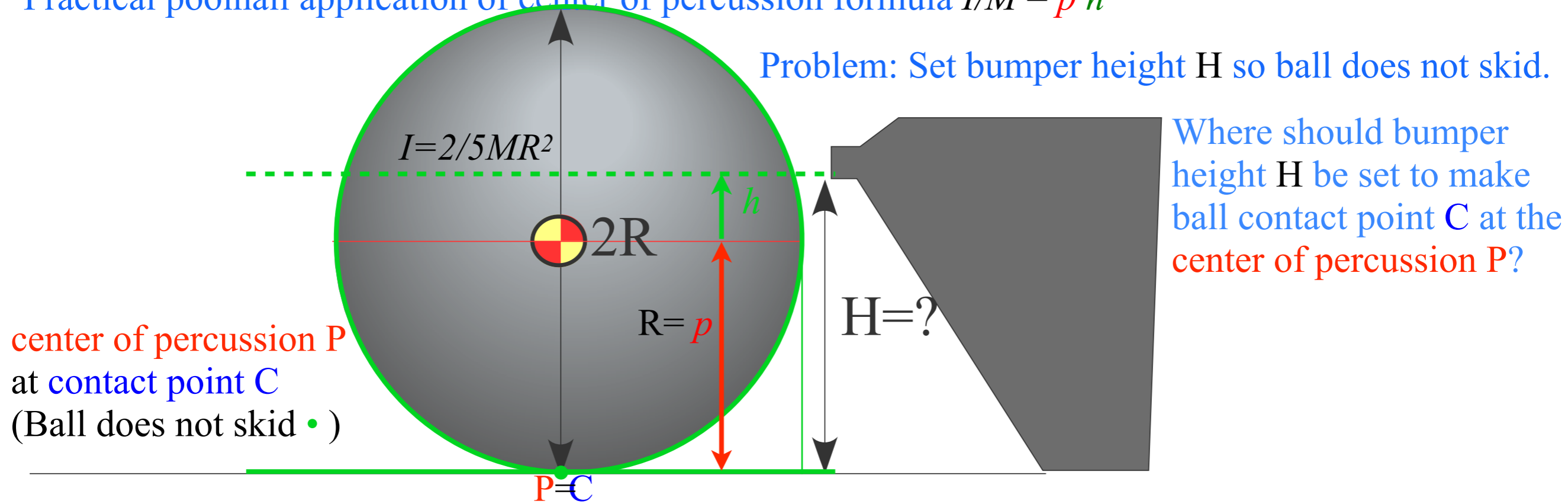
Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P ?

center of percussion P
below contact point C
(Ball skids to left \leftarrow)



Practical poolhall application of center of percussion formula $I/M = p \cdot h$

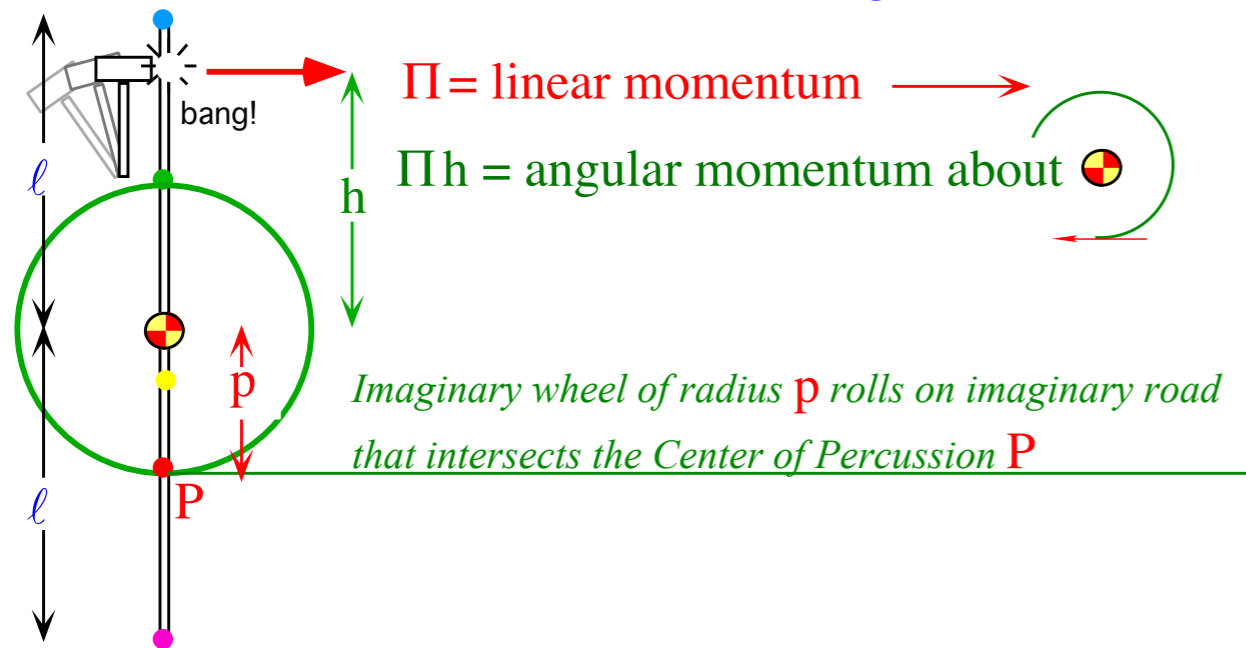
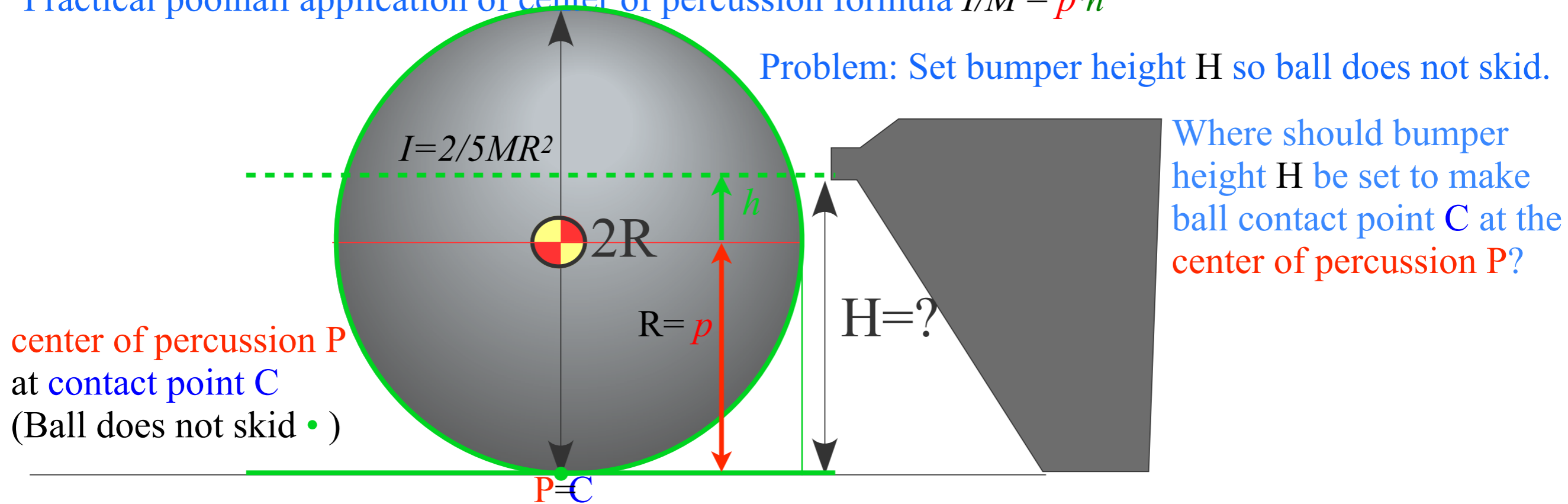


$$I/M = p \cdot h$$

$$h = I/Mp = I/MR$$

(For $R = p$)

Practical poolhall application of center of percussion formula $I/M = p \cdot h$



$$I/M = p \cdot h$$

$$h = I/Mp = I/MR \quad (\text{For } R = p)$$

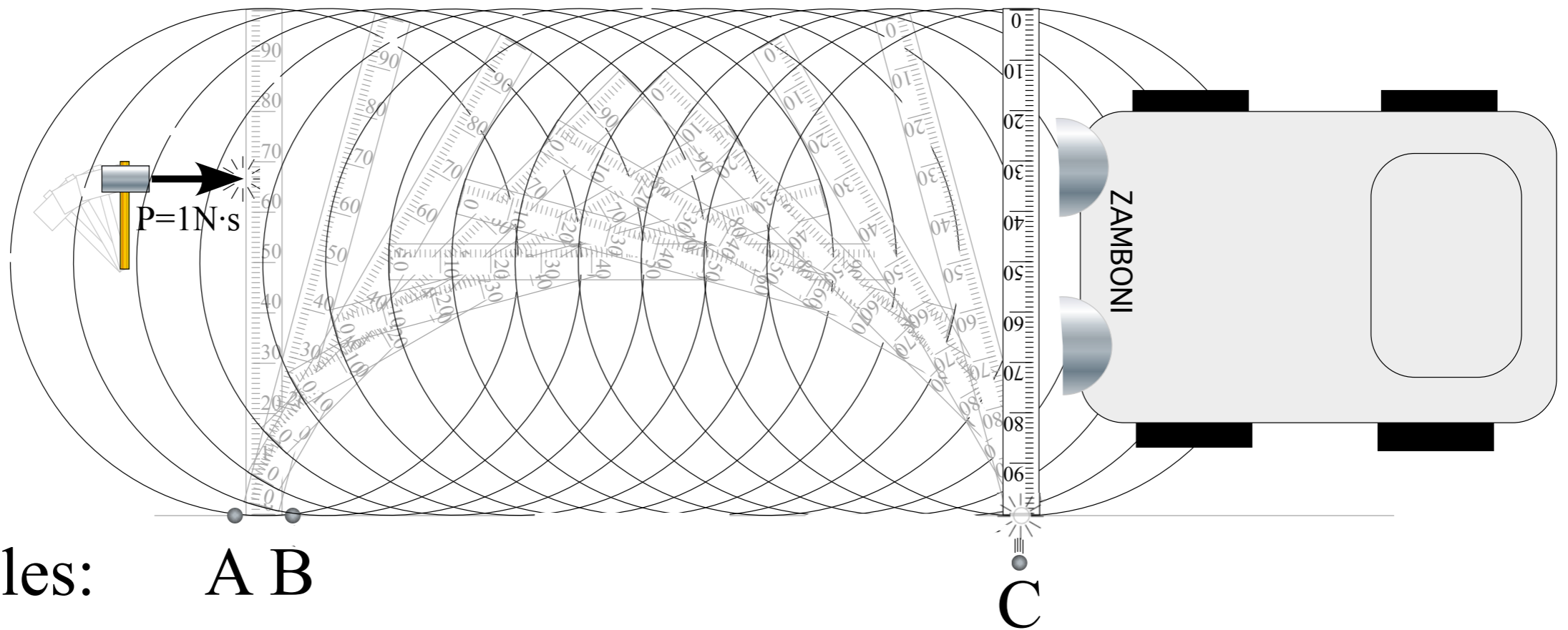
$$= 2/5 MR^2 / MR$$

$$= 2/5 R$$

For: $H = R + h = 7/10(2R)$ ball does not skid.

The Zamboni-Ice-Shot problem

(Assumes frictionless ice rink)



Marbles: A B

Where on a meter-stick do you hit it
so as to not disturb marbles A or B
and...

...knock marble C down as shown.