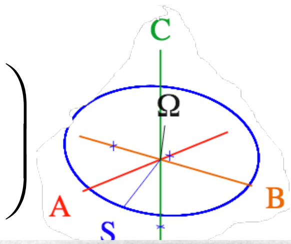
Lecture 23
Wed. 11.07.2018

U(2)~R(3) algebra/geometry in classical or quantum theory

(Classical Mechanics with a BANG! Units 4-6, Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5 and Ch. 7)

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*



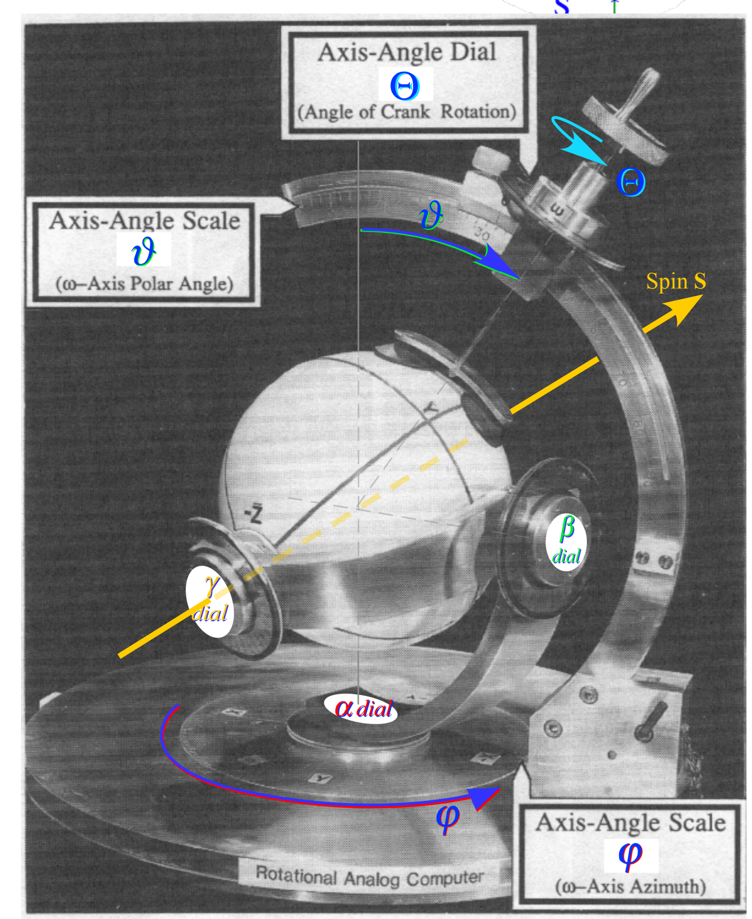
Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes
- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates
- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism
- *Bloch equation for density operator*



A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:
- Sorting ultracold atoms in a 3D optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

AMOP Ch 0 Space-Time Symmetry - 2019

Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Springer AMO Handbook - Ch 32 - Harter-Reimer-2019

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

Older Links from Lectures 14-20

http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
Shooting-range-for-medieval-siege-weapons-Anybody-knows
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
The trebuchet, Chevedden, Sci Am 1995
'*Simple*' *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
'*Cycloid*' *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
Google search on: "Satelite view of Patricia" (Images)
Physics Girl Channel - Fun with Vortex Rings in the Pool
iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2
Mechanical Analog to EM Motion (YouTube video) - https://youtu.be/hTd5FTJ-vRk
CoulIt Web Simulation: Bound-state motion in parabolic coordinates
CoulIt Web Simulation: Bound-state motion in hyperbolic coordinates
OscillIt Web App: Simulations of various types of resonance: 18, 27, 31, 35, 38, 39
Smith Chart
http://nobelprize.org/

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:

https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

Links to supplement Lecture 21

BoxIt Web App:
- Pure A-Type w/Cosine
- Pure B-Type w/Cosine
- Pure B-Type w/Freq ratios
- Mixed AB-Type 2:1 Freq ratio

Wiki on Pafnuty Chebyshev

Links to supplement Lecture 22-23

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
- Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
- Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
- Pure C-Type A,D=4.055, B=0, C=0.1
- Mixed AB-Type w/Cosine
- Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
- Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
- Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
- Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Classical Mechanics with a Bang! 2018
- Lectures 8, 9, 23 page 93
- Text Unit 6, page=27

ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Web based 3D & XR (x∈{A,M,V}, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
  Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
  Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
  Pure C-Type A,D=4.055, B=0, C=0.1
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BoxIt Web App Links:
  Pure A-Type
  Pure B-Type
  Pure C-Type
  Mixed AB-Type w/Stoke's plot & Frequency ratios
2018 Adv AMOP Lecture 9, Pages: 5, 61, 93
In development - Web based Rotational calculator
Babylon.JS 3D Web graphics shim extraordinaire
WebGL API for 3D web graphics modeled after OpenGL

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https://modphys.hosted.uark.edu/markup/BoxItWeb.html?AU2=4.0&BU2=-0.2&CU2=0.0&DU2=4.0&xInitial=1.0&yInitial=0.0&pxInitial=0.0&pyInitial=0.0&wantBoxLines=1&wantCosinePlot=0
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https://modphys.hosted.uark.edu/markup/BoxItWeb.html?AU2=4.055&BU2=-0.0&CU2=0.1&DU2=4.055&xInitial=1.0&yInitial=0.0&pxInitial=0.0&pyInitial=0.0&wantBoxLines=1&wantCosineP

https://modphys.hosted.uark.edu/markup/BoxItWeb.html?AU2=4.833&BU2=0.2403&CU2=0.4162&DU2=4.277&xInitial=0.911&yInitial=-0.244&pxInitial=0.0&pyInitial=0.0&wantBoxLines=0

https://modphys.hosted.uark.edu/markup/BoxItWeb.html?AU2=4.9&BU2=-0.0&CU2=0.0&DU2=4.0&xInitial=1.0&yInitial=1.0&pxInitial=0.0&pyInitial=0.0&wantBoxLines=1&wantCosinePlot=0

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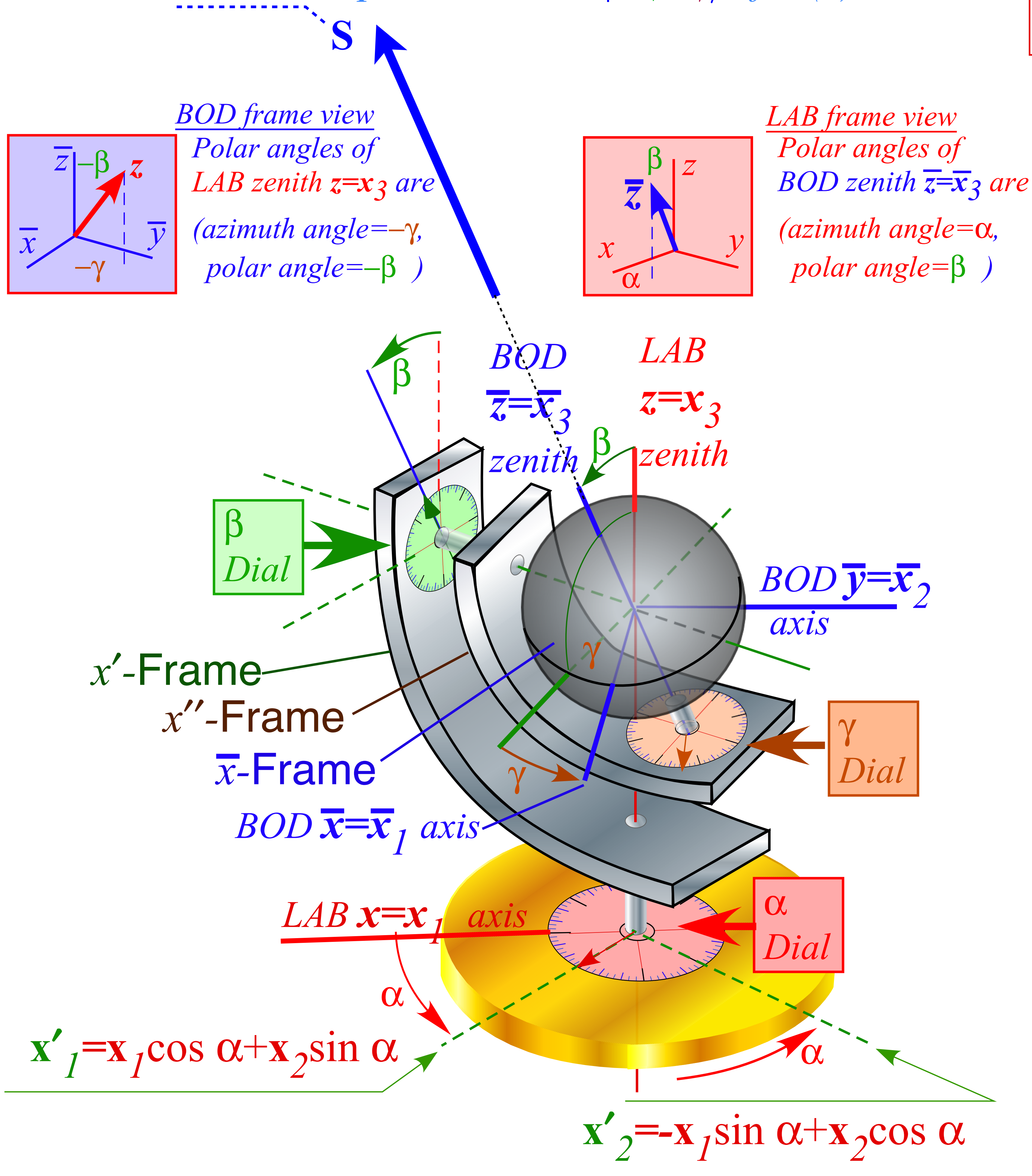
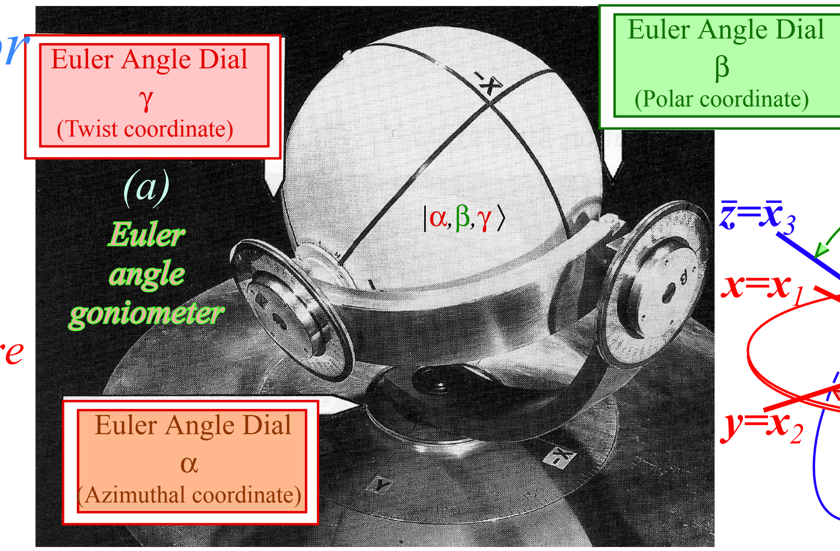
Ellipsometry using U(2) symmetry and related coordinates

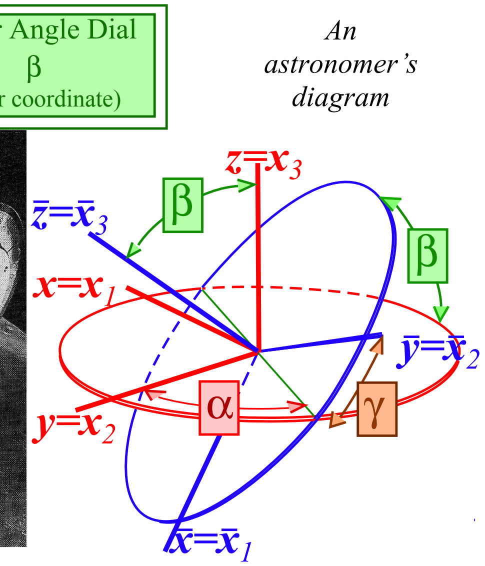
- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

3D-real **S**-*vector represents state* $|\alpha,\beta,\gamma\rangle$ *of U(2) oscillator*







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Euler angles

Development has begun on a web based version of this tool, but much of the App is at present (10/7/2018), in an '*indeterminate state*'. The App's 3D will in future be handled by *Babylon.JS, to act as a* shim to buttress the WebGL (web graphics layer) that is already in place.

Web based U(2) Calculator - Euler State

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha\,\text{about}\,Z]\cdot\mathbf{R}[\beta\,\text{about}\,Y]\cdot\mathbf{R}[\gamma\,\text{about}\,Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

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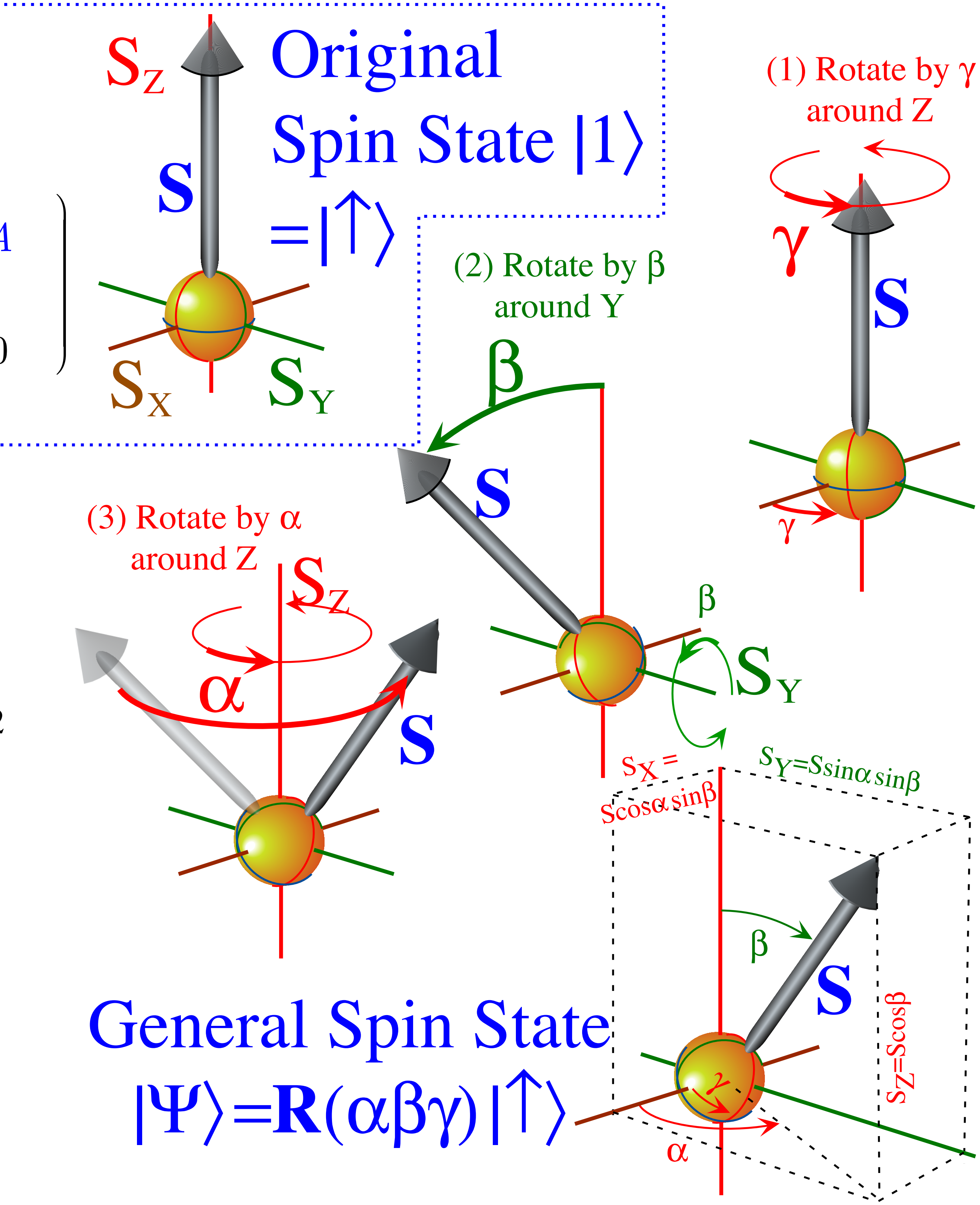

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

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Each point $\{E_1,E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=A\begin{pmatrix}e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}\\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}}$$

Asymmetry $S_A=\frac{1}{2}(a|\sigma_A|a)=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\frac{1}{2}\left[a_1^*a_1-a_2^*a_2\right]=\frac{1}{2}\left[x_1^2+p_1^2-x_2^2-p_2^2\right]=\frac{I}{2}[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}]$ $\boxed{=\frac{I}{2}\cos\beta}$

Balance $S_B=\frac{1}{2}(a|\sigma_B|a)=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\frac{1}{2}\left[a_1^*a_2+a_2^*a_1\right]=\left[p_1p_2+x_1x_2\right]=I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}+\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2}$ $\boxed{=\frac{I}{2}\cos\alpha\sin\beta}$

Chirality $S_C=\frac{1}{2}(a|\sigma_C|a)=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\frac{-i}{2}\left[a_1^*a_2-a_2^*a_1\right]=\left[x_1p_2-x_2p_1\right]=I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}-\cos\frac{\alpha-\gamma}{2}\cdot-\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2}$ $\boxed{=\frac{I}{2}\sin\alpha\sin\beta}$

Three ways to picture U(2) spin or pseudo-spin states

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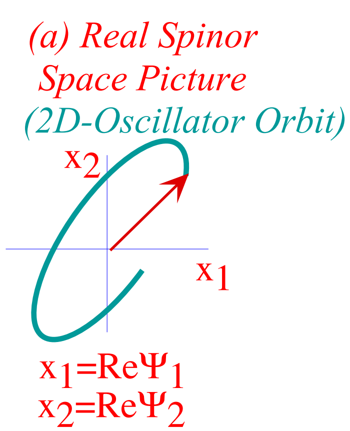
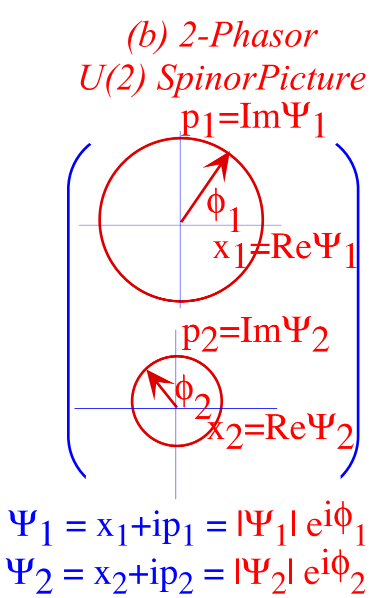
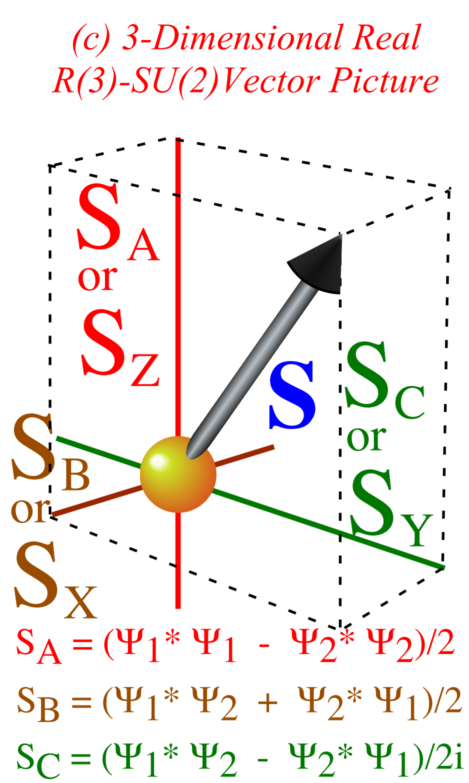
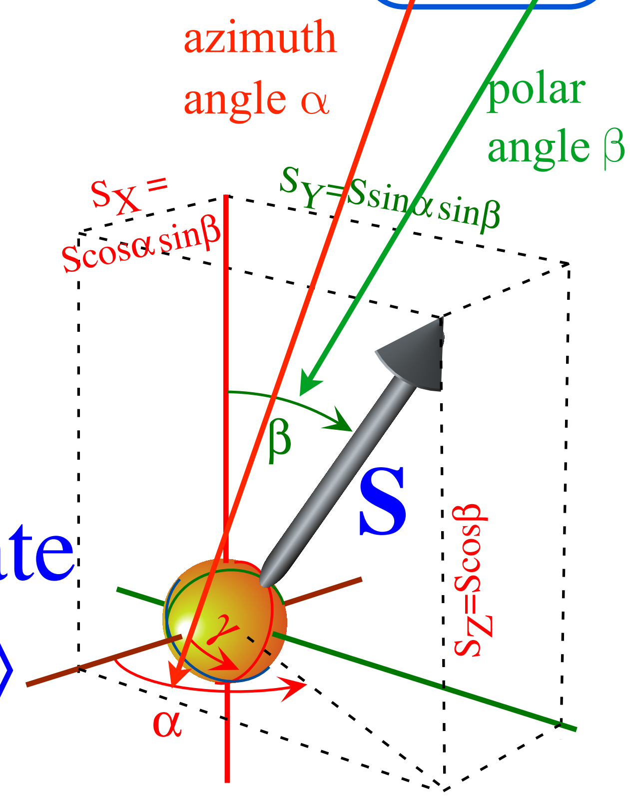






General Spin State

$$|\Psi\rangle=\mathbf{R}(\alpha\beta\gamma)\,|\uparrow\rangle$$



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(a) *Ellipsometry*

(b) *U(2) phasors*

(c) *3D real R(3) vectors*

Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

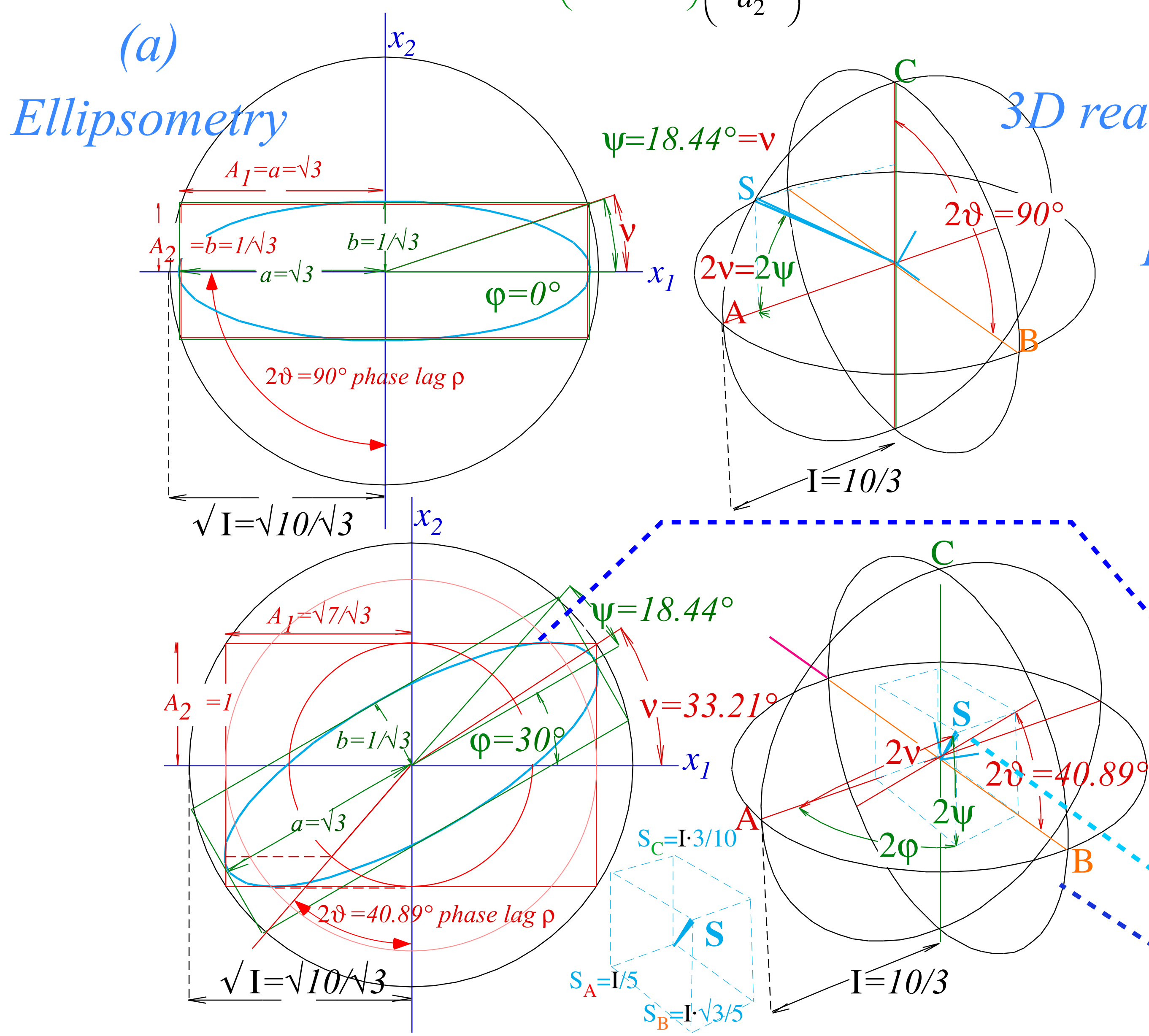
Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

Asymmetry $S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] \quad \boxed{= \frac{I}{2}\cos\beta}$

Balance $S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad \boxed{= \frac{I}{2}\cos\alpha\sin\beta}$

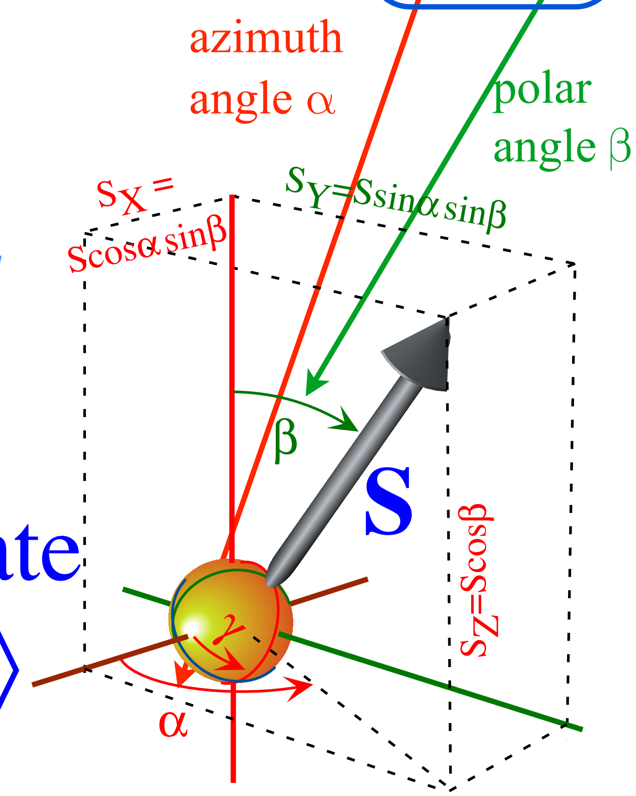
Chirality $S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot - \sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad \boxed{= \frac{I}{2}\sin\alpha\sin\beta}$



Ellipsometry of U(2) states detailed at end of this Lecture

General Spin State
$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$

Complex U(2) ellipse of any state corresponds to a single point **S** in R(3) on the Stoke's sphere



Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).

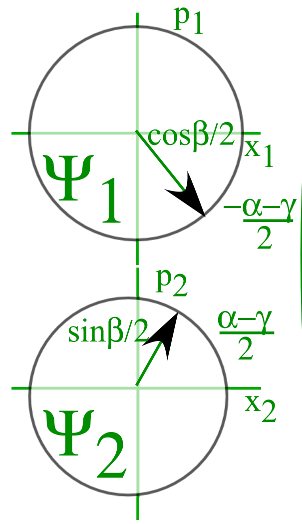
U(2) World : Complex 2D Spinors

Ellipsometry of U(2) states described by Two "Worlds" U(2) or R(3)

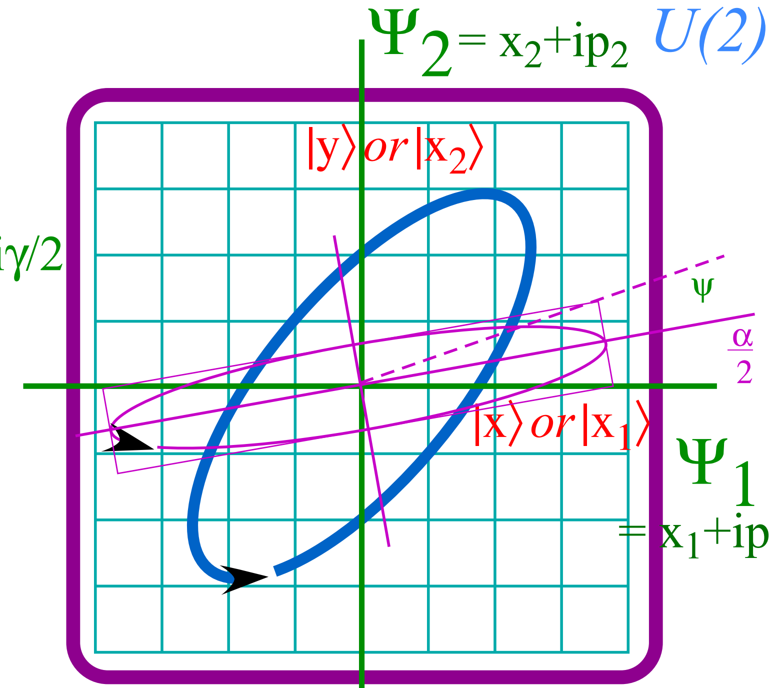
U(2) World labeled by two complex phasors and driven by complex operator

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

2-State ket $|\Psi\rangle=$



$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{N} e^{-i\alpha/2} \cos\beta/2 \\ \sqrt{N} e^{i\alpha/2} \sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

$$\begin{pmatrix} S_B \\ S_C \\ S_A \end{pmatrix} = \begin{pmatrix} N\sin\beta\cos\alpha \\ N\sin\beta\sin\alpha \\ N\cos\beta \end{pmatrix} \frac{1}{2}$$

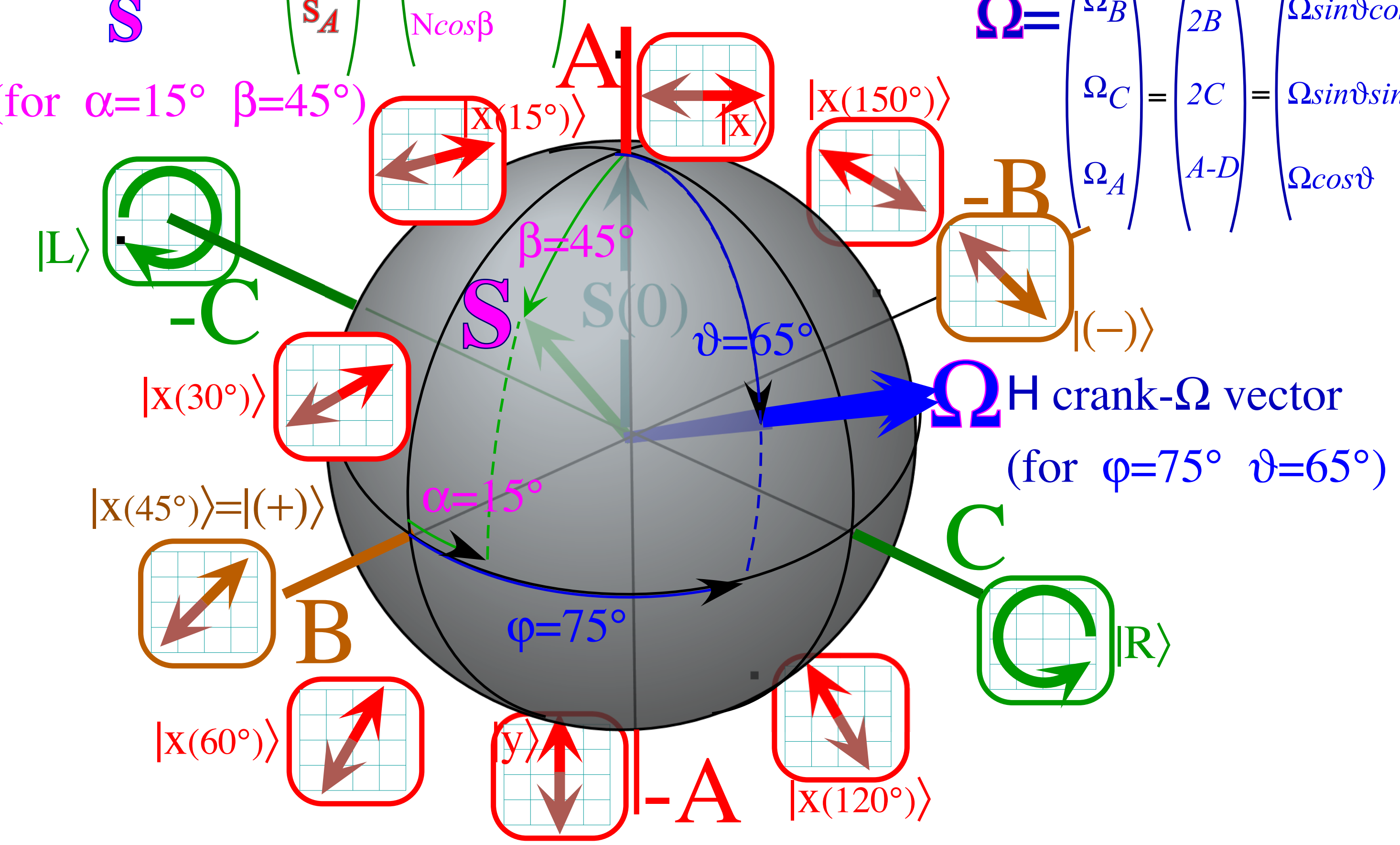
(for α=15° β=45°)

H-Operator $\begin{pmatrix} A & B\text{-}iC \\ B\text{+}iC & D \end{pmatrix}$

Angular velocity

$$\mathbf{\Omega} = \begin{pmatrix} \Omega_B \\ \Omega_C \\ \Omega_A \end{pmatrix} = \begin{pmatrix} 2B \\ 2C \\ A\text{-}D \end{pmatrix} = \begin{pmatrix} \Omega\sin\vartheta\cos\varphi \\ \Omega\sin\vartheta\sin\varphi \\ \Omega\cos\vartheta \end{pmatrix}$$

*R(3) World labeled by real 3-D "spin" vector **S** of angular momentum and driven by real 3-D "spin" vector **Ω** of angular velocity*



Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*

Euler-defined operator **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

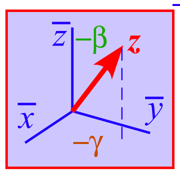
Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

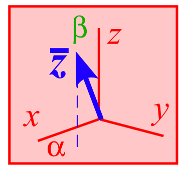
Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Here spin-rotor **S***-polar coordinates are Euler angles*

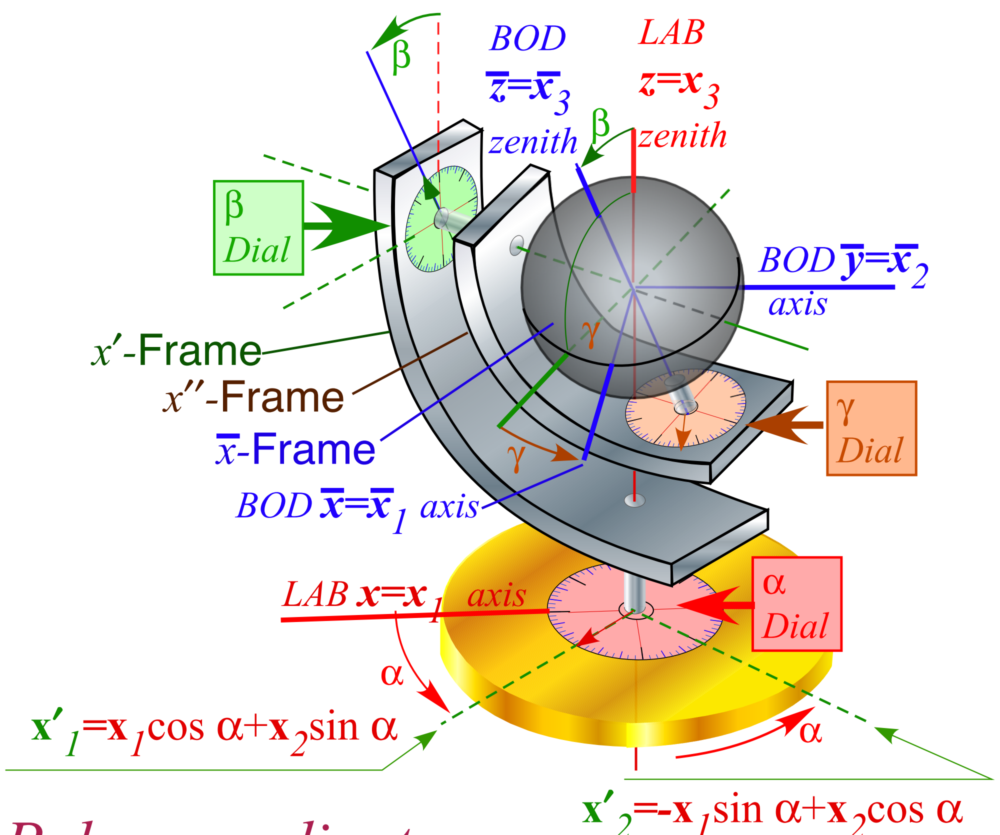
From Lecture 7 page 86



BOD frame view
Polar angles of LAB zenith $z=x_3$ *are (azimuth angle=−γ, polar angle=−β)*



LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are (azimuth angle=α, polar angle=β)*



Polar coordinates for unit Spin vector $\hat{\mathbf{S}}$

$$\hat{S}_X = \cos\alpha \ \sin\beta$$
$$\hat{S}_Y = \sin\alpha \ \sin\beta$$
$$\hat{S}_Z = \cos\beta$$

Spin State&Operator

$$\left|\alpha\beta\gamma\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

by Euler angles

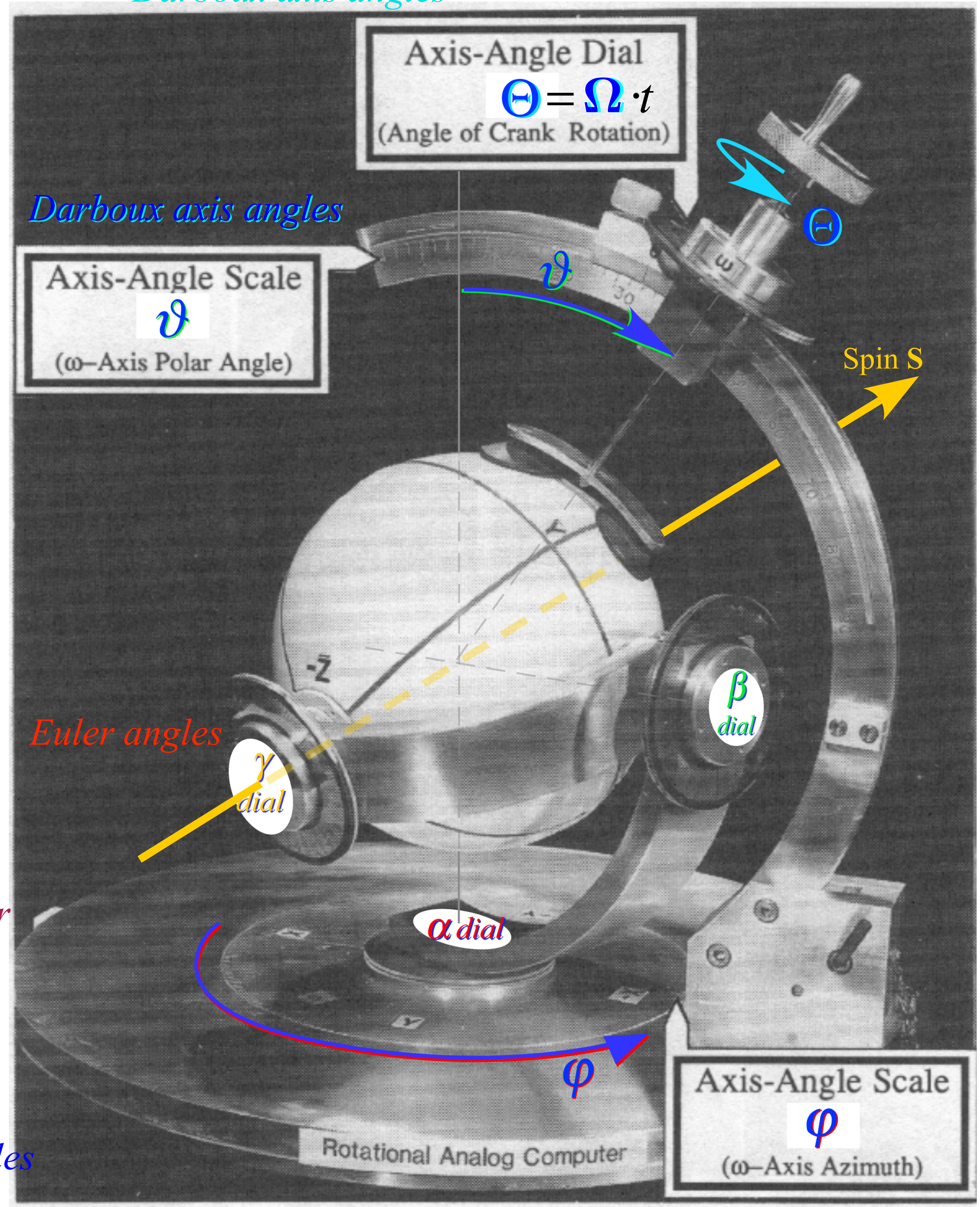
Polar coordinates for unit axis vector $\hat{\boldsymbol{\Theta}}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

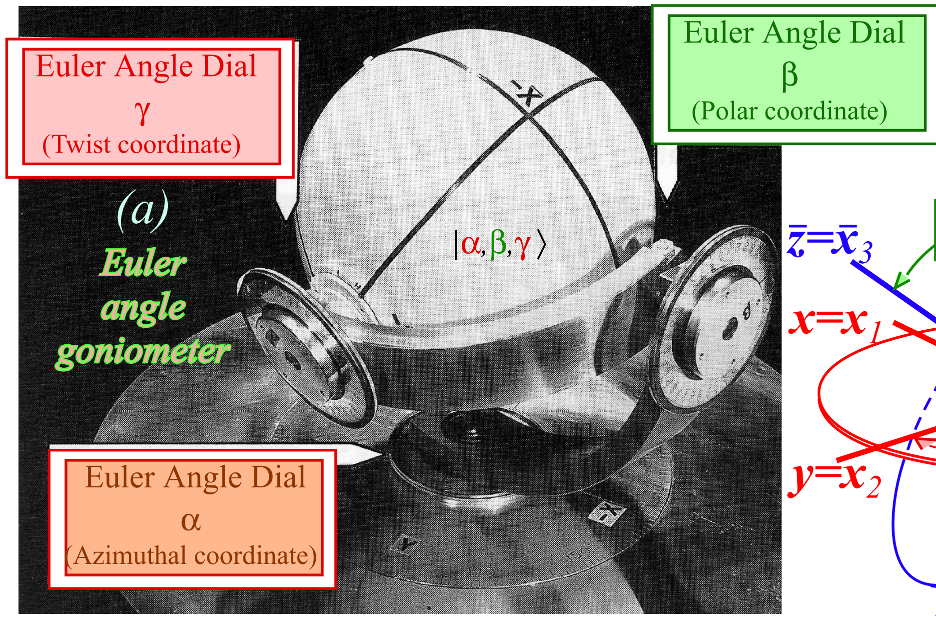
State&Operator

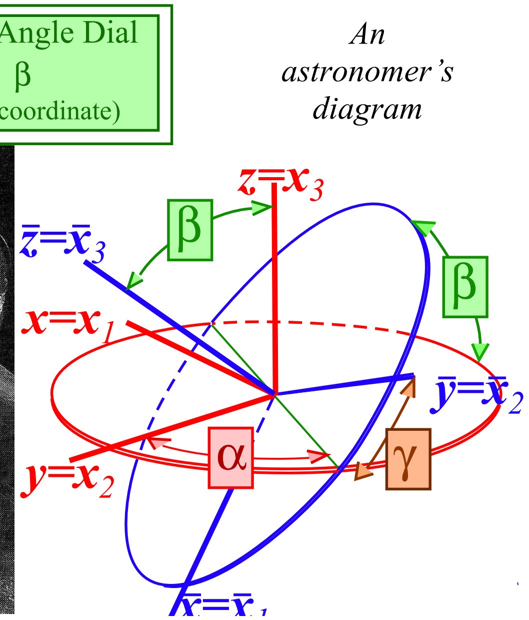
$$\left|[\varphi\vartheta\Theta]\right\rangle = \mathbf{R}[\varphi\vartheta\Theta]\left|\uparrow\right\rangle$$

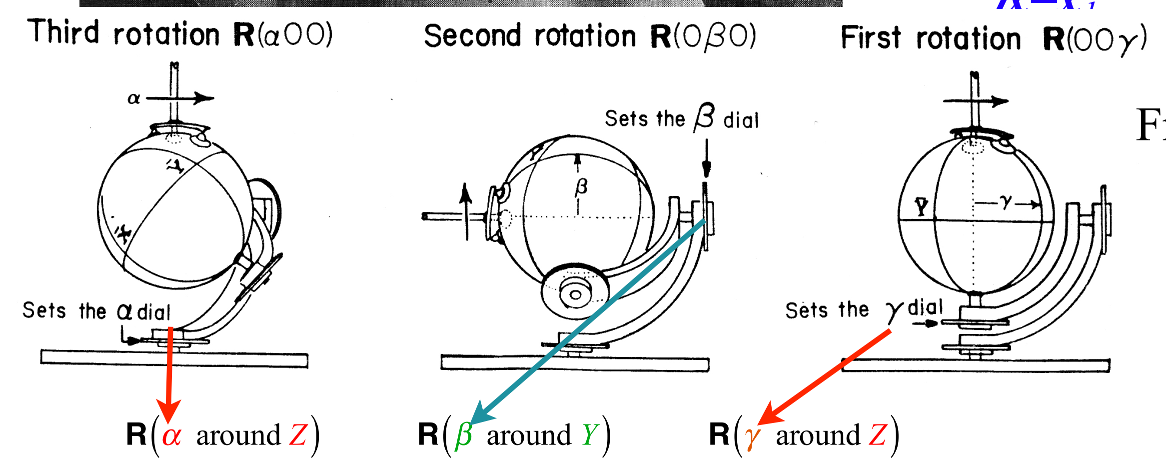
by Darboux axis angles



Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$





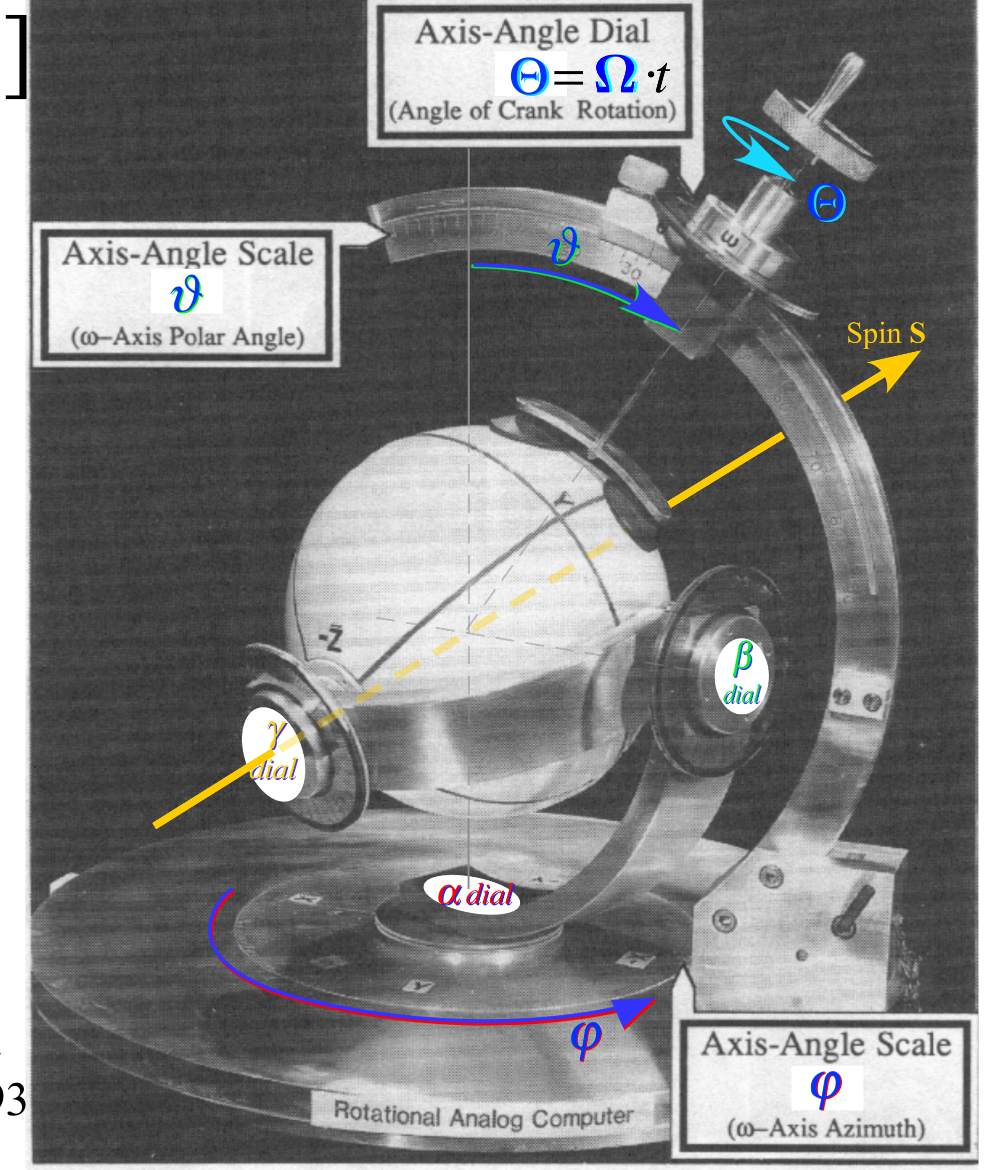


From Lecture 22 page 62 to 70

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.



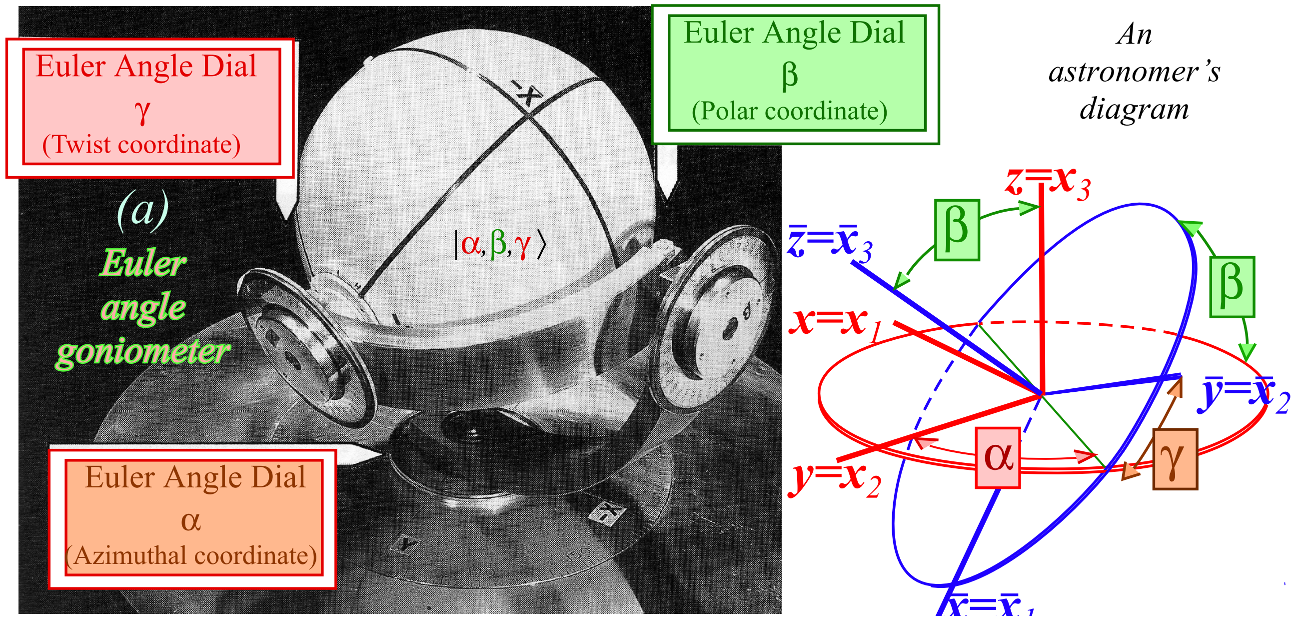
Lecture 22 page 92 to 93

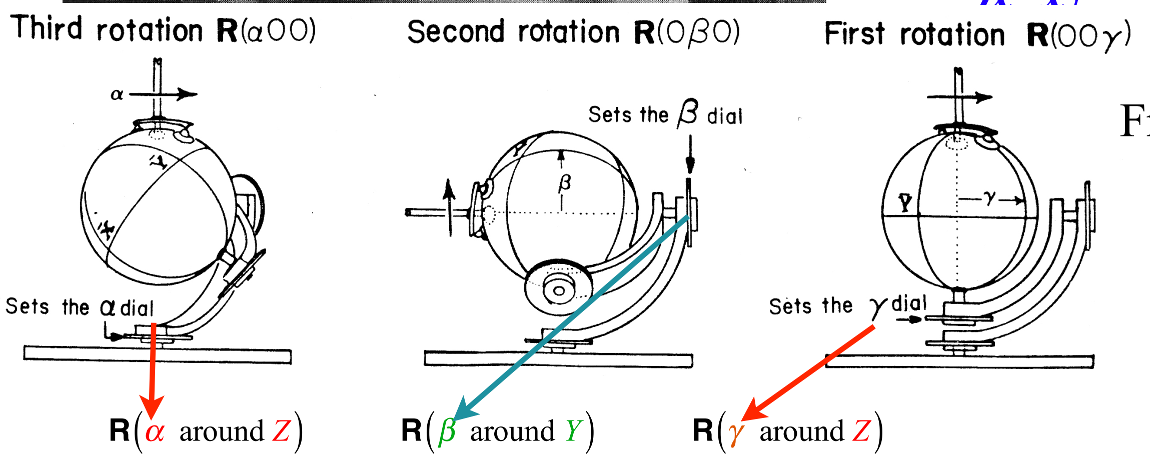
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right)\\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$





From Lecture 22 page 62 to 70

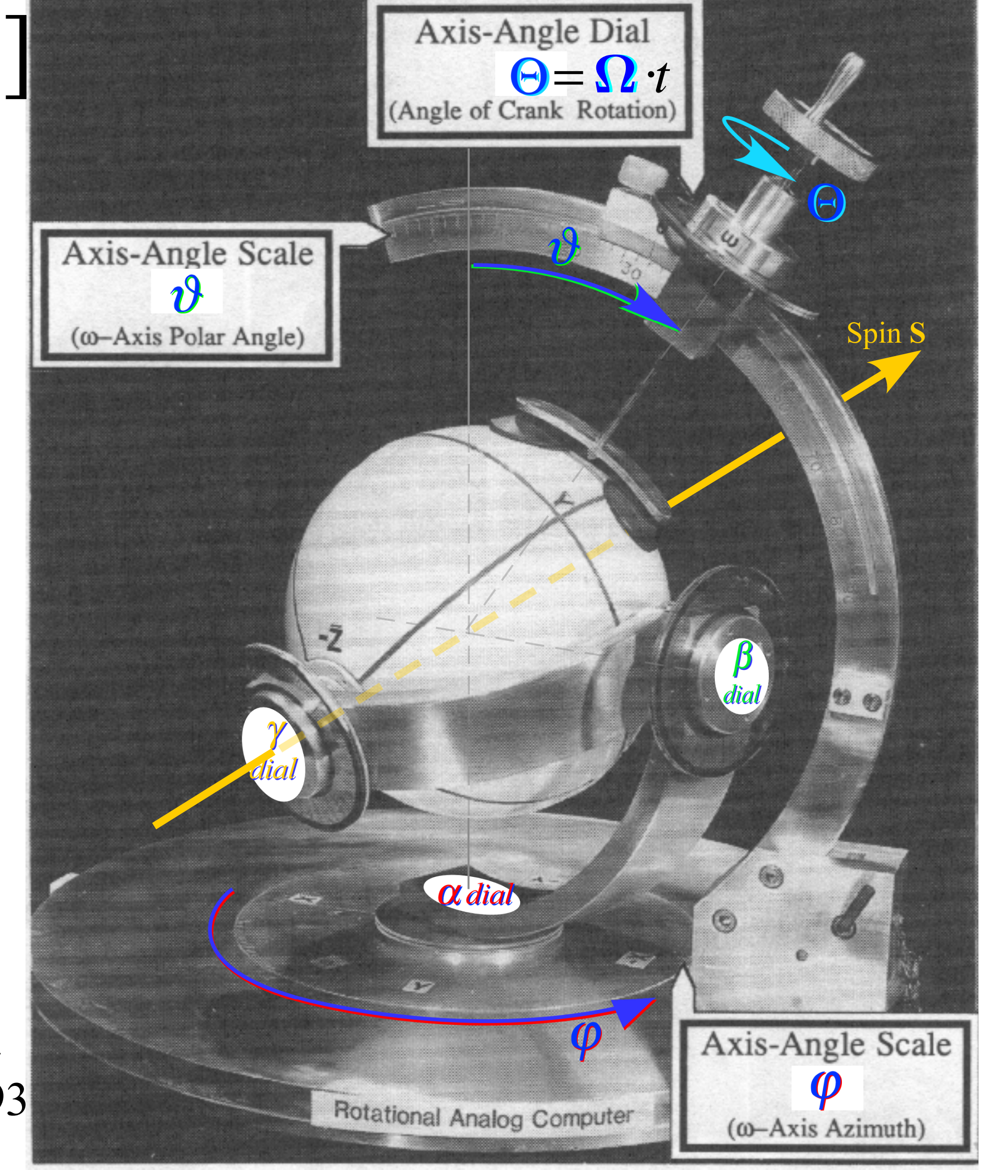
$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition*:

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ ($\alpha\beta\gamma$ *make better coordinates)*



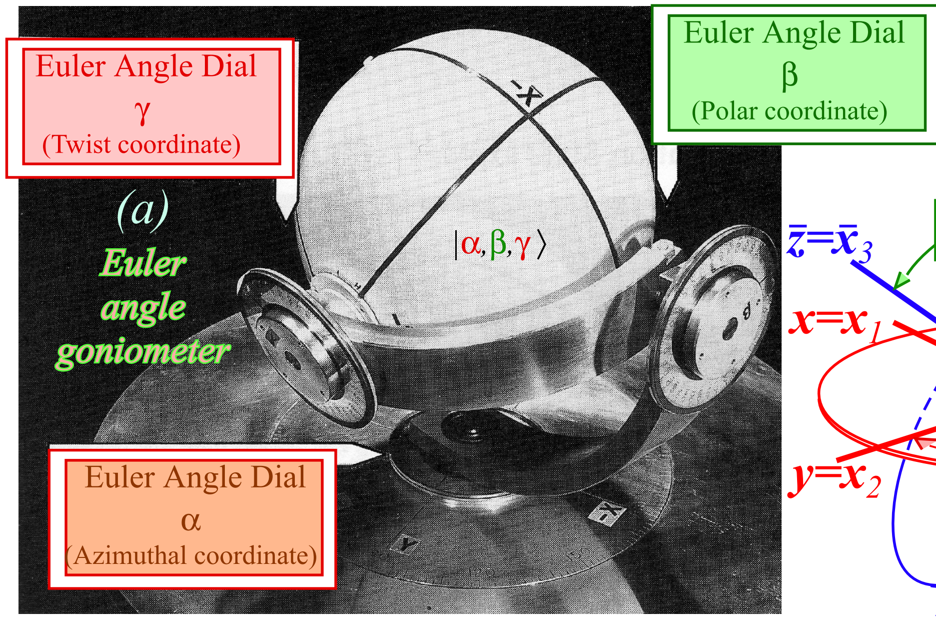
Lecture 22 page 92 to 93

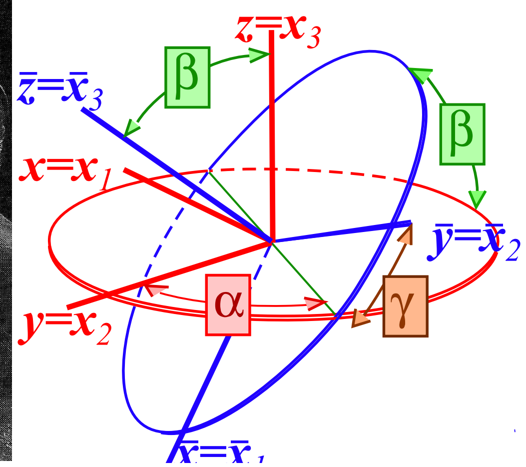
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

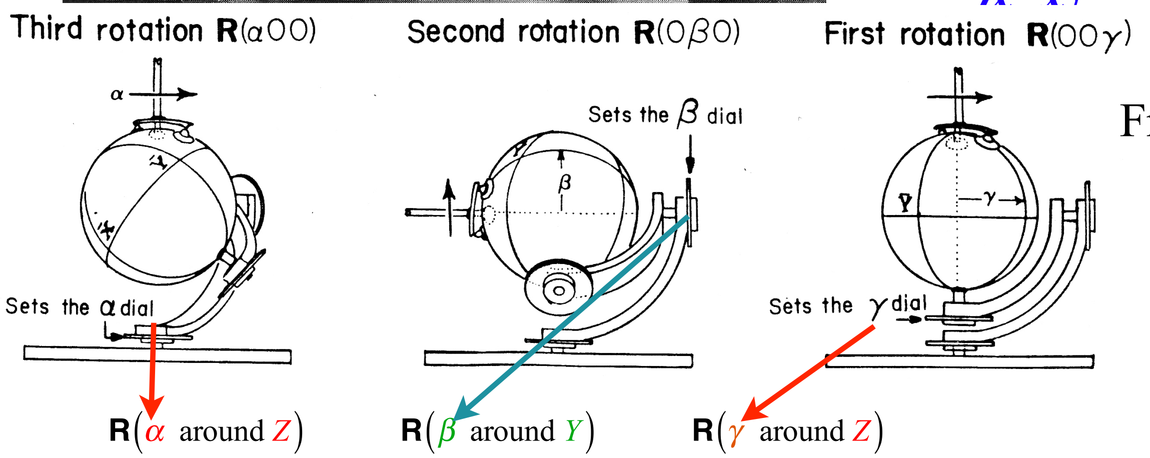
$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]





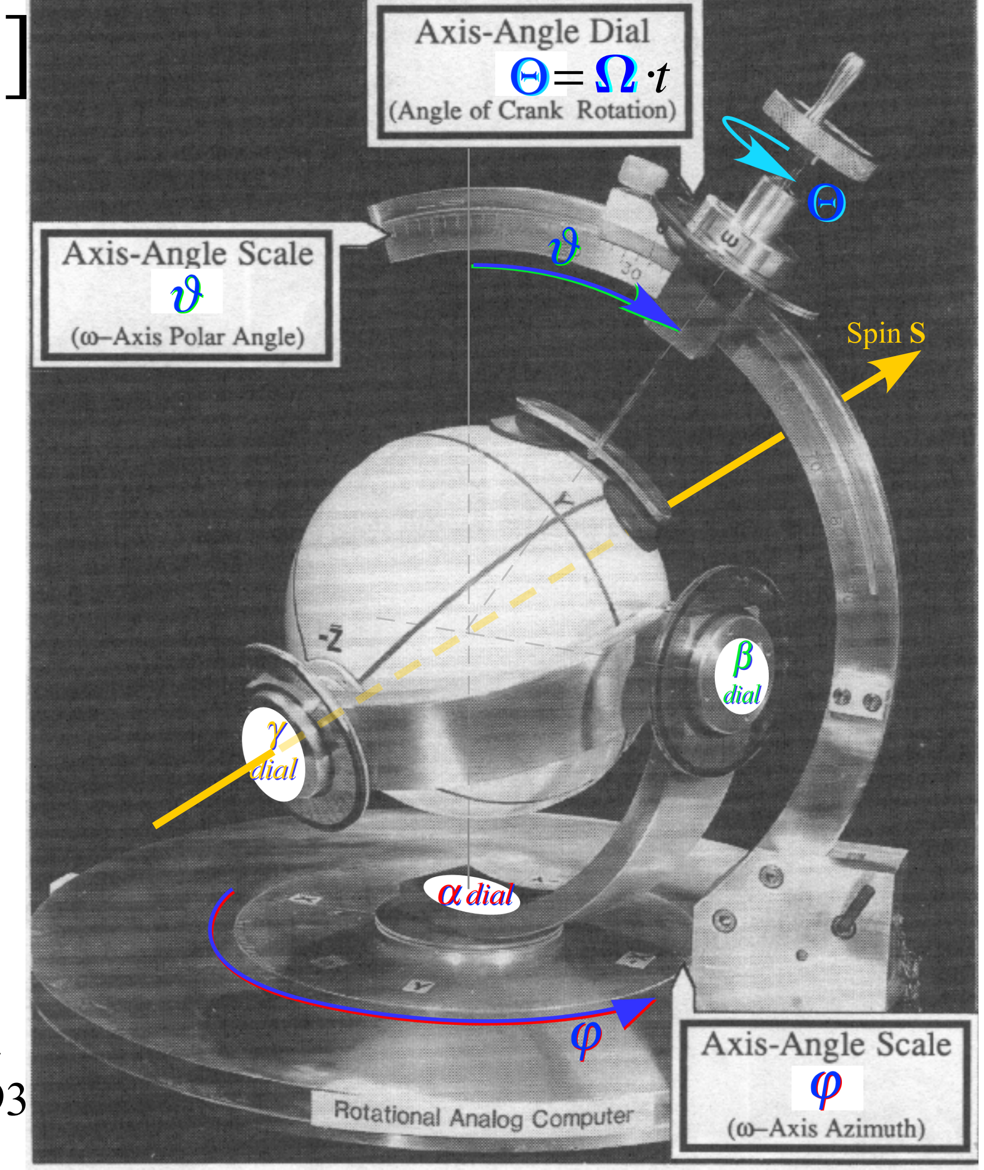


From Lecture 22
page 62 to 70

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].
Euler *state definition* lets us relate R(αβγ) to R[φϑΘ] ...
$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*



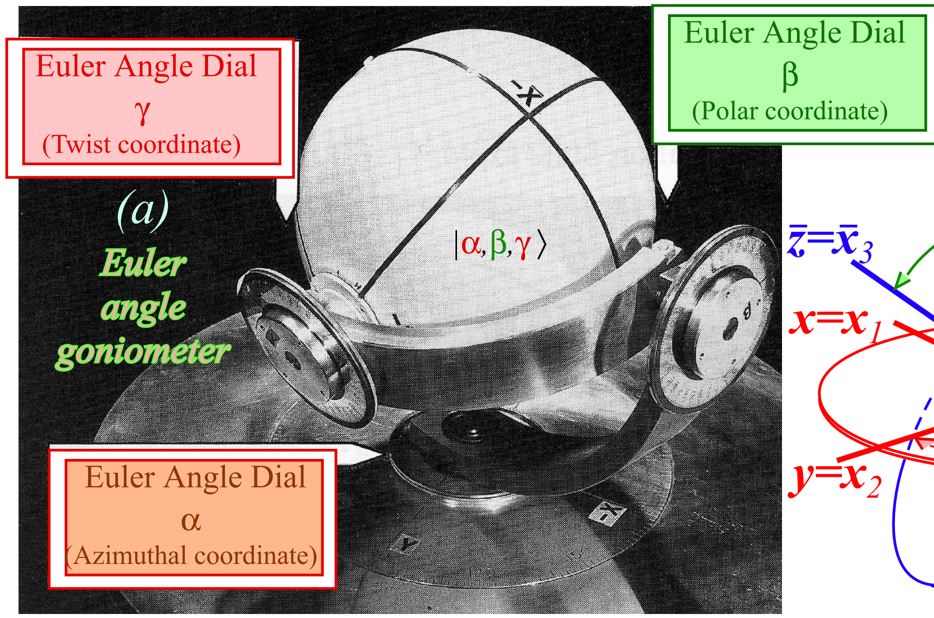
Lecture 22
page 92 to 93

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

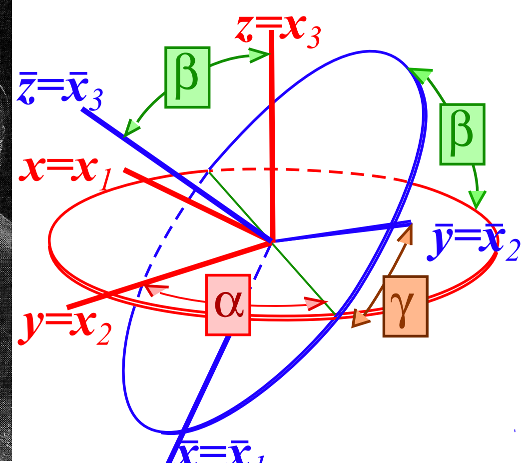
$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

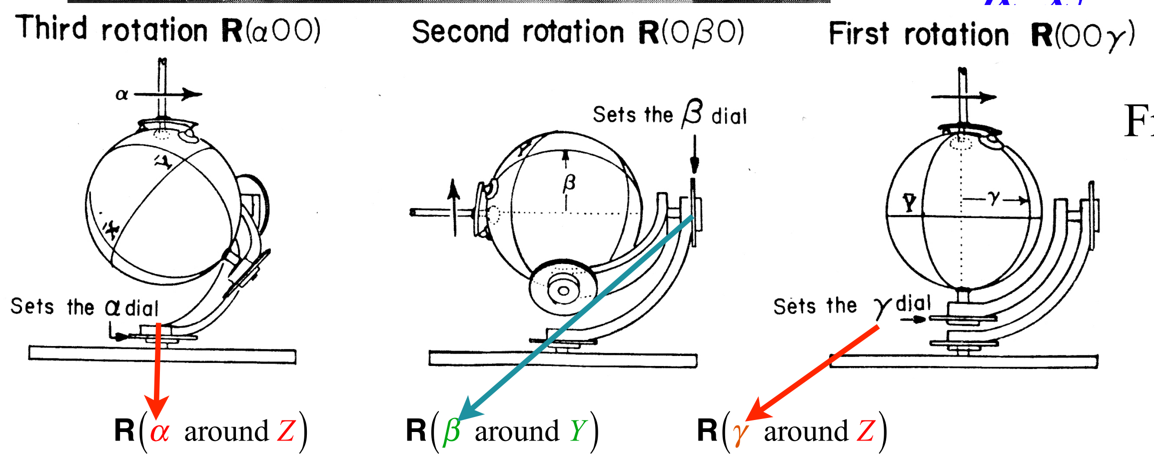
$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right)\\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]



An astronomer's diagram





From Lecture 22 page 62 to 70

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

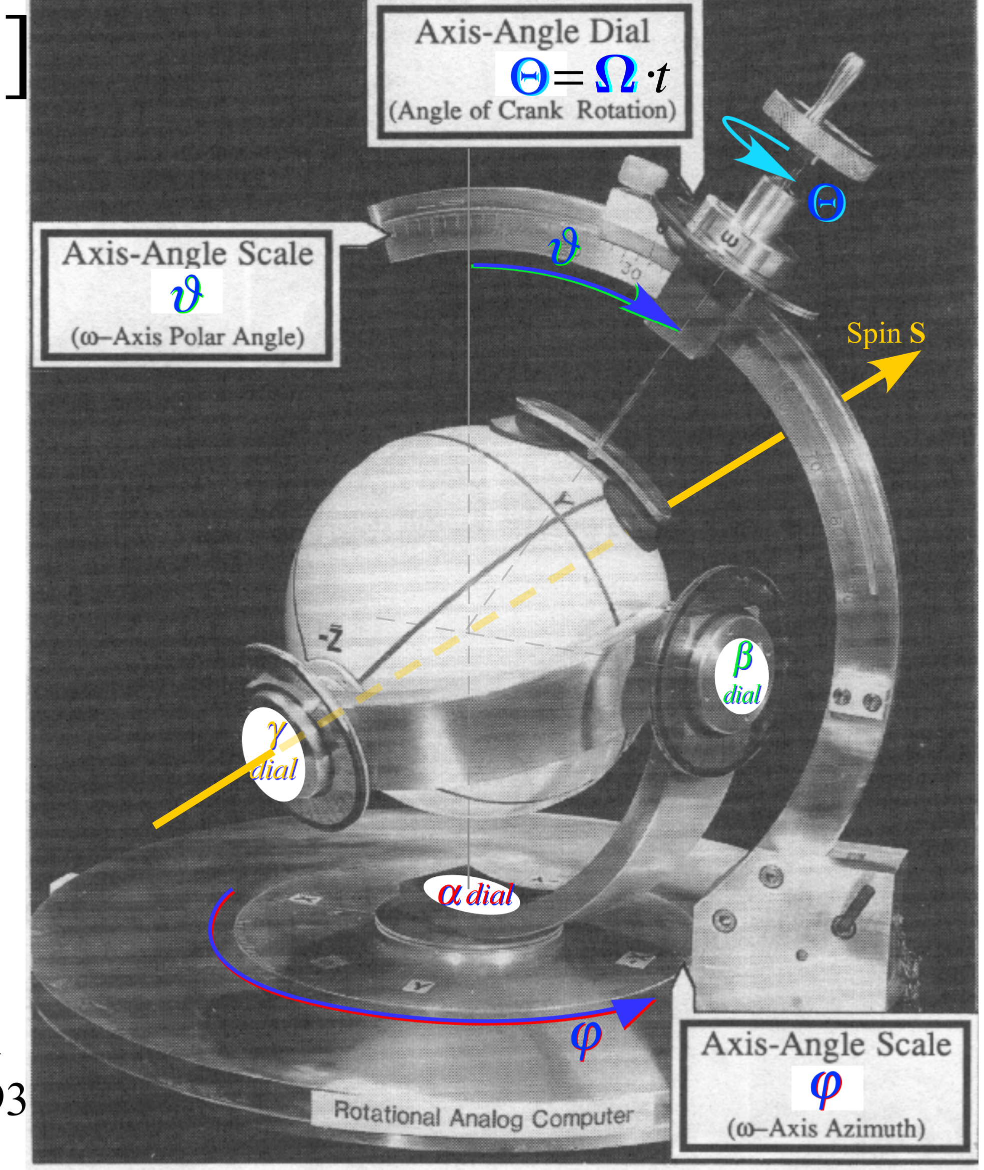
$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) = R[φϑΘ] ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}$$



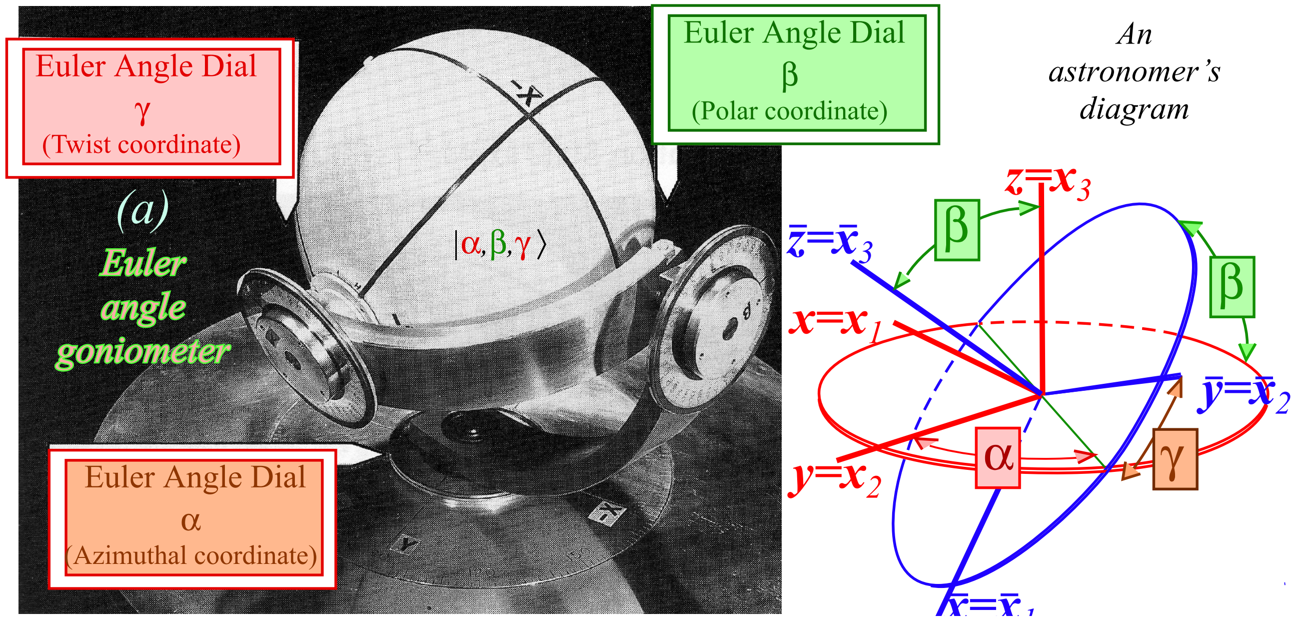
Lecture 22 page 92 to 93

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y)\\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

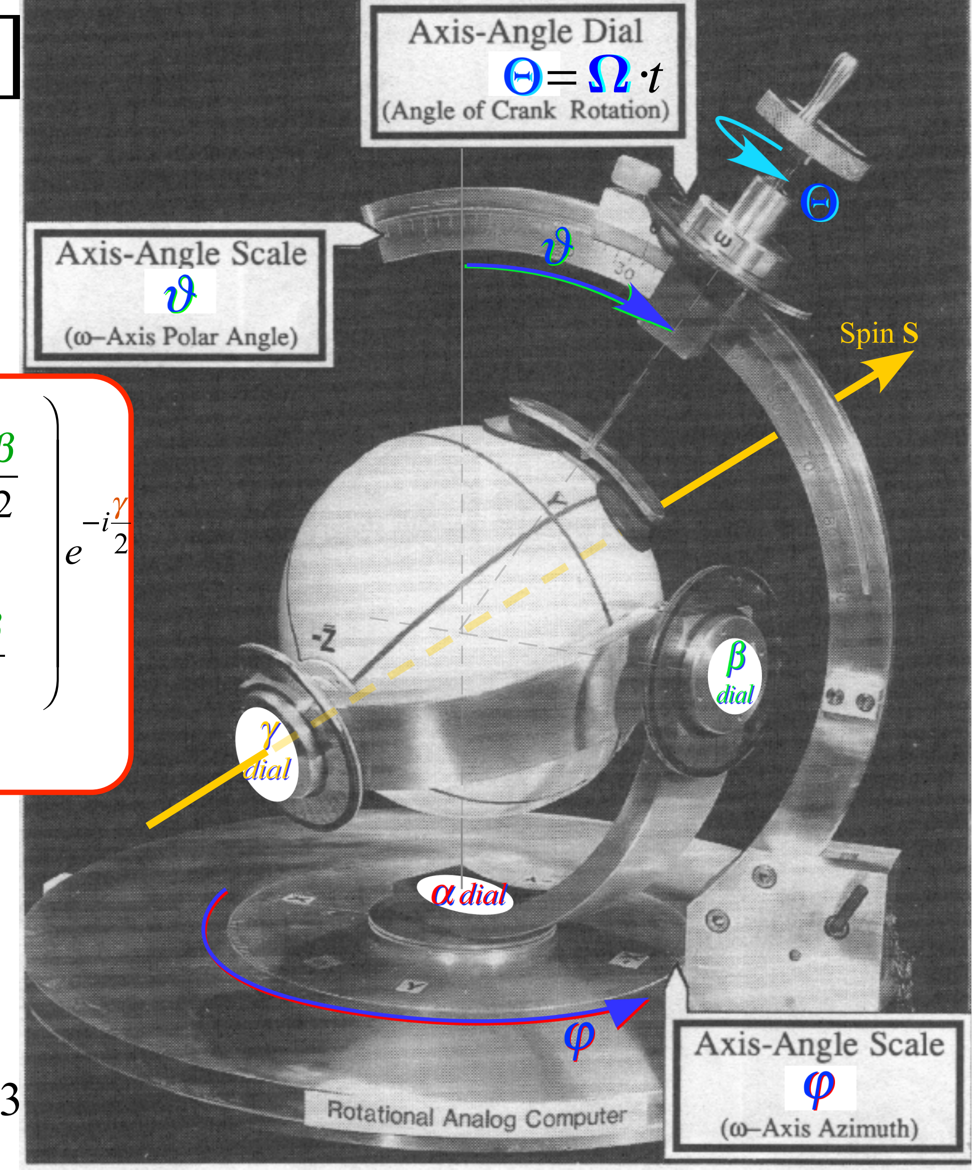
$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

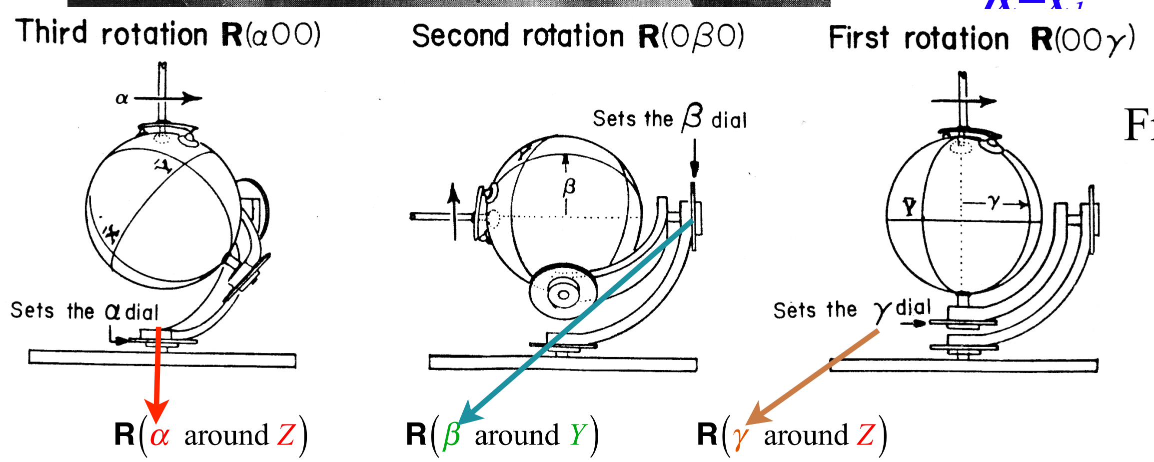
$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta)\\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$





$\mathbf{R}(\alpha$ around $Z)$ $\mathbf{R}(\beta$ around $Y)$ $\mathbf{R}(\gamma$ around $Z)$

From Lecture 22 page 62 to 70

Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

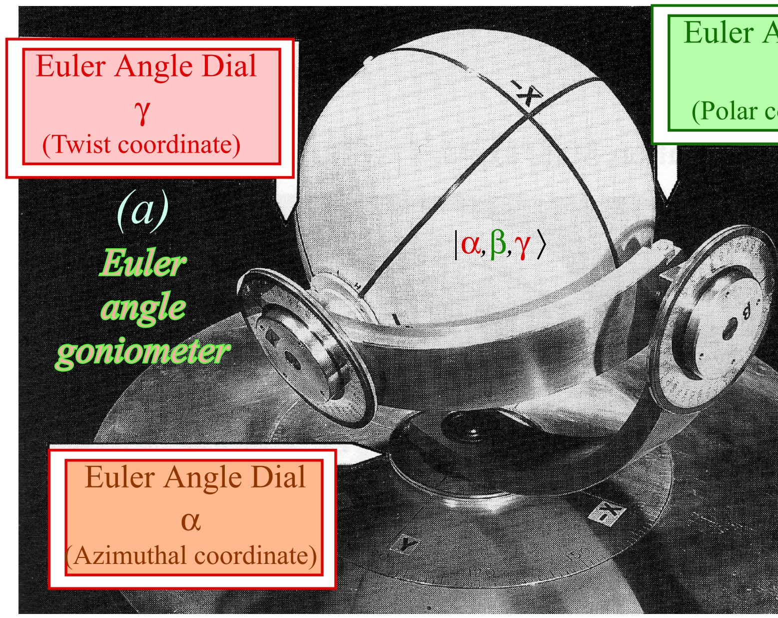
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}}$$

Phase coherence angle

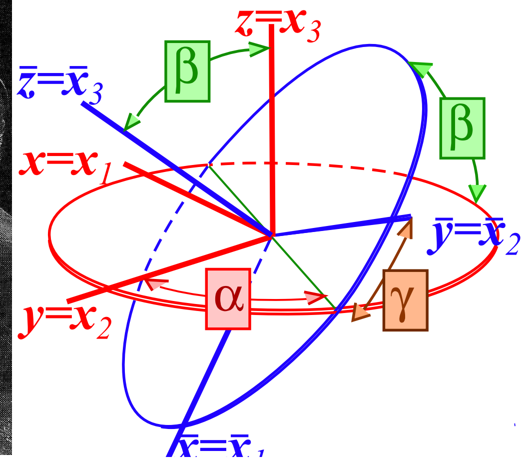
Population inversion angle

Overall phase angle

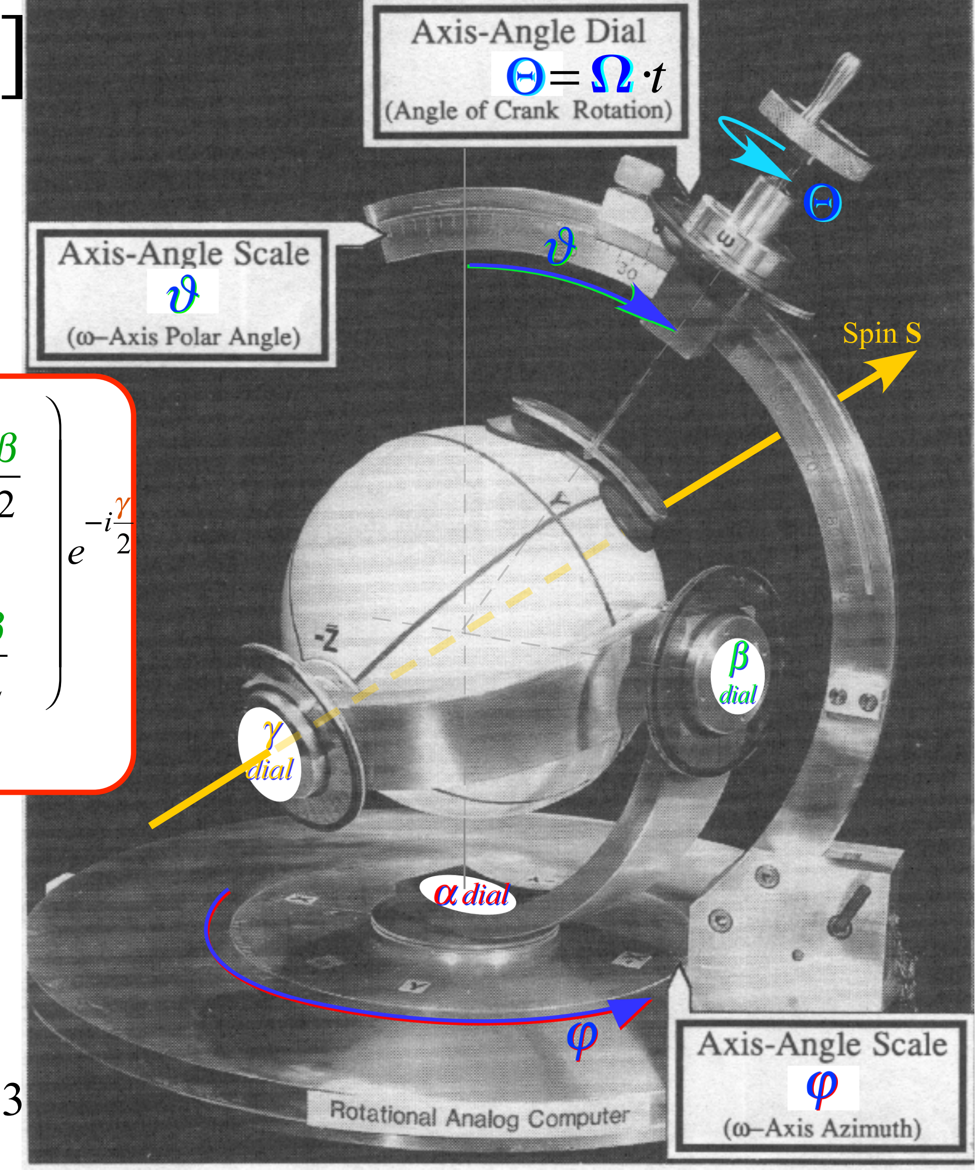
Euler R(αβγ) versus *Darboux* R[φϑΘ]



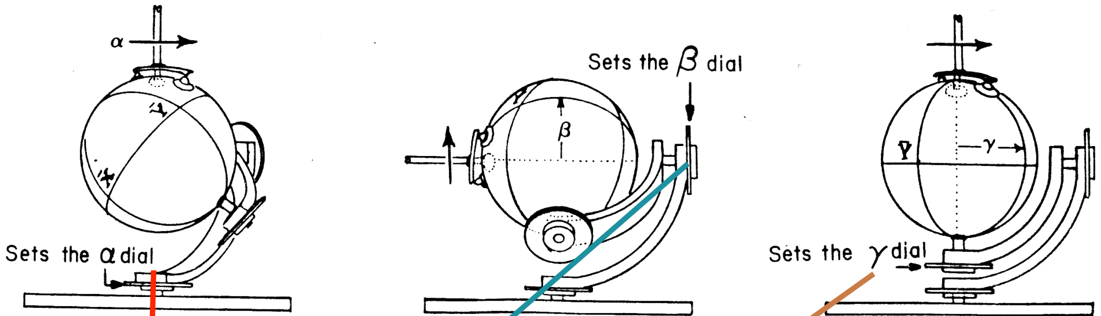
An astronomer's diagram



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Third rotation $\mathbf{R}(\alpha 00)$ Second rotation $\mathbf{R}(0\beta 0)$ First rotation $\mathbf{R}(00\gamma)$



$\mathbf{R}(\alpha$ around $Z)$ $\mathbf{R}(\beta$ around $Y)$ $\mathbf{R}(\gamma$ around $Z)$

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$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}\left[\varphi\vartheta\Theta\right] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

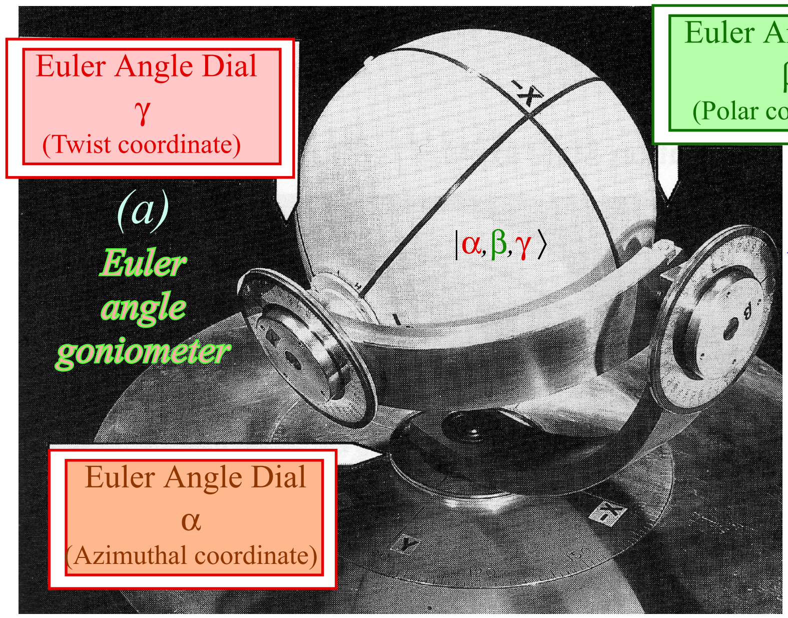
Euler *state definition* lets us relate R(αβγ) = R[φϑΘ] ...

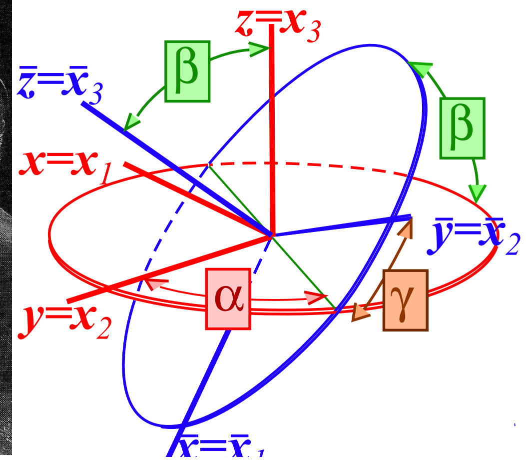
$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

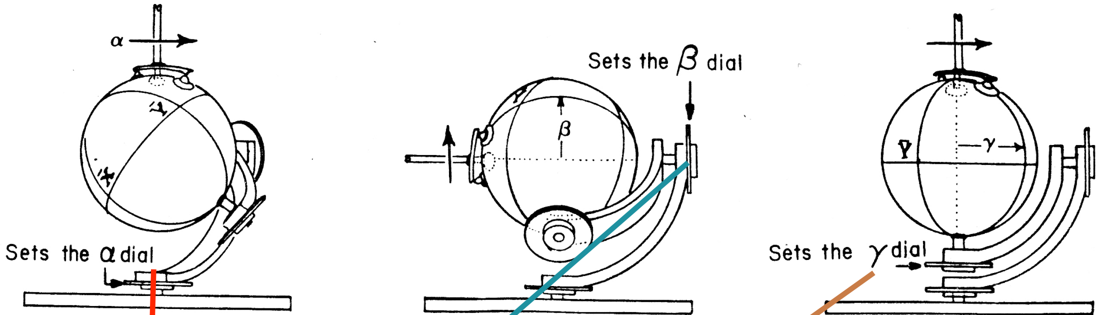
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$





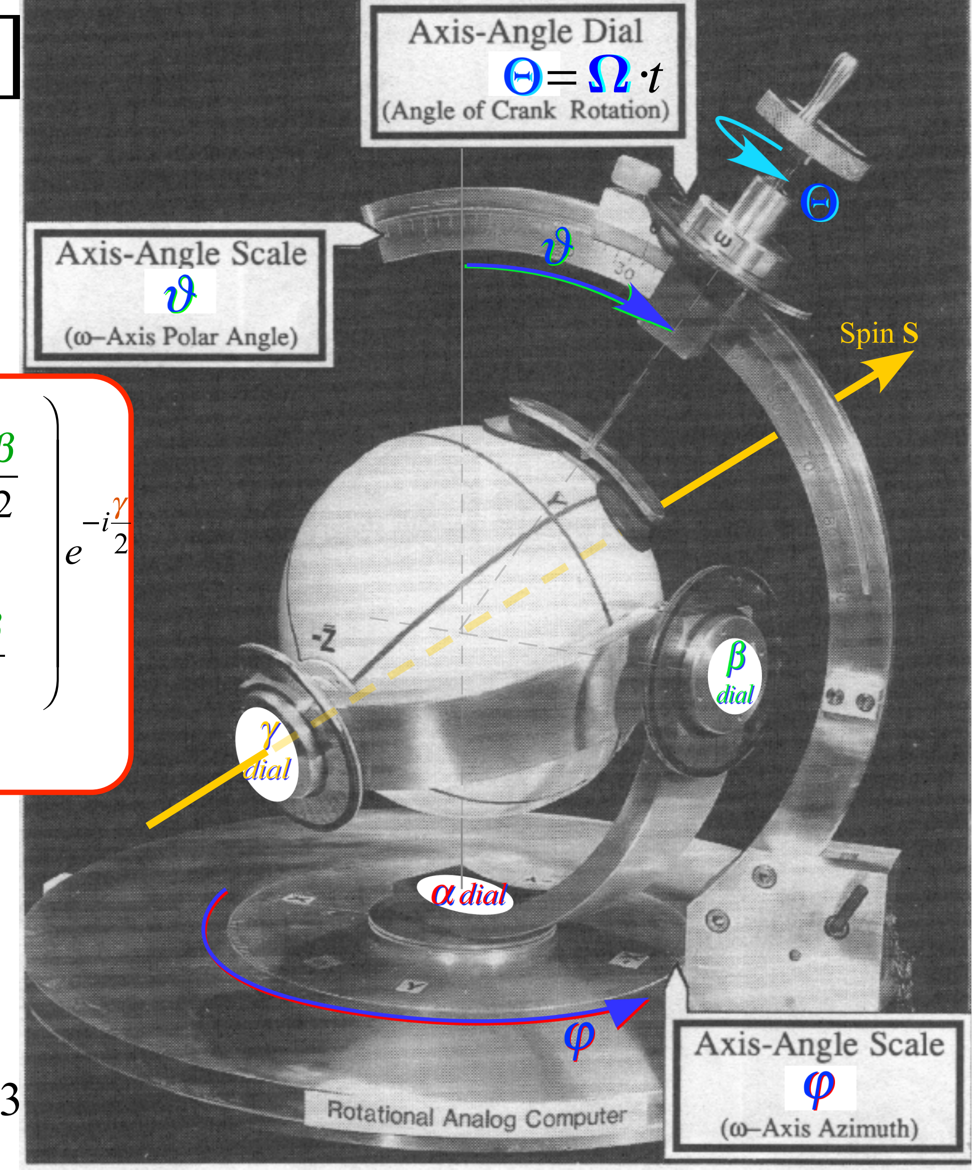
$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

Third rotation $\mathbf{R}(\alpha 00)$ Second rotation $\mathbf{R}(0\beta 0)$ First rotation $\mathbf{R}(00\gamma)$



$\mathbf{R}(\alpha$ around $Z)$ $\mathbf{R}(\beta$ around $Y)$ $\mathbf{R}(\gamma$ around $Z)$

From Lecture 22 page 62 to 70



Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma) = \mathbf{R}[\varphi\vartheta\Theta]$ ...

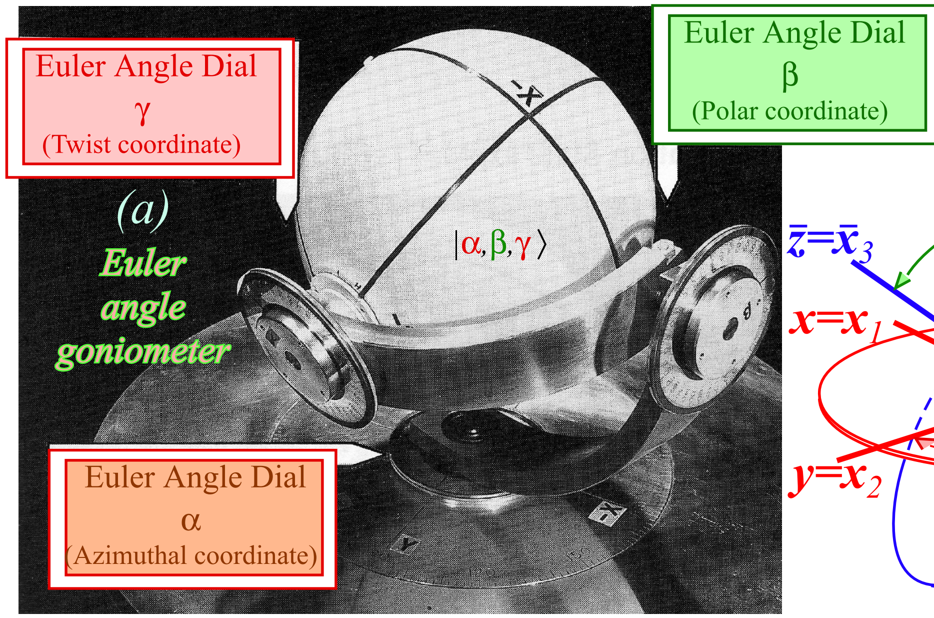
$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

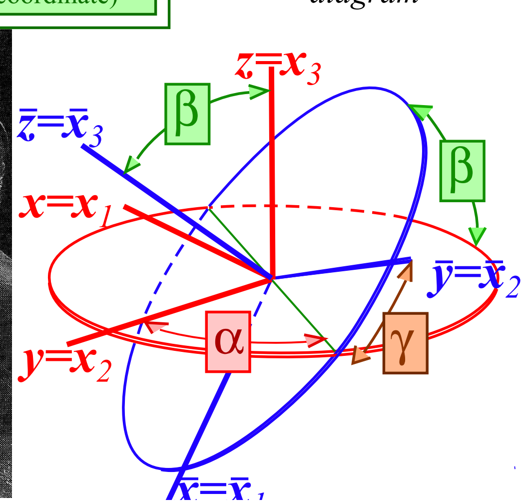
$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

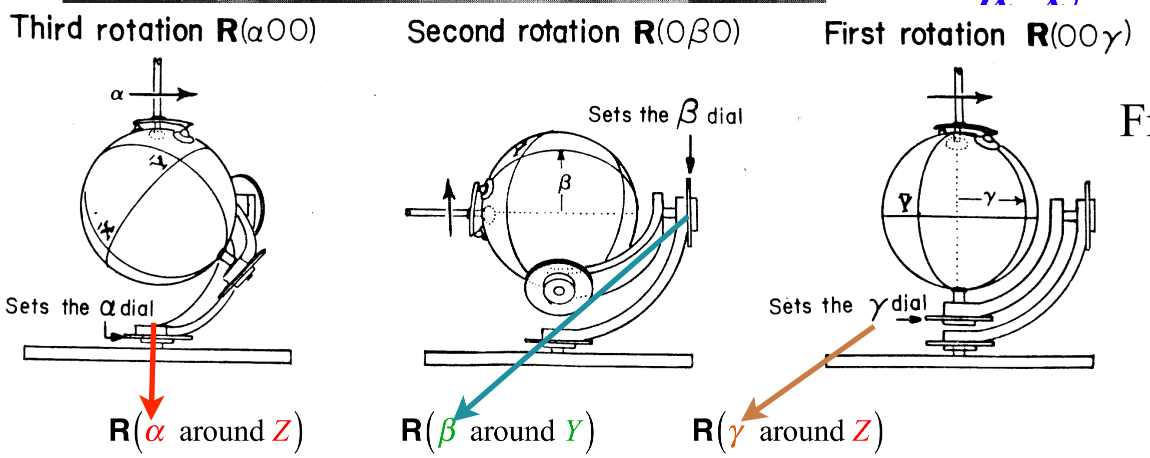
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$



An astronomer's diagram



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



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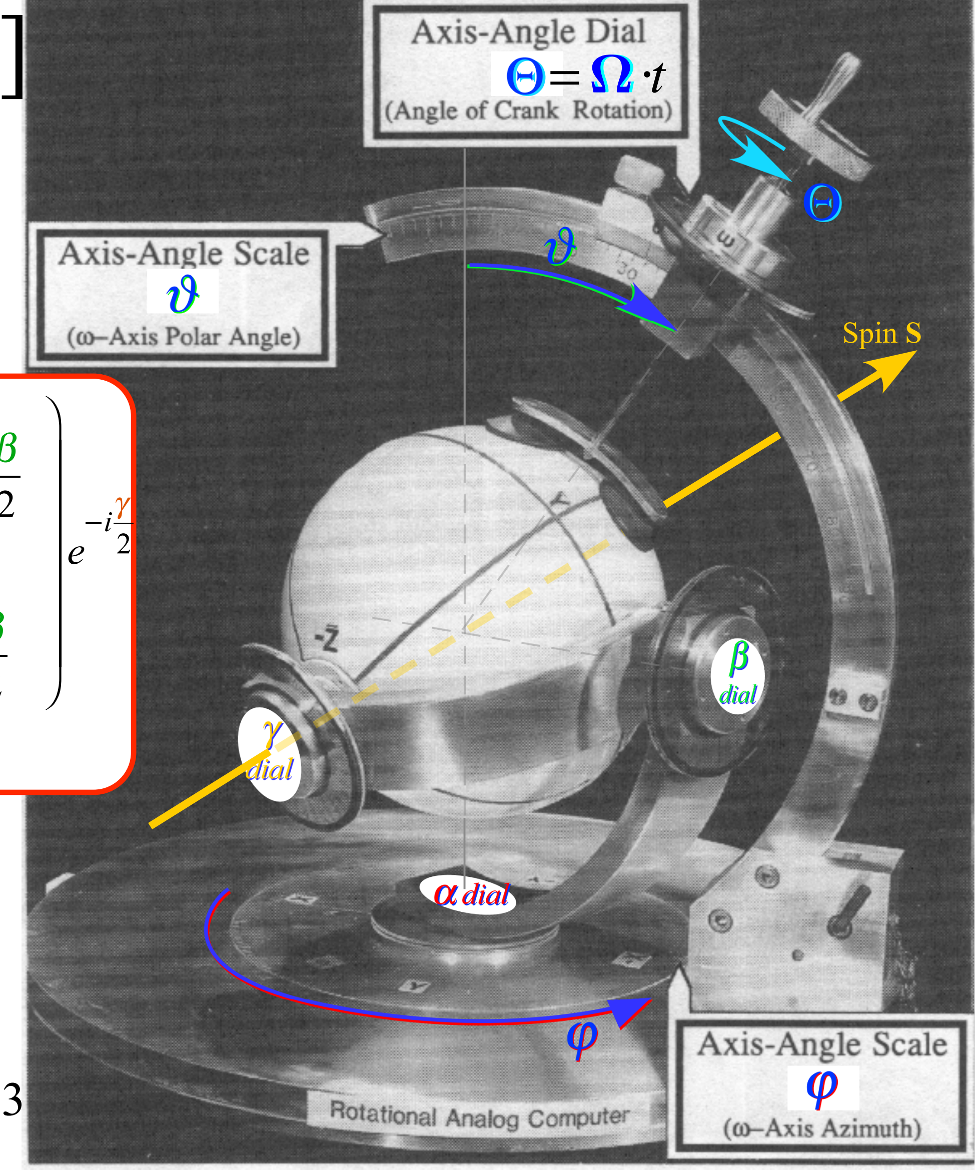
Lecture 22 page 92 to 93



$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix}e^{-i\frac{\alpha}{2}}&0\\0&e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2}&-\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2}&\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}e^{-i\frac{\gamma}{2}}&0\\0&e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix}e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}&-e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}&e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec\Theta]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat\Theta_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat\Theta_X-i\hat\Theta_Y)\\ -i\sin\frac{\Theta}{2}(\hat\Theta_X+i\hat\Theta_Y) & \cos\frac{\Theta}{2}+i\hat\Theta_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat\Theta_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat\Theta_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat\Theta_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta)\\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma) = \mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

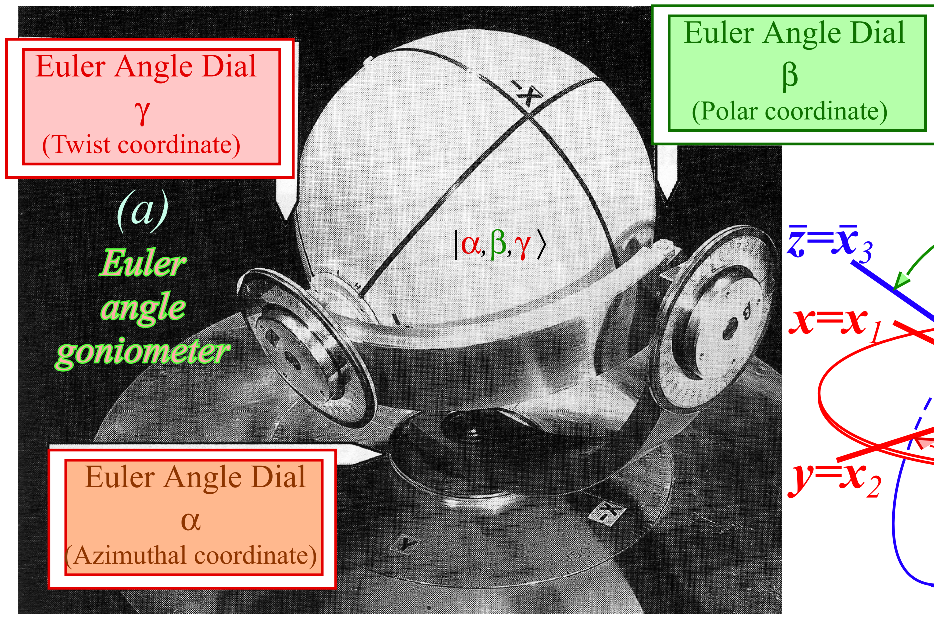
$$\begin{pmatrix}e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}&-e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}&e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

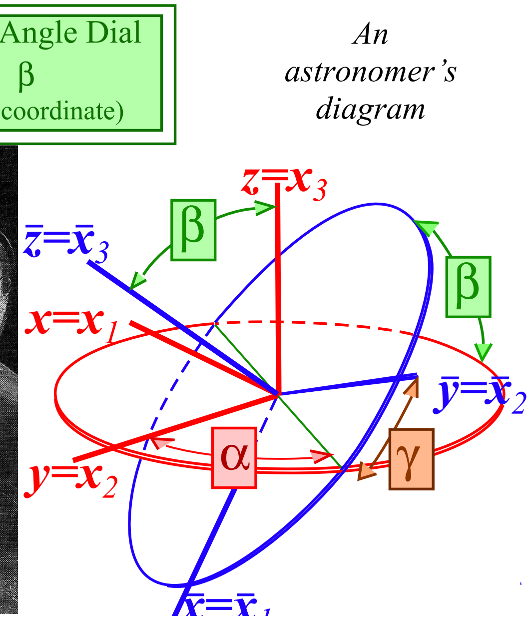
$$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2=\cos\Theta/2$$

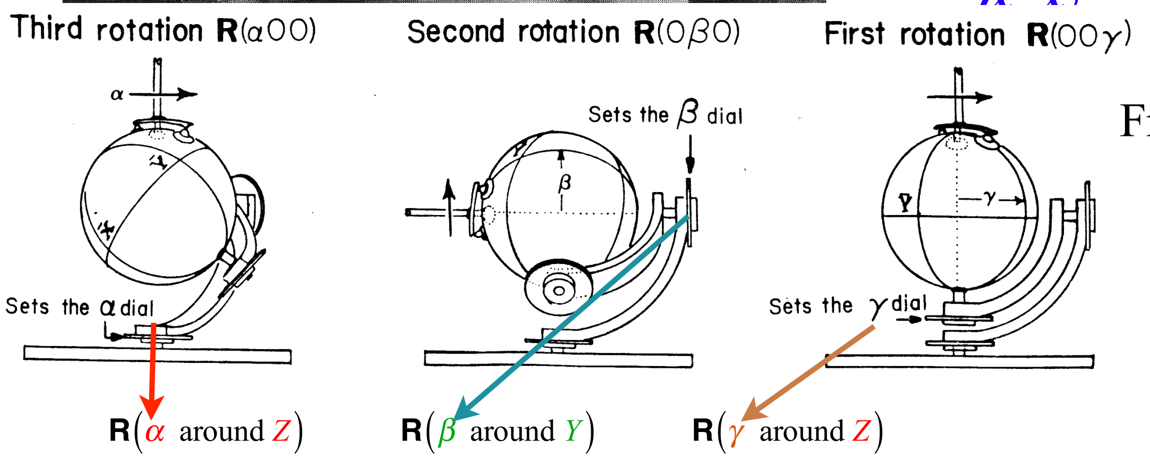
$$-p_2=\sin[(\gamma-\alpha)/2]\sin\beta/2=\hat\Theta_X\sin\Theta/2=\cos\varphi\ \sin\vartheta\ \sin\Theta/2$$

$$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2=\hat\Theta_Y\sin\Theta/2=\sin\varphi\ \sin\vartheta\ \sin\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$







$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

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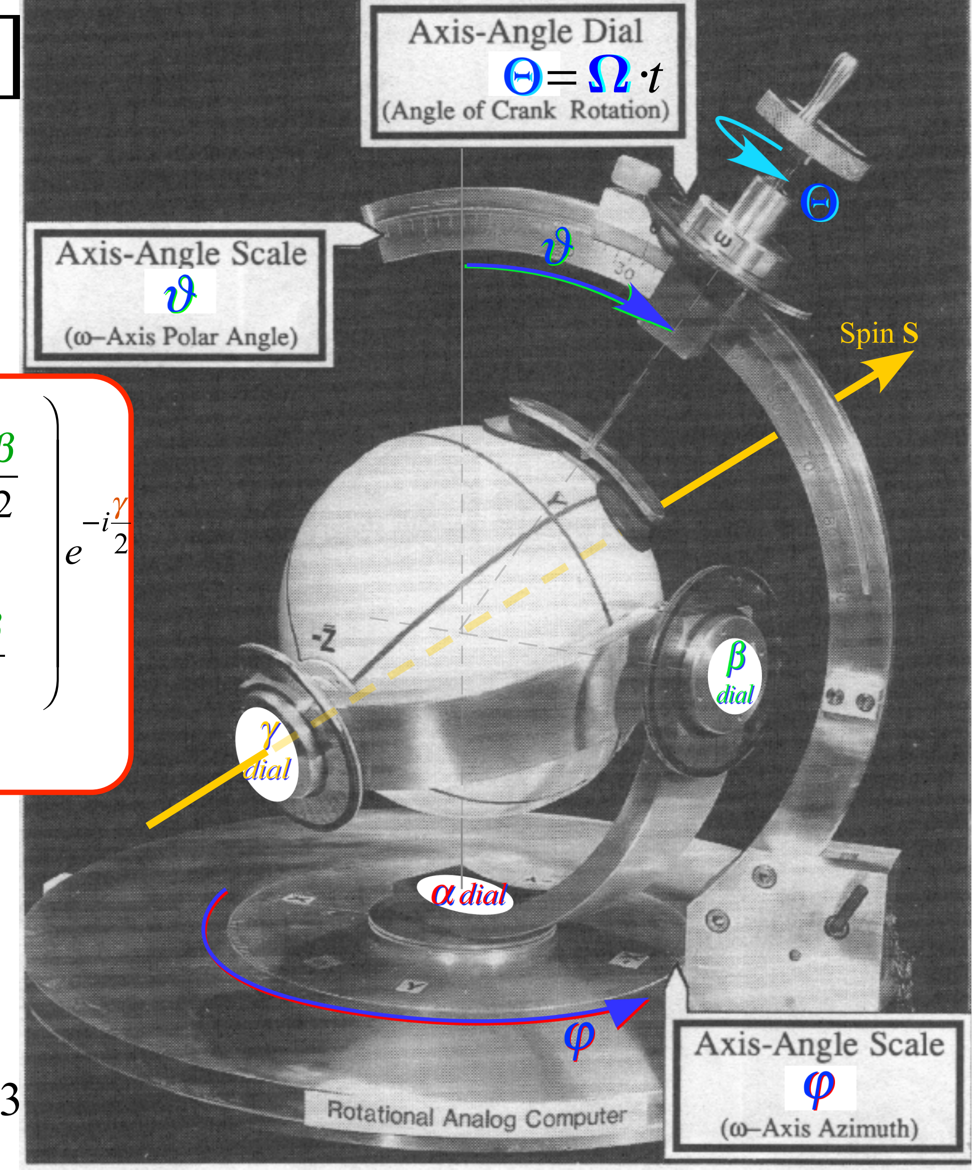
Lecture 22 page 92 to 93



$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \ \sin\Theta/2$$

$U(2)$ *operator vs* (x,p)

$$\begin{pmatrix} x_1+ip_1 & -x_2+ip_2 \\ x_2+ip_2 & x_1-ip_1 \end{pmatrix}$$

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2]\cos\beta/2 = && \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta && \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta && \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta && \sin\Theta/2 \end{aligned}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

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Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:* **R**(αβγ)=**R**[φϑΘ]*)*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

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$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

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$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$

$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$

$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$\varphi = (\alpha-\gamma+\pi)/2$

$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$

$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$

$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

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$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2 \qquad \cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2] \qquad \frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2] \qquad x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

Example: *Euler angles* $(\alpha=50°\ \beta=60°\ \gamma=70°)$

$$\varphi = (50°-70°+180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\ \sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\ \cos(50°+\gamma)/2] = 128.7°$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

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$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* $(\alpha=50°\ \beta=60°\ \gamma=70°)$

$$\varphi = (50° - 70° + 180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$$

Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$

$$\beta = 2\sin^{-1}(\sin 128.7°/2\;\sin 33.7°) = 60.022°$$

$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2)) = 70.007°$$

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

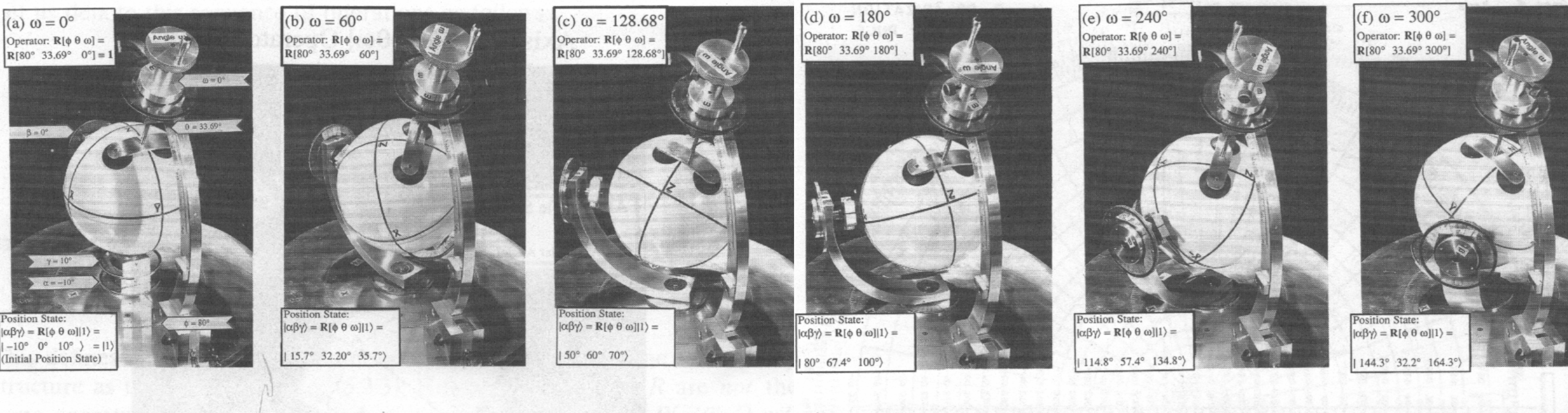
Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

$\Theta=0°$ $\Theta=60°$ $\Theta=128.7°$ $\Theta=180°$ $\Theta=240°$ $\Theta=300°$

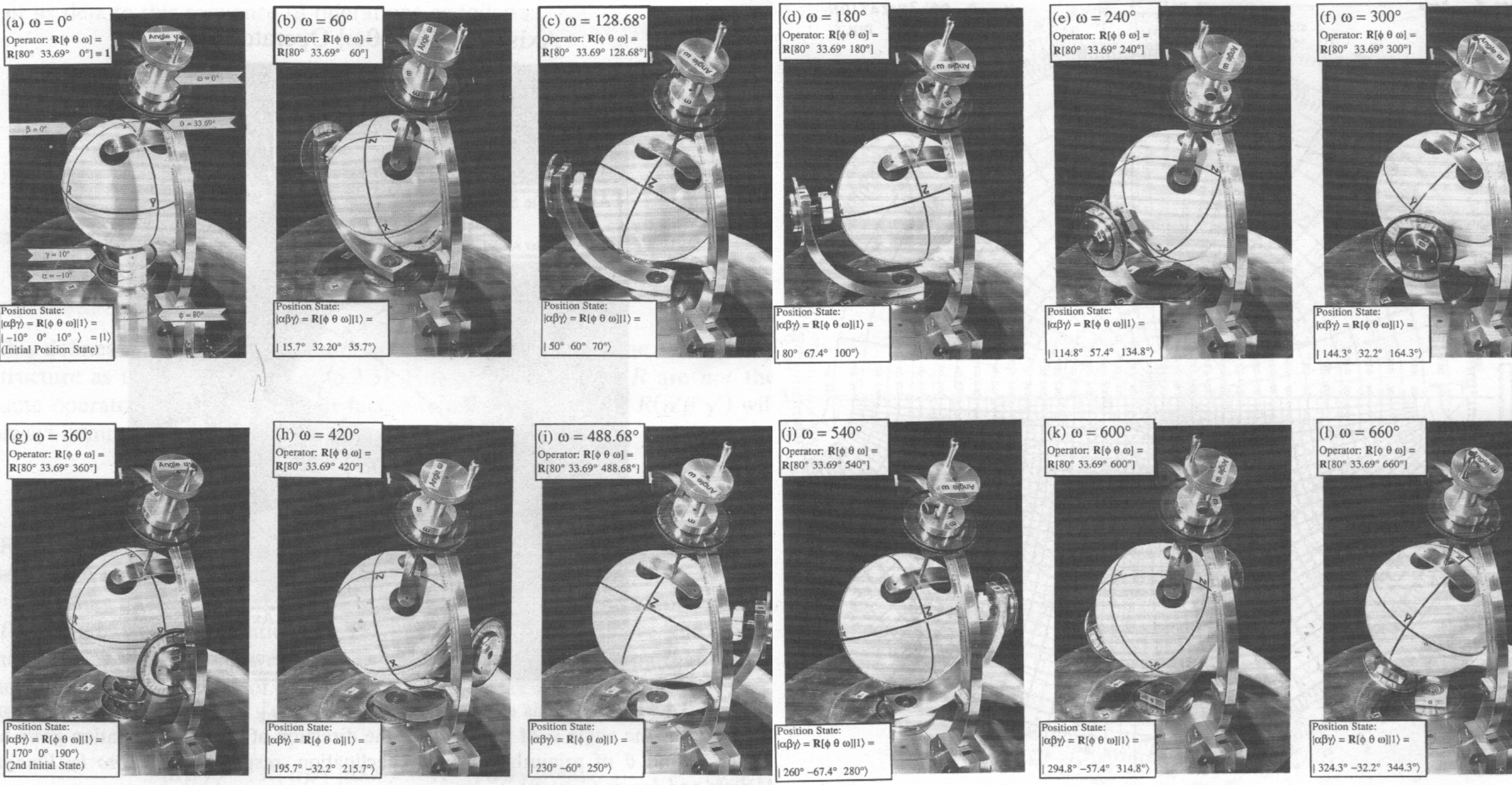


Development has begun on a web based version of this tool,
but much of the App is at present (10/7/2018), in an '*indeterminate state*'.
The App's 3D will in future be handled by *Babylon.JS*, *to act as a* shim
to buttress the WebGL (web graphics layer) that is already in place.

Web based U(2) Calculator - Euler State

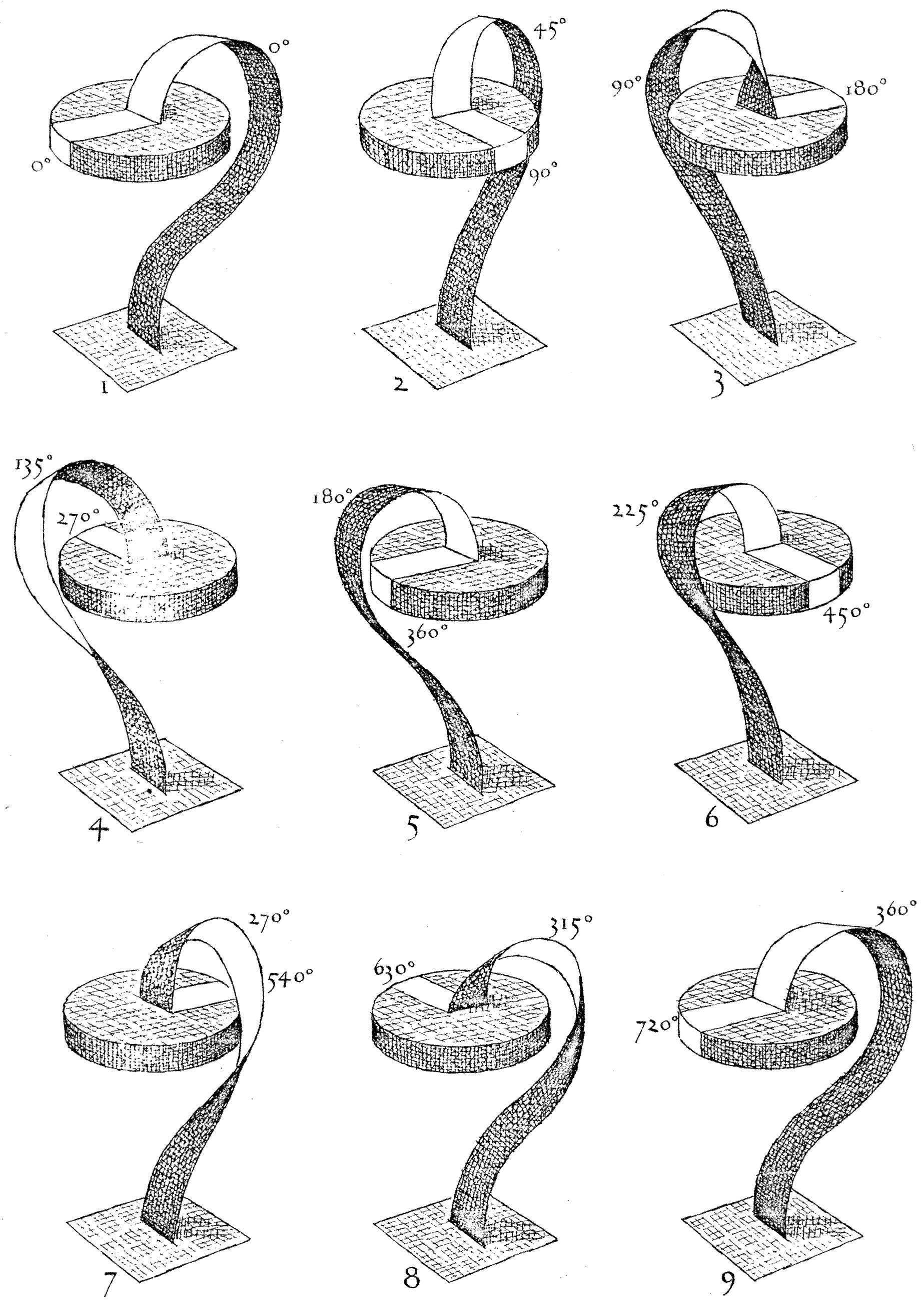
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π-*sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

Some "real-world" applications of
the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

From Scientific American
December 1975-p.120-125

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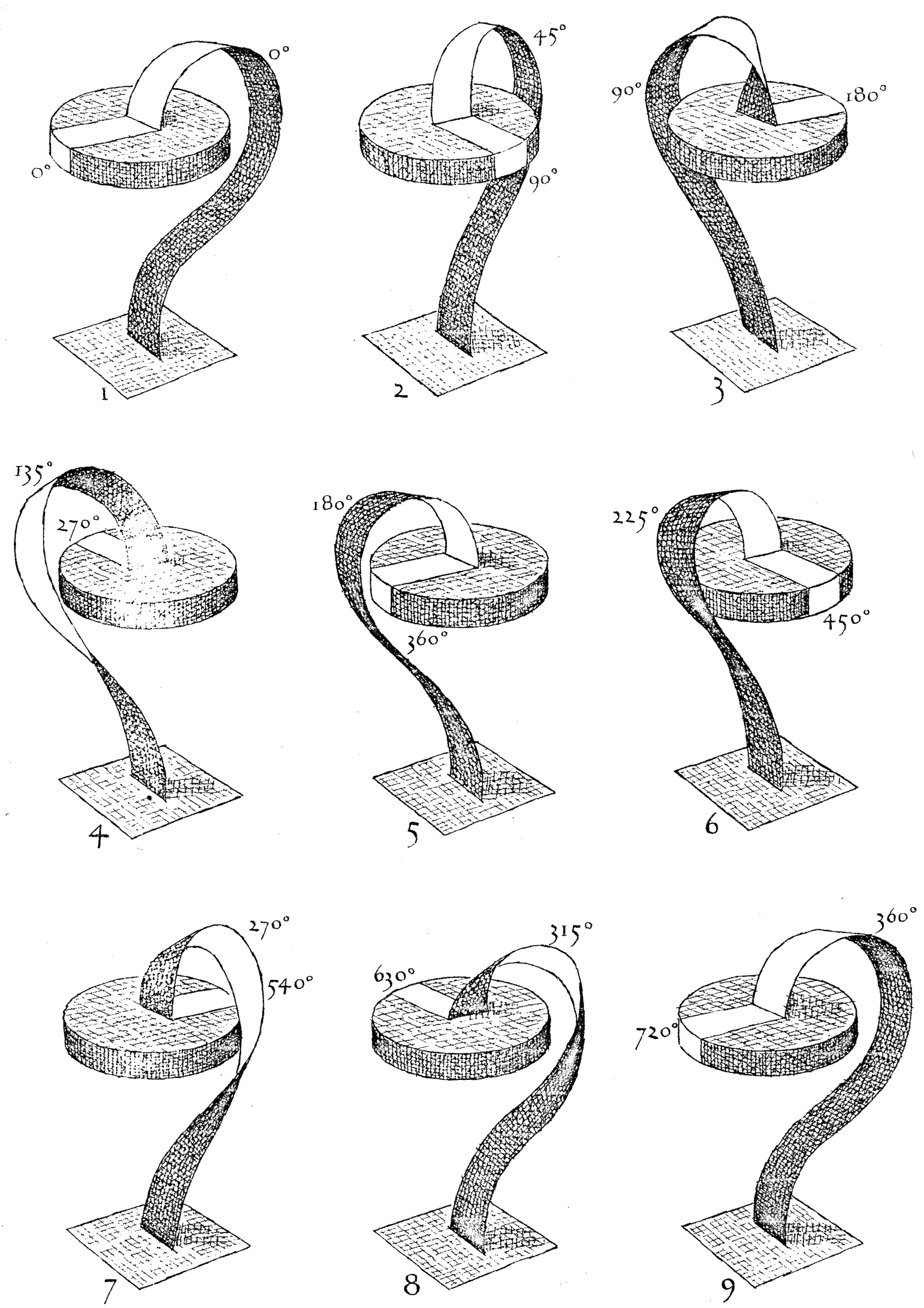
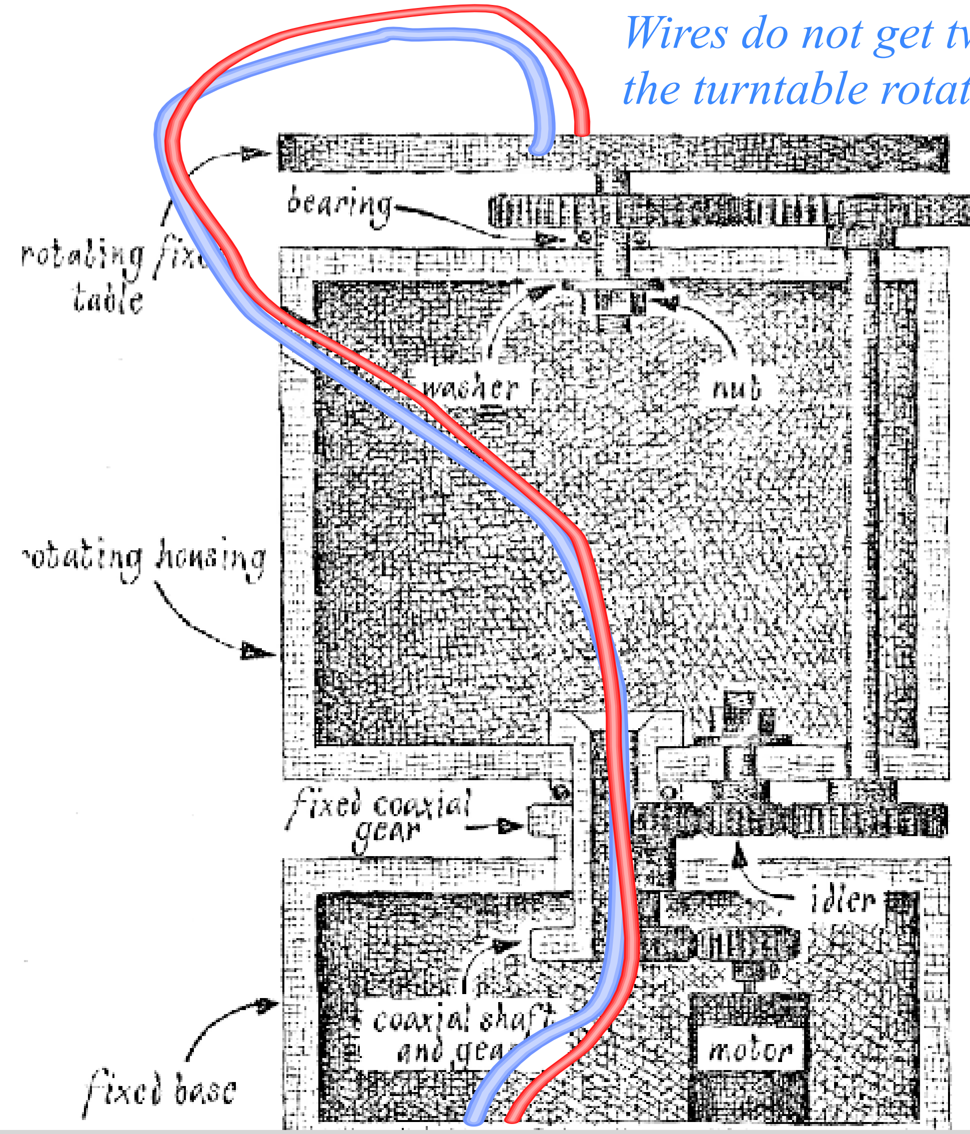


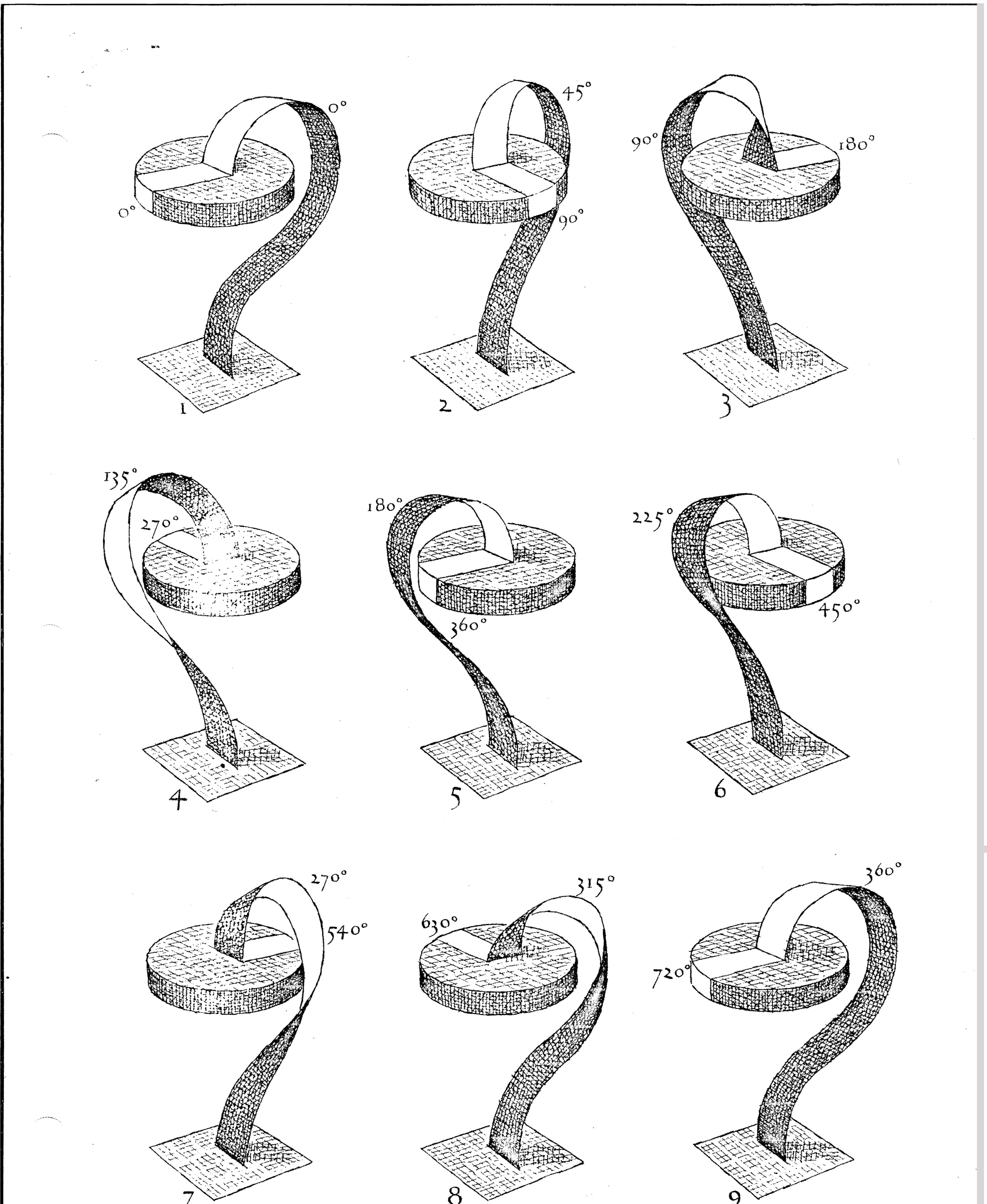
Sequential models of D. A. Adams' antitwister mechanism

From Scientific American December 1975-p.120-125

Wires do not get twisted up as the turntable rotates



Some "real-world" applications of the U(2)-R(3) spinor-vector topology



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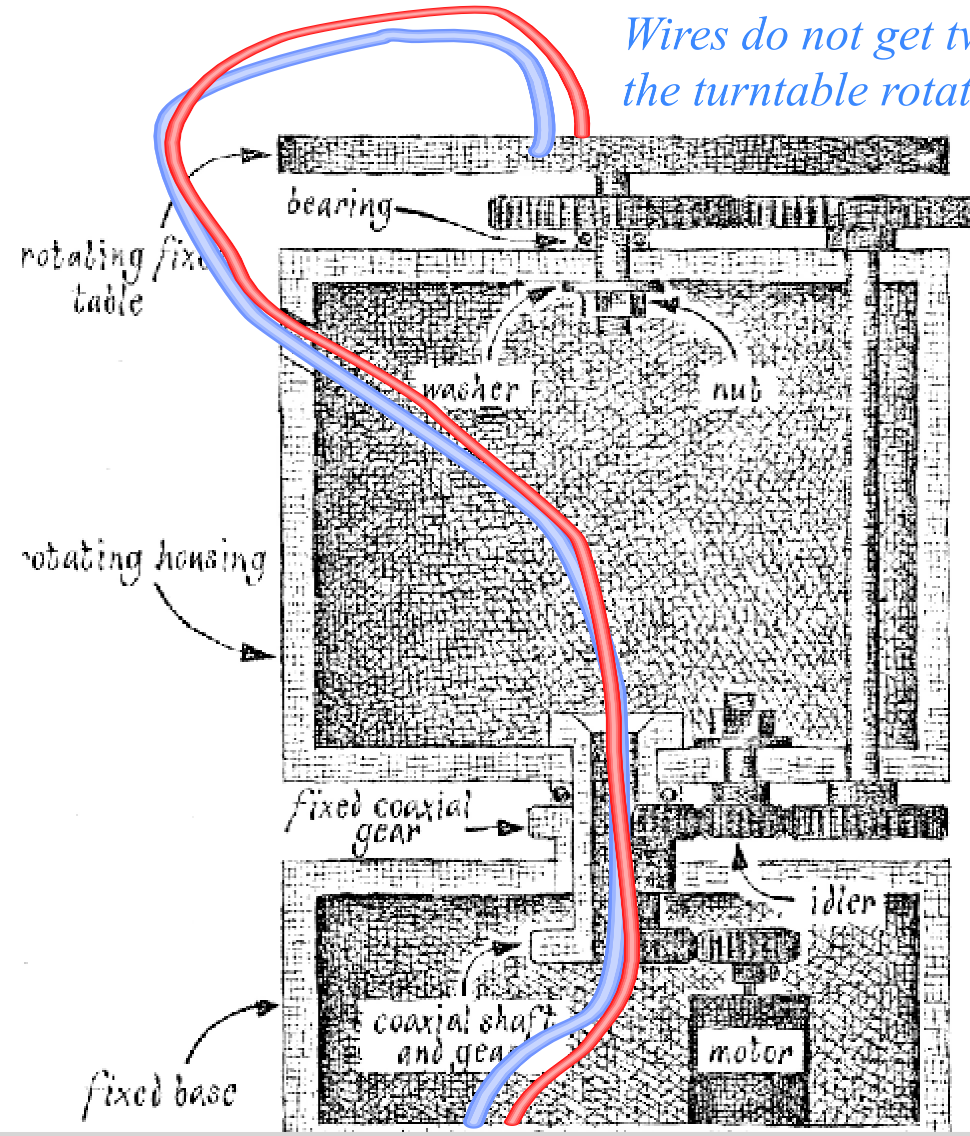

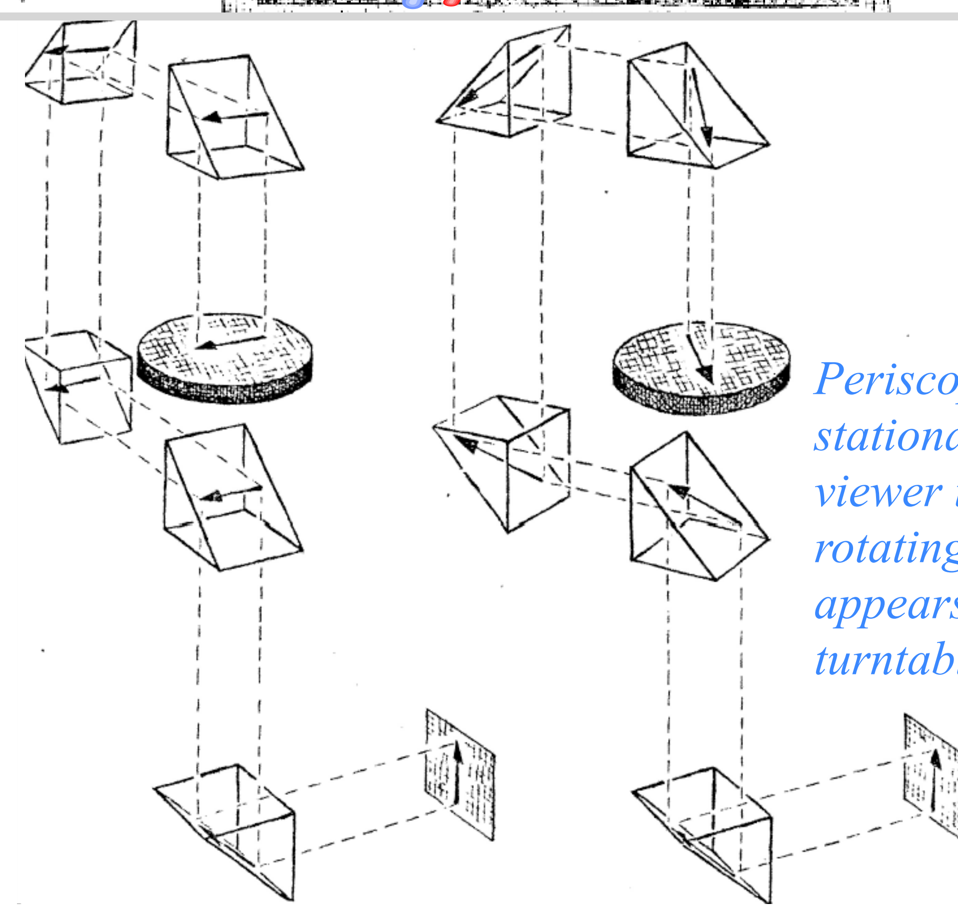

Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

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Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

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- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

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Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

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Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

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$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \boldsymbol{\Omega}_A \quad \mathbf{S}_A \quad + \quad \boldsymbol{\Omega}_B \quad \mathbf{S}_B \quad + \quad \boldsymbol{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\boldsymbol{\Omega}_A{}^2 + \boldsymbol{\Omega}_B{}^2 + \boldsymbol{\Omega}_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \boldsymbol{\Omega}_A \quad \mathbf{S}_A \quad + \quad \boldsymbol{\Omega}_B \quad \mathbf{S}_B \quad + \quad \boldsymbol{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

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Eigenvalues: $\Omega_{\pm} = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

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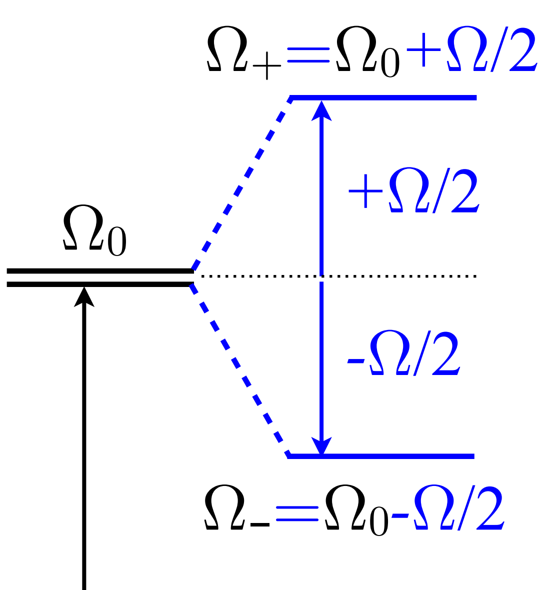

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\mathbf{\Omega}=\mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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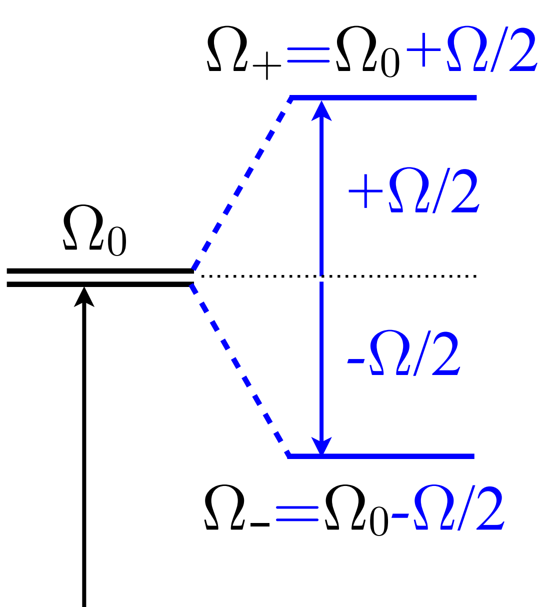
and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2+\Omega_C{}^2}\,]$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

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Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

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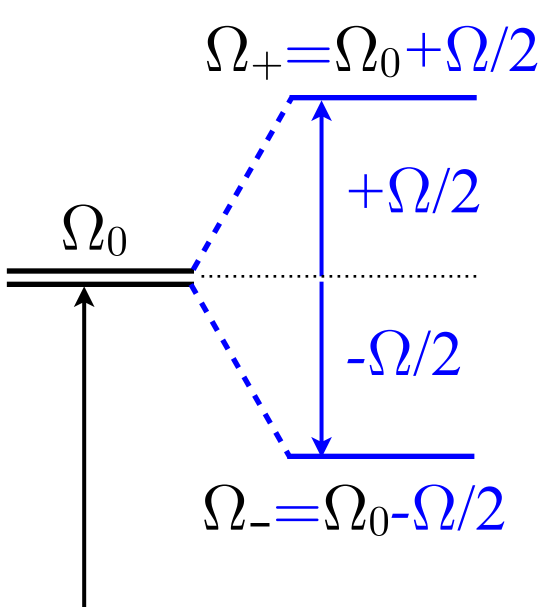
where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\,]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\,]$, $\varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\,]$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \mathbf{\Omega}_A \quad \mathbf{S}_A \quad + \quad \mathbf{\Omega}_B \quad \mathbf{S}_B \quad + \quad \mathbf{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\mathbf{\Omega}} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\mathbf{\Omega}_A{}^2 + \mathbf{\Omega}_B{}^2 + \mathbf{\Omega}_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

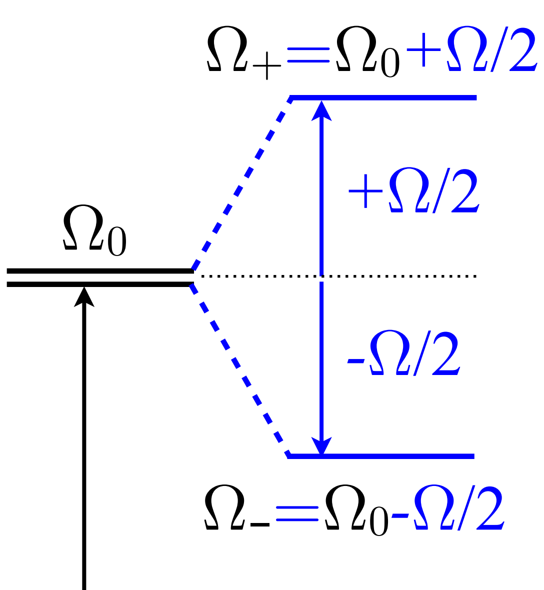
$$\boxed{\textit{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 = \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\mathbf{\Omega}_B{}^2 + \mathbf{\Omega}_C{}^2}\ ]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)\,/\,\sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\qquad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

Step 3.To find eigenvectors replace Euler angles (azimuth α, *polar* β *) of Euler-state*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega \cos\vartheta,\quad \Omega_B = \Omega\cos\varphi \sin\vartheta,\quad \Omega_C = \Omega \sin\varphi \sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A^{\ 2} + \Omega_B^{\ 2} + \Omega_C^{\ 2}} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega \sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B^{\ 2} + \Omega_C^{\ 2}}\ ]$

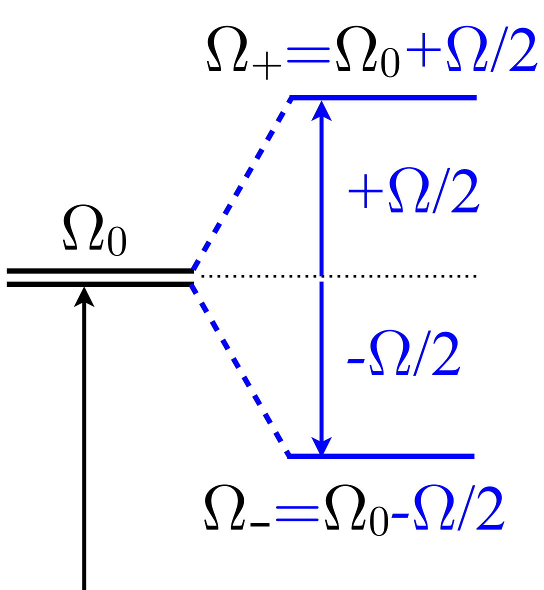
or: $\vartheta = \cos^{-1}[(A\text{-}D) / \sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\quad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

$$\boxed{\textit{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 = \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}}$$

Step 3.To find eigenvectors replace Euler angles (azimuth α, *polar* β *) of Euler-state*

with the Darboux axis polar angles (azimuth φ , *polar* ϑ *) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \; \mathbf{1} + \Omega_A \; \mathbf{S}_A + \Omega_B \; \mathbf{S}_B + \Omega_C \; \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\,]$

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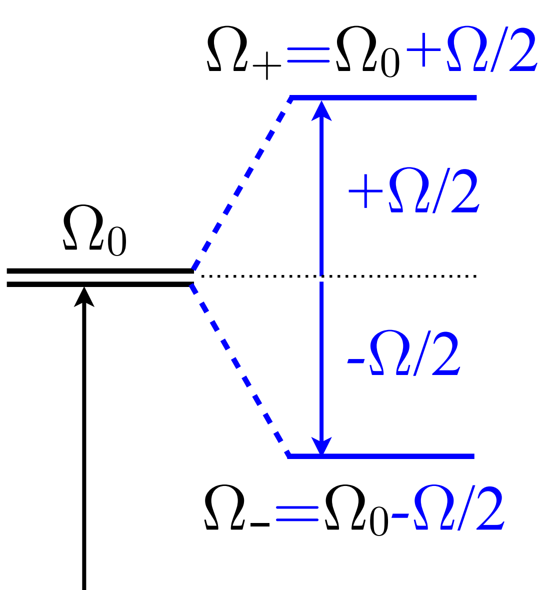
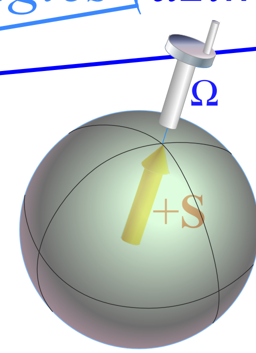
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$



Spin +S
Up-Crank

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\boldsymbol{\Omega}=\boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2} = \sqrt{(A-D)^2+4B^2+4C^2}$

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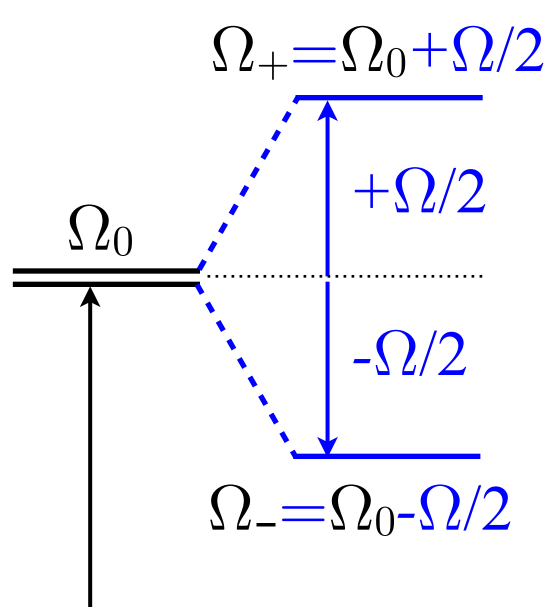
Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$
$$= \frac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}$$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

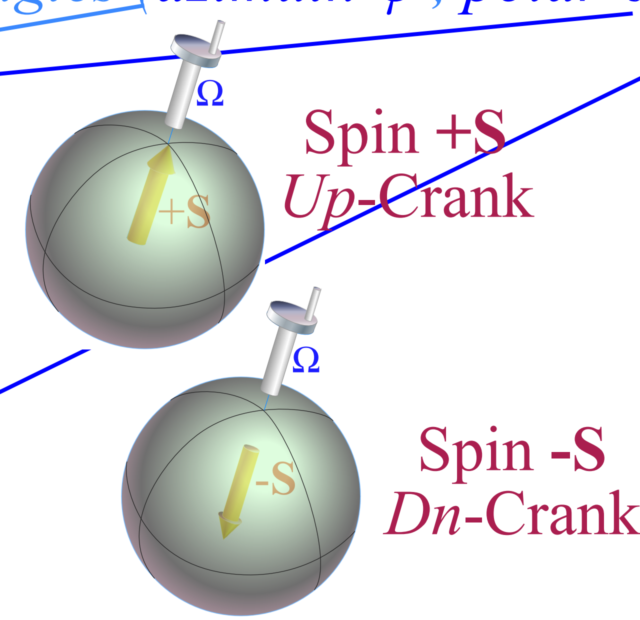
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta\pm\pi$*) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$\left|\Omega_-\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\boldsymbol{\Omega}=\boldsymbol{\Theta}/t$

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$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2} = \sqrt{(A-D)^2+4B^2+4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2+\Omega_C{}^2}\,]$

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$$\boxed{\text{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 = \frac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}}$$

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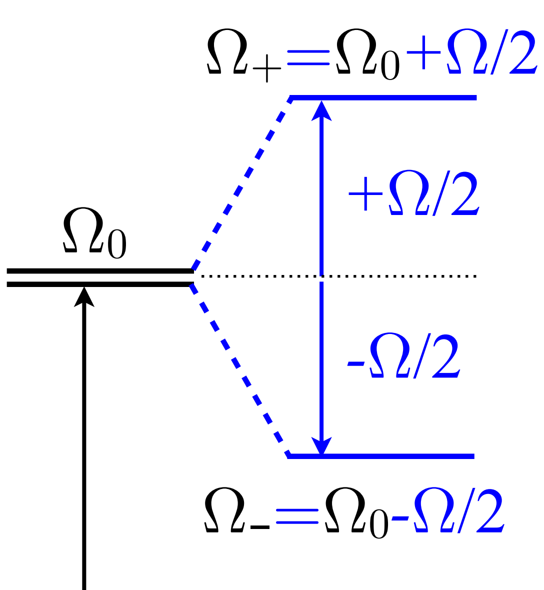
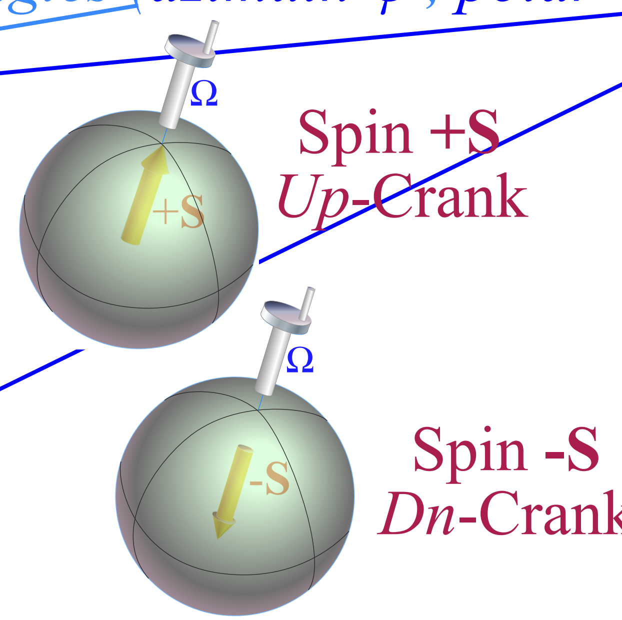
$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)|\uparrow_{000}\rangle$$

with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta\pm\pi$*) of* **H***-matrix*



$$|\Omega_+\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$|\Omega_-\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



More reliable computation:

$\varphi = atan2(C, B)$

[$\tan^{-1}(C\,/\,B)$ *is unreliable*]

$\vartheta = atan2(2\sqrt{B^2+C^2}, A-D)$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

Quick U(2) way example for 2-by-2 **H**

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$$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$$

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Step 2.Convert Cartesian to polar form: ($\Omega_A = \Omega\cos\vartheta$, $\Omega_B = \Omega\cos\varphi\,\sin\vartheta$, $\Omega_C = \Omega\sin\varphi\,\sin\vartheta$)

$$\Omega_0 = \frac{A+D}{2} = 10$$

and: $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2} = \sqrt{(4)^2 + 4\sqrt{6}^2 + 4\sqrt{6}^2} = \sqrt{16+24+24} = \sqrt{64} = 8$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

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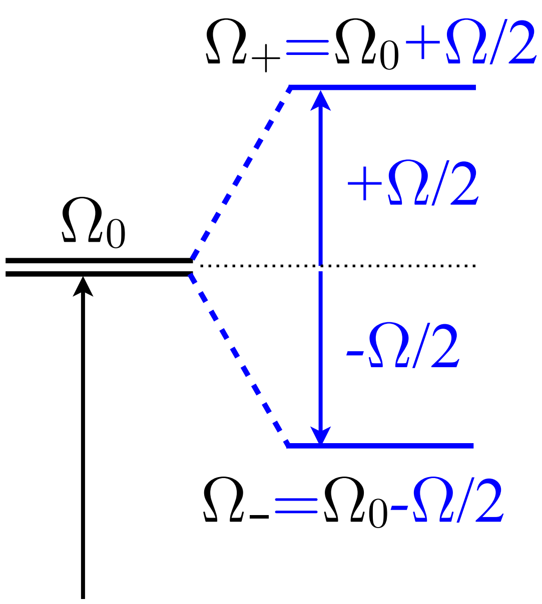


$\underline{|eigenvalue-1}$

$$\omega_\uparrow = 10 + \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 + 4 = 14$$

$\underline{|eigenvalue-2}$

$$\omega_\downarrow = 10 - \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 - 4 = 6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

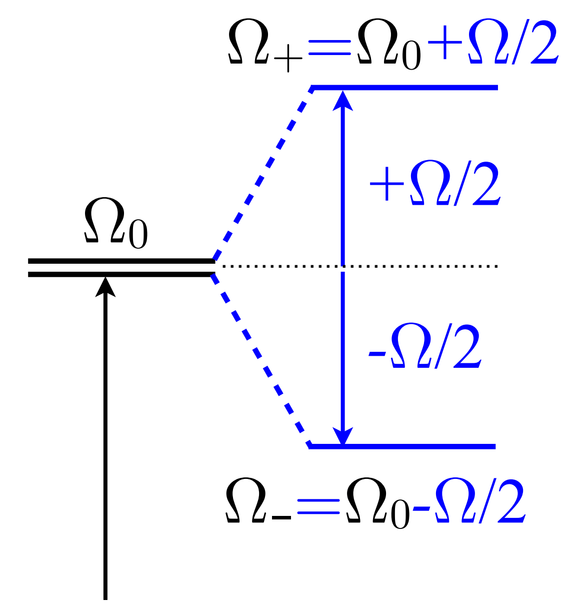
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta$, $\Omega_B=\Omega\cos\varphi\sin\vartheta$, $\Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A-D)/\sqrt{(A-D)^2+4B^2+4C^2}]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



$\boxed{eigenvalue-1}$

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\boxed{eigenvalue-2}$

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta$)

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and: $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

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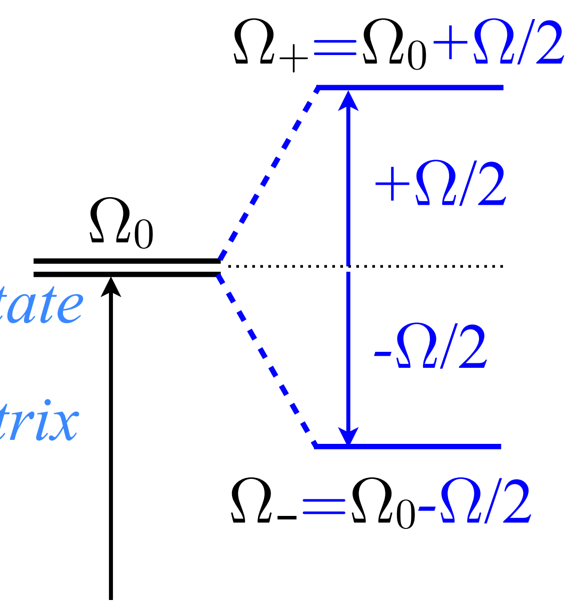
$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\ ]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



Step 3.To find eigenvectors replace Euler angles (azimuth α, *polar* β *) of Euler-state*

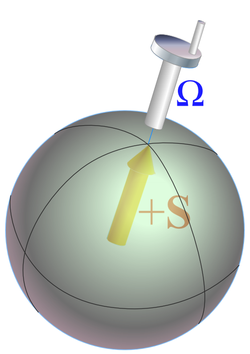
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta\pm\pi$*) of* **H***-matrix*

$\underline{|eigenvalue-1}$

$$\omega_{\uparrow}=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}=10+4=14$$

$\underline{|eigenvector-1}$

$$|\uparrow\rangle=\begin{pmatrix} e^{-i\frac{\pi}{8}}\cos\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\sin\frac{\pi}{6}\end{pmatrix}=\begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3}\end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

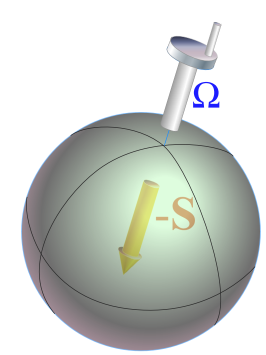


$\underline{|eigenvalue-2}$

$$\omega_{\downarrow}=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}=10-4=6$$

$\underline{|eigenvector-2}$

$$|\downarrow\rangle=\begin{pmatrix} -e^{-i\frac{\pi}{8}}\sin\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\cos\frac{\pi}{6}\end{pmatrix}=\begin{pmatrix} -e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \\ 1\end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$



Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S**-*vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$-*vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_{\mathrm{B}} + C\,\boldsymbol{\sigma}_{\mathrm{C}} + \frac{A-D}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_{\mathrm{B}} + \frac{\Omega_{\mathrm{C}}}{2}\boldsymbol{\sigma}_{\mathrm{C}} + \frac{\Omega_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_{\mathrm{A}} \\ \Omega_B \\ \Omega_{\mathrm{C}} \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boxed{\begin{aligned}\rho &= \tfrac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

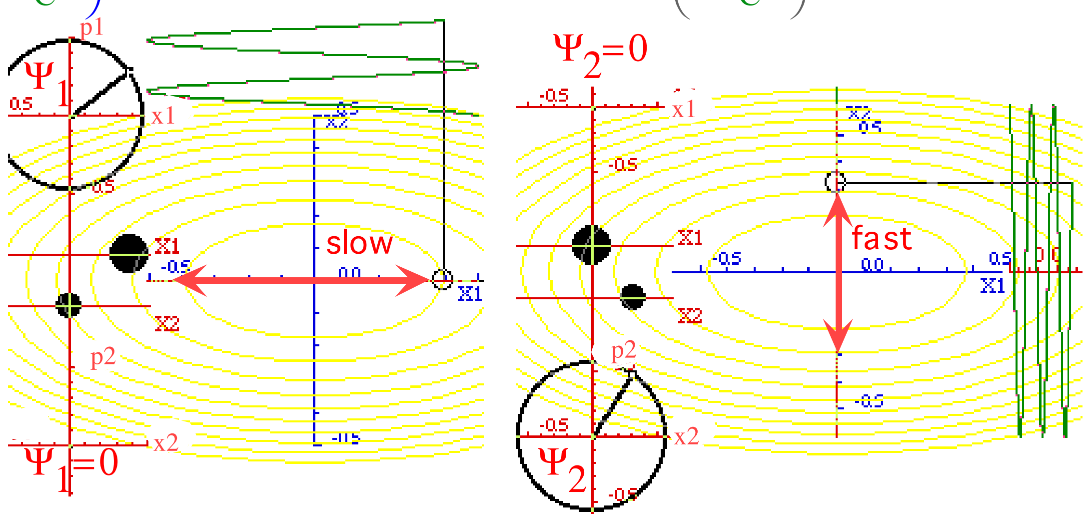
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_A}{2}\sigma_A$$

$$\textit{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix}\qquad \textit{Eigen-Spin}:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)*
$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

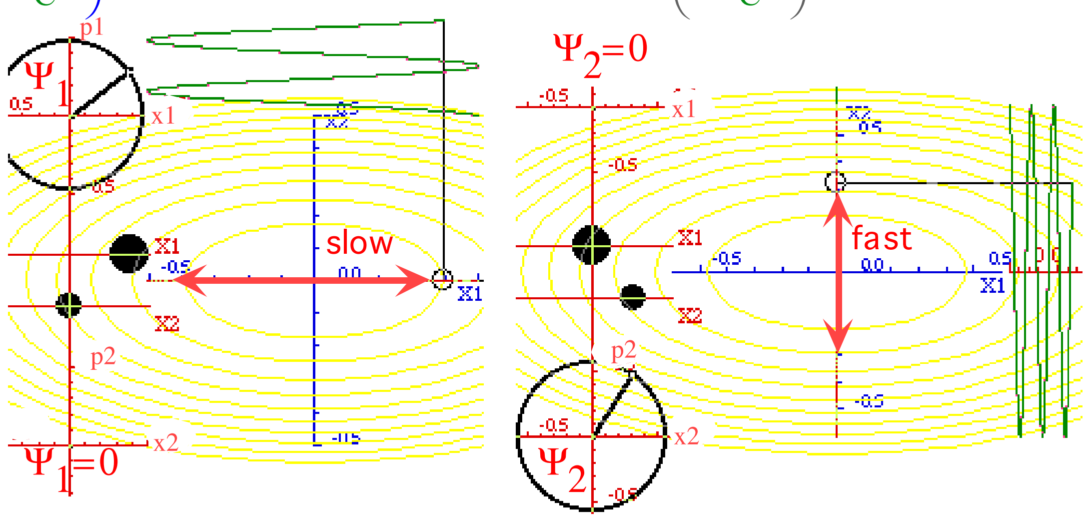
$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\,\boldsymbol{\sigma}_A$$
$$\mathbf{H} = \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\,\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\,\boldsymbol{\sigma}_A$$

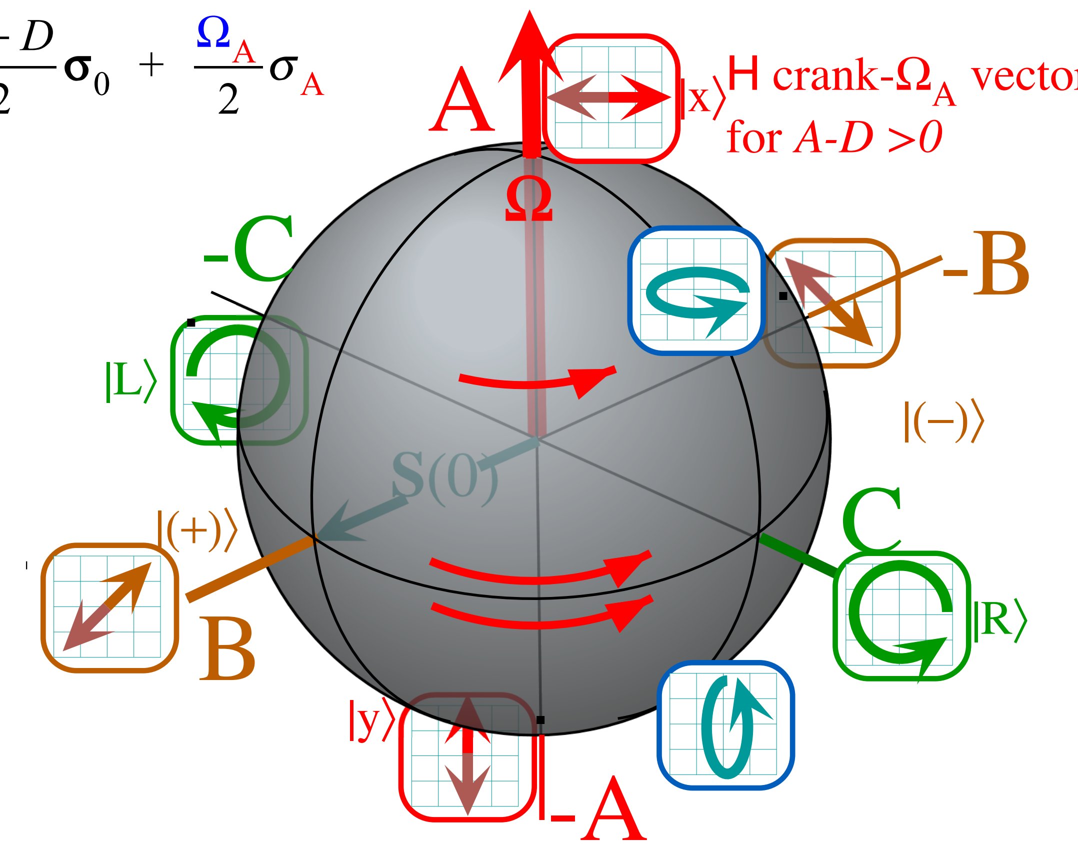
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$





The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow \boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

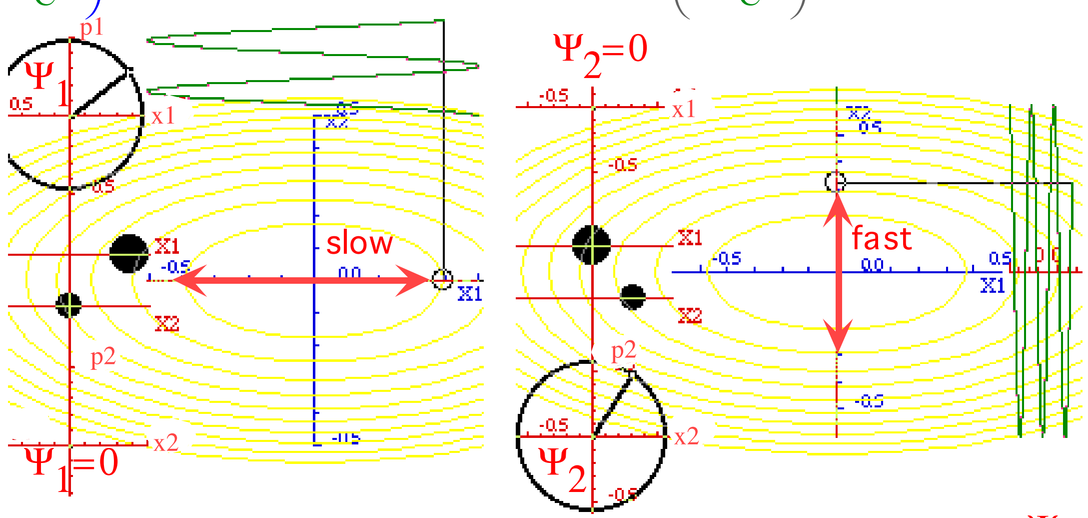
$$\mathbf{H} = \frac{A+D}{2}\,\sigma_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

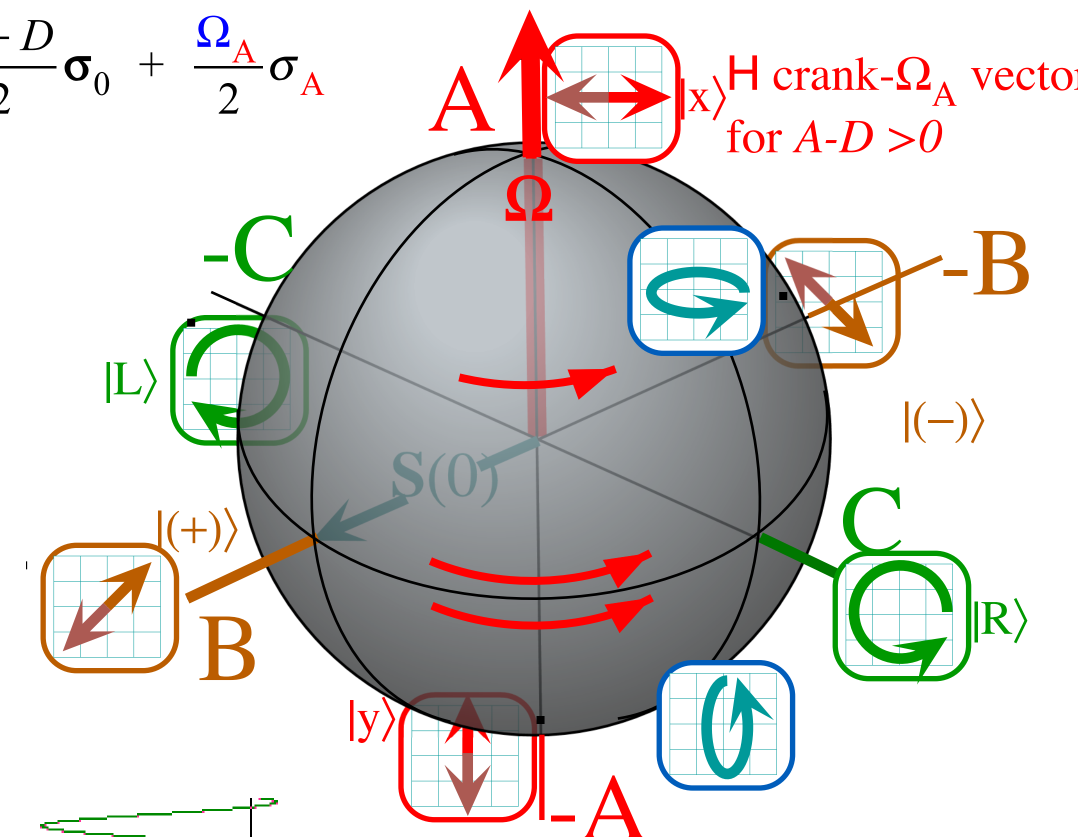
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

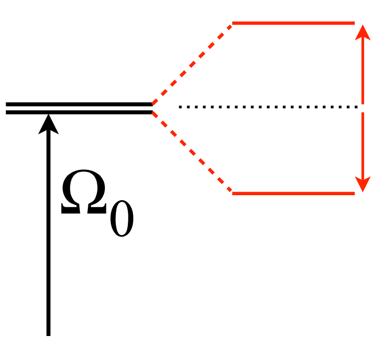
$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_A}{2}\sigma_A$$

$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$

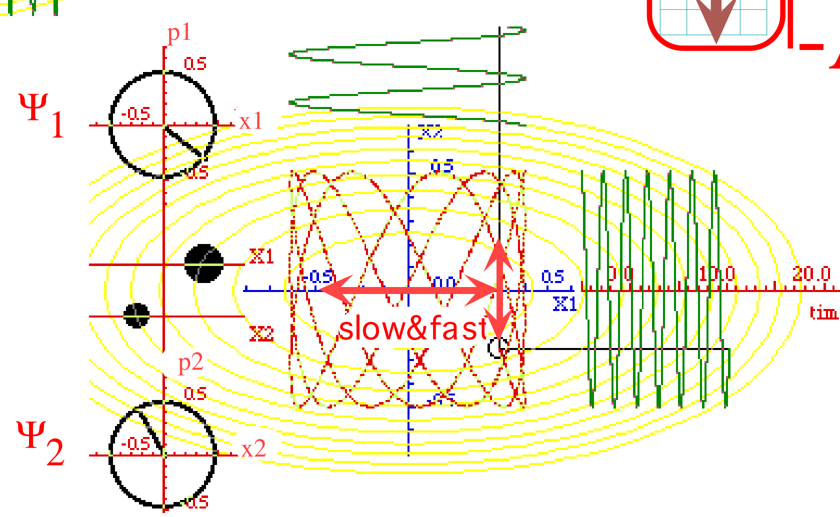




Beat dynamics:

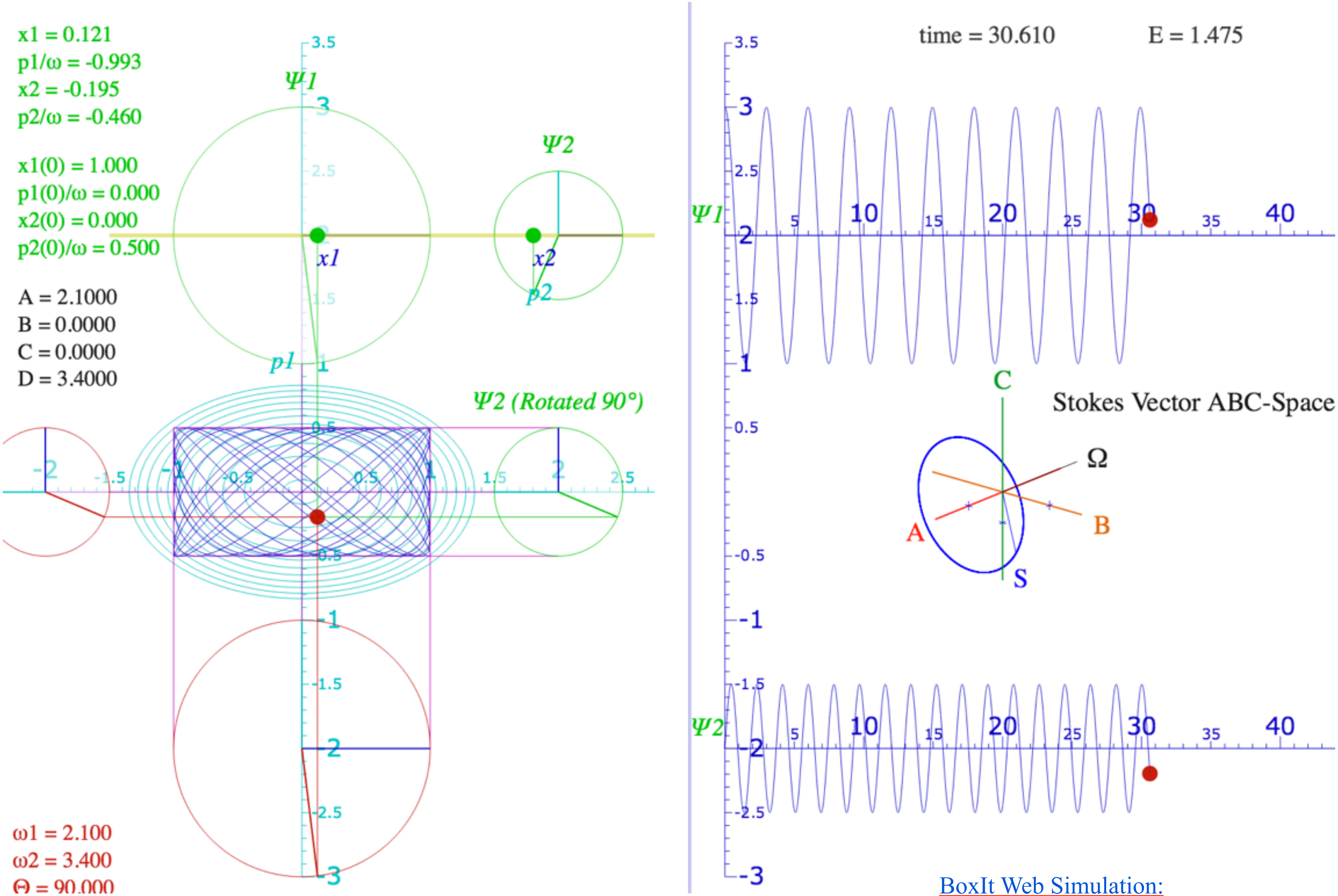


$\frac{\Omega_A}{2}$ $\frac{\Omega_A}{2}$ *Total beat frequency* Ω_A



BoxIt Web Simulation
Pure A-Type

A-Type elliptical polarized motion



[BoxIt Web Simulation:
A-Type with A=2.1, D=3.4]

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_{\mathrm{B}} + C\,\boldsymbol{\sigma}_{\mathrm{C}} + \frac{A-D}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\boldsymbol{\Omega}_B}{2}\boldsymbol{\sigma}_{\mathrm{B}} + \frac{\boldsymbol{\Omega}_{\mathrm{C}}}{2}\boldsymbol{\sigma}_{\mathrm{C}} + \frac{\boldsymbol{\Omega}_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\vec{\boldsymbol{\Omega}} = \begin{pmatrix} \boldsymbol{\Omega}_{\mathrm{A}} \\ \boldsymbol{\Omega}_B \\ \boldsymbol{\Omega}_{\mathrm{C}} \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\boldsymbol{\Omega}_B}{2}\boldsymbol{\sigma}_{\mathrm{B}}$$

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$ $\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$

$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

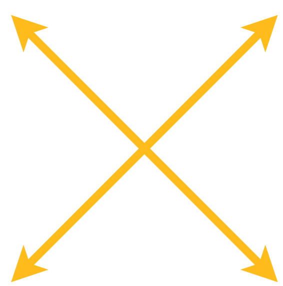
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

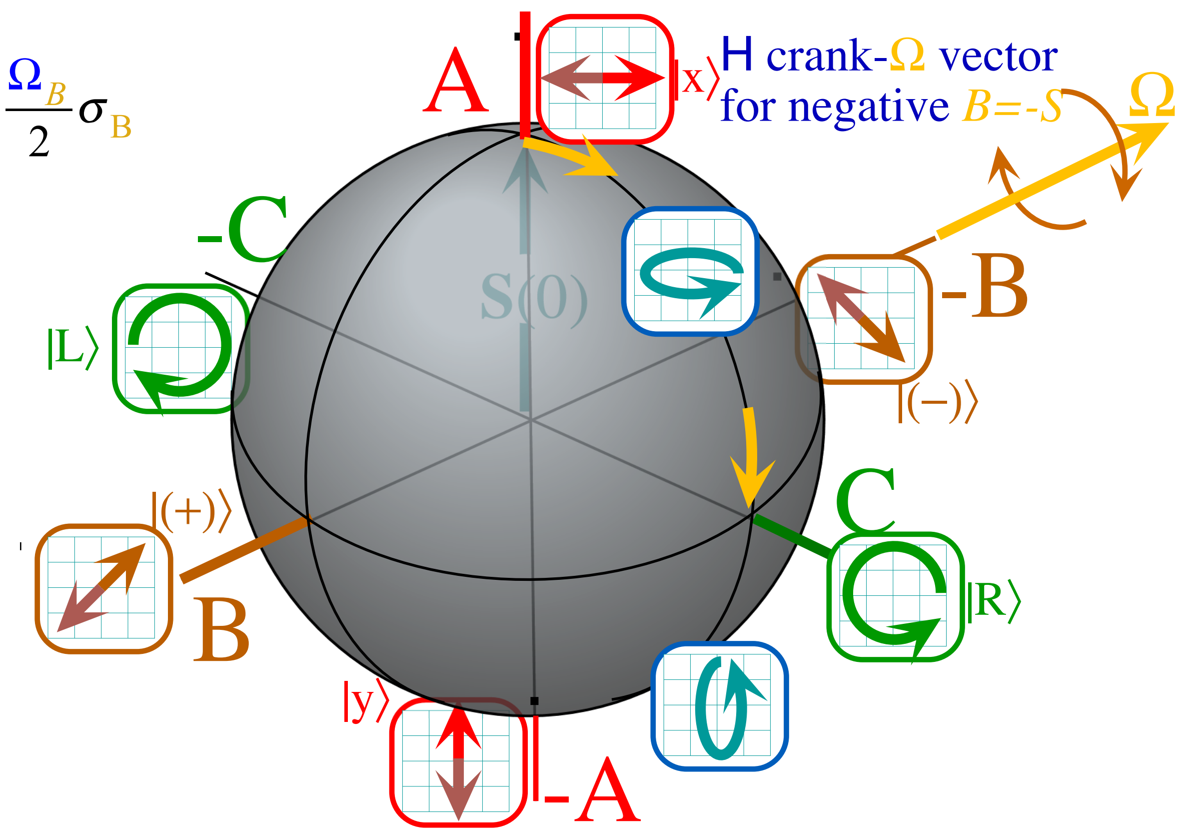
$$\vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B$$

$$\text{Crank}: \vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad \text{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$





The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + B\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + C\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

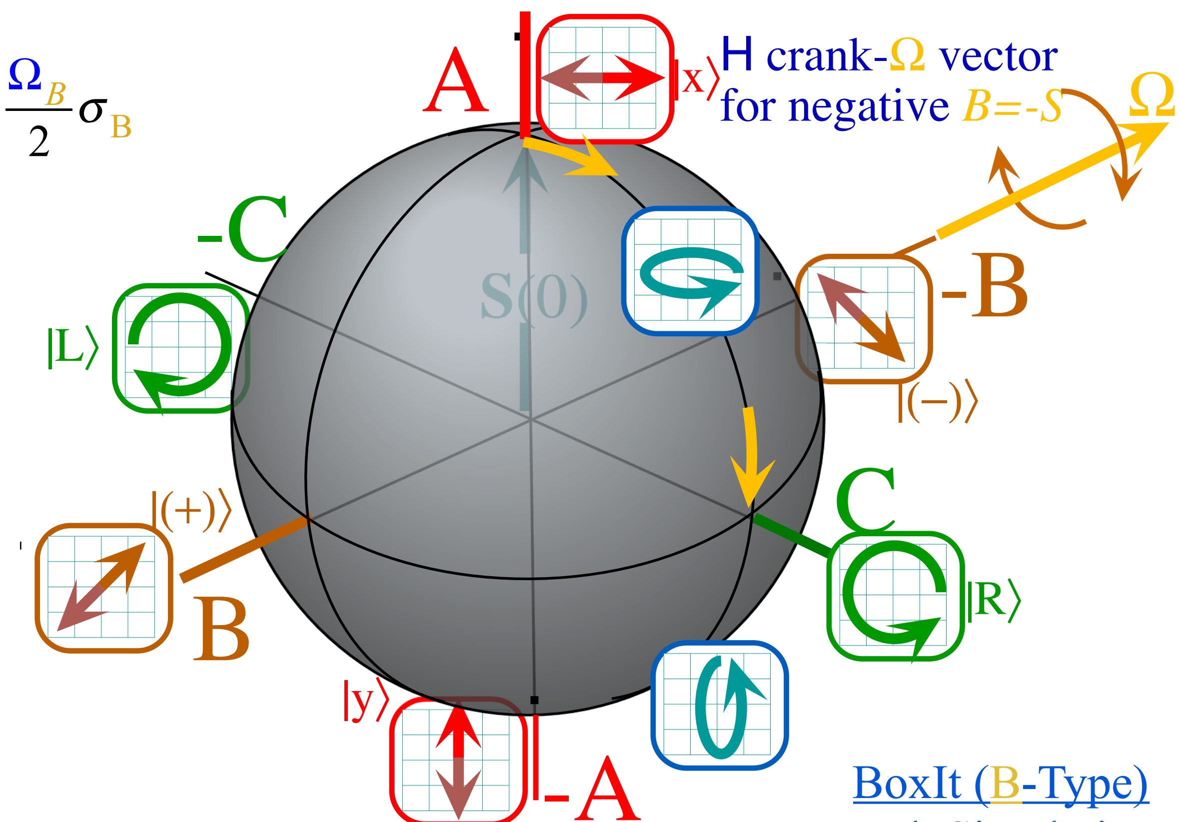
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Bilateral-Balanced B-Type motion

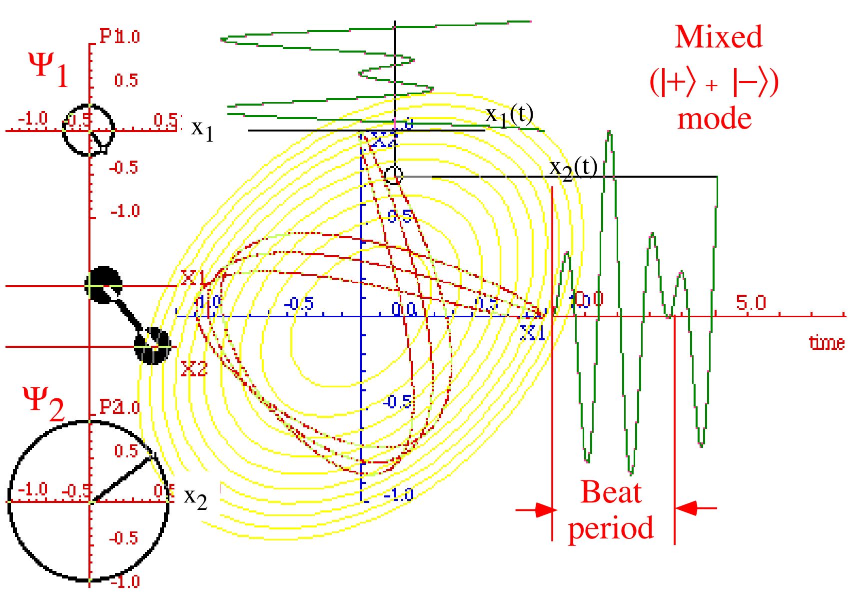
$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + B\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_B}{2}\sigma_B$$

$$\text{Crank}: \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 2B\\ 0\end{pmatrix} \qquad \text{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$



BoxIt (B-Type)
Web Simulation

Beat dynamics:



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

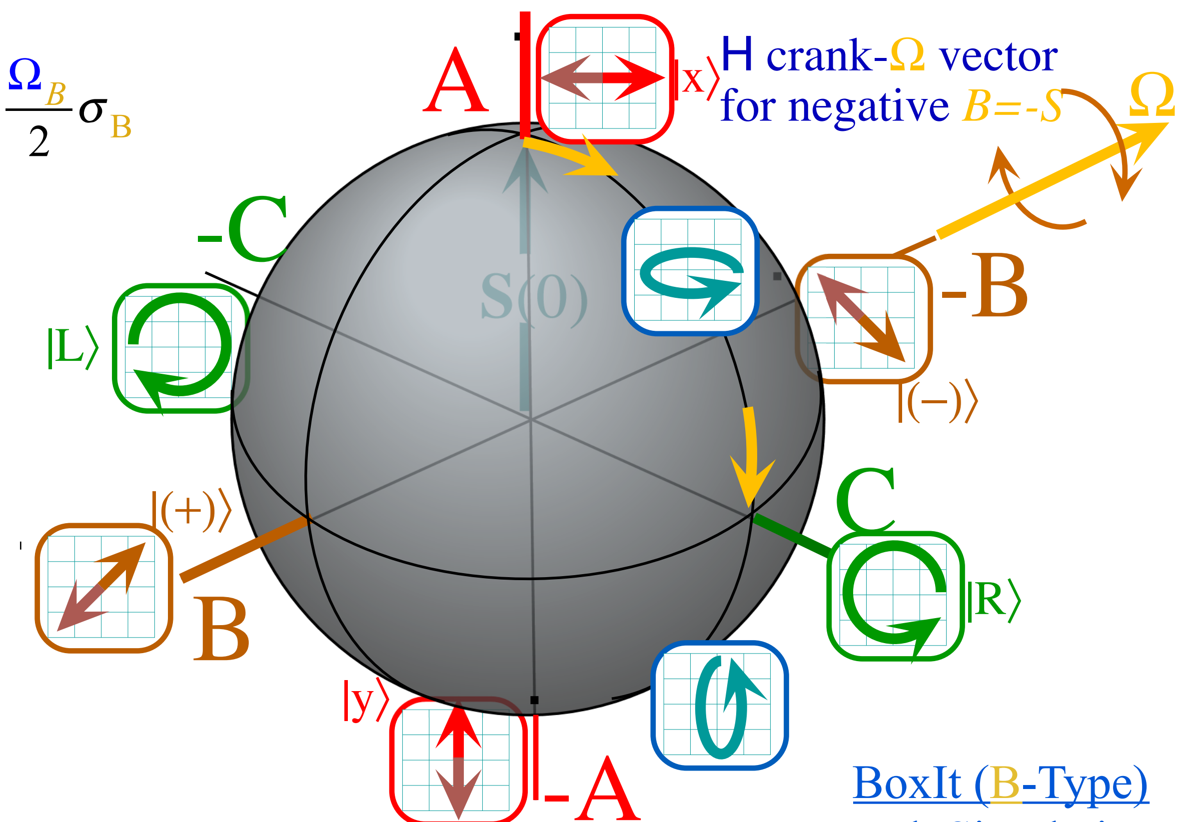
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

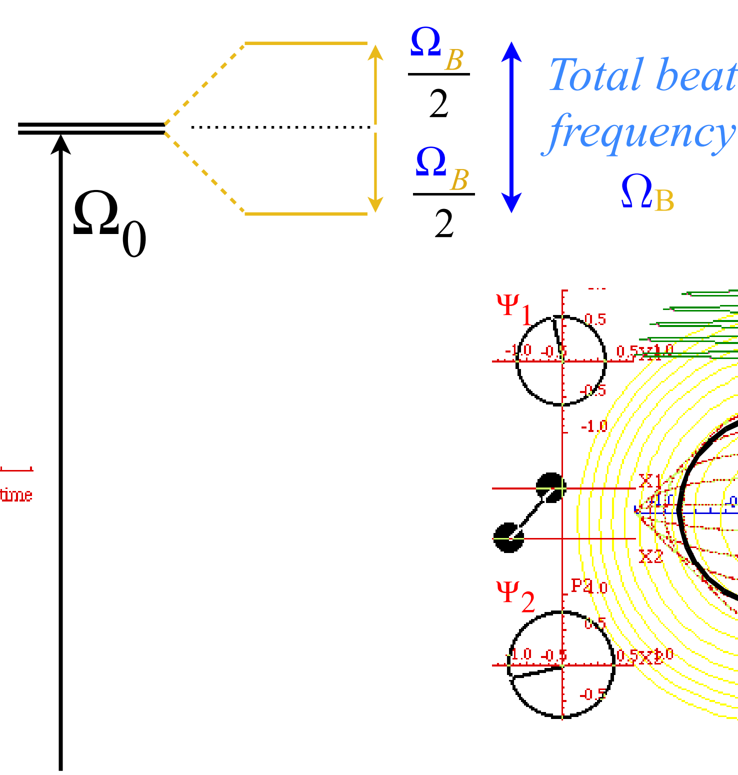
Bilateral-Balanced B-Type motion

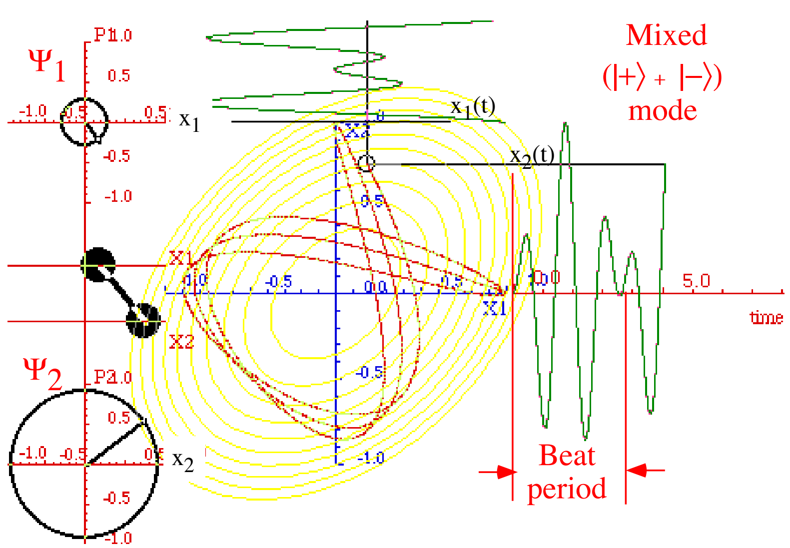
$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_B}{2}\sigma_B$$

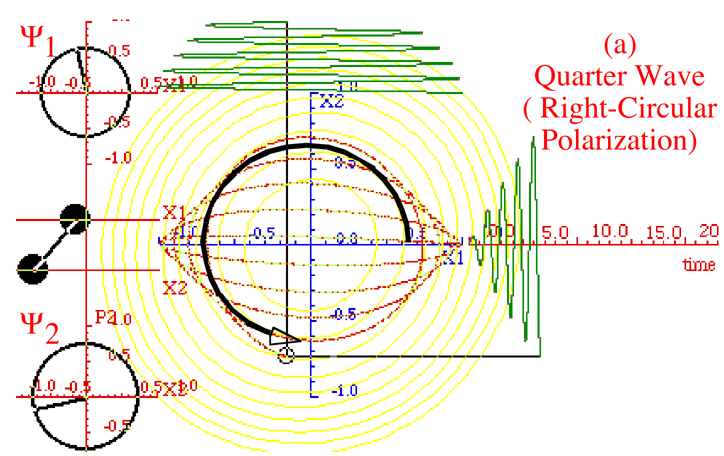
Crank : $\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 2B\\ 0\end{pmatrix}$ *Eigen-Spin :* $\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$

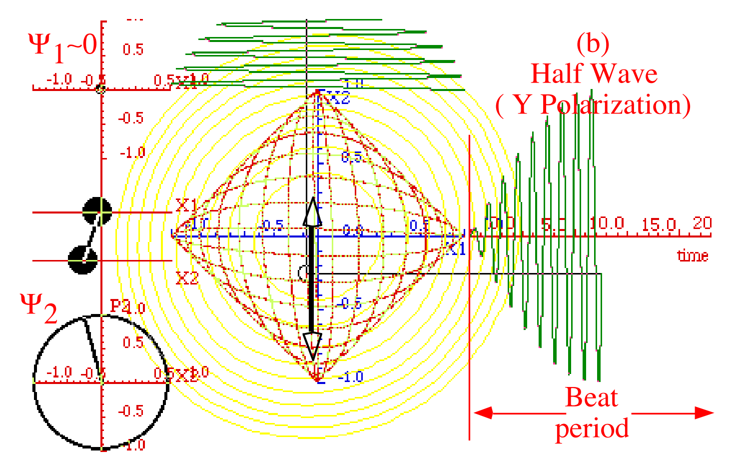


Beat dynamics:



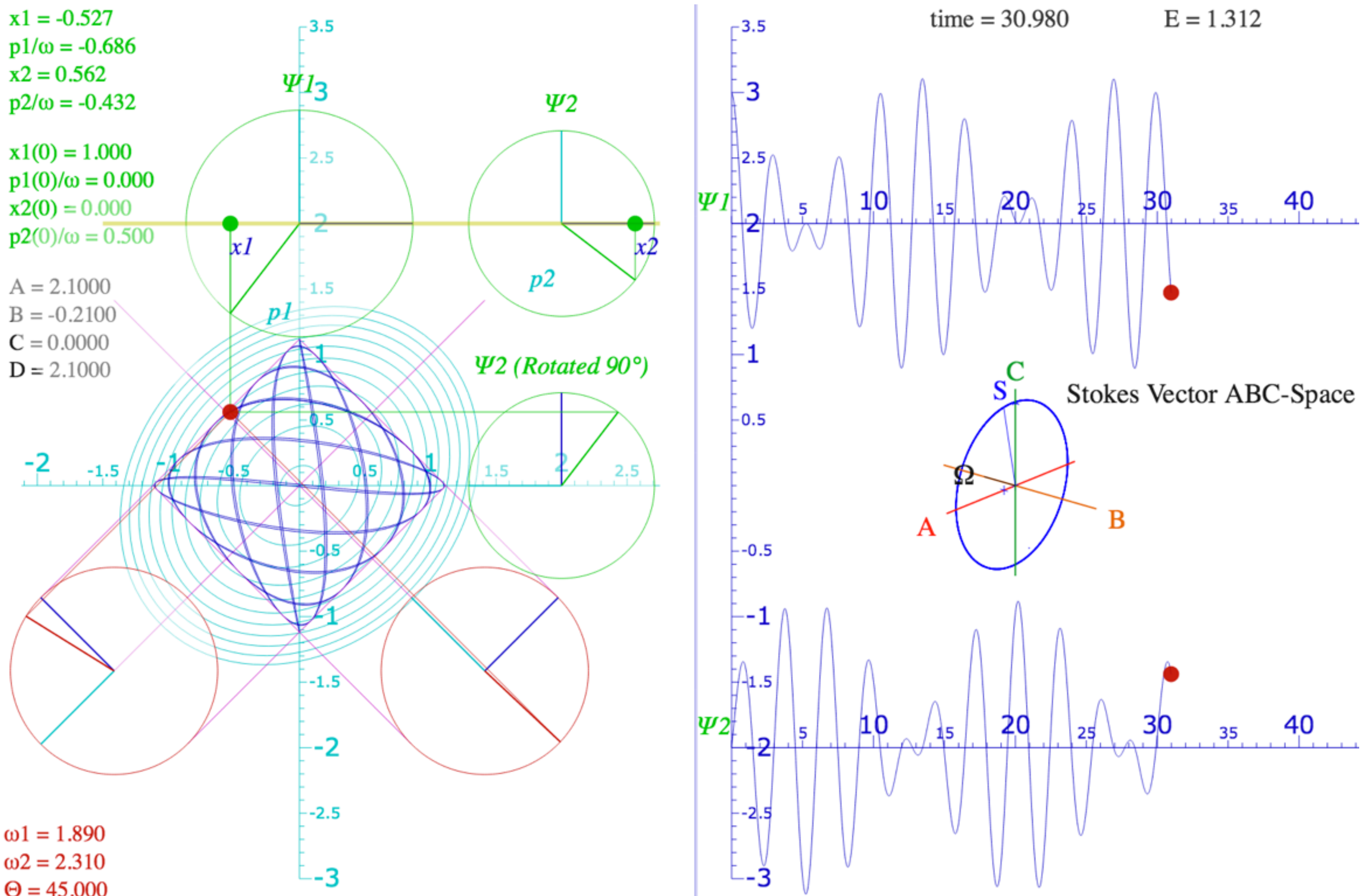






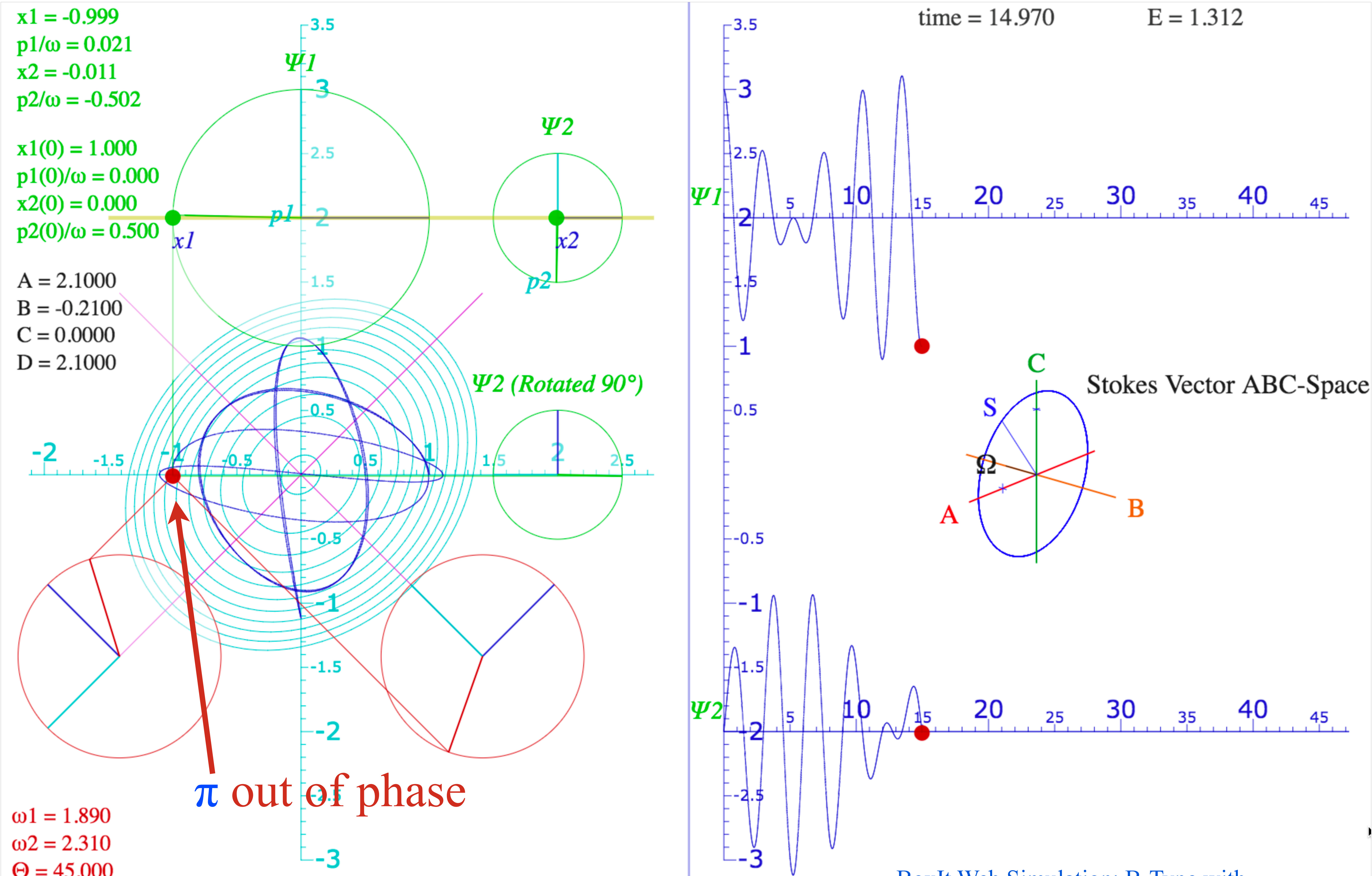
BoxIt (B-Type)
Web Simulation

B-Type elliptical polarized motion



BoxIt Web Simulation:
B-Type with A, D=2.1; B=-0.21

B-Type elliptical polarized motion Note that one 360°=2π rotation of **S** leaves (x_1,x_2) at $-(x_1,x_2)$



BoxIt Web Simulation: B-Type with A, D=2.1; B=-0.21

To assess the rationality of any number we approximate it using successive levels of *continued fractions*.

$$\alpha = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \ddots}}}}$$

Example 1: the number π = 3.1415926..., and recipe for getting n_k

$$A_0 = \alpha = 3.14159265\ldots \qquad n_0 = INT(A_0) = 3 \qquad \pi \cong \quad = 3.000..$$

$$A_1 = \frac{1}{A_0 - n_0} = 7.06\ldots \qquad n_1 = INT(A_1) = 7 \qquad \pi \cong 3 + \frac{1}{7} = \frac{22}{7} = 3.1428$$

$$A_2 = \frac{1}{A_1 - n_1} = 15.99\ldots \qquad n_2 = INT(A_2) = 15 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.141509$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.003\ldots \qquad n_3 = INT(A_3) = 1 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15+1}} = \frac{355}{113} = 3.14159292$$

Example 2: the *Golden Mean G*=(1+√5)/2=1.618033989…

$$A_0 = G = 1.618033989\ldots \qquad n_0 = INT(A_0) = 1 \qquad G \cong \quad = 1.000..$$

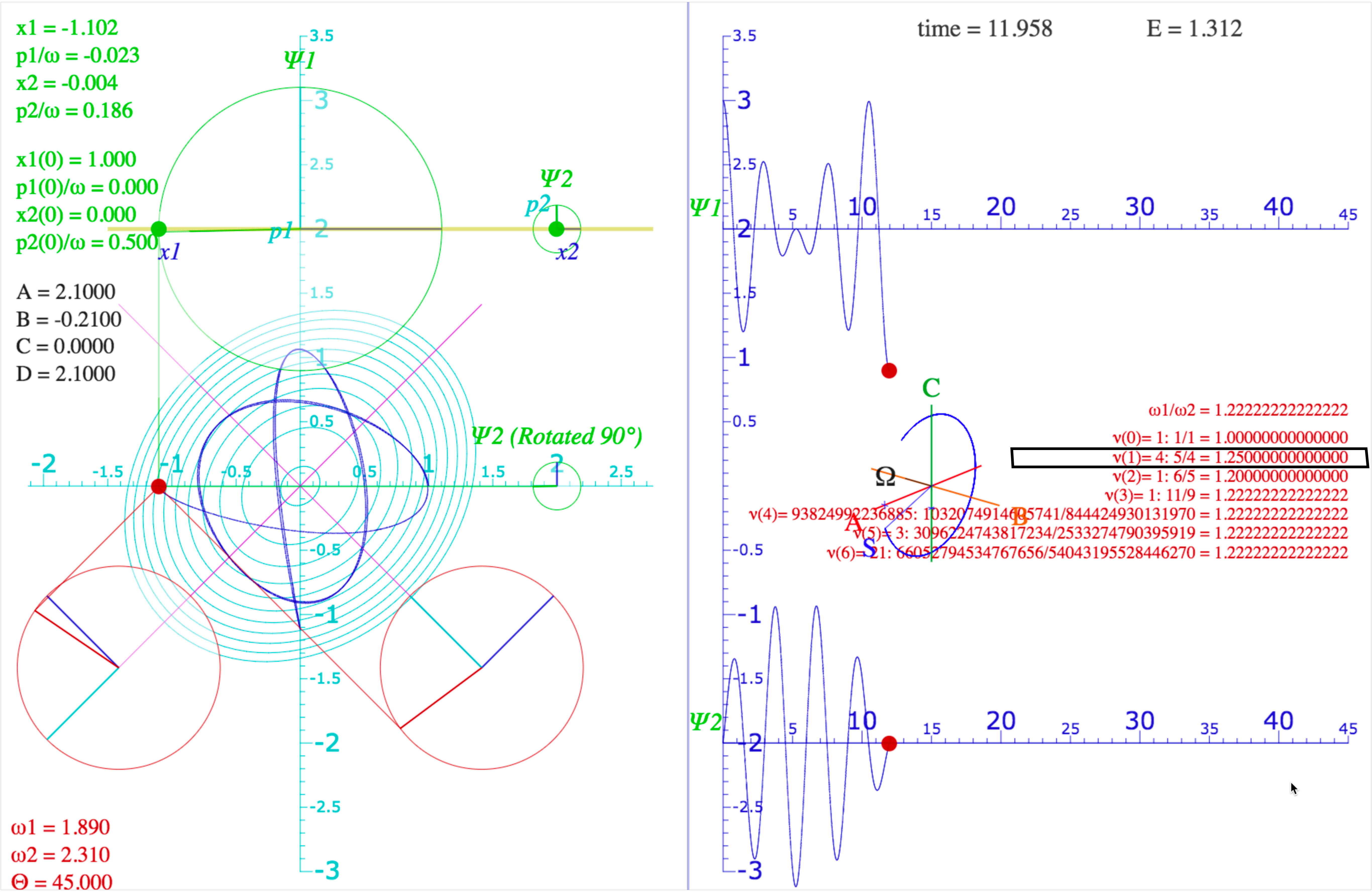
$$A_1 = \frac{1}{A_0 - n_0} = 1.6180\ldots \qquad n_1 = INT(A_1) = 1 \qquad G \cong 1 + \frac{1}{1} = \frac{2}{1} = 2.000$$

$$A_2 = \frac{1}{A_1 - n_1} = 1.6180\ldots \qquad n_2 = INT(A_2) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = 1.500$$

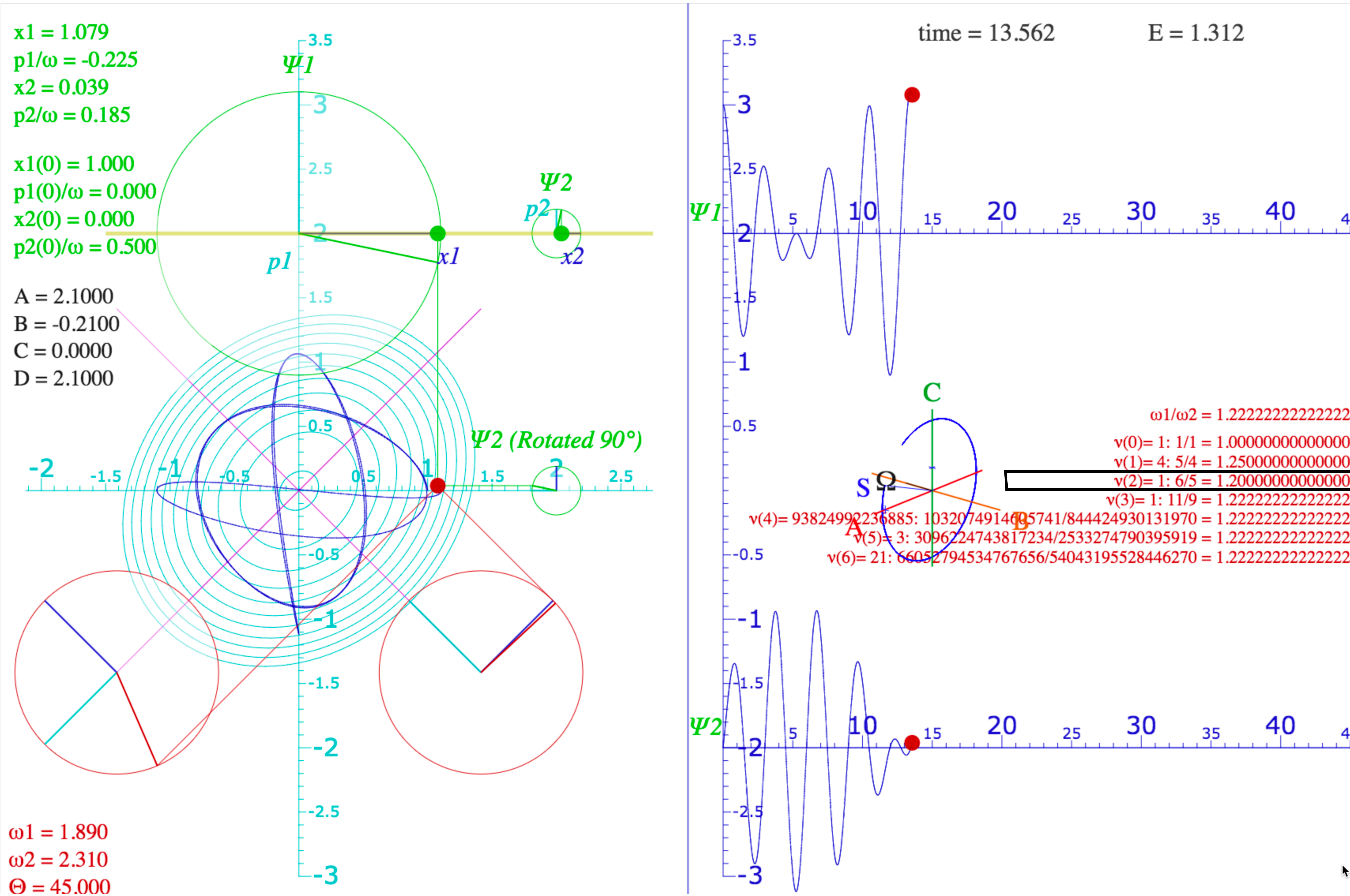
$$A_3 = \frac{1}{A_2 - n_2} = 1.6180\ldots \qquad n_3 = INT(A_3) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1+1}} = \frac{5}{3} = 1.666\ldots$$

Note: *Fibonacci* numbers: 1,2,3,5,8,…

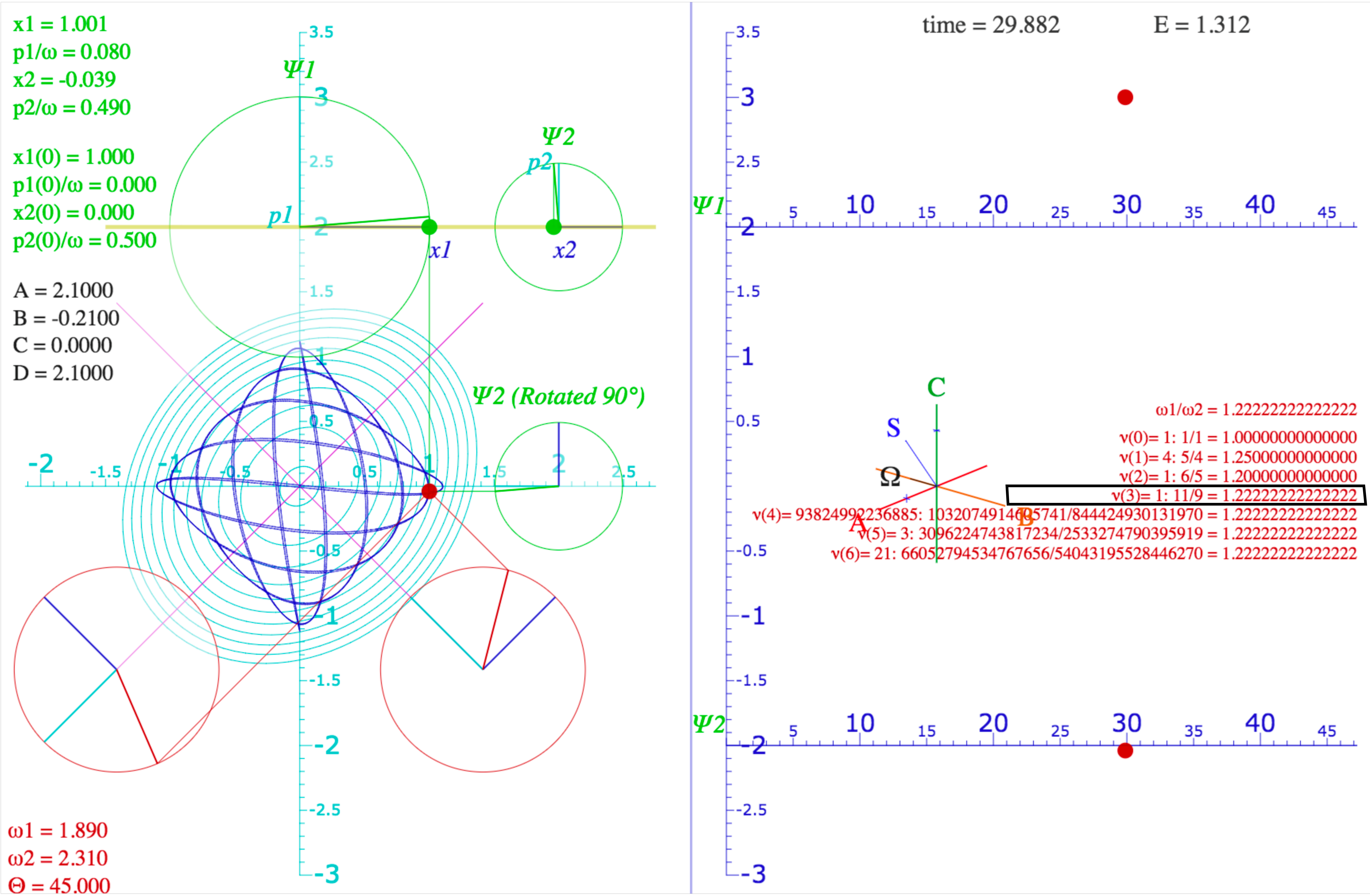
The most irrational number is closest to being rational!



BoxIt Web Simulation: B-Type with A, D=2.1; B=-0.21



[BoxIt Web Simulation: B-Type with A, D=2.1; B=-0.21]



x1 = 1.001
p1/ω = 0.080
x2 = -0.039
p2/ω = 0.490
x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500
A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000
ω1 = 1.890
ω2 = 2.310
Θ = 45.000
Ψ1
Ψ2
p1
p2
x1
x2
Ψ2 (Rotated 90°)
time = 29.882
E = 1.312
C
S
Ω
A
B
ω1/ω2 = 1.22222222222222
ν(0)= 1: 1/1 = 1.00000000000000
ν(1)= 4: 5/4 = 1.25000000000000
ν(2)= 1: 6/5 = 1.20000000000000
ν(3)= 1: 11/9 = 1.22222222222222
ν(4)= 93824992236885: 103207491465741/84442493013197 = 1.22222222222222
ν(5)= 3: 309622474381723 4/25332747903959 19 = 1.22222222222222
ν(6)= 21: 6605279453476765 6/5404319552844627 0 = 1.22222222222222

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes
Asymmetric-Diagonal A-Type motion
Bilateral-Balanced B-Type motion
Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes
AB-Type motion and Wigner's Avoided-Symmetry-Crossings
ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates
Conventional amp-phase ellipse coordinates
Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

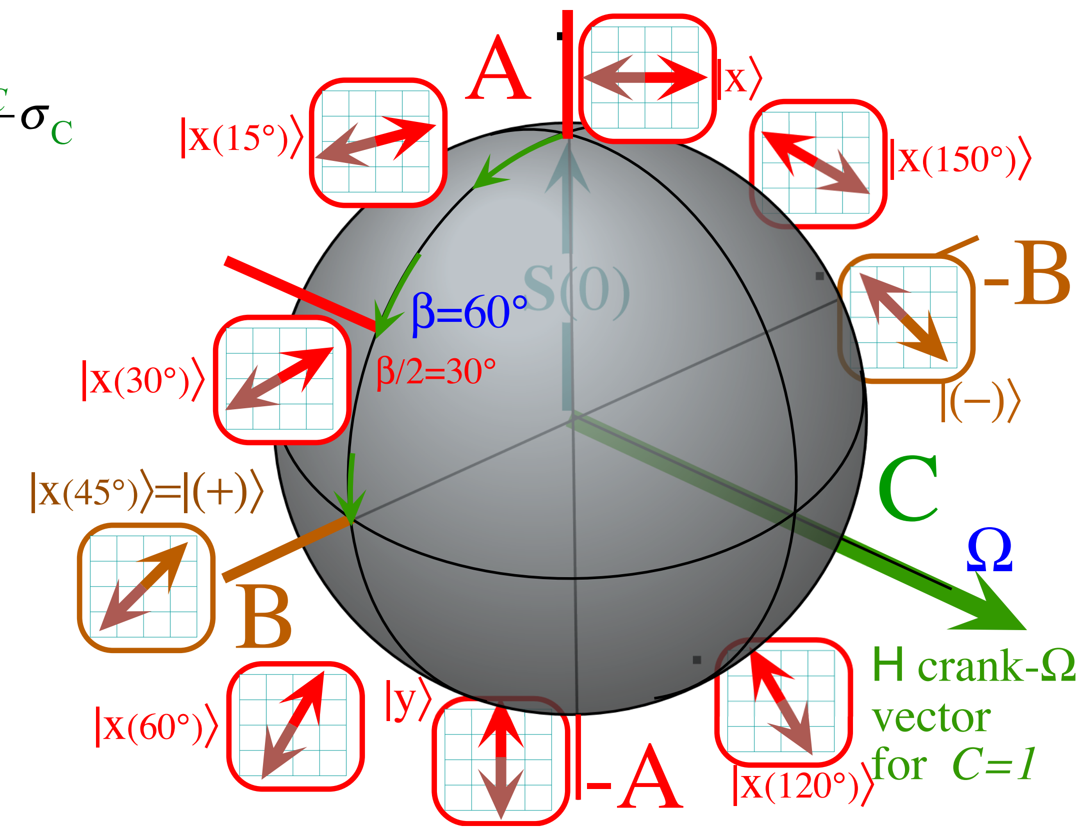
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Circular-Coriolis... C-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle\\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_C}{2}\sigma_C$$

$$\text{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}0\\0\\2C\end{pmatrix}\qquad \text{Eigen}-\text{Spin}:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}0\\0\\ \pm S\end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

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In general:

$$\mathbf{H}=\frac{A+D}{2}\mathbf{1}+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A-D}{2}\boldsymbol{\sigma}_A$$

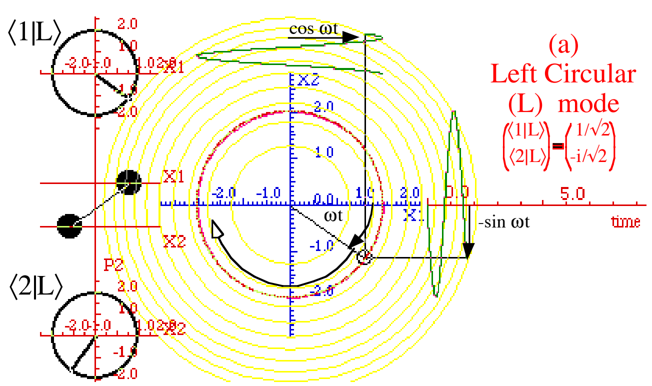
$$\mathbf{H}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\boldsymbol{\sigma}_B+\frac{\Omega_C}{2}\boldsymbol{\sigma}_C+\frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

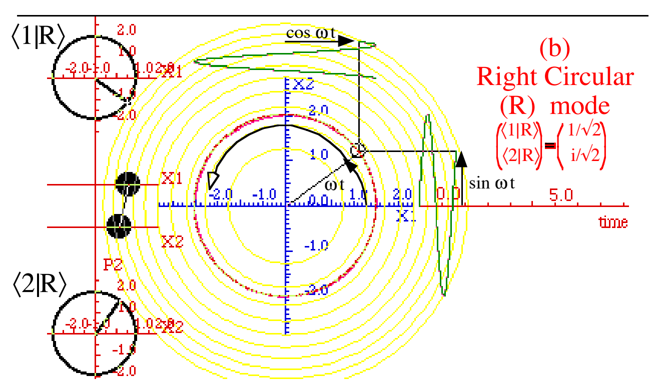
$$\vec{\boldsymbol{\Omega}}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

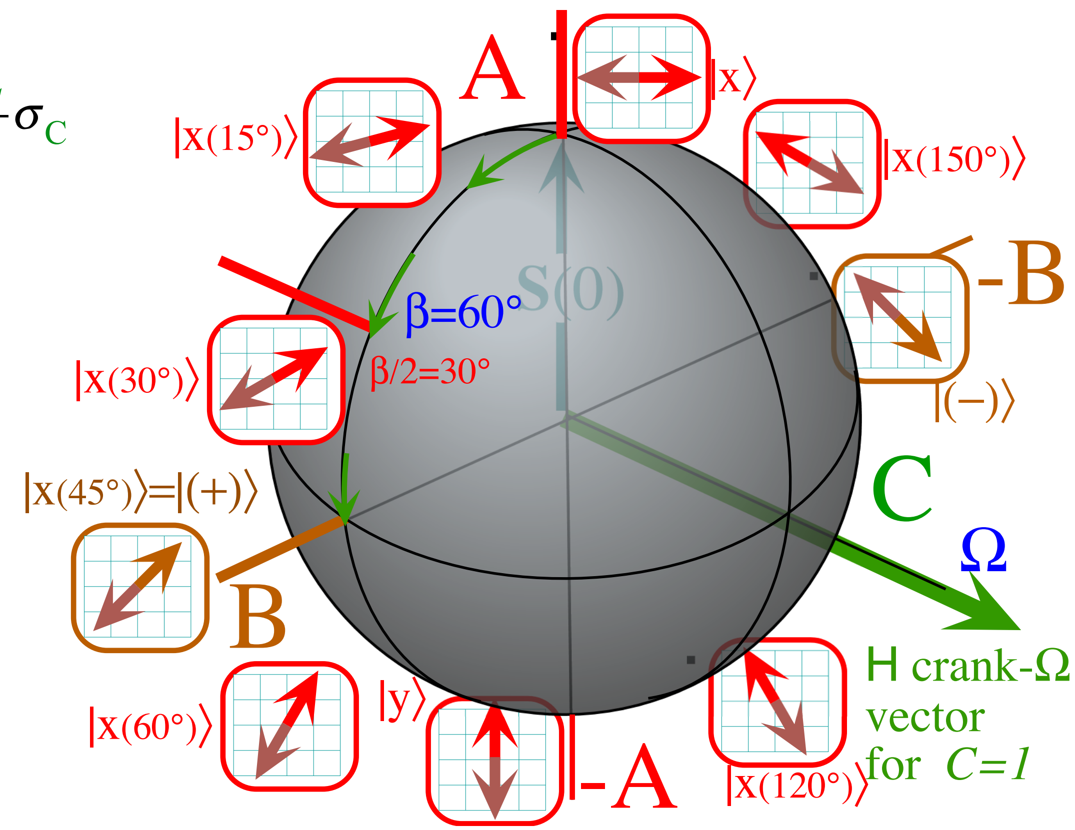
Circular-Coriolis... C-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle\\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_C}{2}\boldsymbol{\sigma}_C$$

$$Crank:\ \vec{\boldsymbol{\Omega}}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}0\\0\\2C\end{pmatrix}\qquad Eigen-Spin:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\S_B\\S_C\end{pmatrix}=\begin{pmatrix}0\\0\\\pm S\end{pmatrix}$$







The ABC's of U(2) dynamics

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$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix} A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

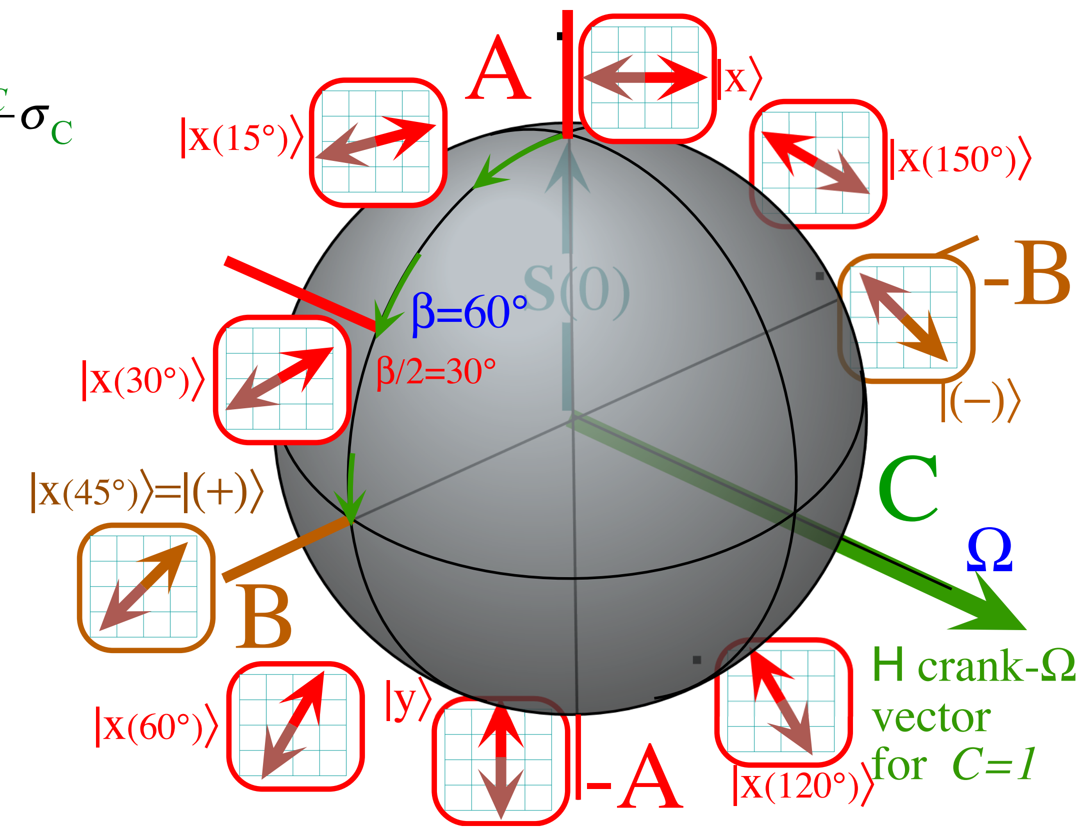
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

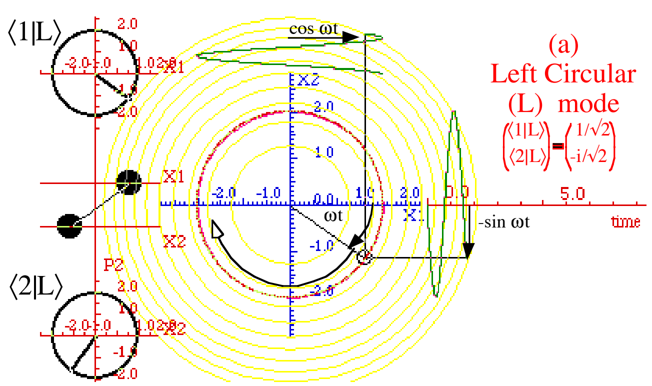
$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

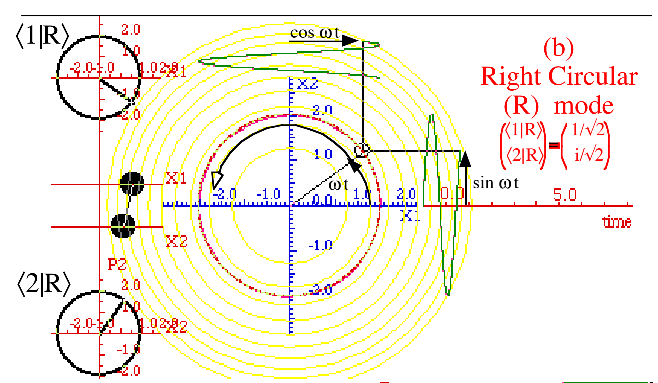
Circular-Coriolis... C-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle\\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

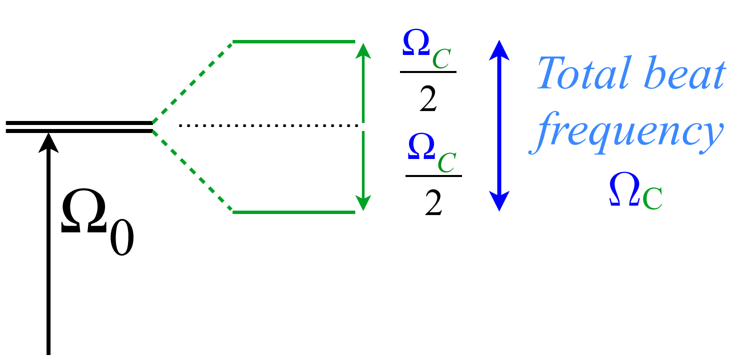
Crank : $\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\0\\2C\end{pmatrix}$ *Eigen – Spin :* $\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\0\\ \pm S\end{pmatrix}$

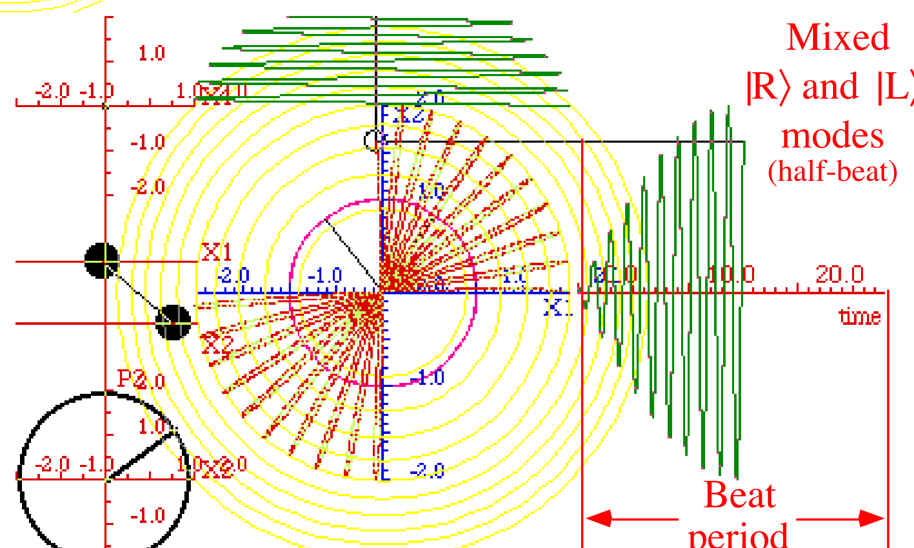






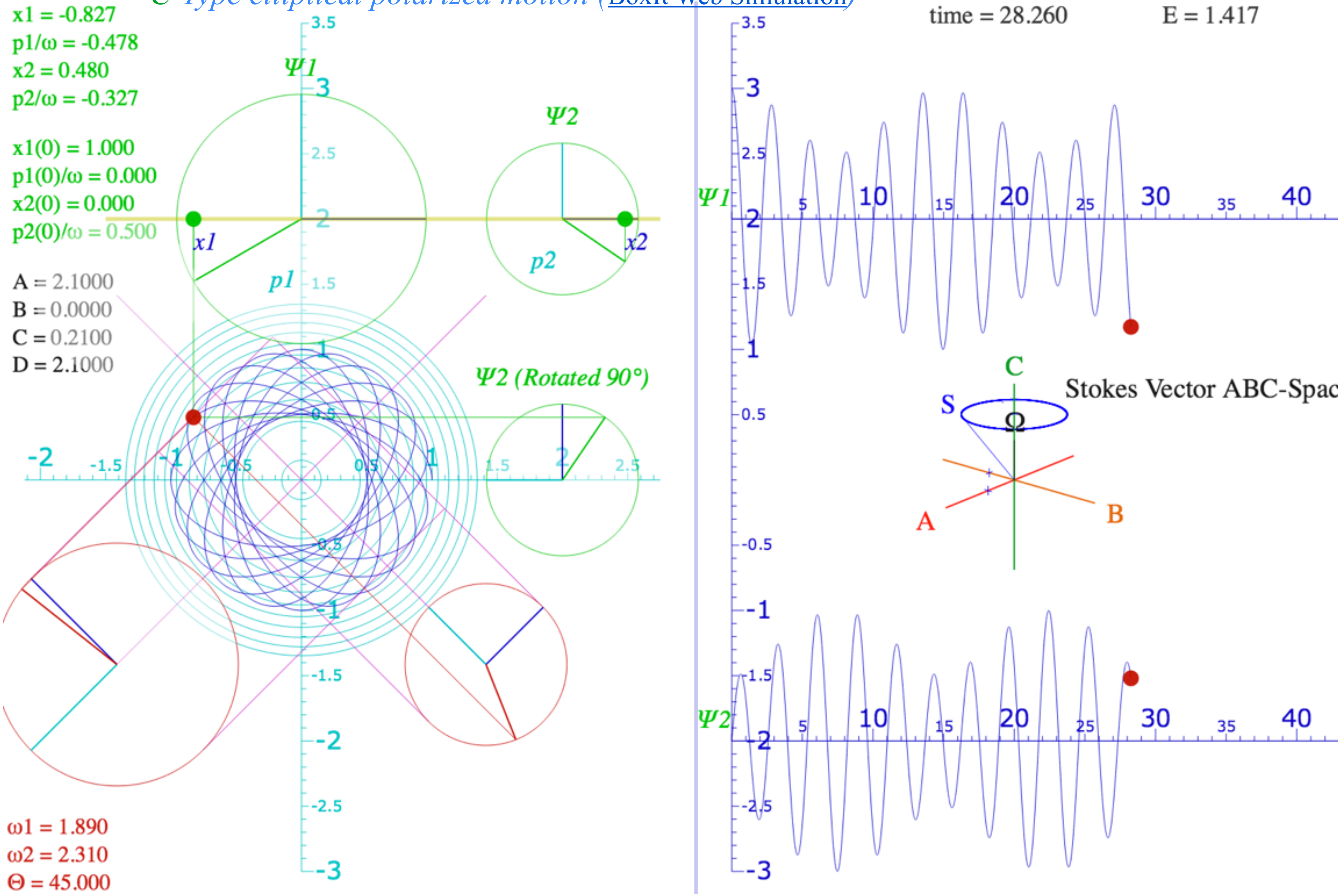
Beat dynamics:





BoxIt (C-Type)
Web Simulation

C-Type elliptical polarized motion (BoxIt Web Simulation)



BoxIt Web Simulation:
C-Type with A, D=2.1; C=-0.21

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S**-*vector, phasors, or ellipsometry*

Darboux defined Hamiltonian **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$-*vector*

Euler-defined operator **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$-*sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics-Mixed modes (AB-Type motion)

$$\rho=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H}=\Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix}\langle1|\mathbf{H}|1\rangle & \langle1|\mathbf{H}|2\rangle\\ \langle2|\mathbf{H}|1\rangle & \langle2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p.60

$$=\frac{A+D}{2}\mathbf{1}+B\,\sigma_B+C\,\sigma_C+\frac{A-D}{2}\sigma_A$$
$$=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C+\frac{\Omega_A}{2}\sigma_A$$

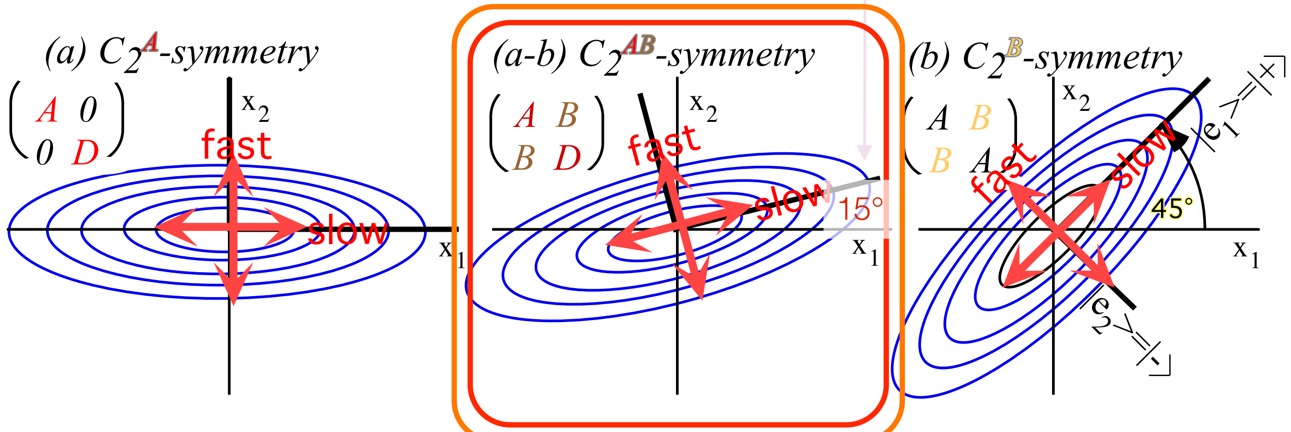
$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Tilted-plane polarization AB-Type motion

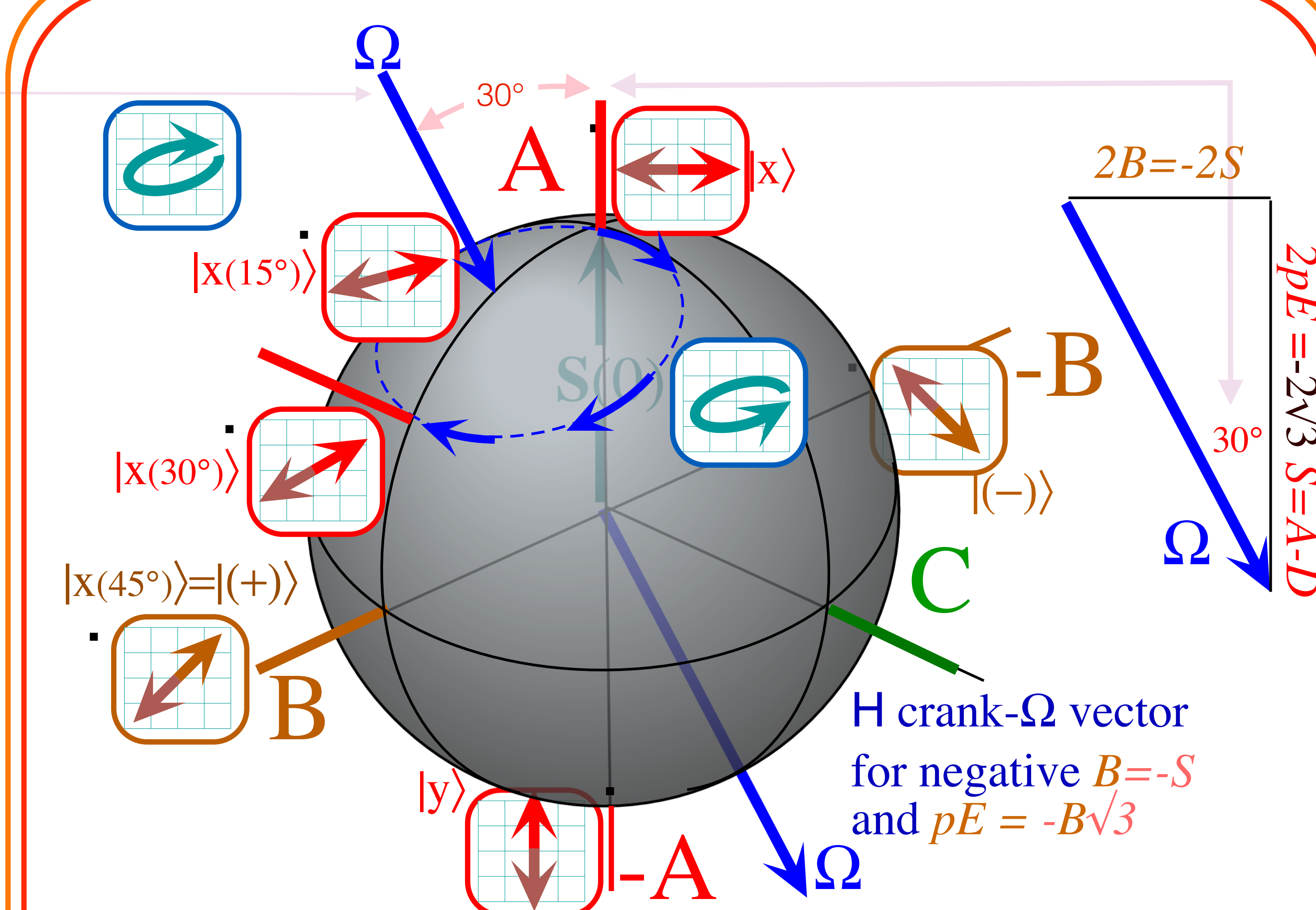
$$\begin{pmatrix}\langle1|\mathbf{H}^{AB}|1\rangle & \langle1|\mathbf{H}^{AB}|2\rangle\\ \langle2|\mathbf{H}^{AB}|1\rangle & \langle2|\mathbf{H}^{AB}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B\\ B & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_A}{2}\sigma_A+\frac{\Omega_B}{2}\sigma_B$$

Crank : $\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 0\end{pmatrix}$ *Eigen–Spin :* $\vec{\mathbf{S}}=\pm S\cdot\hat{\Omega}$

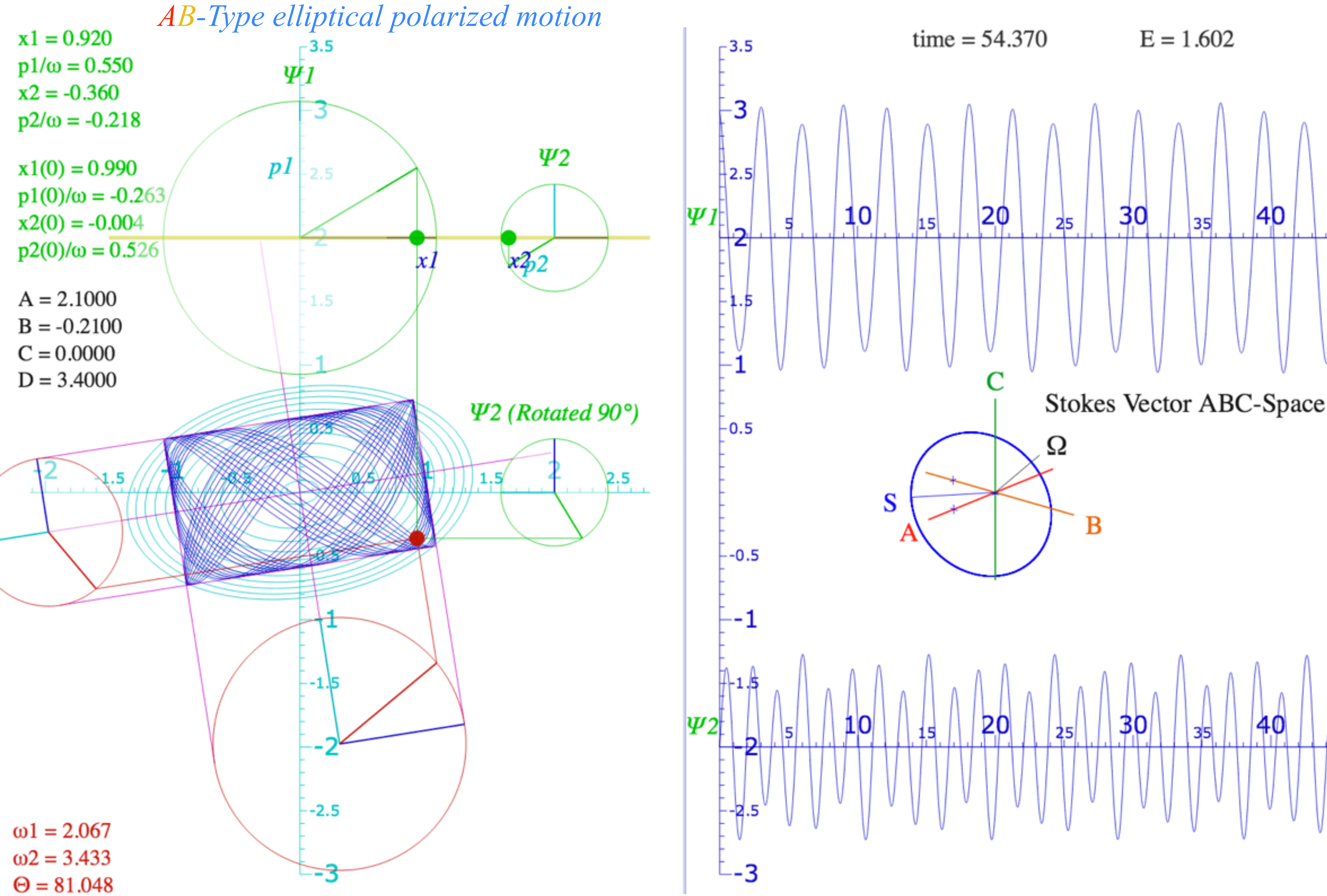
Note the *relative* factor of 1/2



Beat dynamics:



BoxIt Web Simulation: AB-Type Motion
Note: *In the Web App* the C axis is vertical
Older use: AB-Type mixed Motion: A=5.086, B=-0.27, C=0.0, D=2.024



[BoxIt Web Simulation: AB-Type with A=2.1; B=-0.21; D=3.4]

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

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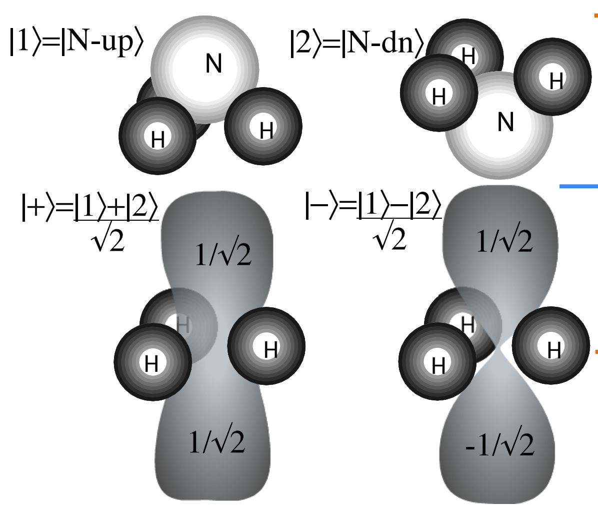

Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

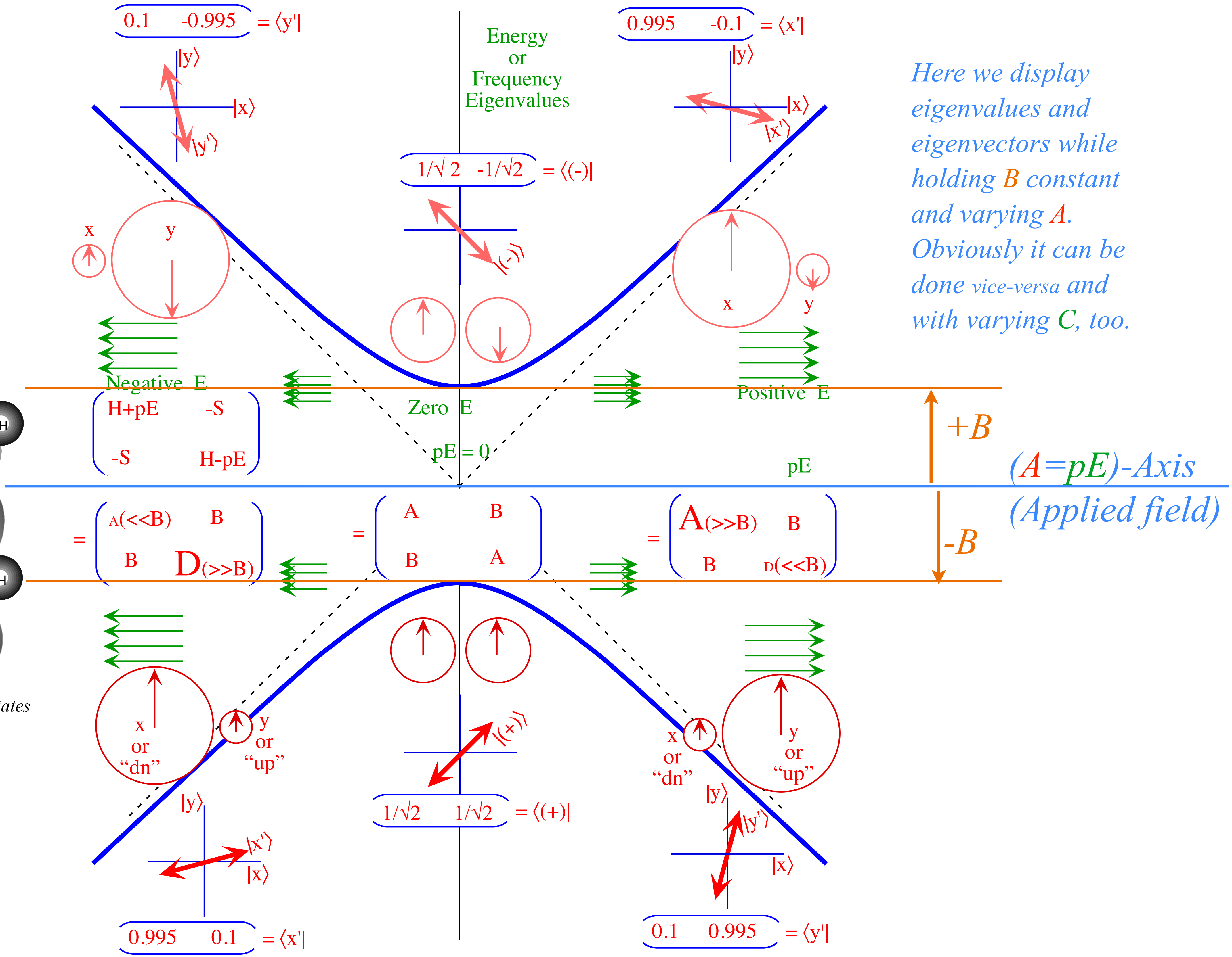

Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

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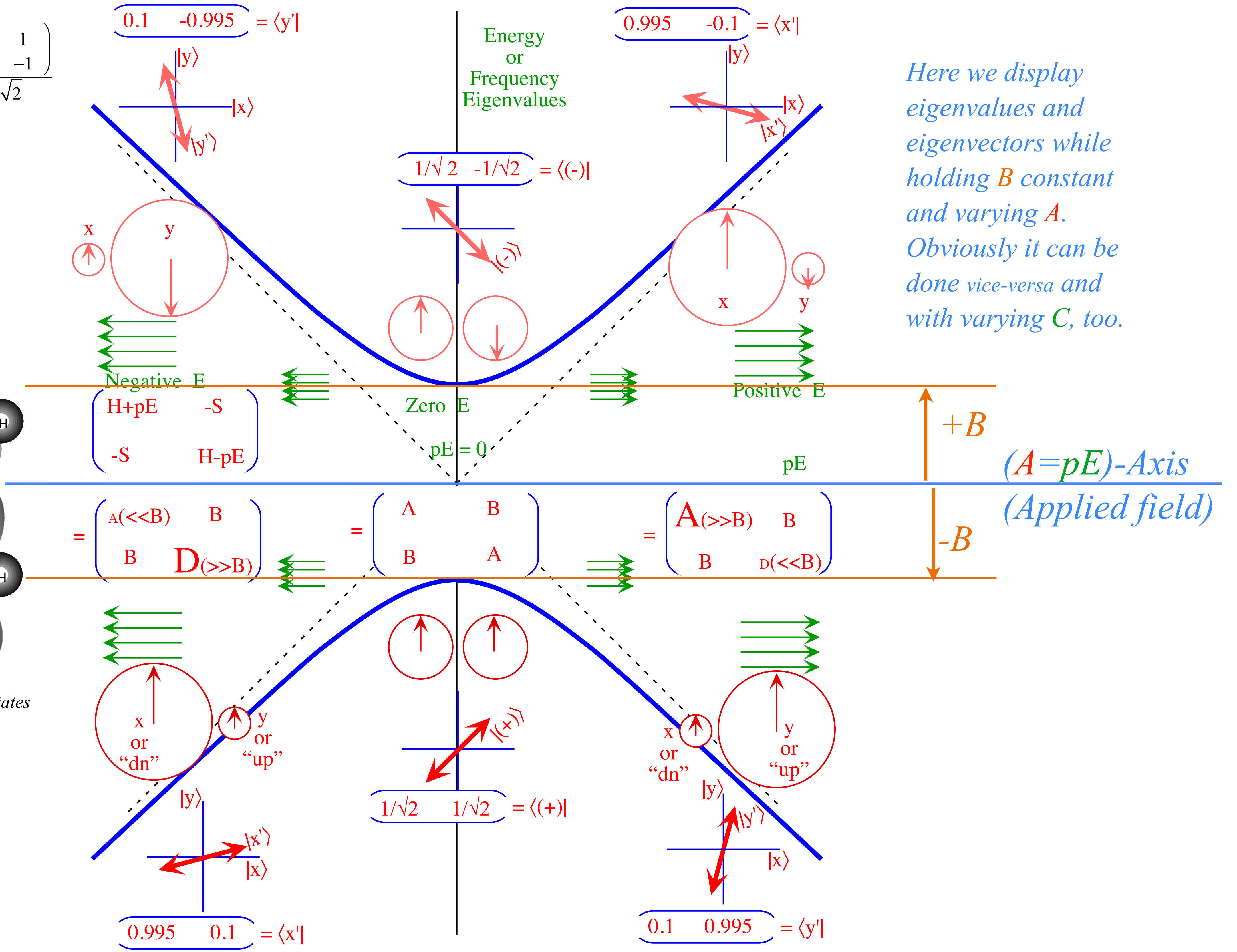
$$\mathbf{H}(B-basis) \qquad \mathbf{H}(A-basis)$$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$



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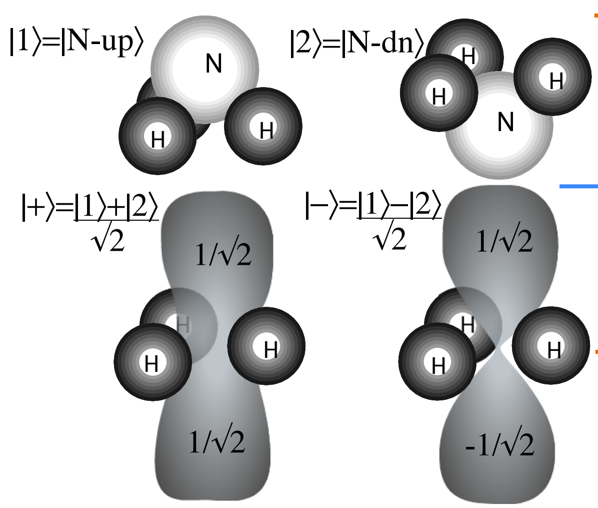

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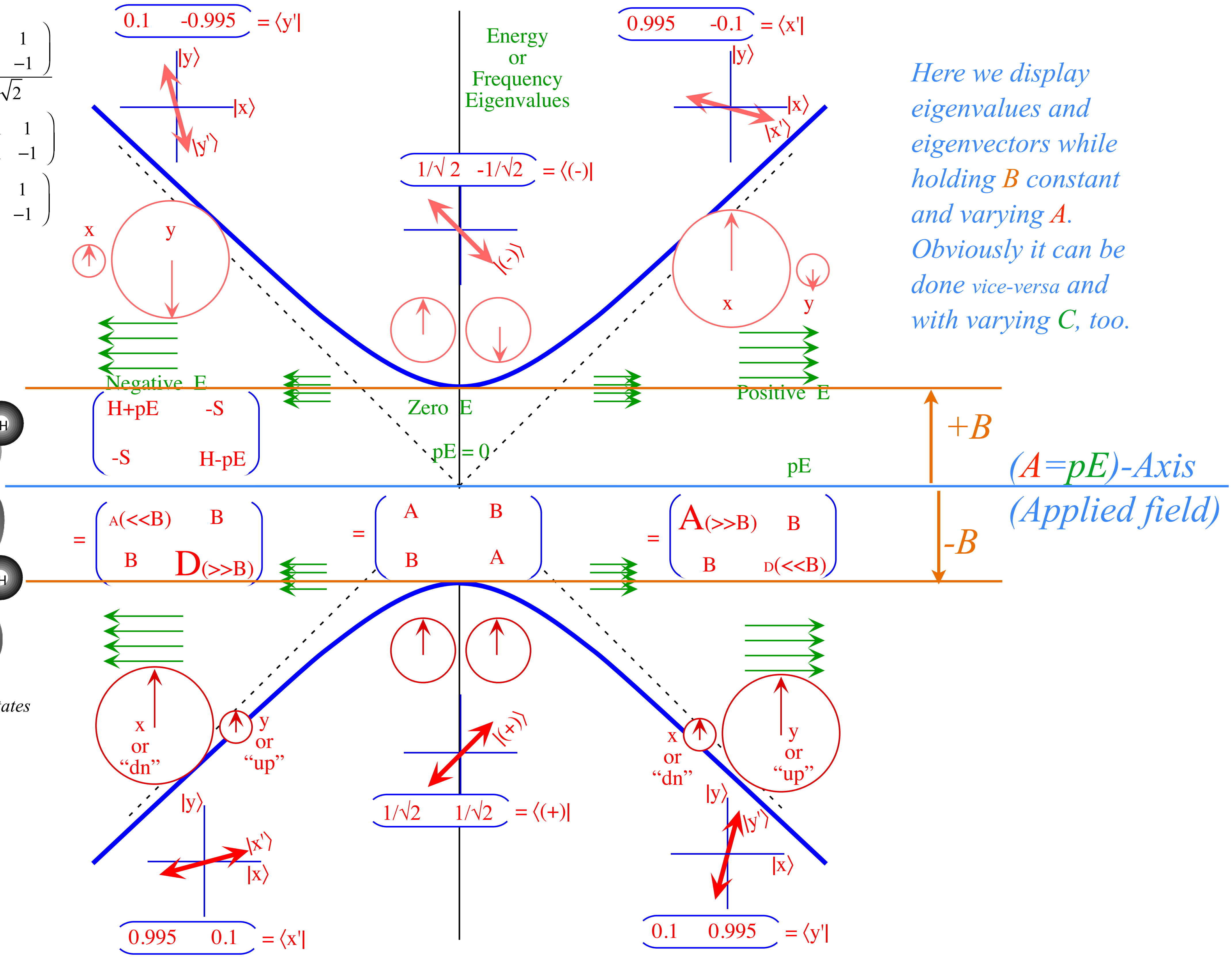
$$=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



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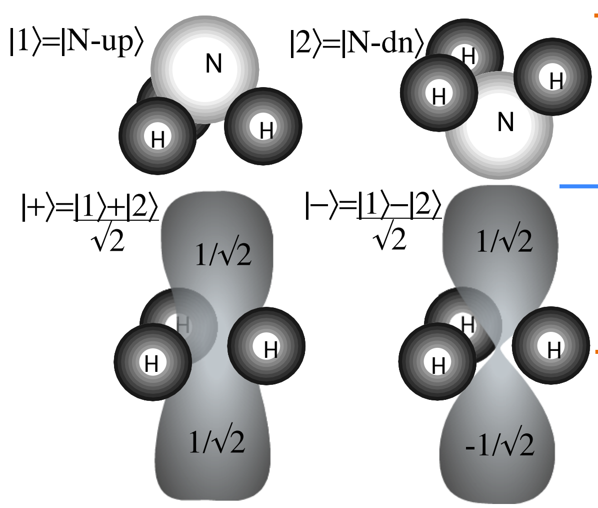

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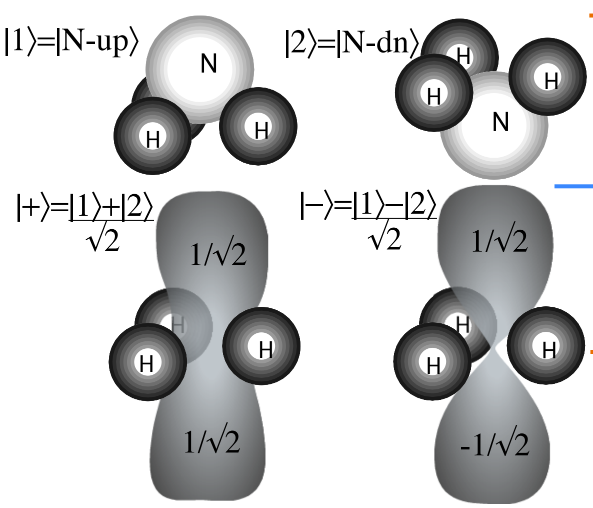
$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$



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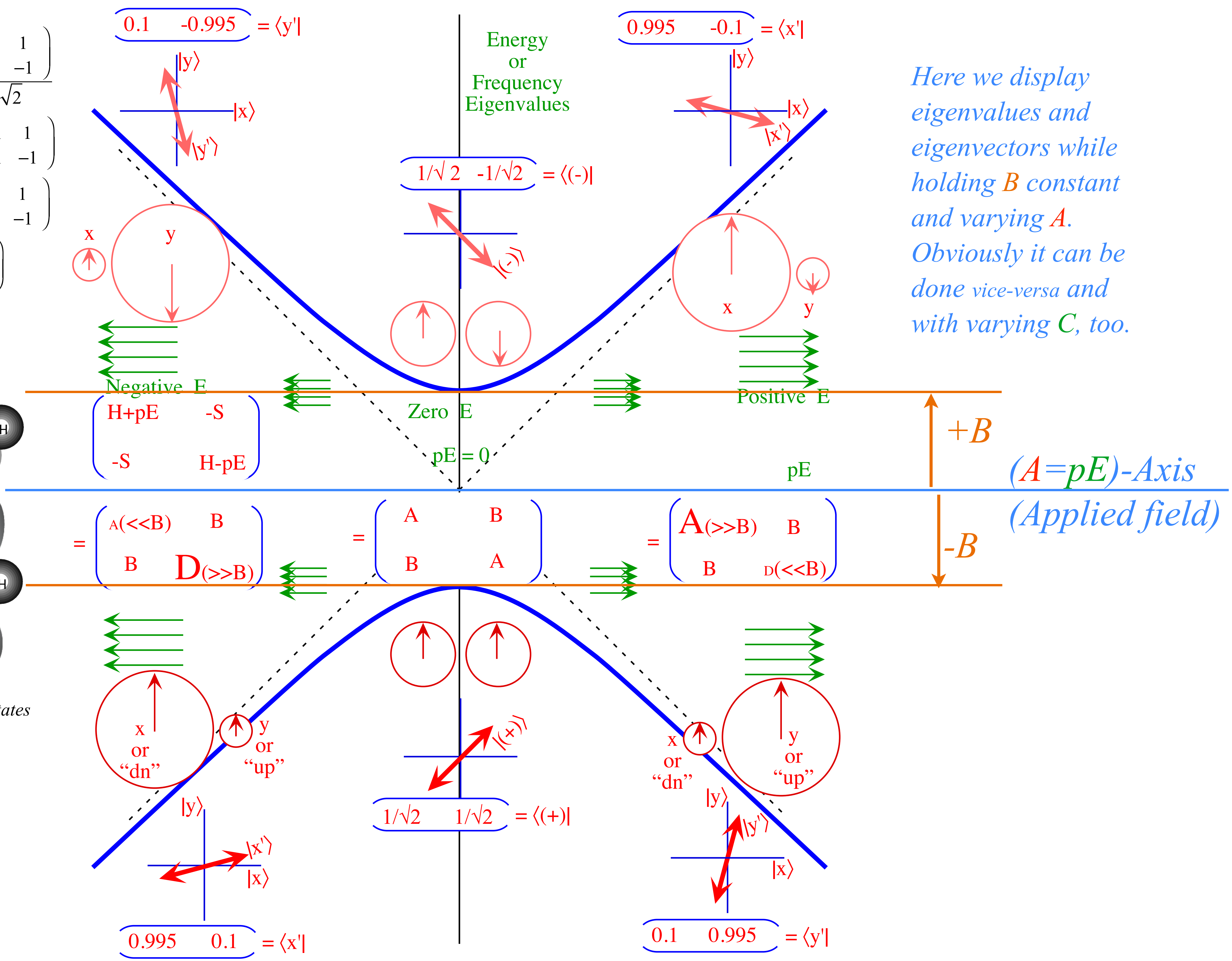
(a) Base states (b) C_2-Eigenstates



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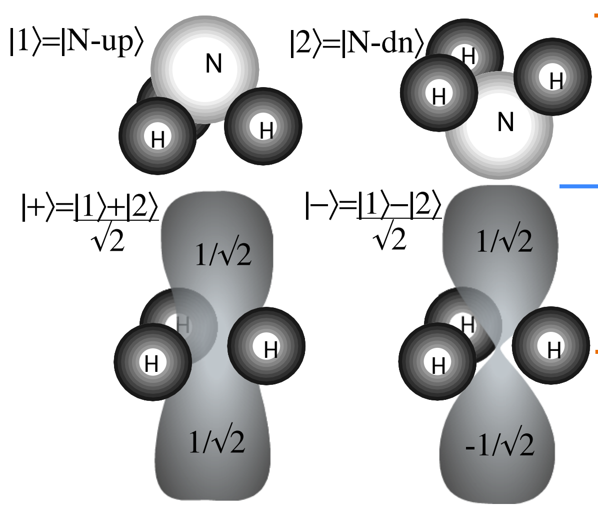

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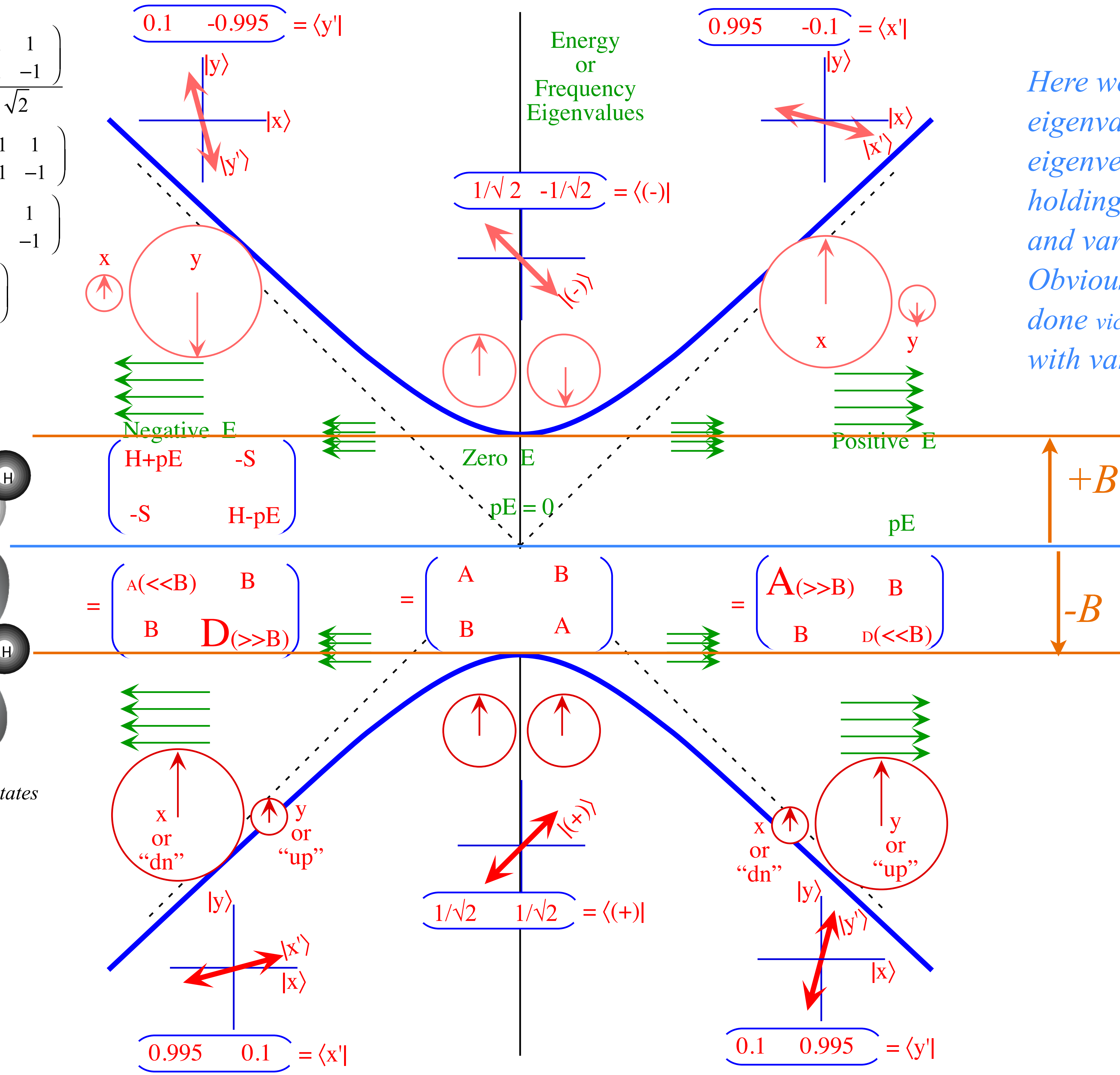


(A=pE)-Axis
(Applied field)

Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

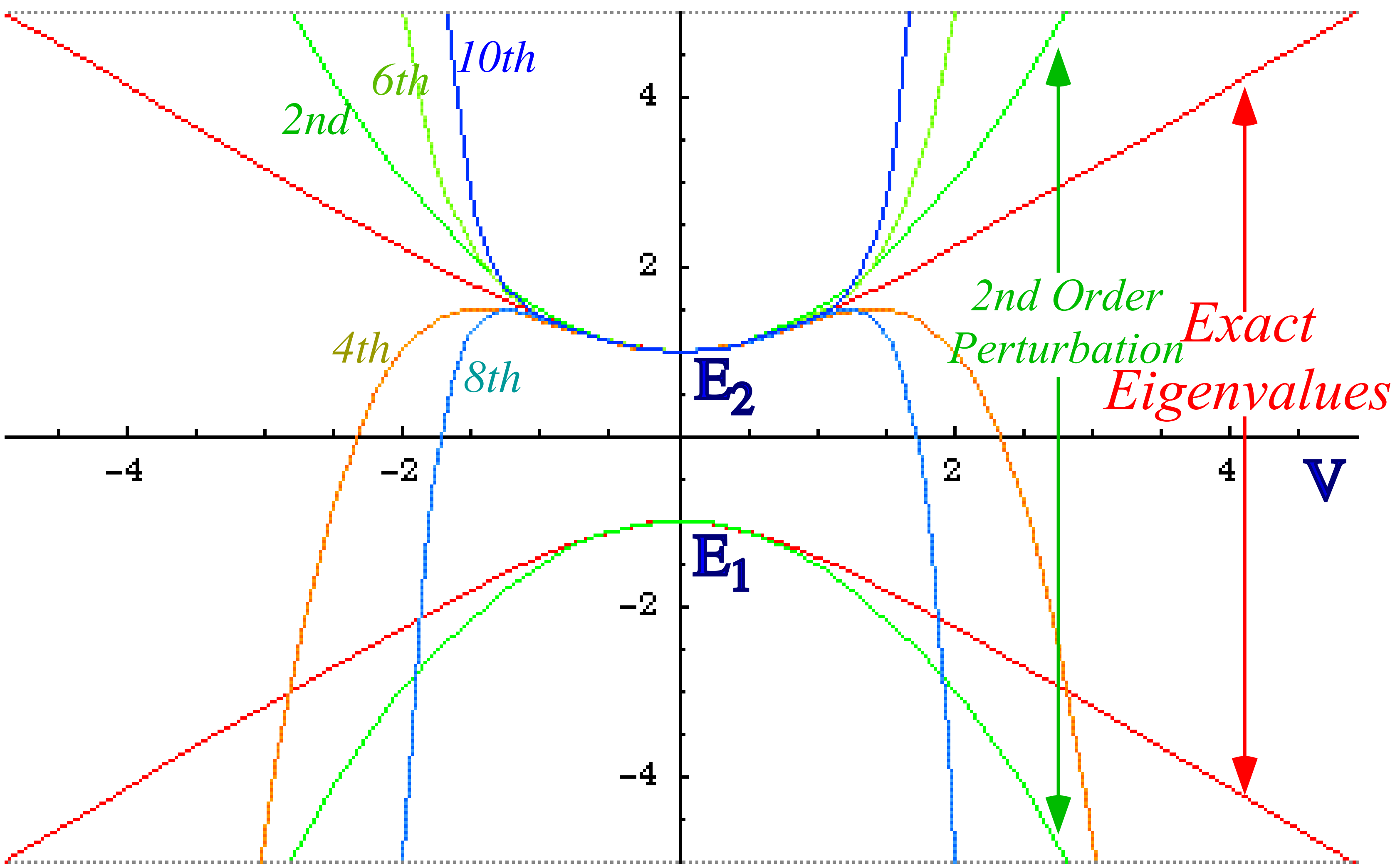
The failure of perturbation methods to get exact hyperbolic eigenvalues

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} \ ,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} \ .$$



$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1+E_2)^2 - 4E_1E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1-E_2)^2 + 4V^2}}{2} \ ,$$

Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \text{ , where: } \Delta = |E_1 - E_2|$$

A view of a conical intersection:

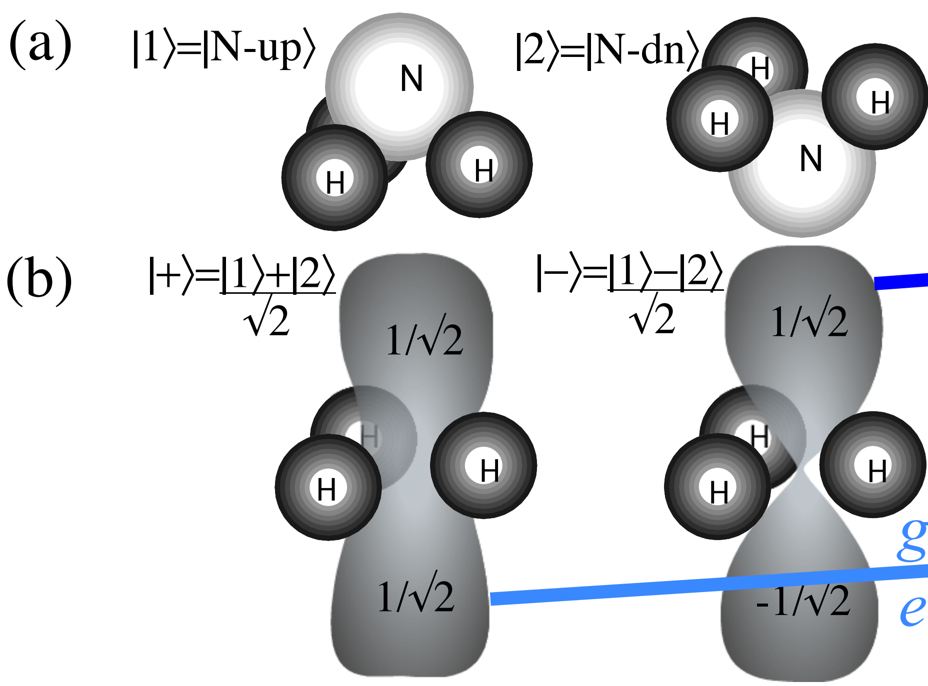
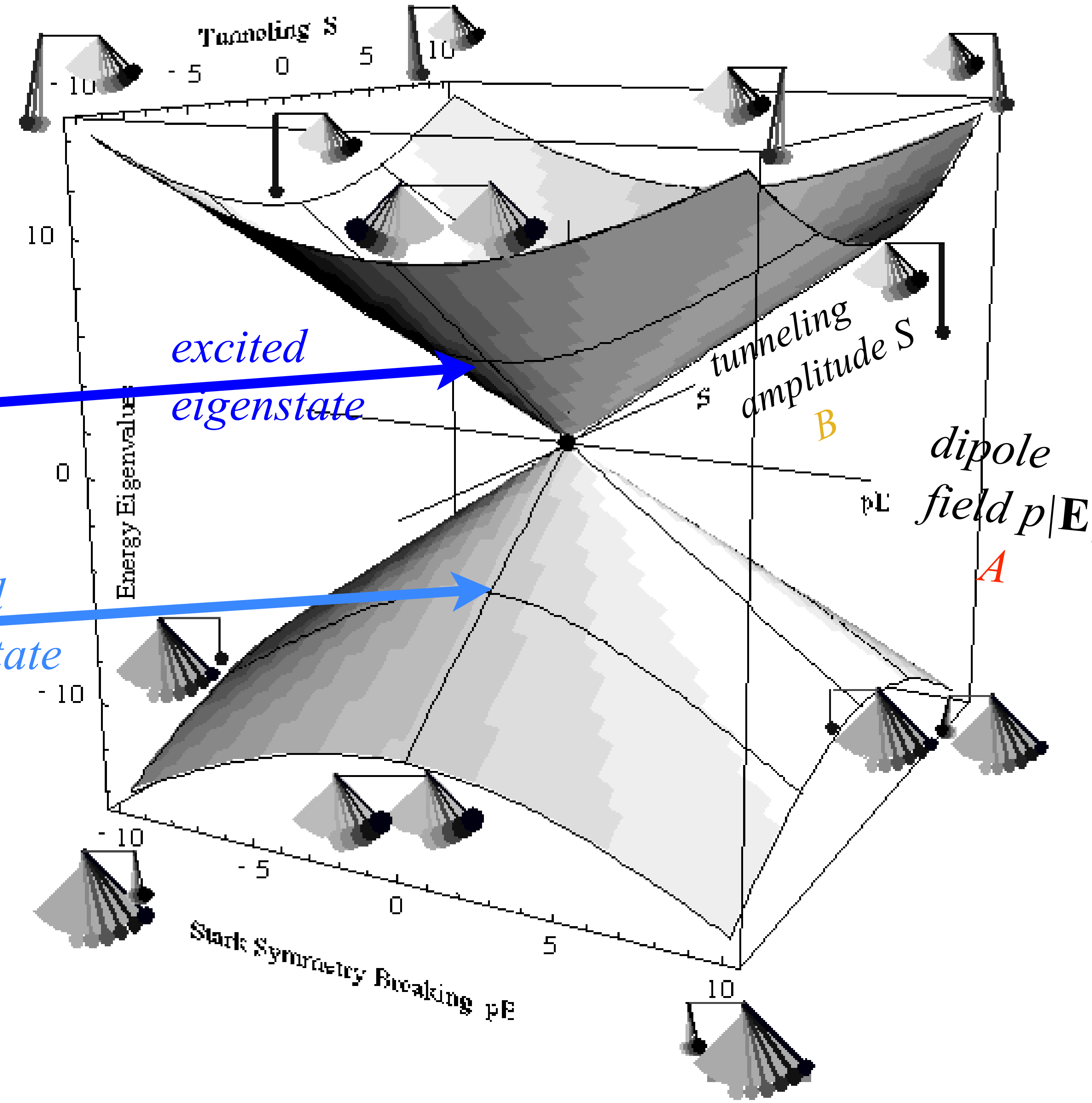


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

(Also known as a “Dirac-point”)

A view of a conical intersection: Any vertical cross-section is hyperbolic avoided-crossing

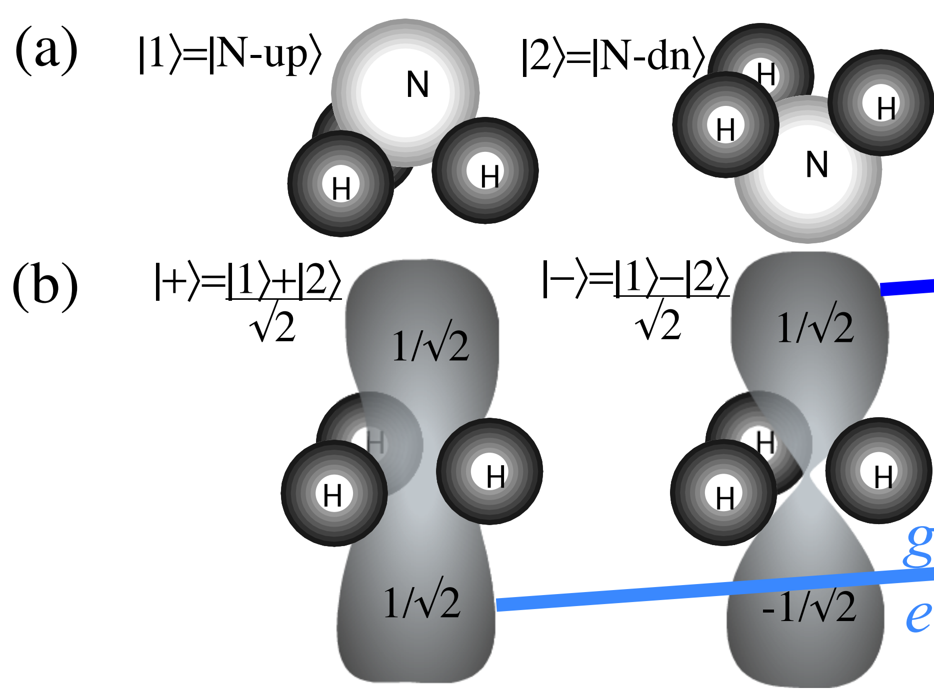


Fig. 10.3.2 Ammonia (NH_3) inversion states
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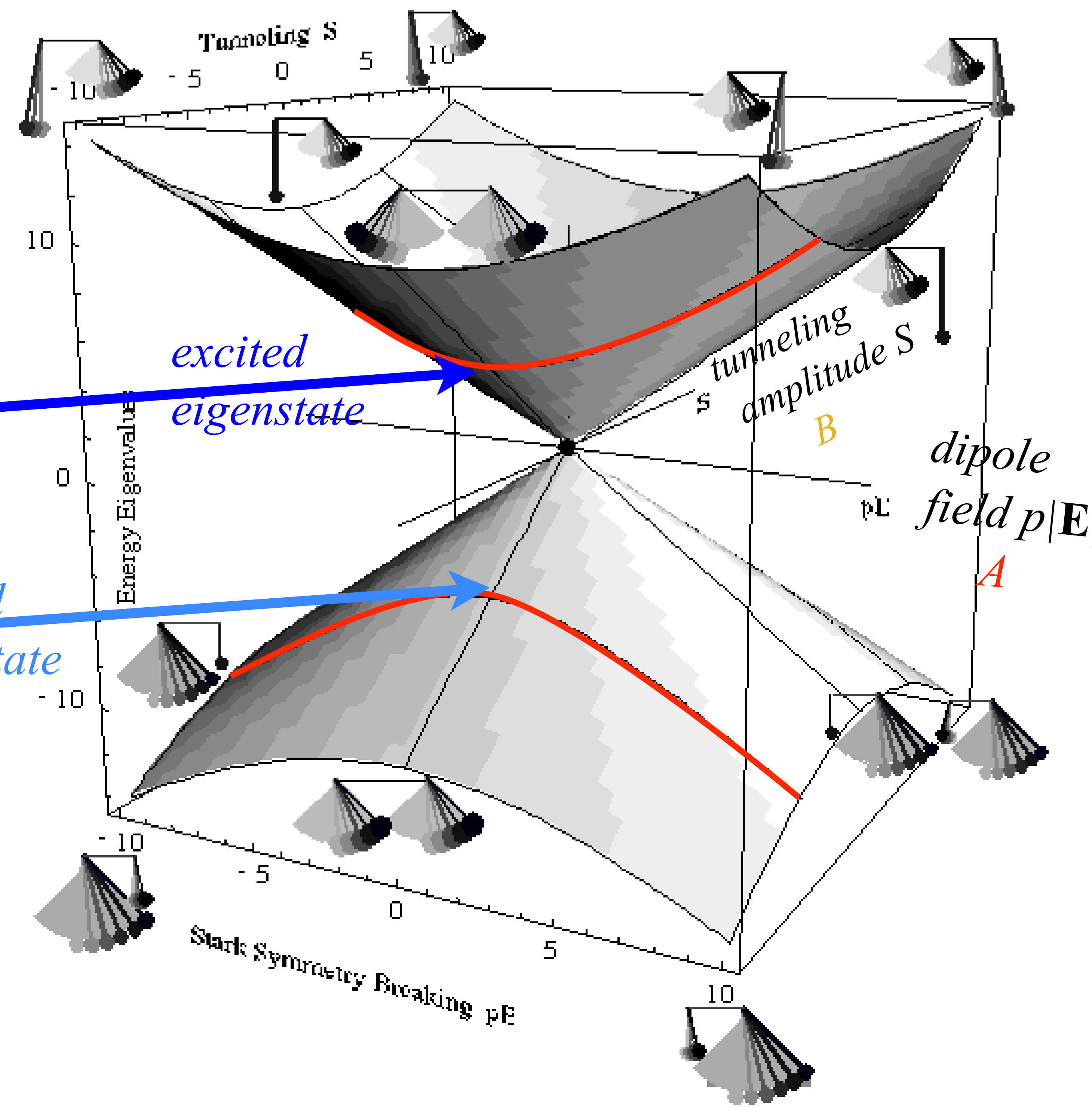

10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

(Also known as a “Dirac-point”)

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

ABC-Type elliptical polarized motion

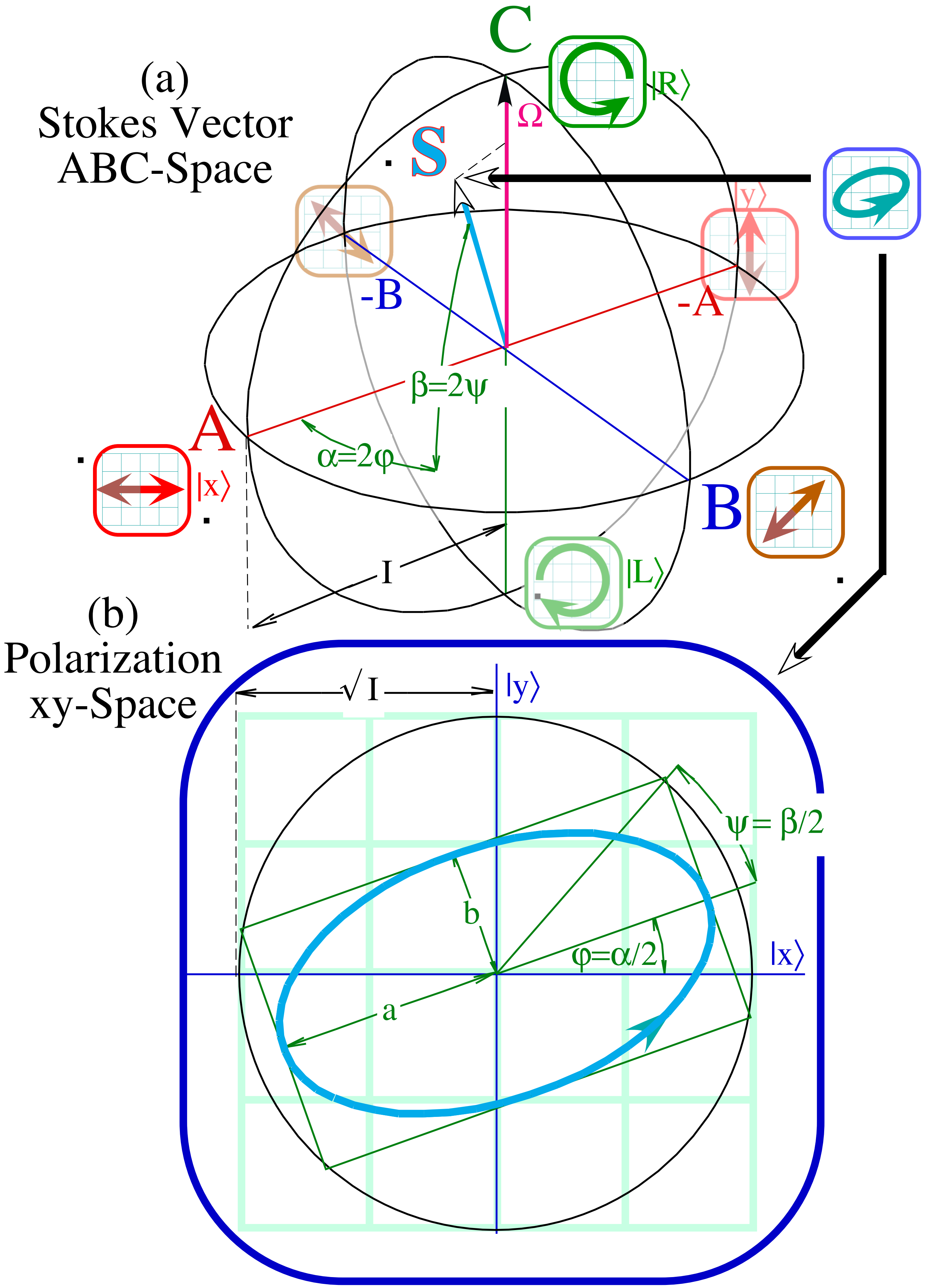


Fig. 10.B.3

Euler-like coordinates for
(a) R(3) spin vector
(b) U(2) polarization ellipse

ABC-Type elliptical polarized motion

(from Principles of Symmetry, Dynamics, and Spectroscopy)

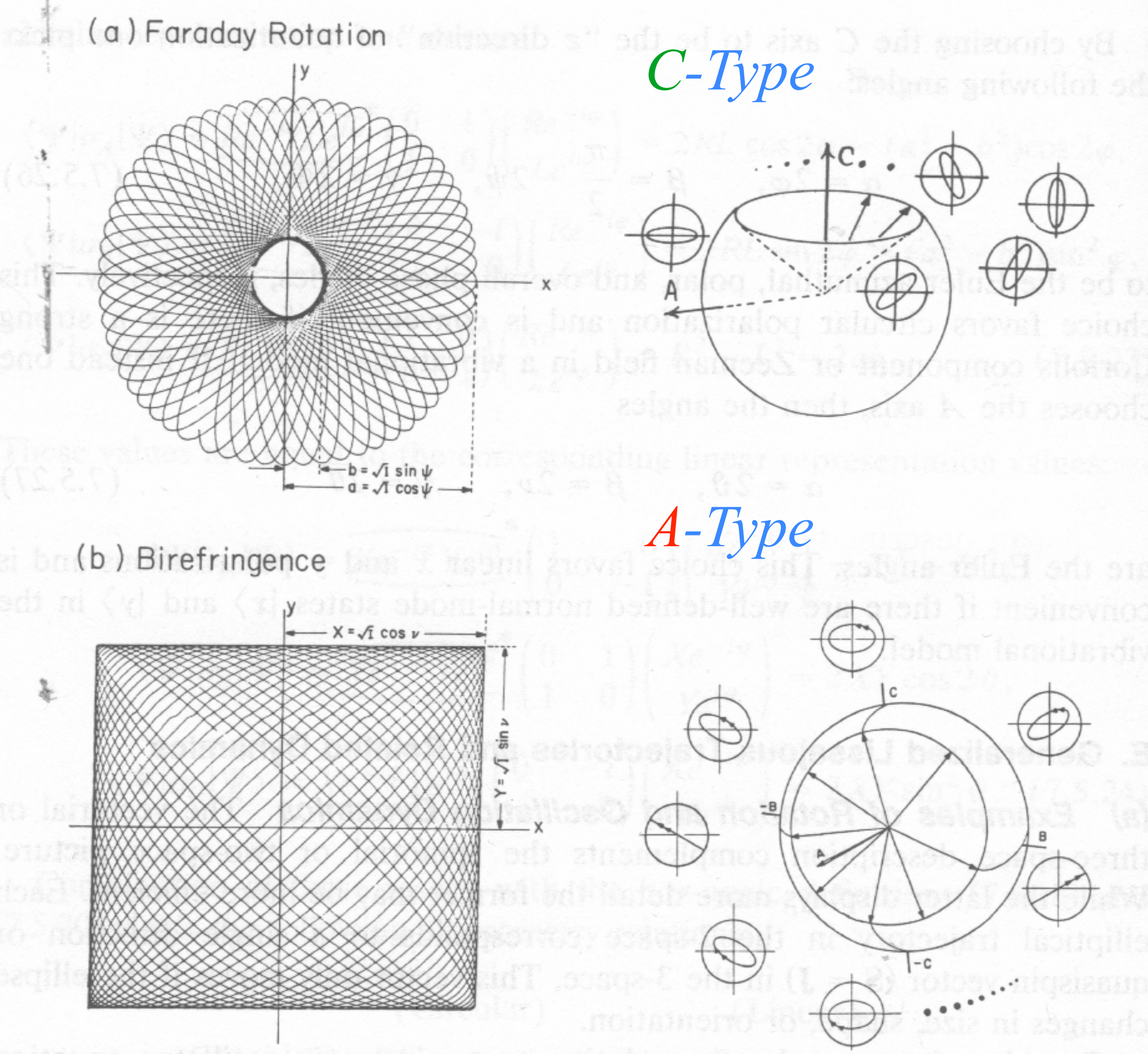
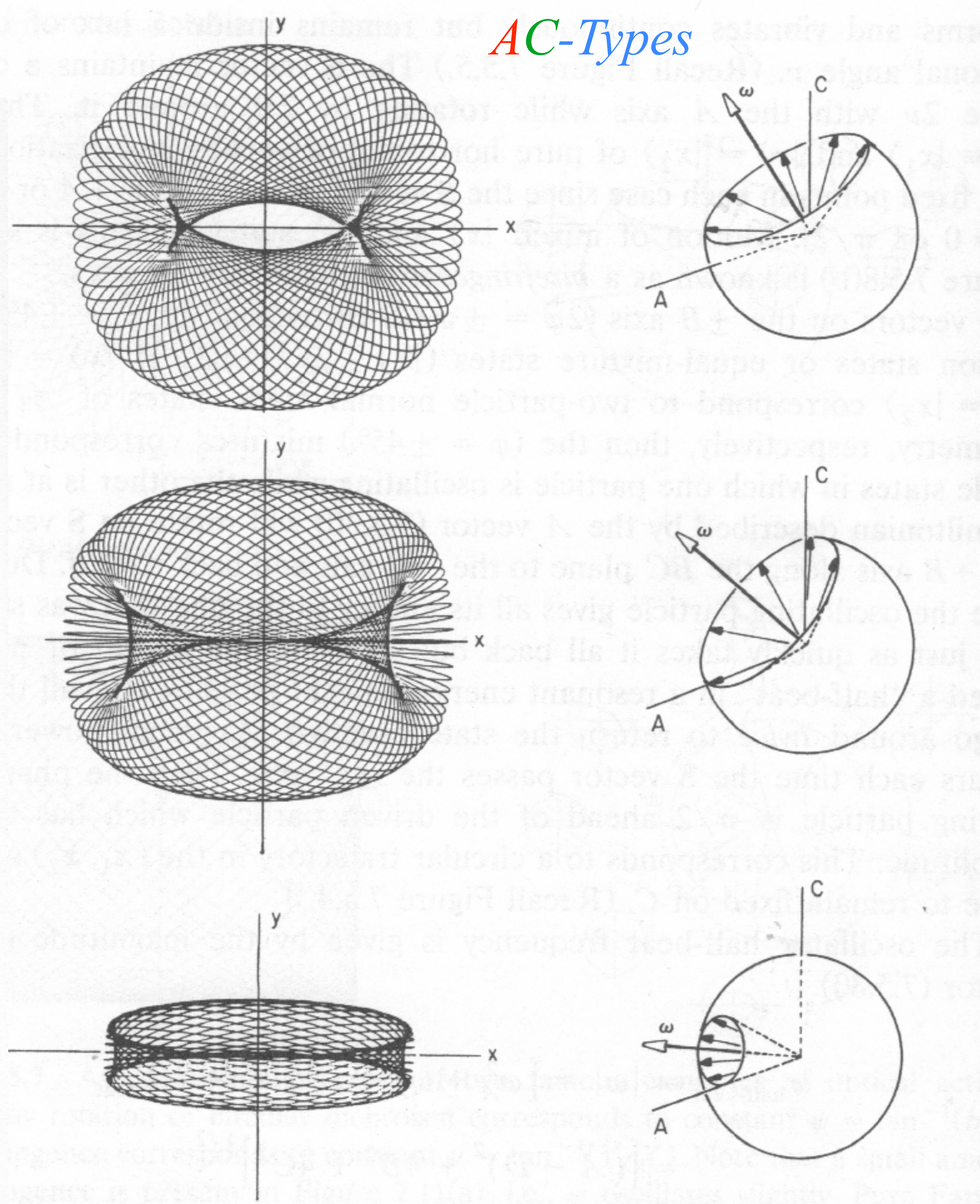
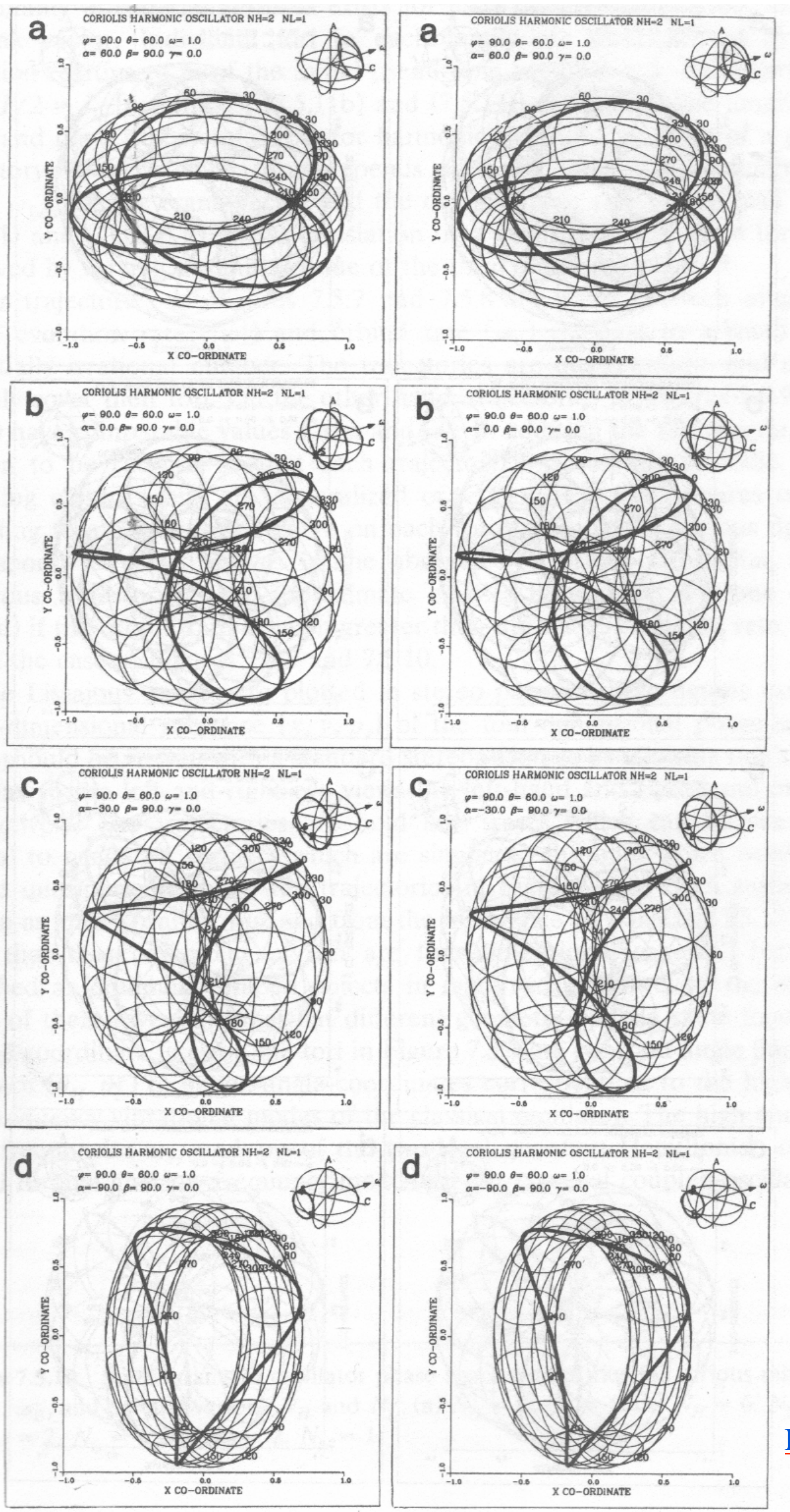


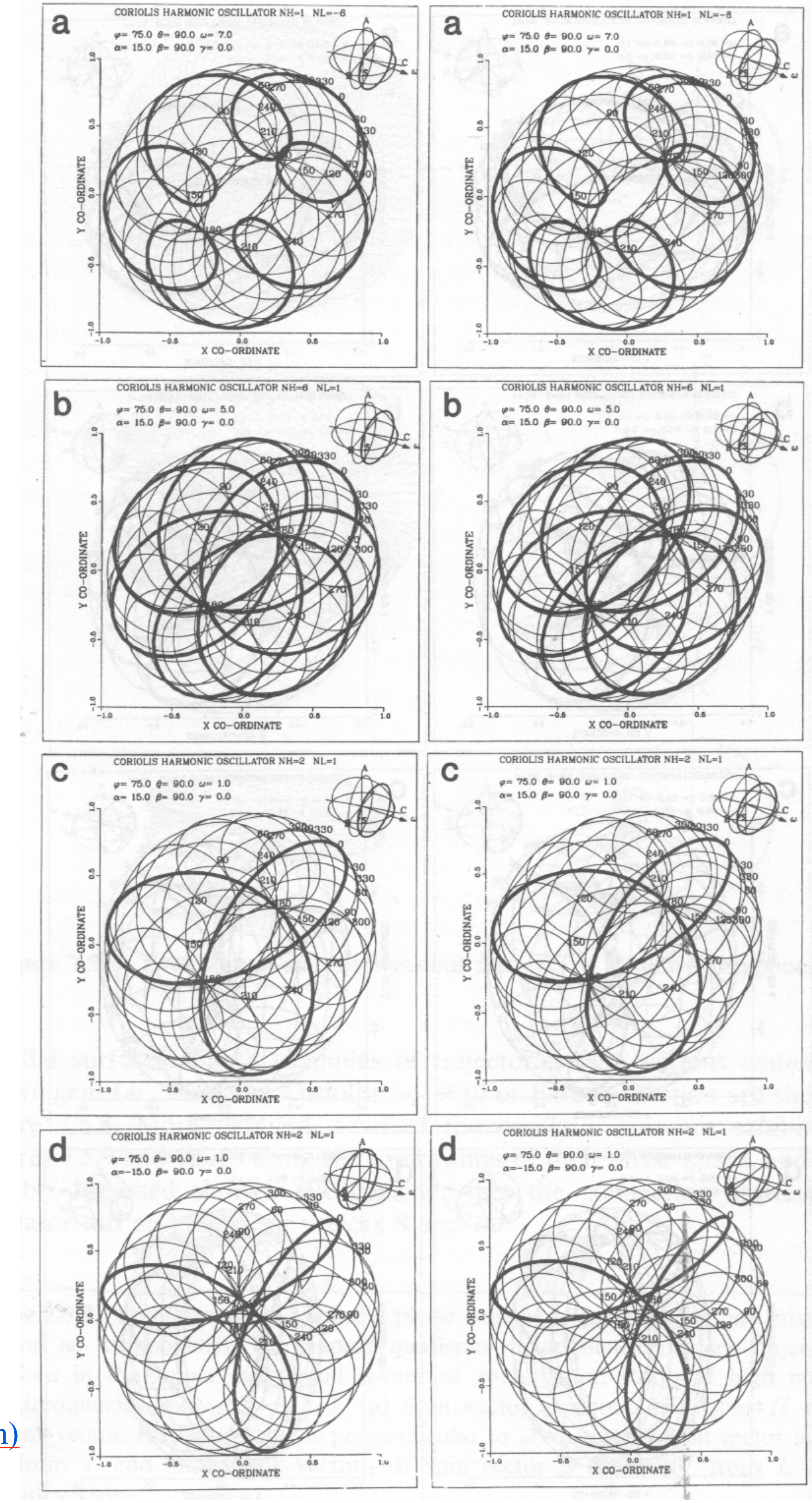
Figure 7.5.7 Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., ψ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.



7.5.8 Evolution of states for various mixtures of A and C components.

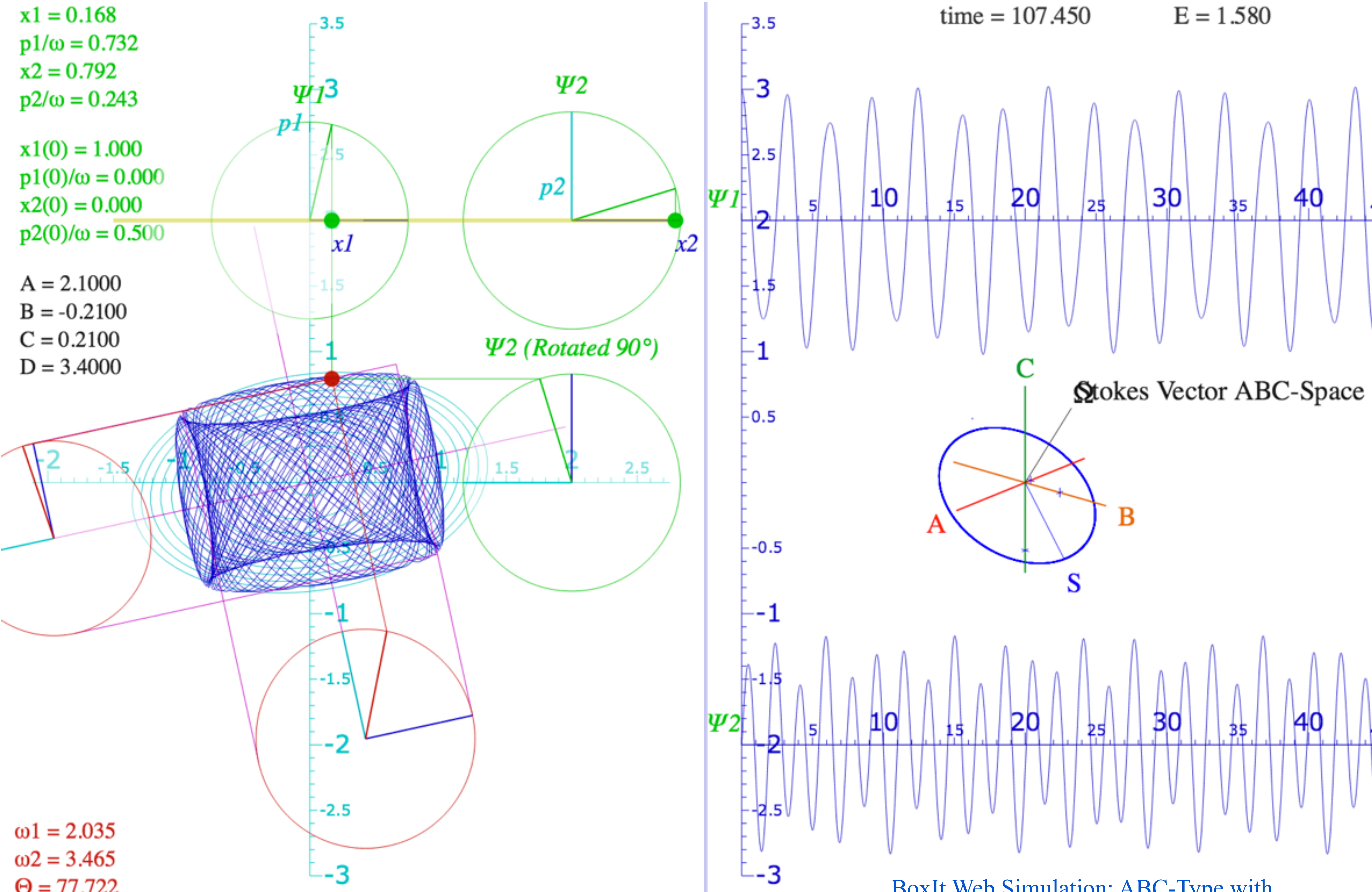


ABC-Type elliptical polarized dynamics



BoxIt (ABC-Motion)
Web Simulation

ABC-Type elliptical polarized motion



[BoxIt Web Simulation: ABC-Type with A=2.1; B=-0.21; C=0.21; D=3.4]

[Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot]

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Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates and related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

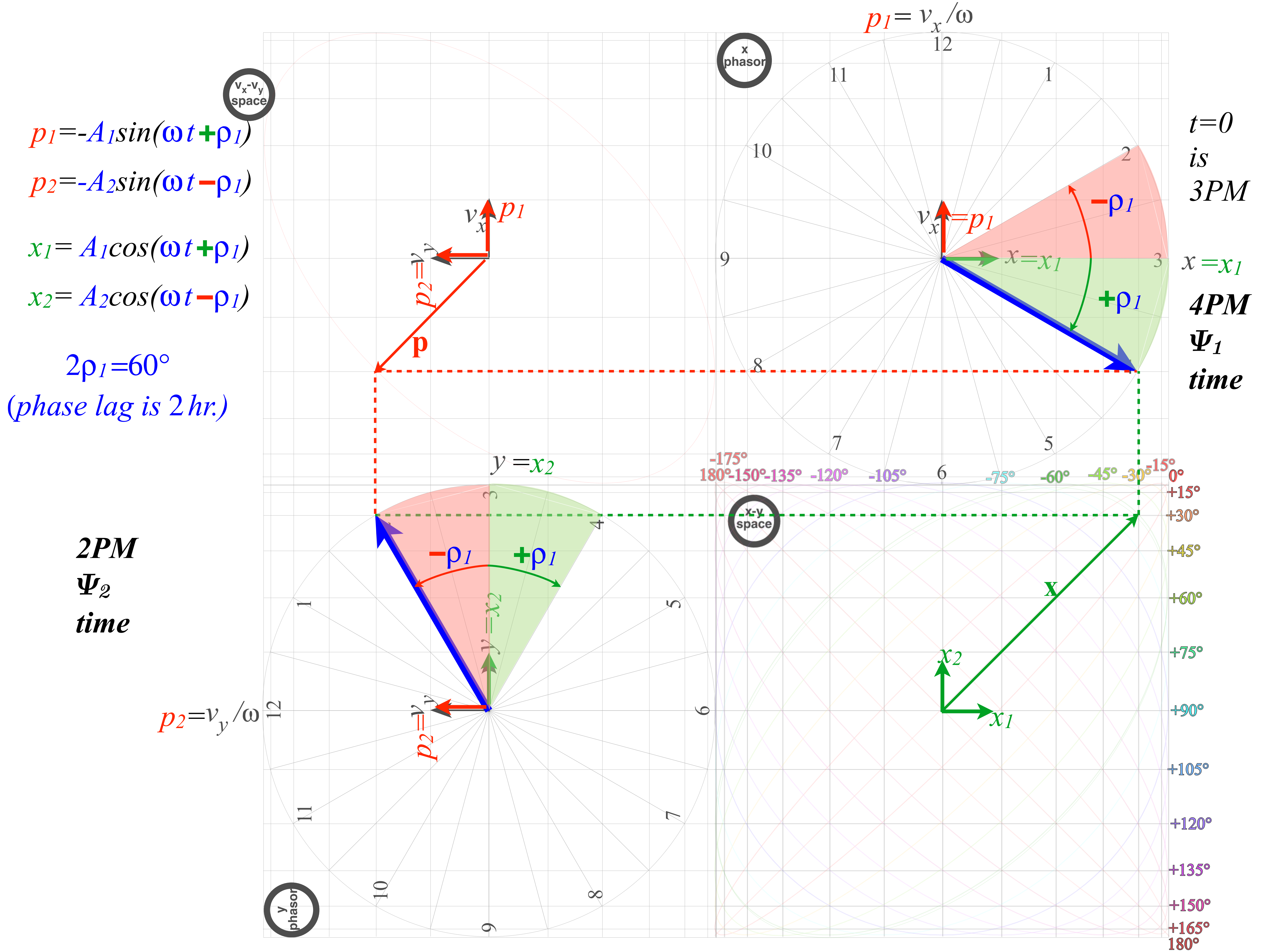
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

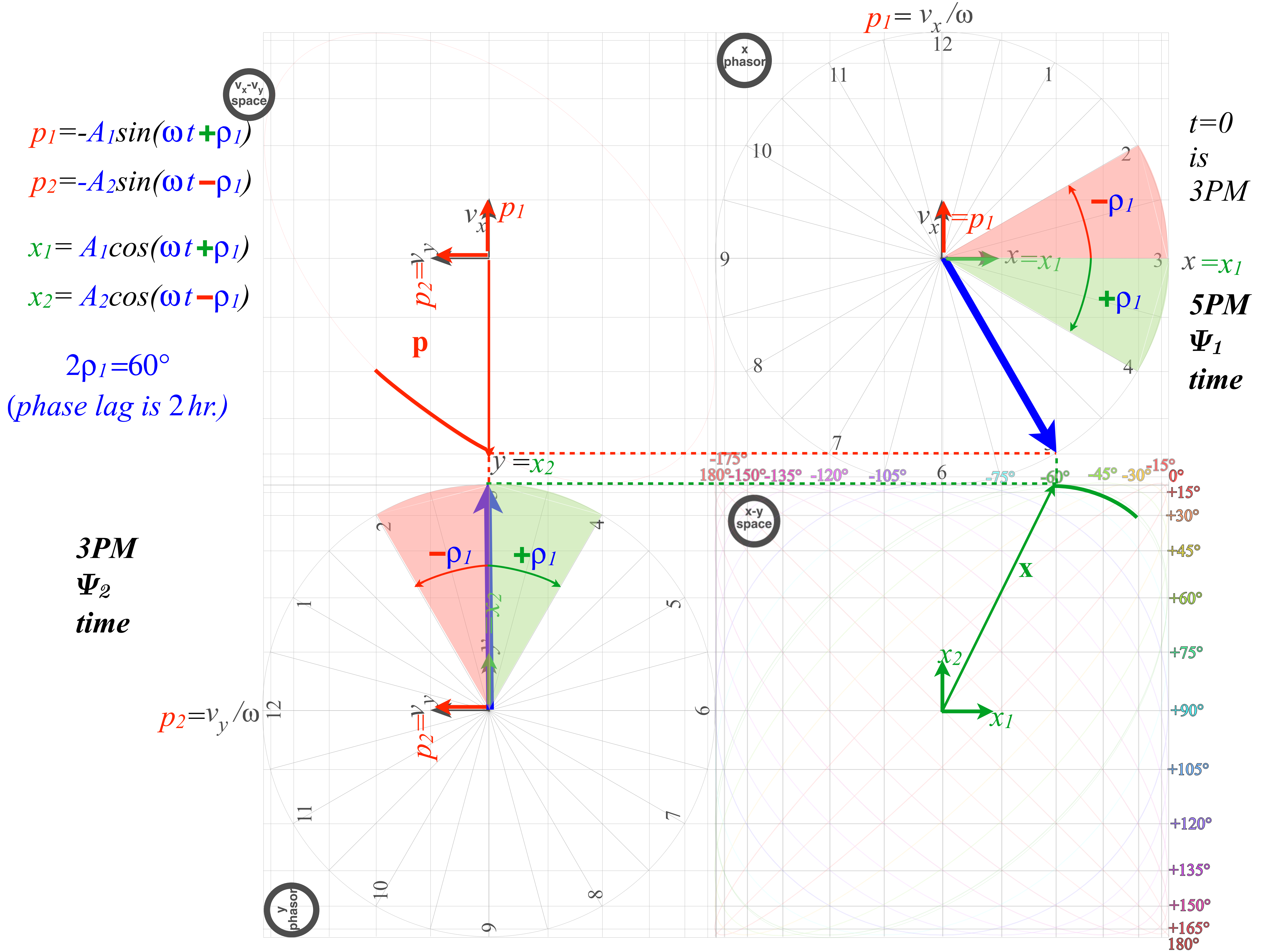
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Amp-phase parameters ($A_1, A_2, \omega t, \rho_1$)

$$\begin{pmatrix} A_1 e^{-i(\omega t + \rho_1)} \\ A_2 e^{-i(\omega t - \rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ \\ x_2 + ip_2 \end{pmatrix}$$



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx /ω
p2=vy /ω
t=0
is
3PM
4PM
Ψ1
time
2PM
Ψ2
time
−ρ1
+ρ1
x =x1
y =x2
vx=p1
vy=p2
x
p
x1
x2
12
1
2
3
4
5
6
7
8
9
10
11
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



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vx-vy space
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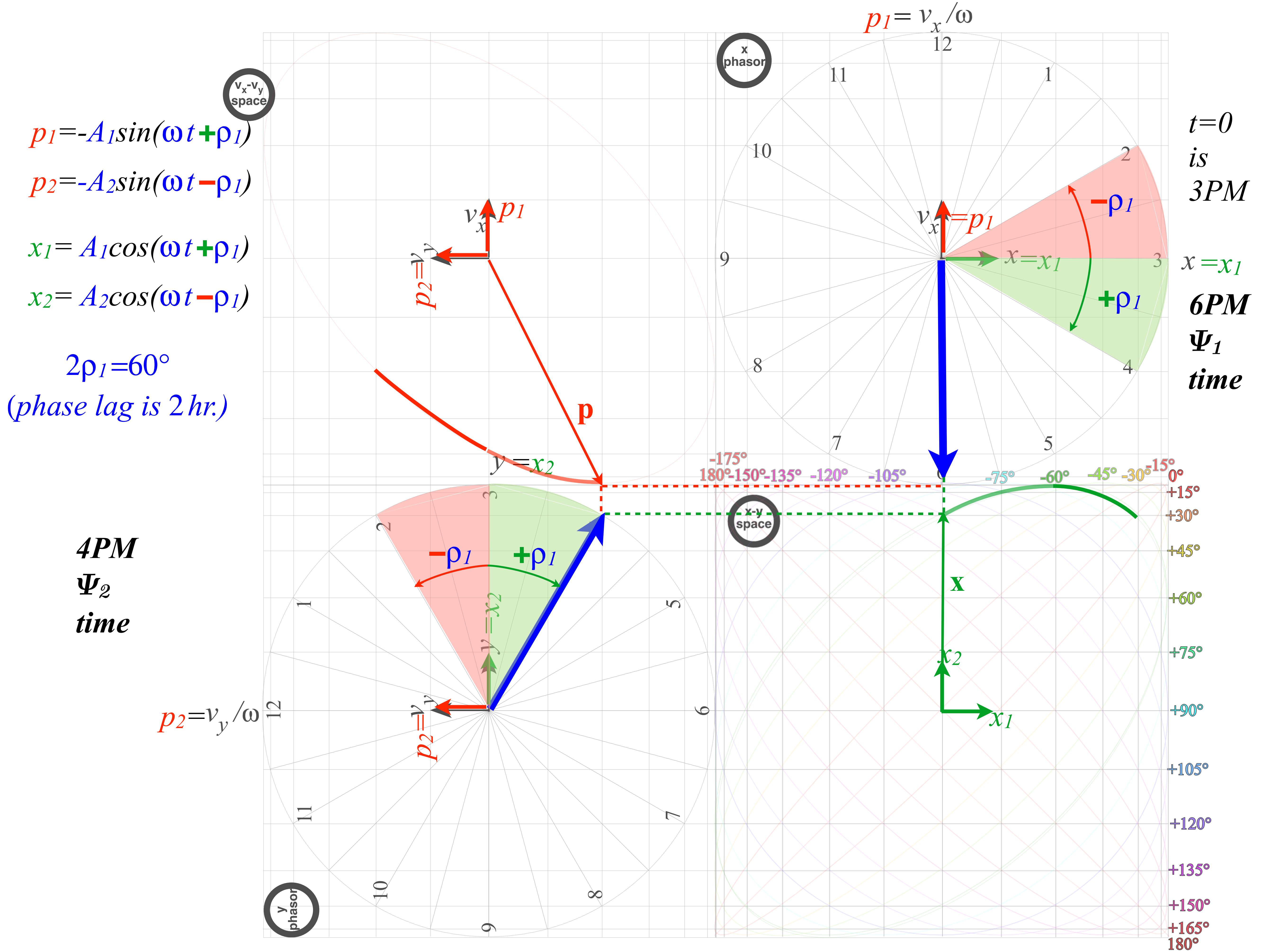
p2=vy/ω
t=0
is
3PM
5PM
Ψ1
time
3PM
Ψ2
time
−ρ1
+ρ1
x =x1
y =x2
p
x
x1
x2



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vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
6PM
Ψ1
time
4PM
Ψ2
time
vx=p1
x=x1
−ρ1
+ρ1
p
y=x2
x
x2
x1
-175°
180°-150°-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
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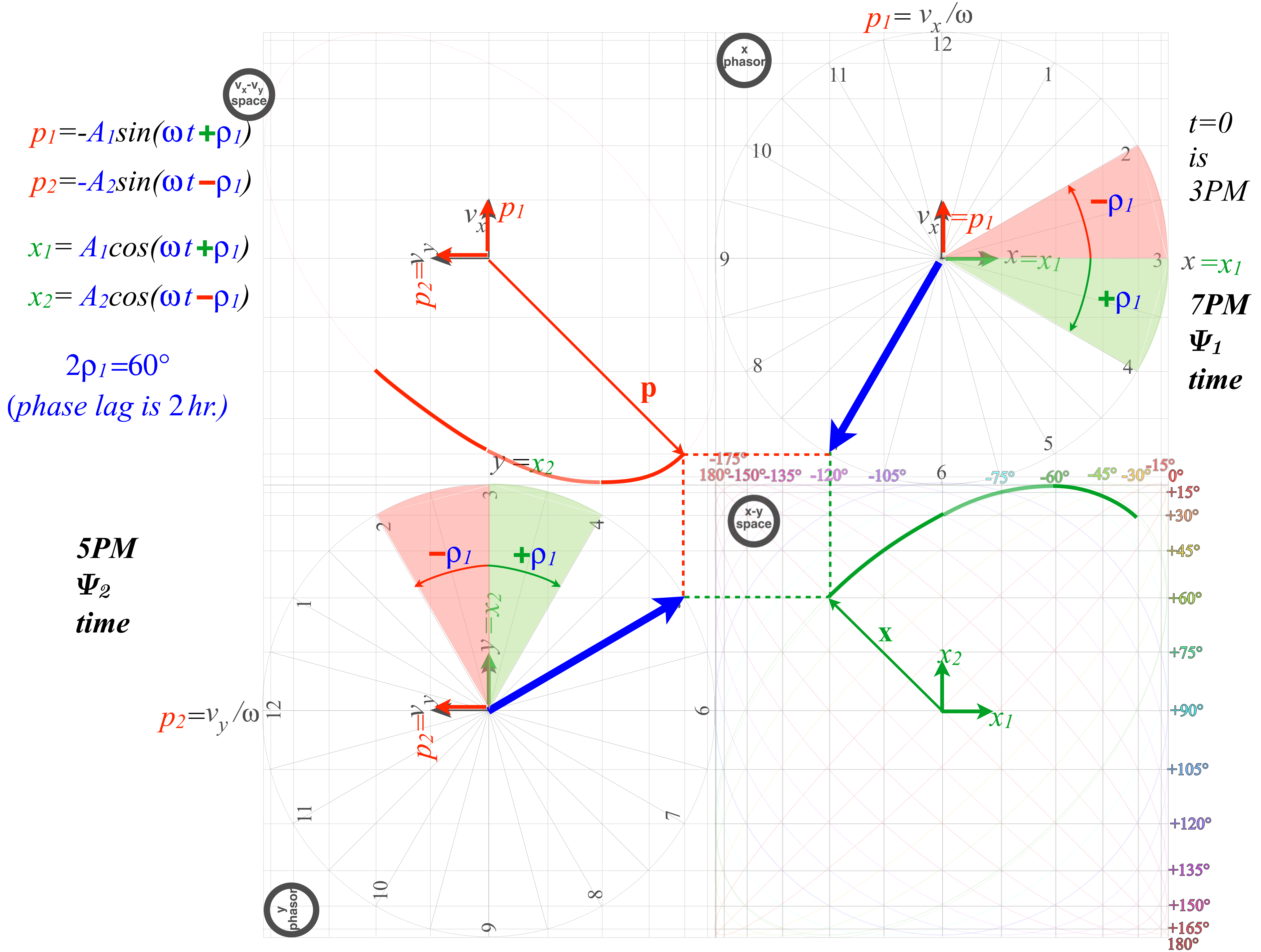
+150°
+165°
180°



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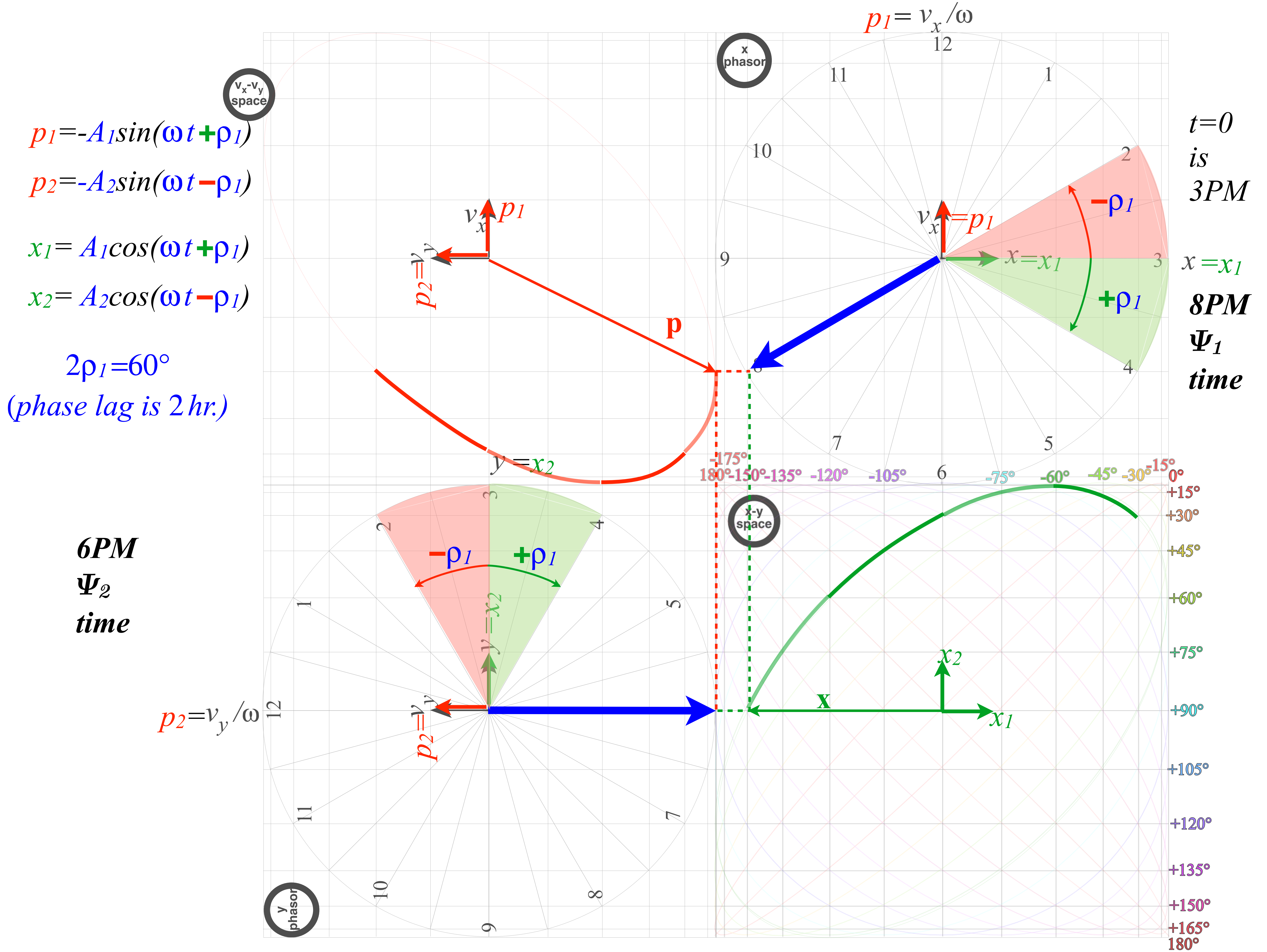
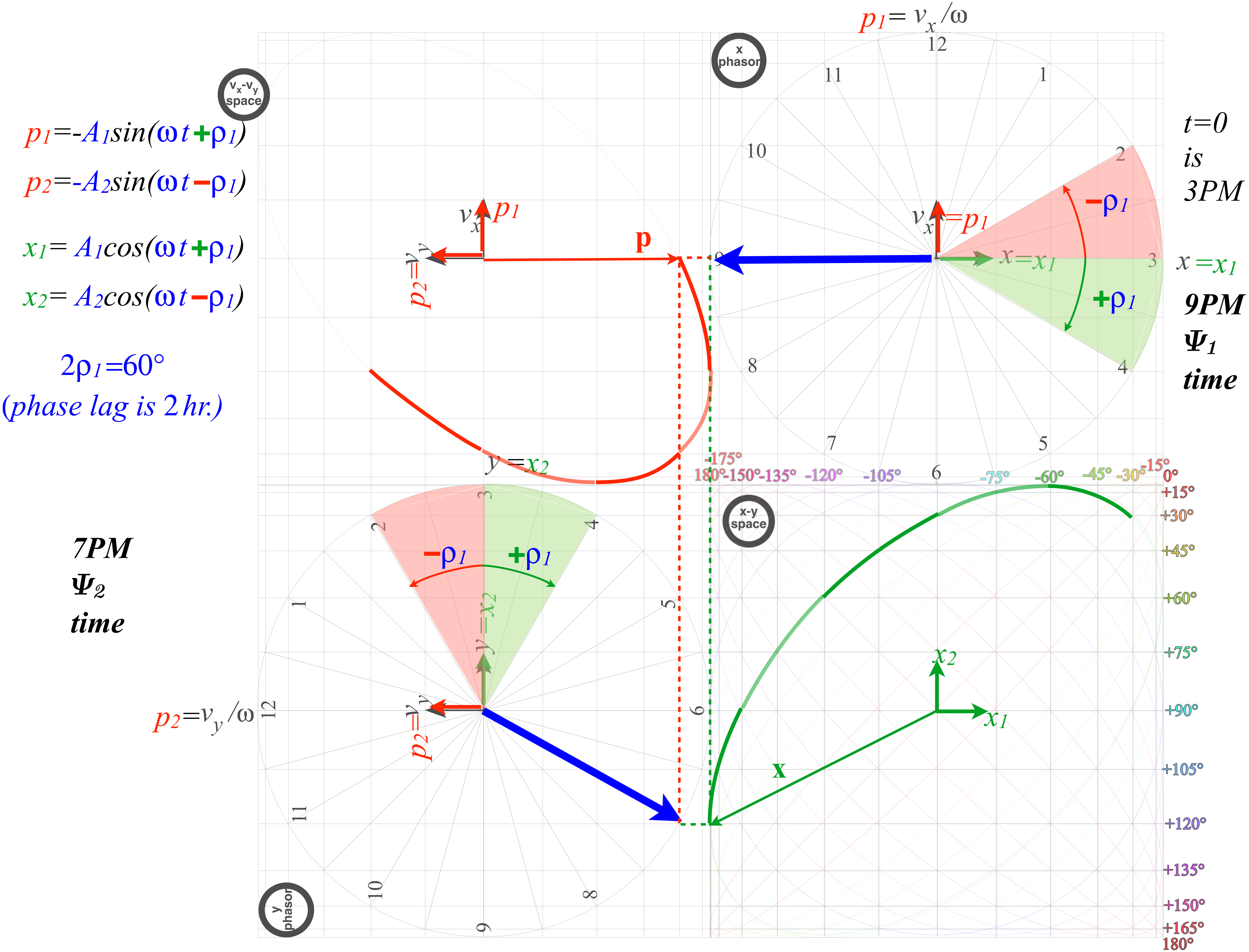
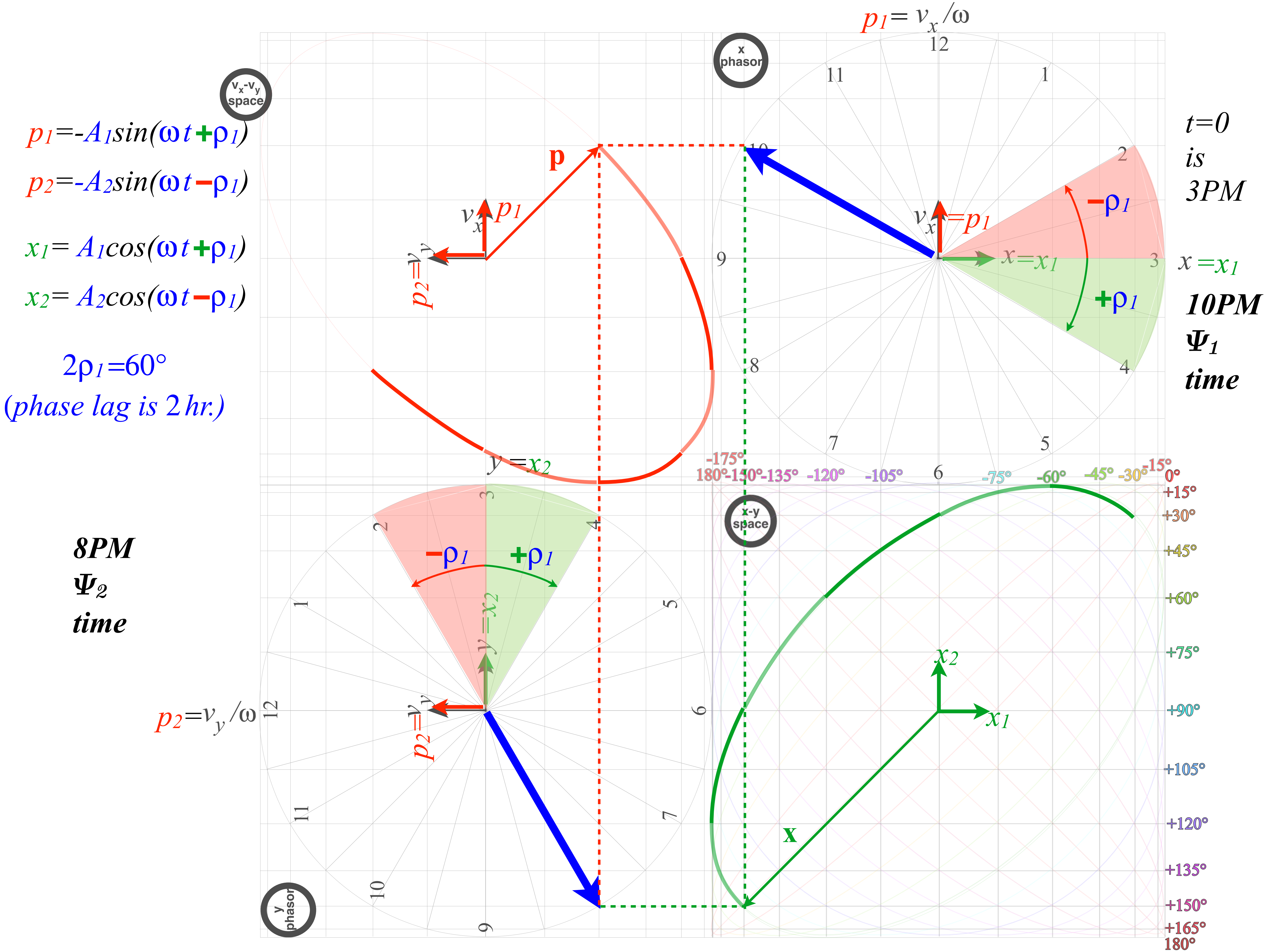
(phase lag is 2 hr.)
vx-vy space
x phasor
p1= vx/ω
t=0
is
3PM
7PM
Ψ1
time
5PM
Ψ2
time
x-y space
y phasor
p2=vy/ω
x=x1
y=x2
−ρ1
+ρ1
p
x
x1
x2



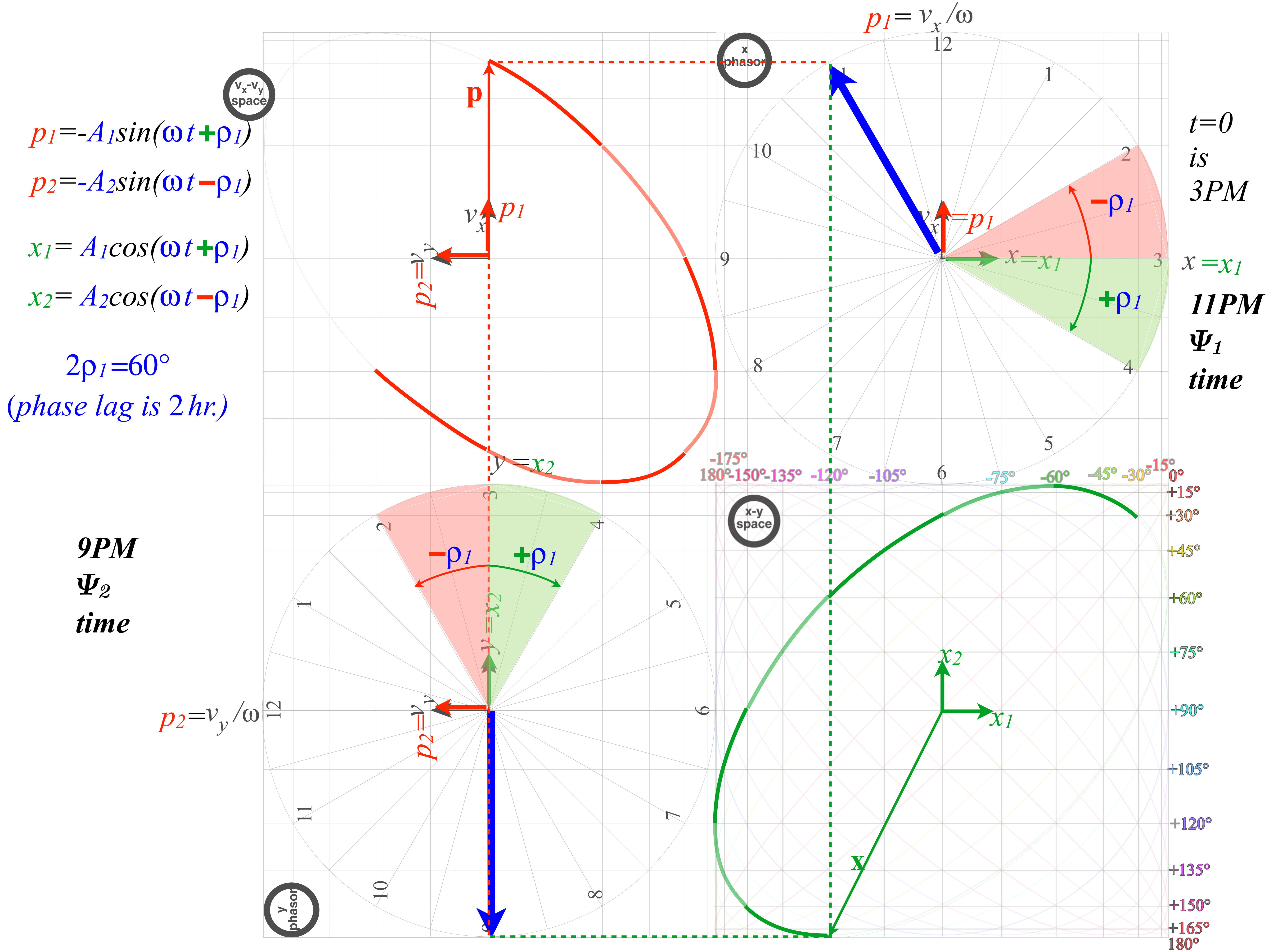
$p_1 = v_x/\omega$
x phasor
v_x-v_y space
$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2 hr.)
$t=0$ is 3PM
$x = x_1$
8PM
Ψ_1
time
$-\rho_1$
$+\rho_1$
$v_x = p_1$
$p_2 = v_y$
p
$y = x_2$
6PM
Ψ_2
time
$p_2 = v_y/\omega$
x-y space
y phasor
x
x_1
x_2
-175° 180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°



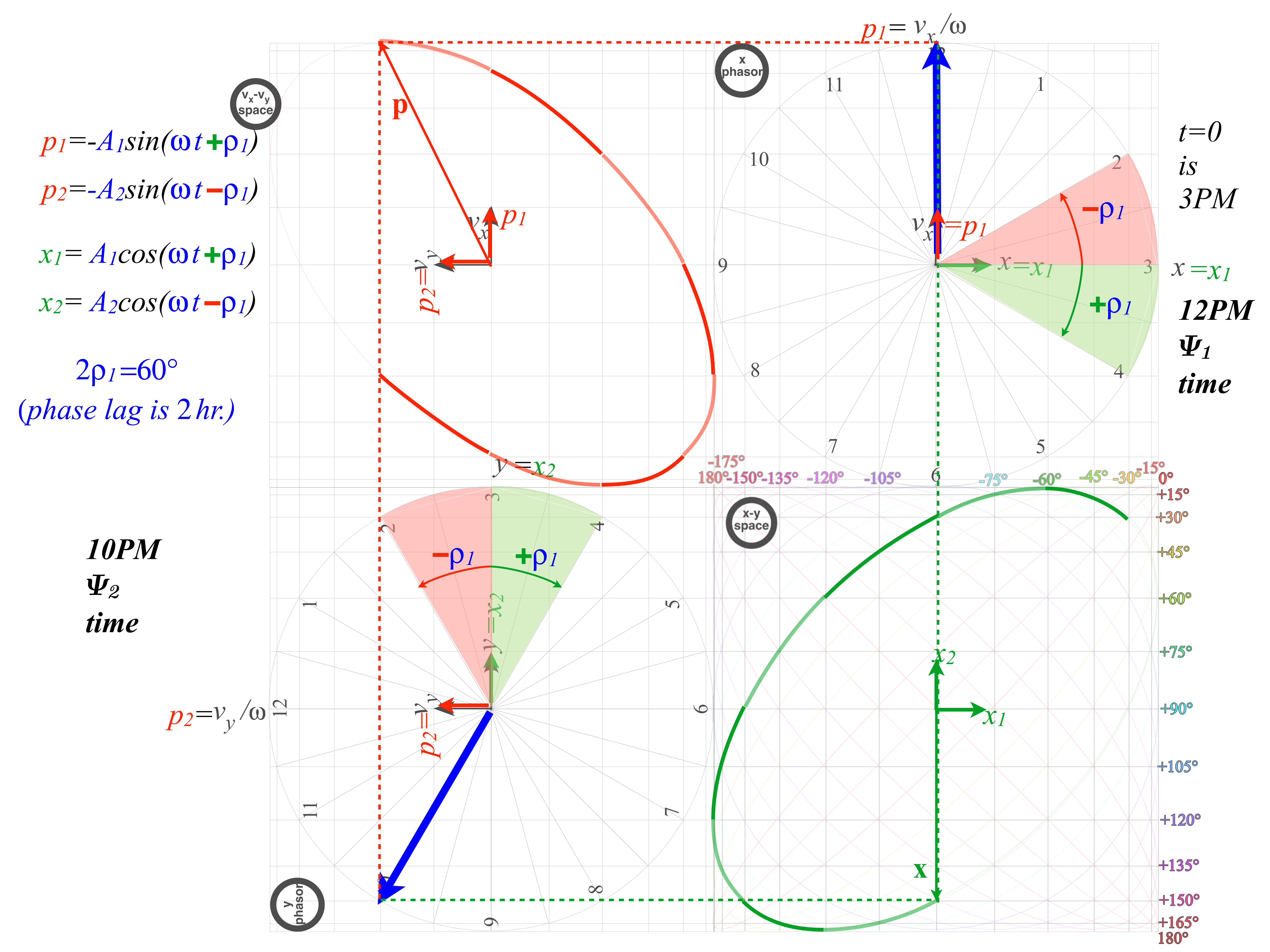
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0 is 3PM
9PM Ψ1 time
7PM Ψ2 time
vx=p1
p2=vy
x=x1
y=x2
p
x
−ρ1
+ρ1



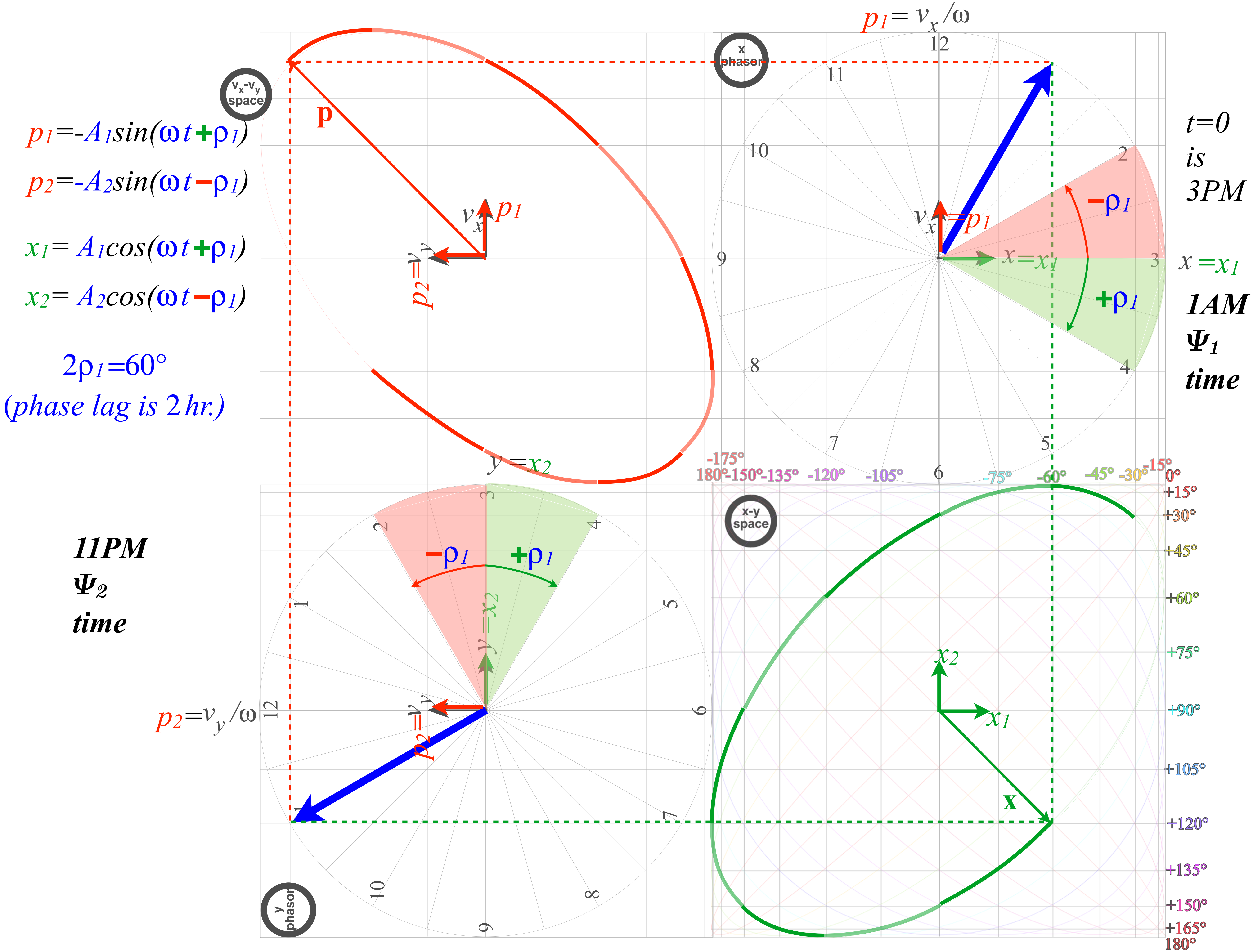
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
x-y space
y phasor
p1= vx/ω
p2=vy/ω
t=0
is
3PM
10PM
Ψ1
time
8PM
Ψ2
time
p
x
−ρ1
+ρ1
x =x1
y =x2



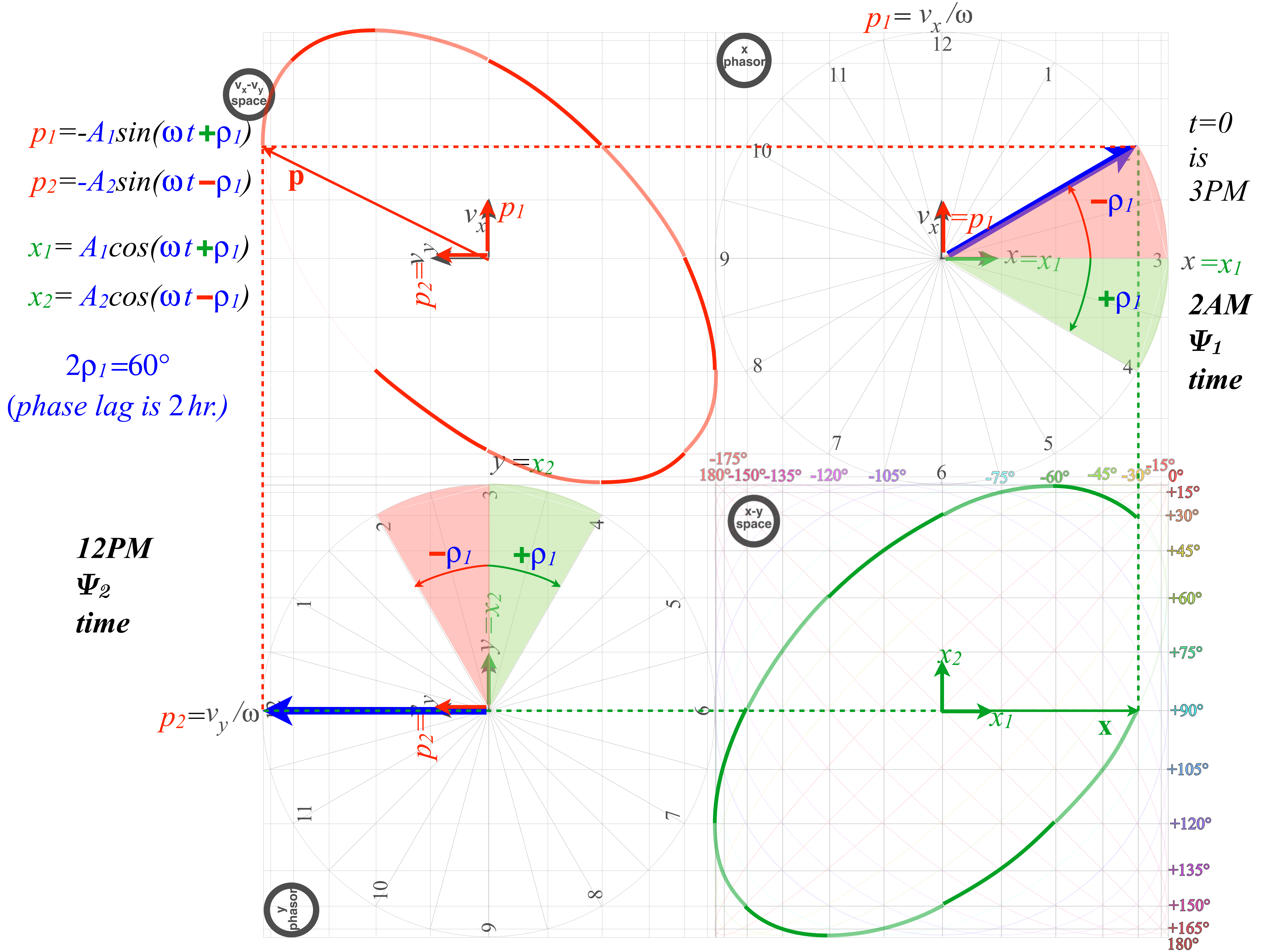
p_1=-A_1sin(ωt+ρ$_1$)
p_2=-A_2sin(ωt−ρ$_1$)
x_1= A_1cos(ωt+ρ$_1$)
x_2= A_2cos(ωt−ρ$_1$)
2ρ$_1$=60°
(phase lag is 2 hr.)
9PM
Ψ$_2$
time
t=0
is
3PM
11PM
Ψ$_1$
time
v_x-v_y space
x phasor
y phasor
x-y space
p_1= v_x/ω
p_2=v_y/ω
−ρ$_1$
+ρ$_1$
x =x_1
y=x_2



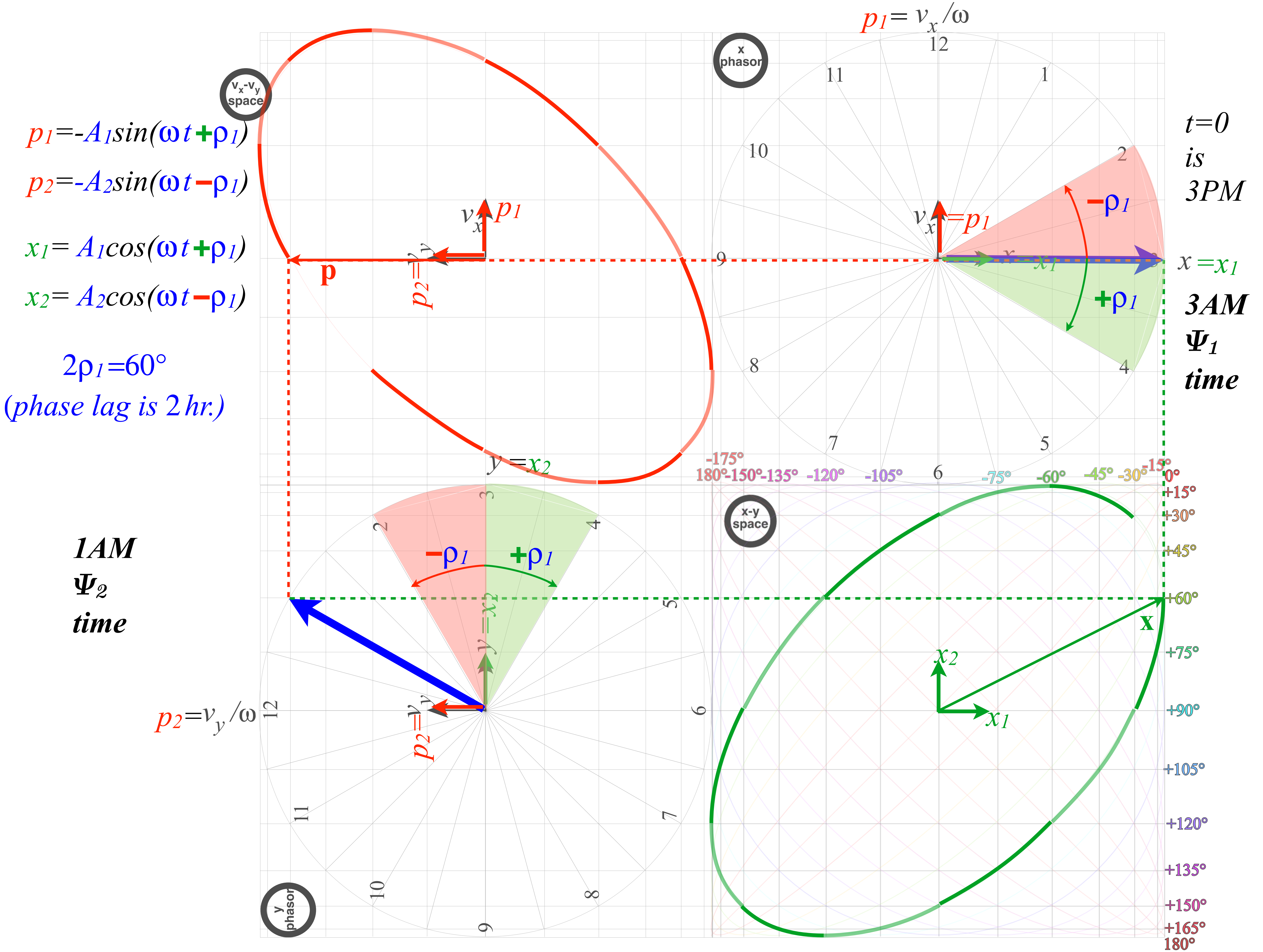
$p_1=-A_1\sin(\omega t+\rho_1)$
$p_2=-A_2\sin(\omega t-\rho_1)$
$x_1= A_1\cos(\omega t+\rho_1)$
$x_2= A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2 hr.)
v_x-v_y space
x phasor
y phasor
x-y space
$p_1= v_x/\omega$
$p_2=v_y/\omega$
p
$v_x=p_1$
$p_2=v_y$
$x=x_1$
$y=x_2$
$-\rho_1$
$+\rho_1$
$t=0$ is 3PM
12PM Ψ_1 time
10PM Ψ_2 time
x_1
x_2
x



$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2 hr.)
v_x-v_y space
p
x phasor
$p_1 = v_x/\omega$
t=0 is 3PM
$x = x_1$
1AM Ψ_1 time
$-\rho_1$
$+\rho_1$
11PM Ψ_2 time
$p_2 = v_y/\omega$
$y = x_2$
y phasor
x-y space
x_1
x_2
x



p1= vx/ω
x phasor
vx-vy space
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
p
vx p1
p2=vy
vx=p1
x=x1
−ρ1
+ρ1
t=0
is
3PM
2AM
Ψ1
time
y=x2
12PM
Ψ2
time
x-y space
y phasor
p2=vy/ω
x2
x1
x
-175°
180°-150°-135° -120° -105° -75° -60° -45° -30° -15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



p1=-A1sin(ωt+ρ1)
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x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
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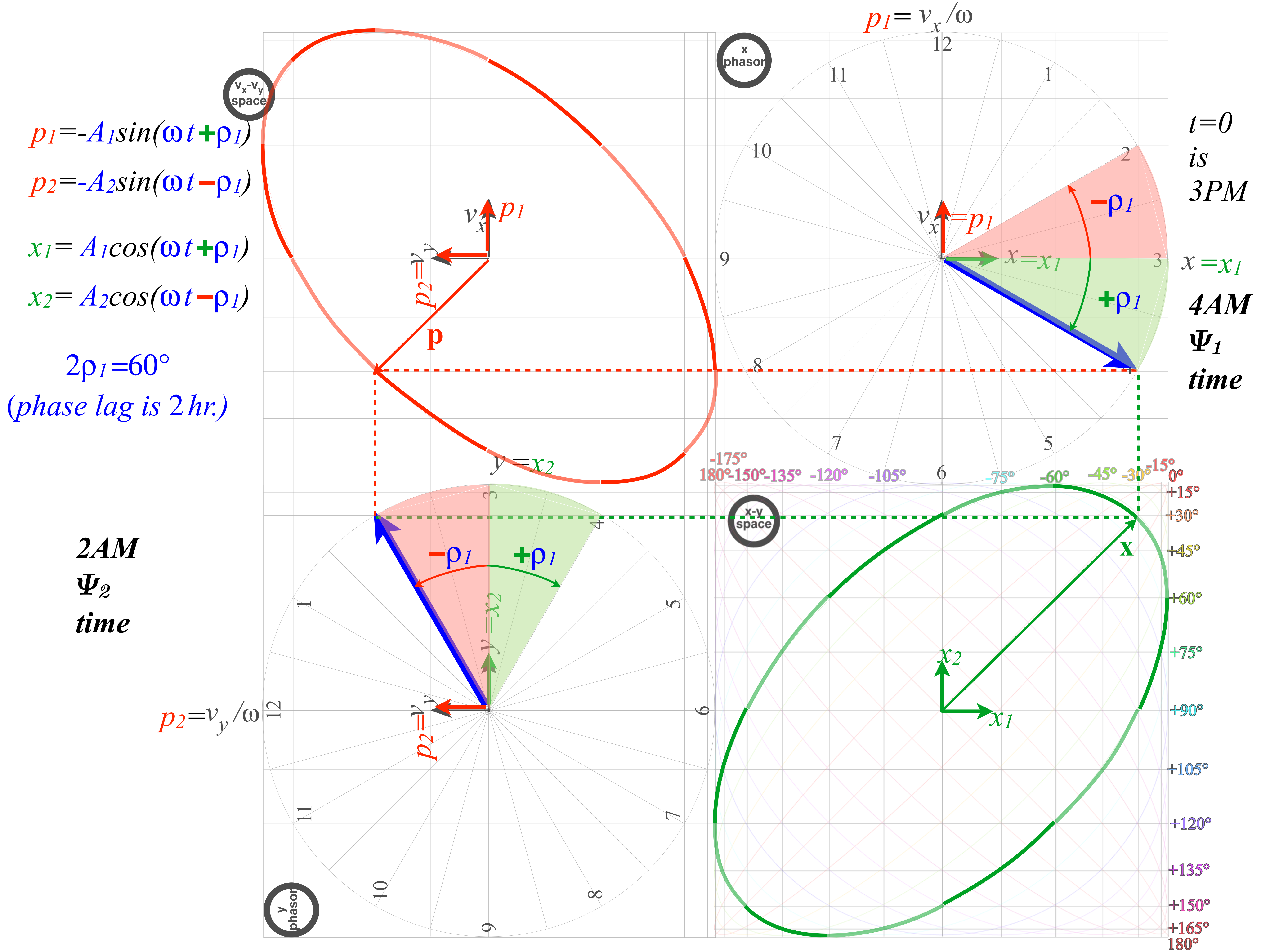
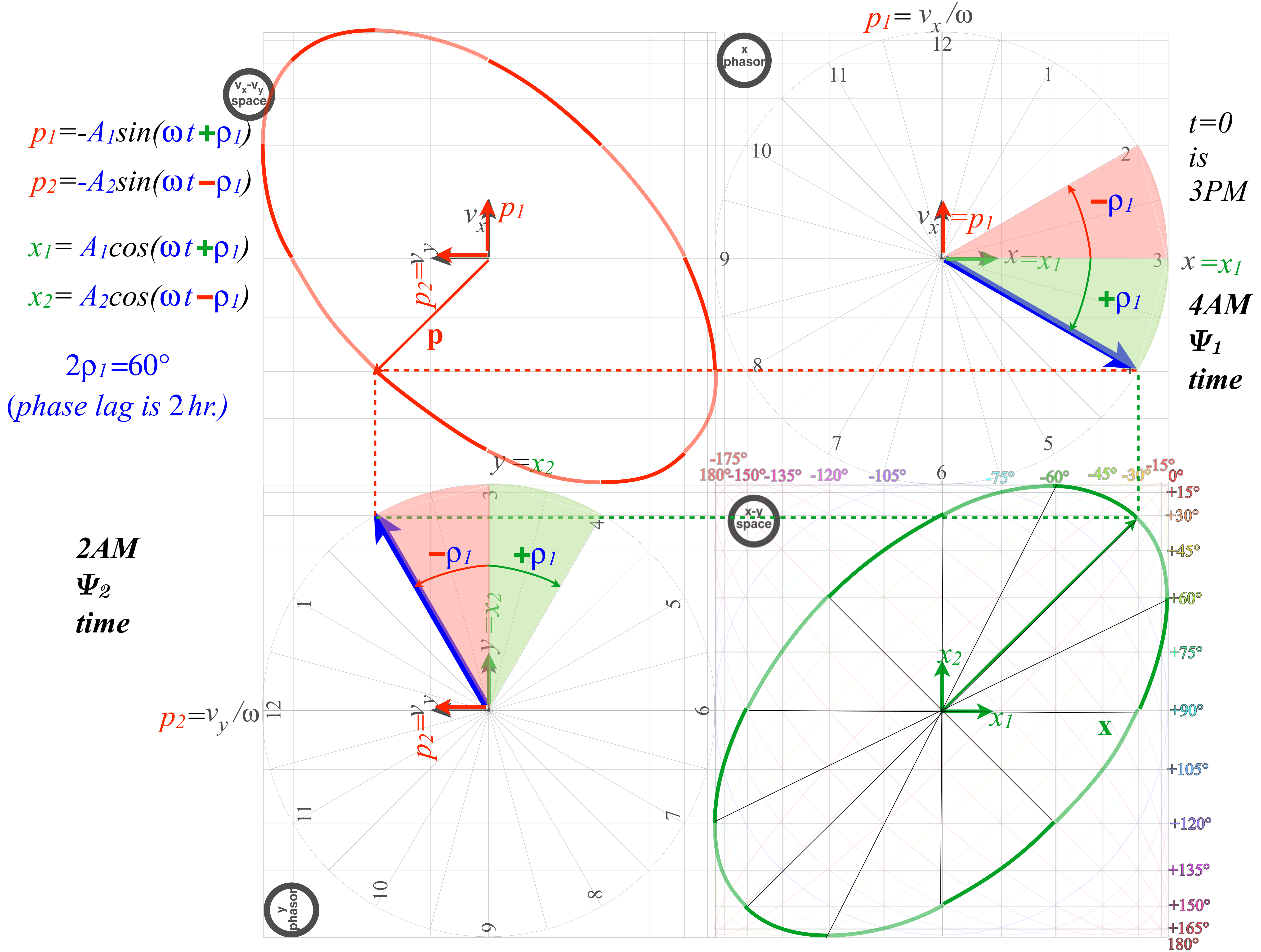
vx-vy space
x phasor
p1= vx/ω
t=0
is
3PM
−ρ1
+ρ1
x =x1
3AM
Ψ1
time
1AM
Ψ2
time
p2=vy/ω
y phasor
x-y space
y =x2
x2
x1
x
p



$p_1 = v_x/\omega$
x phasor
v_x-v_y space
$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2 hr.)
$t=0$ is 3PM
$x = x_1$
4AM
Ψ_1
time
$-\rho_1$
$+\rho_1$
$v_x = p_1$
$v_y = p_2$
p
$y = x_2$
2AM
Ψ_2
time
$p_2 = v_y/\omega$
x-y space
y phasor
x
x_1
x_2



$p_1 = v_x/\omega$
vx-vy space
x phasor
$p_1=-A_1\sin(\omega t+\rho_1)$
$p_2=-A_2\sin(\omega t-\rho_1)$
$x_1= A_1\cos(\omega t+\rho_1)$
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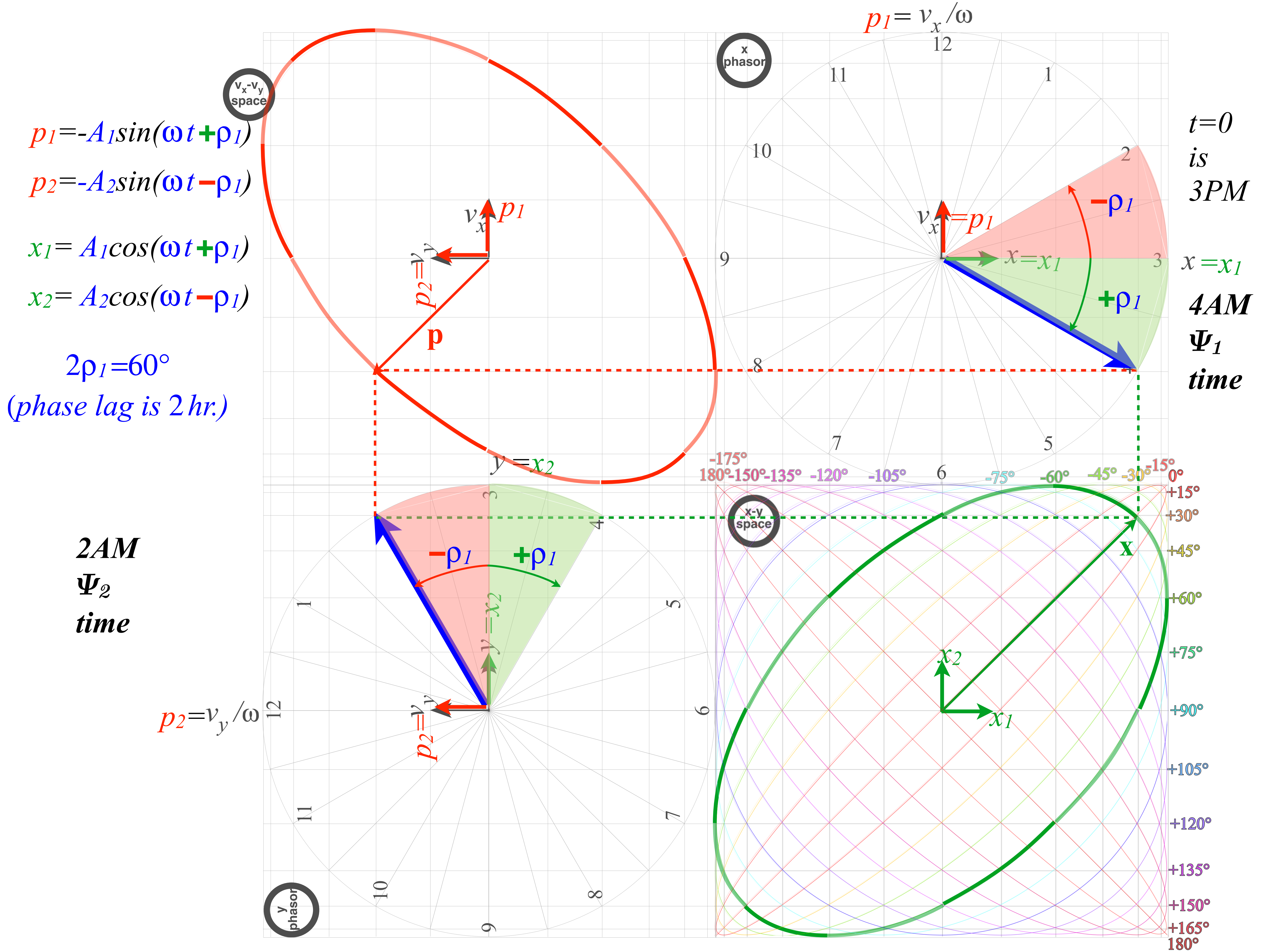
$2\rho_1=60°$
(phase lag is 2 hr.)
t=0 is 3PM
$x=x_1$
4AM
Ψ_1
time
$-\rho_1$
$+\rho_1$
$v_x=p_1$
$p_2=v_y$
p
$y=x_2$
2AM
Ψ_2
time
$p_2=v_y/\omega$
y phasor
x-y space
x_1
x_2
x



p1=-A1sin(ωt+ρ1)
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x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
4AM
Ψ1
time
2AM
Ψ2
time
x =x1
y =x2
−ρ1
+ρ1
p
x

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles ($\alpha\beta\gamma$) and A.

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

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Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$
$$x_2 = A_2 \cos(\omega t - \rho_1)$$
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Let: $A_1 = A\cos\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

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$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

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$$x_1 = A_1 \cos(\omega t + \rho_1)$$
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$A_2 = A\sin\beta/2$

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Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

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Let: $A_1 = A\cos\beta/2$

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$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

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Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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Let: $A_1 = A\cos\beta/2$

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Let: $\omega t + \rho_1 = (\gamma+\alpha)/2$

$\omega t - \rho_1 = (\gamma-\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$\tan\beta/2 = A_2/A_1 \qquad A^2=A_1^2+A_2^2$

$\alpha=2\rho_1 \qquad \gamma=2\omega\cdot t$

Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* ($A_1, A_2, \omega t, \rho_1$)

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

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Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

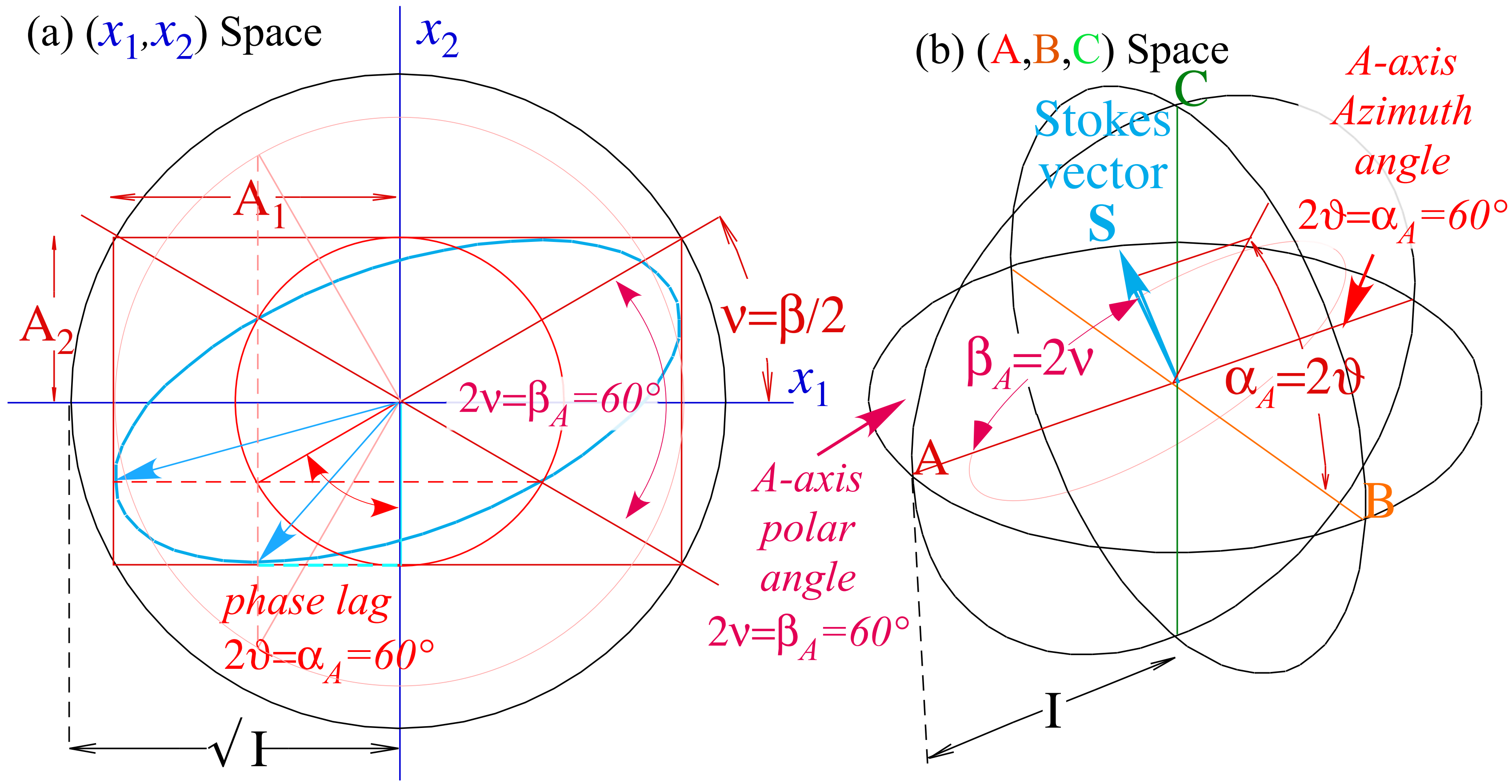
Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The A-view in {x₁,x₂}-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega\cdot t$
define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

$\alpha=\alpha_A=\rho_1-\rho_2=2\rho_1=60°$

$\beta=\beta_A=2\tan^{-1}A_2/A_1=60°$

$\gamma_A=2\omega\cdot t$

The A-view in {x₁,x₂}-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega\cdot t$ define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

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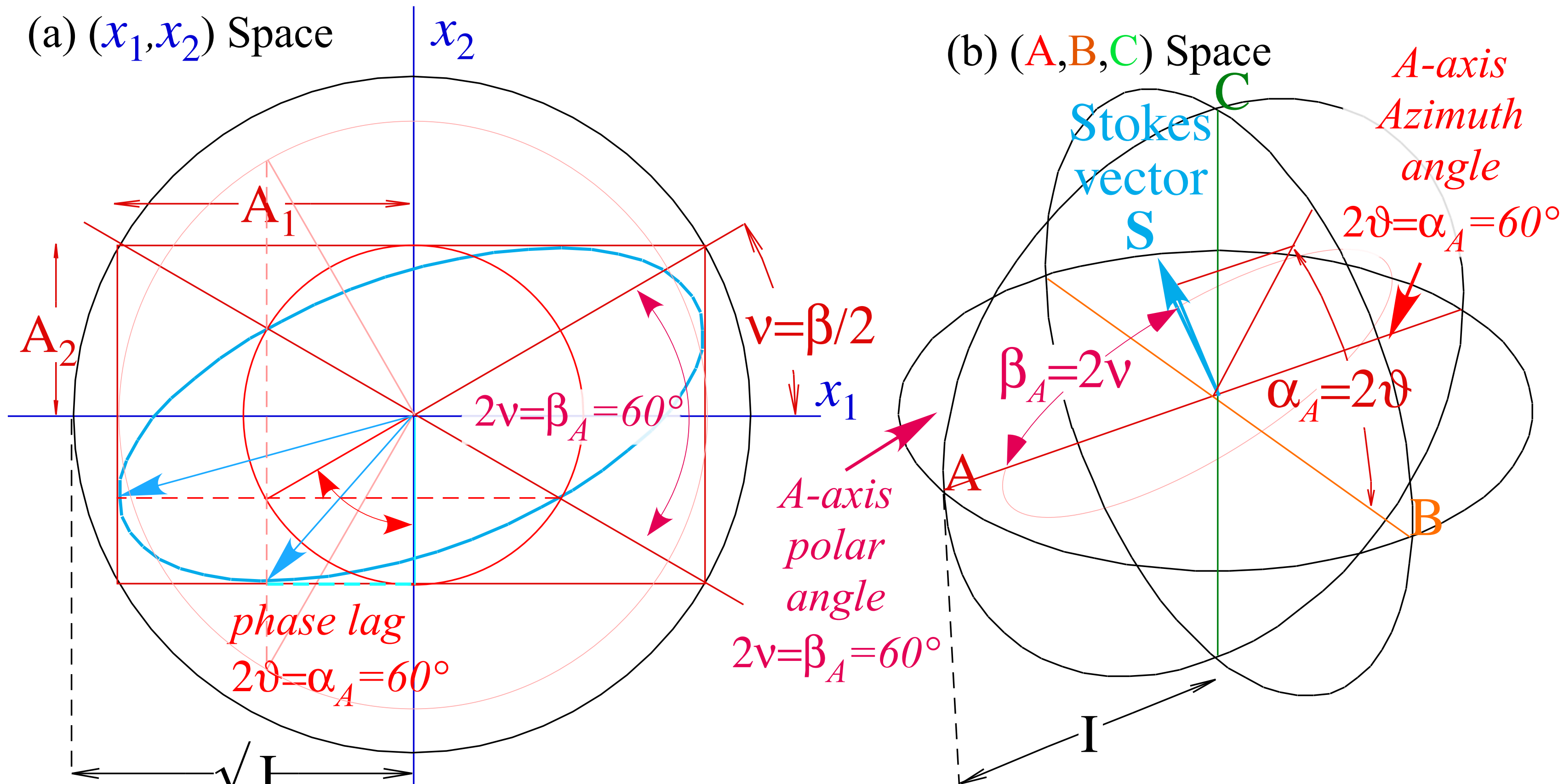


A or *Z*-axis Euler angles

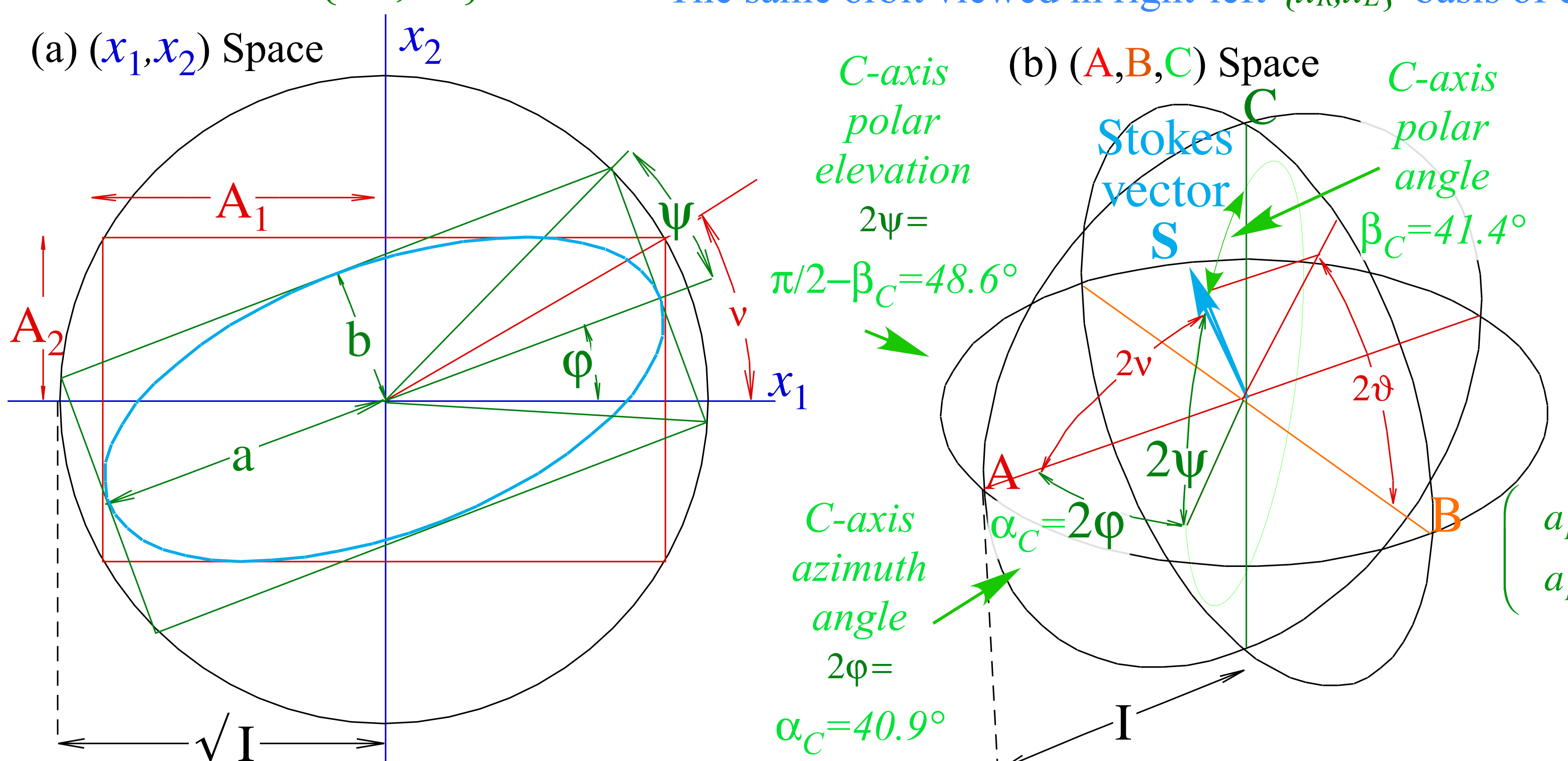
$\alpha = \alpha_A = \rho_1 - \rho_2 = 2\rho_1 = 60°$

$\beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega\cdot t$

The C-view in {x_R,x_L}-basis

The same orbit viewed in right-left $\{x_R, x_L\}$-basis of circular polarization with angles $(\alpha_C, \beta_C, \gamma_C)$.



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

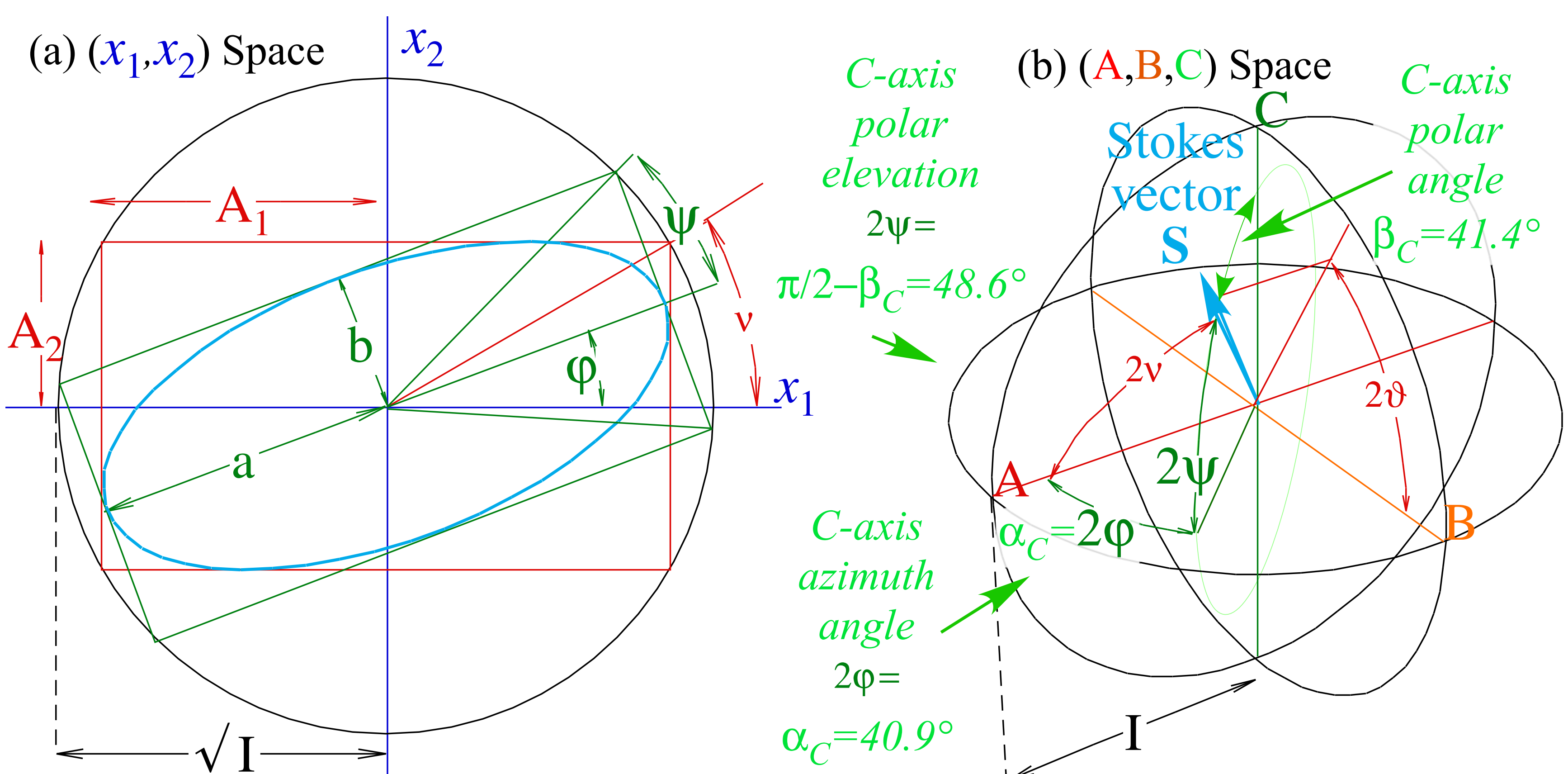
$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R,x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.



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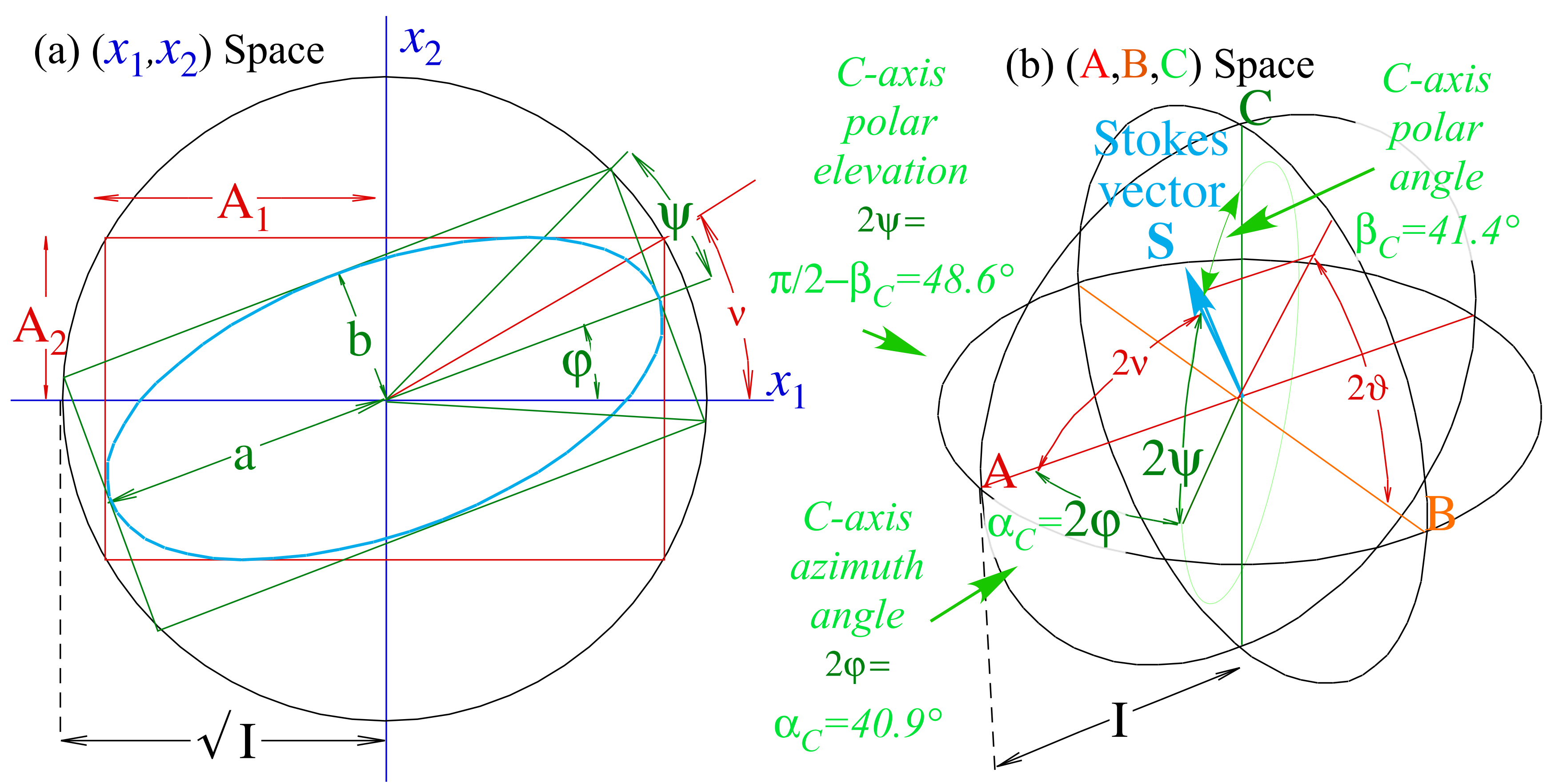
$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R,x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.

Angles (α_C,β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4^\circ$$



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$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C*-view in* $\{x_R, x_L\}$*-basis*

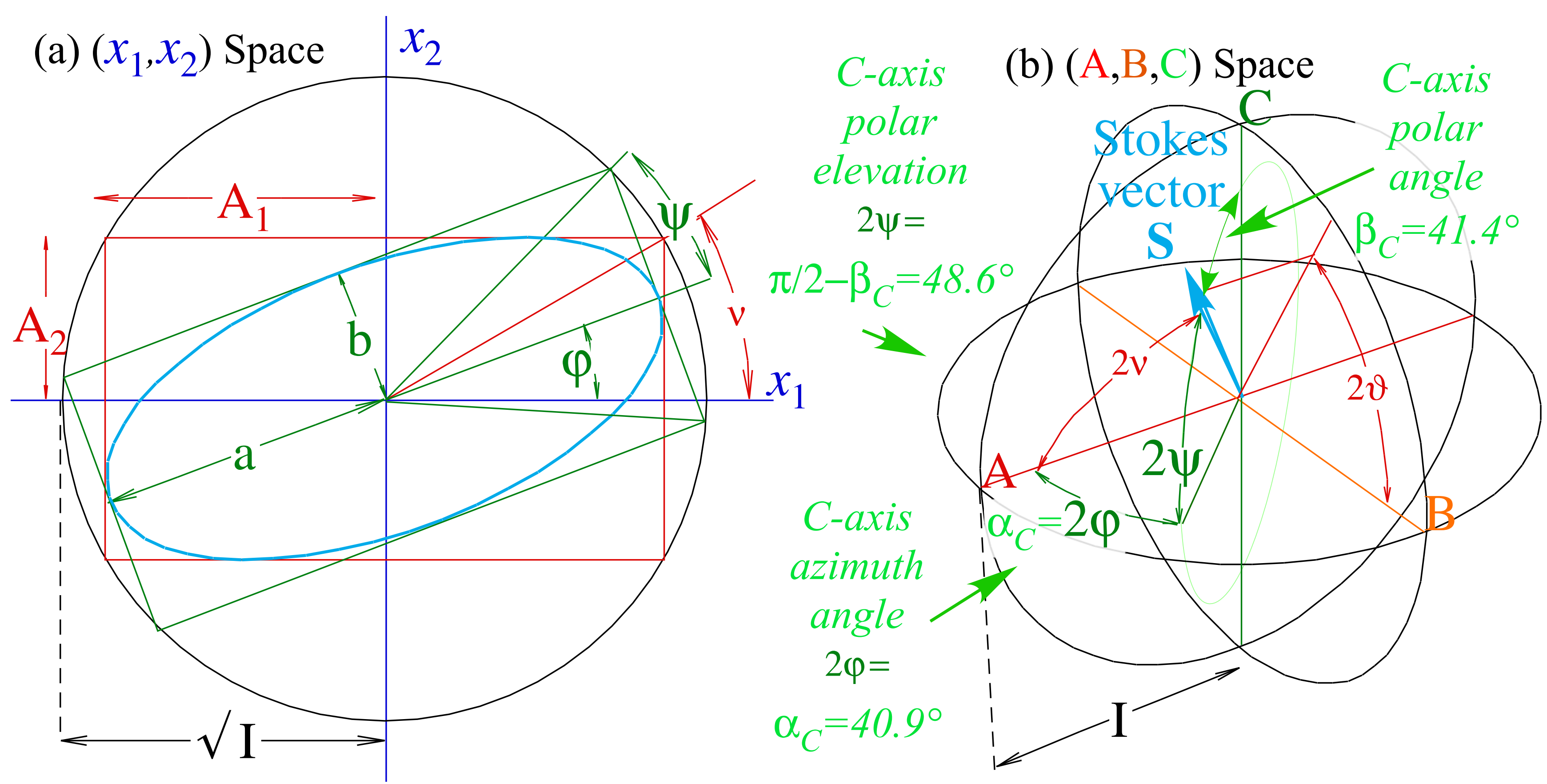
The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

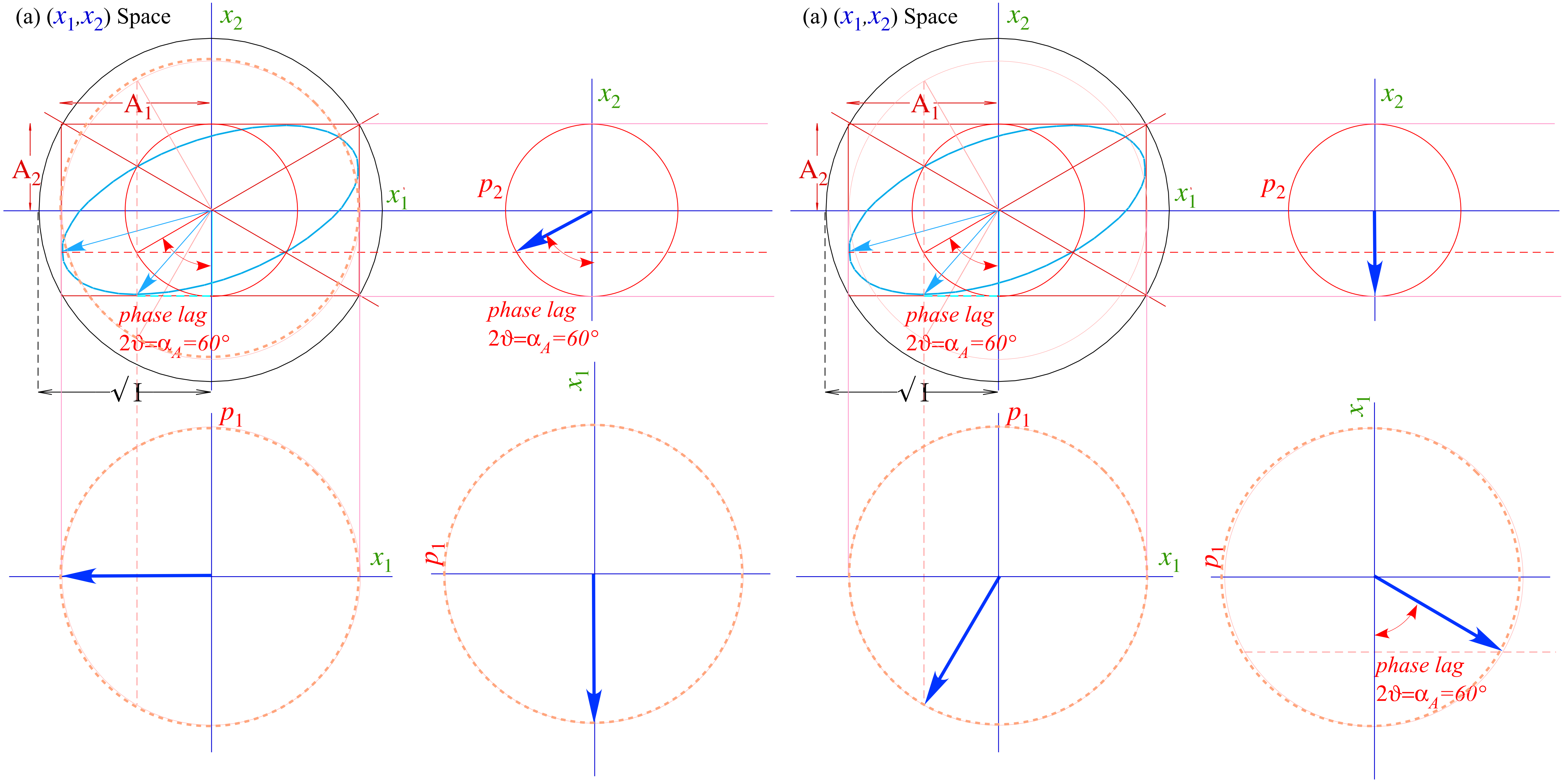
Angles (α_C, β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$

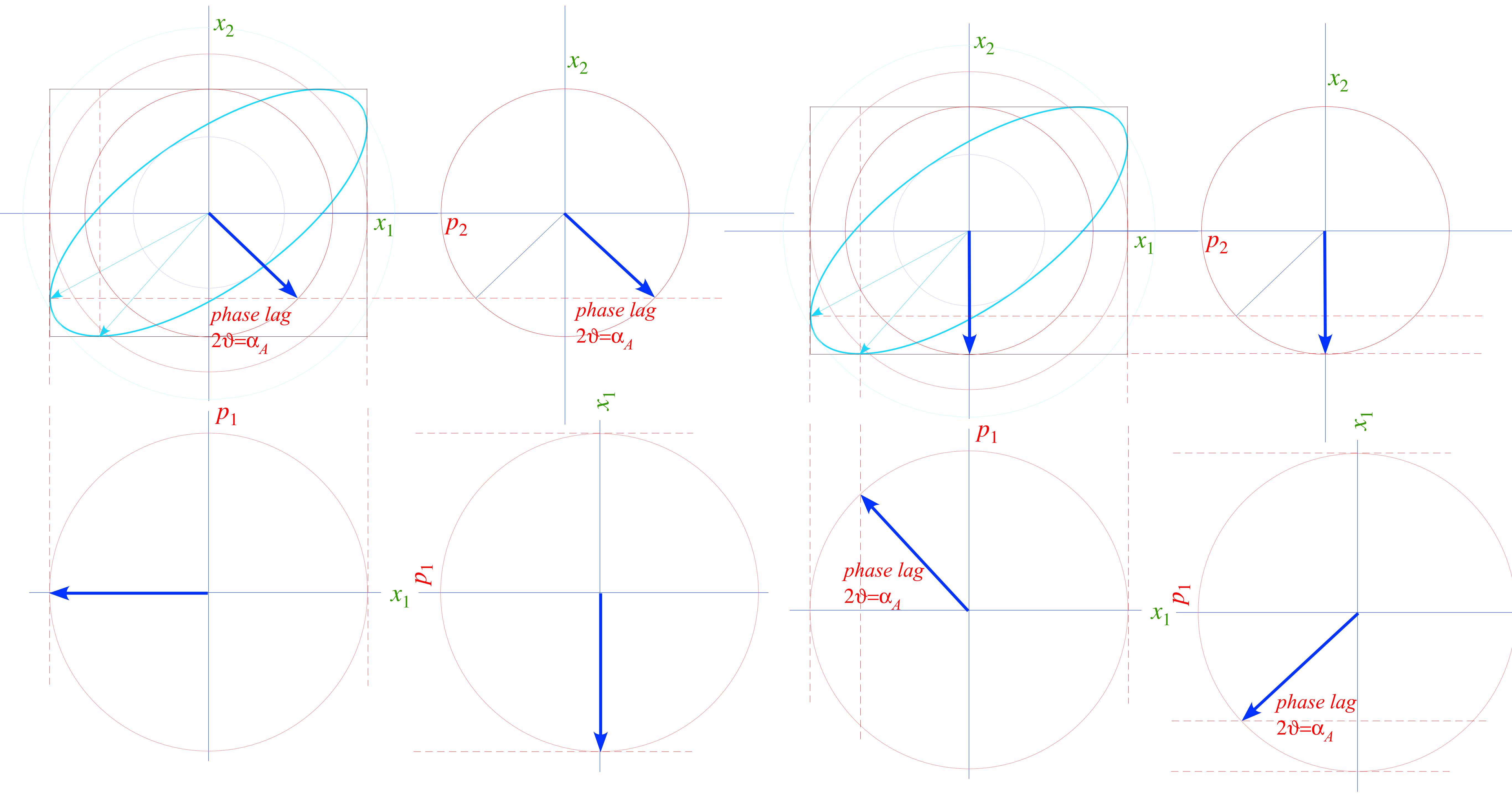
C-axis azimuth angle α_C relates to A-axis angles α_A and β_A . See $\alpha_C = 2\varphi$ below.

$$\frac{\cos\alpha_A \sin\beta_A}{\cos\beta_A} = \tan\alpha_C \quad \text{or:} \quad \alpha_C = ATN2(\cos\alpha_A \sin\beta_A / \cos\beta_A) = ATN2\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2} / \frac{1}{2}\right) = 40.9°$$





(a) (x_1,x_2) Space
x_2
A_1
A_2
x_1
p_2
phase lag
$2\vartheta=\alpha_A=60°$
$\sqrt{I}$
p_1
(a) (x_1,x_2) Space

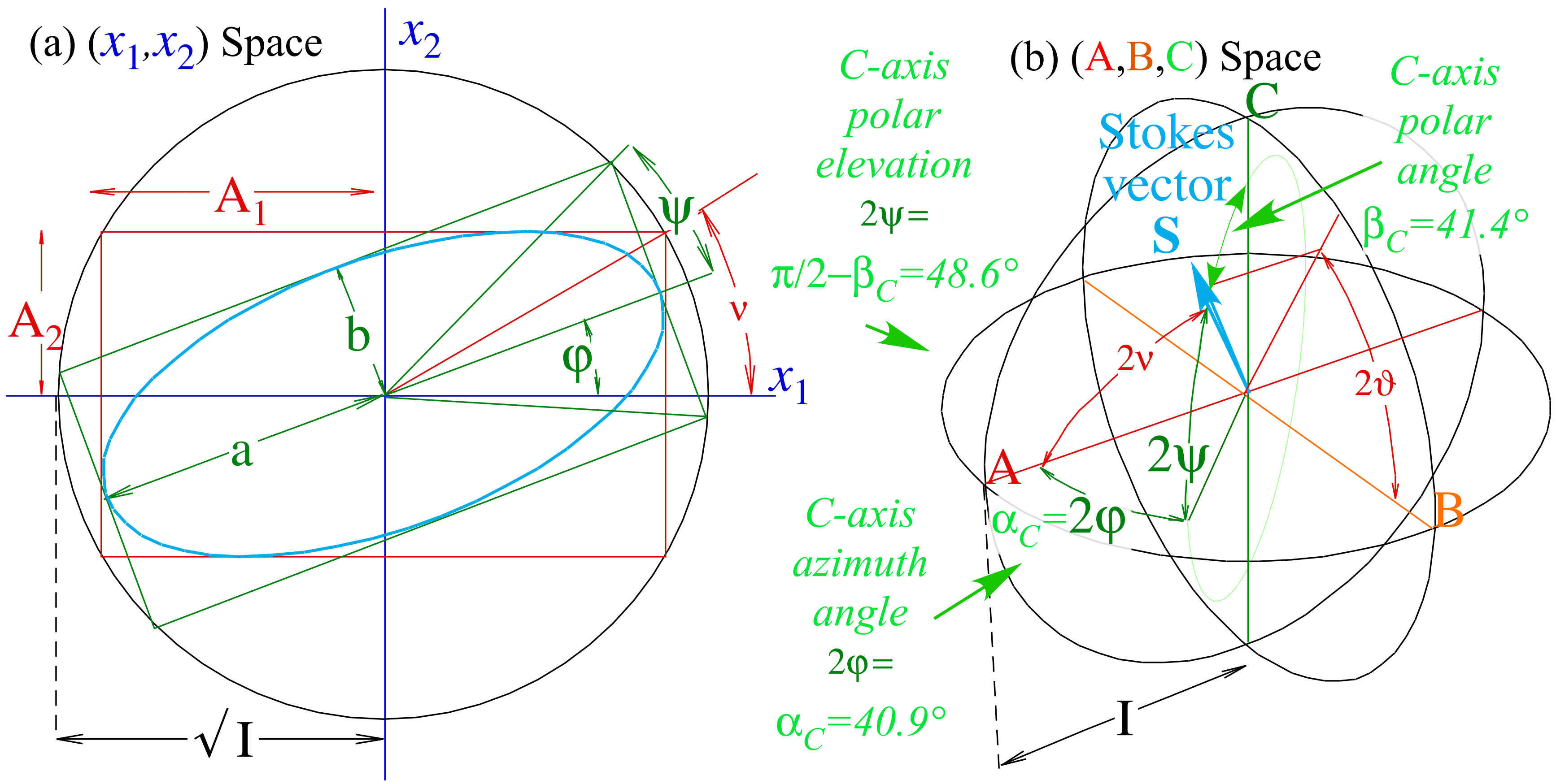


x2
x1
p2
phase lag
2ϑ=αA
p1
x1
p1
x2
x1
p2
phase lag
2ϑ=αA
p1
x1
p1

The C*-view in* $\{x_R, x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



A 90° B –rotation $\mathbf{R}(\pi/4)\left|x_1\right\rangle = \left|x_R\right\rangle$ of axis A into C gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_L + ip_L \end{pmatrix}$$

Polarization ellipse and spinor state dynamics

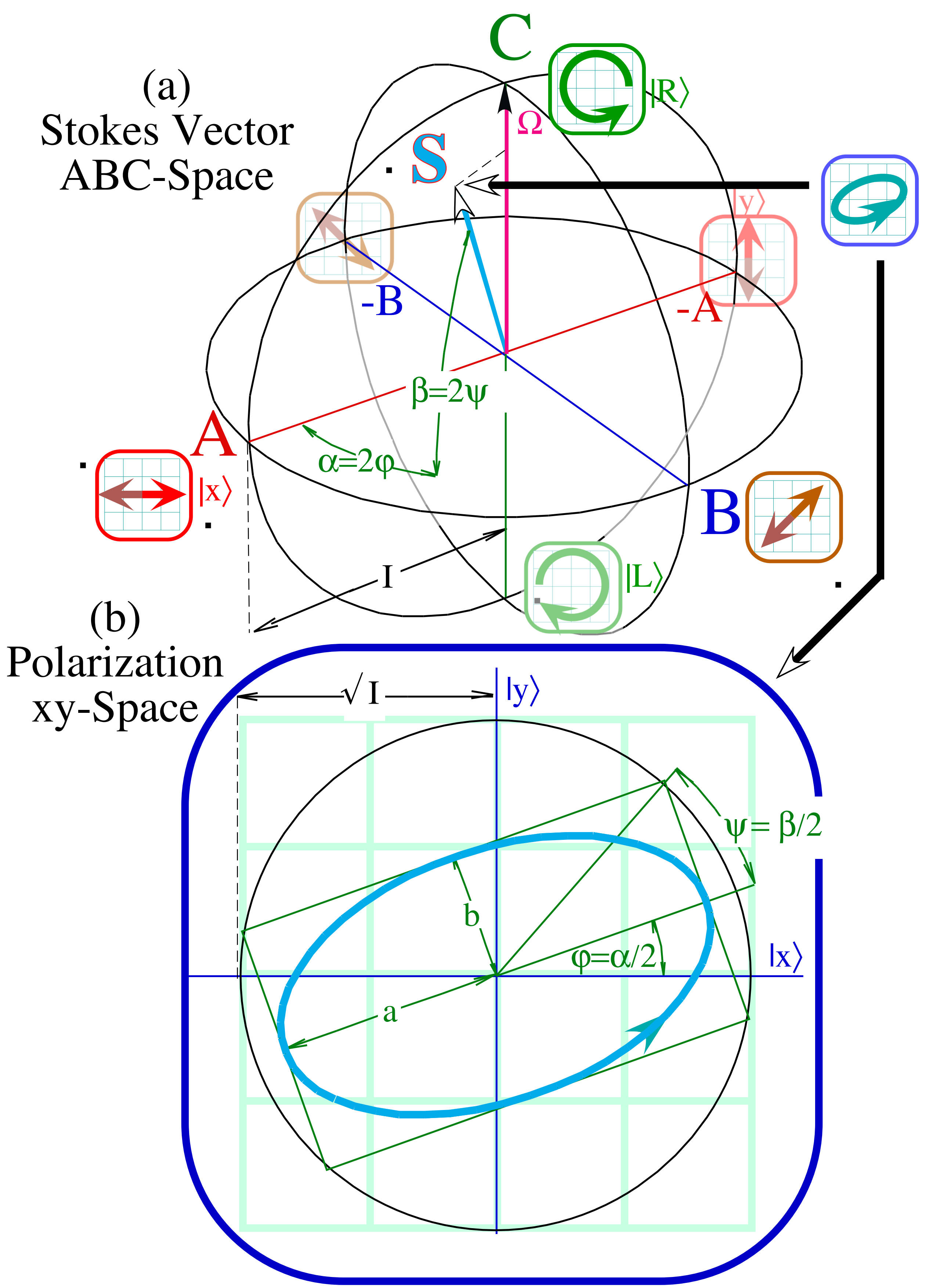


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

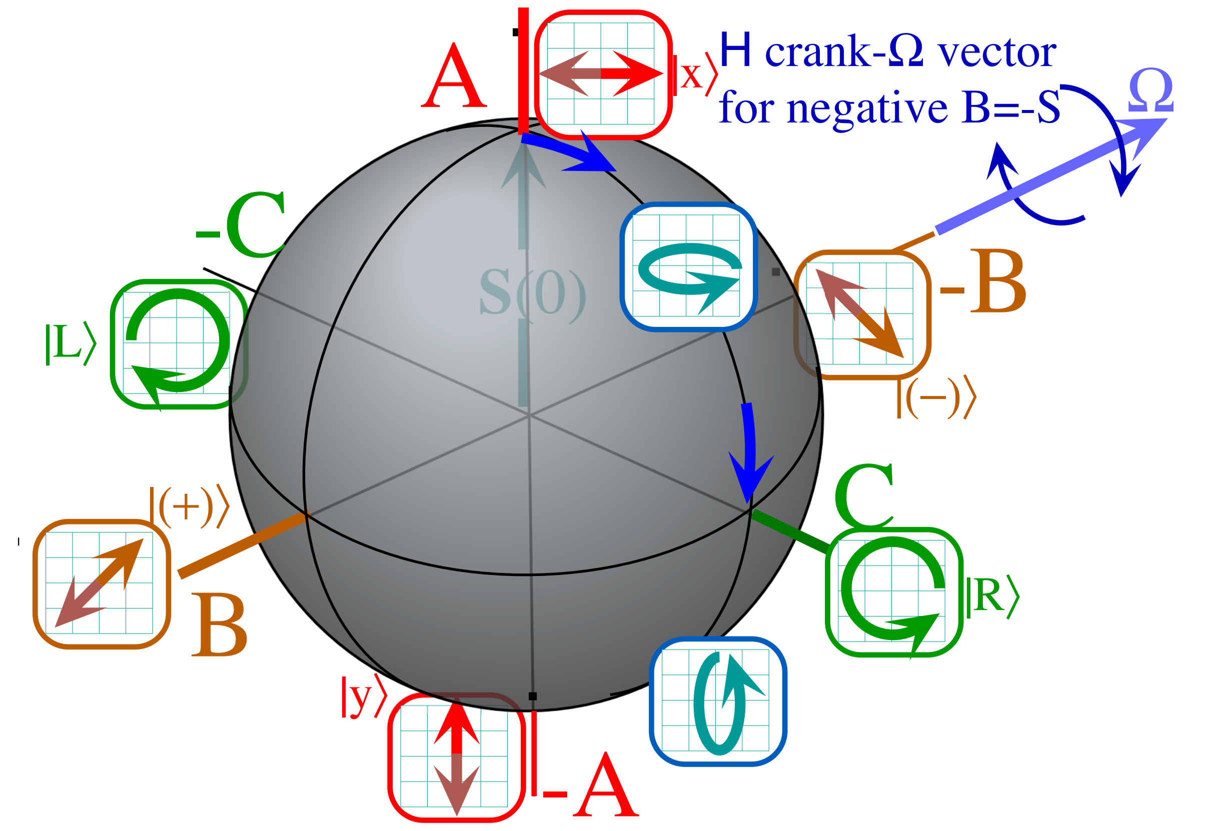
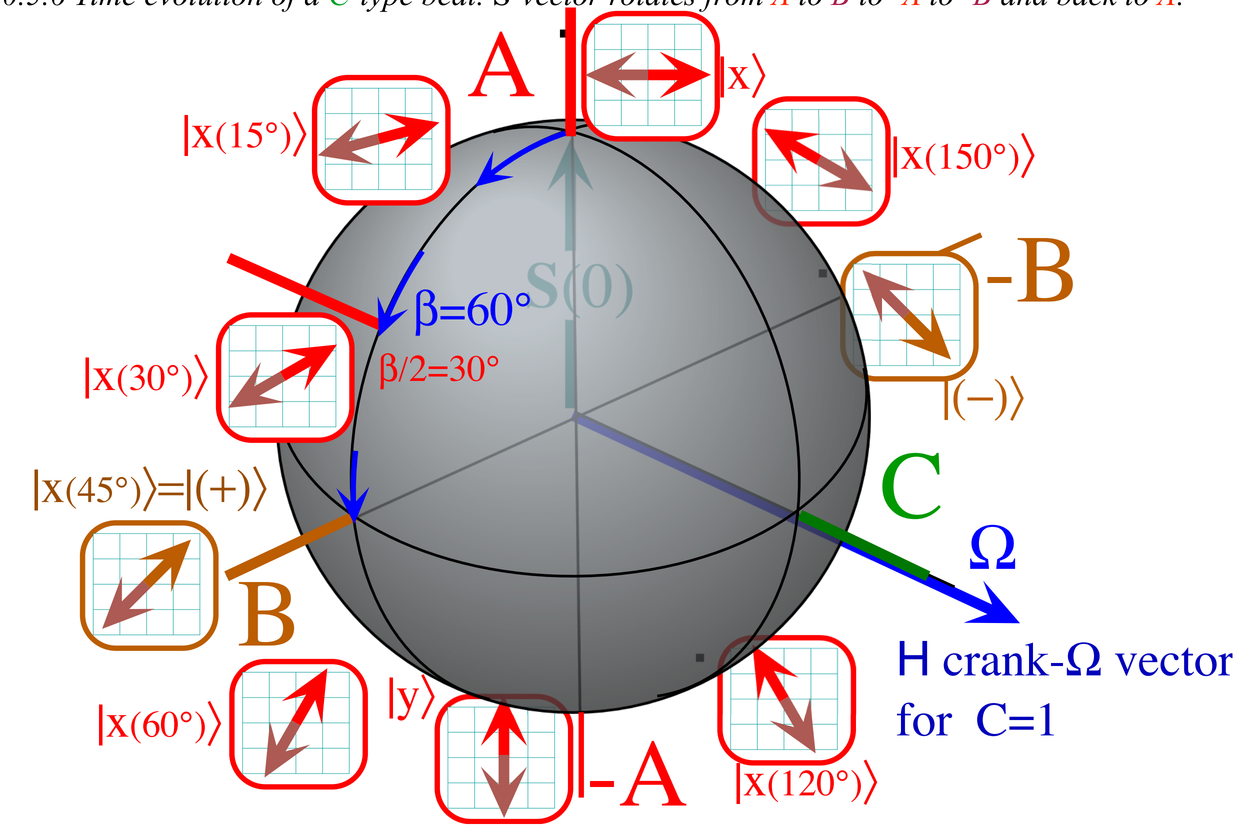


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

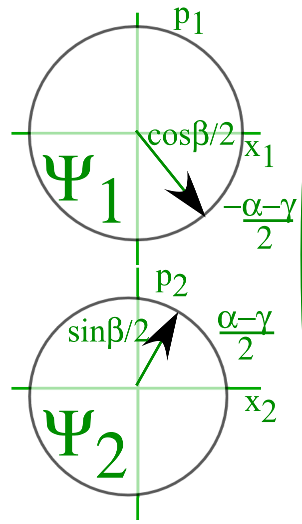
Fig. 10.5.6 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*



U(2) World : Complex 2D Spinors

Ellipsometry of U(2) states described by Two "Worlds" U(2) or R(3)

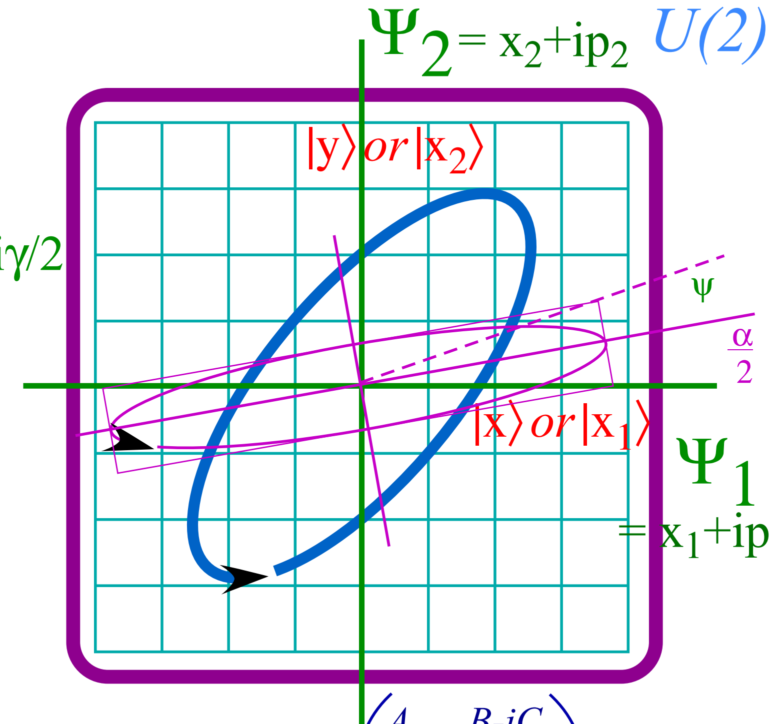
2-State ket $|\Psi\rangle=$



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2\\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$

U(2) World labeled by two complex phasors and driven by complex operator

$$\mathbf{H}=\begin{pmatrix} A & B-iC\\ B+iC & D\end{pmatrix}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

$$\begin{pmatrix}S_B\\ S_C\\ S_A\end{pmatrix} = \begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

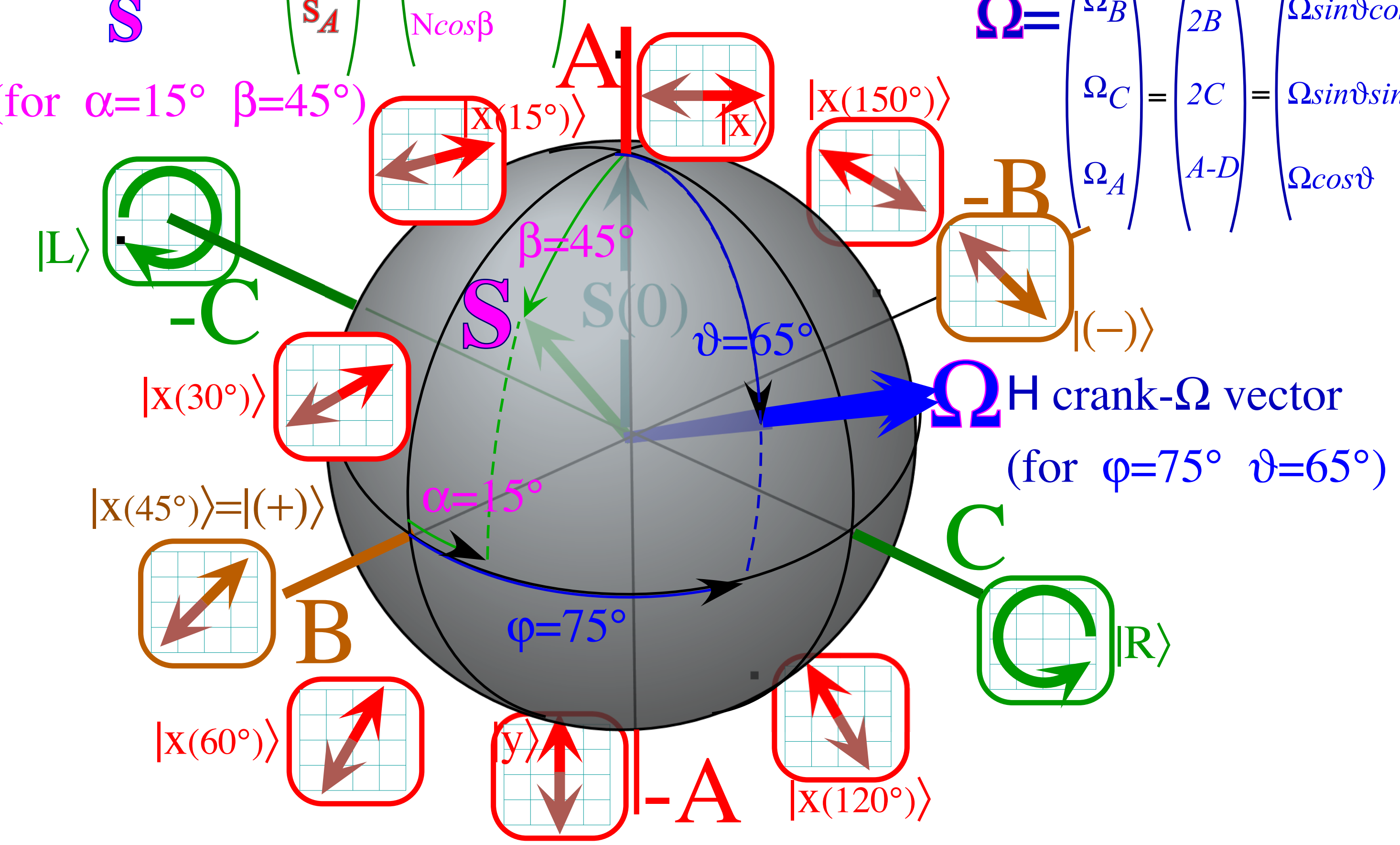
(for α=15° β=45°)

H-Operator $\begin{pmatrix} A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\mathbf{\Omega}=\begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix} = \begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix} = \begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$

*R(3) World labeled by real 3-D "spin" vector **S** of angular momentum and driven by real 3-D "spin" vector **Ω** of angular velocity*



Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

Bloch equation for density operator

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ)
and norm N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right)}_{4D\text{-}norm=1} \quad \textit{scaled by } \tfrac{1}{2}: \quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1{}^2 + x_1{}^2 + p_2{}^2 + x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1{}^2 + x_1{}^2 - p_2{}^2 - x_2{}^2\right)$$

scaled by $\frac{1}{2}$: $S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$

$U(2)$ density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}\quad \textit{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$ *scaled by* $\frac{1}{2}$: $$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$ *scaled by* $\frac{1}{2}$: $$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$ *scaled by* $\frac{1}{2}$: $$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$ *scaled by* $\frac{1}{2}$: $$S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}\quad \textit{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* ρ = |Ψ⟩⟨Ψ|=

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}$$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

...2-by-2 *density operator* ρ

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
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$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\underset{\uparrow\ \rho}{\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\begin{pmatrix}0&1\\1&0\end{pmatrix}+S_Y\begin{pmatrix}0&-i\\i&0\end{pmatrix}+S_Z\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\quad\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$: $\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$
$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y$,
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Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\rho=\frac{1}{2}N\,\mathbf{1}+S_X\,\boldsymbol{\sigma}_X+S_Y\,\boldsymbol{\sigma}_Y+S_Z\,\boldsymbol{\sigma}_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\cdot\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}\quad \textit{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1=S_X-iS_Y,$
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Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$\rho=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma_X}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma_Y}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma_Z}$$

$$=\frac{1}{2}N\,\mathbf{1}+S_X\,\sigma_X+S_Y\,\sigma_Y+S_Z\,\sigma_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma_A}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma_B}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma_C}$$

$$\mathbf{H}=\omega_0\,\sigma_0+\frac{\Omega_A}{2}\sigma_A+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C=\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

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$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

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$\rho_{11}=\Psi_1^*\Psi_1 =\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1 =S_X-iS_Y,$
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Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\rho=\frac{1}{2}N\,\mathbf{1}+S_X\,\sigma_X+S_Y\,\sigma_Y+S_Z\,\sigma_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$\mathbf{H}=\omega_0\,\sigma_0+\frac{\Omega_A}{2}\sigma_A+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C=\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H}=\Omega_0\,\mathbf{1}+\Omega_A\,\mathbf{S}_A+\Omega_B\,\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\mathbf{S}$$

$$\boxed{\begin{aligned}\rho&=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H}&=\Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

Bloch equation for density operator

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}\left(N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

U(2) density operator approach to symmetry dynamics

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$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

U(2) density operator approach to symmetry dynamics

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Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})$$

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U(2) density operator approach to symmetry dynamics

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Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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U(2) density operator approach to symmetry dynamics

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This remains

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

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U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right) = i\hbar\dot{\vec{S}}\bullet\sigma = i\hbar(\vec{\Omega}\times\mathbf{S})\bullet\sigma$$

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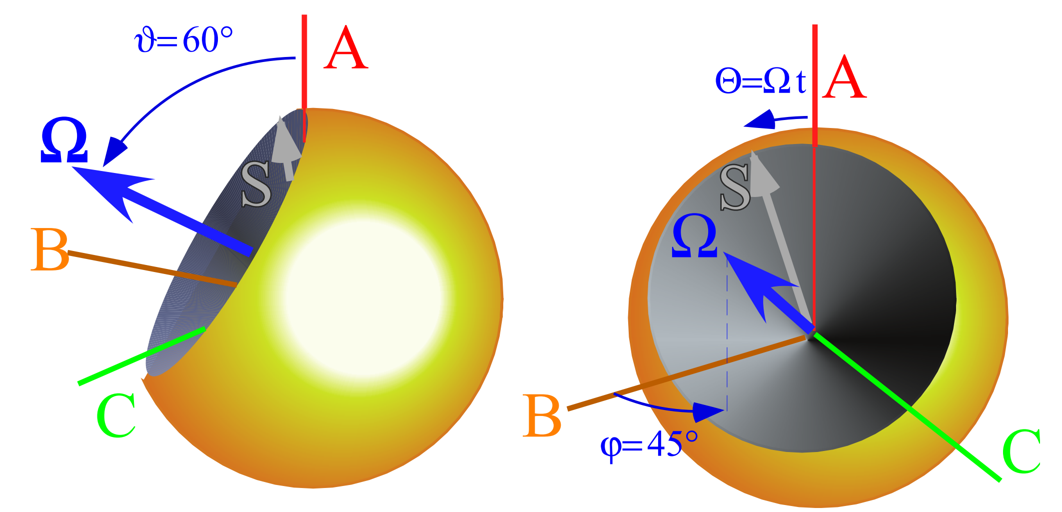
$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathrm{S}}\right)\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}\left(\vec{\mathrm{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = i\hbar\dot{\vec{\mathrm{S}}}\bullet\boldsymbol{\sigma} = i\hbar\left(\vec{\boldsymbol{\Omega}}\times\mathbf{S}\right)\bullet\boldsymbol{\sigma}$$



Factoring out $\bullet\boldsymbol{\sigma}$ gives a classical/quantum *gyro-precession equation*. $\boxed{\frac{\partial\vec{\mathrm{S}}}{\partial t} = \dot{\vec{\mathrm{S}}} = \vec{\boldsymbol{\Omega}}\times\vec{\mathrm{S}}}$