Lecture 19

Self-Study Lecture posted on Mon. 10.29.2018

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

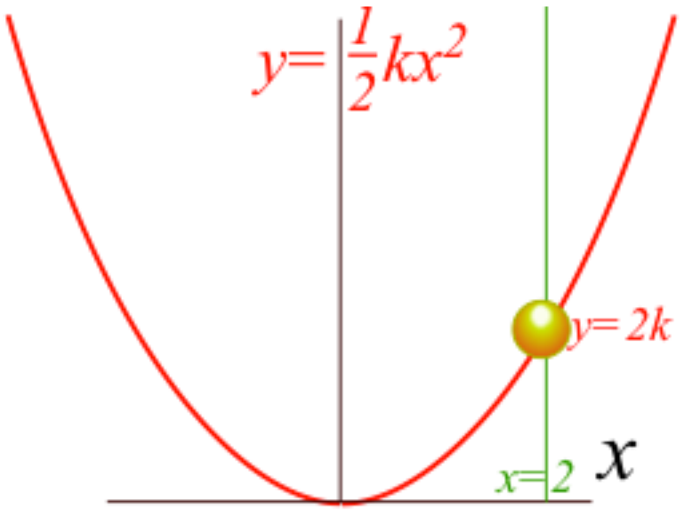
Some Ways to do constraint analysis

- *Way 1. Simple constraint insertion*
- *Way 2. GCC constraint webs*
  - *Find covariant force equations*
  - *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

- *Way 3. OCC constraint webs*
  - *Preview of atomic-Stark orbits*
    - *Classical Hamiltonian separability*
- *Way 4. Lagrange multipliers*
  - *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *"Non-Holonomic" multipliers*

Simple constrained problem...



...and a variety of solutions

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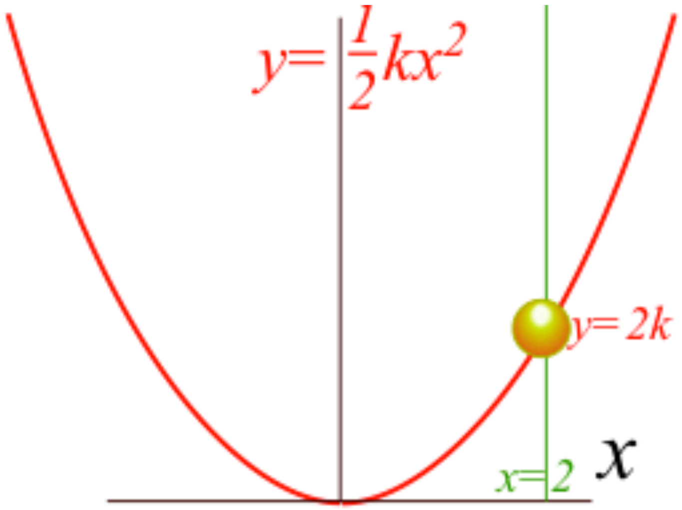

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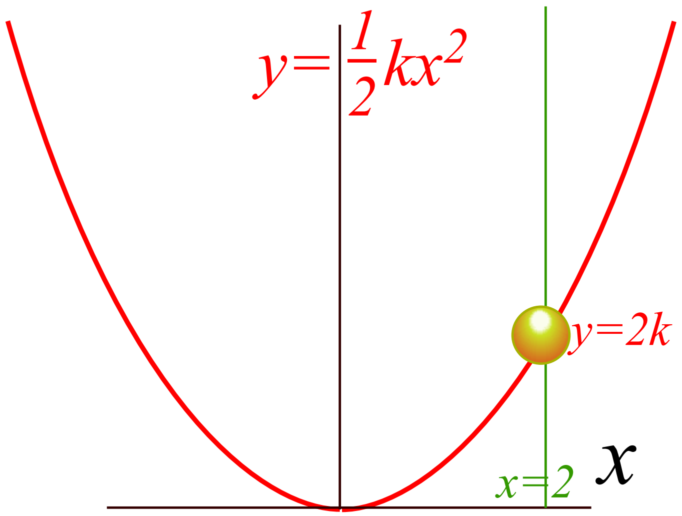
Ways to analyze a particle m constrained to parabola $y=1/2kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L = \overbrace{\frac{1}{2}(m\dot{x}^2 + m\dot{y}^2) - mgy}$$

Let: $y = \frac{1}{2}kx^2$ and: $\dot{y} = kx\dot{x}$

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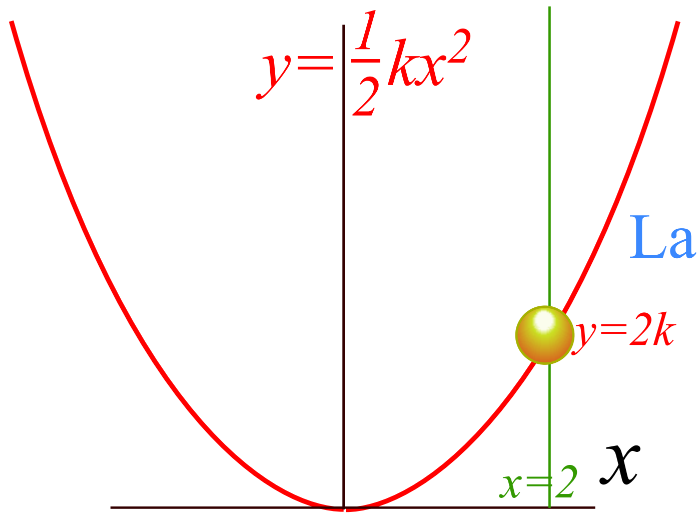

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Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial\dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

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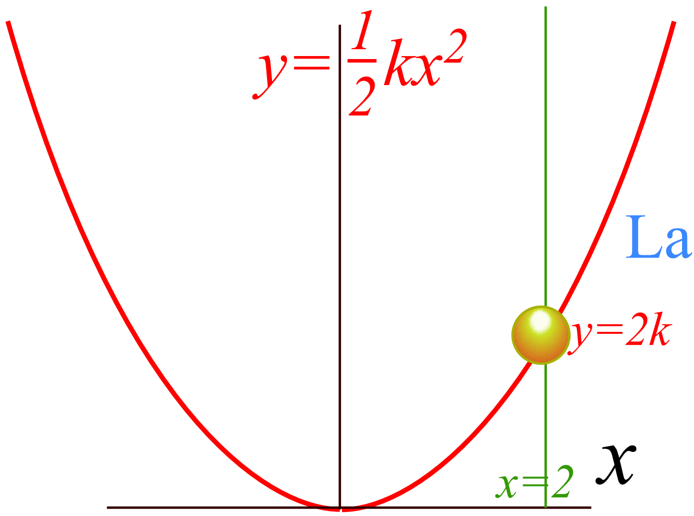

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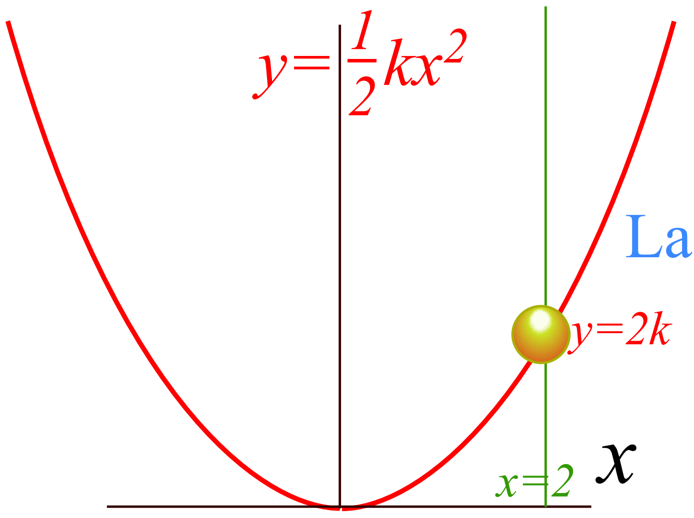

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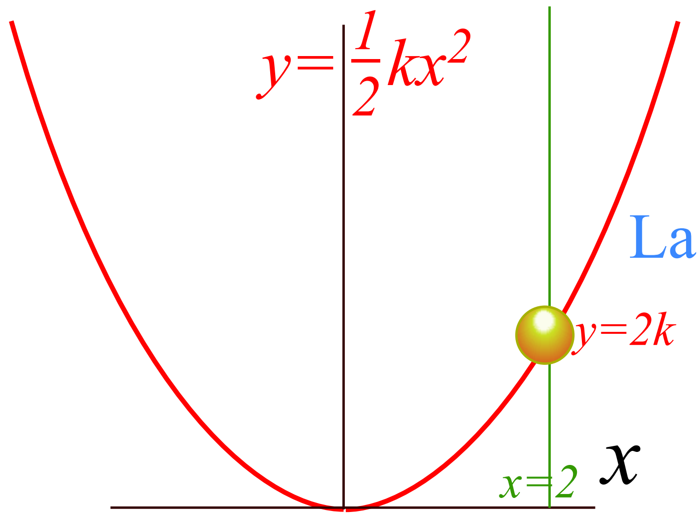

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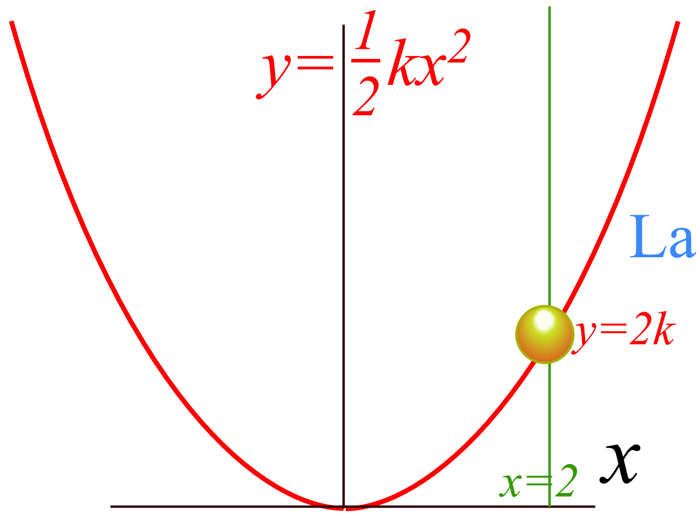

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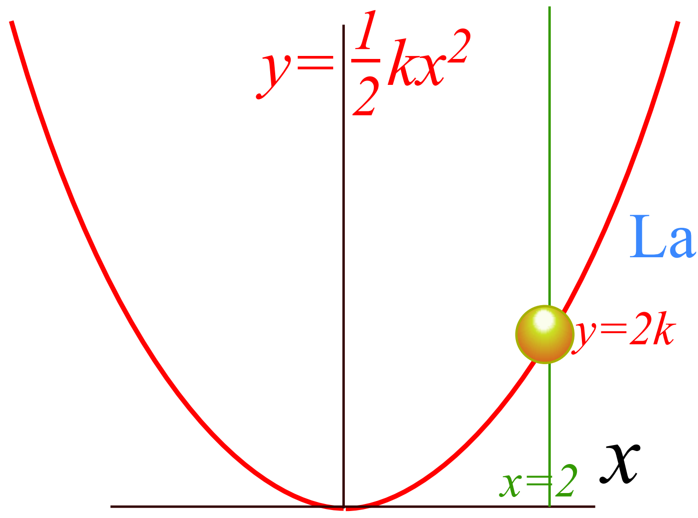

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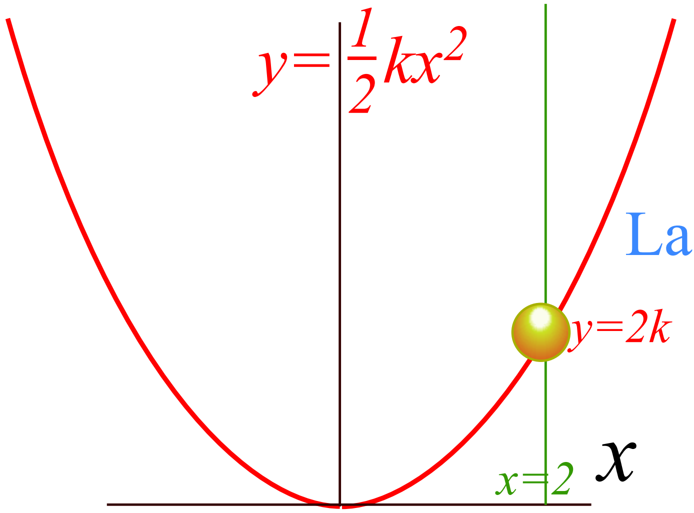

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Ways to analyze a particle *m* constrained to parabola $y=\frac{1}{2}kx^2$ on *(x,y)*-plane with gravitational potential $V(\mathbf{r})=mgy$.

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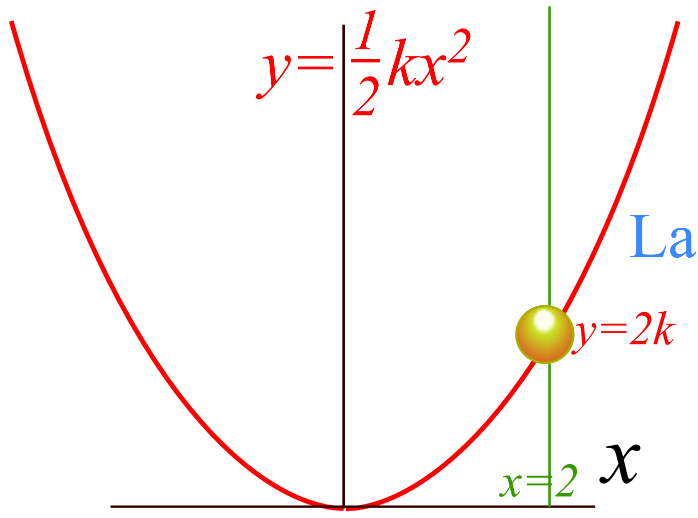

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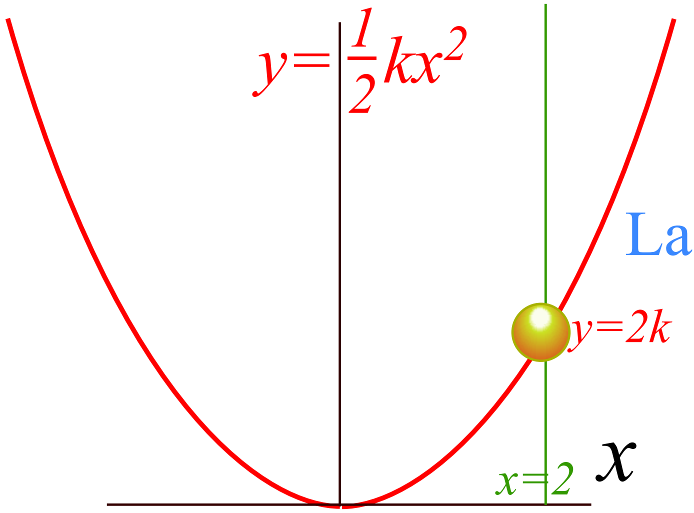

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$$\boxed{\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx}$$

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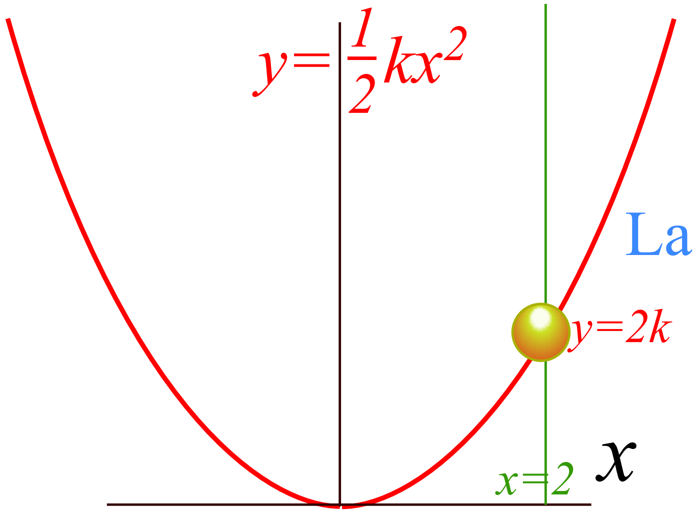

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

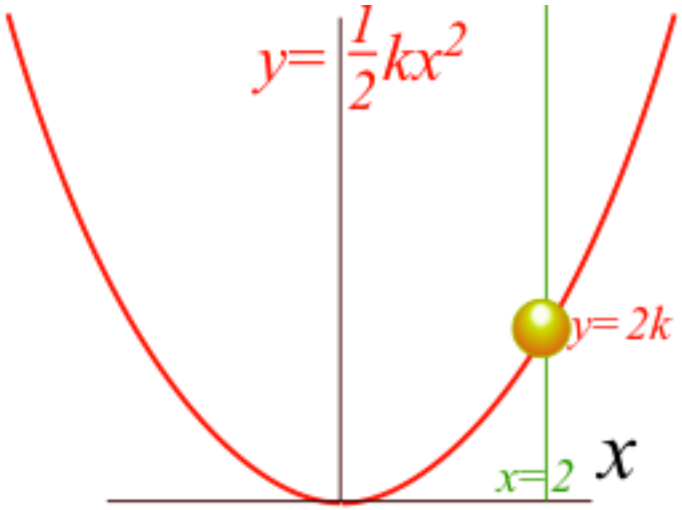
$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$\boxed{m(1+k^2x^2)\ddot{x}}=-mk^2x\dot{x}^2-mgkx=\boxed{-m(k\dot{x}^2+g)kx}$$

$$\boxed{\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx}$$

$$K=\frac{k\dot{x}^2+g}{1+k^2x^2}k\xrightarrow[x\sim0\sim\dot{x}]{}gk$$

Simple constrained problem...



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Way 2. *GCC constraint webs.*

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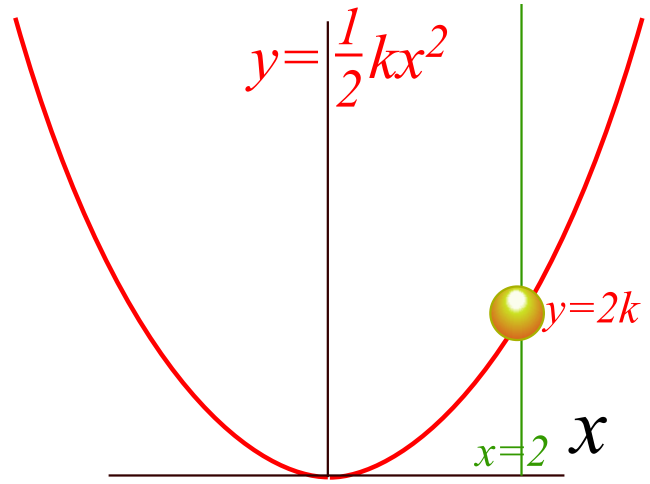
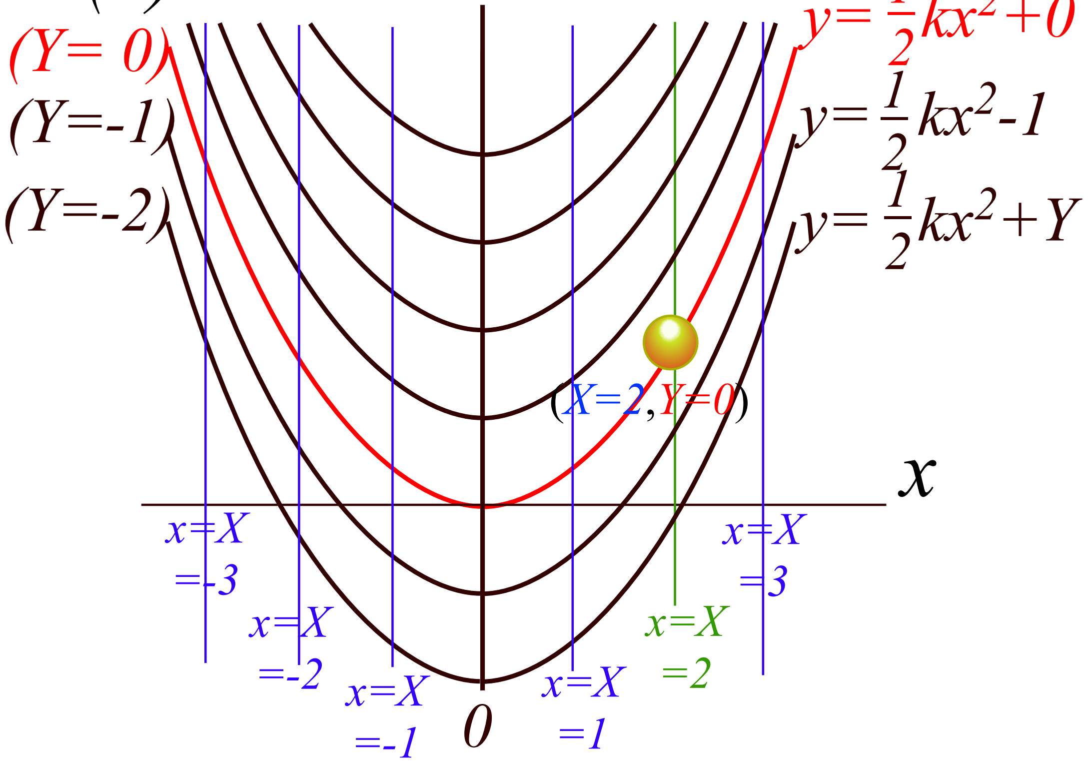


$x = X$

$y=\frac{1}{2}kx^2+Y$

Cartesian (x,y) transform to GCC (X,Y)

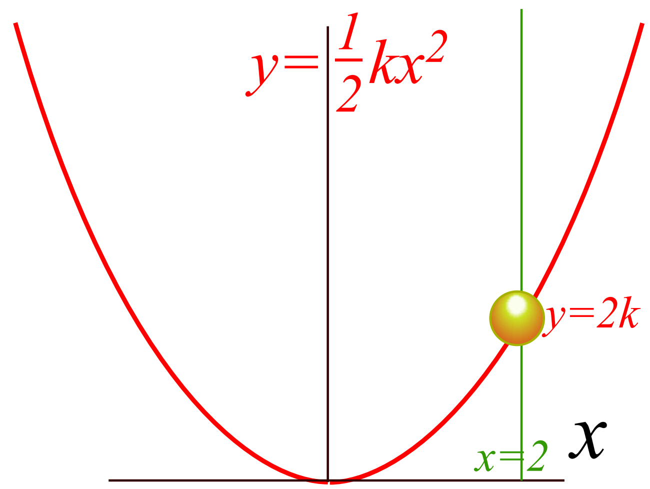
(b) GCC constraint web



Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

Way 2. GCC constraint webs.

(a) Constrained motion



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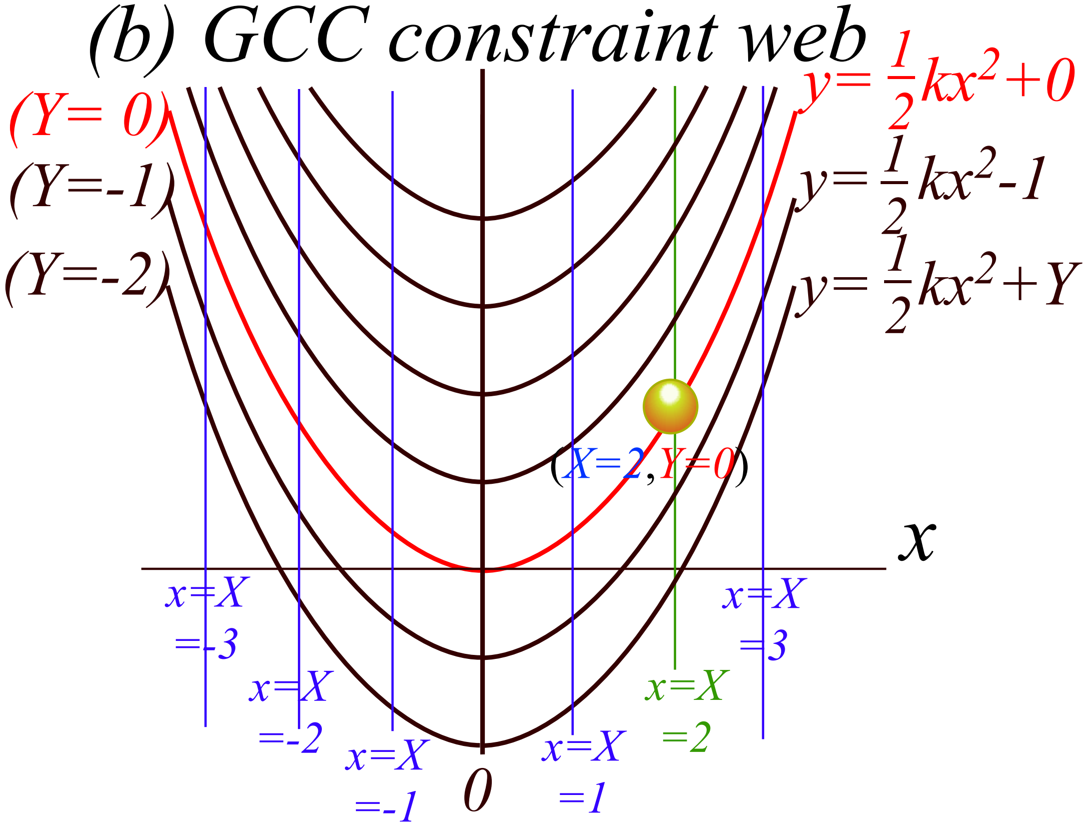

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}^{Indices}$

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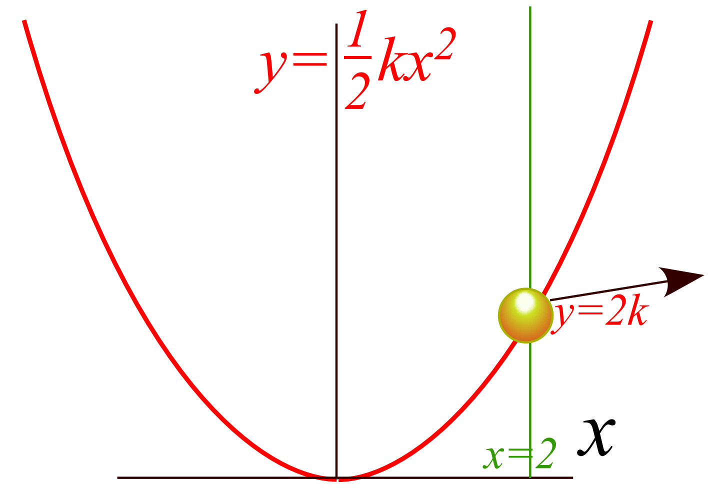
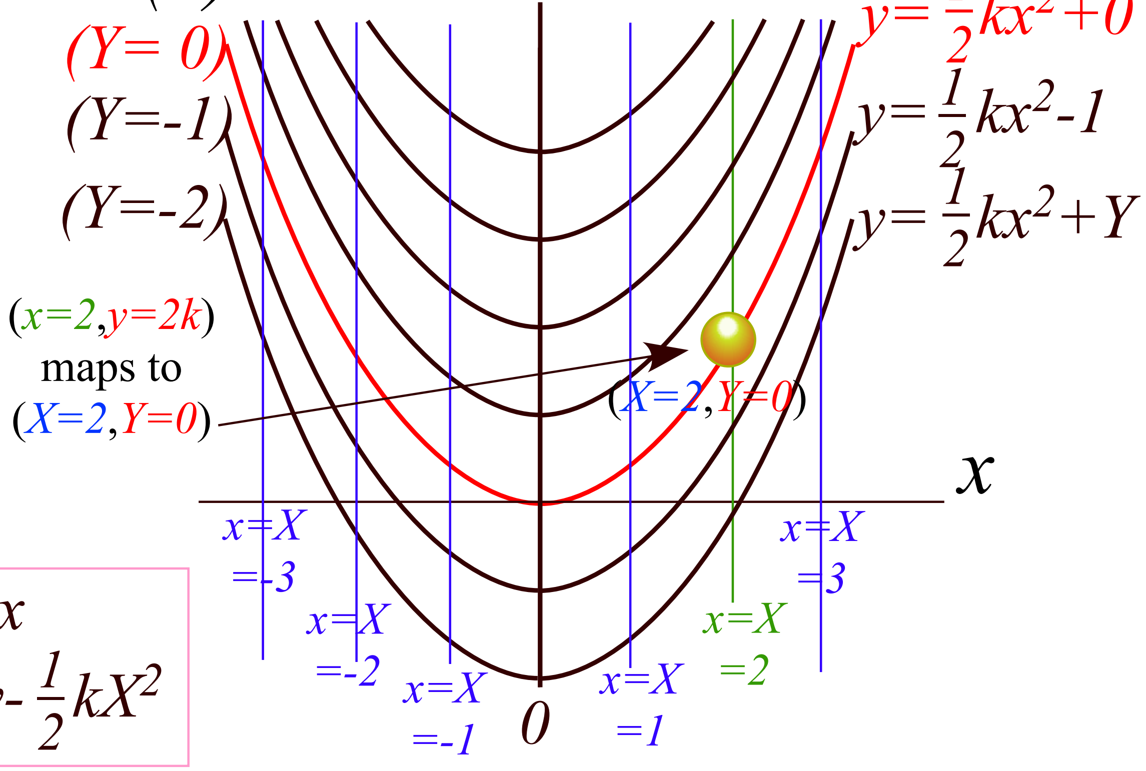
(a) Constrained motion

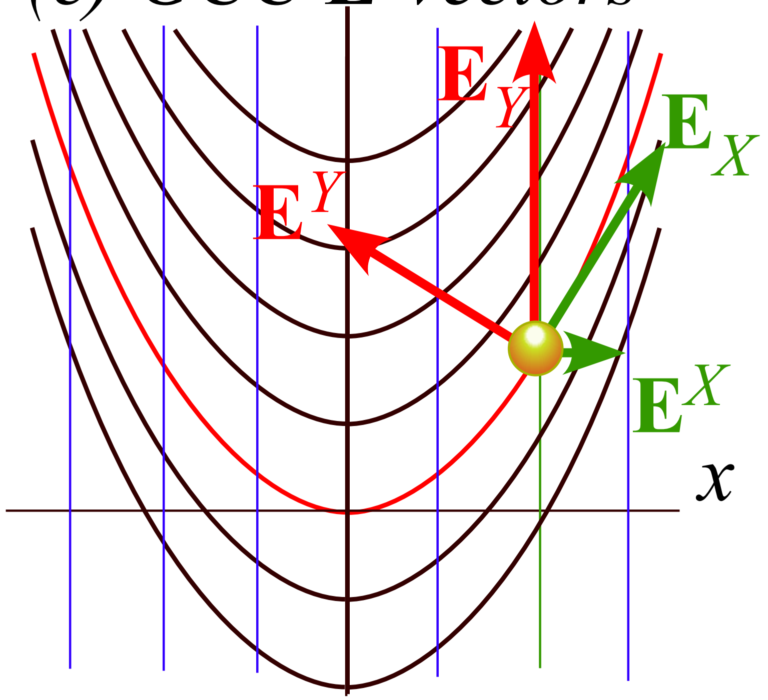


$(x=2, y=2k)$
maps to
$(X=2, Y=0)$

(b) GCC constraint web



(c) GCC **E**-vectors



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Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

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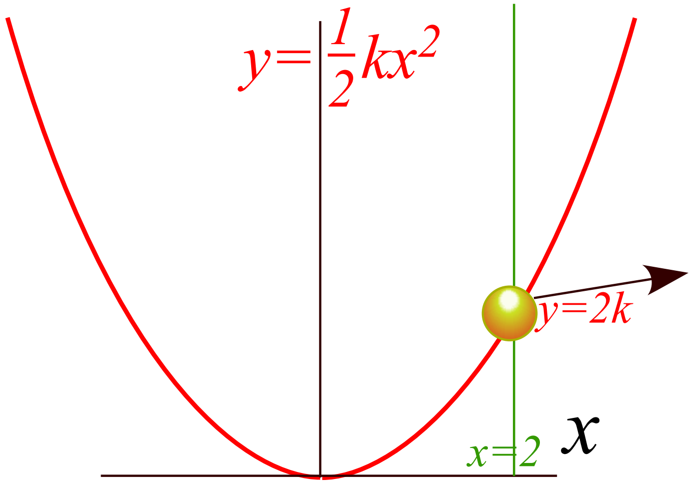
Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Way 2. *GCC constraint webs.*

(a) Constrained motion



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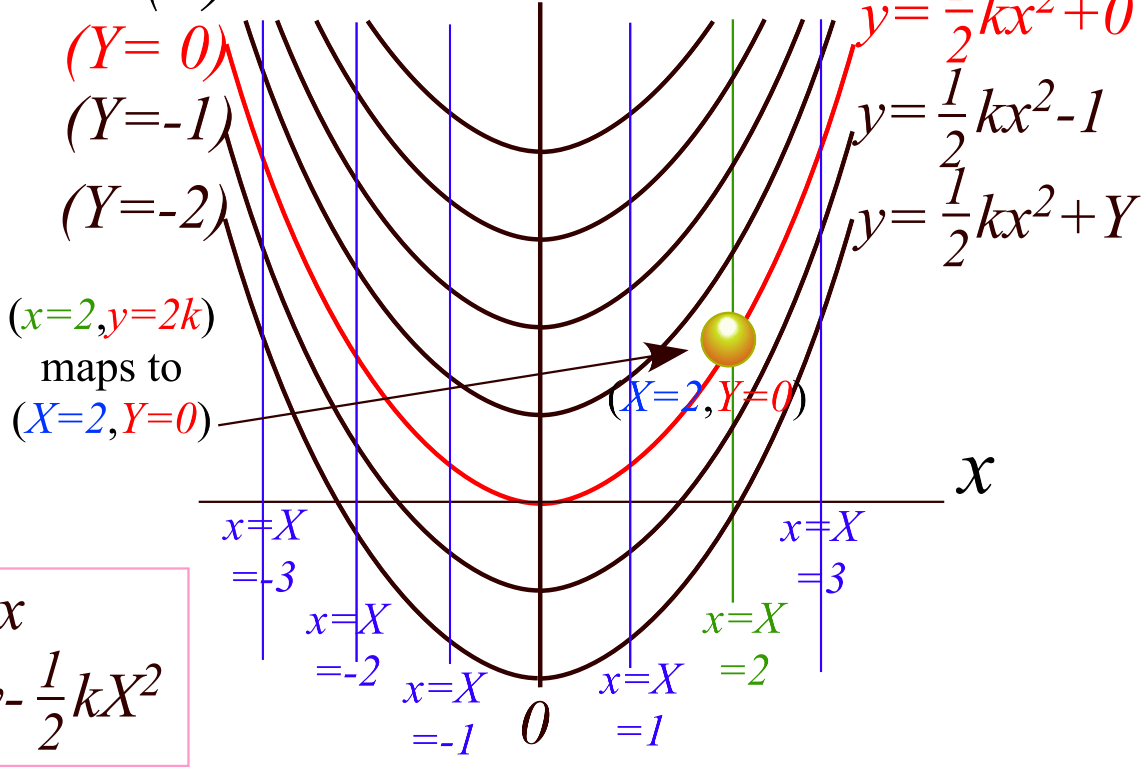

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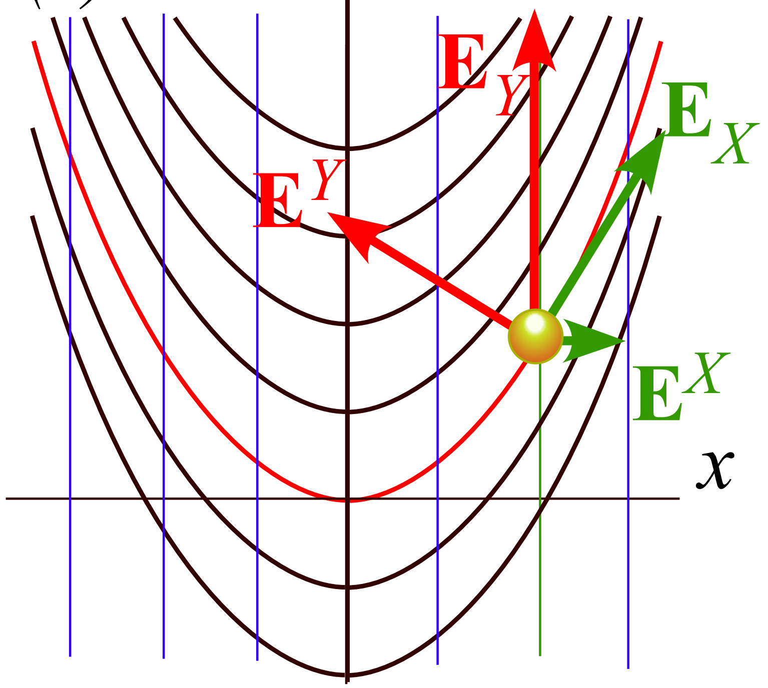

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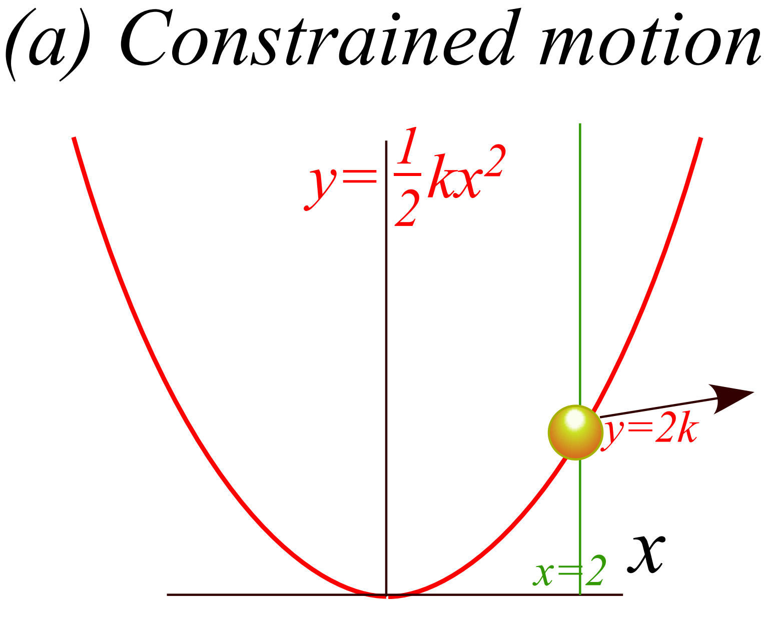
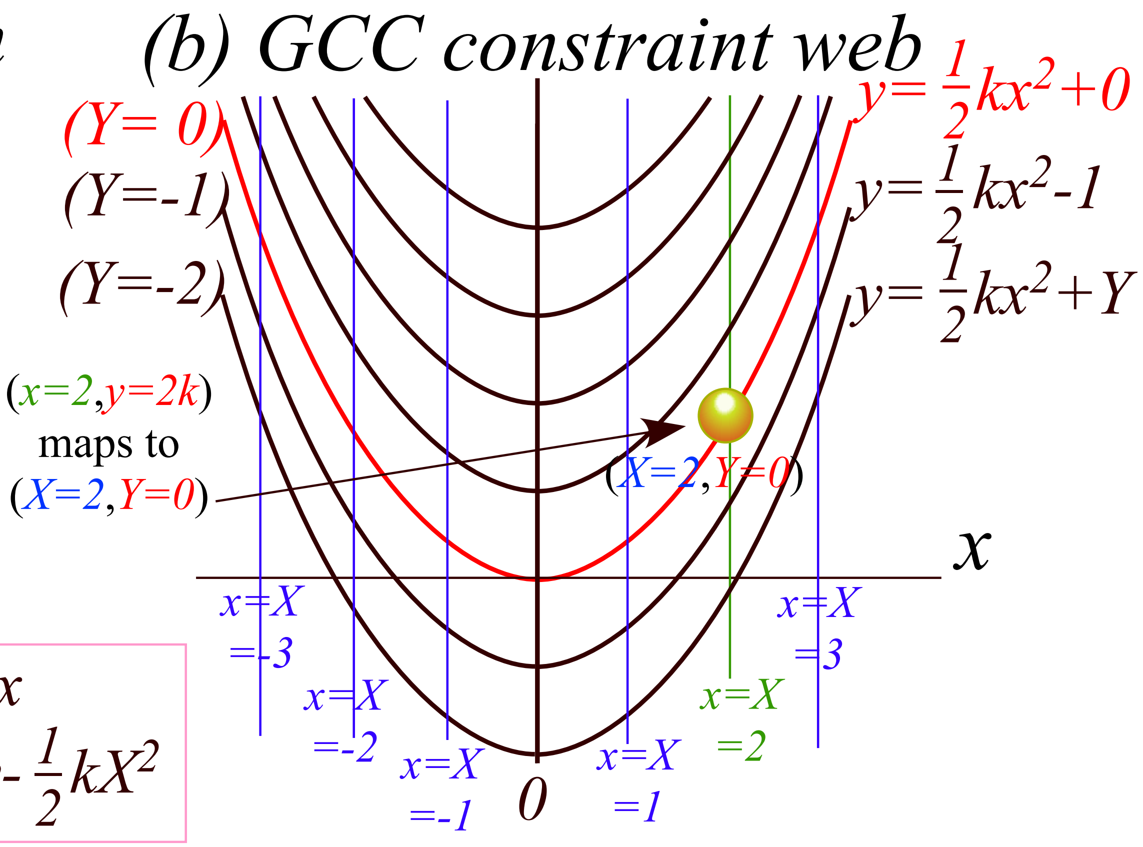
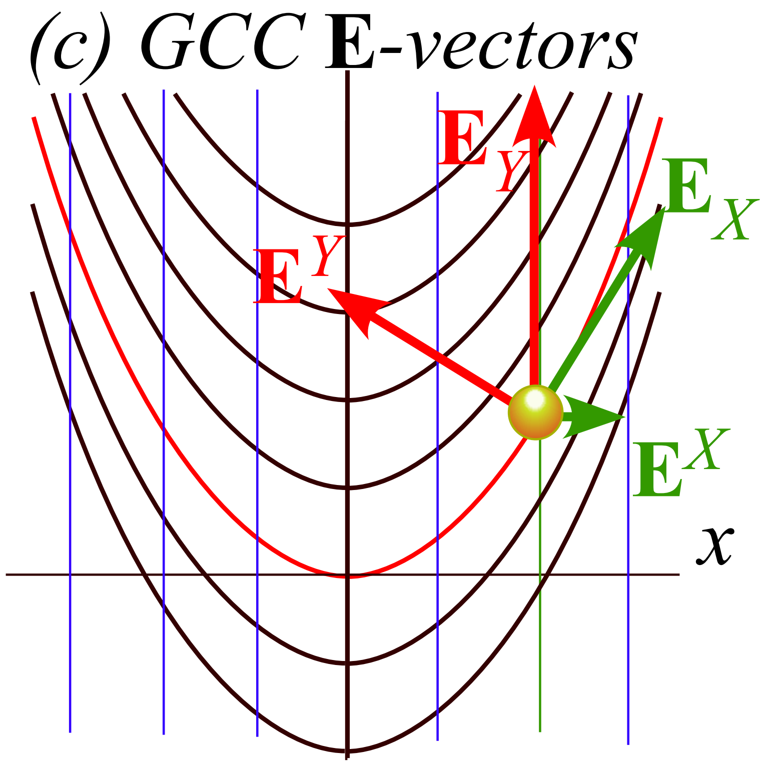
Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \begin{aligned}\mathbf{E}^X&=\begin{pmatrix}1 & 0\end{pmatrix}\\ \mathbf{E}^Y&=\begin{pmatrix}-kx & 1\end{pmatrix}\end{aligned}$$

Way 2. GCC constraint webs.







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Cartesian (x,y) transform to GCC (X,Y)

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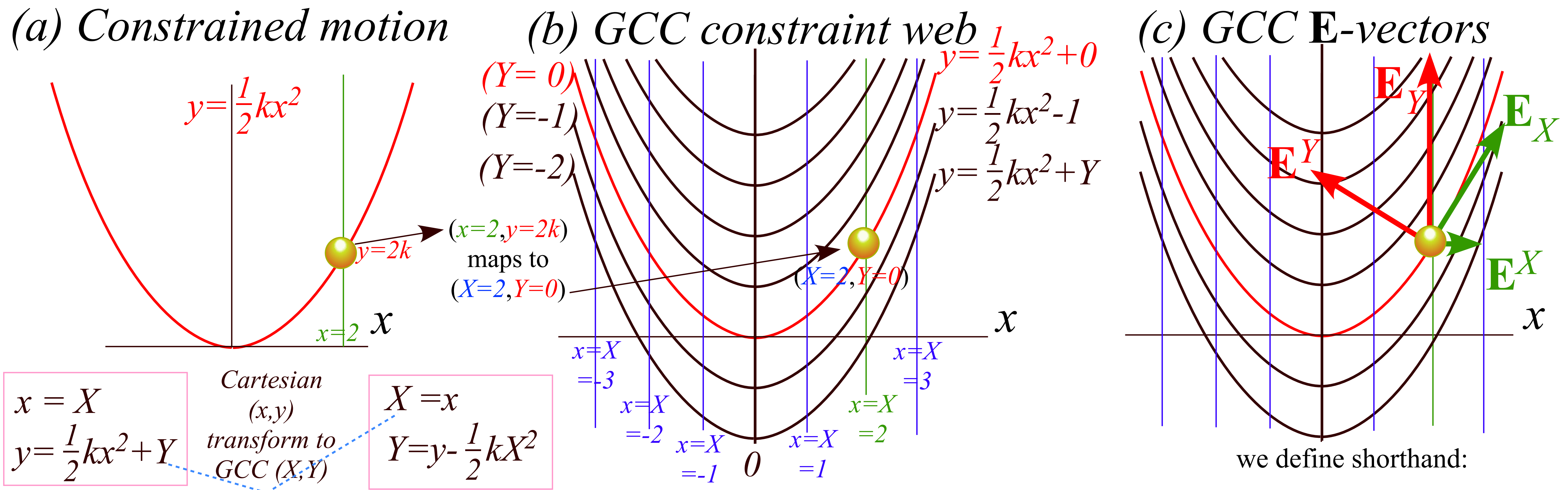

(a) Constrained motion

(b) GCC constraint web

(c) GCC **E**-vectors

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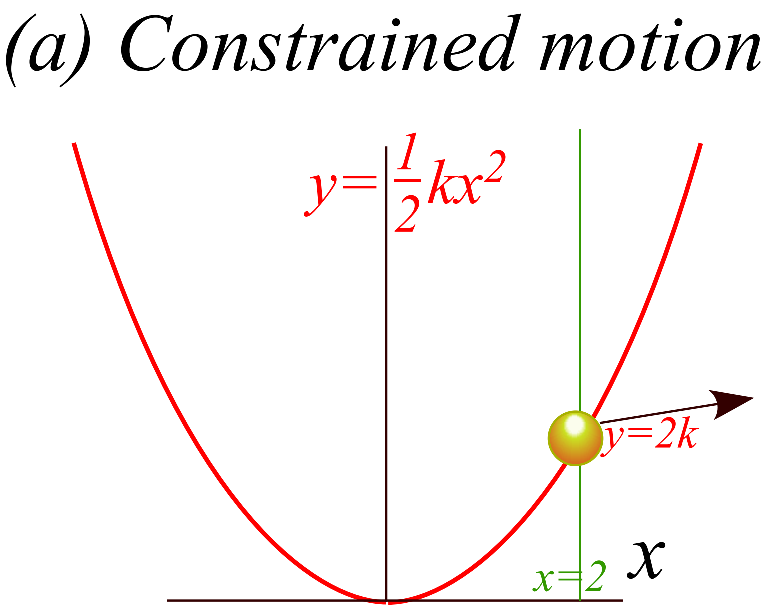

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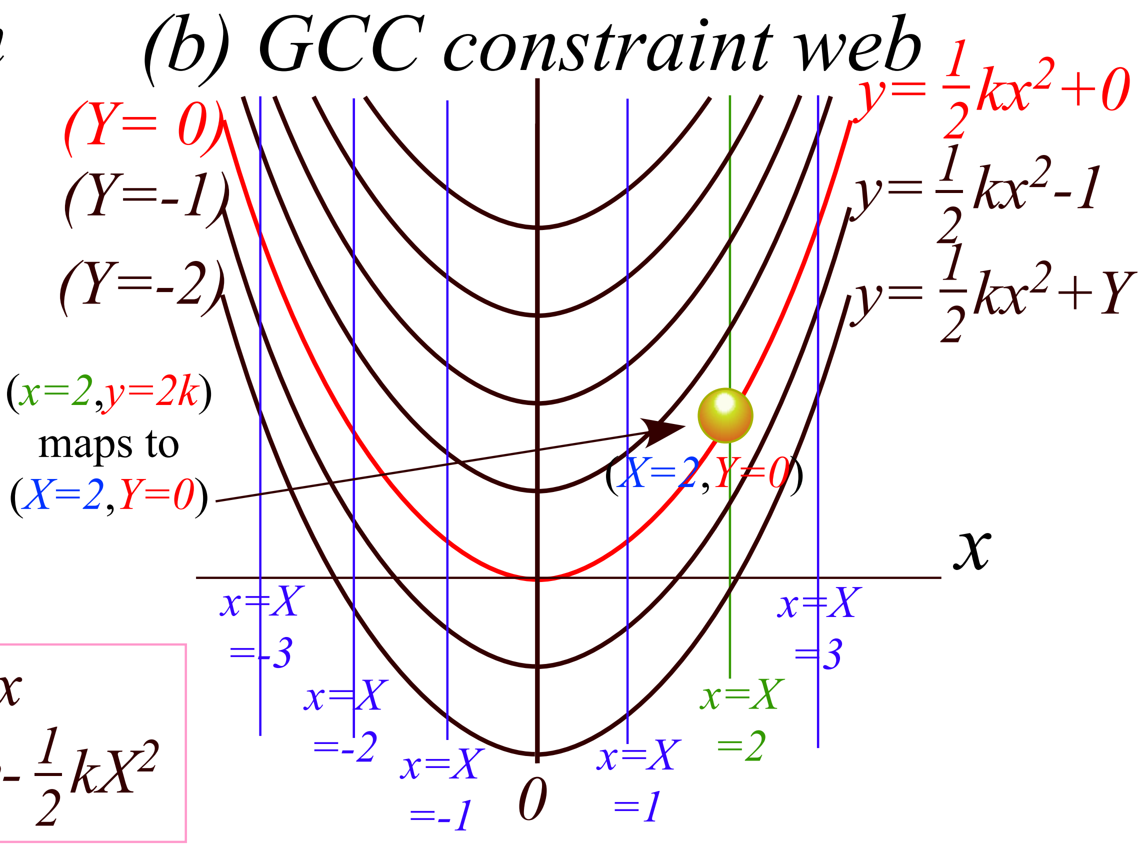
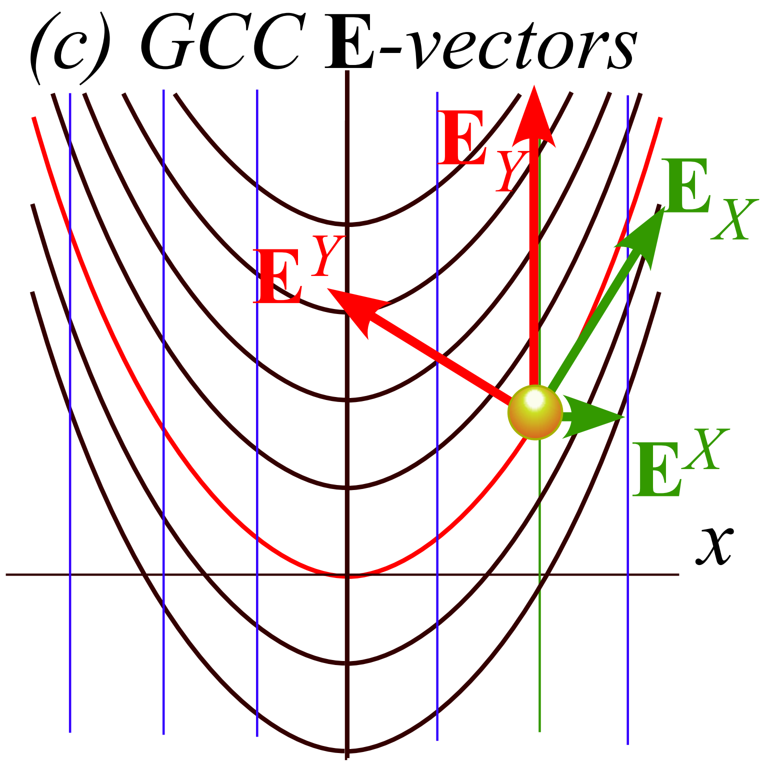


(b) GCC constraint web



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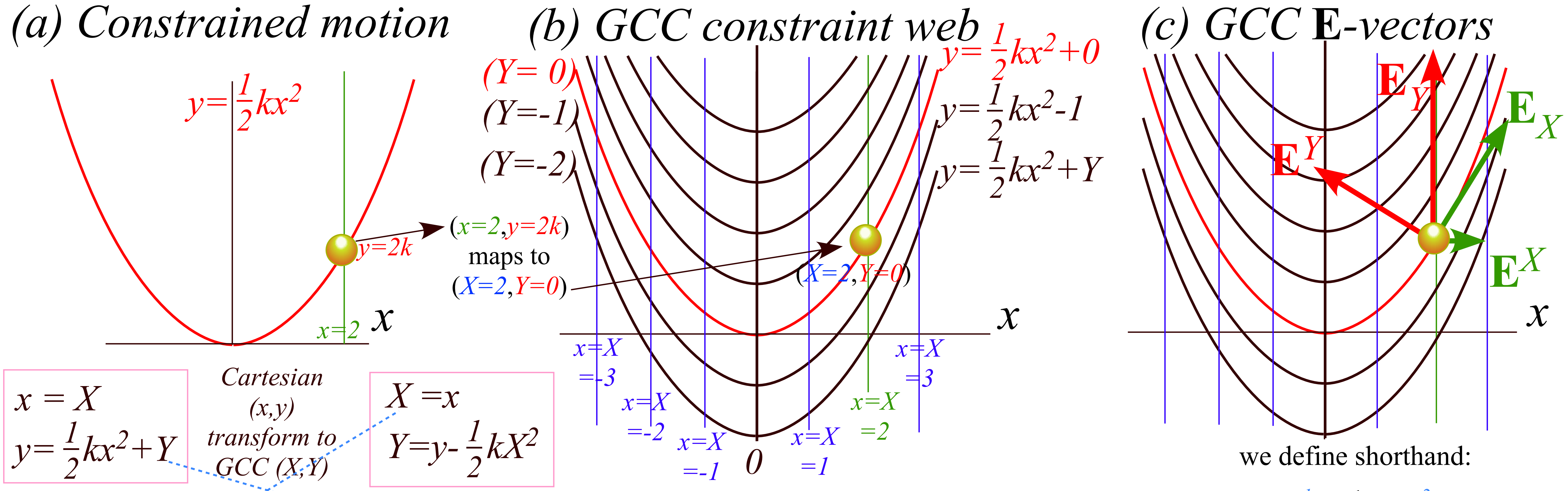

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Way 2. GCC constraint webs.

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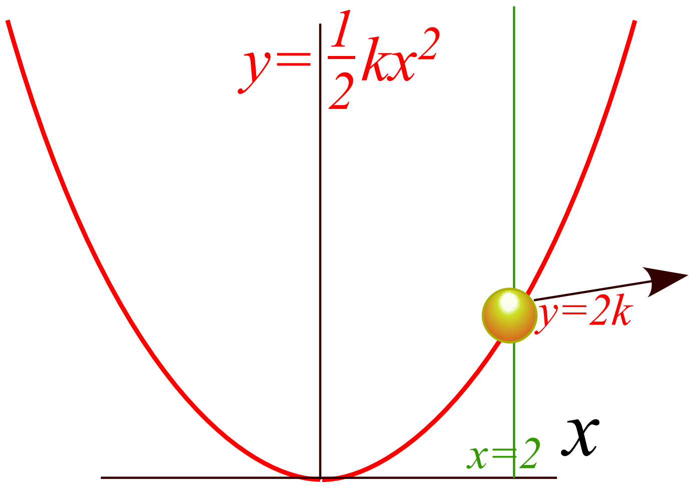
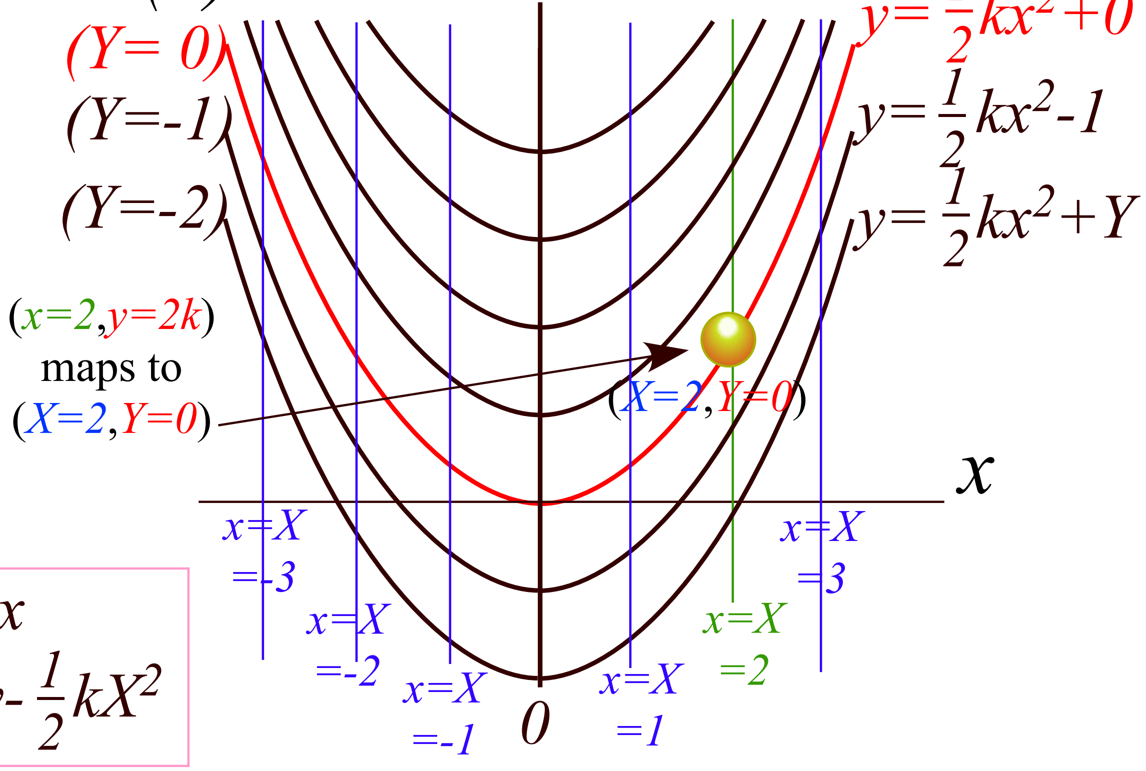
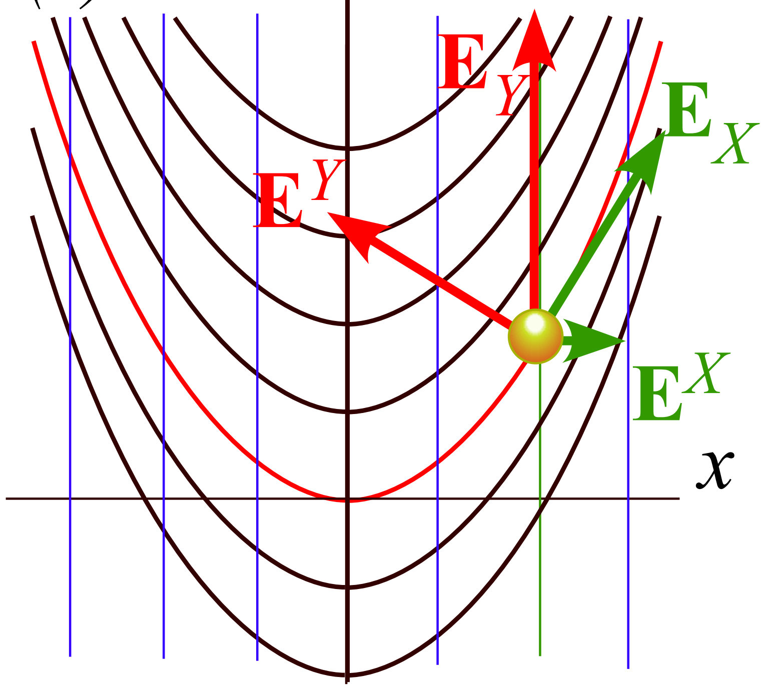


(b) GCC constraint web



(c) GCC **E**-vectors



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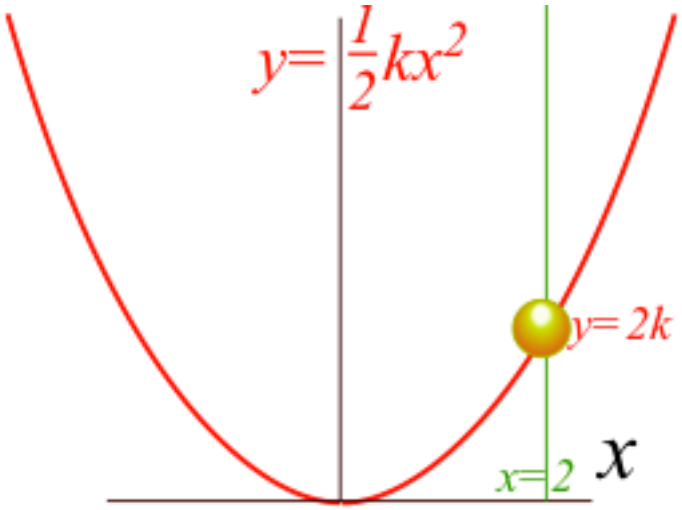
...and Lagrangian:

$$L=T-V=m\left[\frac{1}{2}(1+k^2X^2)\dot X^2+kX\dot X\dot Y+\frac{1}{2}\dot Y^2-gY-\frac{gk}{2}X^2\right]$$

$V=mgy=mg(Y+kX^2/2)$

Simple constrained problem...



...and a variety of solutions

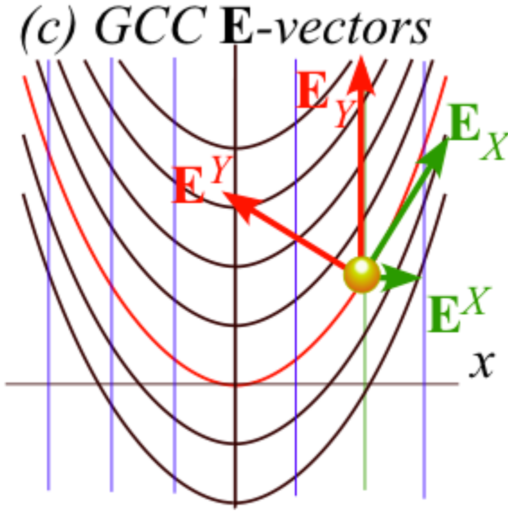
Some Ways to do constraint analysis

Way 1. Simple constraint insertion
Way 2. GCC constraint webs
→ *Find covariant force equations*
Compare covariant vs. contravariant forces

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2 \right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \overset{\text{(metric } \gamma_{AB}\text{)}}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$



(c) GCC **E***-vectors*

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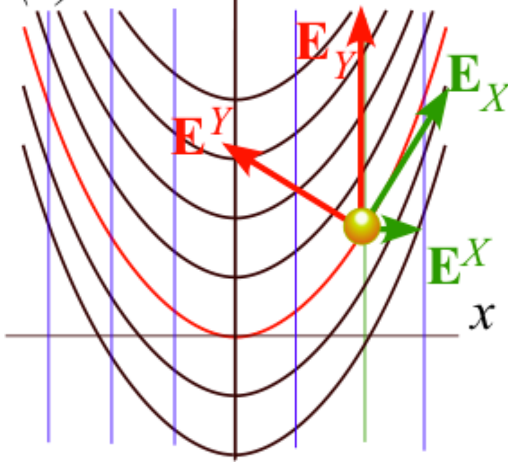

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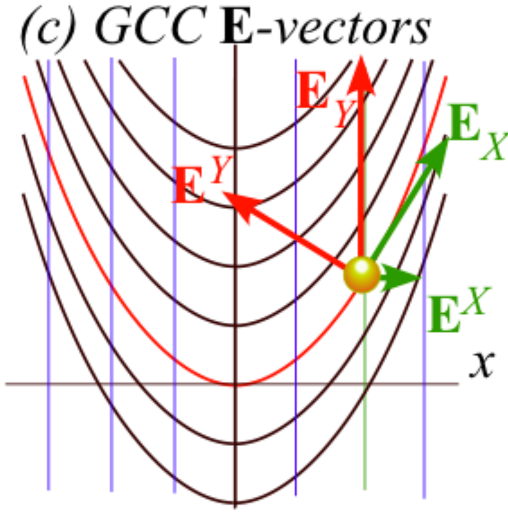
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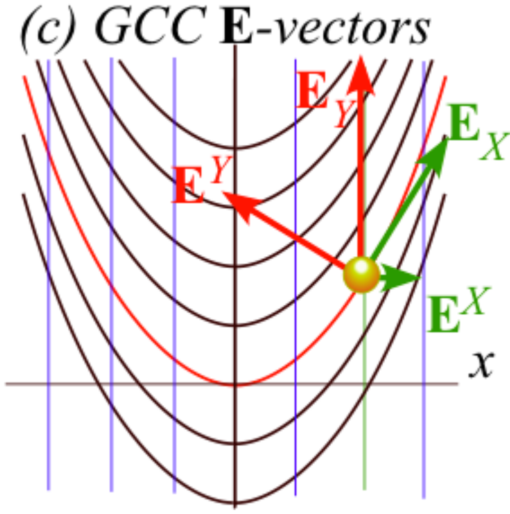
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)



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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\overset{\text{(metric }\gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F^{\text{cov}}_m$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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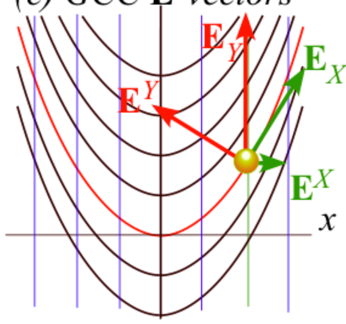
(c) GCC **E**-vectors

Find: Lagrange equations from Lagrangian

$$L = T - V = m\left[\tfrac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \tfrac{1}{2}\dot{Y}^2 - gY - \tfrac{gk}{2}X^2\right]$$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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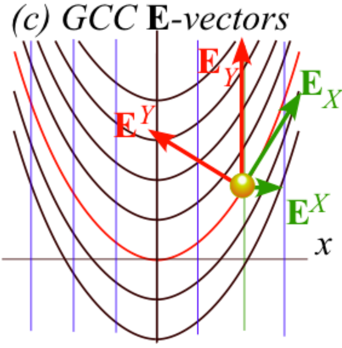
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$


(c) GCC **E**-vectors

Find: Lagrange equations from Lagrangian

$$L = T - V = m\left[\tfrac{1}{2}(1+k^2X^2)\dot X^2 + kX\dot X\dot Y + \tfrac{1}{2}\dot Y^2 - gY - \tfrac{gk}{2}X^2\right]$$

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$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot X \\ \dot Y \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot X} \\ \frac{\partial L}{\partial \dot Y} \end{pmatrix}$$

(1st Lagrange equations) $\quad p_m = \frac{\partial L}{\partial \dot q^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot p_X \\ \dot p_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot X \\ \dot Y \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\quad \dot p_m = \frac{\partial L}{\partial q^m} + F^{cov}_m$

$$\begin{pmatrix} \dot p_X \\ \dot p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot X \\ \dot Y \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot X \\ \dot Y \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot X^2 + k\dot X\dot Y - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F^{cov}_m is zero.($F^{cov}_X = 0 = F^{cov}_Y$)

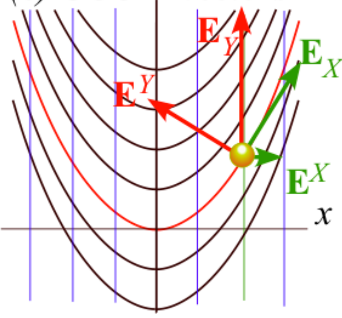
$$\begin{pmatrix} \dot p_X - \frac{\partial L}{\partial X} \\ \dot p_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot X \\ \ddot Y \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot X & k\dot X \\ k\dot X & 0 \end{pmatrix}\begin{pmatrix} \dot X \\ \dot Y \end{pmatrix} - m\begin{pmatrix} k^2X\dot X^2 + k\dot X\dot Y - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot p_X - \frac{\partial L}{\partial X} \\ \dot p_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot X \\ \ddot Y \end{pmatrix} + m\begin{pmatrix} k^2X\dot X^2 + gkX \\ k\dot X^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$

$$\begin{pmatrix} \dot p_X - \frac{\partial L}{\partial X} \\ \dot p_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot X + kX\ddot Y + k^2X\dot X^2 + gkX \\ kX\ddot X + \ddot Y + k\dot X^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$

(c) GCC **E**-*vectors*



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

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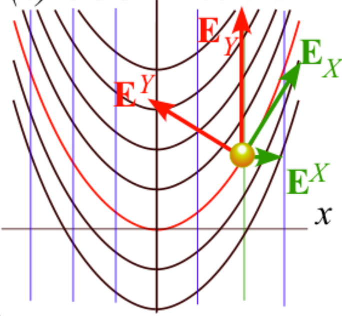
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\overset{\text{(metric } \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\overset{\text{(metric } \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

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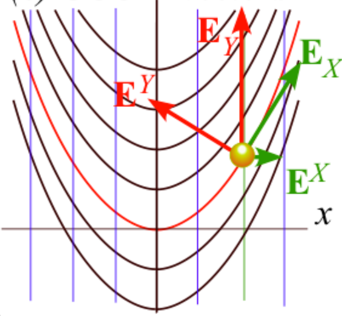
(cancel)

(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

$$\frac{1}{m}\overset{\text{(inverse of } \gamma_{AB})}{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \overset{\text{(inverse of } \gamma_{AB})}{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

Find: Lagrange equations from Lagrangian

$$L = T - V = m\left[\tfrac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \tfrac{1}{2}\dot{Y}^2 - gY - \tfrac{gk}{2}X^2\right]$$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

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(cancel)

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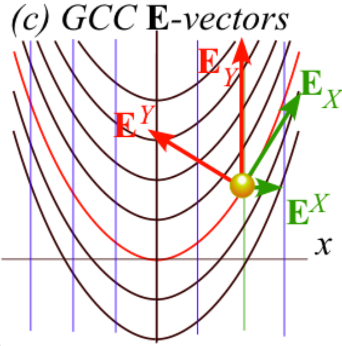

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$$\frac{1}{m}\overset{(inverse\ of\ \gamma_{AB})}{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \overset{(inverse\ of\ \gamma_{AB})}{\begin{pmatrix} \cancel{1} & \cancel{-kX} \\ \cancel{-kX} & \cancel{1+k^2X^2} \end{pmatrix}}\begin{pmatrix} kX(k\dot{X}^2 + g) \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

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(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

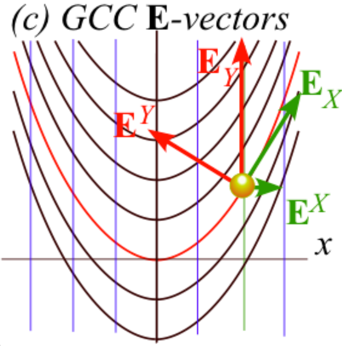
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$


(c) GCC **E**-vectors

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB})

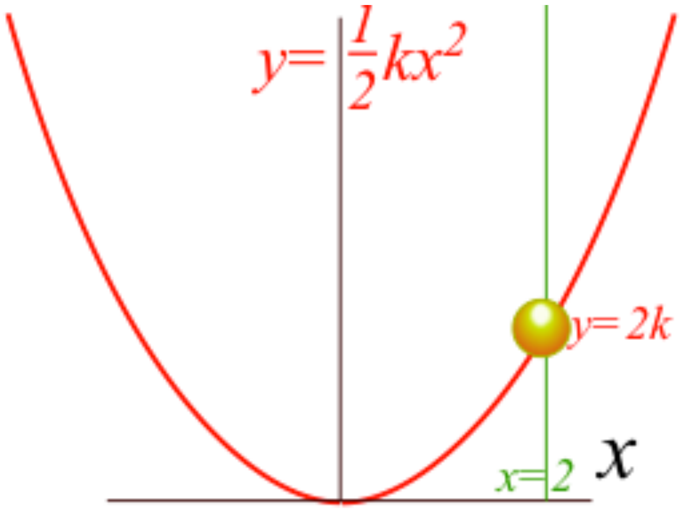
$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} \cancel{1} & \cancel{-kX} \\ \cancel{-kX} & \cancel{1+k^2X^2} \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

(cancelations)

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$\ddot{x} = 0 = \ddot{X}$

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

- *Find covariant force equations*
- → *Compare covariant vs. contravariant forces*

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

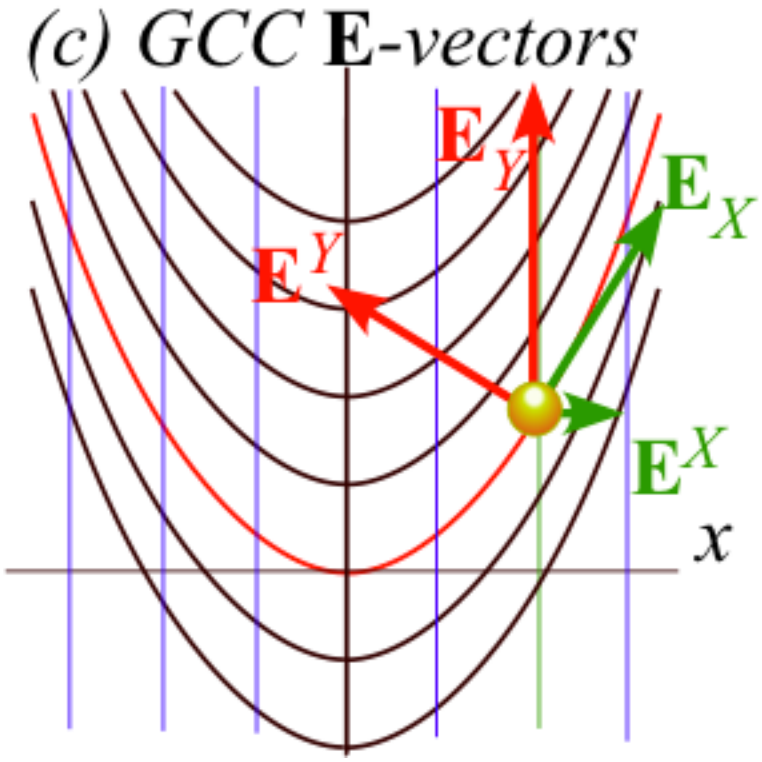
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

General case repeated from p.35

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



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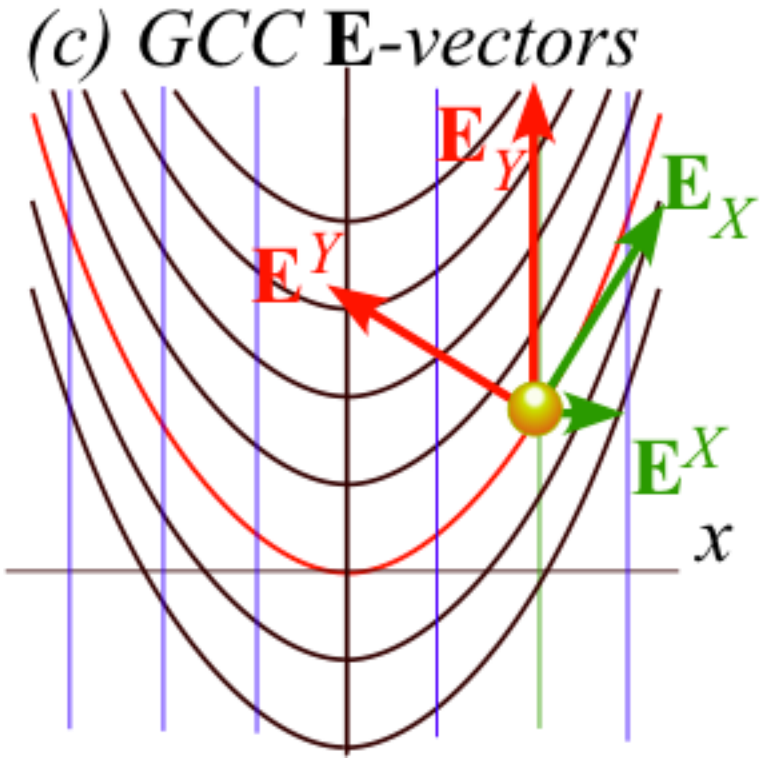
*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇*Y*.)

$$\begin{aligned}\mathbf{F}(Y=const.) &= F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y \\ &= 0\cdot\nabla X + F_Y^{cov}\nabla Y \\ &= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y\end{aligned}$$



General case repeated from p.35

$$\begin{pmatrix}\dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y}\end{pmatrix} = m\begin{pmatrix}(1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}F_X^{cov} \\ F_Y^{cov}\end{pmatrix}$$

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So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

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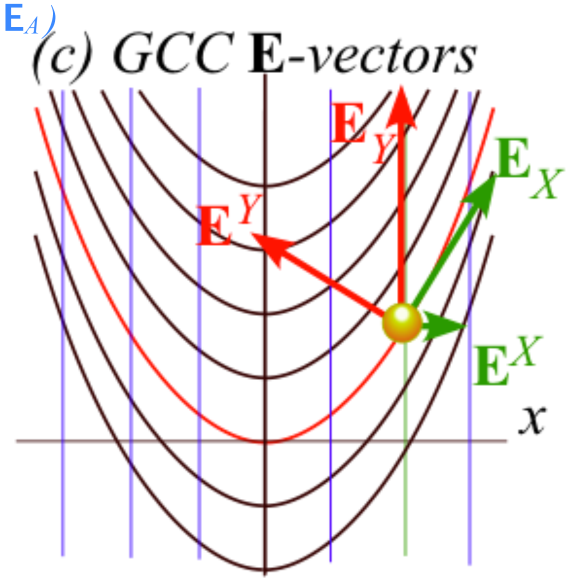

$\dot{Y} = 0 = \ddot{Y}$

General case repeated from p.35

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

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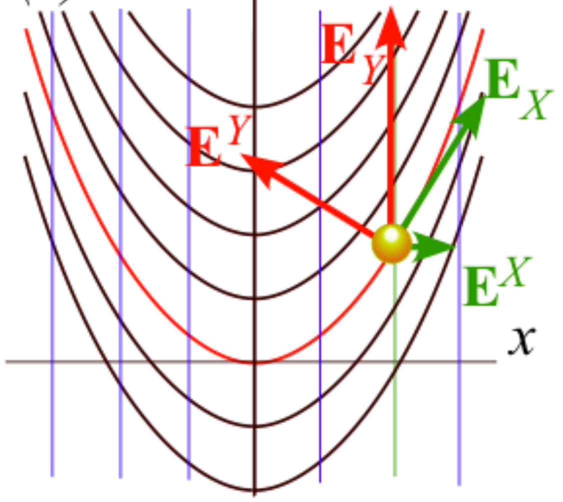
(c) GCC ***E****-vectors*



$\dot{Y} = 0 = \ddot{Y}$

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$$m\begin{pmatrix} (1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ F_Y^{cov} \end{pmatrix} \dashrightarrow \ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

FINALLY ! We get the Way 1. solution of p.12

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.35

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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$$\mathbf{F} = \underbrace{m(kX\ddot{X} + 0 + k\dot{X}^2 + g)}_{F_Y^{cov}}\underbrace{\begin{pmatrix} -kX \\ 1 \end{pmatrix}}_{\mathbf{E}^Y}$$

General case repeated from p.35

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

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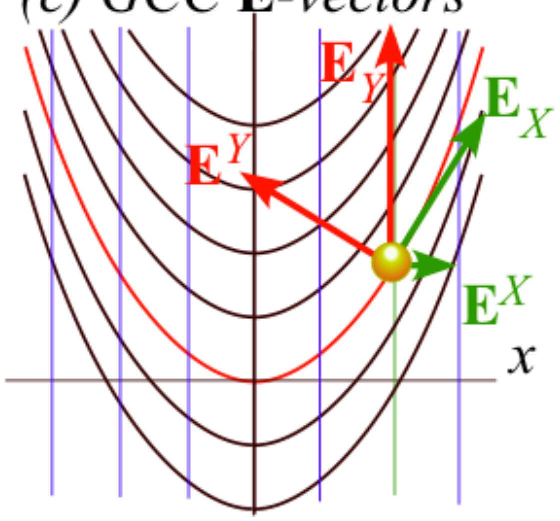

(c) GCC **E**-*vectors*

$$\dot{Y} = 0 = \ddot{Y}$$

Recall: $x \equiv X$

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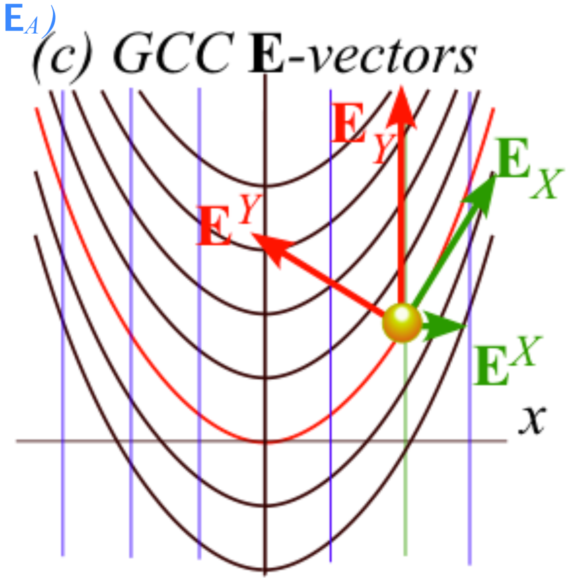
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$$\mathbf{F} = \quad F_Y^{cov} \quad \mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix} -kX \\ 1 \end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix} -kX \\ 1 \end{pmatrix}$$

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$$\mathbf{F} = F^X_{con}\mathbf{E}_X + F^Y_{con}\mathbf{E}_Y = F^X_{con}\frac{\partial \mathbf{r}}{\partial X} + F^Y_{con}\frac{\partial \mathbf{r}}{\partial Y}$$

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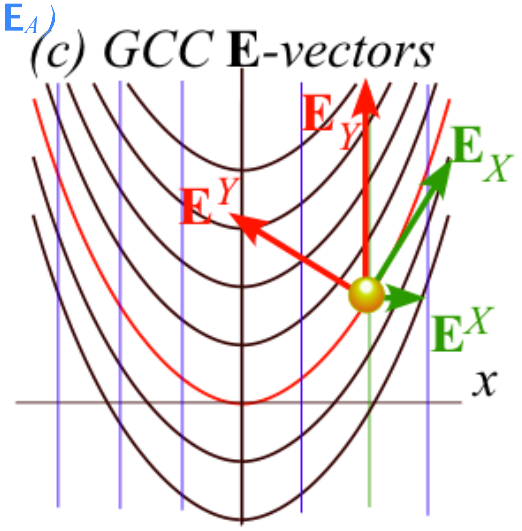
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$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix} -kX \\ 1 \end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix} -kX \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \left(= \begin{pmatrix} 0 \\ mk\dot{X}^2 + mg \end{pmatrix}\right)_{at:X=0}$$

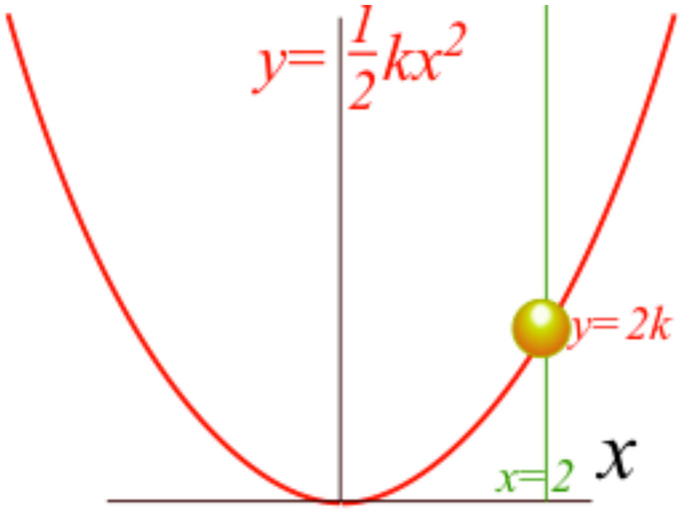
Centripetal
force *mkv²+mg*
(what roller-coaster rider feels at bottom)

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

$$-g = \ddot{y} = \frac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2 + Y\right)$$
$$= k\dot{X}^2 + kX\ddot{X} + \ddot{Y}\left(= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0\right)$$

Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

→ *Way 3. OCC constraint webs*
  - *Preview of atomic-Stark orbits*
    - *Classical Hamiltonian separability*

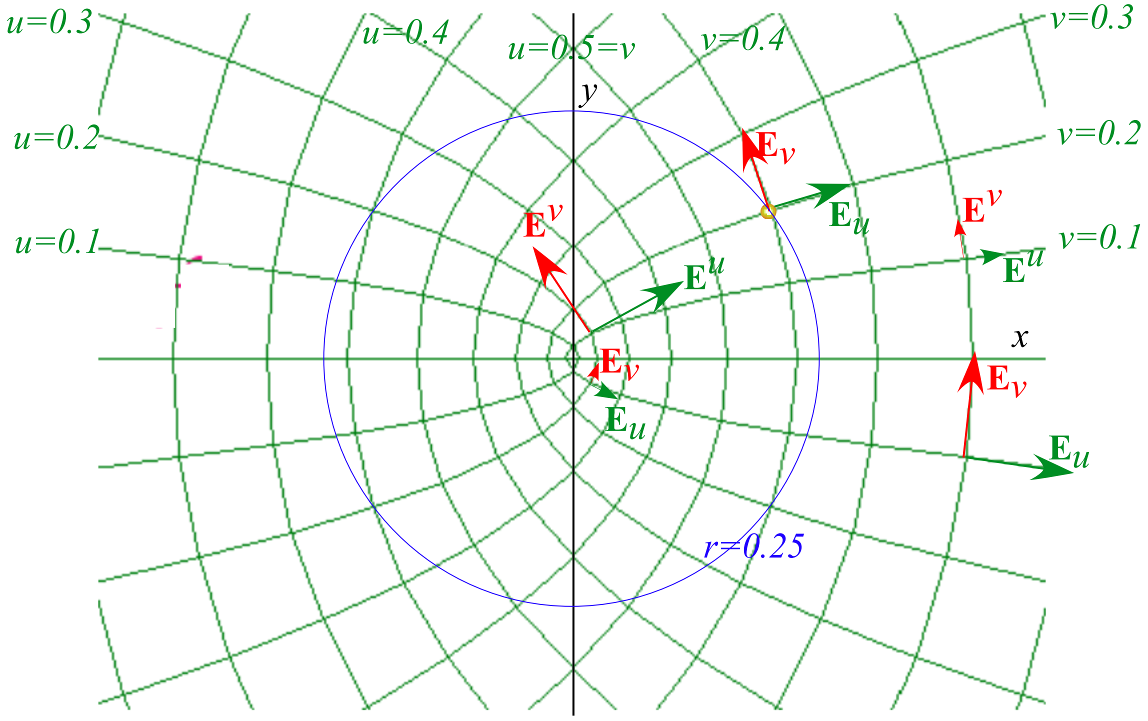
Way 4. Lagrange multipliers
  - *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *"Non-Holonomic" multipliers*

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u + iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^* z = x^2 + y^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2u^2v^2$$



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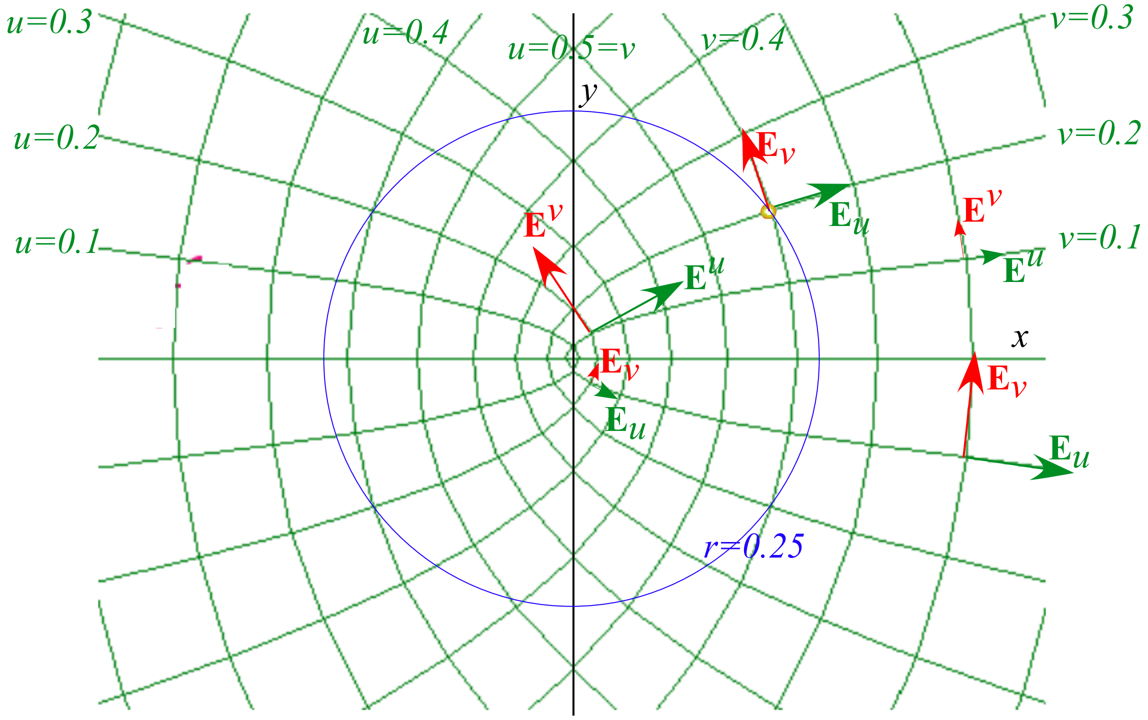
$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$



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$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$

$$2u^2 = r + x = \sqrt{x^2+y^2} + x$$

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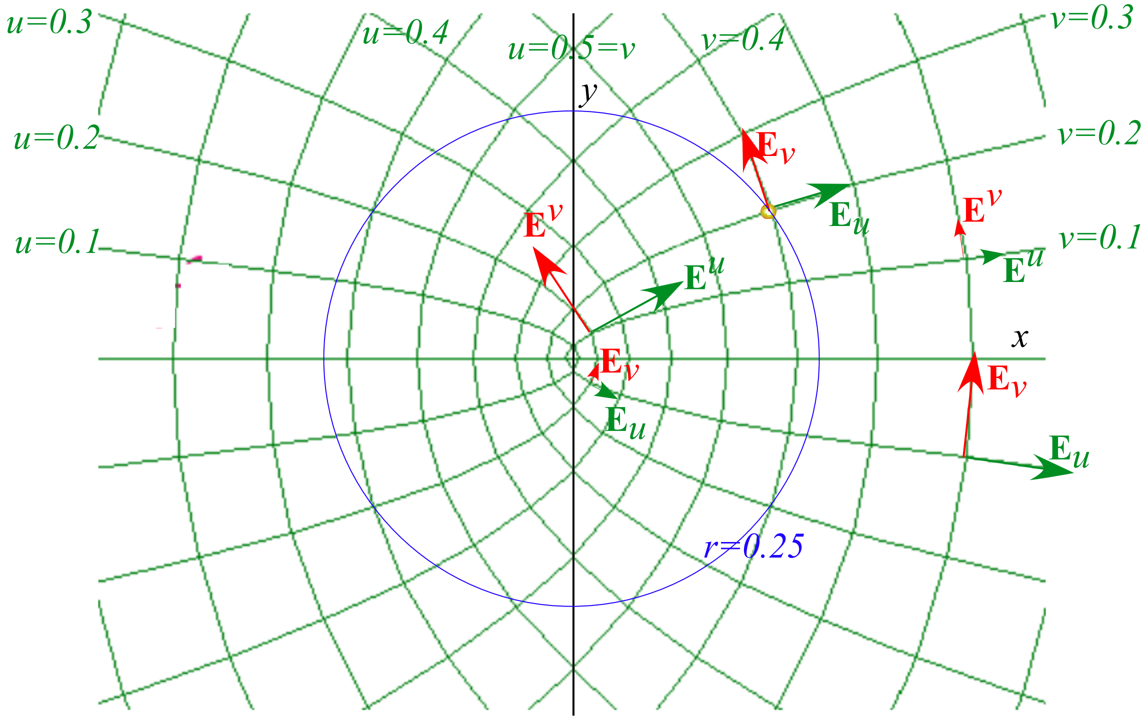

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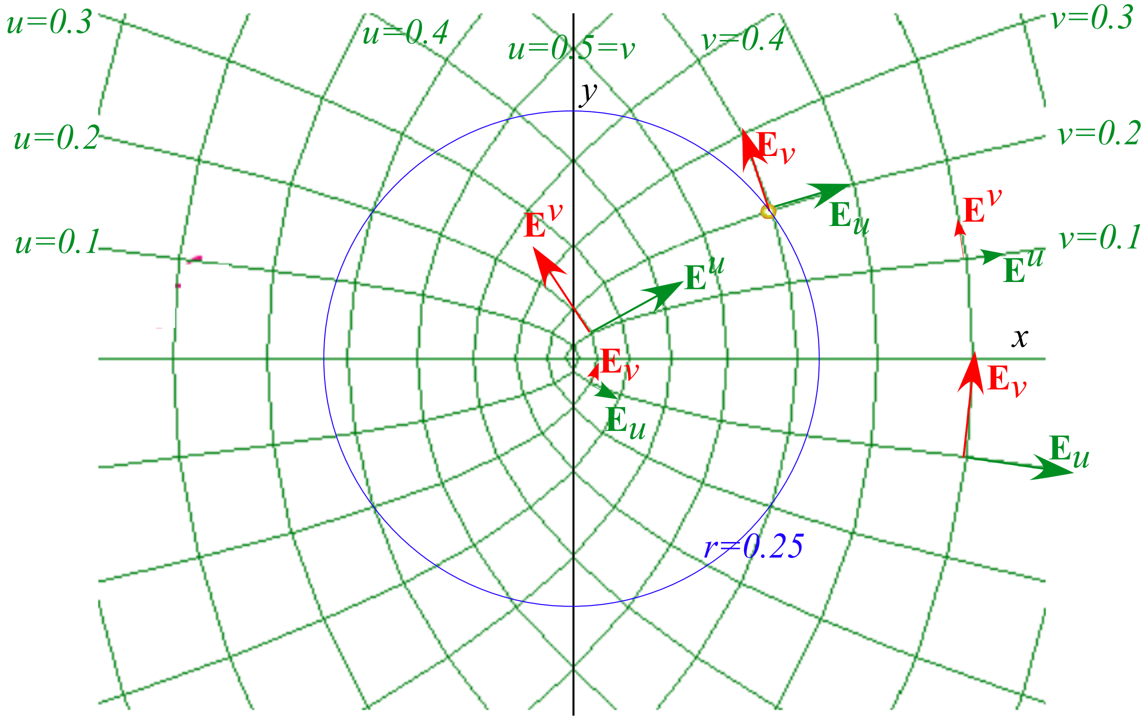
Gives confocal parabolics



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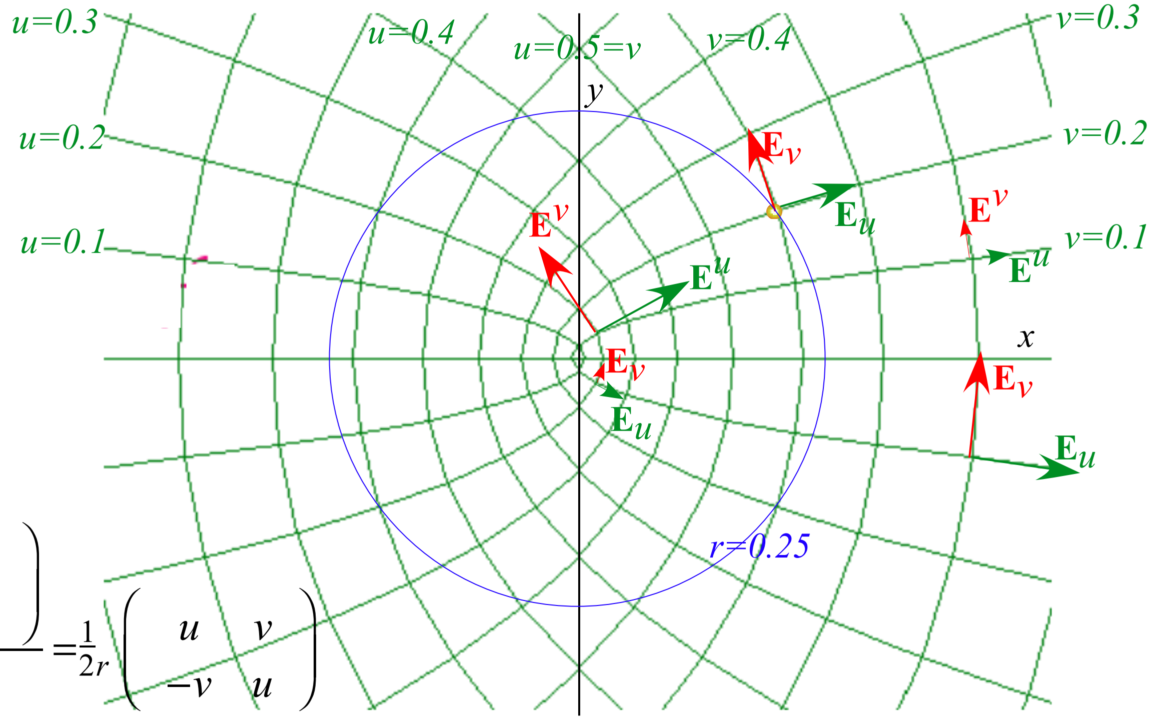

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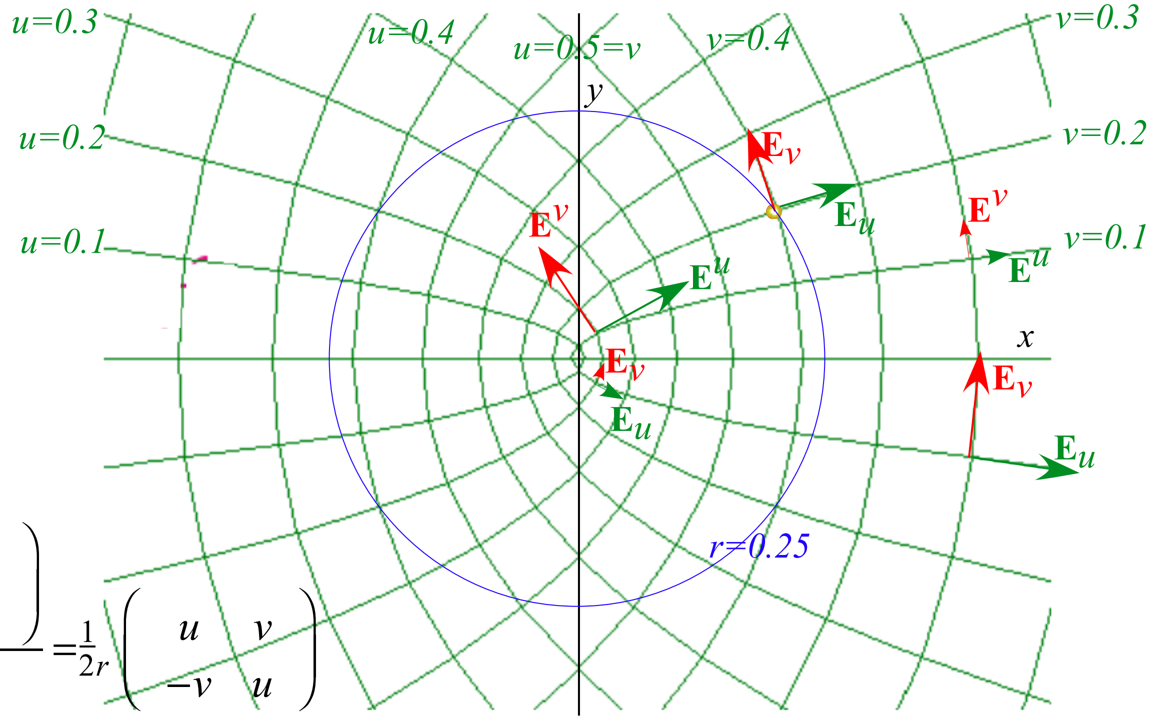

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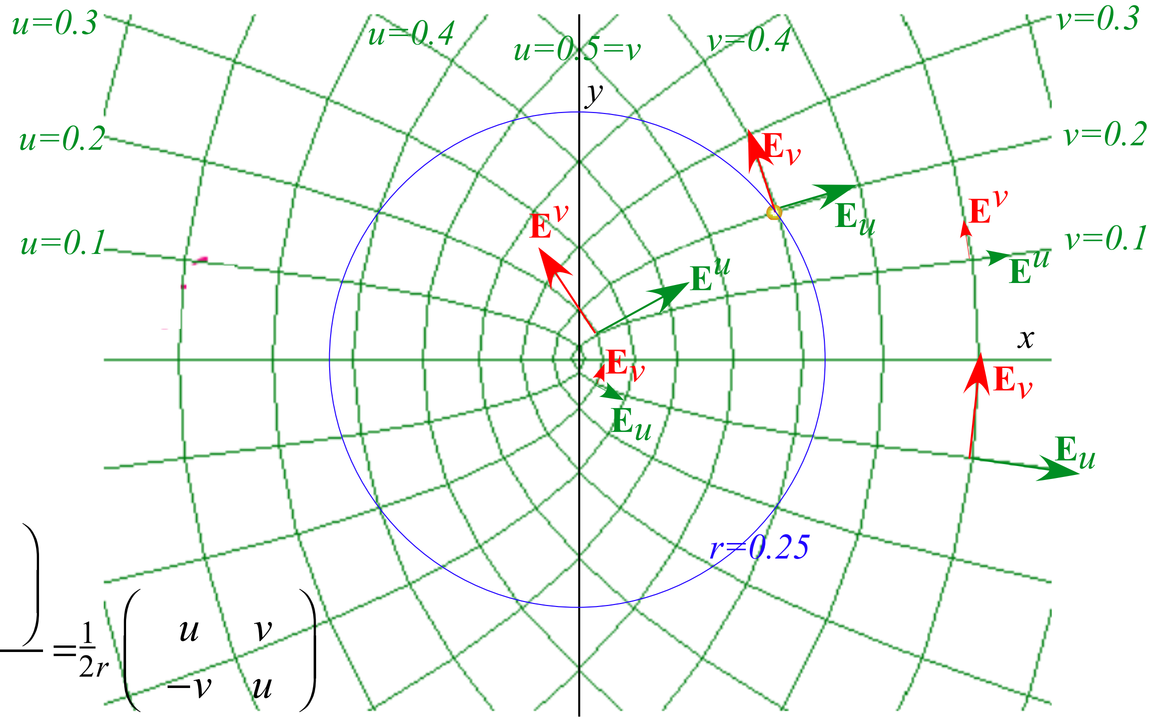

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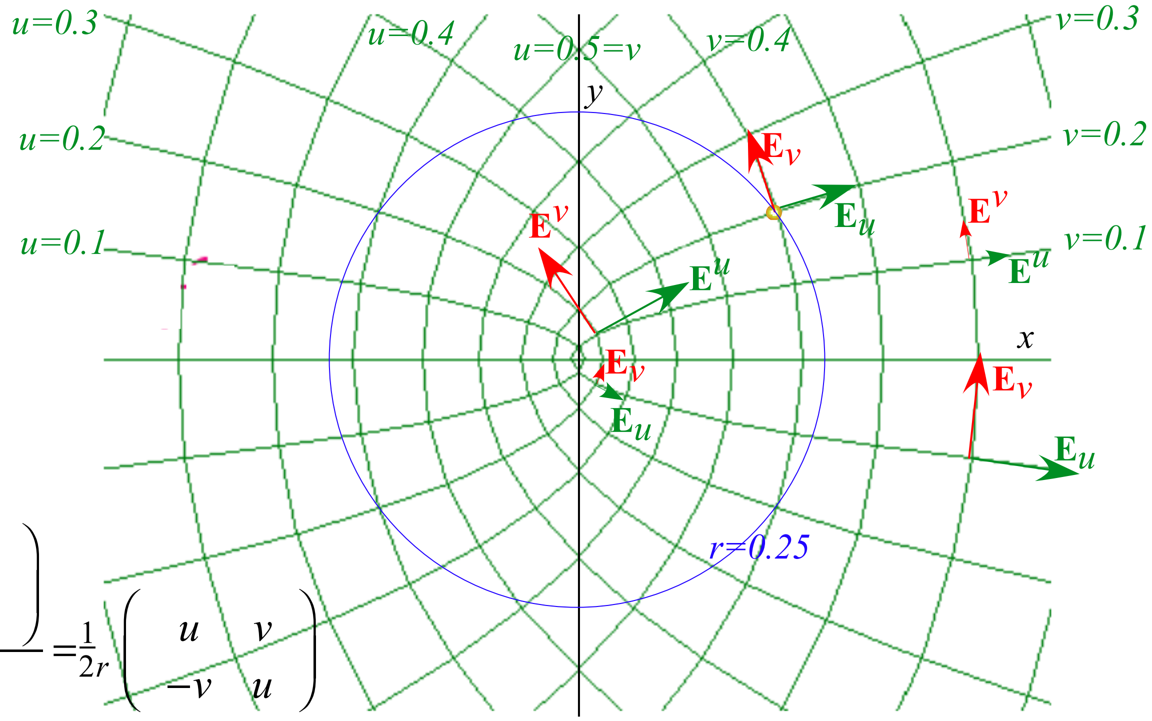

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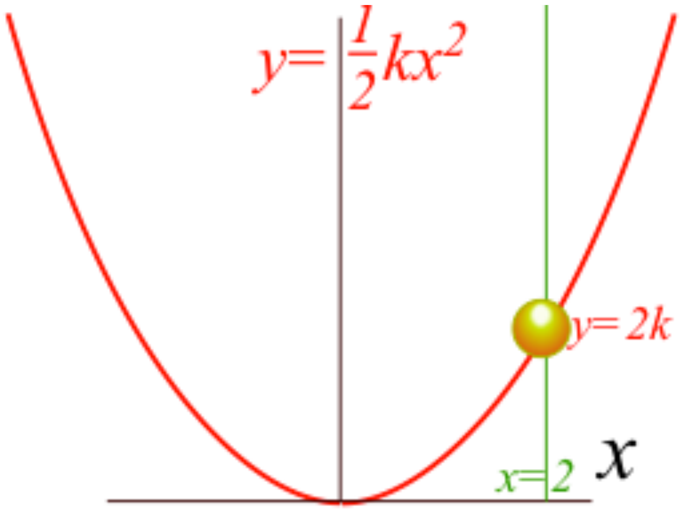
Simple constrained problem...



...and a variety of solutions

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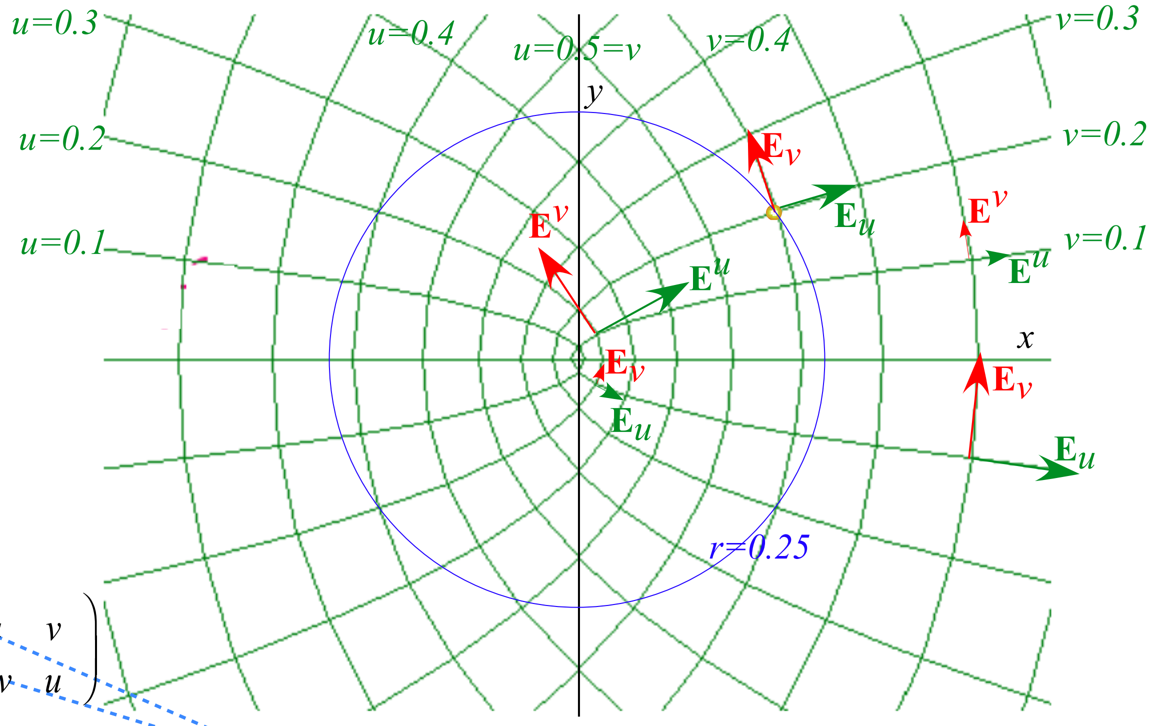

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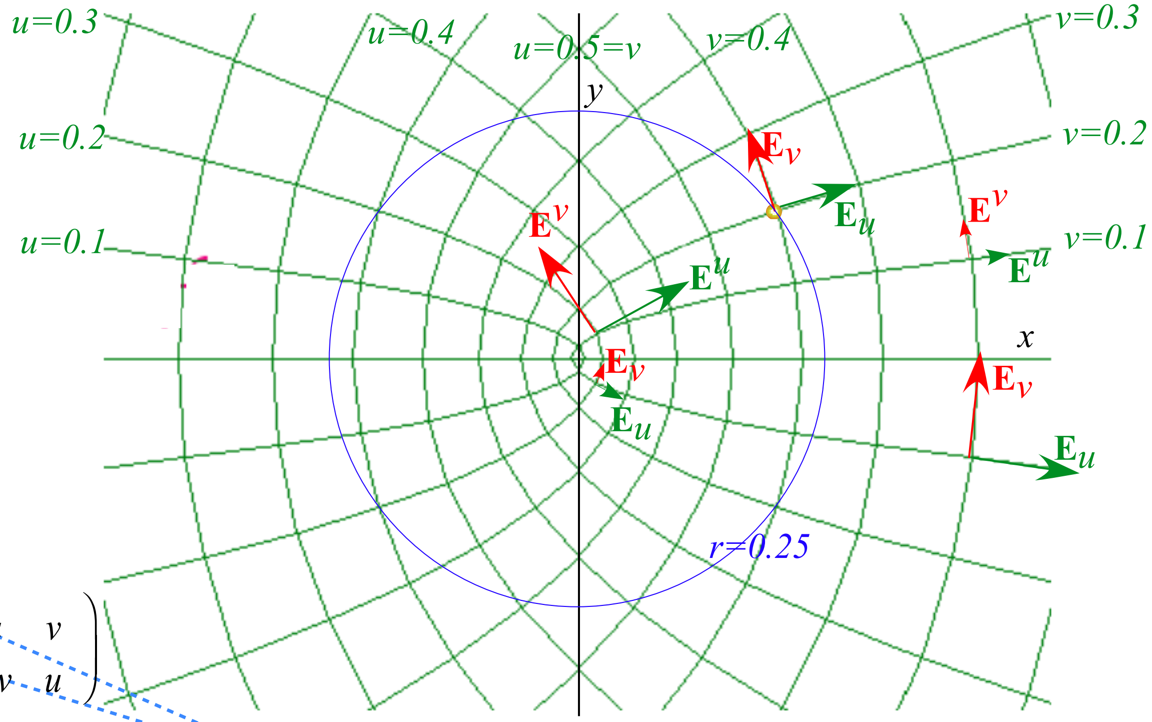

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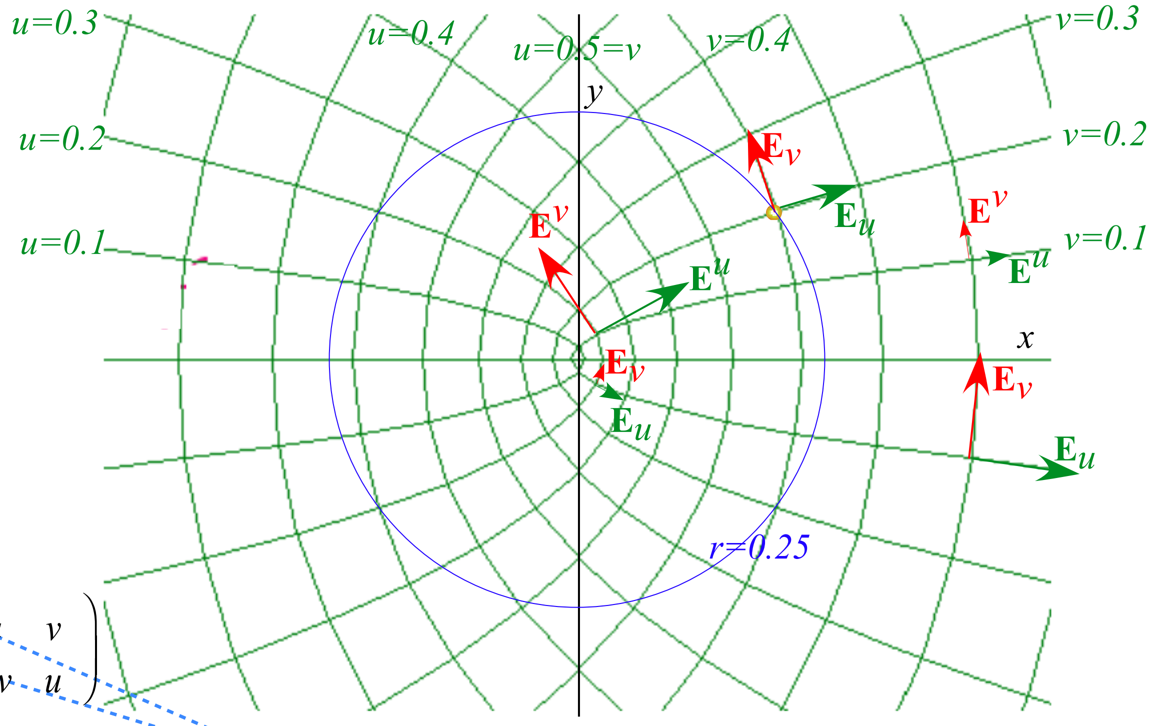

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Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

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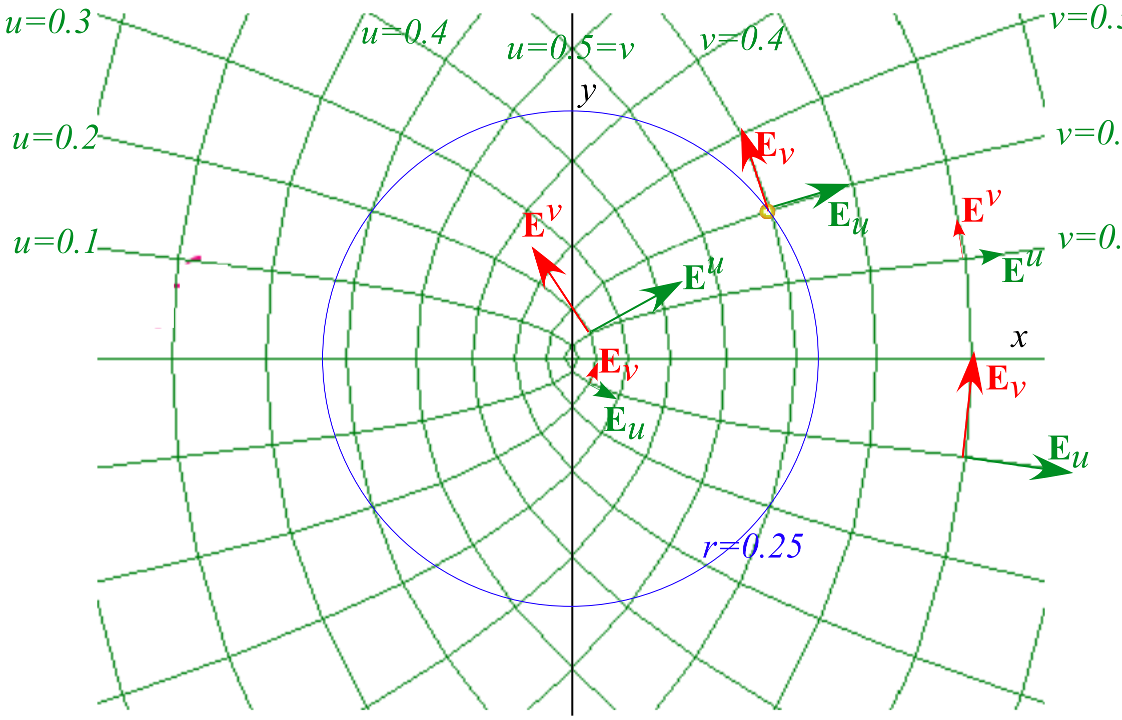

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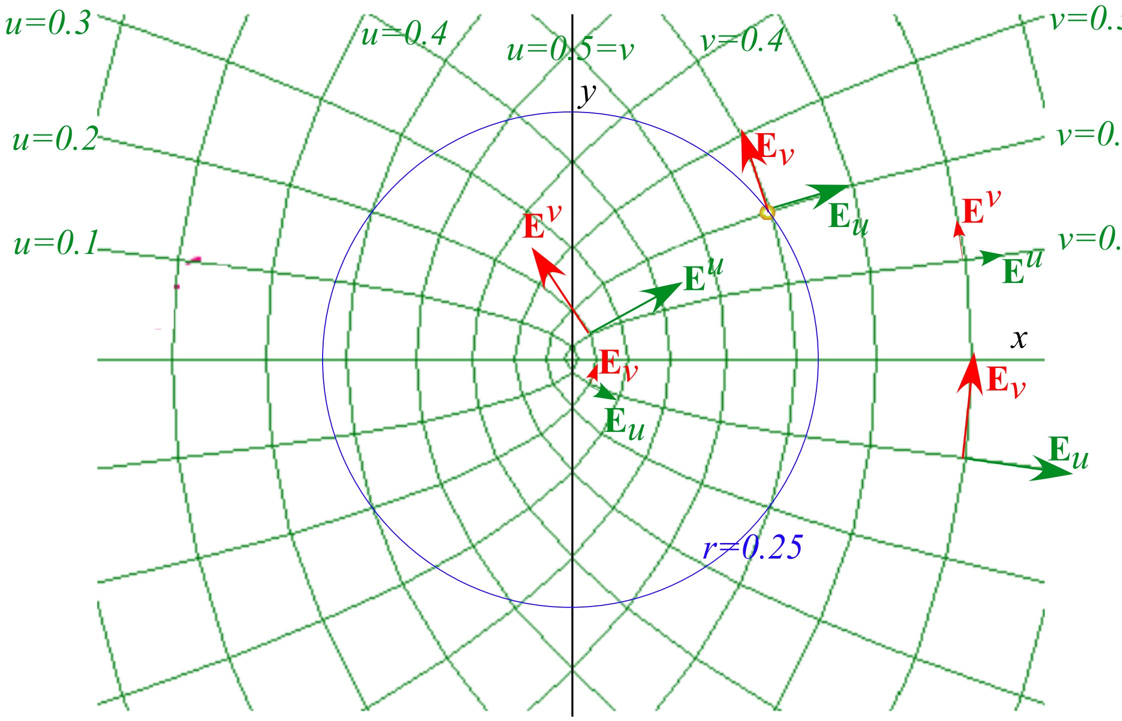

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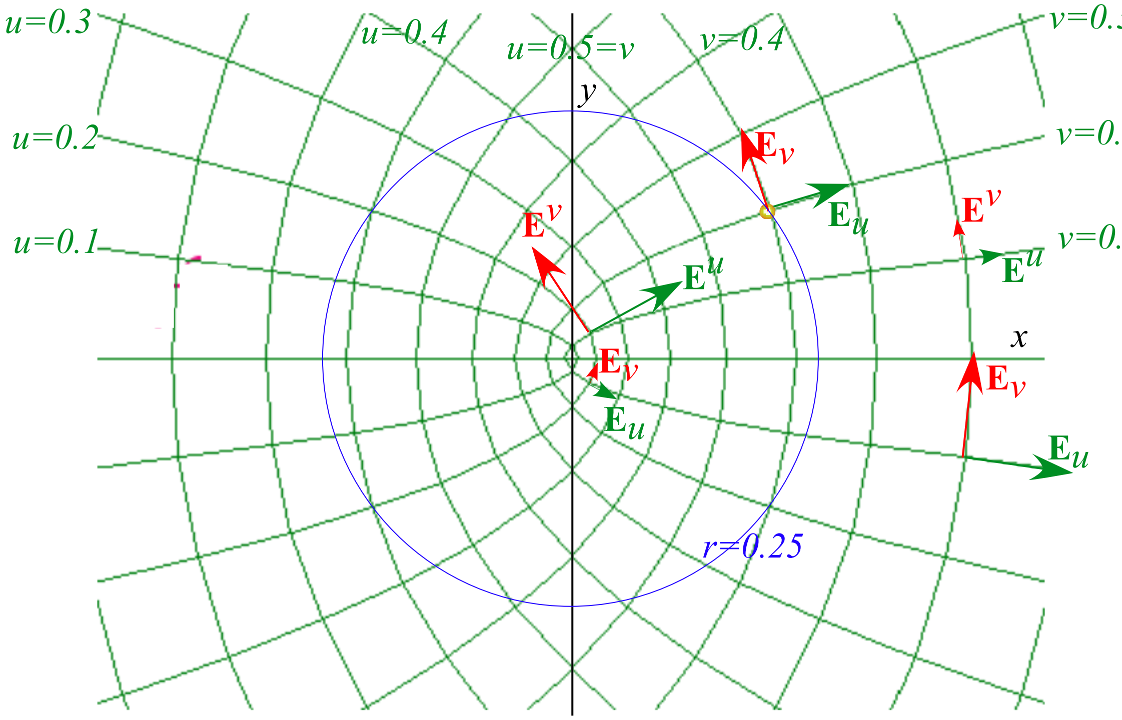
$$4(u^2+v^2)E = \tfrac{1}{2m}\left(p_u^2+p_v^2\right) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H = E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$



Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L=\tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b)-V=\tfrac{m}{2}(g_{uu}\dot{u}^2+g_{vv}\dot{v}^2)-V=2m(\dot{u}^2+\dot{v}^2)(u^2+v^2)-V$$

$$H=\tfrac{1}{2m}(g^{ab}p_ap_b)+V=\tfrac{1}{2m}(g^{uu}p_u^2+g^{vv}p_v^2)+V=\frac{p_u^2+p_v^2}{8m(u^2+v^2)}+V$$

$$V=\varepsilon x+k/r$$

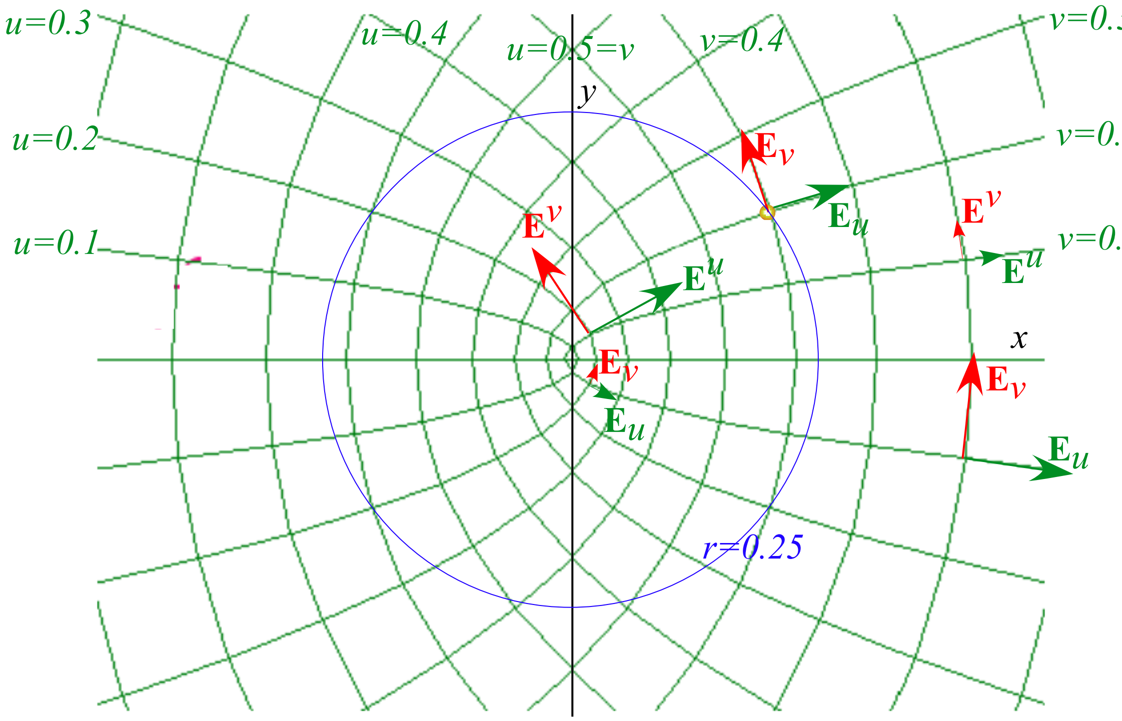
Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E=\tfrac{1}{2m}(p_u^2+p_v^2)+4(u^4-v^4)\varepsilon+4k \quad \text{for: } H=E \text{ and: } V=\varepsilon x+\frac{k}{r}=\varepsilon(u^2-v^2)+\frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0=\tfrac{1}{2m}p_u^2-4Eu^2+4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2-4Ev^2-4\varepsilon v^4+4k \quad = \quad h_u \quad + \quad h_v$$



Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L=\tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b)-V=\tfrac{m}{2}(g_{uu}\dot{u}^2+g_{vv}\dot{v}^2)-V=2m(\dot{u}^2+\dot{v}^2)(u^2+v^2)-V$$

$$H=\tfrac{1}{2m}(g^{ab}p_ap_b)+V=\tfrac{1}{2m}(g^{uu}p_u^2+g^{vv}p_v^2)+V=\frac{p_u^2+p_v^2}{8m(u^2+v^2)}+V$$

$$V=\varepsilon x+k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E=\tfrac{1}{2m}(p_u^2+p_v^2)+4(u^4-v^4)\varepsilon+4k \quad \text{for: } H=E \text{ and: } V=\varepsilon x+\frac{k}{r}=\varepsilon(u^2-v^2)+\frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0=\tfrac{1}{2m}p_u^2-4Eu^2+4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2-4Ev^2-4\varepsilon v^4+4k \quad = \quad h_u \quad + \quad h_v$$

Zero Stark-field $(\varepsilon=0)$ gives h_u or h_v harmonic oscillation if $E<0$. It's unstable or anharmonic otherwise.

$$\dot{p}_u=-\frac{\partial h_u}{\partial u}=-8Eu+16\varepsilon u^3 \qquad \dot{u}=\frac{\partial h_u}{\partial p_u}=p_u/m \qquad \dot{p}_v=-\frac{\partial h_v}{\partial v}=-8Ev-16\varepsilon v^3 \qquad \dot{v}=\frac{\partial h_v}{\partial p_v}=p_v/m$$

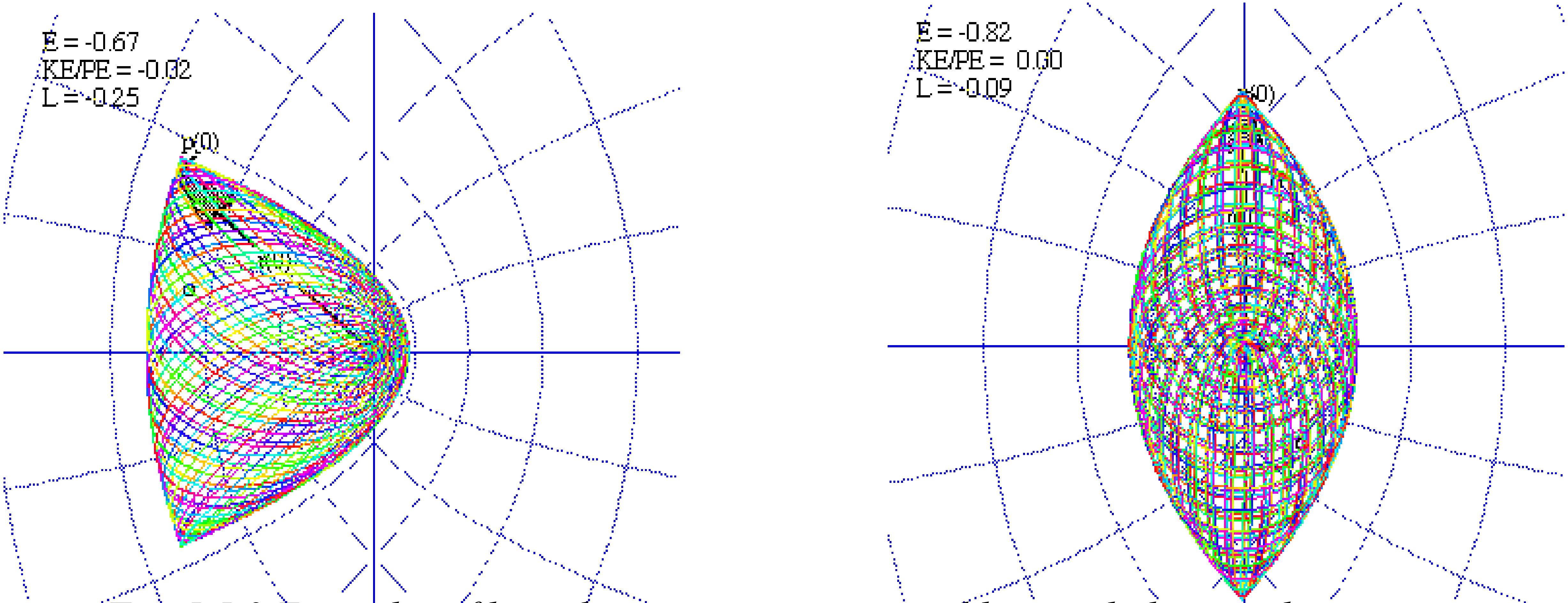


Fig. 5.5.3 Examples of bound-state motion restricted by parabolic coordinates

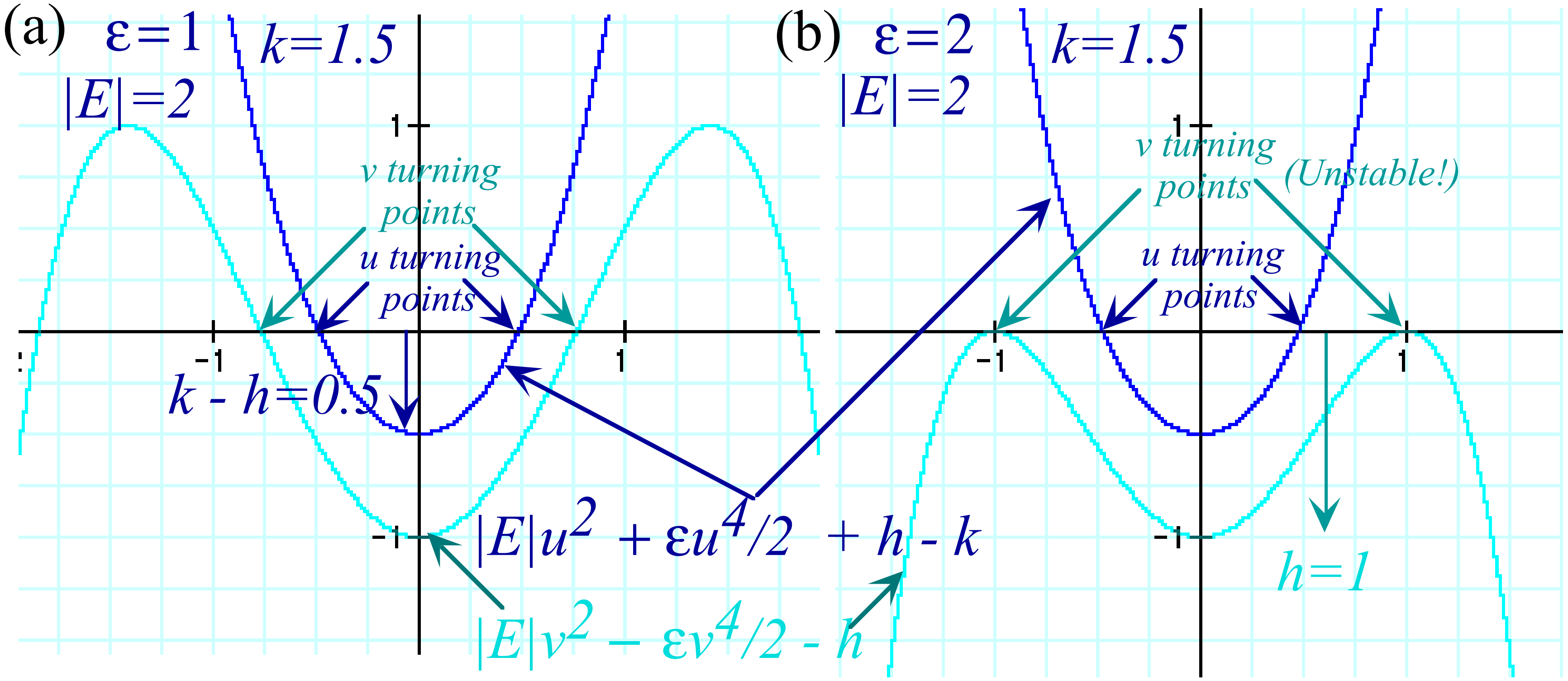
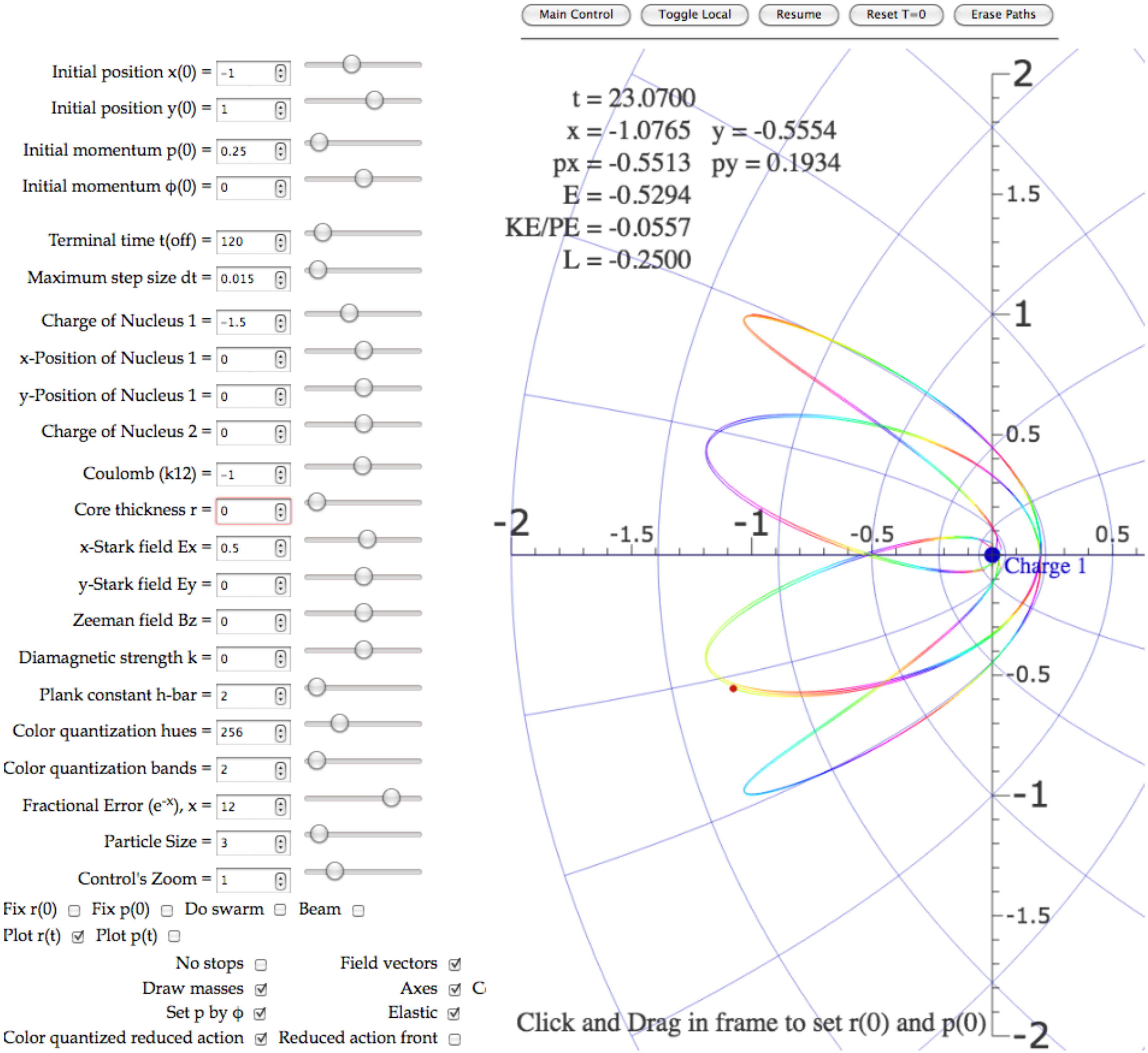


Fig. 5.5.2 Effective potentials for parabolic coordinates

Examples of bound-state motion restricted by parabolic coordinates (H classical electronic Stark-field orbits with color-quantization)



Initial position x(0) = -1

Initial position y(0) = 1

Initial momentum p(0) = 0.25

Initial momentum ϕ(0) = 0

Terminal time t(off) = 120

Maximum step size dt = 0.015

Charge of Nucleus 1 = -1.5

x-Position of Nucleus 1 = 0

y-Position of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = -1

Core thickness r = 0

x-Stark field Ex = 0.5

y-Stark field Ey = 0

Zeeman field Bz = 0

Diamagnetic strength k = 0

Plank constant h-bar = 2

Color quantization hues = 256

Color quantization bands = 2

Fractional Error (e^{-x}), x = 12

Particle Size = 3

Control's Zoom = 1

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐

Plot r(t) ☑ Plot p(t) ☐

No stops ☐ Field vectors ☑

Draw masses ☑ Axes ☑ C

Set p by ϕ ☑ Elastic ☑

Color quantized reduced action ☑ Reduced action front ☐

t = 23.0700

x = -1.0765 y = -0.5554

px = -0.5513 py = 0.1934

E = -0.5294

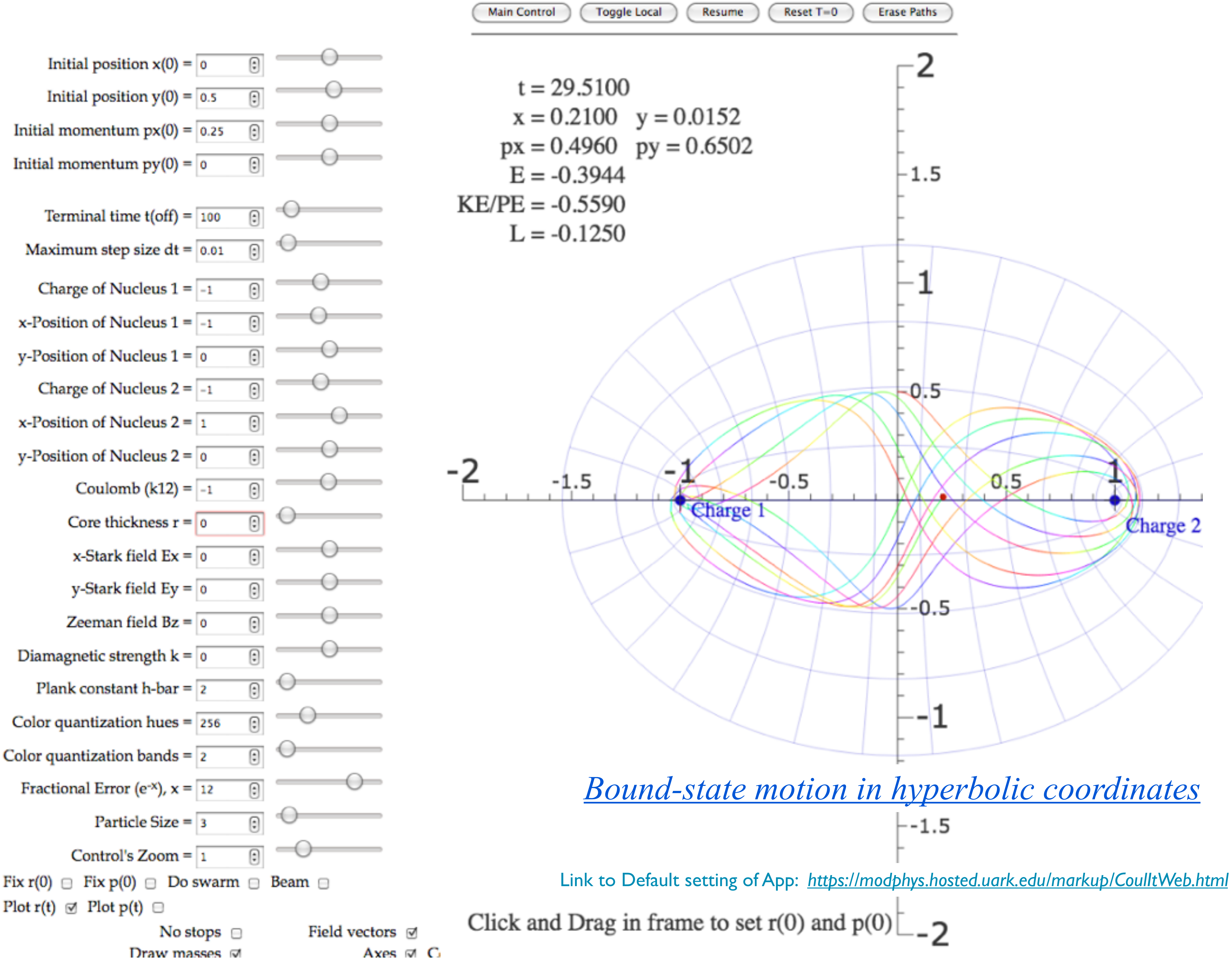
KE/PE = -0.0557

L = -0.2500

Click and Drag in frame to set r(0) and p(0)

Bound-state motion in parabolic coordinates   https://modphys.hosted.uark.edu/markup/CoulItWeb.html

Examples of bound-state motion restricted by hyperbolic-elliptic coordinates (H2+-ion classical electronic orbits with color-quantization)



Initial position x(0) = 0
Initial position y(0) = 0.5
Initial momentum px(0) = 0.25
Initial momentum py(0) = 0
Terminal time t(off) = 100
Maximum step size dt = 0.01
Charge of Nucleus 1 = -1
x-Position of Nucleus 1 = -1
y-Position of Nucleus 1 = 0
Charge of Nucleus 2 = -1
x-Position of Nucleus 2 = 1
y-Position of Nucleus 2 = 0
Coulomb (k12) = -1
Core thickness r = 0
x-Stark field Ex = 0
y-Stark field Ey = 0
Zeeman field Bz = 0
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 256
Color quantization bands = 2
Fractional Error (e^{-x}), x = 12
Particle Size = 3
Control's Zoom = 1

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
No stops ☐ Field vectors ☑
Draw masses ☑ Axes ☑ C

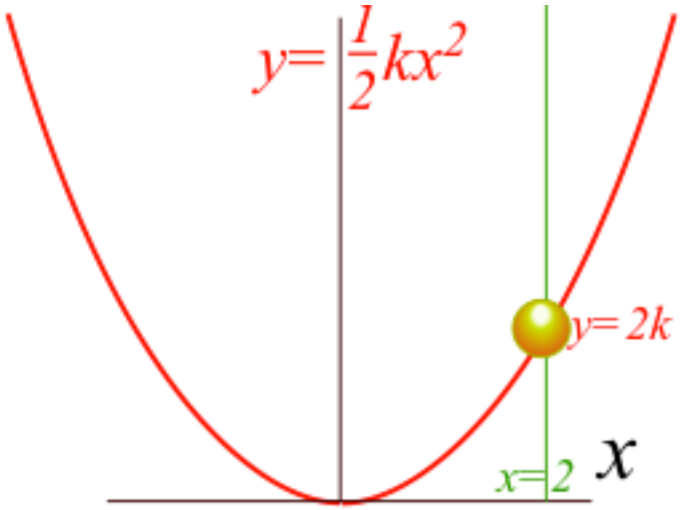
t = 29.5100
x = 0.2100 y = 0.0152
px = 0.4960 py = 0.6502
E = -0.3944
KE/PE = -0.5590
L = -0.1250

Bound-state motion in hyperbolic coordinates

Link to Default setting of App: *https://modphys.hosted.uark.edu/markup/CoulIt Web.html*

Click and Drag in frame to set r(0) and p(0)

Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

Way 3. OCC constraint webs
  - *Preview of atomic-Stark orbits*
    - *Classical Hamiltonian separability*

→ *Way 4. Lagrange multipliers*
  - *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *"Non-Holonomic" multipliers*

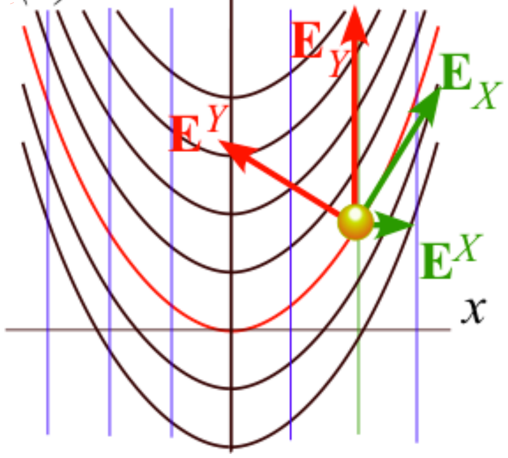
Lagrange multiplier approaches

$\frac{1}{2}$

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)


(c) GCC **E**-*vectors*

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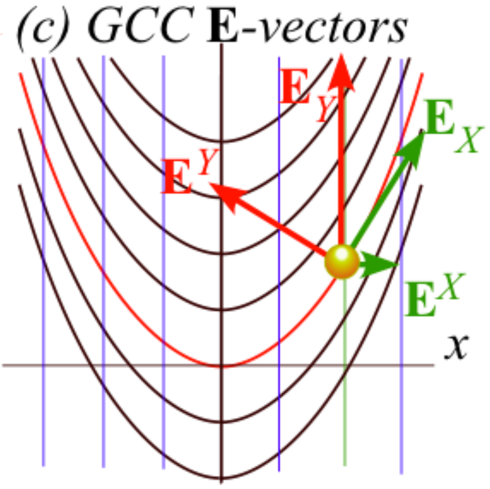
(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



Lagrange multiplier approaches

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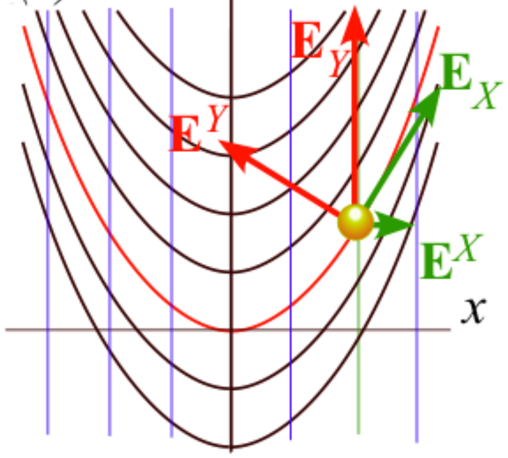
(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

(c) GCC **E**-*vectors*



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(Back to "Stupid-Parabolic" GCC)

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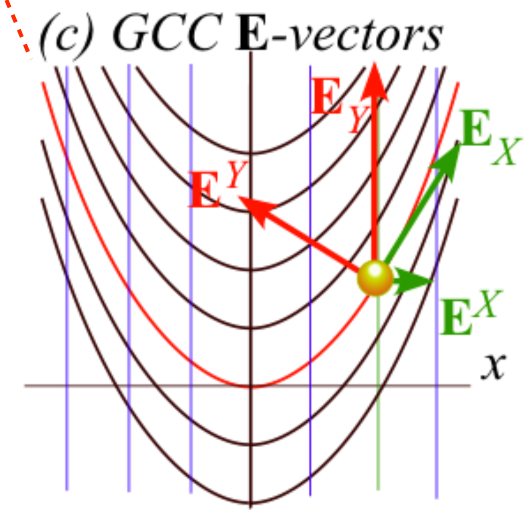

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \dfrac{\partial c^1}{\partial x} \\ \dfrac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola *y=1/2kx²* is defined as follows.

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(Back to "Stupid-Parabolic" GCC)

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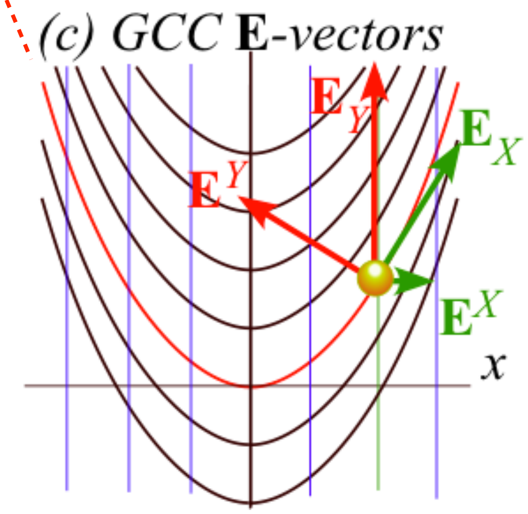

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(Back to "Stupid-Parabolic" GCC)

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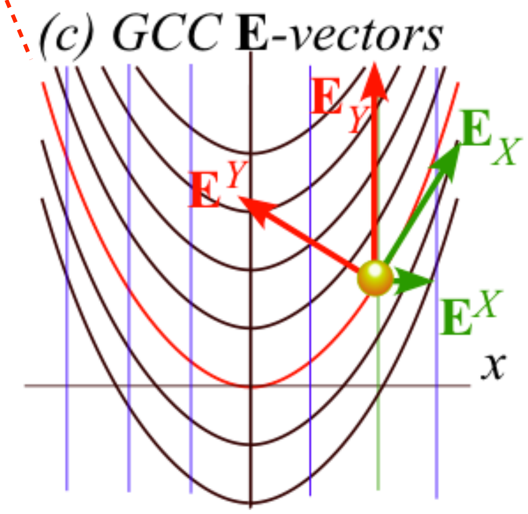
(c) GCC **E**-vectors

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Lagrange multiplier approaches

$\frac{1}{2}$

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

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(Back to "Stupid-Parabolic" GCC)

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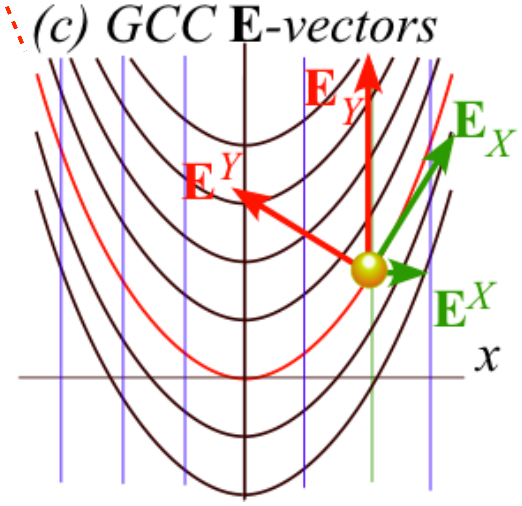
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$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

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(Back to "Stupid-Parabolic" GCC)

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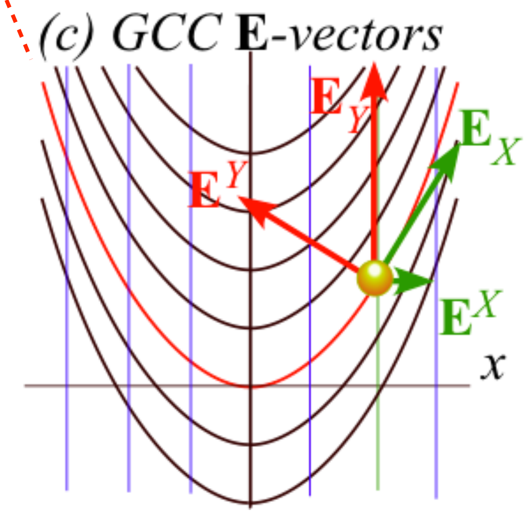
(c) GCC **E**-vectors

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$$\begin{pmatrix}m\ddot{x}\\ m\ddot{y}\end{pmatrix} = \lambda\begin{pmatrix}kx\\ -1\end{pmatrix} - \begin{pmatrix}0\\ mg\end{pmatrix}$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$.

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

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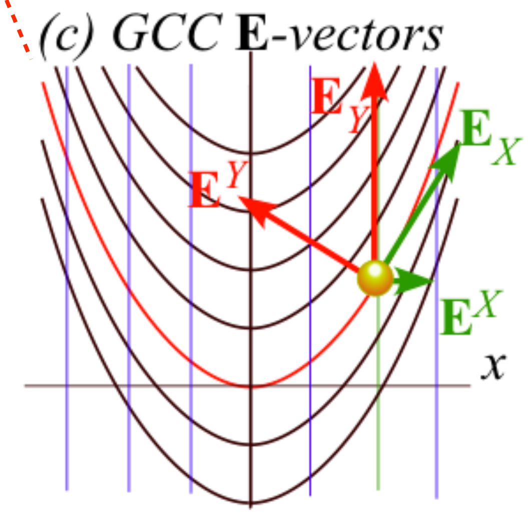
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$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola *y=1/2kx²* is defined as follows.

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(Back to "Stupid-Parabolic" GCC)

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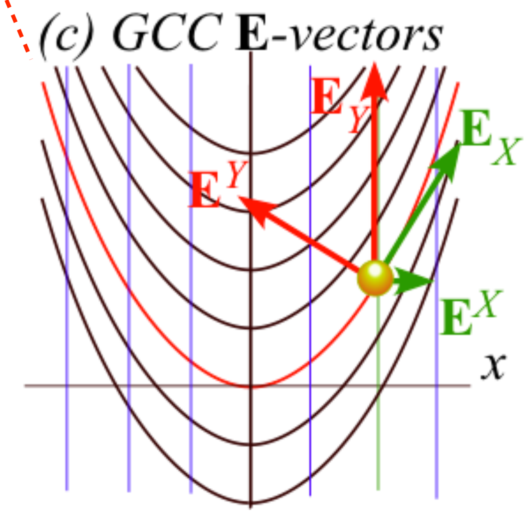

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla\left(\tfrac{1}{2} kx^2 - y\right) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

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$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function *y=1/2kx²* has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to “Stupid-Parabolic” GCC)

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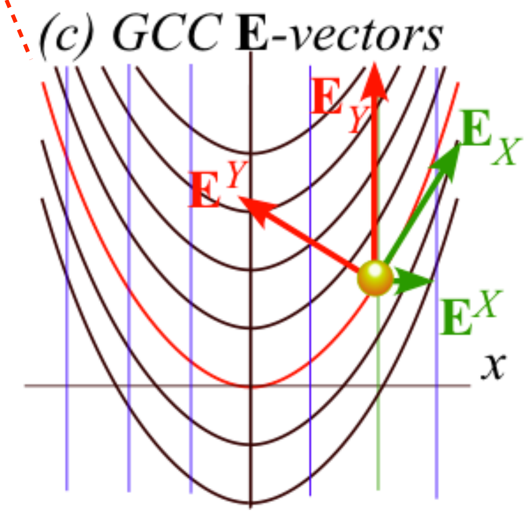
(c) GCC **E**-vectors

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Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola *y=1/2kx²* is defined as follows.

$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

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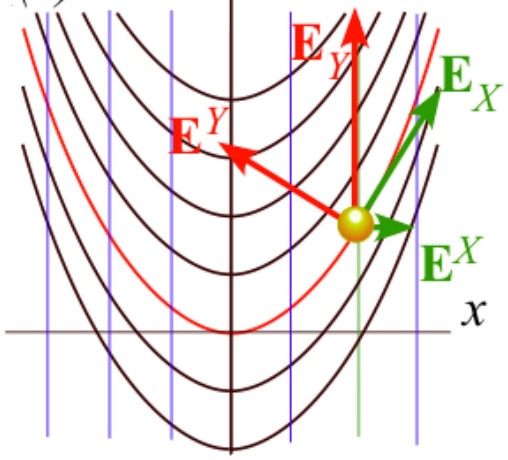
(c) GCC ***E****-vectors*



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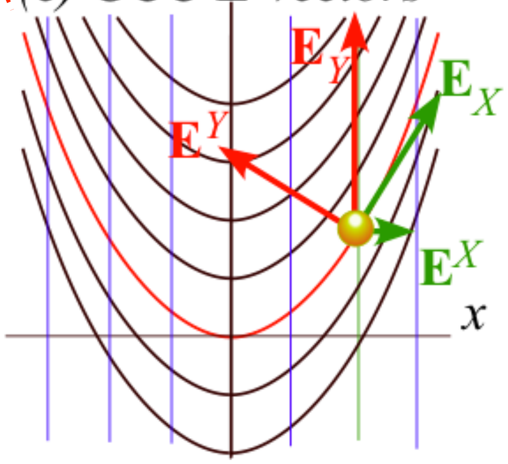

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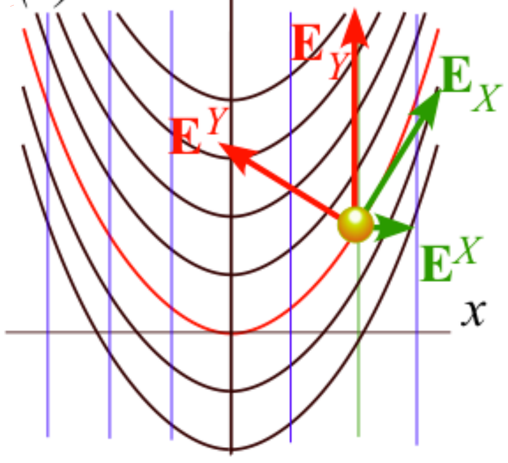

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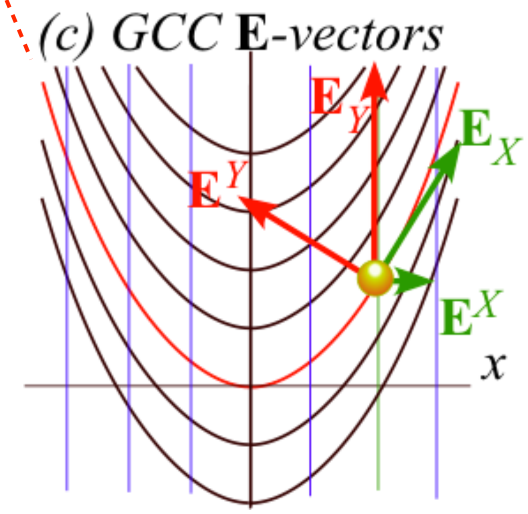
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(Same equation as on p.12)

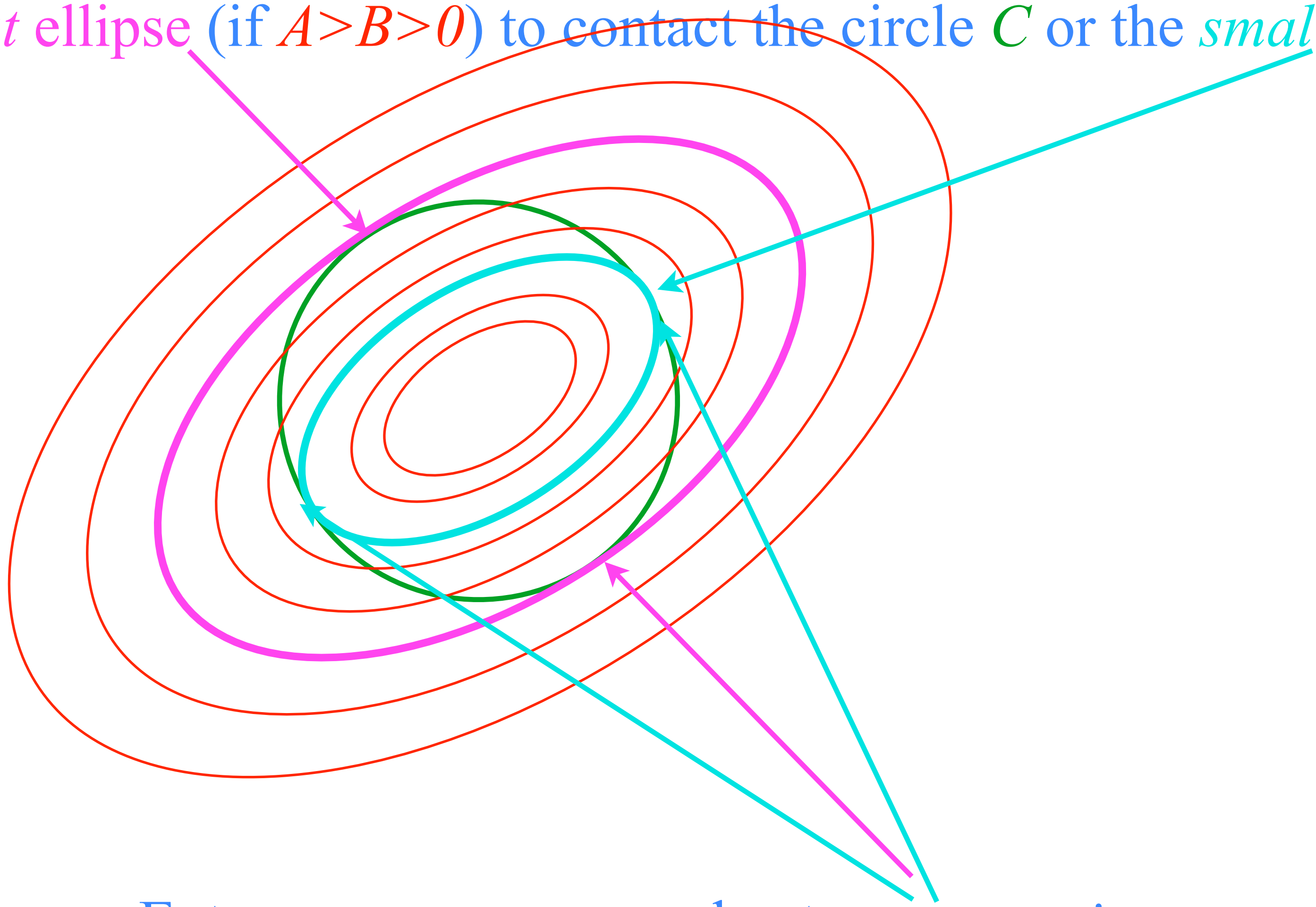
$$\boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2 x^2} kx}$$

Other Ways to do constraint analysis

- Way 3. OCC constraint webs
    - Preview of atomic-Stark orbits
        - Classical Hamiltonian separability
- Way 4. Lagrange multipliers
    - → Lagrange multiplier as eigenvalues
    - Multiple multipliers
    - "Non-Holonomic" multipliers

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



Extreme cases occur only at *contact points*

The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

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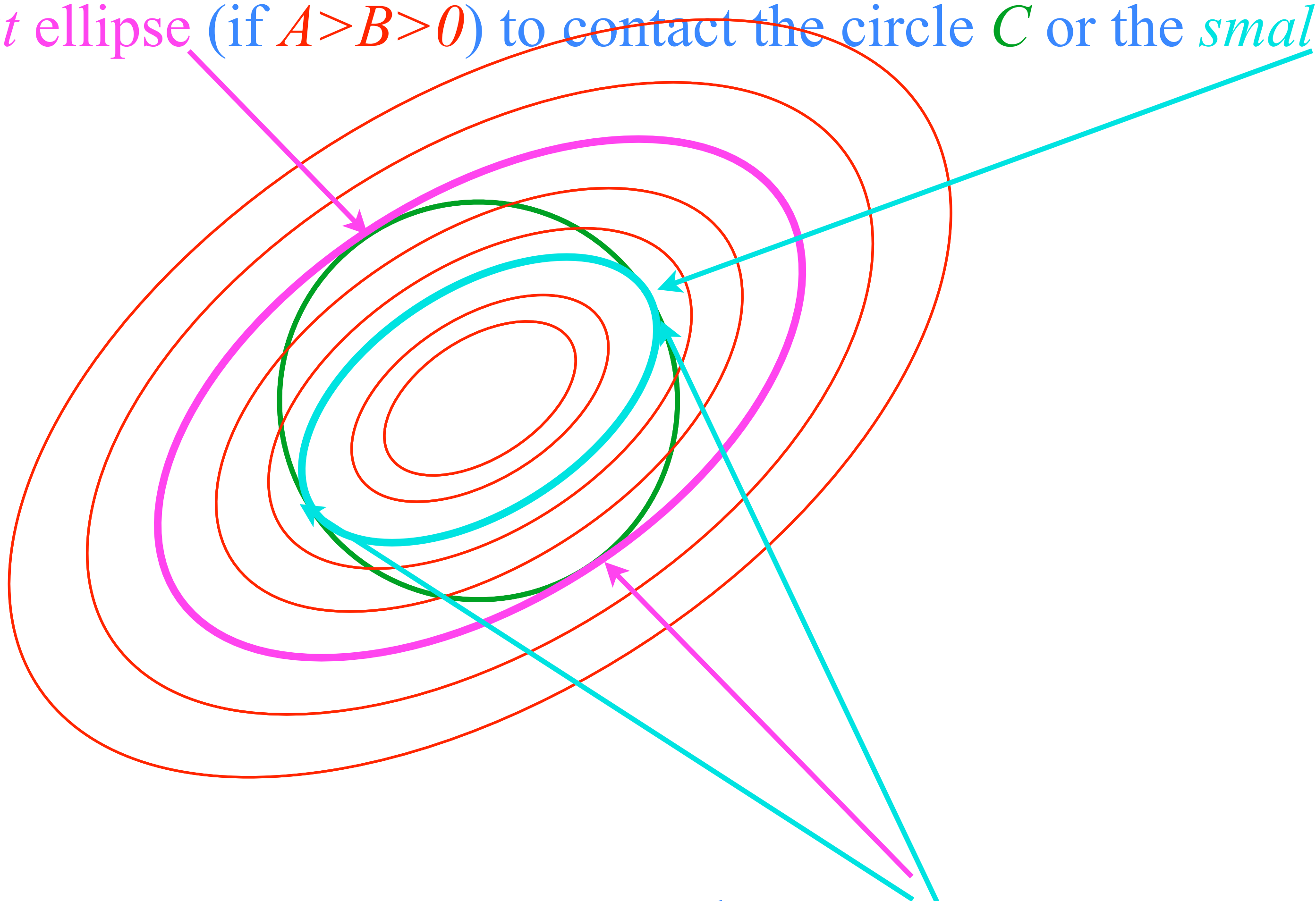

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$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek "L")

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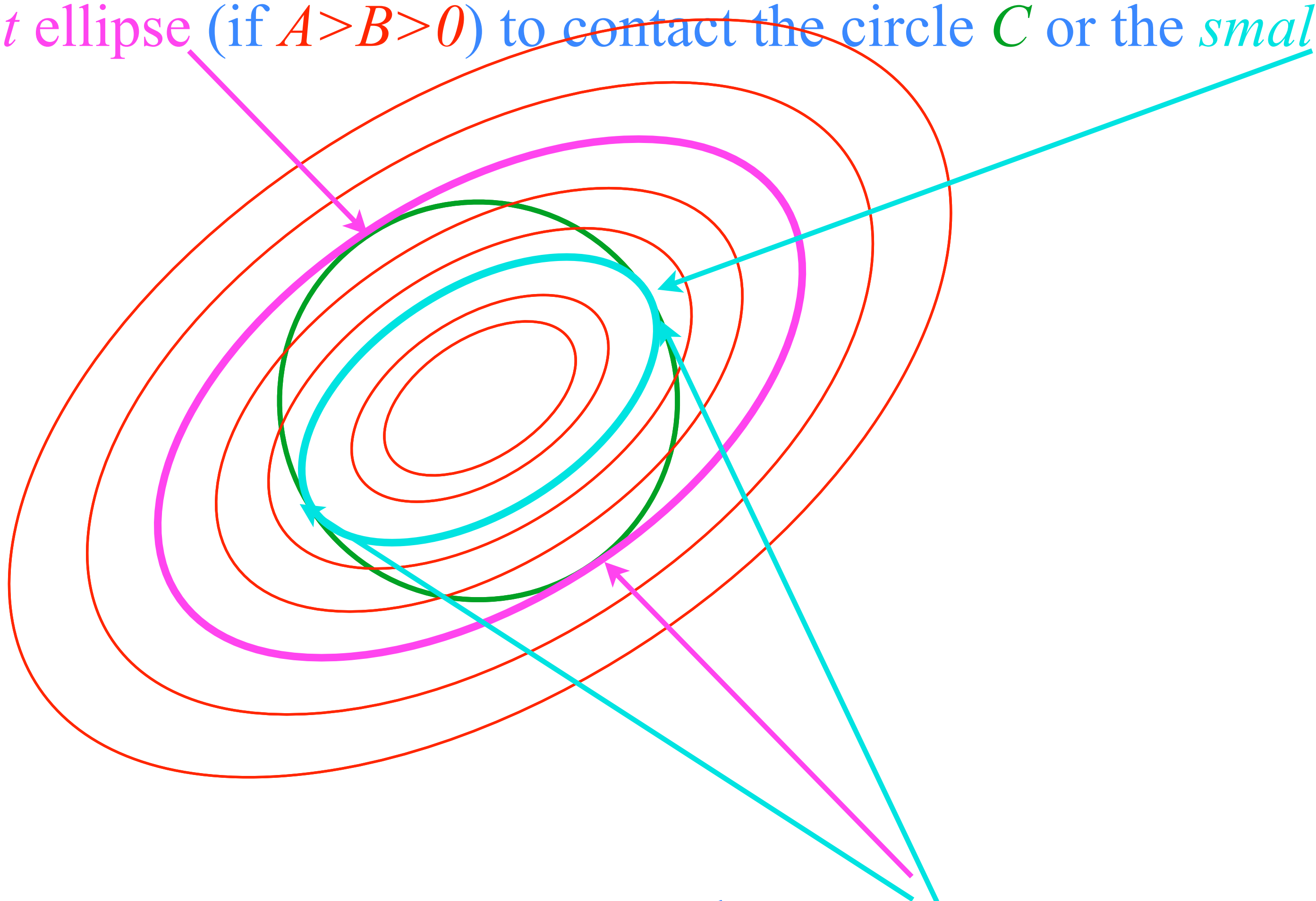

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Eigenvalues λ are *extreme* matrix “own”-values $\langle\psi|\mathrm{M}|\psi\rangle$ subject *Norm-constraint* $\langle\psi|\psi\rangle=1$

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- *Way 3. OCC constraint webs*
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Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{\quad}{\partial q^k}\frac{\partial c}{\quad} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

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Two or more constraints $c^1(q^k) = const., c^2(q^k) = const., \cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs
- *Preview of atomic-Stark orbits*
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- → *"Non-Holonomic" multipliers*

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

<u>Integral constraint differentials</u>

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \dots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \dots$$

$$\vdots \qquad \qquad \vdots$$

<u>General differential constraint relations</u>

$$0 = C_1^1 dq^1 + C_2^1 dq^2 + \dots$$

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Constrained equations of motion

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I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)
A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^{\gamma}}{\partial q^j \partial q^k} = \frac{\partial^2 c^{\gamma}}{\partial q^k \partial q^j}$$

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$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 C_1^1 + \lambda_2 C_1^2 + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 C_2^1 + \lambda_2 C_2^2 + \ldots$$

$$\vdots \qquad \vdots$$

If a differential can't be integrated to give a constraint function it's called a *non-holonomic constraint*.

I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)
A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j}$$

Force components $F_k^\gamma = \frac{\partial c^\gamma}{\partial q^k} = C_k^\gamma$ must satisfy *reciprocity relations* to be gradients of a c^γ function.

Integral constraint differentials

$$\frac{\partial F_k^\gamma}{\partial q^j} = \frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial F_j^\gamma}{\partial q^k}$$

General differential constraint relations

$$\frac{\partial C_k^\gamma}{\partial q^j} \quad \begin{matrix} \textit{may or} \\ \textit{may not be} \end{matrix} \quad \frac{\partial C_j^\gamma}{\partial q^k}$$