Poincare, Lagrange, Hamiltonian, and Jacobi mechanics
(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)

Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)
Clues for future assignments (Web Simulation: CouIIt)

Examples of Hamiltonian mechanics in phase plots
1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))
1D-HO phase-space control (Old Mac OS & Web Simulations of “Catcher in the Eye”)

Exploring phase space and Lagrangian mechanics more deeply
A weird “derivation” of Lagrange’s equations
Poincare identity and Action, Jacobi-Hamilton equations
How Classicists might have “derived” quantum equations
Huygen’s contact transformations enforce minimum action
How to do quantum mechanics if you only know classical mechanics
(“Color-Quantization” simulations: Davis-Heller “Color-Quantization” or “Classical Chromodynamics”)
Parabolic and 2D-KHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)
Some clues for future assignments (Web Simulation: CoulIt)
Link ⇒ Coult - Simulation of the Volcanoes of Io
Parabolic orbital trajectory envelopes

Focus of envelope

(Thales Geometry again) \( \alpha = 30^\circ \)

Focus of orbit trajectory

Line to contact point of \( \alpha = 30^\circ \) orbit with envelope

directrix (for trajectory envelope)

directrix (for trajectories)

Say \( \alpha = 90^\circ \) path rises to 1.0...
Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9, pp. 56-81, and a generalization.)

Some clues for future assignments (Web Simulation: CoulIt)
Exploding-starlet elliptical envelope and contacting elliptical trajectories

(Web Simulation: CouIIt - Exploding*Starlet {IHO Potential})
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))
Circular pendulum dynamics and elliptic functions
Cycloid pendulum dynamics and “sawtooth” functions
1D-HO phase-space control (Old Mac OS & Web Simulations of “Catcher in the Eye”)
1D Pendulum and phase plot

(a) Force geometry

\[ x = R \sin \theta \sim R \theta \]

\[ -MgR \sin \theta = F_\theta \]

\[ = -Mg x \]

(b) Energy geometry

\[ L = KE - PE = T - U \]

\[ = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta \]

\[ PE: \]

\[ V = MgY = -MgR \cos \theta \]

\[ \frac{1}{2} (Mg/R) x^2 \sim Mgh \]

\[ x^2 = h(2R-h) \sim 2hR \]

(Euclid mean)

(c) Time geometry

\[ \varepsilon = \theta/2 \]

\[ \theta \]

\[ M \]

\[ R \]

NOTE: Very common loci of ± sign blunders

Lagrangian function \( L = KE - PE = T - U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)
**1D Pendulum and phase plot**

(a) Force geometry

(b) Energy geometry

(c) Time geometry

Lagrangian function \( L = KE - PE = T - U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[
L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta
\]

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}
\]

NOTE: Very common loci of ± sign blunders
**1D Pendulum and phase plot**

(a) Force geometry

\[ x = R \sin \theta \approx R \theta \]
\[ -MgR \sin \theta = F_\theta \]
\[ = -Mg \cdot x \]

(b) Energy geometry

\[ x^2 = h(2R-h) \sim 2hR \] (Euclid mean)

\[ \frac{1}{2} (Mg/R)x^2 \]

\[ PE: \]
\[ V = MgY \]
\[ = -MgR \cos \theta \]

\[ \frac{1}{2} (Mg/R)x^2 \]
\[ \theta \]
\[ R \]
\[ x \]
\[ h \]
\[ Mg \]

(c) Time geometry

\[ \varepsilon = \theta/2 \]

\[ \theta \]
\[ M \]
\[ R \]

**NOTE:** Very common loci of ± sign blunders

Lagrangian function \( L = KE - PE = T - U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[ L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta \]

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[ H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E \text{ = const.} \]

implies: \( p_\theta = \sqrt{2I(E - MgR \cos \theta)} \)
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space \((\theta, p_\theta)\)

\[
H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR\cos\theta,
\]

or:

\[
p_\theta = \sqrt{2I(E + MgR\cos\theta)}
\]

(unsaddle “balancing” point)
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space $(\theta, p_\theta)$

\[ H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta , \quad \text{or:} \quad p_\theta = \sqrt{2I(E + MgR \cos \theta)} \]

Funny way to look at Hamilton's equations:

\[
\begin{pmatrix}
  \dot{q} \\
  \dot{p}
\end{pmatrix} =
\begin{pmatrix}
  \frac{\partial}{\partial p} H \\
  -\frac{\partial}{\partial q} H
\end{pmatrix} = e_H \times (-\nabla H) = (\text{H-axis}) \times (\text{fall line}), \quad \text{where:} \quad \begin{cases}
  (\text{H-axis}) = e_H = e_q \times e_p \\
  (\text{fall line}) = -\nabla H
\end{cases}
\]
Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))
Circular pendulum dynamics and elliptic functions
Cycloid pendulum dynamics and “sawtooth” functions
1D-HO phase-space control (Old Mac OS & Web Simulations of “Catcher in the Eye”)
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \implies \quad p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}$$

implies: $p_\theta = \sqrt{2I(E + MgR \cos \theta)}$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2$$
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos \theta$

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR\cos \theta = E = \text{const.}
\]

implies: $p_\theta = \sqrt{2I(E + MgR\cos \theta)}$

Let $E = MgY = -MgR\cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

\[
\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = \frac{p_\theta}{I} = \sqrt{2I\left(E + MgR\cos \theta\right)} / I \quad \text{where: } I = MR^2
\]

or: $dt = \frac{d\theta}{\sqrt{2\left(E + MgR\cos \theta\right)} / I}$
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR\cos\theta = E = \text{const.}$$

implies: $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

Let $E=MgY=-MgR\cos\theta_0$ be potential energy where $KE=0$ or $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{dp_\theta}{dt} = \frac{p_\theta}{I} = \sqrt{2I(E + MgR\cos\theta)} / I \quad \text{where: } I = MR^2$$

or: $dt = \frac{d\theta}{\sqrt{2(E + MgR\cos\theta)}} / I$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_q} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_q} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = \frac{p_\theta}{I} = \sqrt{2I(E + MgR \cos \theta)}/I$$ where: $I = MR^2$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = \text{(Travel time 0 to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon,$$
Circular pendulum dynamics and elliptic functions

**Hamiltonian function** $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies:} \quad p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE=0$ or $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = \frac{p_\theta}{I} = \sqrt{2I(\text{E} + MgR \cos \theta)} / I \quad \text{where:} \quad I = MR^2 \quad \text{or:} \quad dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\epsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \epsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \epsilon_0 - 2 \sin^2 \epsilon$$
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_{\theta}, \theta) = \frac{1}{2I} p_{\theta}^2 + U(\theta) = \frac{1}{2I} p_{\theta}^2 - MgR\cos\theta = E = \text{const.}$$

Let $E = Mg\gamma = -MgR\cos\theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_{\theta}} = \dot{\theta} = \frac{d\theta}{dt} = \frac{p_{\theta}}{I} = \frac{\sqrt{2I(E + MgR\cos\theta)}}{I}$$

where: $I = MR^2$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2MgR \cos\theta - \cos\theta_0}} = \int_{0}^{\theta} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\epsilon = \theta/2$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\epsilon, \quad \cos\theta - \cos\theta_0 = 2\sin^2\epsilon_0 - 2\sin^2\epsilon$$

$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_{0}^{\epsilon_0} \frac{d\epsilon}{\sqrt{\sin^2\epsilon_0 - \sin^2\epsilon}}$$

(Thales Geometry again)
Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}$$

implies: $p_\theta = \sqrt{2I(E + MgR \cos \theta)}$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta) / I}$$

where: $I = MR^2$

or: $\frac{dt}{d\theta} = \frac{1}{\sqrt{2(E + MgR \cos \theta) / I}}$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{2MgR \cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\epsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \epsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \epsilon_0 - 2 \sin^2 \epsilon$$

$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\epsilon_0} \frac{d\epsilon}{\sqrt{\sin^2 \epsilon_0 - \sin^2 \epsilon}} = \sqrt{\frac{R}{g}} \int_0^{\epsilon_0} \frac{k d\epsilon}{\sqrt{1 - k^2 \sin^2 \epsilon}}$$

, where:

$$\frac{1}{k} = \sin \epsilon_0 = \sin \frac{\theta_0}{2}$$

$$I = MR^2$$
Circular pendulum dynamics and elliptic functions

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}
\]

implies: \( p_\theta = \sqrt{2I(E + MgR \cos \theta)} \)

Let \( E = MgY = -MgR \cos \theta_0 \) be potential energy where \( KE = 0 \) or \( p_\theta = 0 \)

\[
\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = \frac{p_\theta}{I} = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2
\]

or: \( dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I} \)

Quadrature integral gives quarter-period \( \tau_{1/4} \):

\[
\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}
\]

Uses a half-angle coordinate \( \varepsilon = \frac{\theta}{2} \)

\[
\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon
\]

\[
\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{k d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}}, \quad \text{where: }
\]

\[
\begin{align*}
1/k & = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\
I & = MR^2
\end{align*}
\]

The integral is an elliptic integral of the first kind: \( F(k, \varepsilon_0) = \text{am}^{-1} \) or the "inverse amu" function.

\[
F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv \text{am}^{-1}(k, \varepsilon_0)
\]
Circular pendulum dynamics and elliptic functions

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}
\]

implies: \( p_\theta = \sqrt{2I(E + MgR \cos \theta)} \)

Let \( E = MgY = -MgR \cos \theta_0 \) be potential energy where \( KE = 0 \) or \( p_\theta = 0 \)

\[
\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{I} = \frac{\sqrt{2I(E + MgR \cos \theta)}}{I} \quad \text{where: } I = MR^2
\]

Quadrature integral gives quarter-period \( \tau_{1/4} \):

\[
\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = \left( \text{Travel time 0 to } \theta_0 \right) = \tau_{1/4}
\]

Uses a half-angle coordinate \( \varepsilon = \theta/2 \)

\[
\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \frac{\varepsilon}{2}, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \frac{\varepsilon_0}{2} - 2 \sin^2 \varepsilon
\]

\[
\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{kd\varepsilon}{\sqrt{1-k^2 \sin^2 \varepsilon}}, \quad \text{where:}
\]

\[
\left\{\begin{array}{l}
1/k = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\
I = MR^2
\end{array}\right.
\]

The integral is an elliptic integral of the first kind: \( F(k, \varepsilon_0) = \text{am}^{-1} \) or the "inverse amu" function.

\[
F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1-k^2 \sin^2 \varepsilon}} = \text{am}^{-1}(k, \varepsilon_0)
\]

\[
\tau_{1/4} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \left[ \text{sin}^{-1} \frac{\varepsilon}{\varepsilon_0} \right]_0^{\varepsilon_0} = \sqrt{\frac{R \pi}{g}} = \frac{2\pi}{4}
\]

For low amplitude \( \varepsilon \ll 1 \): \( \sin \varepsilon_0 \simeq \varepsilon_0 \) reduces \( \tau_{1/4} \) to \( \frac{2\pi}{4} \)
Circular pendulum dynamics and elliptic functions

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E \text{ = const.}
\]

implies: \( p_\theta = \sqrt{2I(E + MgR \cos \theta)} \)

Let \( E = MgY = -MgR \cos \theta_0 \) be potential energy where \( KE = 0 \) or \( p_\theta = 0 \)

\[
\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = \frac{p_\theta}{I} = \sqrt{2I(E + MgR \cos \theta)}/I \quad \text{where: } I = MR^2
\]

Quadrature integral gives quarter-period \( \tau_{1/4} \):

\[
\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = \left(\text{Travel time 0 to } \theta_0\right) = \tau_{1/4}
\]

Uses a half-angle coordinate \( \varepsilon = \theta/2 \)

\[
\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon
\]

\[
\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{kd\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}}, \text{ where:}
\]

\[
\frac{1}{k} = \sin \varepsilon_0 = \sin \frac{\theta_0}{2}
\]

\[
I = MR^2
\]

The integral is an elliptic integral of the first kind: \( F(k, \varepsilon_0) = am^{-1} \) or the "inverse amu" function.

\[
F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0)
\]

\[
\tau_{1/4} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon(t)} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon(t)} \frac{d\varepsilon}{\varepsilon_0}
\]

low \( \varepsilon \ll I \): \( \sin \varepsilon_0 \simeq \varepsilon_0 \) reduces \( \tau_{1/4} \) to \( \tau \frac{2\pi}{4} \)
Circular pendulum dynamics and elliptic functions

**Hamiltonian function** \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR\cos\theta \)

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR\cos\theta = E = \text{const.}
\]

implies: \( p_\theta = \sqrt{2I(E + MgR\cos\theta)} \)

Let \( E=MgY=-MgR\cos\theta_0 \) be potential energy where \( KE=0 \) or \( p_\theta=0 \)

\[
\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{dt}{d\theta} = \frac{p_\theta}{I} = \frac{\sqrt{2I(E + MgR\cos\theta)}}{I} \quad \text{where: } I = MR^2 \quad \text{or } \quad dt = \frac{d\theta}{\sqrt{2(E + MgR\cos\theta)}}/I
\]

Quadrature integral gives quarter-period \( \tau_{1/4} \):

\[
\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = \text{(Travel time 0 to } \theta_0) = \tau_{1/4}
\]

Uses a half-angle coordinate \( \varepsilon = \theta/2 \)

\[
\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\varepsilon, \quad \cos\theta - \cos\theta_0 = 2\sin^2\varepsilon_0 - 2\sin^2\varepsilon
\]

\[
\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0 - \sin^2\varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{k d\varepsilon}{1 - k^2 \sin^2\varepsilon}, \quad \text{where:}
\]

\[
\left\{
\begin{aligned}
1/k &= \sin\varepsilon_0 = \sin\frac{\theta_0}{2} \\
I &= \frac{MR^2}{2}
\end{aligned}
\right.
\]

The integral is an **elliptic integral of the first kind**: \( F(k,\varepsilon_0) = \text{am}^{-1} \) or the "inverse amu" function.

\[
F(k,\varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2\varepsilon}} \equiv \text{am}^{-1}(k,\varepsilon_0)
\]

..reduces to sine...

\[
\varepsilon(t) = \varepsilon_0 \sin \sqrt{\frac{g}{R}} t = \varepsilon_0 \sin \omega t, \quad \text{where: } \omega = \sqrt{\frac{g}{R}}
\]

For low amplitude \( \varepsilon \ll I \): \( \sin\varepsilon_0 \simeq \varepsilon_0 \) reduces \( \tau_{1/4} \) to \( \tau_{2\pi/4} \)
Circular pendulum dynamics and elliptic functions

(See also: Simulation of cycloidally constrained pendulum)
Circular pendulum dynamics and elliptic functions

(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))
Circular pendulum dynamics and elliptic functions
Cycloid pendulum dynamics and “sawtooth” functions
1D-HO phase-space control (Old Mac OS & Web Simulations of “Catcher in the Eye”)
Cycloid Pendulum
Follows cycloid arc by wrapping on upper cycloid.

Circular Pendulum
Follows circular arc
(hanging freely)
Cycloid pendulum dynamics and "sawtooth" functions

momentum $p_\Theta$
and
coordinate $\Theta$
Fourier transformed

*(Simulations of cycloidally constrained pendulum)*
Cycloid pendulum dynamics and “sawtooth” functions

(Simulations of cycloidally constrained pendulum)
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot *(Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*

Circular pendulum dynamics and elliptic functions

Cycloid pendulum dynamics and “sawtooth” functions

1D-HO phase-space control *(Old Mac OS & Web Simulations of “Catcher in the Eye”)*
Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control

\[ F(Y) = -kY - Mg \]
\[ U(Y) = \frac{1}{2}kY^2 + MgY \]

\[ u(y) = \frac{1}{4}y^2 + y \]
\[ f(y) = -(1/2)y - 1 \]

\[ U_{shift} = -\frac{(Mg)^2}{2k} \]
\[ Y_{shift} = \frac{-Mg}{k} \]

Unit 1
Fig. 7.4
Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations

Poincare identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations

Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics
A strange “derivation” of Lagrange’s equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): \( S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t) \).

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. (1)

\[
\delta q(t_0) = 0 = \delta q(t_1)
\]

1st order \( L(q + \delta q) \) approximate:

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]
\]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)
An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all.

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t).$$

An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all. (1)

1st order $L(q+\delta q)$ approximate:

$$\delta q(t_0) = 0 = \delta q(t_1) \quad (1)$$

Replace $\frac{\partial L}{\partial \dot{q}} \delta q$ with $\frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q - \frac{1}{\partial q} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q$
An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. (1)

**1st order \( L(q+\delta q) \) approximate:**

\[
S(q+\delta q) = \int_{t_0}^{t_1} dt \left[ L(q,q,t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]
\]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)

Replace \( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \) with \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \)
A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \, L(q(t), \dot{q}(t), t). \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. (1)

1st order \( L(q+\delta q) \) approximate:

\[ \delta q(t_0) = 0 = \delta q(t_1) \quad (1) \]

\[ u \frac{dv}{dt} = \frac{d}{dt} (uv) - \frac{du}{dt} v \]
A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[
S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t).
\]

An arbitrary but small variation function \(\delta q(t)\) is allowed at every point \(t\) in the figure along the curve except at the end points \(t_0\) and \(t_1\). There we demand it not vary at all. (1)

1st order \(L(q+\delta q)\) approximate:

\[
\delta q(t_0) = 0 = \delta q(t_1)
\]

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]
\]

where: \(\delta \dot{q} = \frac{d}{dt} \delta q\)

Replace \(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} \) with \(\frac{d}{dt} \left( \frac{\partial L}{\partial q} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right)\)

Third term vanishes by (1). This leaves first order variation: \(\delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\)

Extreme value (actually \(\text{minimum}\) value) of \(S(q)\) occurs \(\text{if and only if}\) Lagrange equation is satisfied!

\[\delta S = 0 \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}\]
A weird “derivation” of Lagrange’s equations
Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \, L(q(t), \dot{q}(t), t). \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. \( 1 \)

1st order \( L(q + \delta q) \) approximate:

\[
\delta q(t_0) = 0 = \delta q(t_1) \quad (1)
\]

Third term vanishes by \( (1) \). This leaves first order variation:

\[
\delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right]
\]

Extreme value (actually minimum value) of \( S(q) \) occurs if and only if Lagrange equation is satisfied!

\[
\delta S = 0 \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}
\]

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian \( L = T - U \)???
Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations

Poincaré identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations

Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes \textit{Poincare's invariant differential} if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$$

$$(v = \frac{dr}{dt} \text{ implies: } v \cdot dt = dr)$$
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes *Poincare’s invariant differential* if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot d\mathbf{r} - H \cdot dt$$

($v = \frac{d\mathbf{r}}{dt}$ implies: $v \cdot dt = d\mathbf{r}$)

This is the time differential $dS$ of *action* $S = \int L \cdot dt$ whose time derivative is rate $L$ of *quantum phase*.

$$dS = L \cdot dt = p \cdot d\mathbf{r} - H \cdot dt$$  where:  $L = \frac{dS}{dt}$
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes Poincare’s invariant differential if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt \quad \quad v = \frac{dr}{dt}$$

This is the time differential $dS$ of action $S = \int L \cdot dt$ whose time derivative is rate $L$ of quantum phase.

$$dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

Unit 8 shows **DeBroglie law** $p = \hbar k$ and **Planck law** $H = \hbar \omega$ make quantum plane wave phase $\Phi$:

$$\Phi = \frac{S}{\hbar} = \int \frac{L \cdot dt}{\hbar}$$
Legendre-Poincare identity and Action

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes Poincare’s invariant differential if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

This is the time differential \( dS \) of action \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of quantum phase.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 8 shows DeBroglie law \( p = \hbar k \) and Planck law \( H = \hbar \omega \) make quantum plane wave phase \( \Phi \):

\[
\psi(r, t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega t)}
\]

Legendre-Poincare identity and Action
**Legendre-Poincare identity and Action**

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes Poincare’s invariant differential if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

This is the time differential \( dS \) of **action** \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of quantum phase.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 8 shows **DeBroglie law** \( p = \hbar k \) and **Planck law** \( H = \hbar \omega \) make quantum plane wave phase \( \Phi \):

\[
\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}
\]

Q: When is the **Action**-differential \( dS \) integrable?

A: A differential \( dW = f_x(x,y)dx + f_y(x,y)dy \) is **integrable** to a \( W(x,y) \) if: \( f_x = \frac{\partial W}{\partial x} \) and: \( f_y = \frac{\partial W}{\partial y} \)
**Legendre-Poincare identity and Action**

Legendre transform $L(v) = p \cdot v - H(p)$ becomes *Poincare’s invariant differential* if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$$

This is the time differential $dS$ of *action* $S = \int L \cdot dt$ whose time derivative is rate $L$ of *quantum phase*.

$$dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

Unit 8 shows *DeBroglie law* $p=\hbar k$ and *Planck law* $H = \hbar \omega$ make *quantum plane wave phase* $\Phi$:

$$\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}$$

Q: When is the *Action*-differential $dS$ integrable?

A: Differential $dW = f_x(x,y)dx + f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

That condition is *no curl allowed*: $\nabla \times f = 0$ or *partial-symmetry* of $W$:

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial f_y}{\partial x}$$

**Similar to conditions for integrating work differential $dW = f \cdot dr$ to get potential $W(r)$**.
**Legendre-Poincare identity and Action**

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes *Poincare’s invariant differential* if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

\( v = \frac{dr}{dt} \)

This is the time differential \( dS \) of *action* \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of *quantum phase*.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 8 shows **DeBroglie law** \( p = \hbar k \) and **Planck law** \( H = \hbar \omega \) make *quantum plane wave phase* \( \Phi \):

\[
\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}
\]

Q: When is the *Action*-differential \( dS \) integrable?

A: Differential \( dW = f_x(x,y)dx + f_y(x,y)dy \) is *integrable* to a \( W(x,y) \) if: \( f_x = \frac{\partial W}{\partial x} \) and: \( f_y = \frac{\partial W}{\partial y} \)

\( dS \) is integrable if:

\[
\frac{\partial S}{\partial r} = p \quad \text{and:} \quad \frac{\partial S}{\partial t} = -H
\]

*These conditions are known as Jacobi-Hamilton equations*
Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations
Poincaré identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations
Huygen’s contact transformations enforce minimum action
How to do quantum mechanics if you only know classical mechanics
How Jacobi-Hamilton could have “derived” Schrodinger equations

Given “quantum wave”

$$\psi(r, t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}$$

$dS$ is integrable if: \[
\frac{\partial S}{\partial r} = p \quad \text{and:} \quad \frac{\partial S}{\partial t} = -H
\]

These conditions are known as Jacobi-Hamilton equations.
How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

\[ \psi(r, t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)} \]

\( dS \) is integrable if: \[ \frac{\partial S}{\partial r} = p \] and: \[ \frac{\partial S}{\partial t} = -H \]

These conditions are known as Jacobi-Hamilton equations

Try 1st r-derivative of wave \( \psi \)

\[ \frac{\partial}{\partial r} \psi(r, t) = \frac{\partial}{\partial r} e^{iS/\hbar} = \frac{\partial}{\partial r} (iS/\hbar) e^{iS/\hbar} = (i/\hbar) \frac{\partial S}{\partial r} \psi(r, t) \]

\[ \frac{\partial}{\partial r} \psi(r, t) = (i/\hbar) \mathbf{p} \psi(r, t) \text{ or: } \frac{\hbar \partial}{i \partial r} \psi(r, t) = \mathbf{p} \psi(r, t) \]
How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p} \cdot \mathbf{r} - \mathbf{H} \cdot t)/\hbar} = e^{i(k \cdot \mathbf{r} - \omega \cdot t)}$$

$dS$ is integrable if: $\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\frac{\partial S}{\partial t} = -\mathbf{H}$

These conditions are known as Jacobi-Hamilton equations

Try 1$^{st}$ $\mathbf{r}$-derivative of wave $\psi$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}} e^{iS/\hbar} = \frac{\partial (iS/\hbar)}{\partial \mathbf{r}} e^{iS/\hbar} = (i/\hbar) \frac{\partial S}{\partial \mathbf{r}} \psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r},t) = (i/\hbar) \mathbf{p} \psi(\mathbf{r},t) \quad \text{or:} \quad \hbar \frac{\partial}{i \partial \mathbf{r}} \psi(\mathbf{r},t) = \mathbf{p} \psi(\mathbf{r},t) \quad \text{Momentum Operator}$$

or $\mathbf{p}$-op in $\mathbf{r}$-basis

$$\mathbf{p} \Rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}}$$
How Jacobi-Hamilton could have “derived” Schrodinger equations

Given “quantum wave”

$$\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}$$

$dS$ is integrable if: $\frac{\partial S}{\partial r} = p$ and: $\frac{\partial S}{\partial t} = -H$

These conditions are known as Jacobi-Hamilton equations

Try 1\textsuperscript{st} r-derivative of wave $\psi$

$$\frac{\partial}{\partial r} \psi(r,t) = \frac{\partial}{\partial r} e^{iS/\hbar} = \frac{\partial (iS/\hbar)}{\partial r} e^{iS/\hbar} = \left(\frac{i}{\hbar}\right) \frac{\partial S}{\partial r} \psi(r,t)$$

or: $\frac{\hbar}{i} \frac{\partial}{\partial r} \psi(r,t) = p \psi(r,t)$

Momentum Operator or $p$-op in $r$-basis

$p \Rightarrow \frac{\hbar}{i} \frac{\partial}{\partial r}$

Try 1\textsuperscript{st} t-derivative of wave $\psi$

$$\frac{\partial}{\partial t} \psi(r,t) = \frac{\partial}{\partial t} e^{iS/\hbar} = \frac{\partial (iS/\hbar)}{\partial t} e^{iS/\hbar} = \left(\frac{i}{\hbar}\right) \frac{\partial S}{\partial t} \psi(r,t)$$

$$= \left(\frac{i}{\hbar}\right)(-H) \psi(r,t) \text{ or: } i\hbar \frac{\partial}{\partial t} \psi(r,t) = H \psi(r,t)$$

or

$$\frac{\partial}{\partial t} \psi(r,t) = H \psi(r,t)$$
How Jacobi-Hamilton could have “derived” Schrödinger equations

(Given “quantum wave”)

\[ \psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)} \]

\[ dS \text{ is integrable if: } \frac{\partial S}{\partial r} = p \text{ and: } \frac{\partial S}{\partial t} = -H \]

These conditions are known as Jacobi-Hamilton equations

Try 1st r-derivative of wave \( \psi \)

\[ \frac{\partial}{\partial r} \psi(r,t) = \frac{\partial}{\partial r} e^{iS/\hbar} = \frac{\partial}{\partial r} \left( \frac{iS}{\hbar} \right) e^{iS/\hbar} = \left( \frac{i}{\hbar} \right) \frac{\partial S}{\partial r} \psi(r,t) \]

\[ \frac{\partial}{\partial r} \psi(r,t) = \left( \frac{i}{\hbar} \right) p \psi(r,t) \text{ or: } \frac{\hbar}{i} \frac{\partial}{\partial r} \psi(r,t) = p \psi(r,t) \]

Try 1st t-derivative of wave \( \psi \)

\[ \frac{\partial}{\partial t} \psi(r,t) = \frac{\partial}{\partial t} e^{iS/\hbar} = \frac{\partial}{\partial t} \left( \frac{iS}{\hbar} \right) e^{iS/\hbar} = \left( \frac{i}{\hbar} \right) \frac{\partial S}{\partial t} \psi(r,t) \]

\[ = \left( \frac{i}{\hbar} \right) (-H) \psi(r,t) \text{ or: } \frac{i\hbar}{\partial t} \psi(r,t) = H \psi(r,t) \]
Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations
Poincare identity and Action, Jacobi-Hamilton equations
How Classicists might have “derived” quantum equations

**Huygen’s contact transformations enforce minimum action**
How to do quantum mechanics if you only know classical mechanics
Huygen’s contact transformations enforce minimum action

Each point \( r_k \) on a wavefront “broadcasts” in all directions. Only **minimum action** path interferes constructively

Time-independent action (Hamilton’s *reduced action*) is a purely spatial integral.

\[
S_H = \int_{r_0}^{r_f} p \cdot dr
\]

\[
S_H(r_0;r) = 30
\]

\[
S_H(r_0;r) = 20
\]

\[
S_H(r_0;r) = 10
\]

Time-dependent action (Hamilton’s *principle action*) is space-time integral.

\[
S_p = \int_{r_0}^{r_f} \left( p \cdot dr - H \cdot dt \right)
\]

**Unit 1**

Fig. 12.12
Huygen’s contact transformations enforce minimum action

Each point \( \mathbf{r}_k \) on a wavefront “broadcasts” in all directions.
Only minimum action path interferes constructively

Time-independent action (Hamilton’s reduced action)

\[
S_H = \int_{r_0}^{r_1} \mathbf{p} \cdot d\mathbf{r}
\]

is a purely spatial integral.

\[
\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0, \mathbf{r}_1)/\hbar}
\]

Feynman’s path-sum closure relation

\[
\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \equiv \sum_{\mathbf{r}'} e^{i \left( S_H(\mathbf{r}_0, \mathbf{r}') + S_H(\mathbf{r}', \mathbf{r}_1) \right)/\hbar} = e^{i S_H(\mathbf{r}_0, \mathbf{r}_1)/\hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle
\]

...because action is quantum wave phase

Time-dependent action (Hamilton’s principle action)

\[
S_p = \int_{r_0}^{r_{f_1}} \left( \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \right)
\]

is space-time integral.

\[
\langle \mathbf{r}_{f_1} | \mathbf{r}_{f_0}, t_0 \rangle = e^{i S(\mathbf{r}_{f_0}, t_0; \mathbf{r}_{f_1}, t_{f_1})/\hbar}
\]

Unit 1

Fig. 12.12
Exploring phase space and Lagrangian mechanics more deeply

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How to do quantum mechanics if you only know classical mechanics

Davis-Heller “Color-Quantization” or “Classical Chromodynamics”
Bohr quantization requires quantum phase \( s_H/\hbar \) in amplitude to be an integral multiple \( n \) of \( 2\pi \) after a closed loop integral \( S_H(r_0 : r_0) = \int_{r_0}^{r_0} p \cdot dr \). The integer \( n \) (\( n = 0, 1, 2,... \)) is a quantum number.

\[
I = \langle r_0 | r_0 \rangle = e^{iS_H(r_0 : r_0)/\hbar} = e^{i\Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi \hbar n = \hbar n
\]

Numerically integrate Hamilton's equations and Lagrangian \( L \). Color the trajectory according to the current accumulated value of action \( S_H(0 : r)/\hbar \). Adjust energy to quantized pattern (if closed system*)

\[
S_H(0 : r) = S_p(0 , 0 : r , t ) + Ht = \int_0^t L \, dt + Ht.
\]
Bohr quantization requires quantum phase \( s_{\hbar}/\hbar \) in amplitude to be an integral multiple \( n \) of \( 2\pi \) after a closed loop integral \( S_{\hbar}(r_0: r_0) = \int_{r_0}^{r_0} p \cdot dr \). The integer \( n \) \((n = 0, 1, 2,...)\) is a quantum number.

\[
I = \langle r_0 \mid r_0 \rangle = e^{i \frac{S_H(r_0: r_0)}{\hbar}} = e^{i \frac{\Sigma}{\hbar}} = 1 \quad \text{for: } \Sigma = 2\pi \hbar n = \hbar n
\]

Numerically integrate Hamilton's equations and Lagrangian \( L \). Color the trajectory according to the current accumulated value of action \( S_A(0 : r)/\hbar \). Adjust energy to quantized pattern (if closed system*)

\[
S_A(0 : r) = S_p(0, 0 : r, t) + Ht = \int_0^t L \, dt + Ht.
\]

The hue should represent the phase angle \( S_A(0 : r)/\hbar \) modulo \( 2\pi \) as, for example, 
\( 0 = \text{red}, \pi/4 = \text{orange}, \pi/2 = \text{yellow}, 3\pi/4 = \text{green}, \pi = \text{cyan} \) (opposite of red), \( 5\pi/4 = \text{indigo}, 3\pi/2 = \text{blue}, 7\pi/4 = \text{purple}, \) and \( 2\pi = \text{red} \) (full color circle). Interpolating action on a palette of 32 colors is enough precision for low quanta.

**How to do quantum mechanics if you only know classical mechanics**

Unit 1

Fig. 12.13

*closed system

Standing wave has

only two phases(\( \pm \))

\( \text{cyan and red} \)

Quantum dynamical tunneling in bound states - Wavepacket and Color-quantization -

The Semiclassical Way to Molecular Spectroscopy:
Bohr quantization requires quantum phase $s_h/\hbar$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $s_h(r_0: r_0) = \int_{r_0}^{r_0} p \cdot dr$. The integer $n$ ($n = 0, 1, 2, \ldots$) is a quantum number.

$$I = \langle r_0 | r_0 \rangle = e^{i \frac{S_H(r_0 \cdot r_0)}{\hbar}} = e^{i \frac{\Sigma_H}{\hbar}} = 1 \text{ for: } \Sigma_H = 2\pi \hbar n = \hbar n$$

Numerically integrate Hamilton's equations and Lagrangian $L$. Color the trajectory according to the current accumulated value of action $S_H(0 : r)/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(0 : r) = S_p(0, 0 : r, t) + Ht = \int_0^t L \, dt + Ht.$$

The hue should represent the phase angle $S_H(0 : r)/\hbar$ modulo $2\pi$ as, for example, $0=$red, $\pi/4=$orange, $\pi/2=$yellow, $3\pi/4=$green, $\pi=$cyan (opposite of red), $5\pi/4=$indigo, $3\pi/2=$blue, $7\pi/4=$purple, and $2\pi=$red (full color circle). Interpolating action on a palette of 32 colors is enough precision for low quanta.

*open system has continuous energy

Simulation by “Color U(2)”
Unit 1
Fig. 12.13
*closed system has quantized E
Standing wave has only two phases(±)
cyan and red

Unit 1
Fig. 12.14
A moving wave has a *quantum phase velocity* found by setting $S=\text{const.}$ or $dS(0,0;r,t)=0=p \cdot dr - H dt$.

$$V_{\text{phase}} = \frac{dr}{dt} = \frac{H}{p} = \frac{\omega}{k}$$

**Quantum “phase wavefronts”**

(a) $S_H=0.3$

(b) $S_H=0.35$

(c) $S_H=0.4$

(d) $S_H=0.9$

**CoulIt Web Simulation with "Quantum phase front"**
A moving wave has a \textit{quantum phase velocity} found by setting $S=\text{const.}$ or $dS(0,0;r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt$.

This is quite the opposite of classical particle velocity which is \textit{quantum group velocity}.

\[
\text{V}_{\text{phase}} = \frac{d\mathbf{r}}{dt} = \frac{H}{p} = \frac{\omega}{k}
\]

\[
\text{V}_{\text{group}} = \frac{d\mathbf{r}}{dt} = \dot{r} = \frac{\partial H}{\partial p} = \frac{\partial \omega}{\partial k}
\]

Note: This is Hamilton’s 1st Equation.

Quantum “phase wavefronts”

(a) $S_H=0.3$

(b) $S_H=0.35$

(c) $S_H=0.4$

(d) $S_H=0.9$

Unit 1
Fig. 12.15

quantum phase velocity

CoulIt Web Simulation with "Quantum phase front"
A moving wave has a **quantum phase velocity** found by setting \( S = \text{const.} \) or \( dS(0,0:r,t)=0=p\cdot dr-Hdt \).

\[
\nabla S_H = p
\]

This is quite the opposite of classical particle velocity which is **quantum group velocity**.

\[
\begin{align*}
V_{\text{phase}} &= \frac{dr}{dt} = \frac{H}{p} = \frac{\omega}{k} \\
V_{\text{group}} &= \frac{dr}{dt} = \dot{r} = \frac{\partial H}{\partial p} = \frac{\partial \omega}{\partial k}
\end{align*}
\]

Note: This is Hamilton’s 1st Equation

**Quantum “phase wavefronts”**

(a) \( S_H=0.3 \)  
(b) \( S_H=0.35 \)  
(c) \( S_H=0.4 \)  
(d) \( S_H=0.9 \)

“cat ears” scoot outward..

Unit 1  
Fig. 12.15

quantum phase velocity

After a while ... nothing left but a smile!

From *Alice’s Adventures in Wonderland* by Lewis Carrol (1865)
A moving wave has a *quantum phase velocity* found by setting \( S = \text{const.} \) or \( dS(0,0;r,t) = 0 = p \cdot dr - H dt \).

\[
\nabla S_H = p
\]

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

\[
V_{\text{phase}} = \frac{dr}{dt} = \frac{H}{p} = \frac{\omega}{k}
\]

\[
V_{\text{group}} = \frac{dr}{dt} = \dot{r} = \frac{\partial H}{\partial p} = \frac{\partial \omega}{\partial k}
\]

Note: This is Hamilton’s 1\textsuperscript{st} Equation

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**Quantum “phase wavefronts”**

(a) \( S_H = 0.3 \)  
(b) \( S_H = 0.35 \)  
(c) \( S_H = 0.4 \)  
(d) \( S_H = 0.9 \)

**Classical “blast wavefronts”**

(a) \( T = 0.4 \)  
(b) \( T = 1.0 \)  
(c) \( T = 2.3 \)

higher \( V_{\text{phase}} \) up here  
quantum *phase* velocity  
lower \( V_{\text{phase}} \) down here  

...not to be confused with...

...quantum *group* velocity...  
that is classical particle velocity

lower \( V_{\text{group}} \) up here  
higher \( V_{\text{group}} \) down here

---

Unit 1  
Fig. 12.15
The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took the prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chladni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.
Check out the Heller Galleries

http://jalbum.net/en/browse/user/album/1696720

**National Science Foundation (NSF)**
Arlington, VA

September-November 2002

Selected images.


**University Museum, University of Arkansas, Fayetteville, AK**

October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

**Bessel 21**
Check out the Heller Galleries

http://jalbum.net/en/browse/user/album/1696720

National Science Foundation (NSF)
Arlington, VA

September-November 2002

Selected images.


University Museum, University of Arkansas, Fayetteville, AK

October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

*UAF Museum closed after this exhibit

Bessel 21

Lecture 11 ends here
Wed. 9.26.2018
A running collection of links to course-relevant sites and articles

**Physics Web Resources**
- Comprehensive Harter-Soft Resource Listing
- UAF Physics YouTube channel
- LearnIt Physics Web Applications

**Texts**
- Classical Mechanics with a Bang!
- Quantum Theory for the Computer Age
- Principles of Symmetry, Dynamics, and Spectroscopy
- Modern Physics and its Classical Foundations

**Classes**
- 2014 AMOP
- 2017 Group Theory for QM
- 2018 AMOP
- 2018 Adv Mechanics

Neat external material to start the class:
- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:
- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell’s demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

_Slightly* Older ones:
- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

“Relawavity” and quantum basis of *Lagrangian & Hamiltonian* mechanics:
- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App
- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018