Lecture 7
Tue. 9.12.2017

Kepler Geometry of IHO (Isotropic Harmonic Oscillator) Elliptical Orbits

(Ch. 9 and Ch. 11 of Unit 1)

Constructing 2D IHO orbital phasor "clock" dynamics in uniform-body

Constructing 2D IHO orbits using Kepler anomaly plots
- *Mean-anomaly and eccentric-anomaly geometry*
- *Calculus and vector geometry of IHO orbits*
- *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better in Ch. 12)*

Some Kepler's "laws" for <u>all</u> central (isotropic) force F(r) fields
- *Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$* *(Derived here)*
- *Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$* *(Derived in Unit 5)*
- *Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$* *(Derived here)*
- *Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$* *(Derived in Unit 5)*

Introduction to dual matrix operator contact geometry (based on IHO orbits)
- *Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ vs.inverse form ellipse $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$*
  - *Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)*
  - *Q-Ellipse tangents $\mathbf{r}'$ normal to dual Q^{-1}-ellipse position $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)*
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*

Q:Where is this headed? A: Lagrangian-Hamiltonian duality

Link ⇒ BoxIt simulation of IHO orbits
Link ⟶ IHO orbital time rates of change
Link ⟶ IHO Exegesis Plot

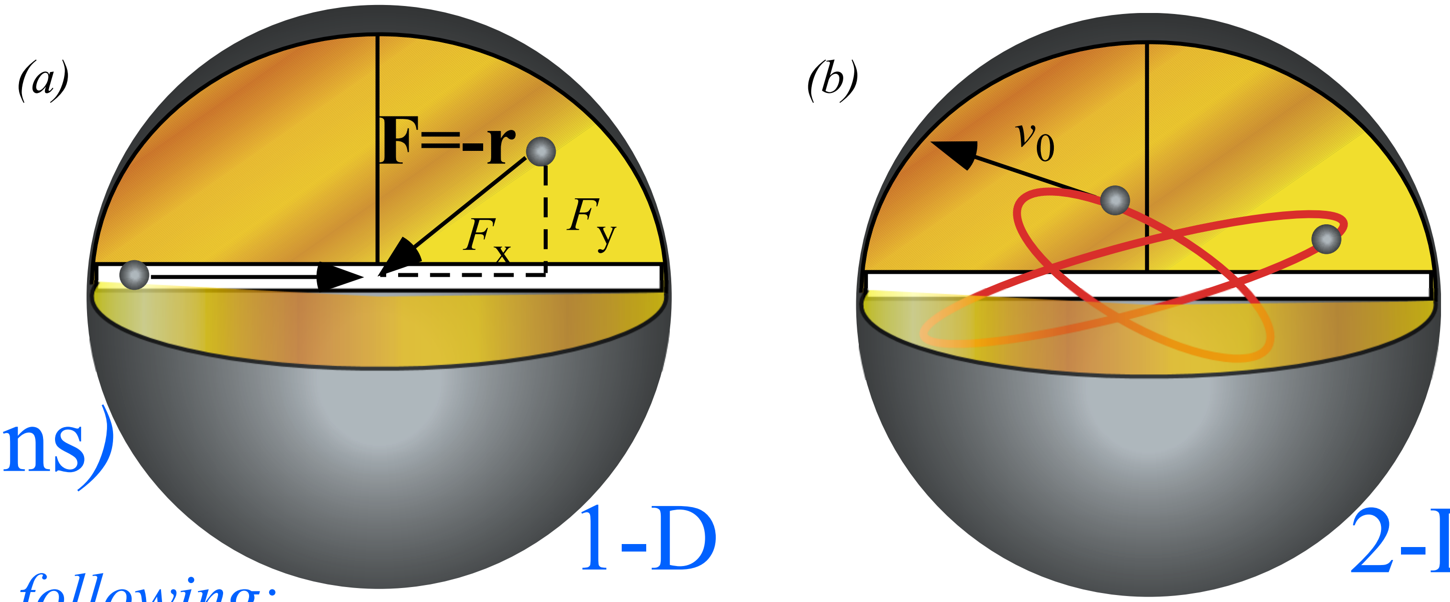
→ *Review of IHO orbital phasor “clock” dynamics in uniform-body with two “movie” examples*

Review of IHO orbital phase dynamics in uniform-body

I.H.O. Force law

F =-x (1-Dimension)

F =-r (2 or 3-Dimensions)



1-D

2-D or 3-D

(Paths are *always* 2-D ellipses if viewed right!)

Unit 1
Fig. 9.10

Each dimension x, y, or z obeys the following:

$$Total\ E = KE + PE = \frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = const.$$

Equations for x-motion [x(t) and v_x=v(t)] are given first. They apply as well to dimensions [y(t) and v_y=v(t)] and [z(t) and v_z=v(t)] in the ideal isotropic case.

$$1 = \frac{mv^2}{2E} + \frac{kx^2}{2E} = \left(\frac{v}{\sqrt{2E/m}}\right)^2 + \left(\frac{x}{\sqrt{2E/k}}\right)^2$$

$$1 = \frac{mv^2}{2E} + \frac{kx^2}{2E} = (\cos\theta)^2 + (\sin\theta)^2$$

Another example of the old "scale-a-circle" trick...

velocity: *position:* *angular velocity:*

$$Let: \textbf{(1)}\ v = \sqrt{2E/m}\cos\theta, \quad and: \textbf{(2)}\ x = \sqrt{2E/k}\sin\theta \qquad def.\ \textbf{(3)}\ \omega = \frac{d\theta}{dt}$$

velocity:

$$\underbrace{\sqrt{\frac{2E}{m}}\cos\theta = v}_{by\ (1)} = \frac{dx}{dt} = \underbrace{\frac{d\theta}{dt}\frac{dx}{d\theta}}_{by\ def.\ (3)} = \omega \underbrace{\frac{dx}{d\theta} = \omega\sqrt{\frac{2E}{k}}}_{by\ (2)}\cos\theta$$

by def. **(3)**

$$\underbrace{\omega = \frac{d\theta}{dt} = \sqrt{\frac{k}{m}}}_{divide\ this\ by\ (1)}$$

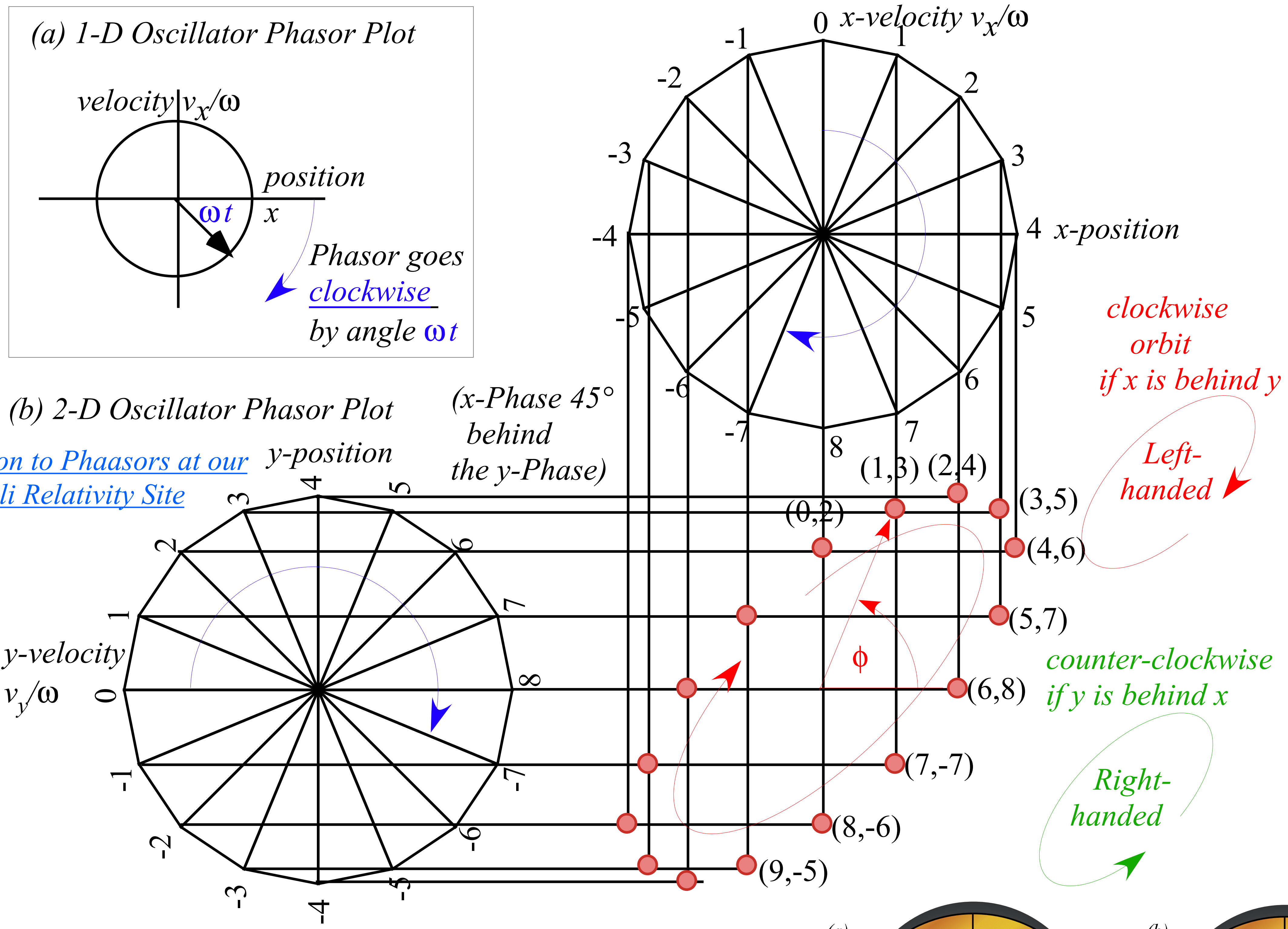
by integration given constant ω:

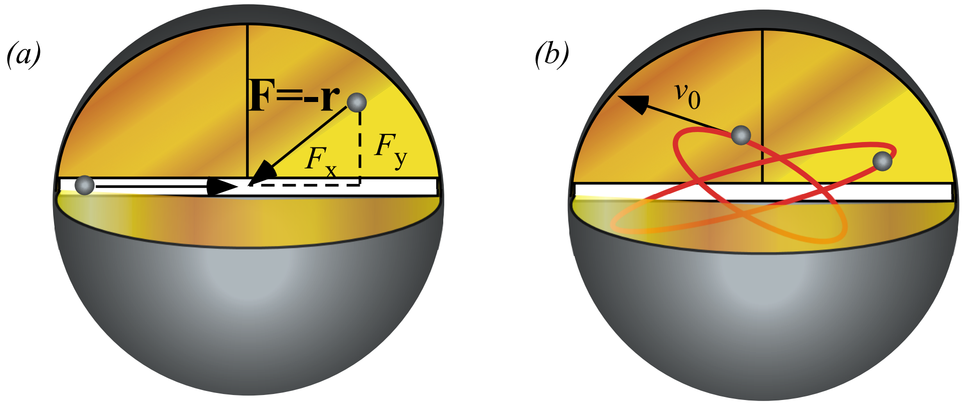
$$\theta = \int \omega \cdot dt = \omega \cdot t + \alpha$$

Review of IHO orbital phase dynamics in uniform-body

Unit 1
Fig. 9.10

Introduction to Phaasors at our Pirelli Relativity Site

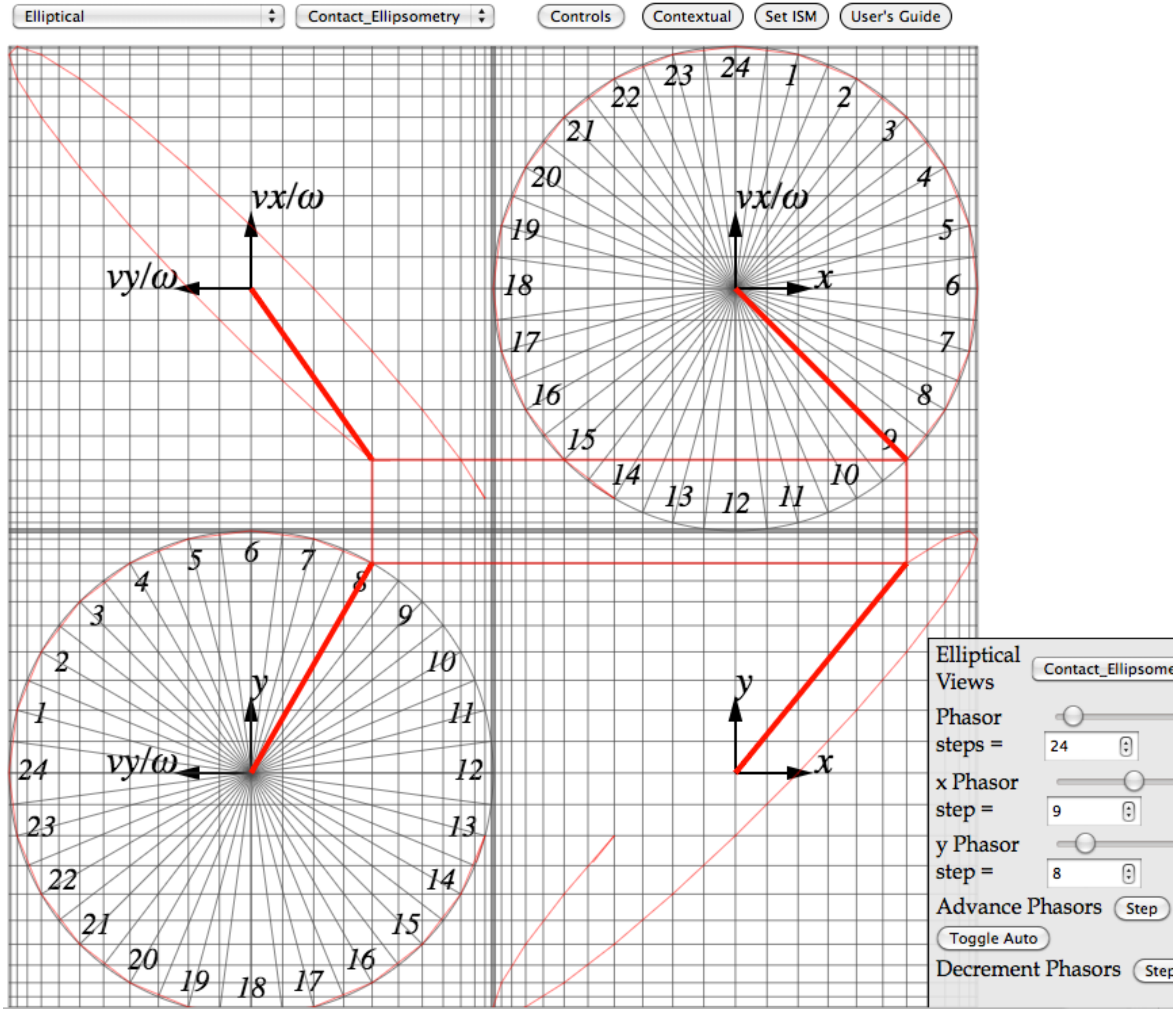




BoxIt web simulation - *With y-Phasor is on other side of xy plot*

RelaWavity web simulation - Contact ellipsometry

RelaWavity
ellipsometry
web-app



Elliptical
Contact_Ellipsometry

Elliptical Views: Contact_Ellipsome
Phasor steps = 24
x Phasor step = 9
y Phasor step = 8

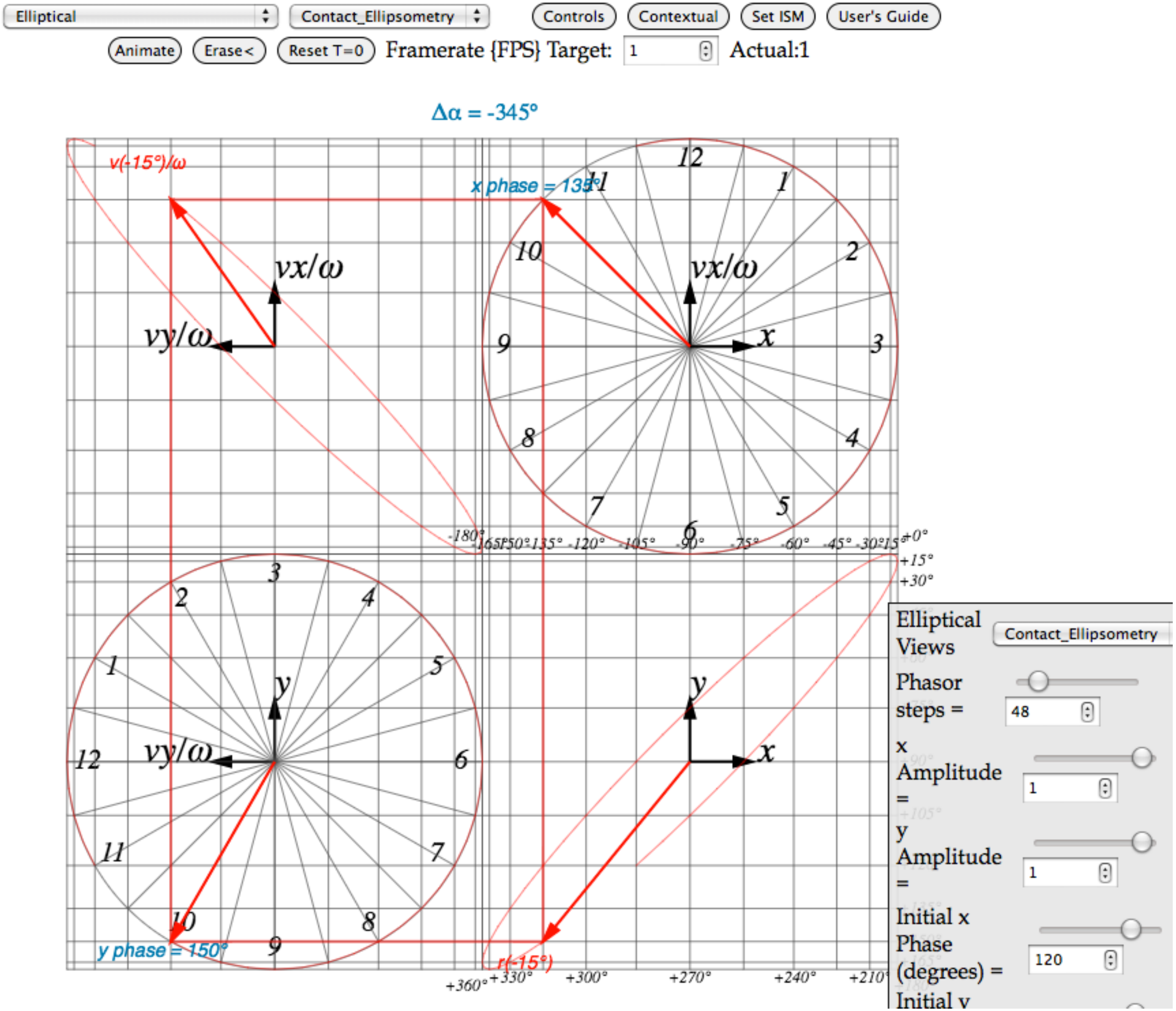
RelaWavity Web Simulation
Ellipsometry

Geometry of Kepler anomalies for vectors $[\mathbf{r}(\phi), \mathbf{v}(\phi)]$ *in coordinate (x,y) space rendered by animation web-apps BoxIt and RelaWavity described below after p.70.*

http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?phasorSteps=24&use12Steps=0&plotType=1l2

RelaWavity
ellipsometry
web-app

[RelaWavity Web Simulation](http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?plotType=1l2)
[Ellipsometry](http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?plotType=1l2)



Framerate {FPS} Target: 1 Actual:1

Phasor steps = 48

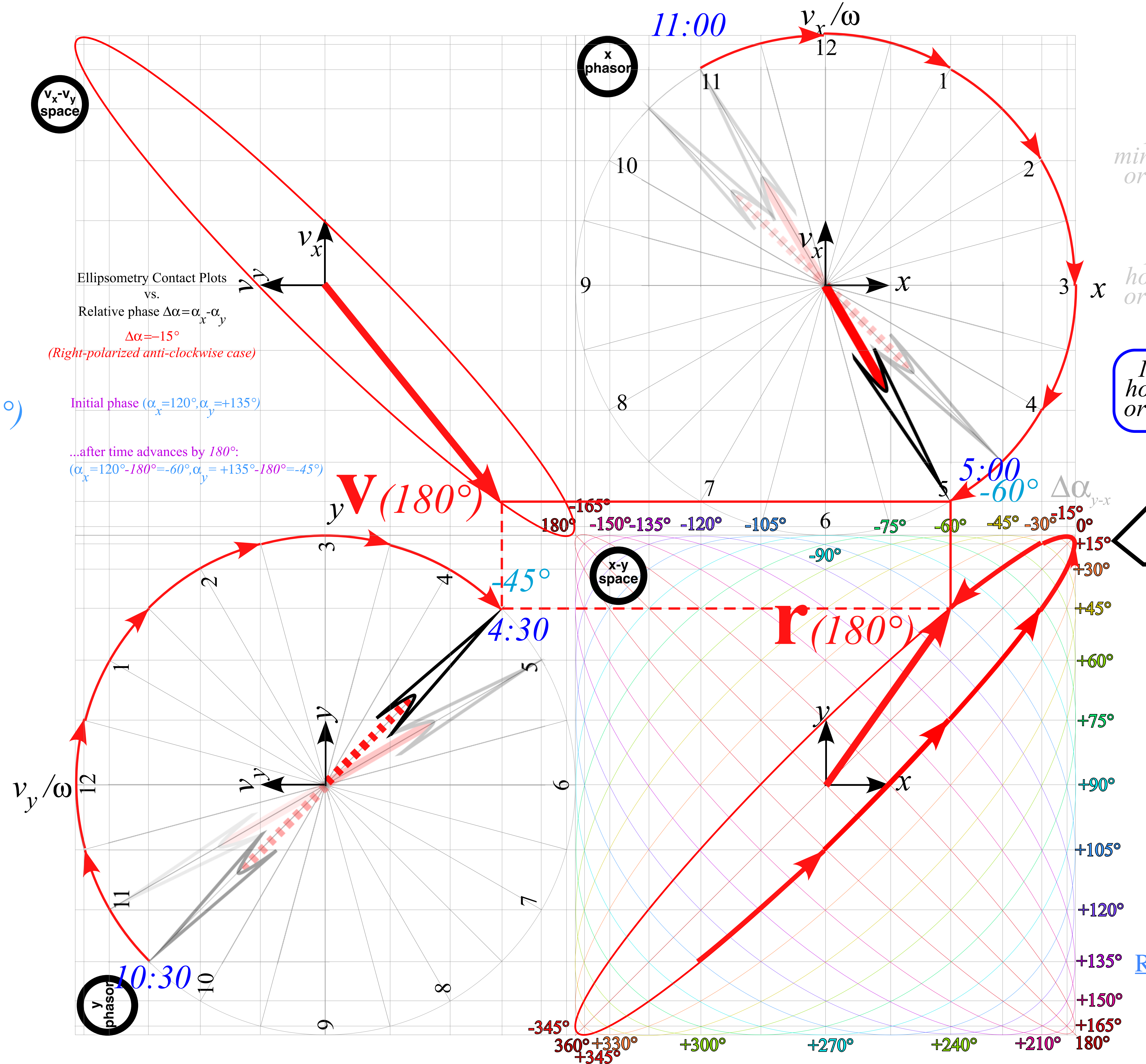
x Amplitude = 1

y Amplitude = 1

Initial x Phase (degrees) = 120

Geometry of Kepler anomalies for vectors [$\mathbf{r}(\phi)$, $\mathbf{v}(\phi)$] *in coordinate (x,y) space rendered by animation web-apps BoxIt and RelaWavity described below after p.7 and p.17.*

http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?plotType=1l2



Ellipsometry Contact Plots
vs.
Relative phase $\Delta\alpha=\alpha_x-\alpha_y$

$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)

Initial phase ($\alpha_x=120°,\alpha_y=+135°$)

...after time advances by *180°*:
($\alpha_x=120°-180°=-60°,\alpha_y = +135°-180°=-45°$)

phase lag:
$\Delta\alpha=\alpha_x -\alpha_y =15°$

1 minute orbit *(2.5 seconds for second hand)*

or

1 hour orbit *(2.5 minutes for minute hand)*

or

12 hour orbit *(1/2 hour for hour hand)*

RelaWavity Web Simulation
Ellipsometry

Constructing 2D IHO orbits using Kepler anomaly plots

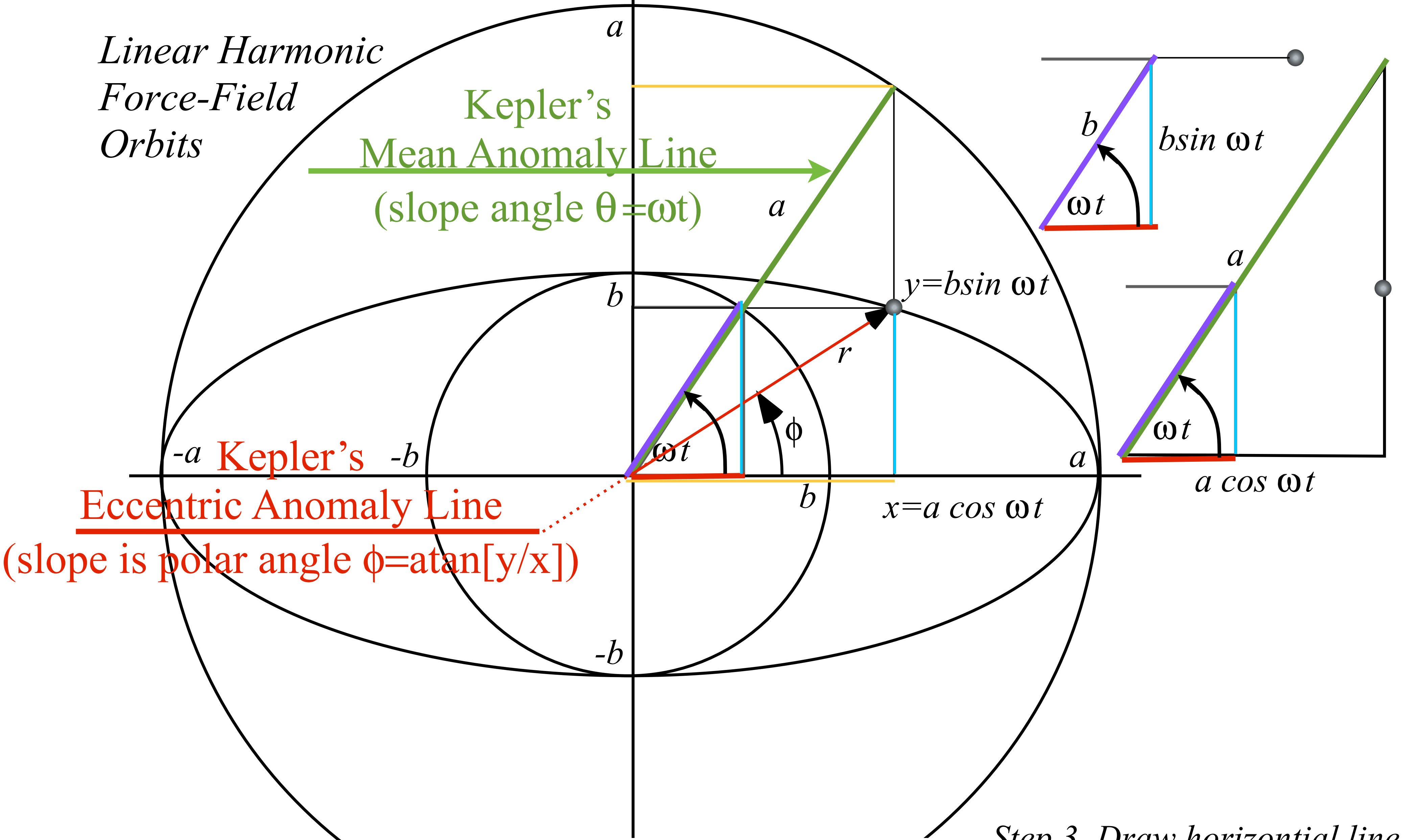
→ Mean-anomaly and eccentric-anomaly geometry

Calculus and vector geometry of IHO orbits

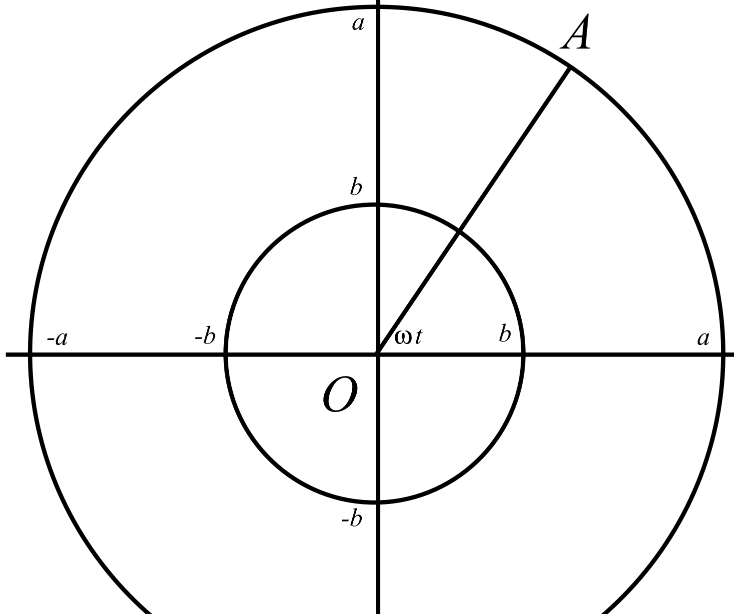
A confusing introduction to Coriolis-centrifugal force geometry (Derived better in Ch. 12)

Linear Harmonic Force-Field Orbits

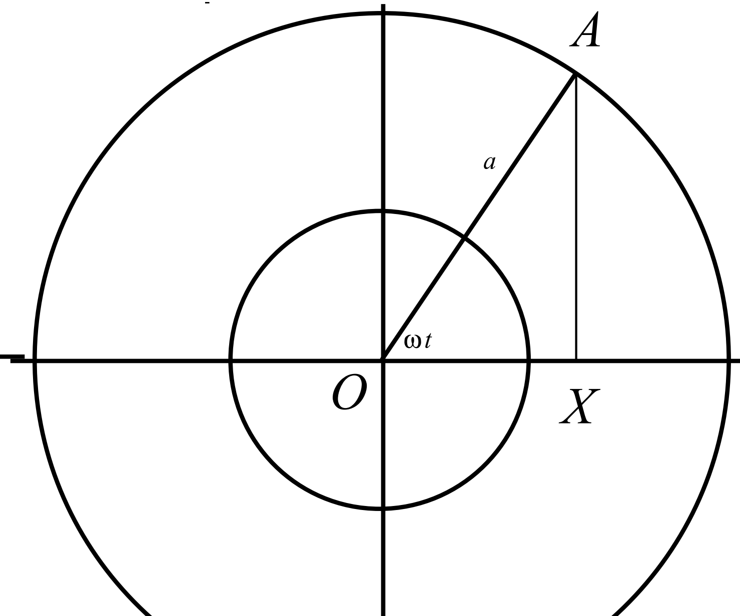
Unit 1
Fig. 11.1
(top 2/3's)



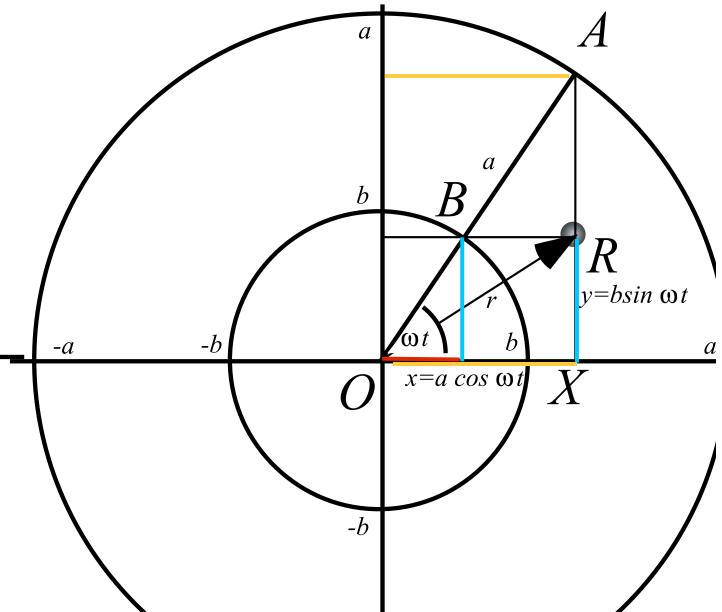
Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt



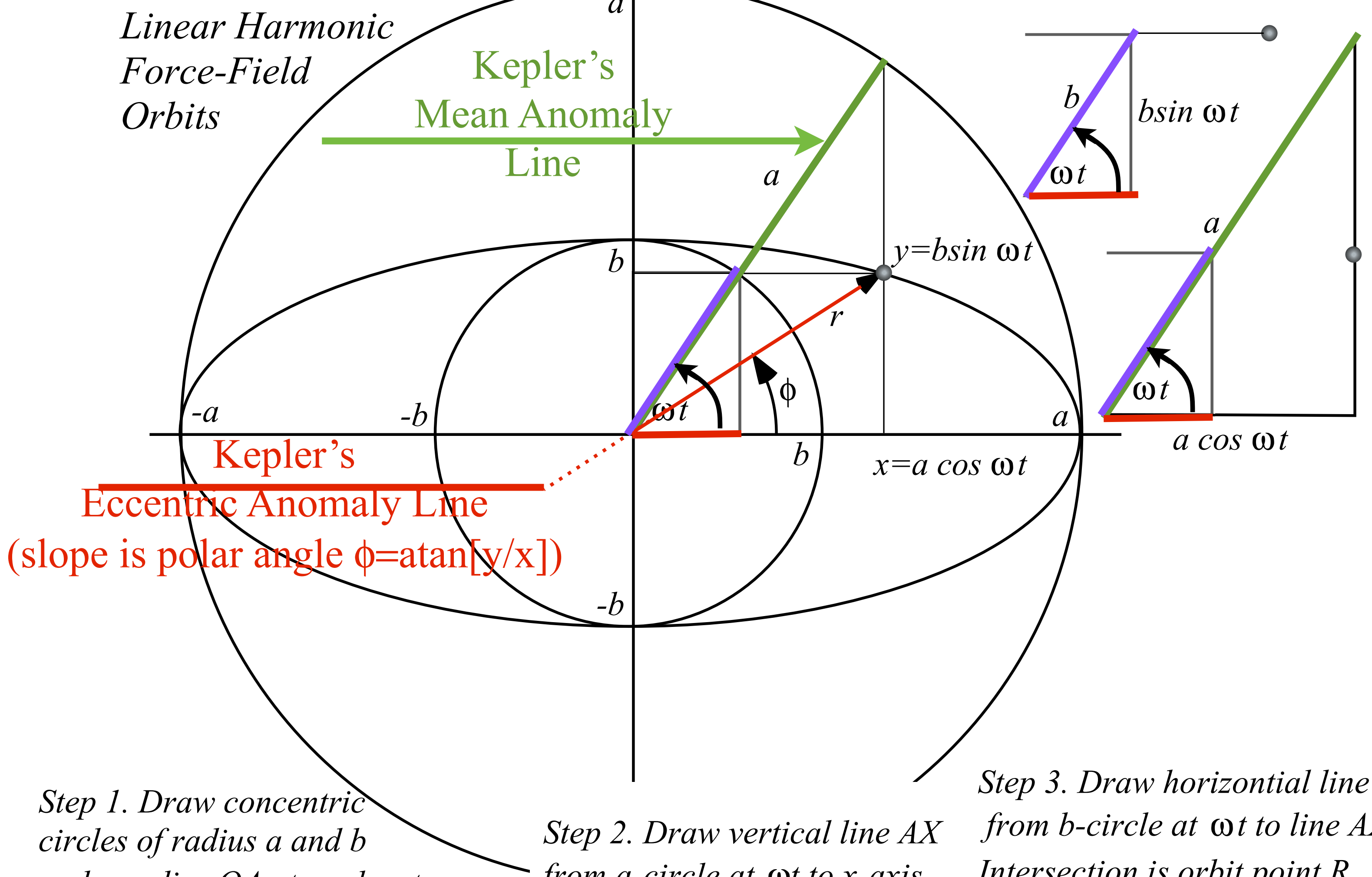
Step 2. Draw vertical line AX from a-circle at ωt to x-axis



Step 3. Draw horizontial line BR from b-circle at ωt to line AX. Intersection is orbit point R.



Linear Harmonic Force-Field Orbits



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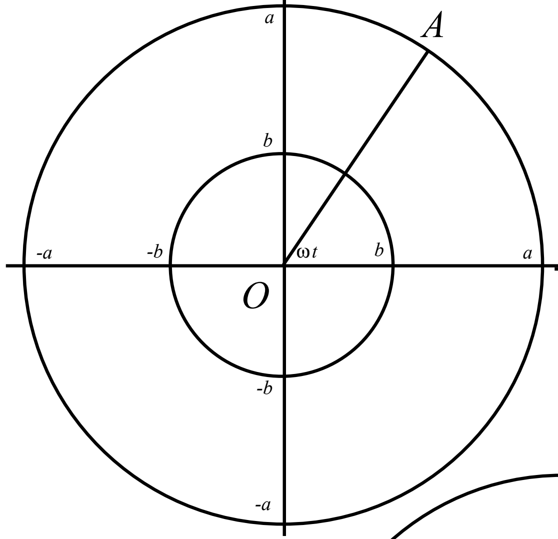

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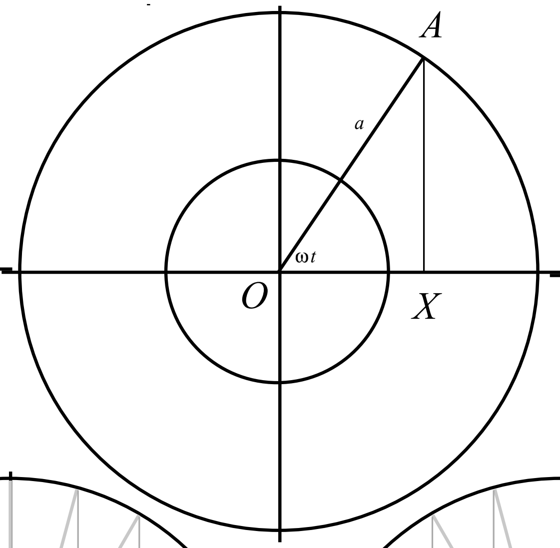

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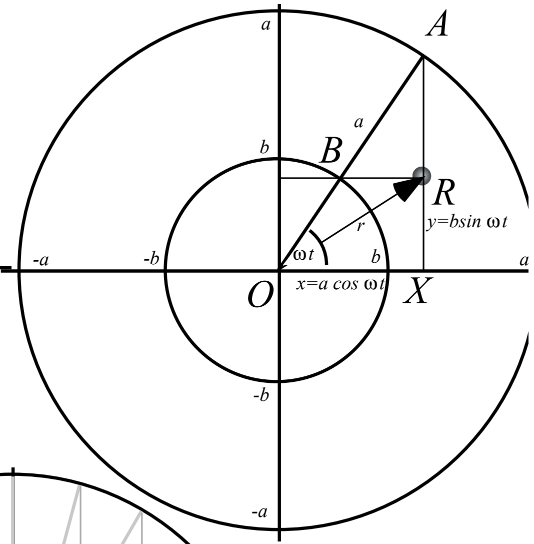
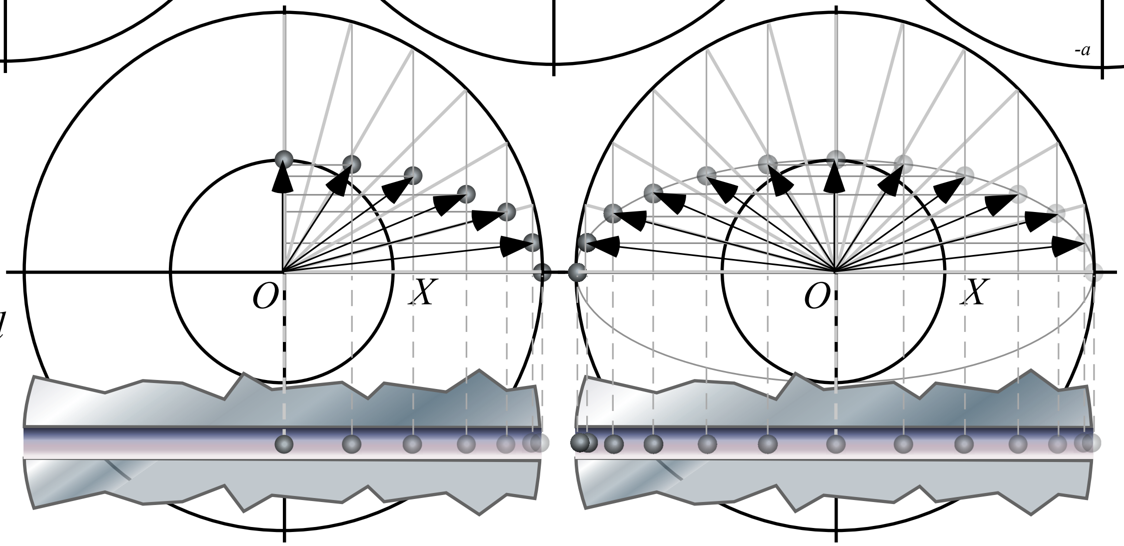


Step 4-N Repeat as often as needed



Unit 1
Fig. 11.1

Constructing 2D IHO orbits using Kepler anomaly plots

- Mean-anomaly and eccentric-anomaly geometry
- → Calculus and vector geometry of IHO orbits
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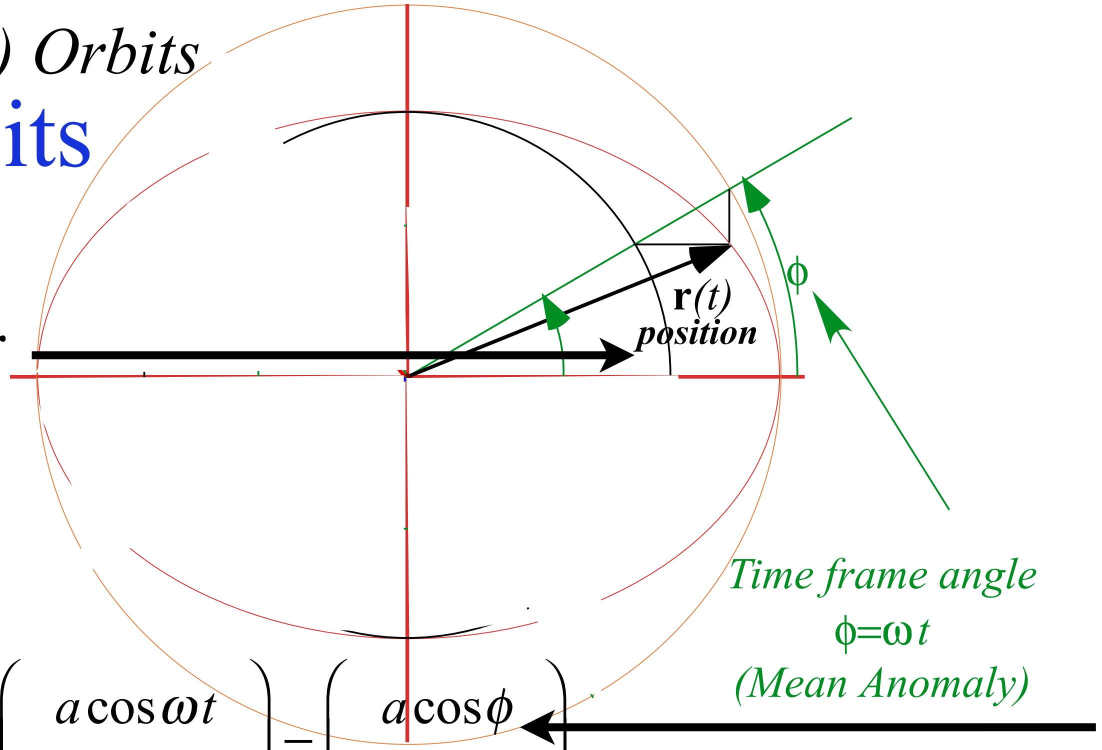
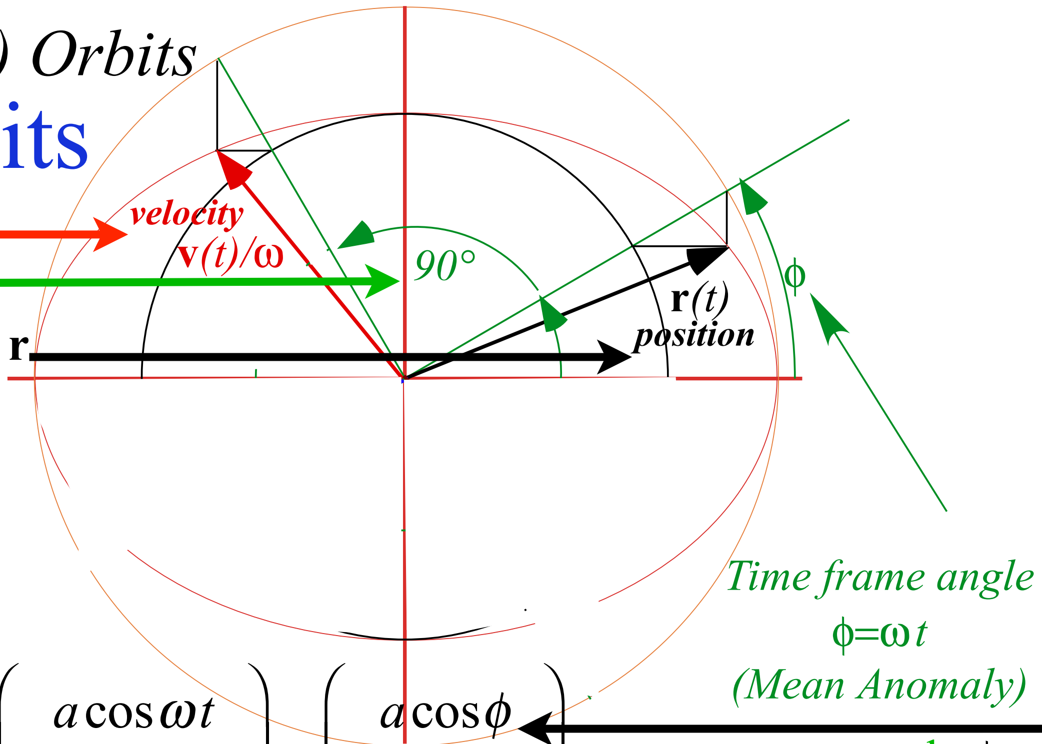
Calculus of IHO orbits



$$\text{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

Unit 1
Fig. 11.5

Calculus of IHO orbits



To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**

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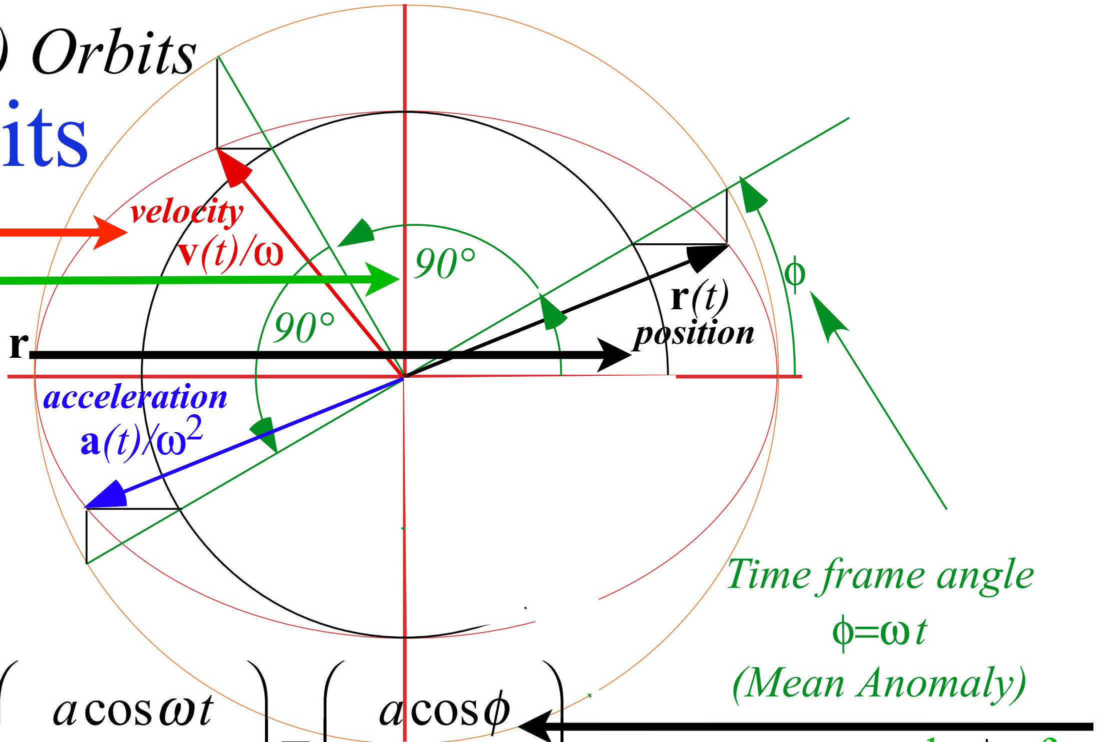
mean-anomaly ϕ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

$$\text{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\mathit{for}\ \omega = 1)$$

Unit 1
Fig. 11.5

Calculus of IHO orbits



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Unit 1
Fig. 11.5

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m.a. ϕ+π/2 of vector **v** rotated by
another π/2 is *m.a.* of vector **a**

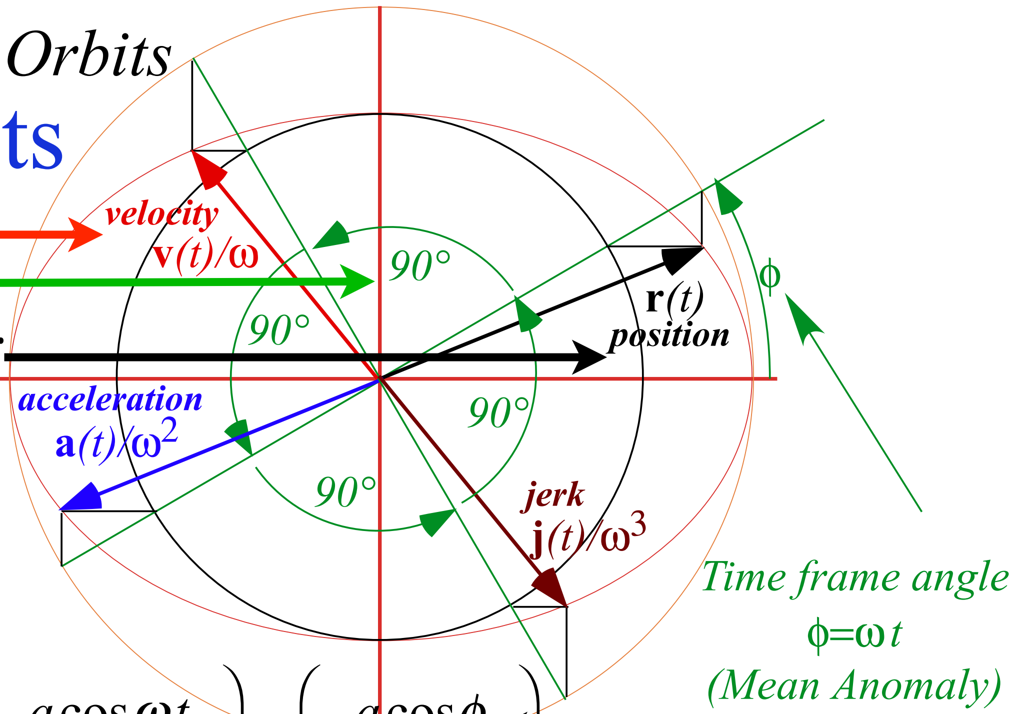
acceleration or force vector : $$\frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

Calculus of IHO orbits

(a) Orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**



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mean-anomaly ϕ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\textit{velocity vector}:\mathbf{v}=\begin{pmatrix}v_x\\ v_y\end{pmatrix}=\begin{pmatrix}-a\omega\sin\omega t\\ b\omega\cos\omega t\end{pmatrix}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\begin{pmatrix}a\cos\left(\phi+\frac{\pi}{2}\right)\\ b\sin\left(\phi+\frac{\pi}{2}\right)\end{pmatrix}(\mathit{for}\ \omega=1)$$

m.a. ϕ+π/2 of vector **v** rotated by
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$$\textit{acceleration or force vector}:\frac{\mathbf{F}}{m}=\mathbf{a}=\begin{pmatrix}a_x\\ a_y\end{pmatrix}=\begin{pmatrix}-a\omega^2\cos\omega t\\ -b\omega^2\sin\omega t\end{pmatrix}=\frac{d\mathbf{v}}{dt}=\dot{\mathbf{v}}=\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2}=\begin{pmatrix}a\cos\left(\phi+\frac{2\pi}{2}\right)\\ b\sin\left(\phi+\frac{2\pi}{2}\right)\end{pmatrix}$$

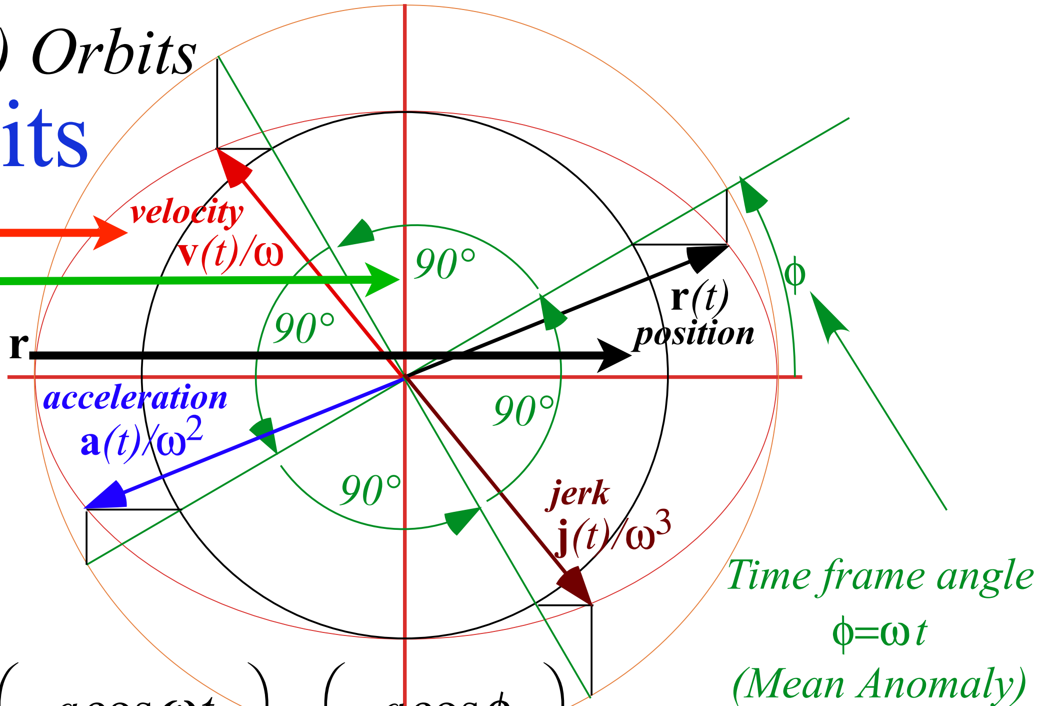
or change of velocity

$$\textit{jerk or change of acceleration}:\mathbf{j}=\begin{pmatrix}j_x\\ j_y\end{pmatrix}=\begin{pmatrix}+a\omega^3\sin\omega t\\ -b\omega^3\cos\omega t\end{pmatrix}=\frac{d\mathbf{a}}{dt}=\dot{\mathbf{a}}=\ddot{\mathbf{v}}=\dddot{\mathbf{r}}=\frac{d^3\mathbf{r}}{dt^3}=\begin{pmatrix}a\cos\left(\phi+\frac{3\pi}{2}\right)\\ b\sin\left(\phi+\frac{3\pi}{2}\right)\end{pmatrix}$$

...and so forth...

Calculus of IHO orbits

(a) Orbits



To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**

$$\text{radius vector}:\mathbf{r}=\begin{pmatrix} r_x \\ r_y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}=\begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\text{velocity vector}:\mathbf{v}=\begin{pmatrix} v_x \\ v_y \end{pmatrix}=\begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\begin{pmatrix} a\cos\left(\phi+\frac{\pi}{2}\right) \\ b\sin\left(\phi+\frac{\pi}{2}\right) \end{pmatrix}(\text{for } \omega=1)$$

m.a. φ+π/2 of vector **v** rotated by
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...and so forth...

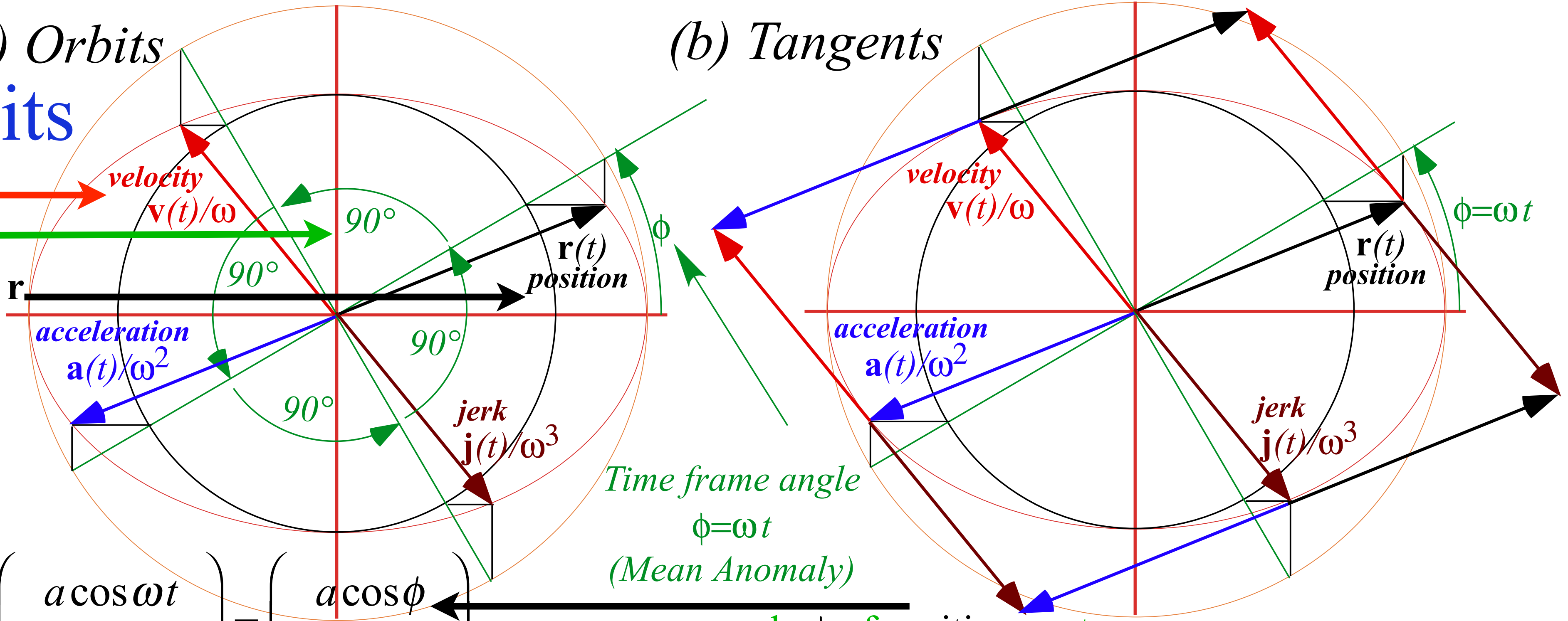
$$\text{inauguration or change of jerk}:\mathbf{i}=\begin{pmatrix} i_x \\ i_y \end{pmatrix}=\begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix}=\frac{d\mathbf{j}}{dt}=\dot{\mathbf{j}}=\ddot{\mathbf{a}}=\dddot{\mathbf{v}}=\ddddot{\mathbf{r}}=\frac{d^4\mathbf{r}}{dt^4}=\begin{pmatrix} a\cos\left(\phi+\frac{4\pi}{2}\right) \\ b\sin\left(\phi+\frac{4\pi}{2}\right) \end{pmatrix}$$

...and so on...
...But, now it
repeats after 4
t-derivatives

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**

Link ⇒ BoxIt simulation of IHO orbits



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mean-anomaly ϕ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

Link → IHO Exegesis Plot

Link → IHO orbital time rates of change

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

m.a. ϕ+π/2 of vector **v** rotated by
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or change of velocity

...and so forth...

$$\textit{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi + \frac{3\pi}{2}\right) \\ b\sin\left(\phi + \frac{3\pi}{2}\right) \end{pmatrix}$$

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...and so on...
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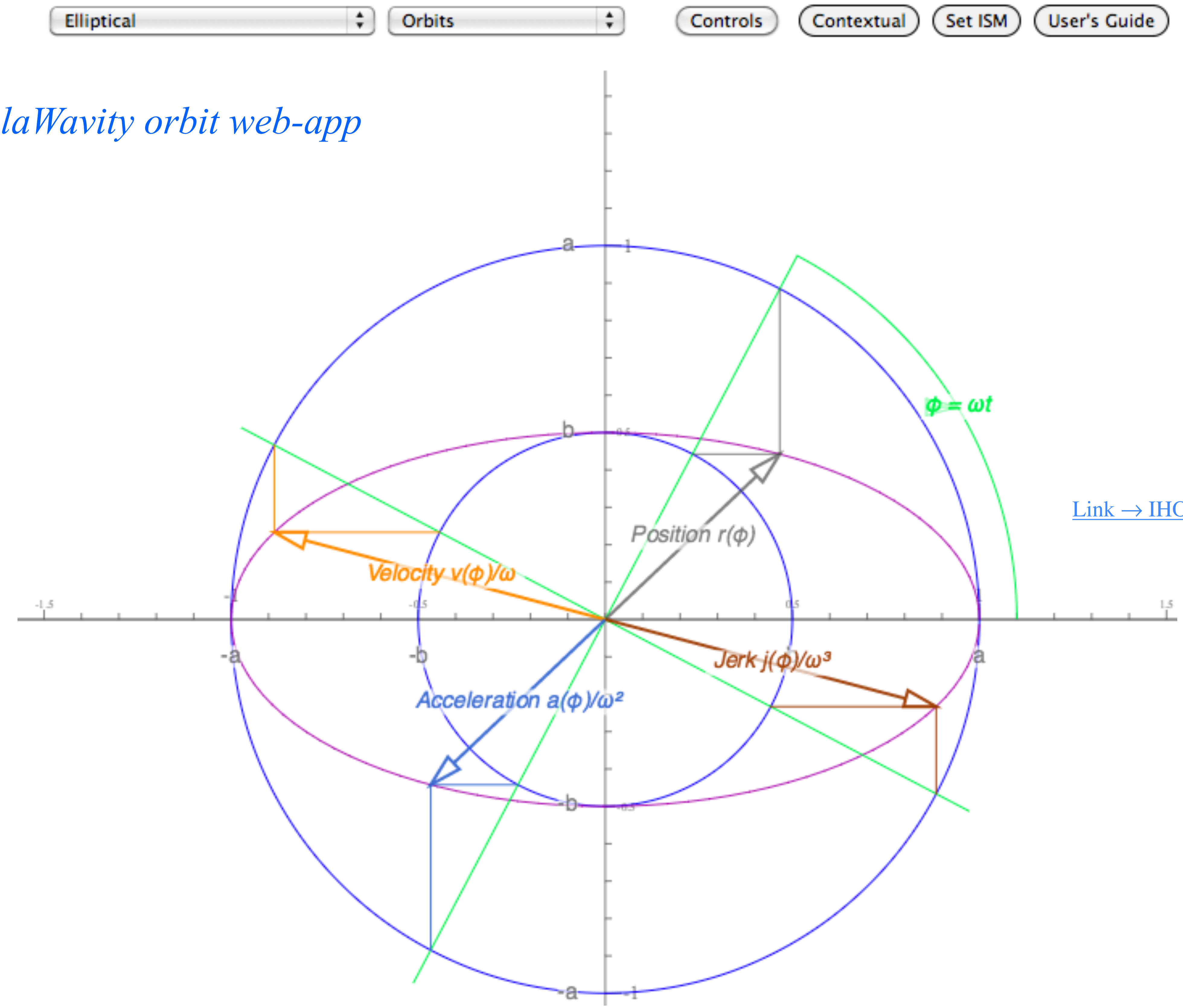
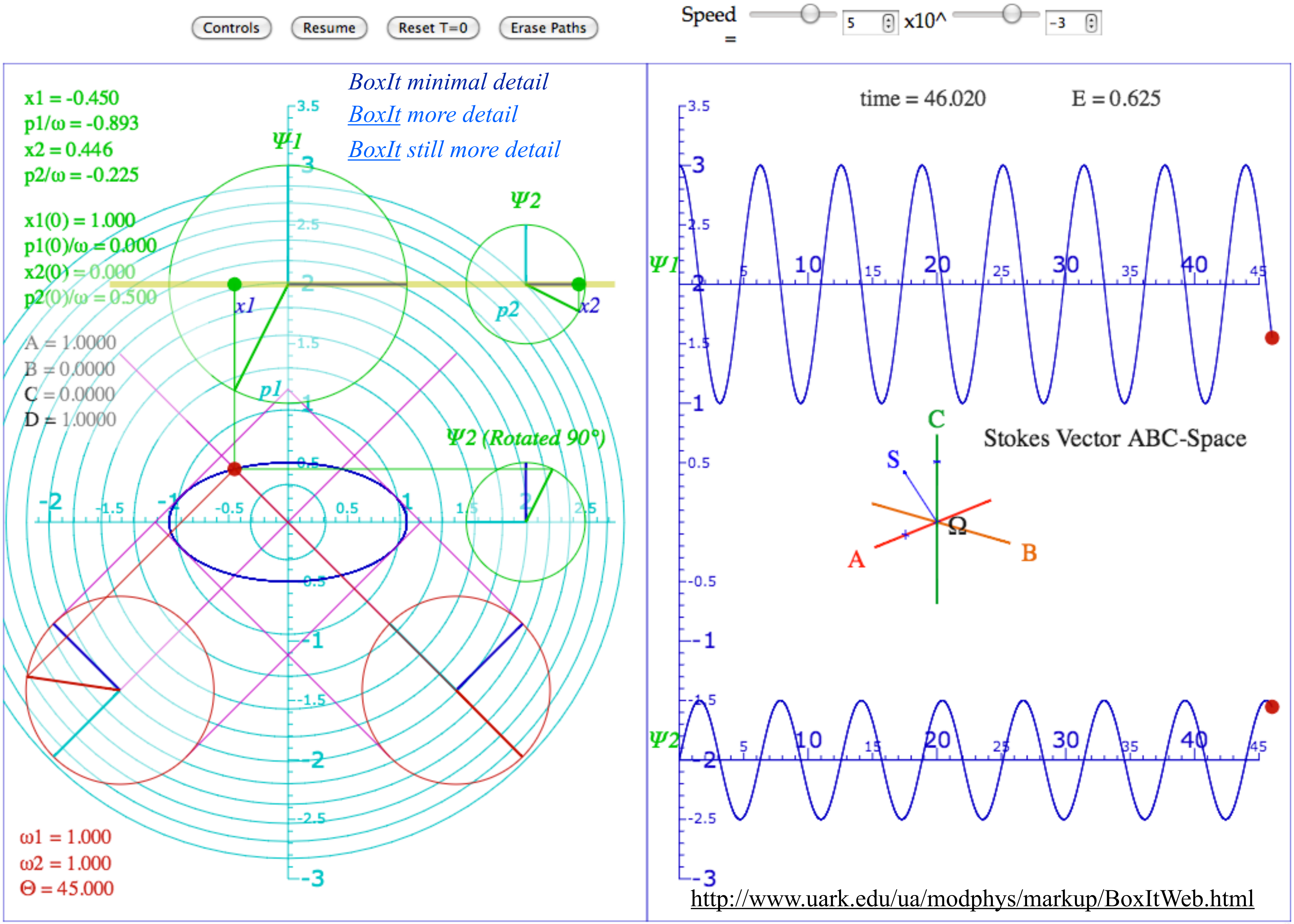
RelaWavity orbit web-app



Link → IHO orbital time rates of change

Geometry of Kepler anomalies for vectors $[\mathbf{r}(\phi), \mathbf{v}(\phi), \mathbf{a}(\phi), \mathbf{j}(\phi),]$ *in coordinate (x,y) space rendered by animation web-apps BoxIt and RelaWavity.*



Speed = 5 x10^ -3

x1 = -0.450
p1/ω = -0.893
x2 = 0.446
p2/ω = -0.225

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 1.0000
B = 0.0000
C = 0.0000
D = 1.0000

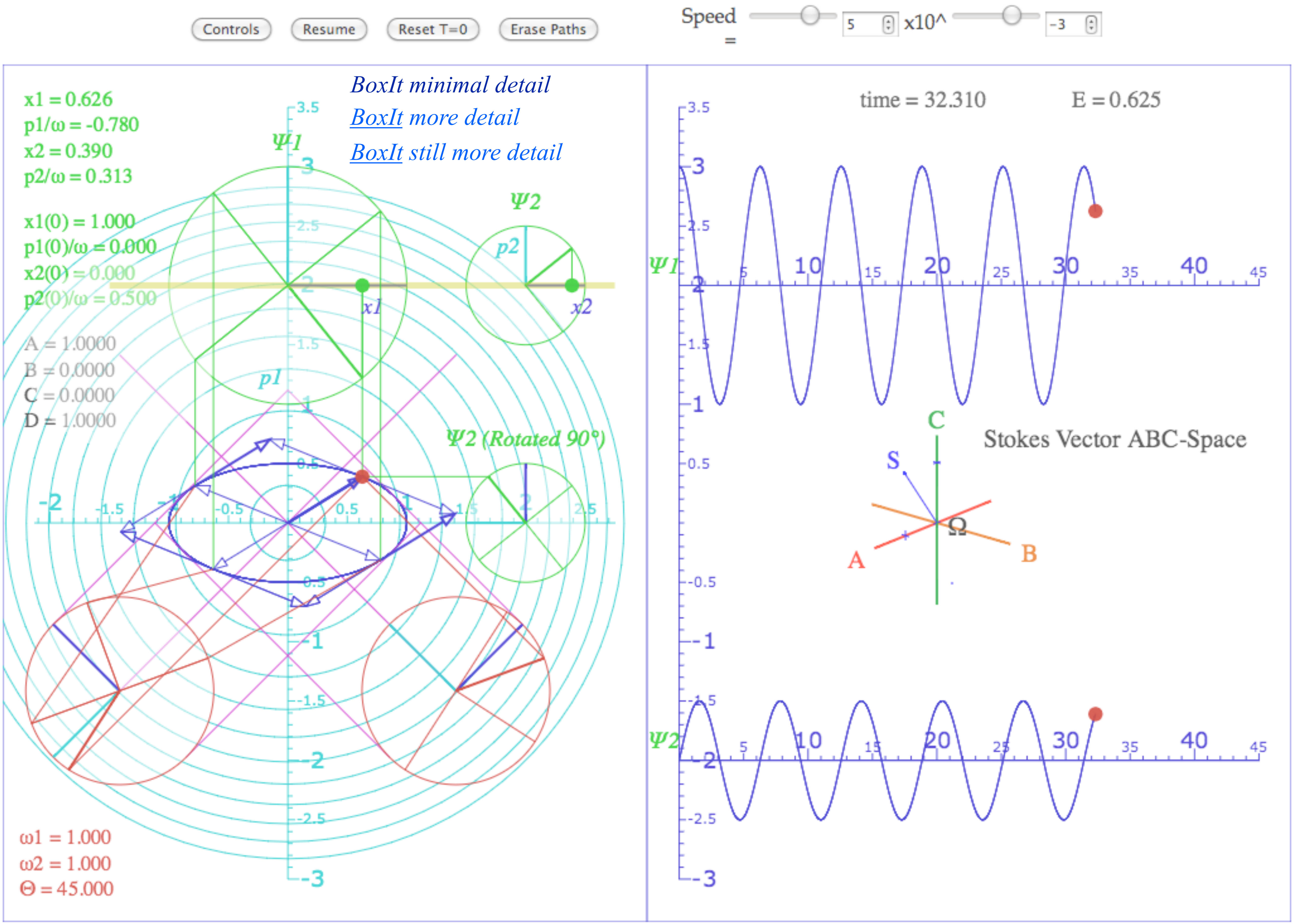
ω1 = 1.000
ω2 = 1.000
Θ = 45.000

time = 46.020 E = 0.625

http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

Geometry of Kepler anomalies for vectors [**r**(ϕ)] *in coordinate (x,y) space and 2-particle* (x_1,x_2) *space rendered by animation web-apps BoxIt.*

BoxIt Web Stokes Simulation



x1 = 0.626
p1/ω = -0.780
x2 = 0.390
p2/ω = 0.313

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 1.0000
B = 0.0000
C = 0.0000
D = 1.0000

ω1 = 1.000
ω2 = 1.000
Θ = 45.000

time = 32.310
E = 0.625

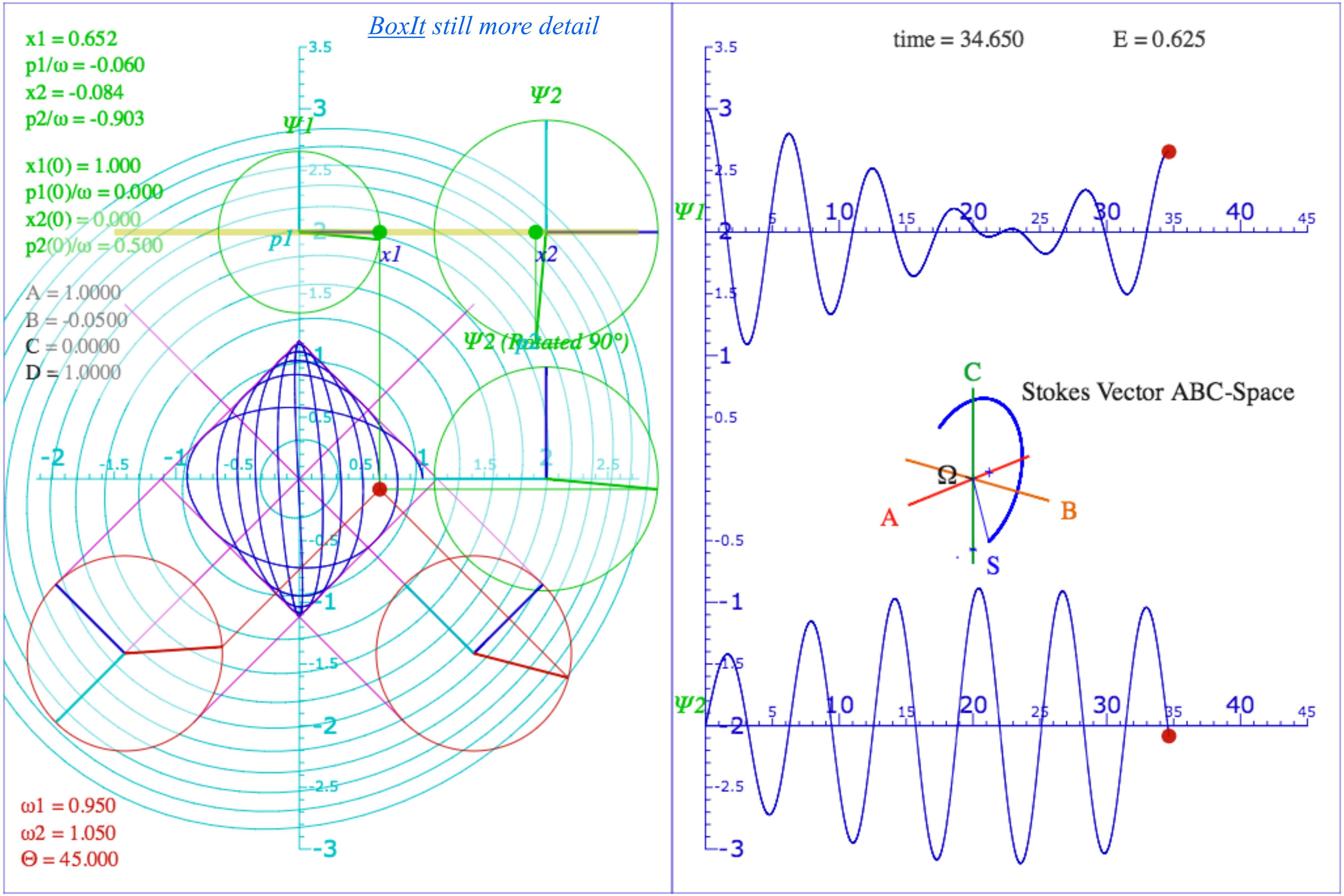
Geometry of Kepler anomalies for vectors [**r**(ϕ), **v**(ϕ), **a**(ϕ), **j**(ϕ),] *in coordinate (x,y) space and 2-particle* (x_1,x_2) *space rendered by animation web-apps BoxIt.*

BoxIt Web Simulation - *w/Derivatives*

BoxIt minimal detail

BoxIt more detail

BoxIt still more detail



x1 = 0.652
p1/ω = -0.060
x2 = -0.084
p2/ω = -0.903

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 1.0000
B = -0.0500
C = 0.0000
D = 1.0000

ω1 = 0.950
ω2 = 1.050
Θ = 45.000

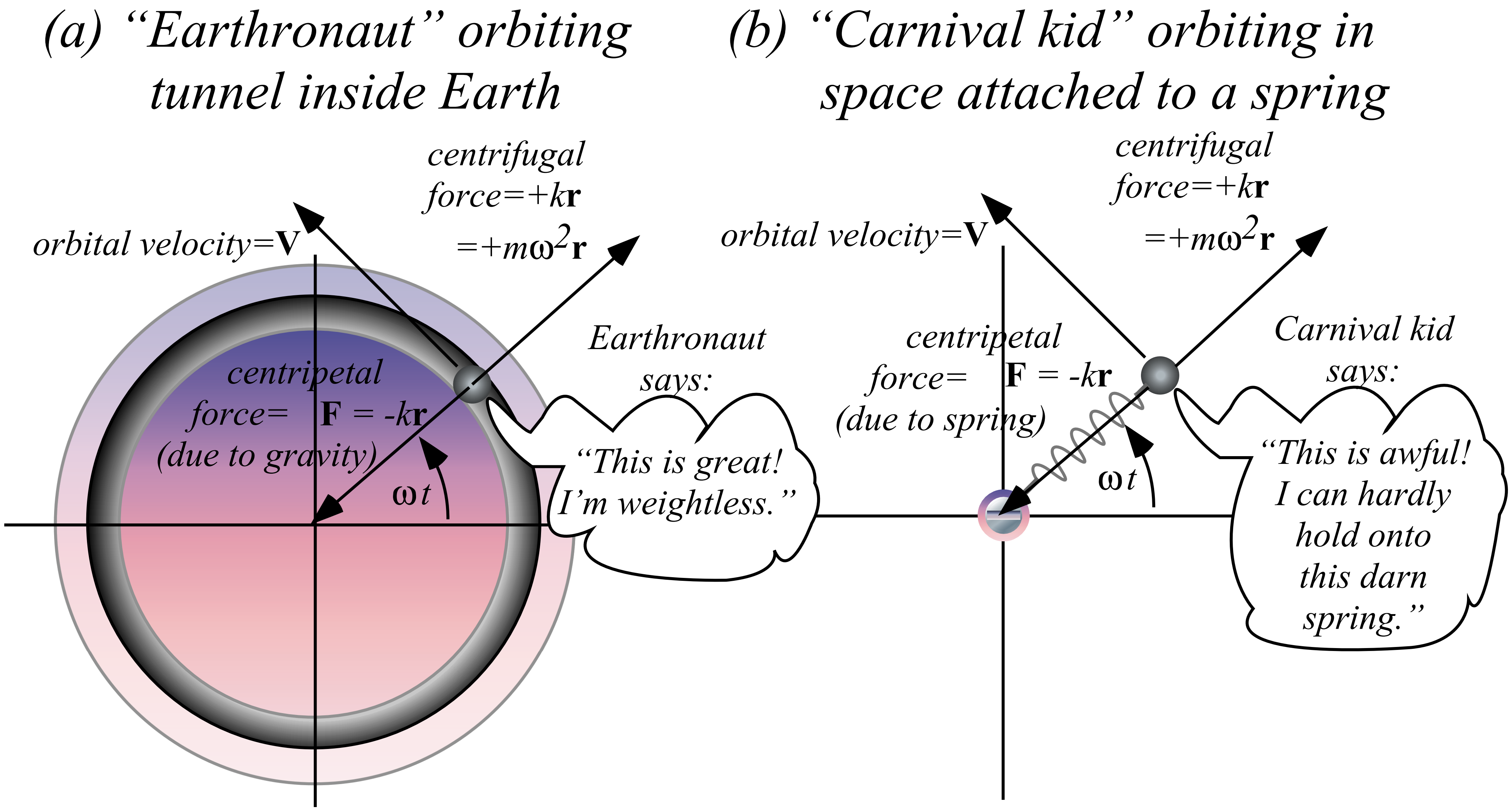
time = 34.650
E = 0.625

Geometry of vectors [**r**(ϕ), **p**(ϕ)] *and quantum spin S-space*
and 2-particle (x_1,x_2) *space rendered by animation web-apps BoxIt.*

BoxIt Web Simulation - *B-Type Motion*

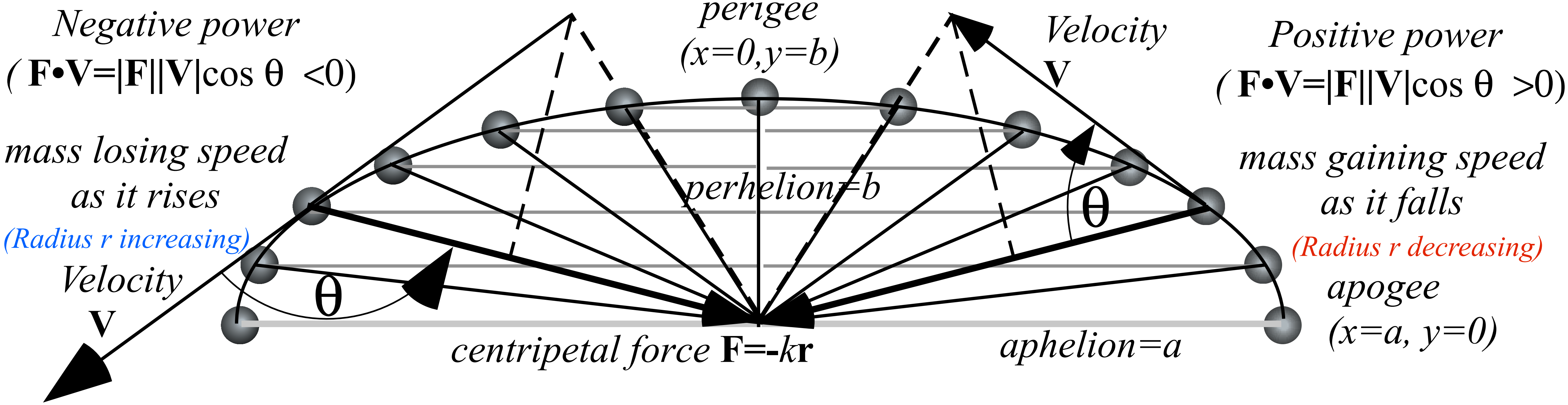
Constructing 2D IHO orbits using Kepler anomaly plots

- *Mean-anomaly and eccentric-anomaly geometry*
- *Calculus and vector geometry of IHO orbits*
- → *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better in Ch. 12)*

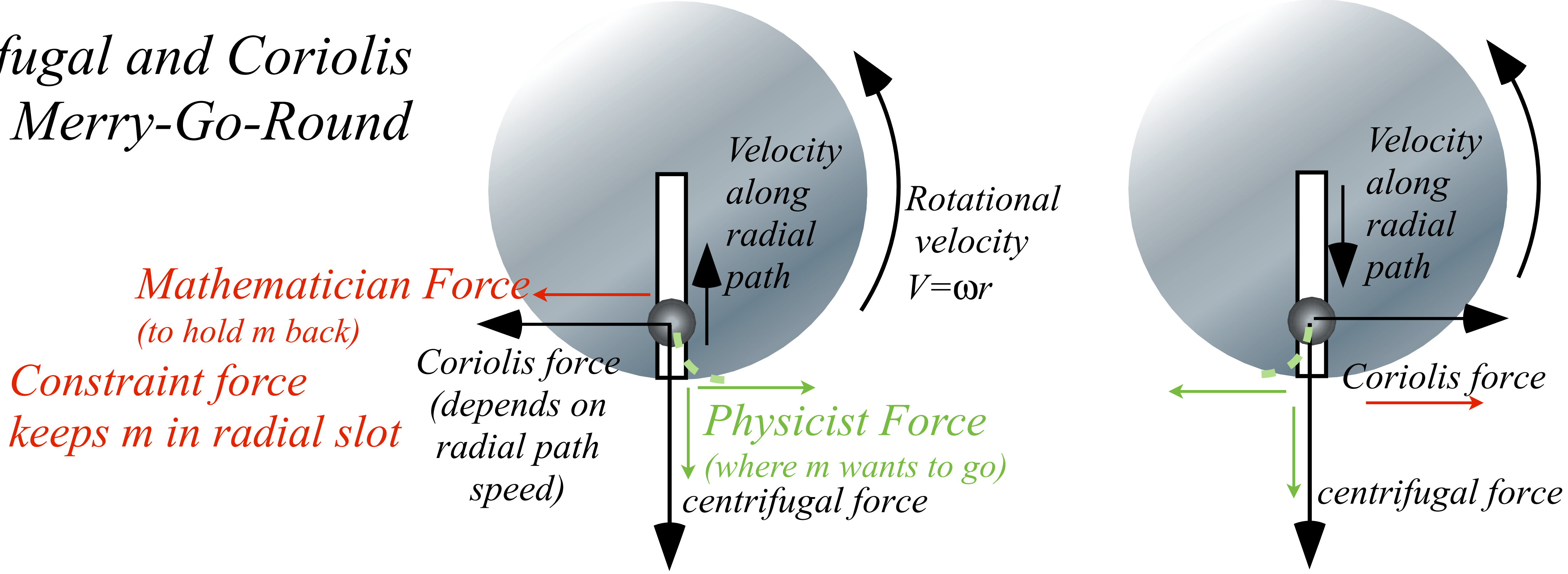


Unit 1
Fig. 11.2

Unit 1
Fig. 11.3



(a) Centrifugal and Coriolis Forces on Merry-Go-Round



Velocity along radial path

Rotational velocity $V=\omega r$

Mathematician Force (to hold m back)

Constraint force keeps m in radial slot

Coriolis force (depends on radial path speed)

Physicist Force (where m wants to go)

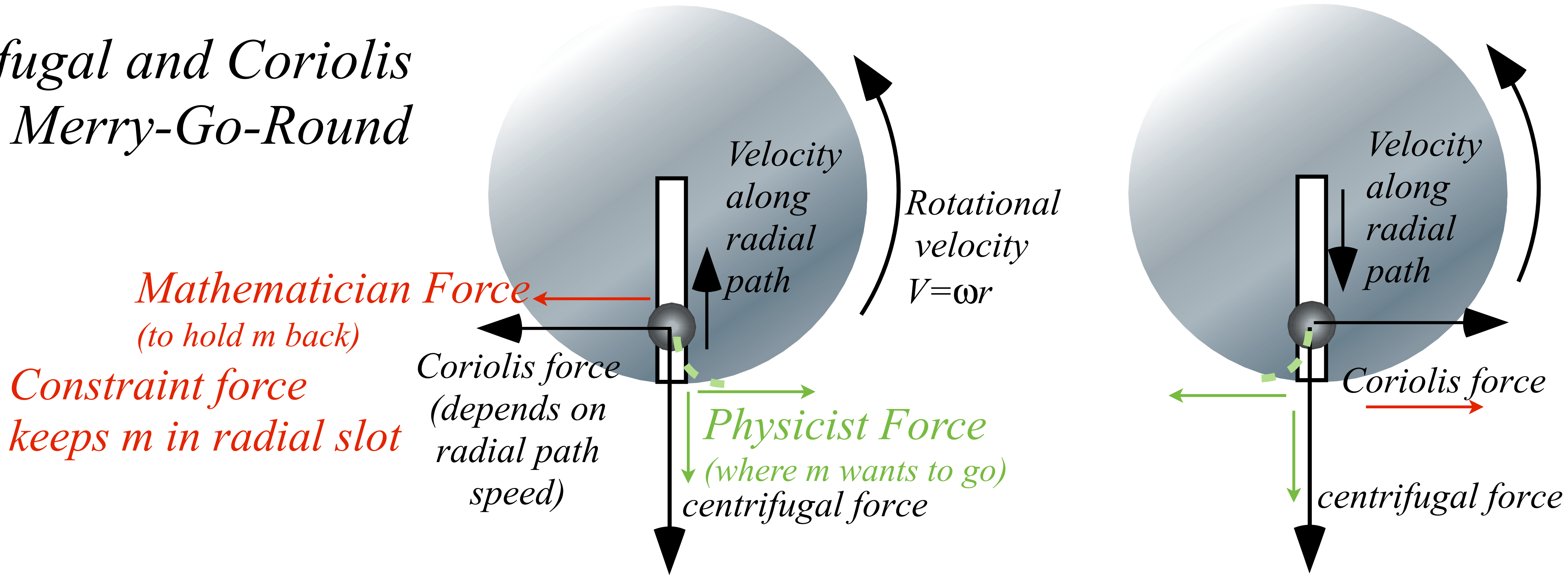
centrifugal force

Velocity along radial path

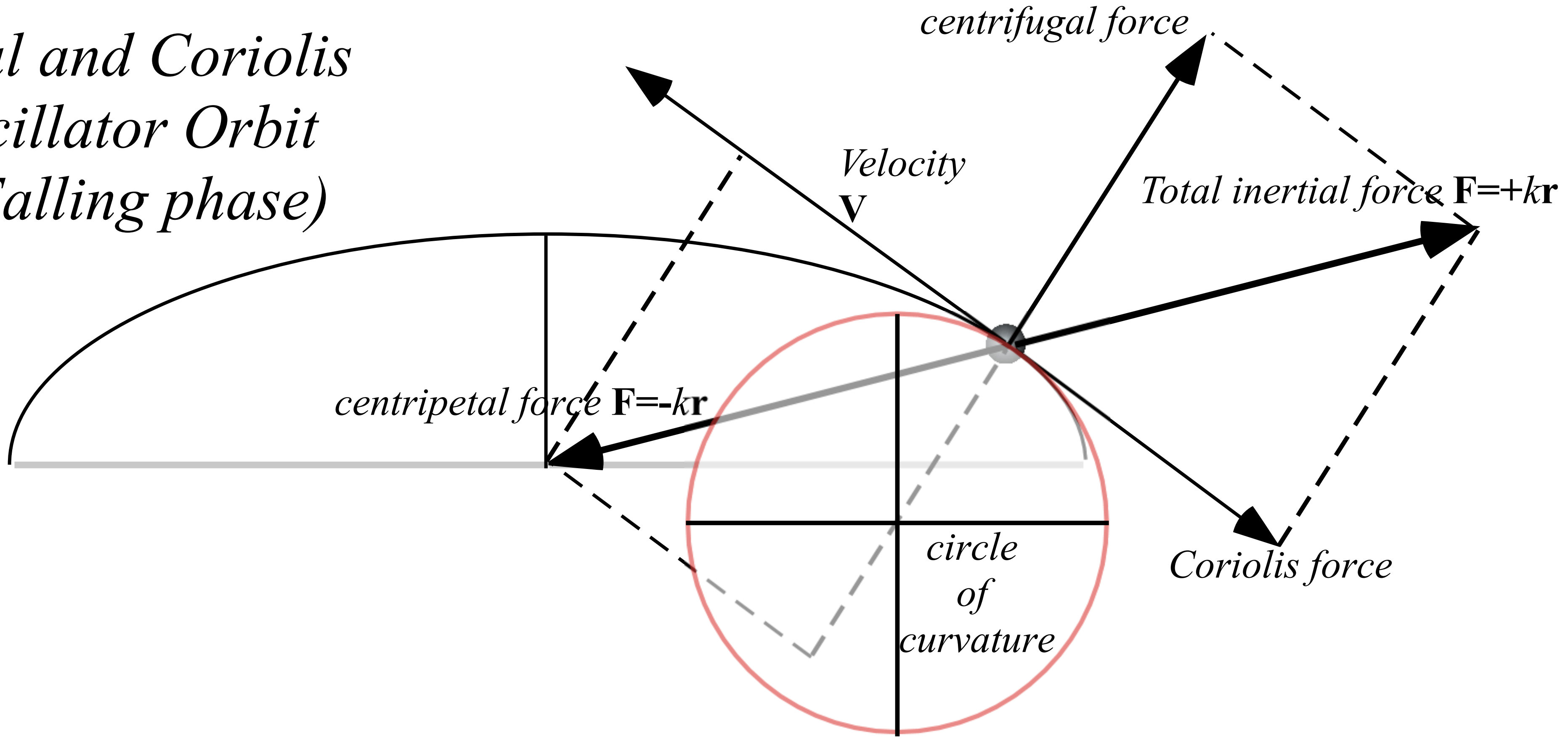
Coriolis force

centrifugal force

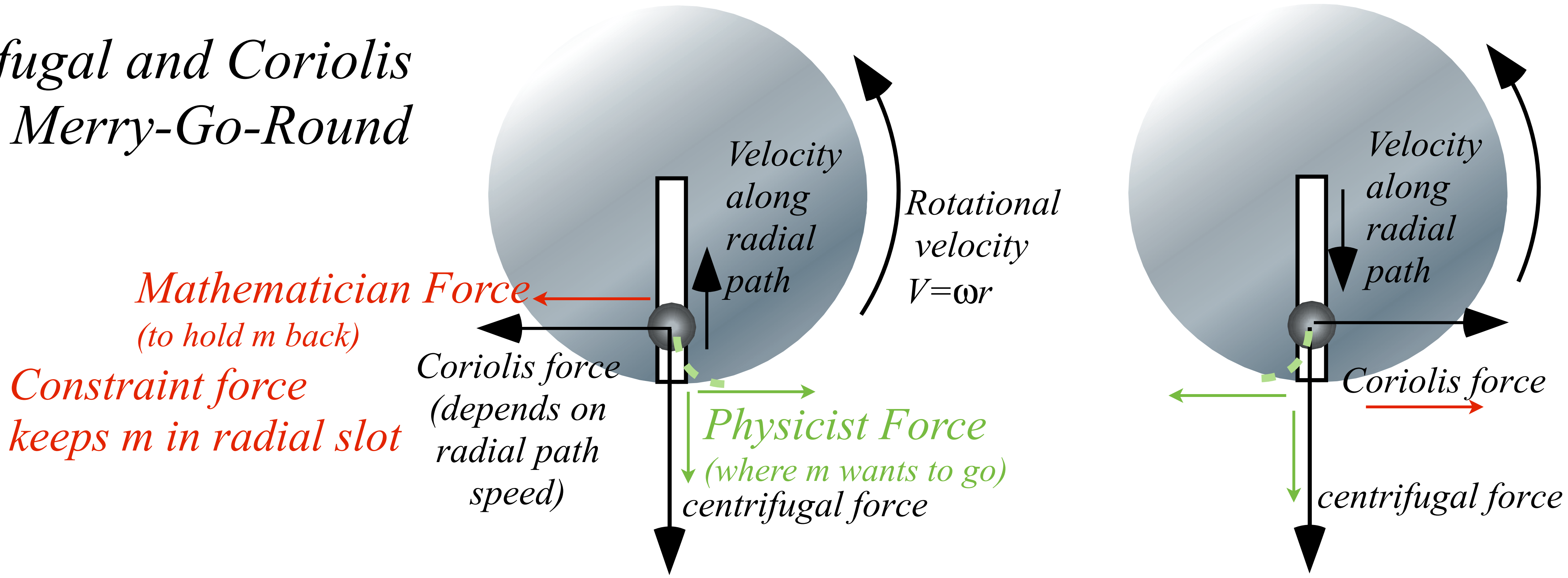
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



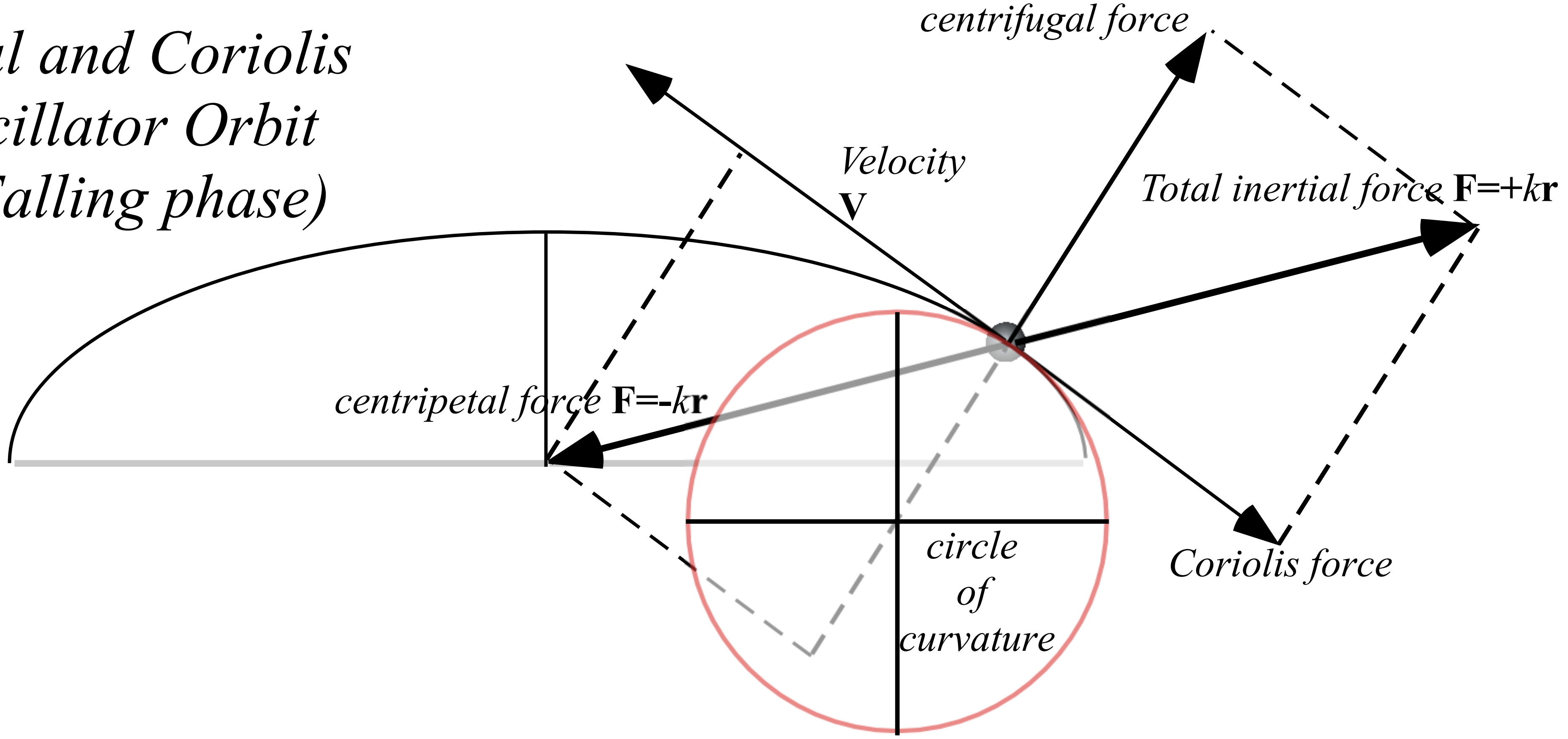
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)



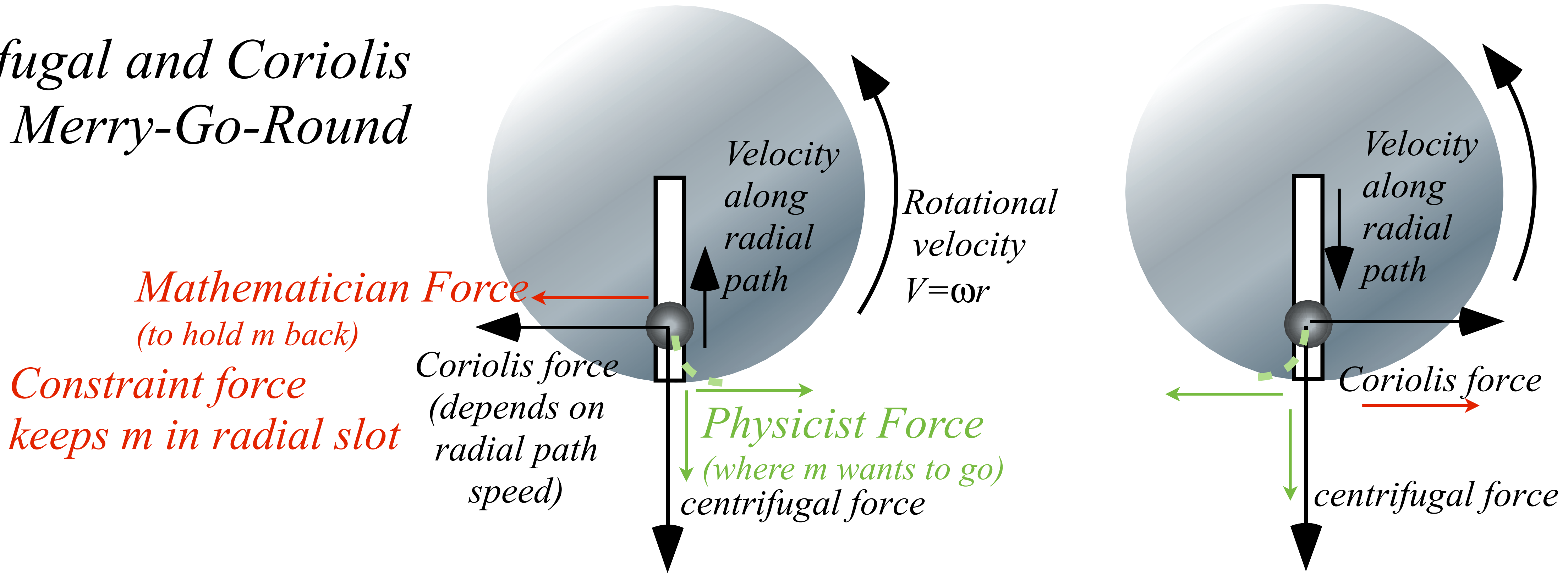
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



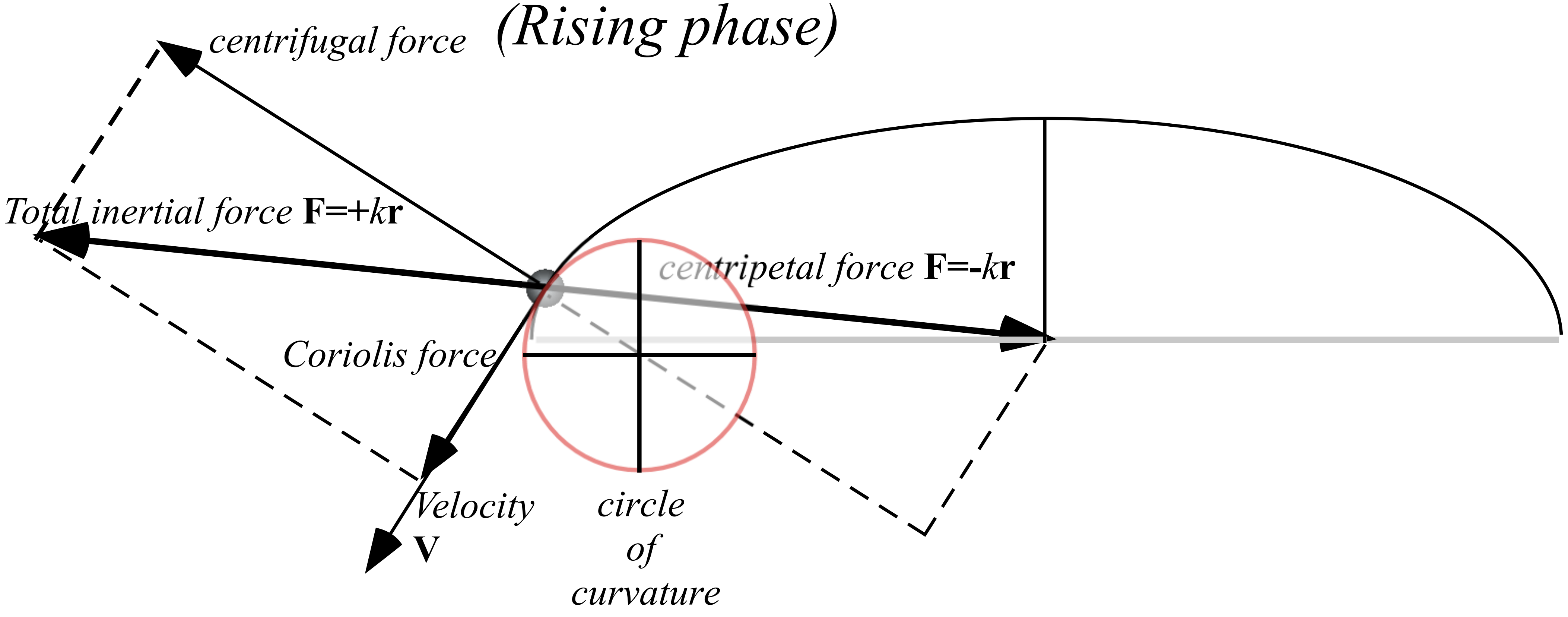
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)

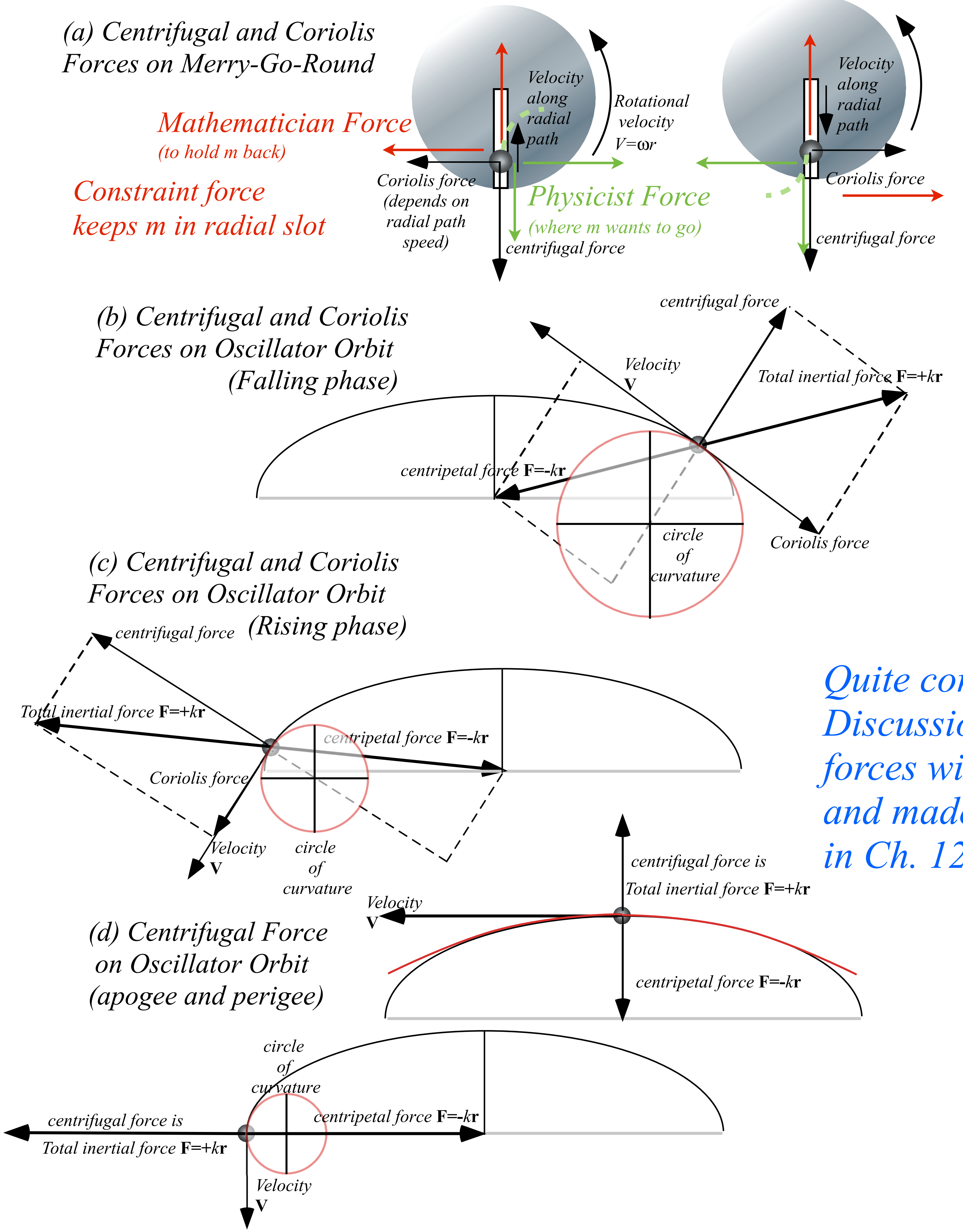


(a) Centrifugal and Coriolis Forces on Merry-Go-Round



(c) Centrifugal and Coriolis Forces on Oscillator Orbit (Rising phase)





Unit 1
Fig. 11.4
a-d

Quite confusing?
Discussion of Coriolis forces will be done more elegantly and made more physically intuitive in Ch. 12 of Unit1 and in Unit 6.

Some Kepler's "laws" for all central (isotropic) force F(r) fields

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*

Total energy $E=KE+PE$ invariance of IHO: $F(r)=-k\cdot r$ *(Derived here)*

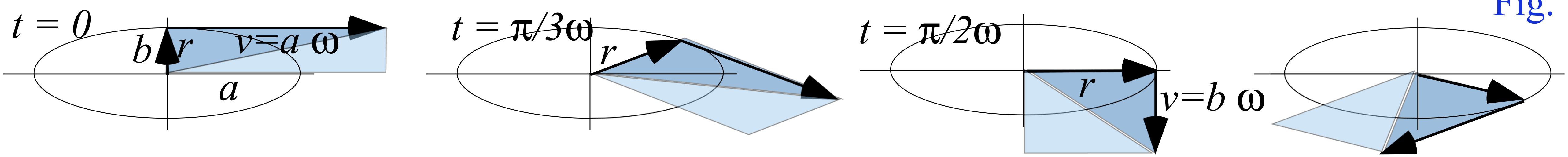
Total energy $E=KE+PE$ invariance of Coulomb: $F(r)=-GMm/r^2$ *(Derived in Unit 5)*

Some Kepler's "laws" for central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot\left(b\omega\cos\omega t\right) - b\sin\omega t\cdot\left(-a\omega\sin\omega t\right) = ab\cdot\omega\left(\cos^2\omega t+\sin^2\omega t\right)$$

✓ *for IHO*

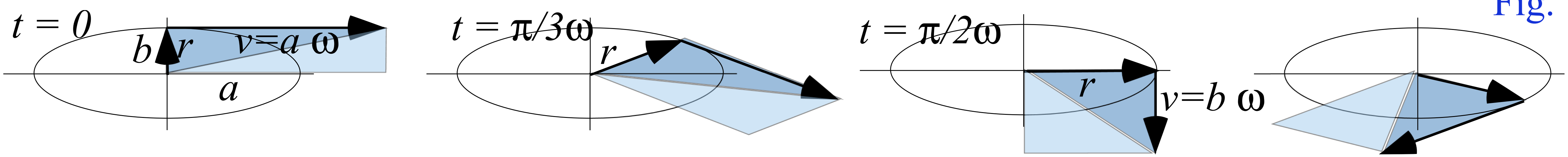
$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} \qquad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



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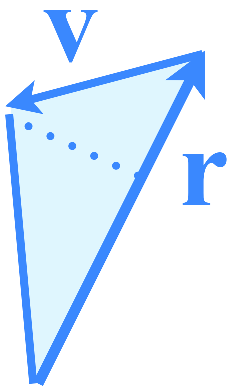
✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

✓ *for IHO*

$|\mathbf{r}\times\mathbf{v}| = r\cdot v\cdot \sin\measuredangle_r^v$



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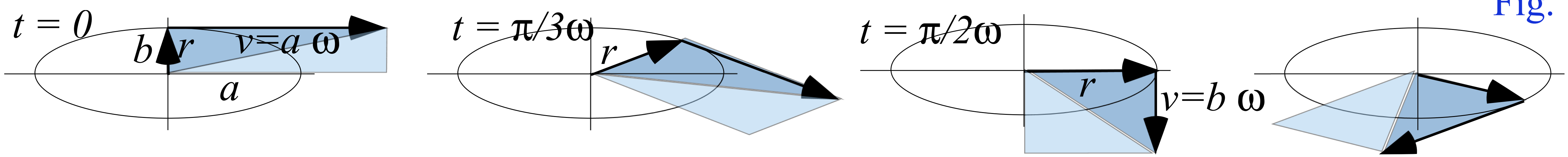
Unit 1
Fig. 11.8



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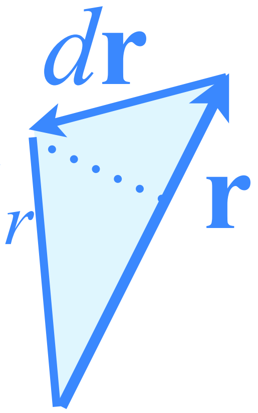
✓ *for IHO*

3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{\underset{\boxed{\text{by 2.}}}{2m}}\int_0^T dt = \frac{L}{2m}T$$

✓ *for IHO*

$$|\mathbf{r}\times d\mathbf{r}| = r\cdot dr\cdot \sin\measuredangle_{r}^{dr}$$

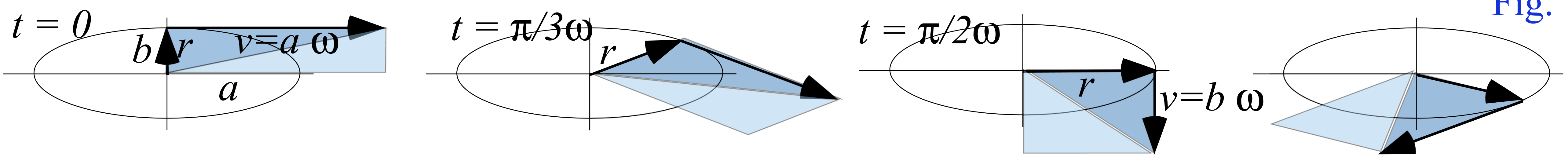


Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2

(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



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✓ *for IHO*

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega = m\cdot ab\cdot\frac{2\pi}{\tau}$$

✓ *for IHO*

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✓ *for IHO*

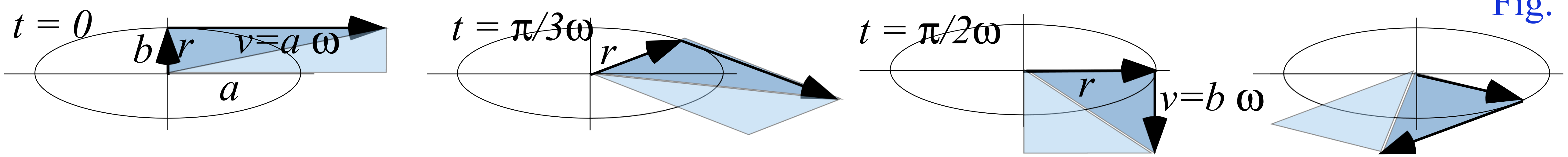
In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit$)$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



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In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit)

(Recall from Lecture 6: $\omega = \sqrt{k/m} = \sqrt{G\rho_{\oplus}4\pi/3}$)

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" for all central (isotropic) force $F(r)$ fields

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived here)

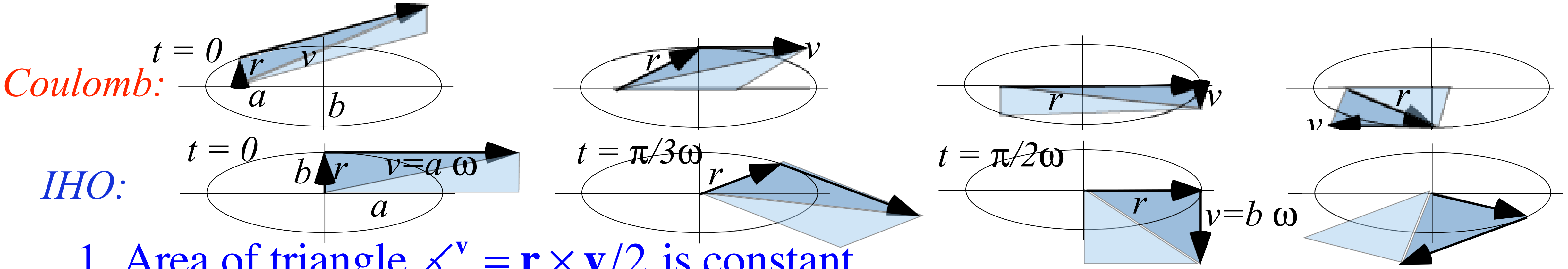
→ Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived in Unit 5)

Total energy $E=KE+PE$ invariance of IHO: $F(r)=-k\cdot r$ (Derived here)

Total energy $E=KE+PE$ invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived in Unit 5)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = \begin{cases} ab\cdot\sqrt{G\rho_{\oplus}4\pi/3} & \text{for IHO} \\ a^{-1/2}b\sqrt{GM_{\oplus}} & \text{for Coul.} \end{cases}$$

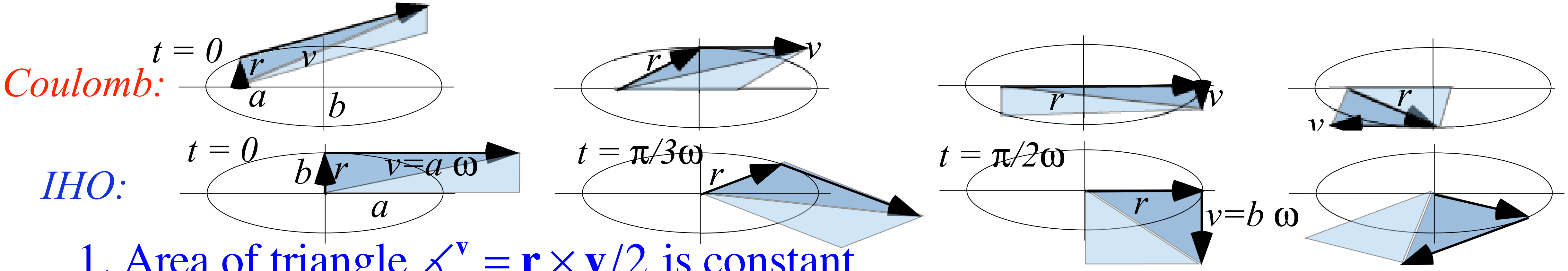
✓ for IHO

(Derived in Unit 5) ✓ for Coul.

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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✓ *for IHO*

(Derived in Unit 5) ✓ *for Coul.*

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(... in Unit 5)

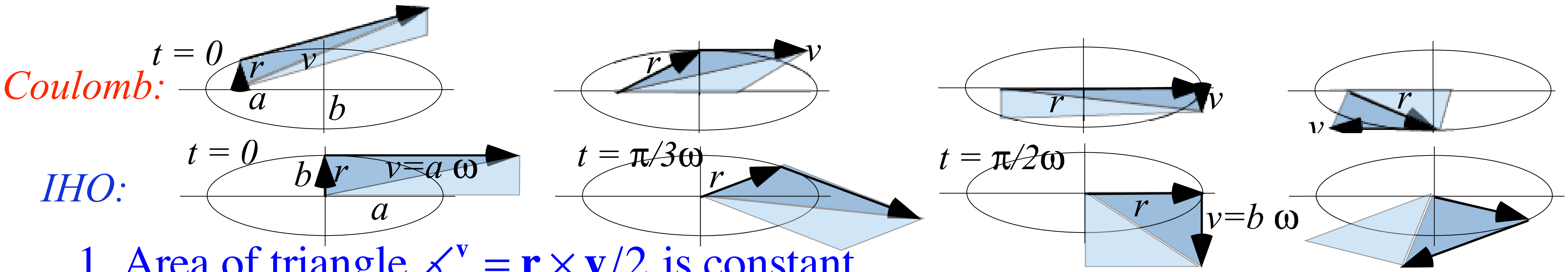
✓ *for IHO*

✓ *for Coul.*

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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✓ for IHO

(Derived in Unit 5) ✓ for Coul.

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

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(... in Unit 5)

✓ for IHO

✓ for Coul.

3. Equal area is swept by radius vector in each equal time interval T

In one period:

$$\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L} = \frac{2m\cdot ab\cdot\pi}{L} = \begin{cases} \dfrac{2m\cdot ab\cdot\pi}{m\cdot ab\cdot\sqrt{G\rho_\oplus 4\pi/3}} \\ \dfrac{2m\cdot ab\cdot\pi}{m\cdot a^{-1/2}b\sqrt{GM_\oplus}} \end{cases}$$

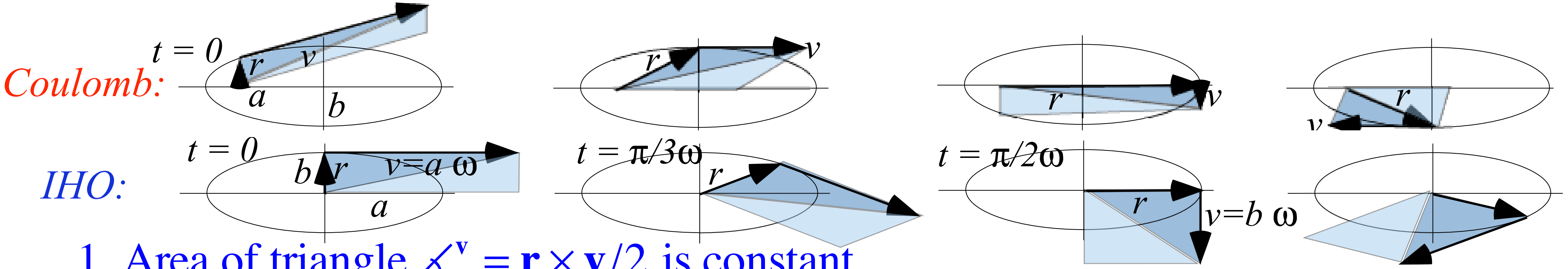
Applies to any central F(r)

Applies to IHO and Coulomb

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



1. Area of triangle $\measuredangle^{\mathbf{v}}_{\mathbf{r}} = \mathbf{r}\times\mathbf{v}/2$ is constant

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✓ for IHO

(Derived in Unit 5) ✓ for Coul.

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✓ for IHO

(... in Unit 5) ✓ for Coul.

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(not a function of *a* or *b*)

that is ω_{IHO}

that is ω_{Coul}

(not a function of *b*)

Applies to any central F(r)

Applies to IHO and Coulomb

Some Kepler's "laws" for all central (isotropic) force F(r) fields

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→ *Total energy* $E=KE+PE$ *invariance of* IHO: $F(r)=-k\cdot r$ *(Derived here)*

Total energy $E=KE+PE$ *invariance of* Coulomb: $F(r)=-GMm/r^2$ *(Derived in Unit 5)*

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}=\begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

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$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$$Given:k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_{\oplus}4\pi/3}$ or: $m\omega^2=k$

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" for <u>all</u> central (isotropic) force $F(r)$ fields

- Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived here)
- Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived in Unit 5)
- Total energy $E=KE+PE$ invariance of IHO: $F(r)=-k\cdot r$ (Derived here)
- → Total energy $E=KE+PE$ invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived in Unit 5)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE = \frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$= \frac{1}{2}m\omega^2(a^2+b^2) \qquad Given: k = m\omega^2$$

$E = KE + PE = \frac{1}{2}m\omega^2(a^2+b^2) = \frac{1}{2}k(a^2+b^2)$ since: $\omega = \sqrt{\frac{k}{m}} = \sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2 = k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

(G IHO formulas from Lect. 6 p.70-79)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=*KE* + *PE* is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y\end{pmatrix}\bullet\begin{pmatrix} m & 0\\ 0 & m\end{pmatrix}\bullet\begin{pmatrix} v_x\\ v_y\end{pmatrix}+\begin{pmatrix} r_x & r_y\end{pmatrix}\bullet\begin{pmatrix} k & 0\\ 0 & k\end{pmatrix}\bullet\begin{pmatrix} r_x\\ r_y\end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)\qquad Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)\quad \text{since: }\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_{\oplus}4\pi/3}\quad \text{or: }m\omega^2=k$$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

$$E=KE+PE=\frac{1}{2}mv^2{}_x+\frac{1}{2}mv^2{}_y-\frac{k}{r}=\frac{1}{2}mv^2{}_x+\frac{1}{2}mv^2{}_y-\frac{GM_{\oplus}m}{r}=-\frac{GM_{\oplus}m}{a}$$

Introduction to dual matrix operator contact geometry (based on IHO orbits)

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations (* $\mathbf{r}\bullet\mathbf{p}=1$*)*
  - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r***=1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\overbrace{\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} \qquad = 1} = 1 = \overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}} \bullet \overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

A inverse matrix Q^{-1} generates an ellipse by **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

$$\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} \qquad = 1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}} = a^2 p_x^2 + b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *is called* positive-definite (*if* $\mathbf{r}\bullet Q\bullet\mathbf{r}$ *always* >0)

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\overbrace{\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}}\bullet\begin{pmatrix} x \\ y \end{pmatrix}\overset{=1}{=1}=\overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}}\bullet\overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}=\mathbf{p}}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Defined mapping between ellipses

A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\overbrace{\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix}\overset{=1}{=1}=\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}}\bullet\overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}=\mathbf{r}}=a^2 p_x^2+b^2 p_y^2$$

Introduction to dual matrix operator contact geometry (based on IHO orbits)

→ *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

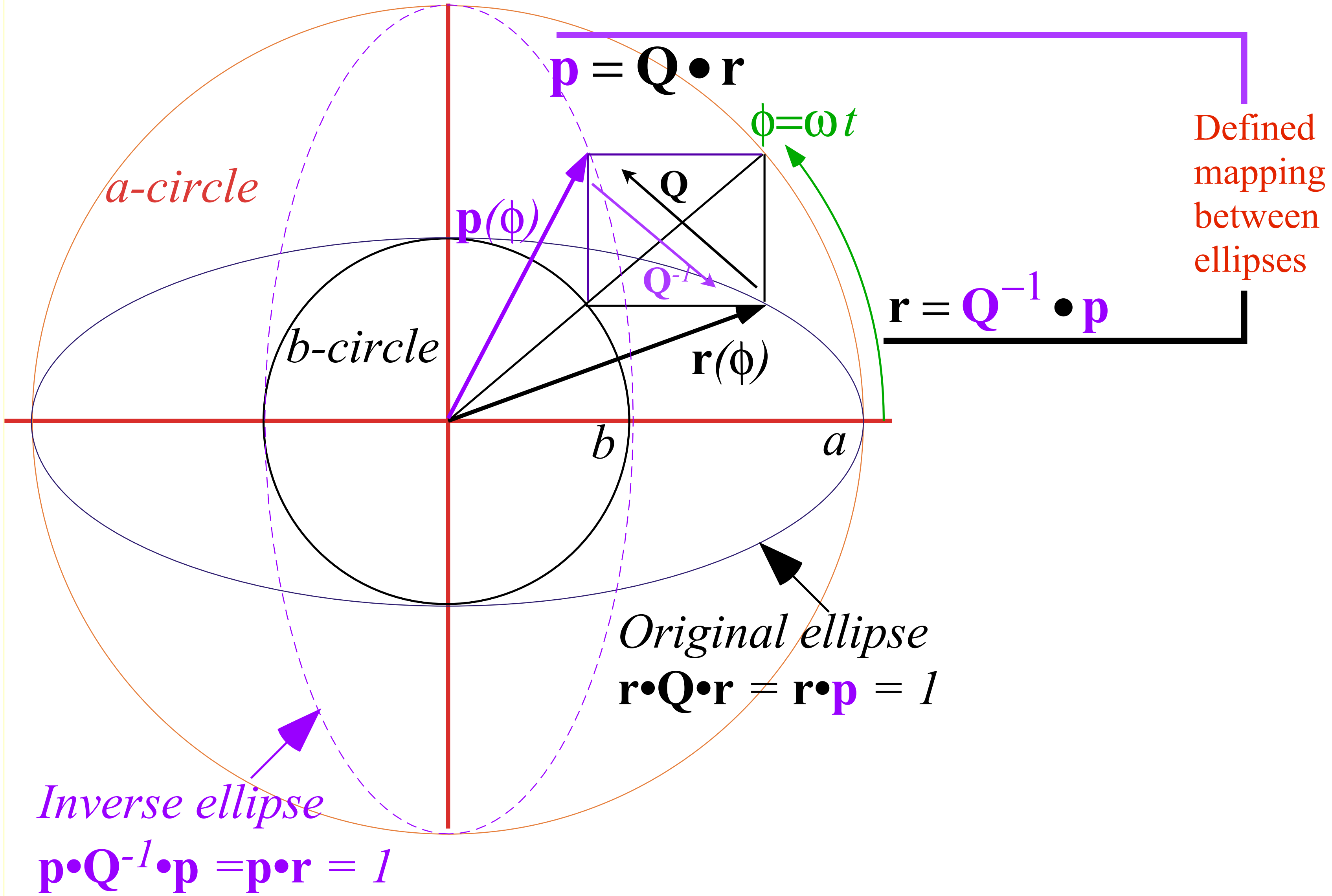
Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

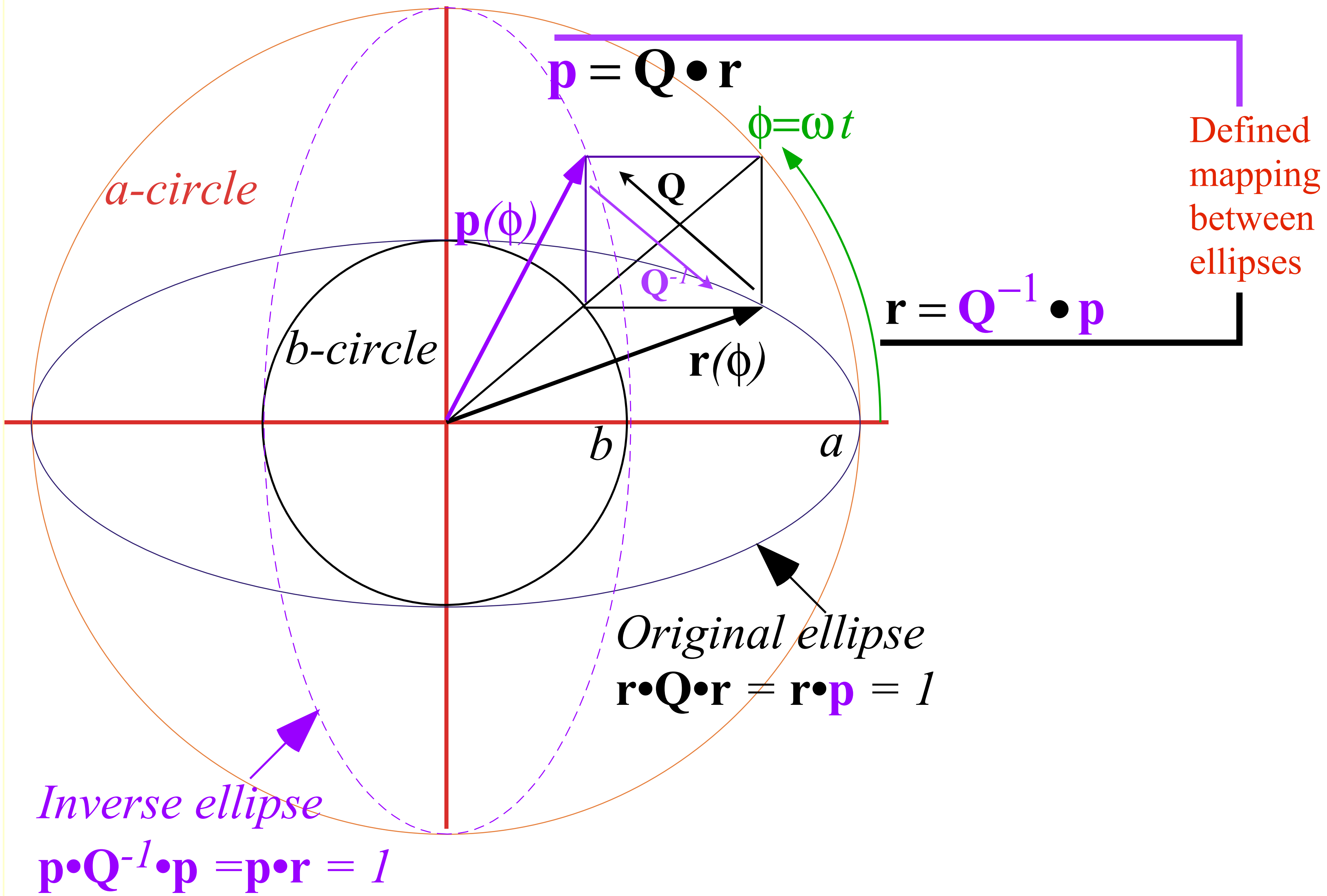
(a) Quadratic form ellipse and Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Here plot of **p**-ellipse is re-scaled by scalefactor $S=a \cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

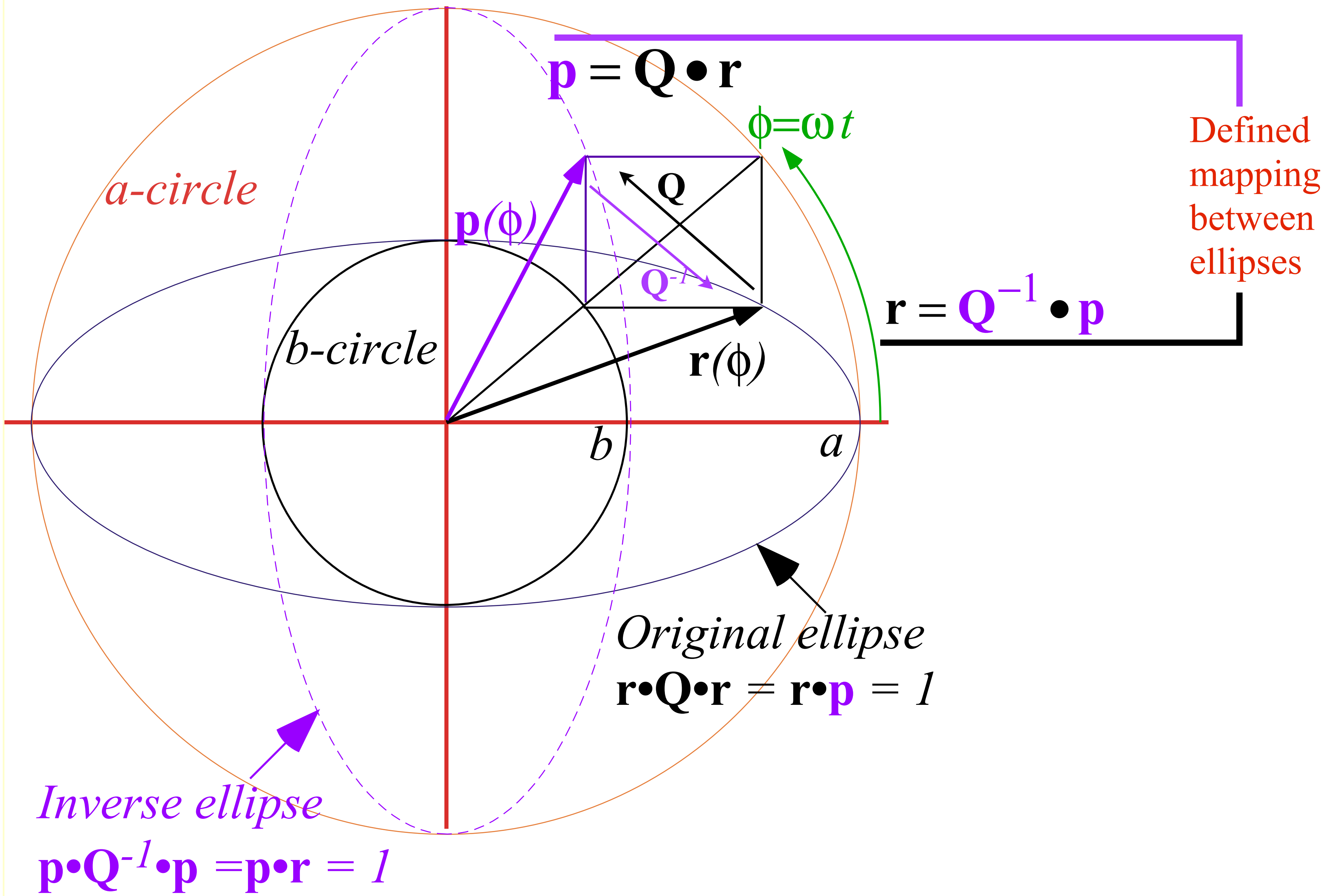
p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Introduction to dual matrix operator geometry (based on IHO orbits)

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
    - → *Duality norm relations* $(\mathbf{r}\bullet\mathbf{p}=1)$
    - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ $(\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}')$
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
    - *Vector calculus of tensor operation*

(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

based on
Unit 1
Fig. 11.6



Quadratic form $\mathbf{r} \bullet \mathbf{Q} \bullet \mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p} \bullet \mathbf{Q}^{-1} \bullet \mathbf{p} = 1 = \mathbf{p} \bullet \mathbf{r}$

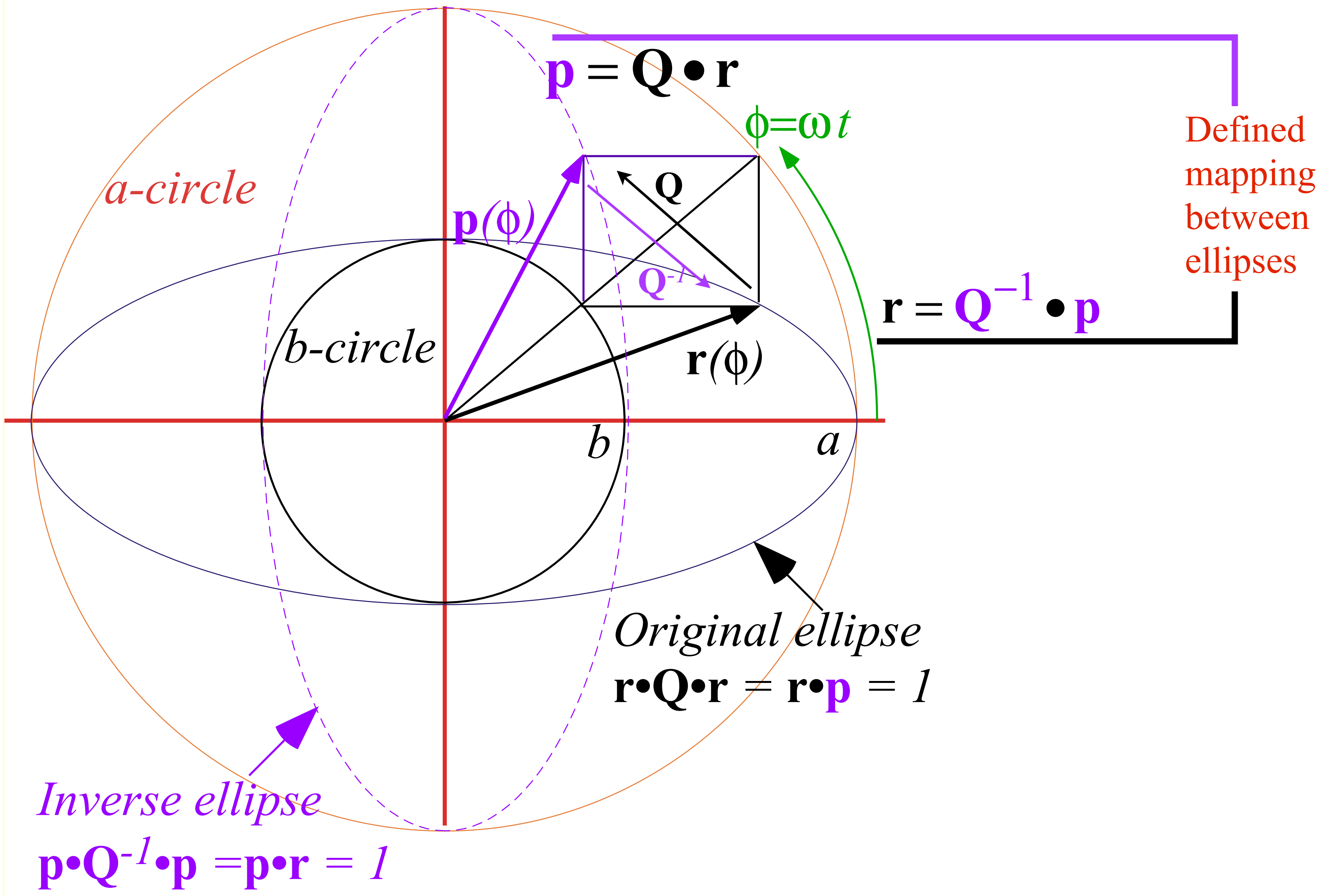
Here plot of **p**-ellipse is re-scaled by scalefactor $S = a \cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a) = b$ (=1 for $a=2$, $b=1$)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b) = a$ (=2 for $a=2$, $b=1$)

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p} = \mathbf{Q} \bullet \mathbf{r} = \begin{pmatrix} \overbrace{1/a^2 \quad\quad 0}^{\mathbf{Q}} \\ 0 \quad\quad 1/b^2 \end{pmatrix} \bullet \begin{pmatrix} \overbrace{x}^{\mathbf{r}} \\ y \end{pmatrix} = \begin{pmatrix} \overbrace{x/a^2}^{\mathbf{p}} \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where:} \quad \begin{matrix} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi \;\; = b\sin\omega t \end{matrix} \quad \text{so:} \quad \boxed{\mathbf{p}\bullet\mathbf{r} = 1}$$

Here plot of **p**-ellipse is re-scaled by scalefactor $S=a \cdot b$

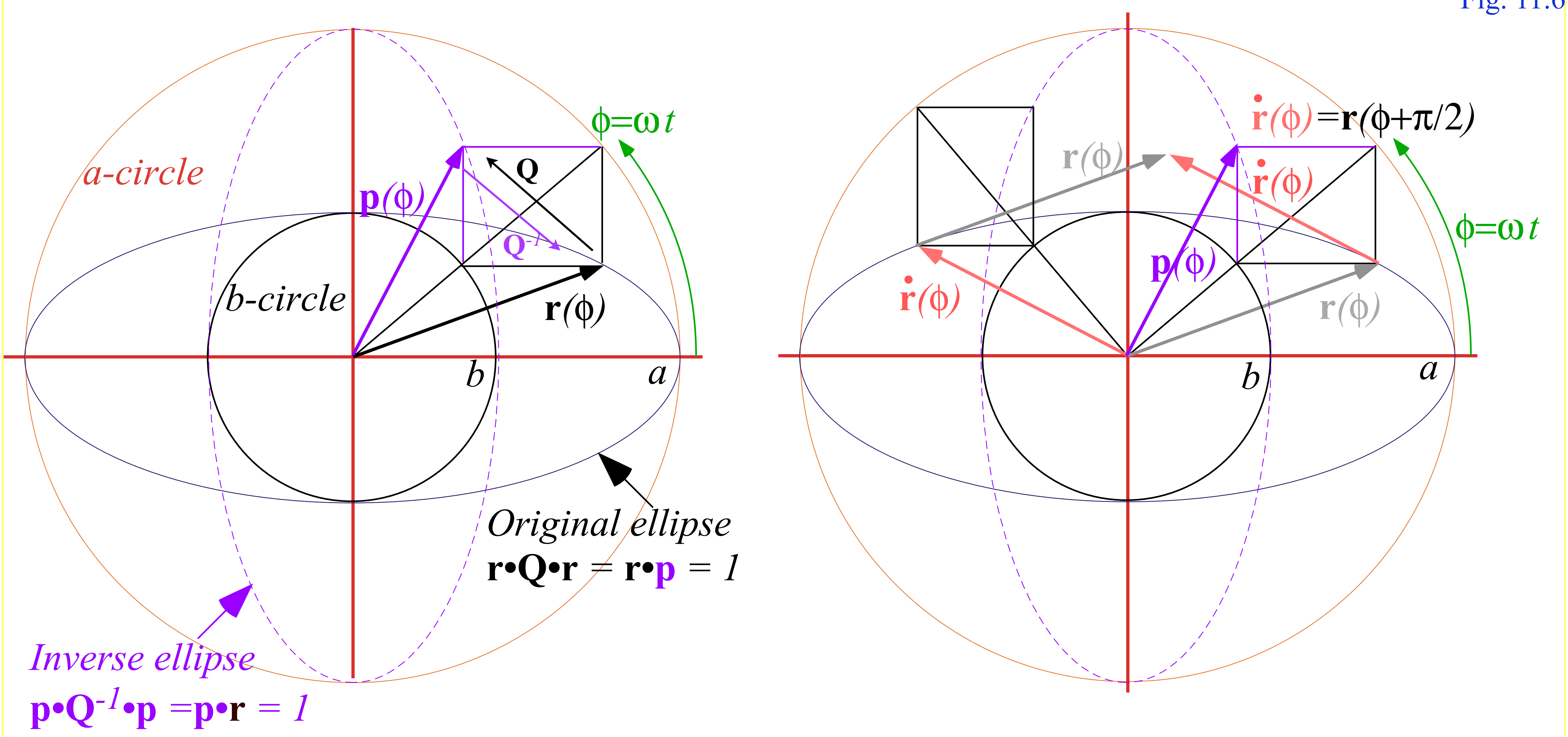
p-ellipse x-radius=1/a plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

- *Introduction to dual matrix operator geometry (based on IHO orbits)*
  - *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
    - *Duality norm relations (* $\mathbf{r}\bullet\mathbf{p}=1$*)*
    - → *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*
  - *Operator geometric sequences and eigenvectors*
  - *Alternative scaling of matrix operator geometry*
    - *Vector calculus of tensor operation*

based on
Unit 1
Fig. 11.6


(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

(b) Ellipse tangents

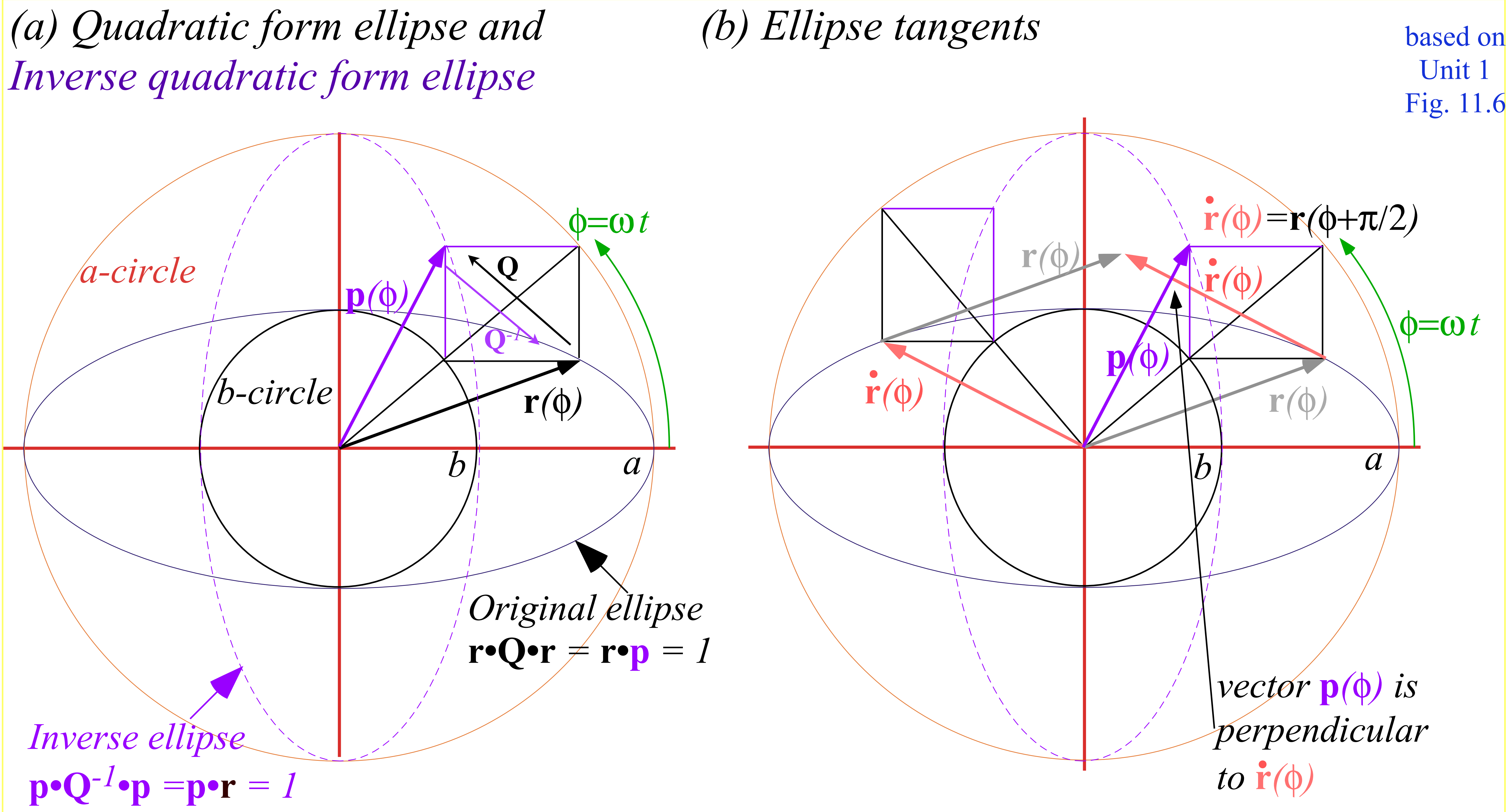
Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}=\begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{matrix} x=r_x=a\cos\phi=a\cos\omega t \\ y=r_y=b\sin\phi \;\; =b\sin\omega t \end{matrix} \quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)



(a) *Quadratic form ellipse and Inverse quadratic form ellipse*

(b) *Ellipse tangents*

based on Unit 1 Fig. 11.6

Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}=1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}=1=\mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $x=r_x=a\cos\phi=a\cos\omega t$

$y=r_y=b\sin\phi \quad =b\sin\omega t$

so: $\boxed{\mathbf{p}\bullet\mathbf{r}=1}$

$\mathbf{p}$ *is perpendicular to velocity* $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

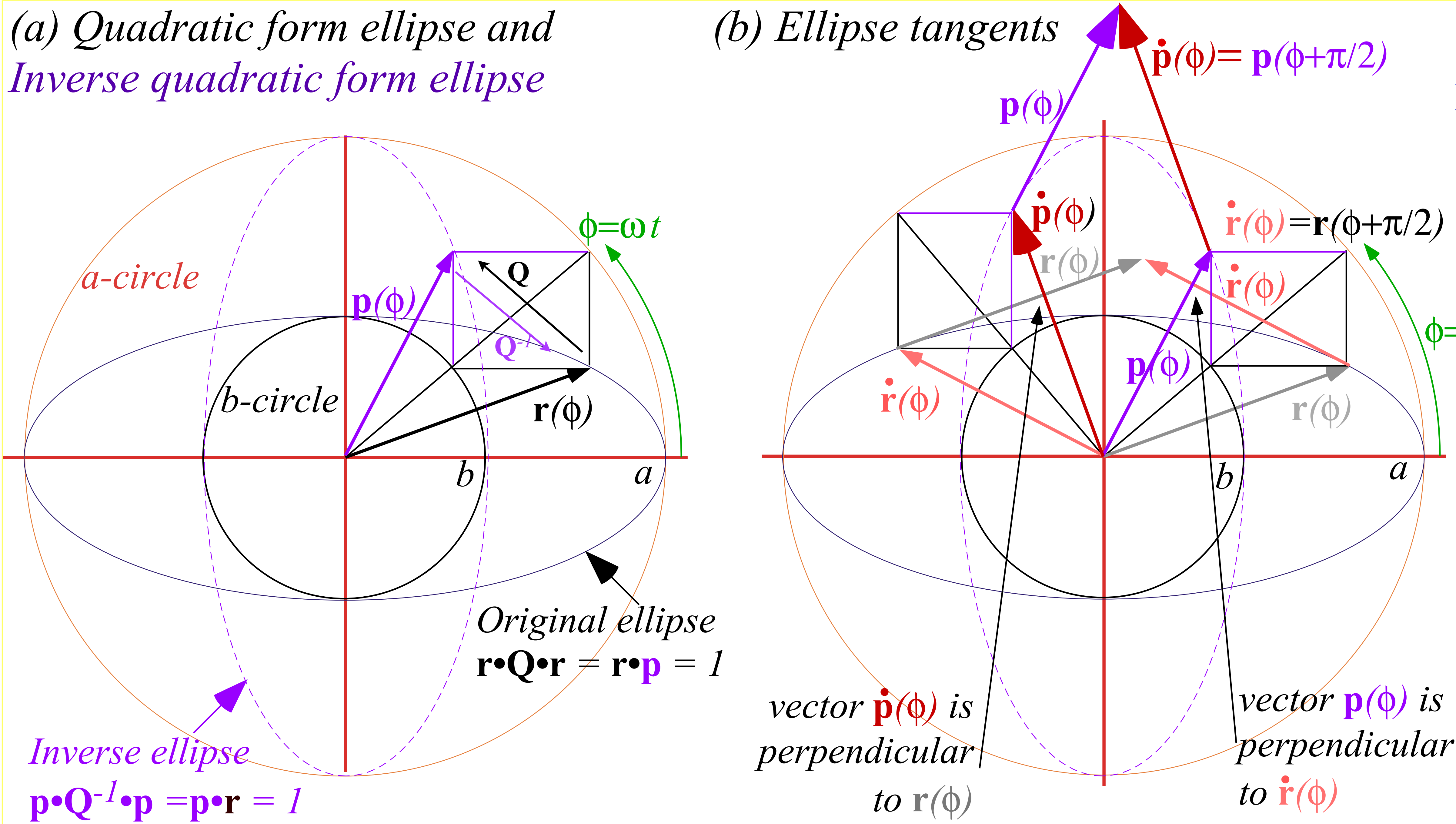
where: $\dot{r}_x=-a\sin\phi$

$\dot{r}_y=b\cos\phi$

and: $p_x=(1/a)\cos\phi$

$p_y=(1/b)\sin\phi$

Unit 1
Fig. 11.6



Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q⁻¹•p** *=1*

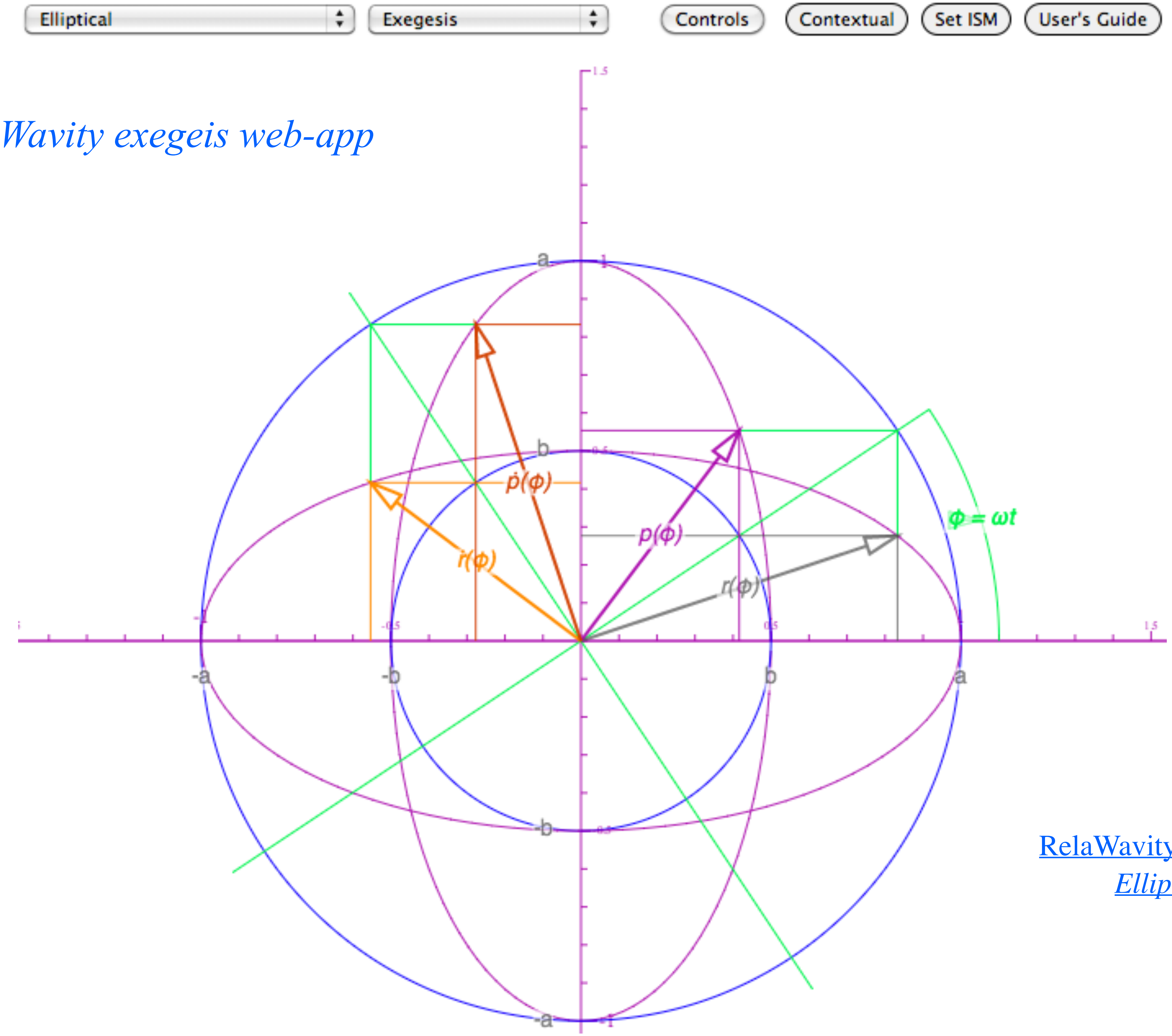
unit mutual projection

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad\text{where: }\begin{array}{l}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ \ =b\sin\omega t\end{array}\quad\text{so: }\boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

p *is perpendicular to velocity* **v**=$\dot{\mathbf{r}}$ *, a mutual orthogonality. So is* **r** *perpendicular to* $\dot{\mathbf{p}}$*:* $\boxed{\dot{\mathbf{p}}\bullet\mathbf{r}=0}$

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad\text{where: }\begin{array}{l}\dot{r}_x=-a\sin\phi\\ \dot{r}_y=b\cos\phi\end{array}\quad\text{and: }\begin{array}{l}p_x=(1/a)\cos\phi\\ p_y=(1/b)\sin\phi\end{array}$$

RelaWavity exegeis web-app



RelaWavity Web Simulation

Ellipse/Exegesis

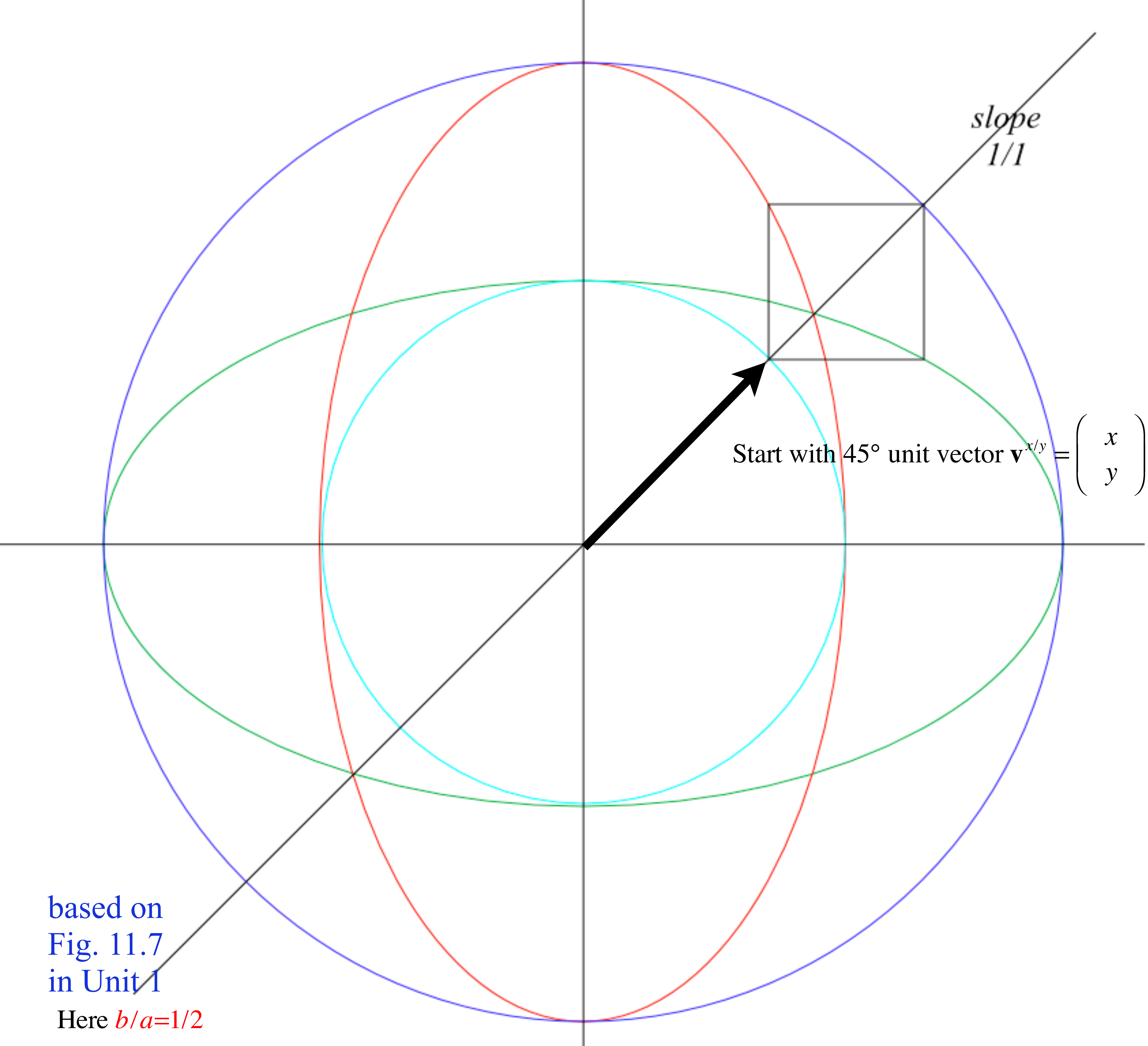
Geometry of dual ellipse Kepler anomalies for vectors $[\mathbf{r}(\phi), \mathbf{p}(\phi)]$ and ${}^{d}/_{dt}[\mathbf{r}(\phi), \mathbf{p}(\phi),]$ *in coordinate (x,y) space rendered by animation web-app in RelaWavity and described in Lect. 12-advanced.*

Introduction to dual matrix operator geometry (based on IHO orbits)

- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
    - *Duality norm relations* ($\mathbf{r}\bullet\mathbf{p}=1$)
    - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

→ *Operator geometric sequences and eigenvectors*

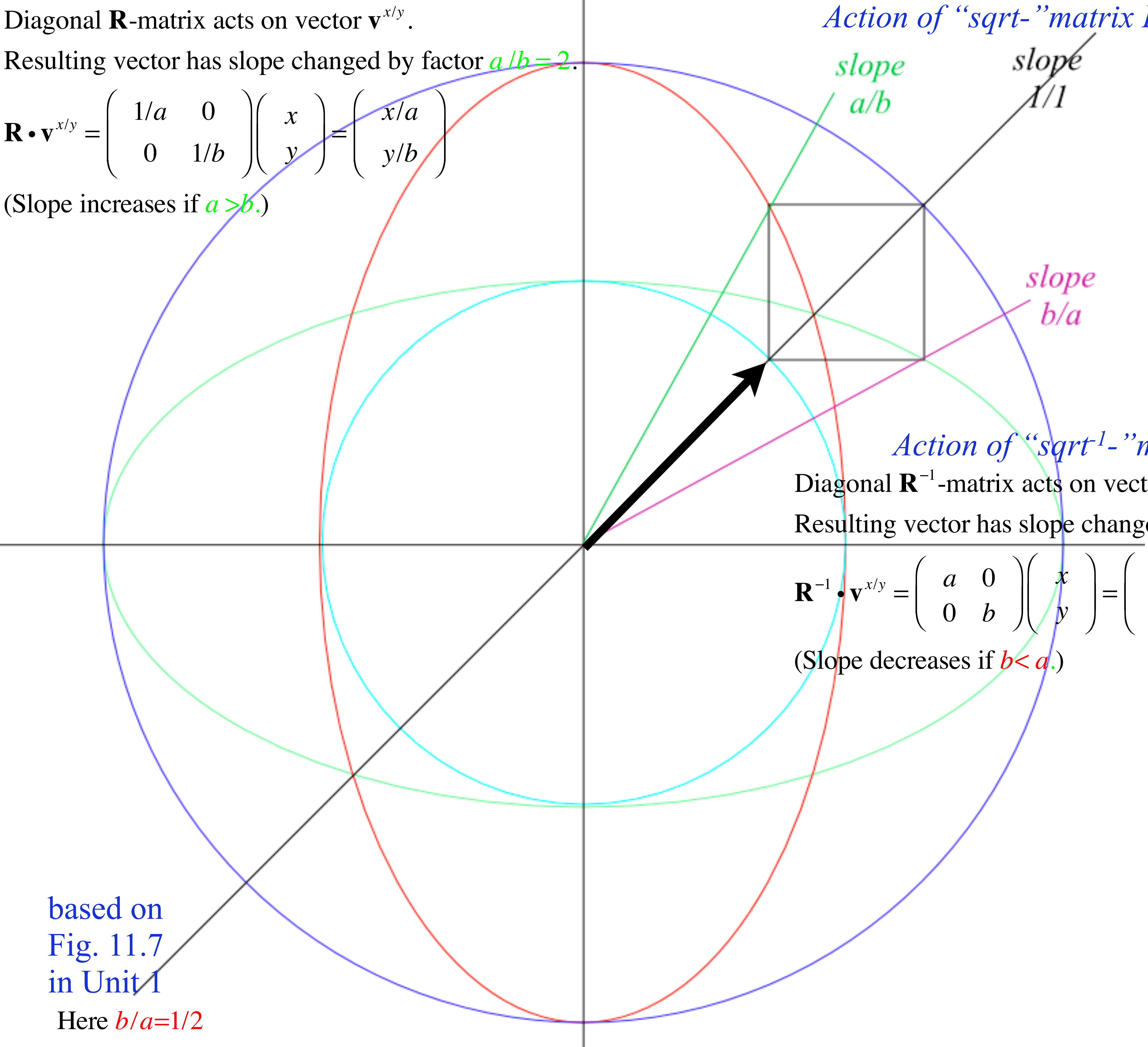
- *Alternative scaling of matrix operator geometry*
    - *Vector calculus of tensor operation*



Start with 45° unit vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

based on
Fig. 11.7
in Unit 1

Here $b/a=1/2$



Action of "sqrt-"matrix $R=\sqrt{Q}$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(Slope increases if $a>b$.)

Action of "sqrt^{-1}-"matrix $R^{-1}=\sqrt{Q}^{-1}$

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b/a.

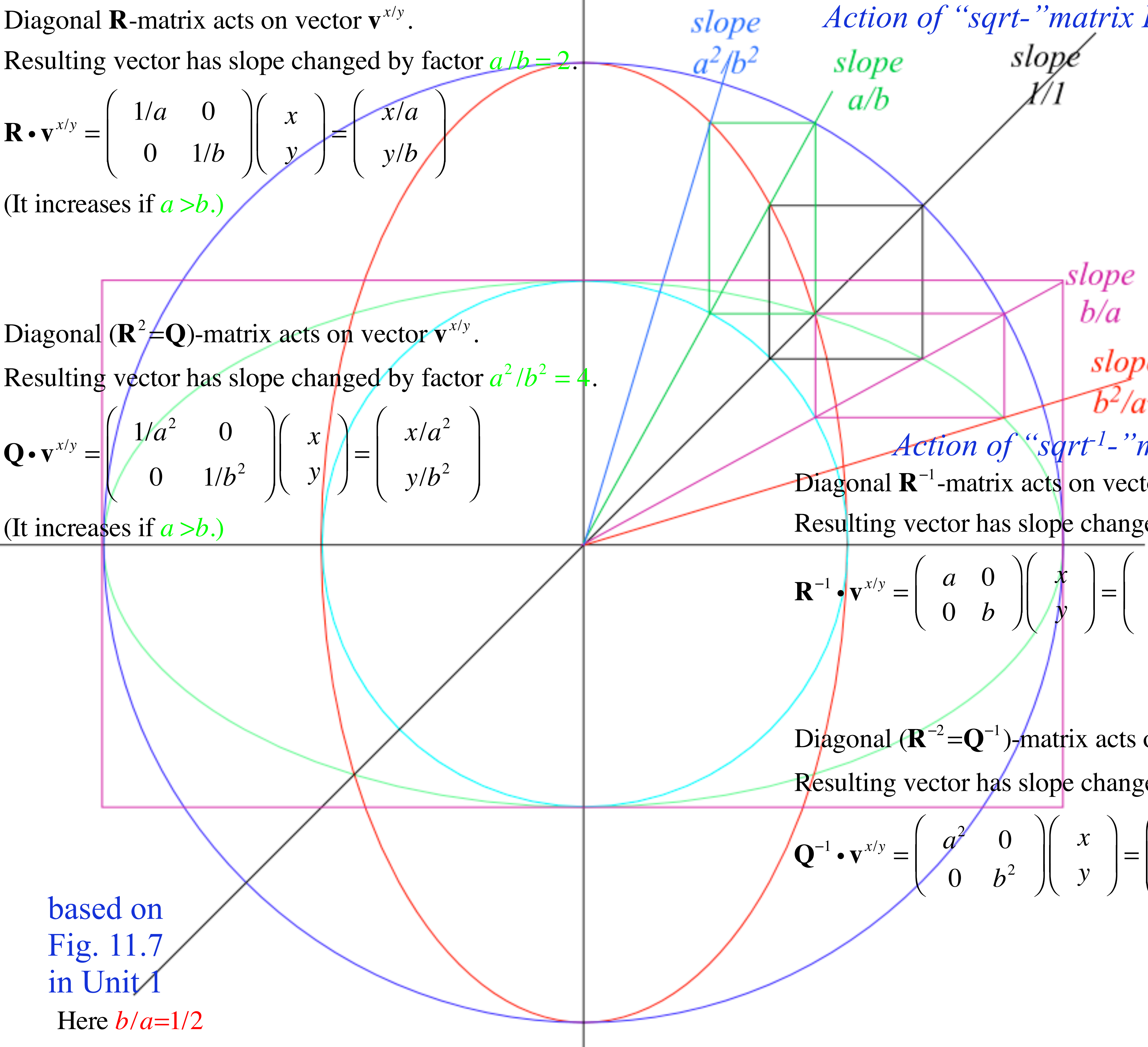
$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a \\ y\cdot b \end{pmatrix}$$

(Slope decreases if $b<a$.)

based on
Fig. 11.7
in Unit 1

Here $b/a=1/2$

Action of "sqrt-"matrix $R=\sqrt{Q}$



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

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$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)

Action of "sqrt^{-1}-"matrix $R^{-1} = \sqrt{Q}^{-1}$

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b/a=1/2.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a \\ y\cdot b \end{pmatrix}$$

Diagonal ($\mathbf{R}^{-2}$=$\mathbf{Q}^{-1}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b^2/a^2=1/4.

$$\mathbf{Q}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a^2 \\ y\cdot b^2 \end{pmatrix}$$

based on
Fig. 11.7
in Unit 1

Here b/a=1/2

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)

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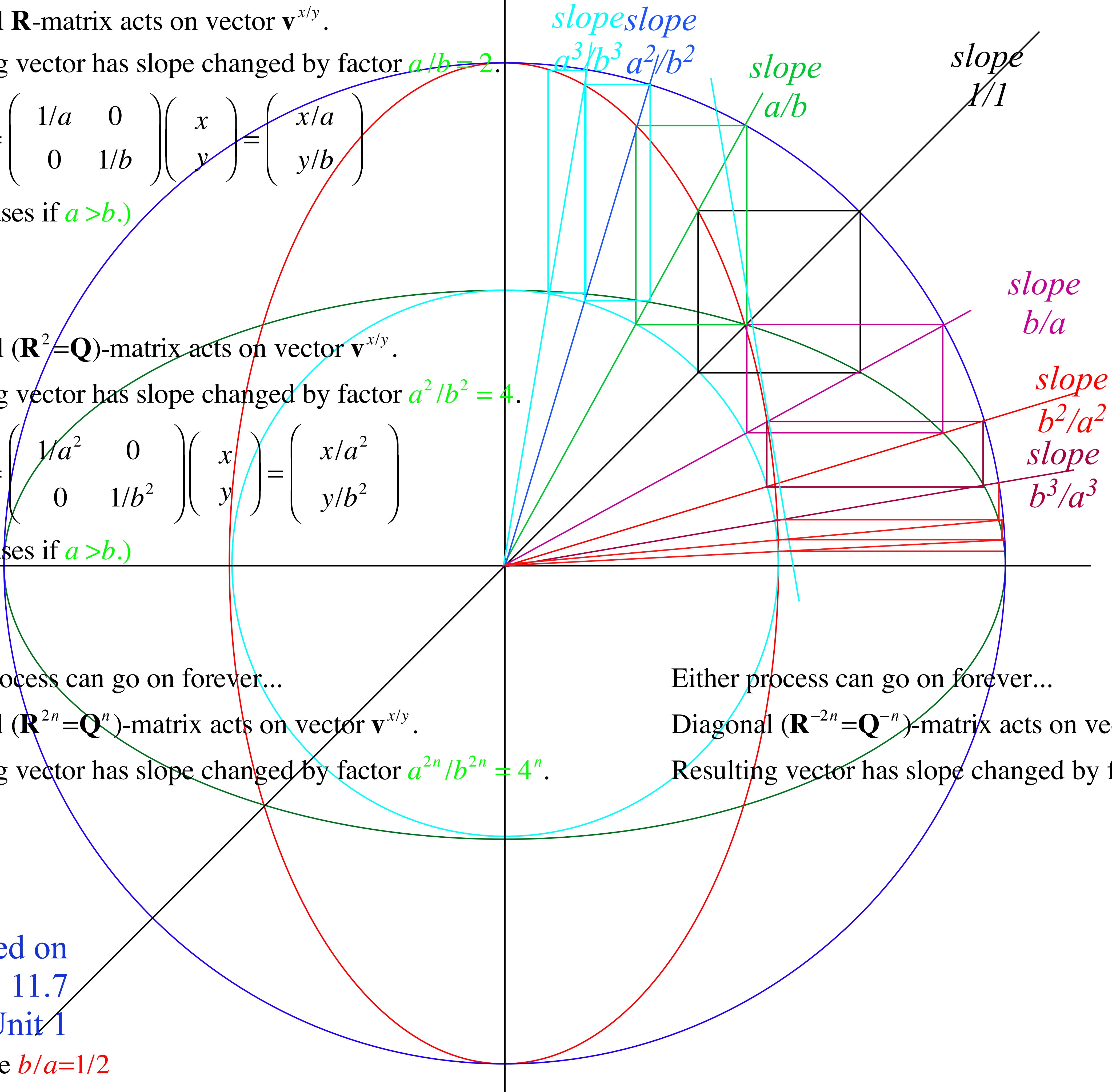
(It increases if $a>b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n}=4^n$.

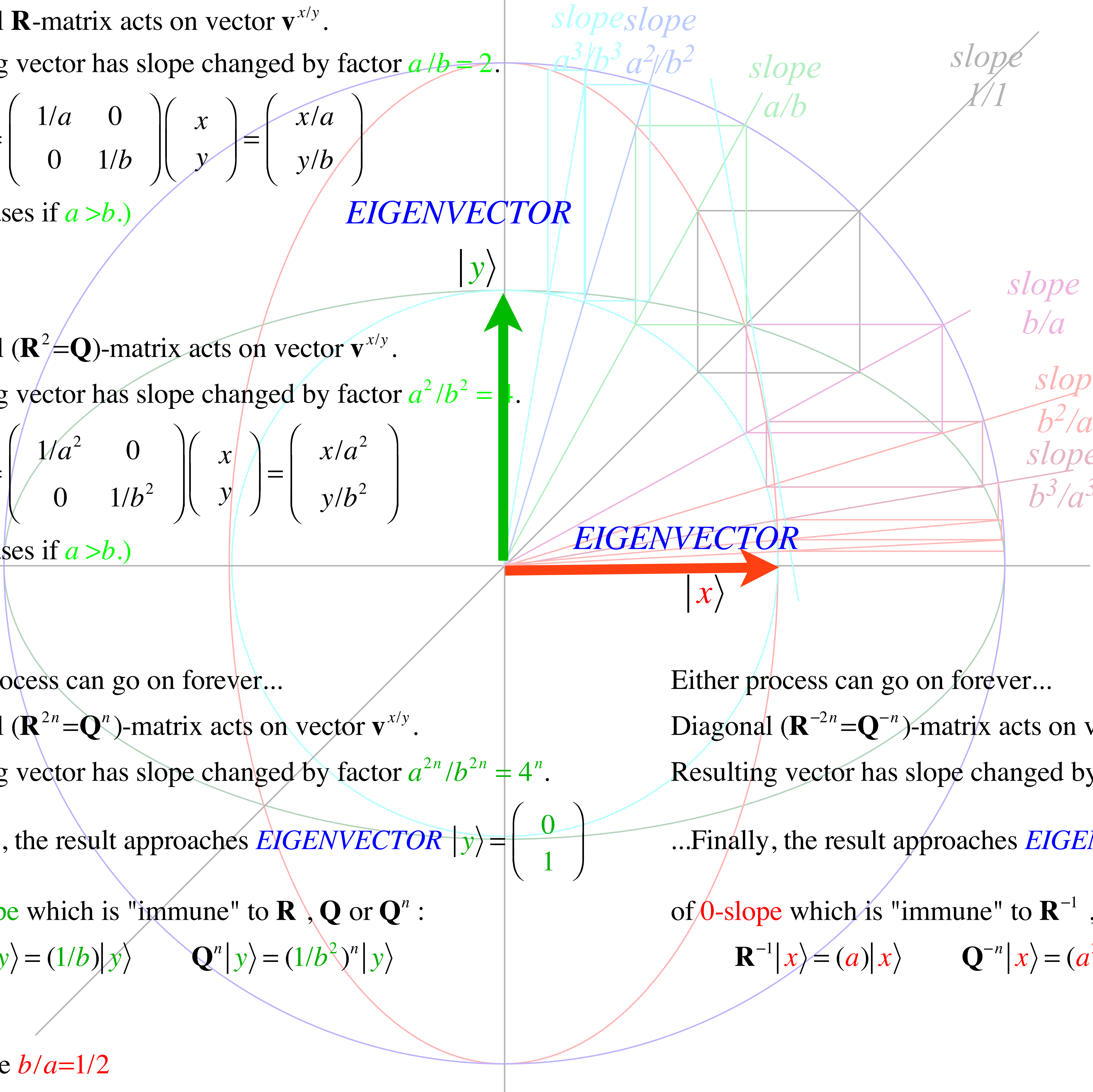
Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n}=4^{-n}$.

based on
Fig. 11.7
in Unit 1

Here b/a=1/2



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

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(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

$\mathbf{R}|y\rangle = (1/b)|y\rangle$ $\mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle$ $\mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$

Here b/a=1/2

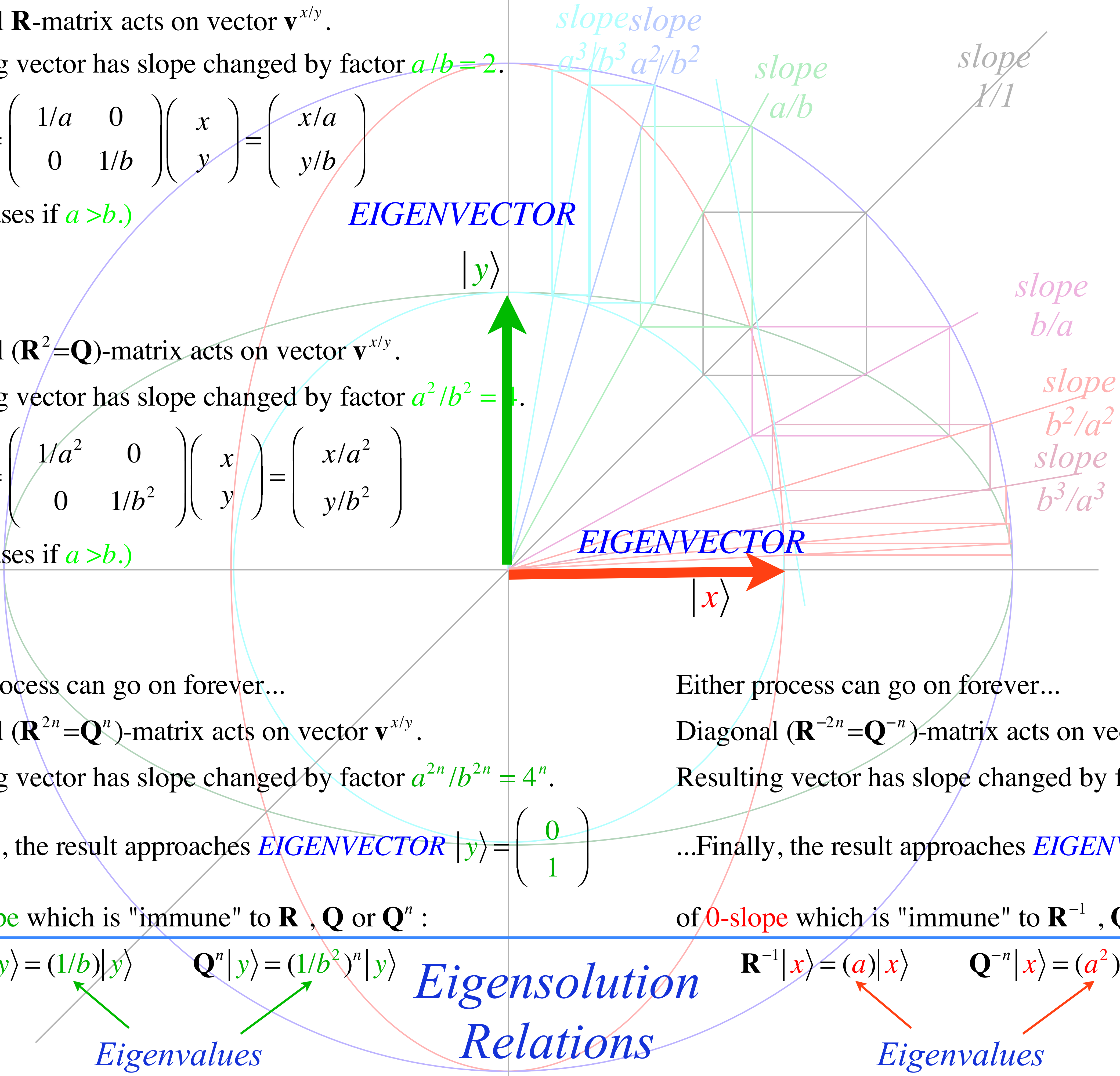
Here $b/a=1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a > b$.)



Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$\mathbf{R}|y\rangle = (1/b)|y\rangle$ $\mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$

Eigenvalues

Eigensolution Relations

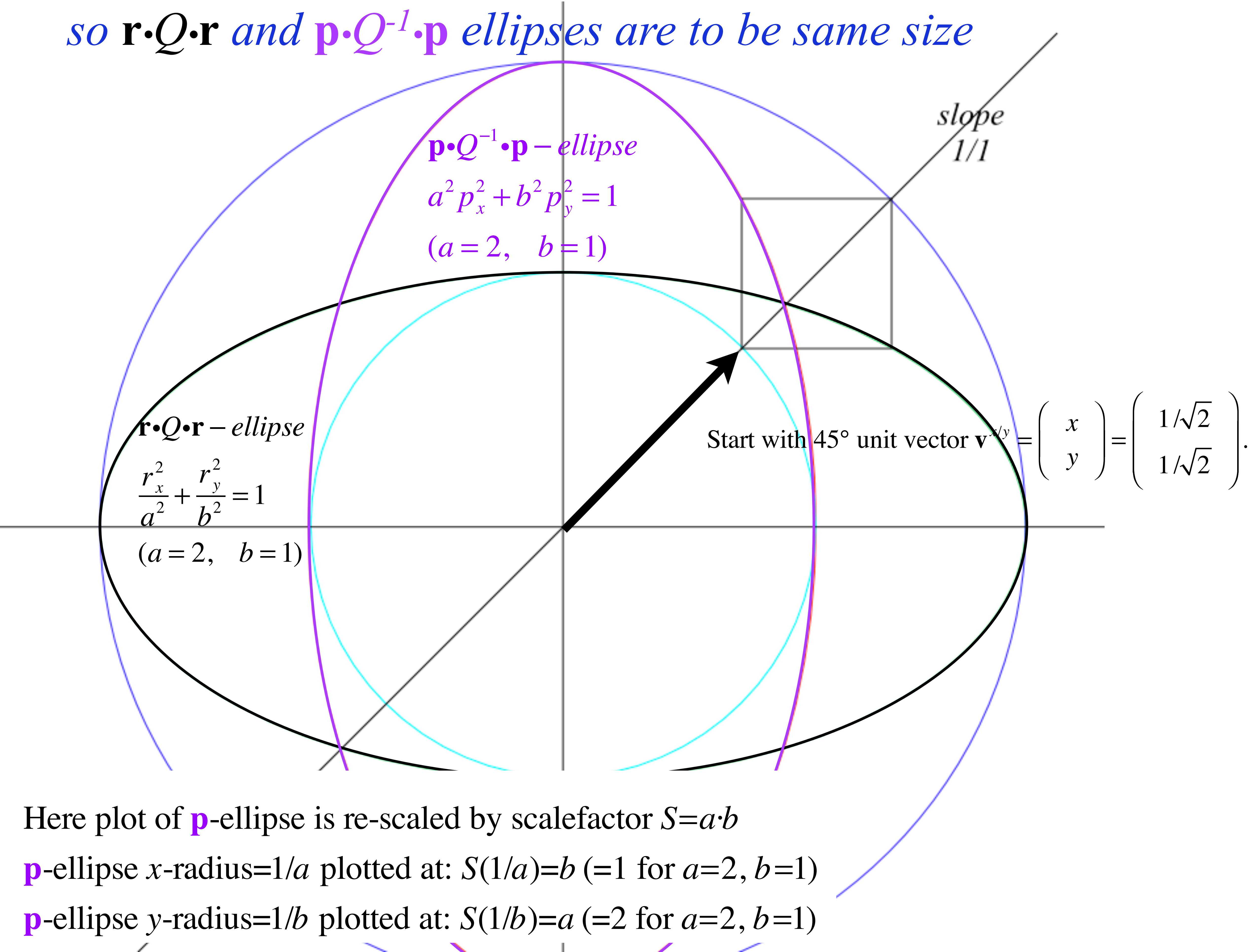
$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle$ $\mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$

Eigenvalues

- *Introduction to dual matrix operator geometry (based on IHO orbits)*
    - *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
        - *Duality norm relations* ($\mathbf{r}\bullet\mathbf{p}=1$)
        - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)
    - *Operator geometric sequences and eigenvectors*
    - → *Alternative scaling of matrix operator geometry*
        - *Vector calculus of tensor operation*

Here $b/a=1/2$

You may rescale **p***-plot by scale factor* $S=(a\cdot b)$ *so* $\mathbf{r}\cdot Q\cdot\mathbf{r}$ *and* $\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ *ellipses are to be same size*



Start with 45° unit vector $\mathbf{v}^{x/y}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

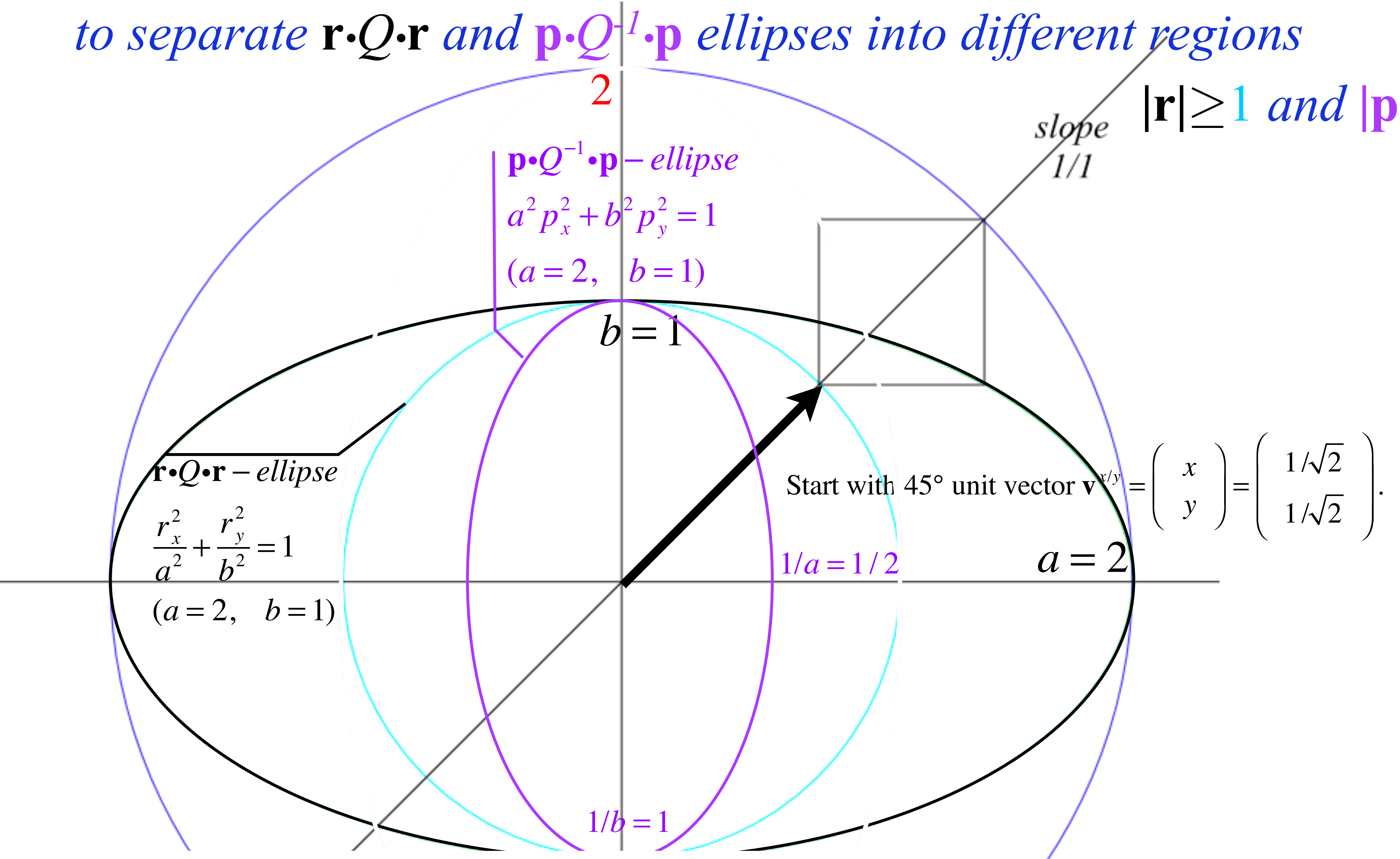
Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for $a=2$, $b=1$)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for $a=2$, $b=1$)

..or else rescale **p**-plot by scale factor *S*=*b* to separate **r**·*Q*·**r** and **p**·Q^{-1}·**p** ellipses into different regions

Here $b/a=1/2$



$\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ – *ellipse*

$$a^2p_x^2+b^2p_y^2=1$$

$(a=2,\quad b=1)$

$\mathbf{r}\cdot Q\cdot\mathbf{r}$ – *ellipse*

$$\frac{r_x^2}{a^2}+\frac{r_y^2}{b^2}=1$$

$(a=2,\quad b=1)$

Start with 45° unit vector $\mathbf{v}^{x/y}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

Here plot of **p**-ellipse is re-scaled by scalefactor *S*=*b*

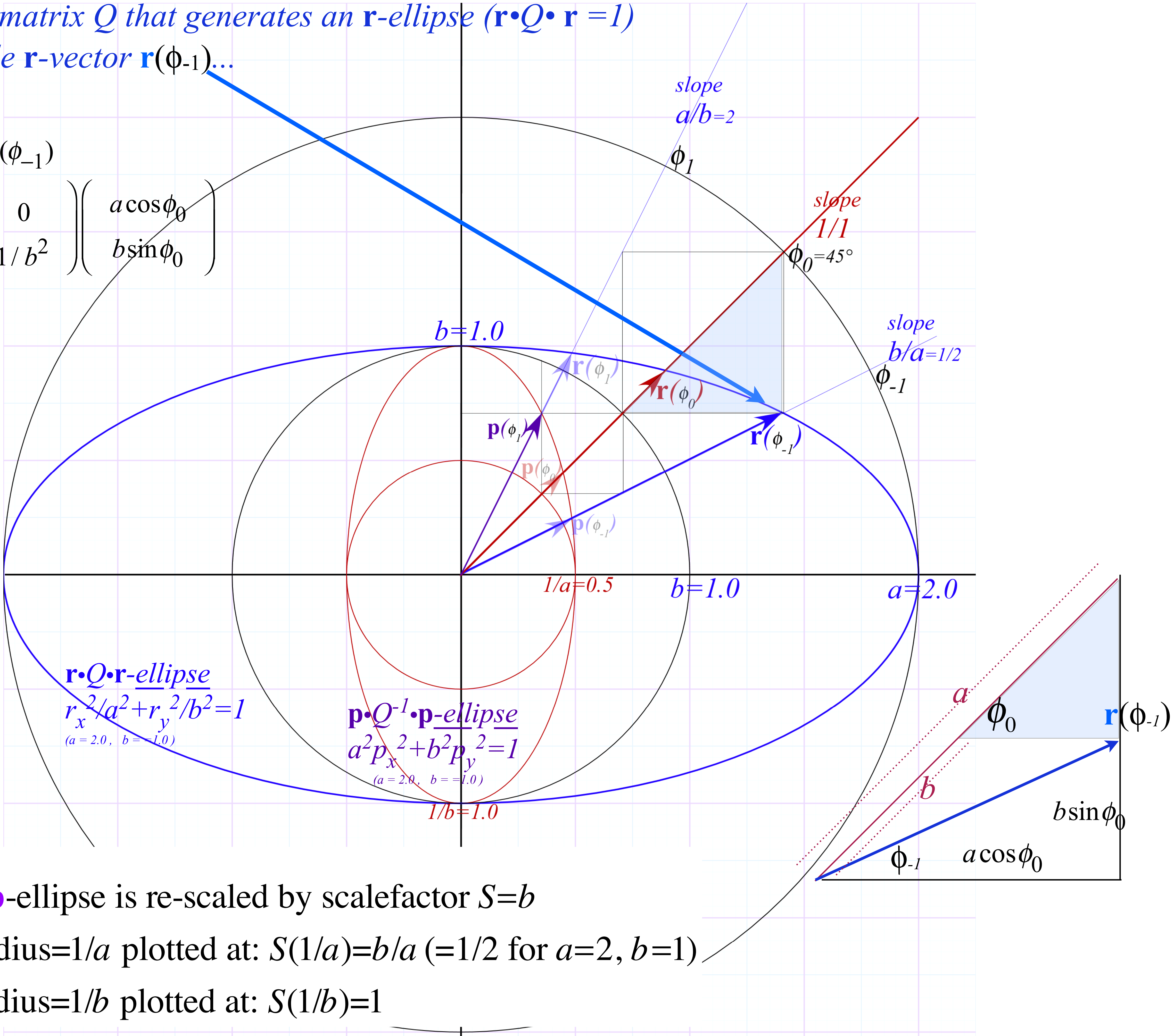
p-ellipse *x*-radius=1/*a* plotted at: *S*(1/*a*)=*b*/*a* (=1/2 for *a*=2, *b*=1)

p-ellipse *y*-radius=1/*b* plotted at: *S*(1/*b*)=1

Action of matrix Q that generates an **r**-*ellipse* (**r**•*Q*• **r** =1) *on a single* **r**-*vector* **r**(ϕ-1)...

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius=1/a plotted at: $S(1/a)=b/a$ (=1/2 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)=1$

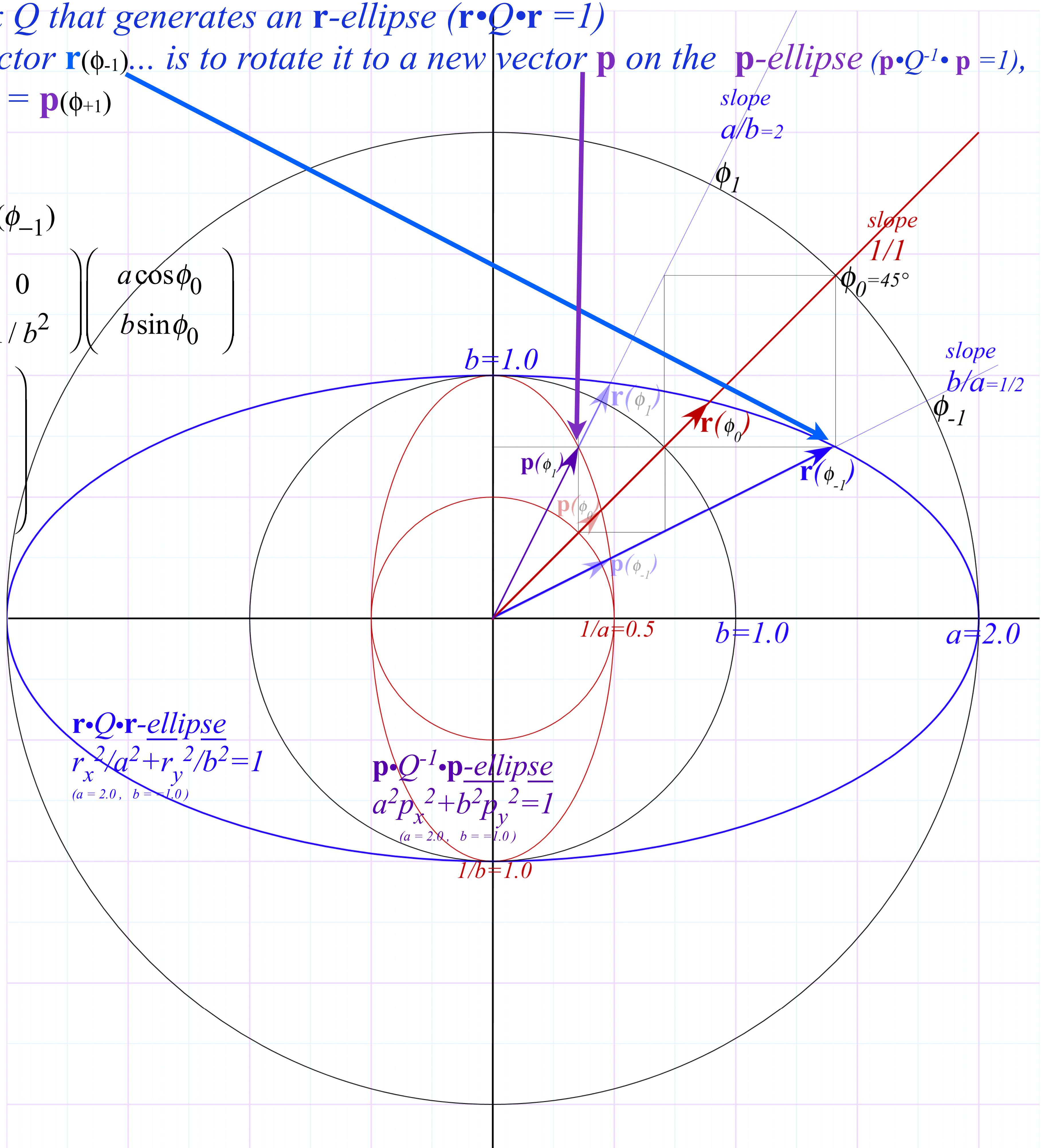
Action of matrix Q *that generates an* **r***-ellipse* ($\mathbf{r}\bullet Q\bullet\mathbf{r}=1$)
on a single **r***-vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* ($\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$),
that is, $Q\bullet\mathbf{r}(\phi_{-1})=\mathbf{p}(\phi_{+1})$

$$\mathbf{p}(\phi_1)=\mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}a\cos\phi_0\\ b\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{a}\cos\phi_0\\ \frac{1}{b}\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2}\frac{1}{\sqrt{2}}\\ \frac{1}{1}\frac{1}{\sqrt{2}}\end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

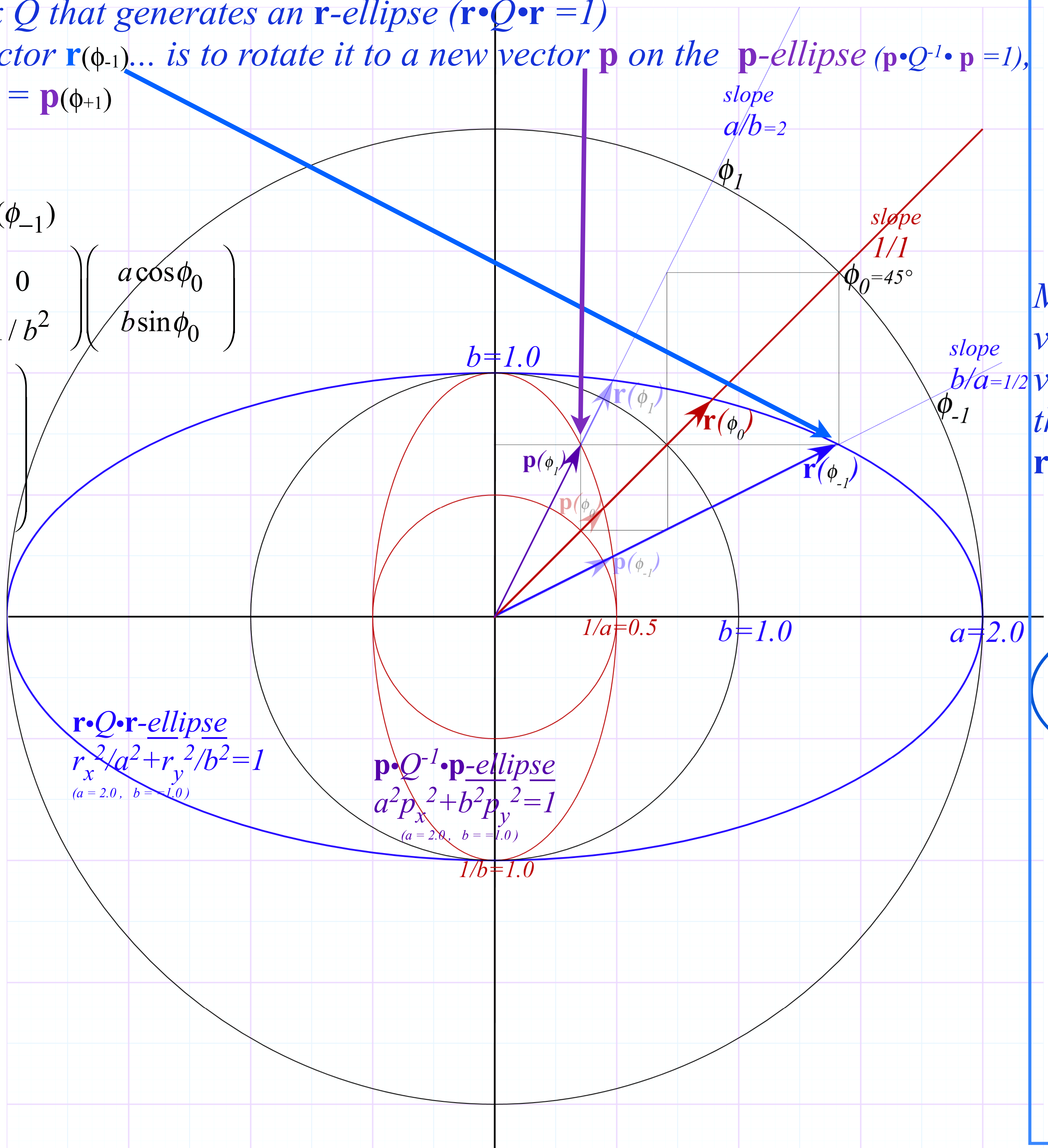
Action of matrix Q that generates an **r***-ellipse* (**r**•Q•**r** $=1$) *on a single* **r***-vector* **r**(ϕ_{-1})*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* (**p**•Q^{-1}• **p** $=1$), *that is,* Q•**r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

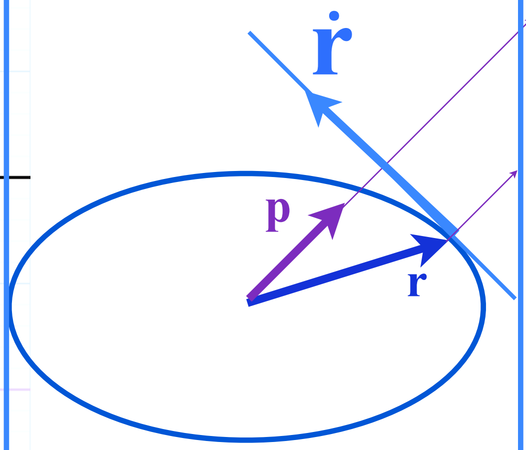
$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* **ṙ** *to its* **r**•Q•**r***-ellipse.*



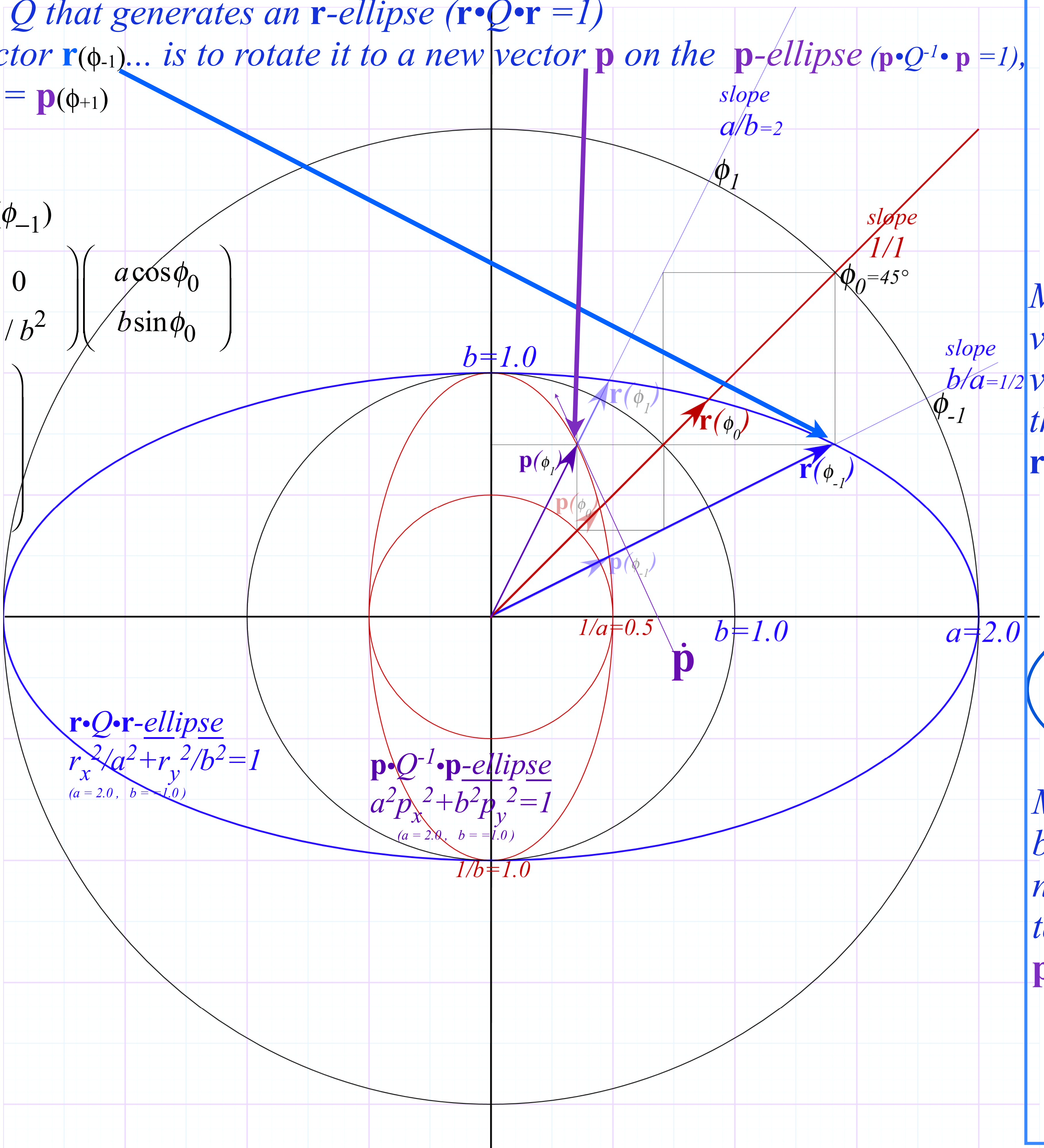
Action of matrix Q that generates an **r**-*ellipse* ($\mathbf{r}\bullet Q\bullet\mathbf{r}=1$) *on a single* **r**-*vector* $\mathbf{r}(\phi_{-1})$... *is to rotate it to a new vector* **p** *on the* **p**-*ellipse* ($\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$), *that is,* $Q\bullet\mathbf{r}(\phi_{-1})=\mathbf{p}(\phi_{+1})$

$$\mathbf{p}(\phi_1)=\mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}a\cos\phi_0\\ b\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{a}\cos\phi_0\\ \frac{1}{b}\sin\phi_0\end{pmatrix}$$

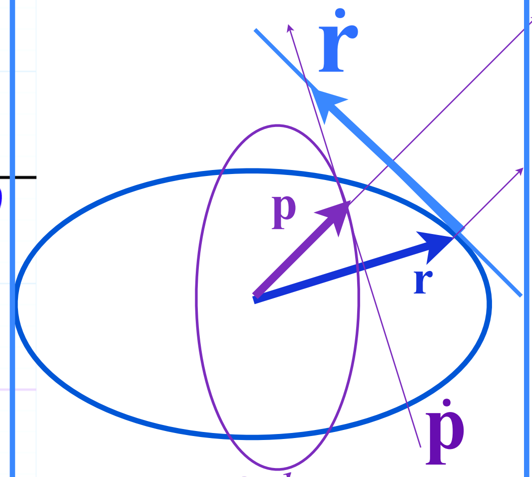
$$=\begin{pmatrix}\frac{1}{2}\frac{1}{\sqrt{2}}\\ \frac{1}{1}\frac{1}{\sqrt{2}}\end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Key points of matrix geometry:

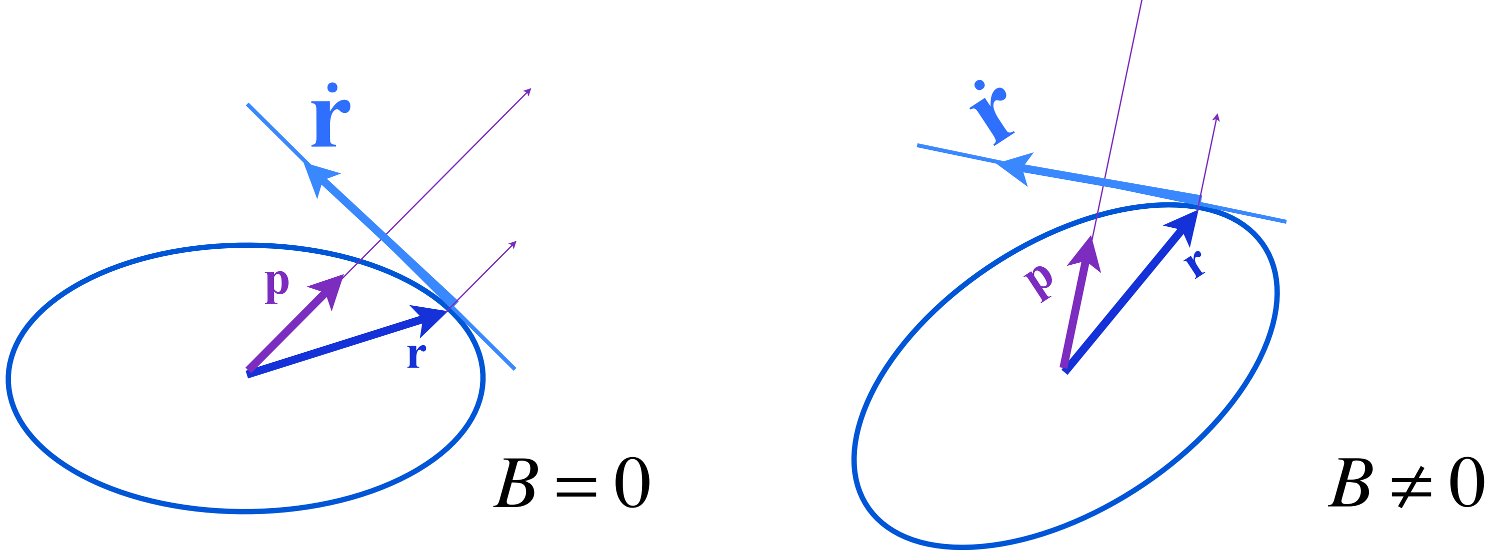
Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* $\dot{\mathbf{r}}$ *to its* $\mathbf{r}\bullet Q\bullet\mathbf{r}$-*ellipse.*



Matrix Q^{-1} maps **p** *back to* **r** *that is normal to the tangent* $\dot{\mathbf{p}}$ *to its* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}$-*ellipse.*

Introduction to dual matrix operator geometry (based on IHO orbits)

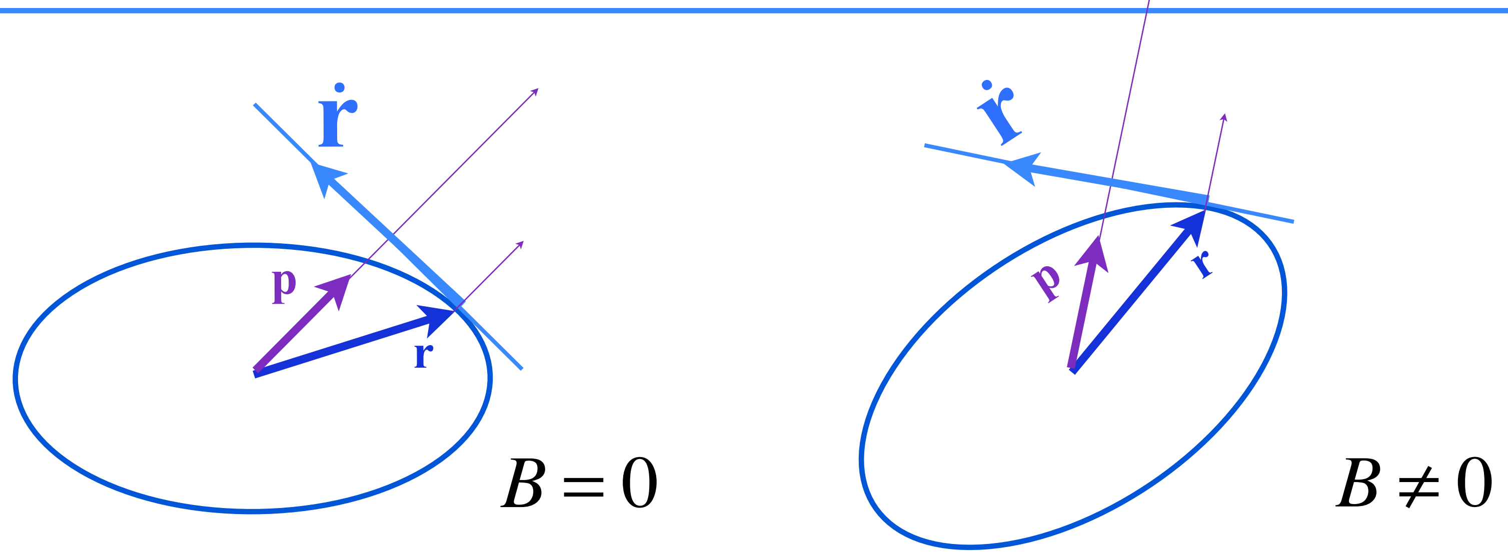
- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations* ($\mathbf{r}\bullet\mathbf{p}=1$)
  - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - → *Vector calculus of tensor operation*



Derive matrix “normal-to-ellipse”geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1 = \mathbf{r} \bullet Q \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$



Derive matrix “normal-to-ellipse”geometry by vector calculus:

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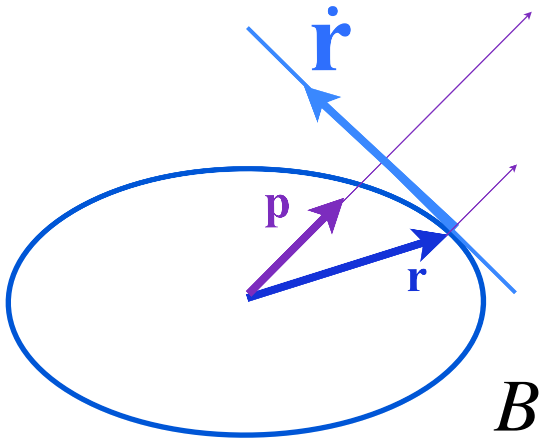
Compare operation by Q on vector **r** *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

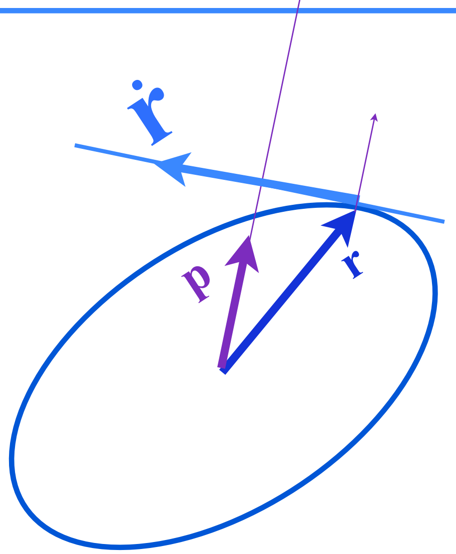
$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r})=\nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2+2B\cdot xy+D\cdot y^2\right)=\begin{pmatrix} 2A\cdot x+2B\cdot y \\ 2B\cdot x+2D\cdot y \end{pmatrix}$$



$B = 0$



$B \neq 0$

Derive matrix “normal-to-ellipse”geometry by vector calculus:

Let matrix Q = $\begin{pmatrix} A & B \\ B & D \end{pmatrix}$

*define the ellipse 1=***r**•*Q*•**r** $=\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$

Compare operation by Q on vector **r** *with* *vector derivative or gradient of* **r**•*Q*•**r**

$$\frac{\partial}{\partial \mathbf{r}}(\mathbf{r} \bullet Q \bullet \mathbf{r}) = \nabla(\mathbf{r} \bullet Q \bullet \mathbf{r})$$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \left(A \cdot x^2 + 2B \cdot xy + D \cdot y^2\right) = \begin{pmatrix} 2A \cdot x + 2B \cdot y \\ 2B \cdot x + 2D \cdot y \end{pmatrix}$$

Very simple result:

$$\frac{\partial}{\partial \mathbf{r}}\left(\frac{\mathbf{r} \bullet Q \bullet \mathbf{r}}{2}\right) = \nabla\left(\frac{\mathbf{r} \bullet Q \bullet \mathbf{r}}{2}\right) = Q \bullet \mathbf{r}$$

Introduction to dual matrix operator geometry (based on IHO orbits)

- Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ vs.inverse form ellipse $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)
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- (Still more) Operator geometric sequences and eigenvectors
- Alternative scaling of matrix operator geometry
  - Vector calculus of tensor operation

Action of "sqrt-"matrix $R=\sqrt{Q}$ *(R generates another ellipse* $\mathbf{r}\bullet R\bullet\mathbf{r}=1$ *not shown)*

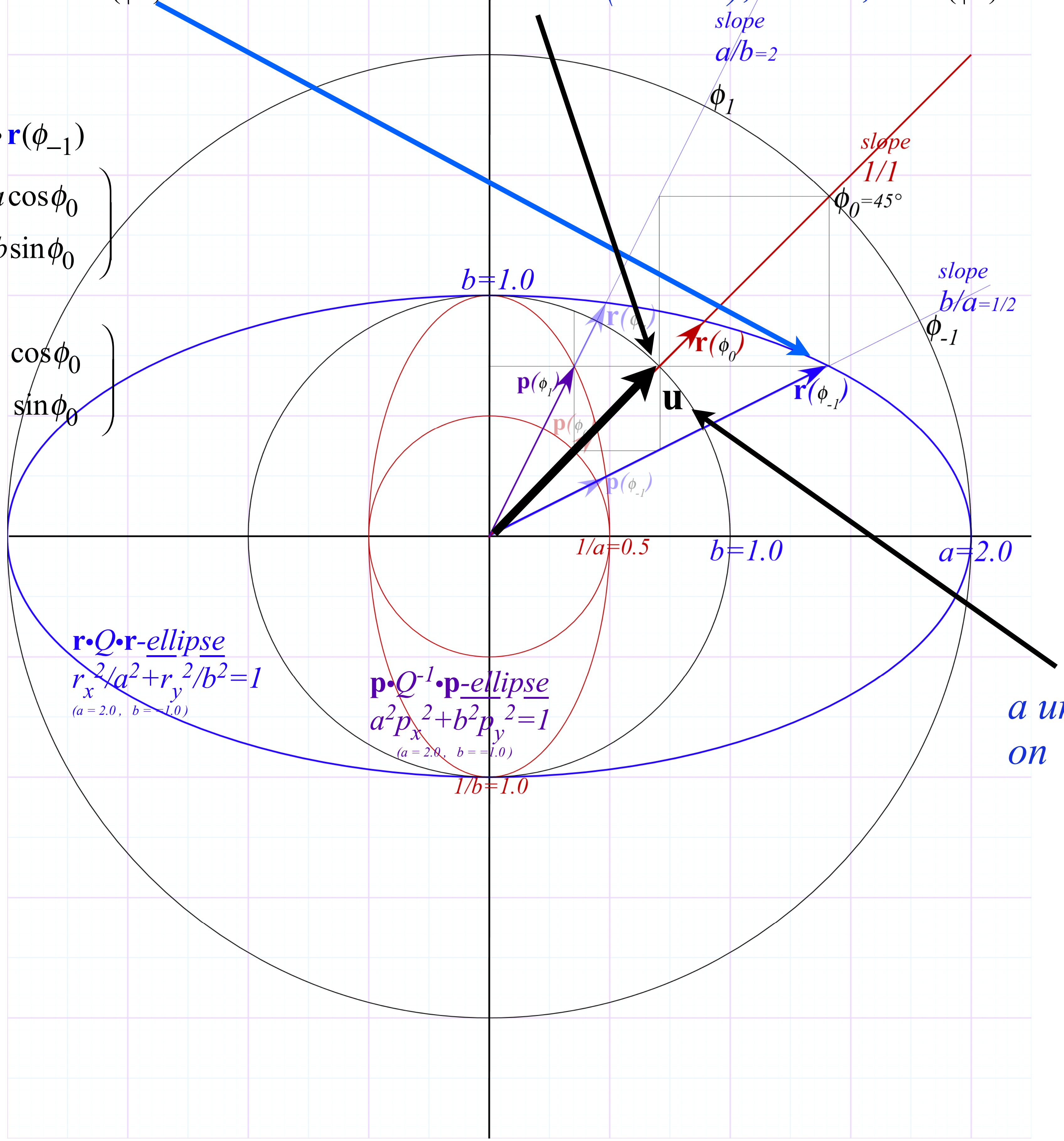
on a single **r**-*vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to* **u**-*circle* ($\mathbf{u}\bullet\mathbf{u}=1$), *that is,* $R\bullet\mathbf{r}(\phi_{-1}) = \mathbf{u} = (const.)\mathbf{r}(\phi_0)$

$$\mathbf{u} = \sqrt{\mathbf{Q}} \bullet \mathbf{r}(\phi_{-1}) = \mathbf{R} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

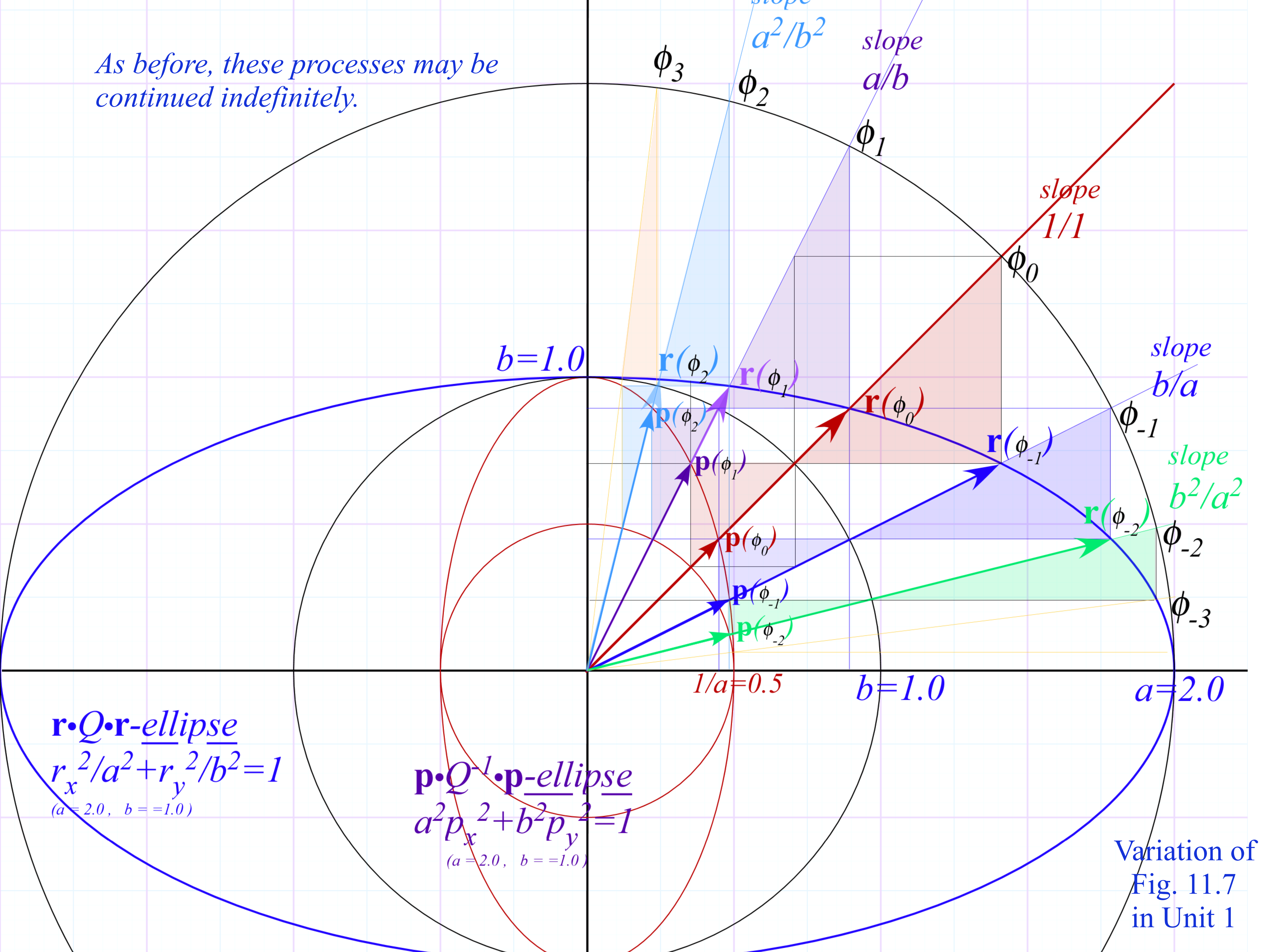
$$= \begin{pmatrix} \frac{1}{a}a\cos\phi_0 \\ \frac{1}{b}b\sin\phi_0 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



a unit vector
on unit-circle

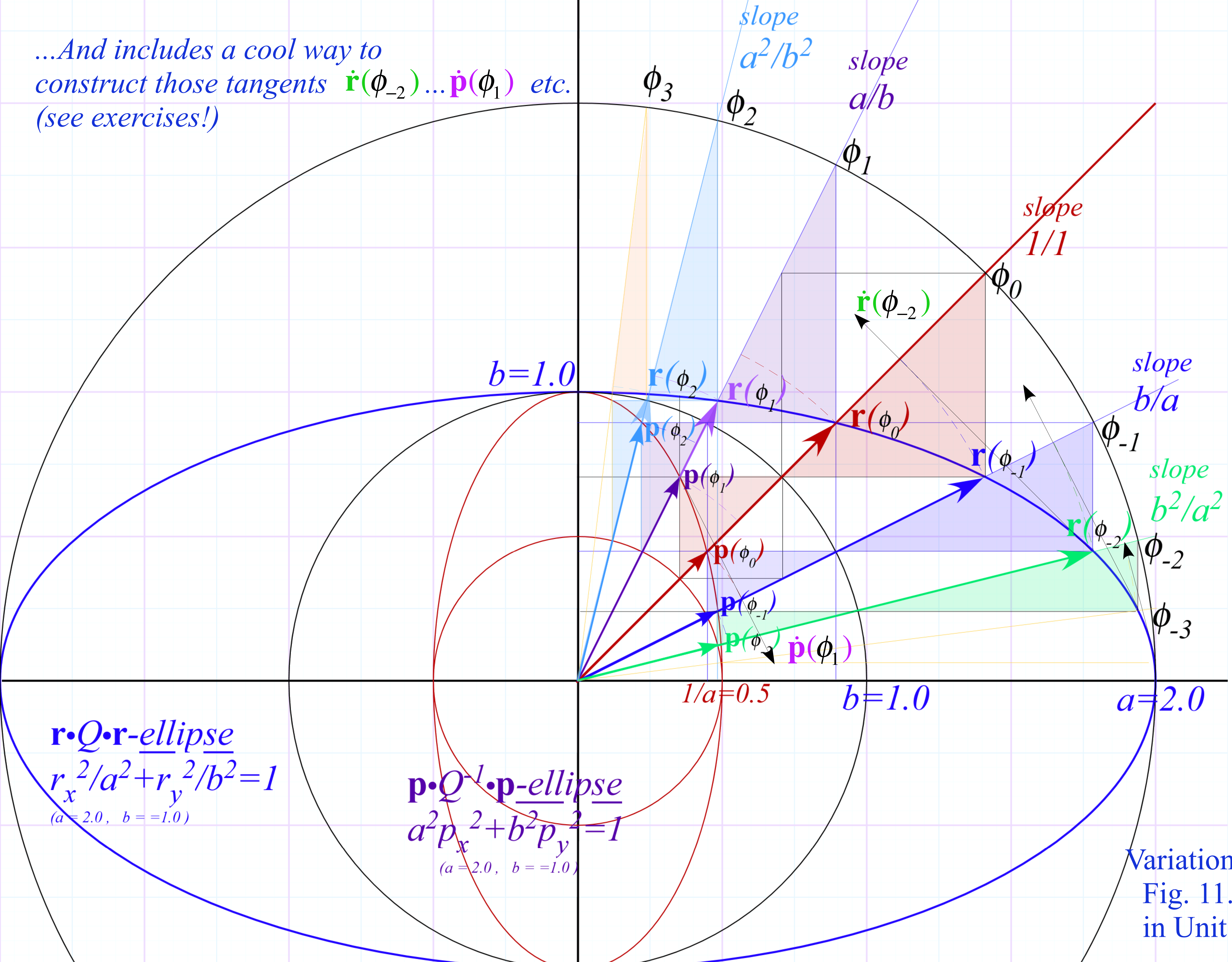
Variation of
Fig. 11.7
in Unit 1



As before, these processes may be continued indefinitely.

Variation of
Fig. 11.7
in Unit 1

...And includes a cool way to construct those tangents $\dot{\mathbf{r}}(\phi_{-2})\ldots\dot{\mathbf{p}}(\phi_1)$ *etc. (see exercises!)*



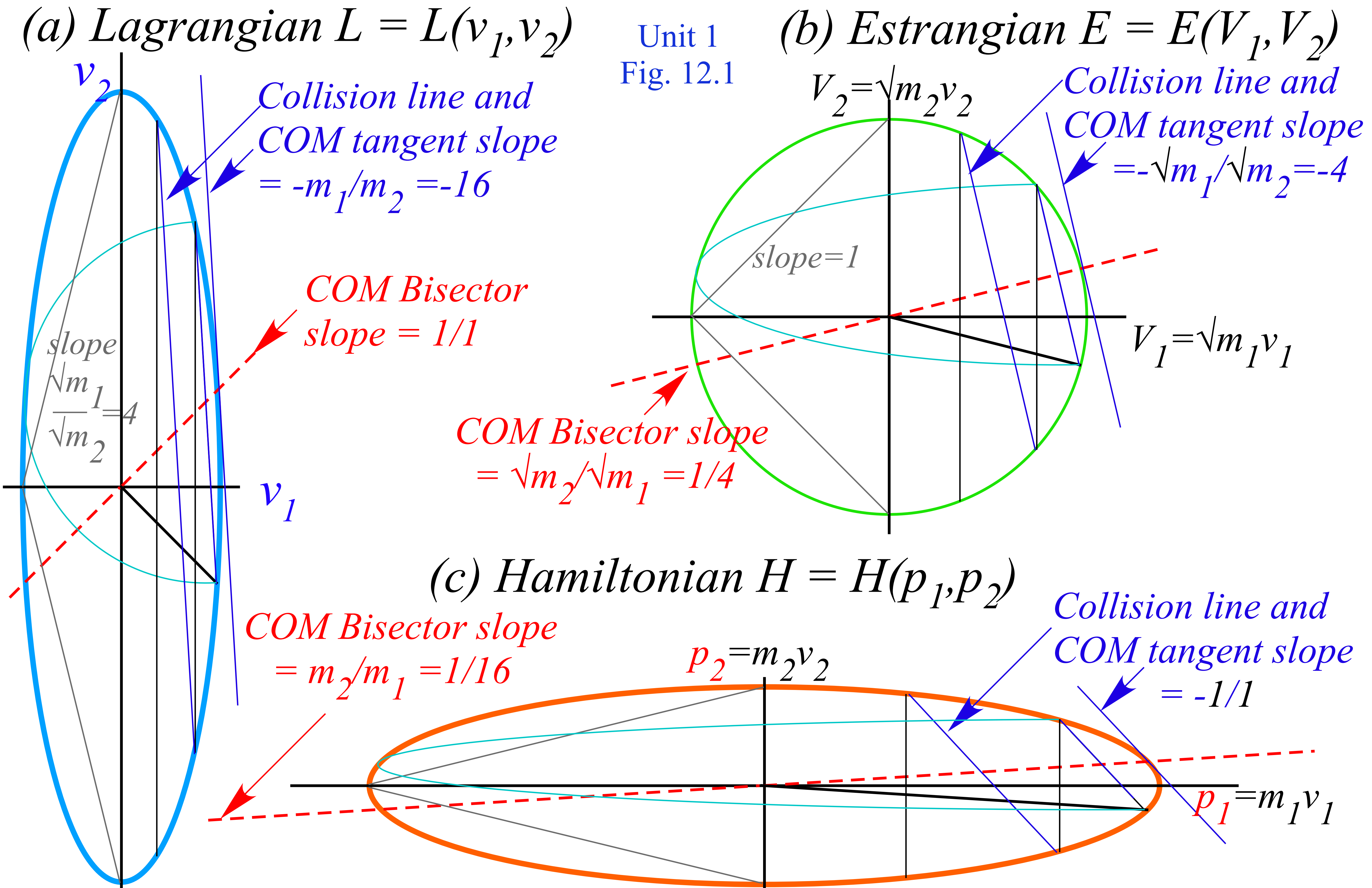
Variation of Fig. 11.7 in Unit 1

Q:Where is this headed? *A: Lagrangian-Hamiltonian duality*

Preview of Lecture 8

The R and Q matrix transformations are like the mechanics rescaling matrices $\sqrt{\mathbf{M}}$ *and* $\mathbf{M}$:

Like $Q=R^2$: $\mathbf{M}=\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}=\mathbf{R}^2$ *Like* $\sqrt{Q}=R$: $\sqrt{\mathbf{M}}=\begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}=\mathbf{R}$ *Like* $Q^{-1}=R^{-2}$: $\mathbf{M}^{-1}=\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}=\mathbf{R}^{-2}$



Unit 1
Fig. 12.1

(a) Lagrangian $L = L(v_1,v_2)$

Collision line and COM tangent slope $= -m_1/m_2 = -16$

COM Bisector slope = 1/1

(b) Estrangian $E = E(V_1,V_2)$

Collision line and COM tangent slope $=-\sqrt{m_1}/\sqrt{m_2}=-4$

COM Bisector slope $= \sqrt{m_2}/\sqrt{m_1} =1/4$

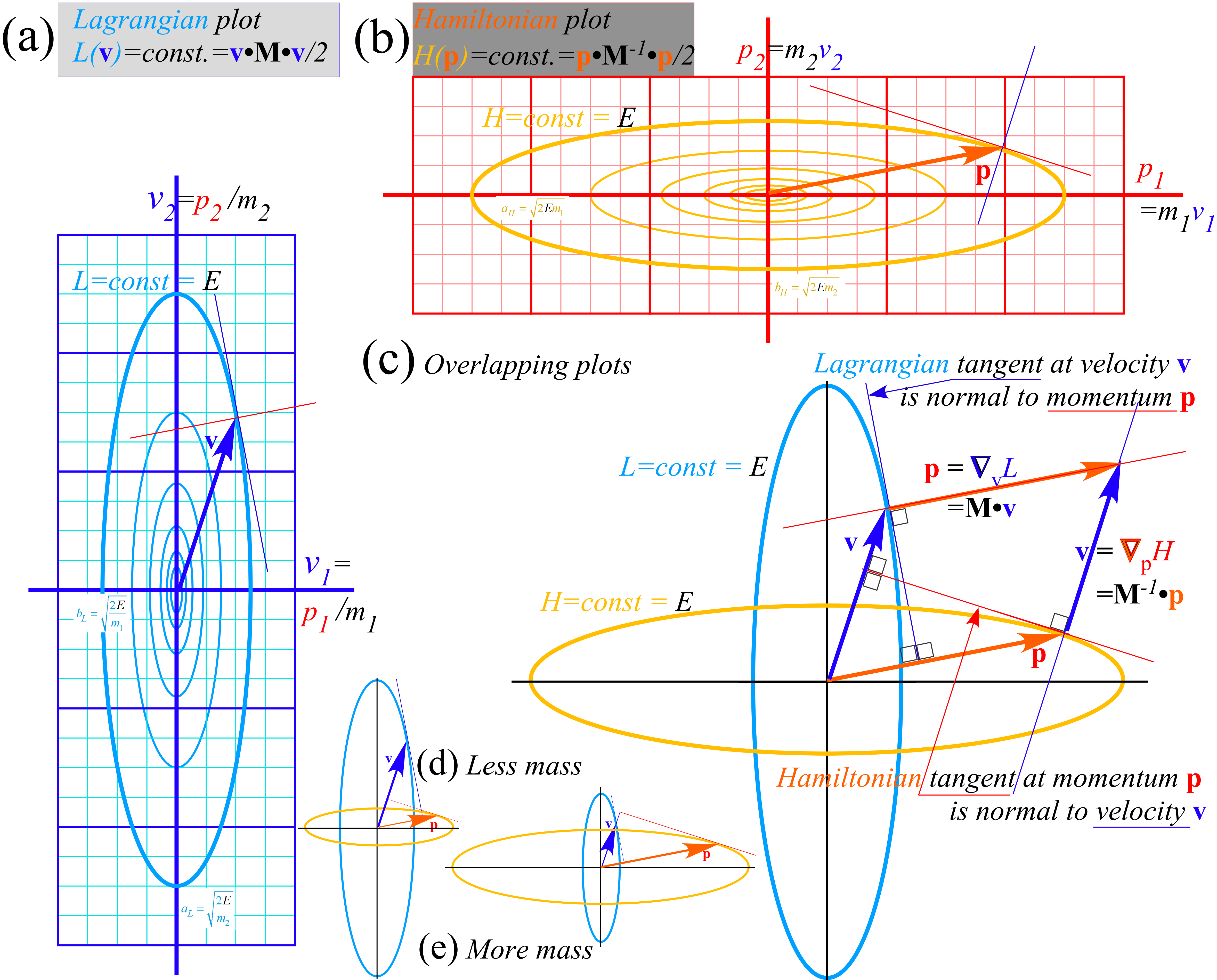
(c) Hamiltonian $H = H(p_1,p_2)$

Collision line and COM tangent slope $= -1/1$

COM Bisector slope $= m_2/m_1 =1/16$

Unit 1

Fig. 12.2



(a) *Lagrangian plot* $L(\mathbf{v})=const.=\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}/2$

(b) *Hamiltonian plot* $H(\mathbf{p})=const.=\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}/2$

(c) *Overlapping plots*

(d) *Less mass*

(e) *More mass*