Lecture 26
Thur. 11.30.2017

Geometry and Symmetry of Coulomb Orbital Dynamics

(Ch. 2-4 of Unit 5 11.30.17)

Rutherford scattering and hyperbolic orbit geometry
- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Projection **ε•r** *geometry of* **ε***-vector and orbital radius* **r**
- *Review and connection to usual orbital algebra (previous lecture)*

Projection **ε•p** *geometry of* **ε***-vector and momentum* **p**$=m$**v**

General geometric orbit construction using **ε***-vector and (γ,R)-parameters*
- *Derivation of* **ε***-construction by analytic geometry*

Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* **p**$=m$**v**
- *Example of complete (***r***,***p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

➤ *Rutherford scattering and hyperbolic orbit geometry*

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Projection **ε•r** *geometry of* ***ε-vector*** *and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

Projection **ε•p** *geometry of* ***ε-vector*** *and momentum* **p**=*m***v**

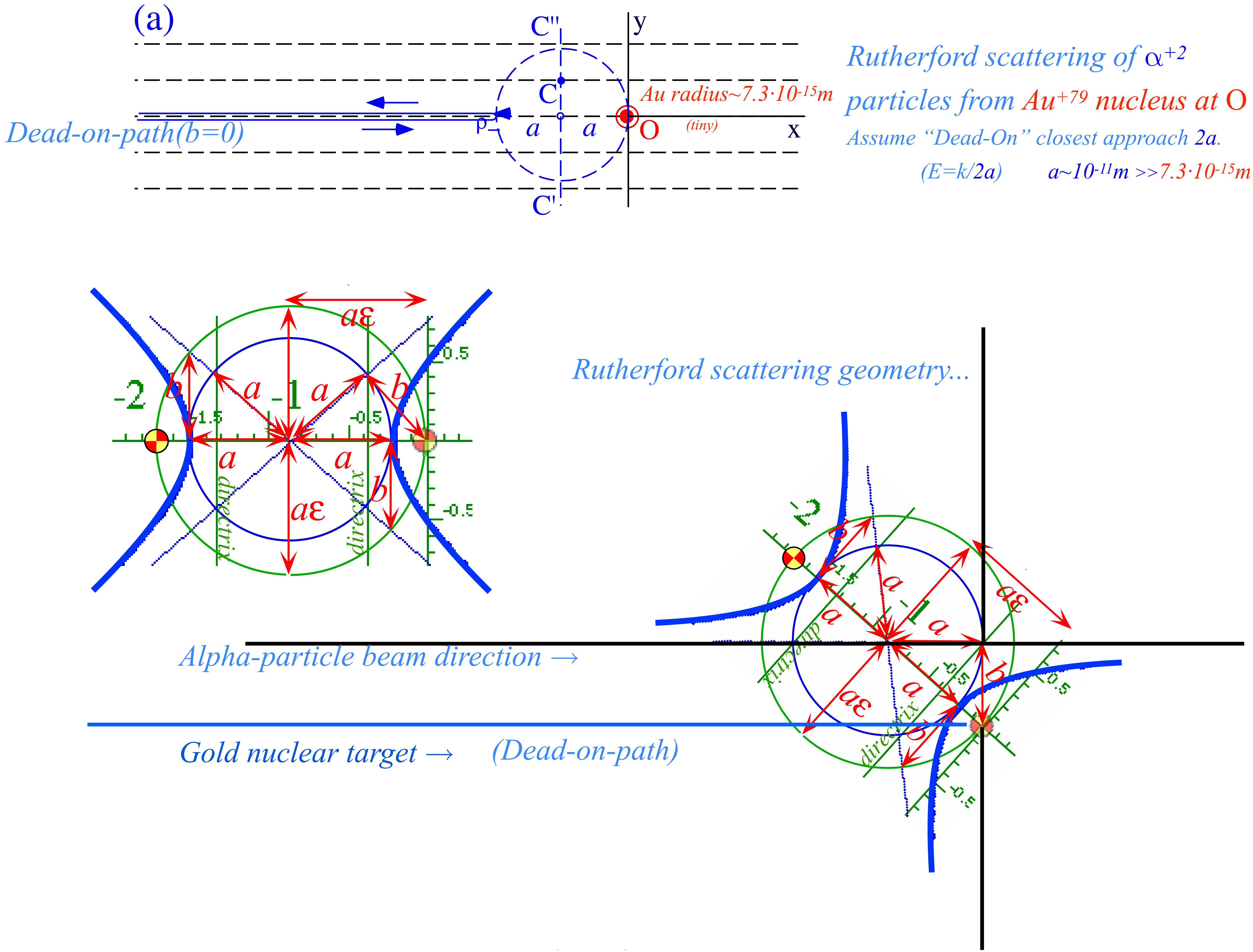
General geometric orbit construction using ***ε-vector*** *and (γ,R)-parameters*

Derivation of **ε**-*construction by analytic geometry*

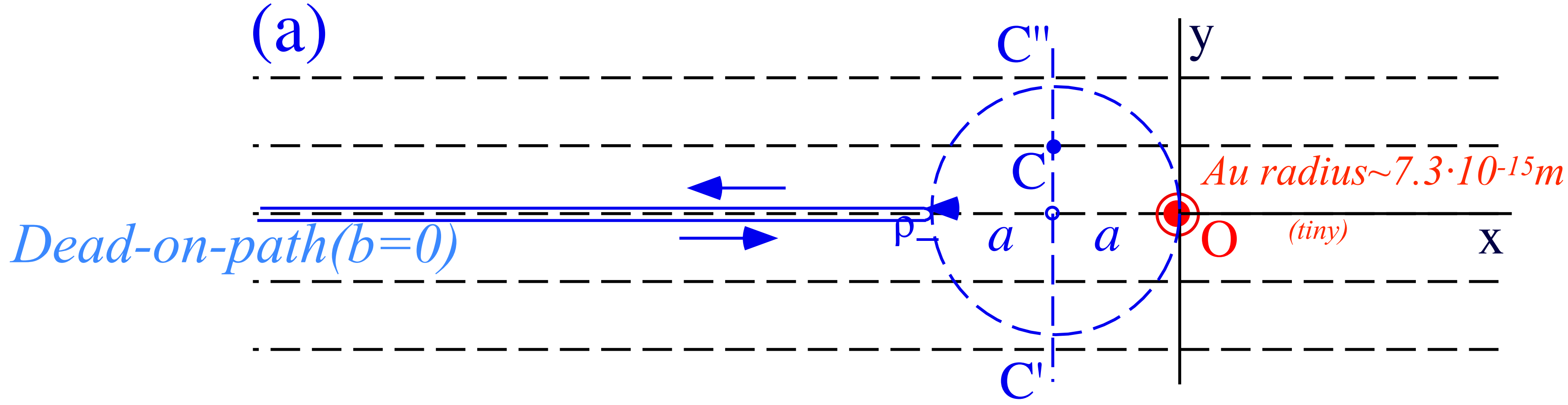
Coulomb orbit algebra of ***ε-vector*** *and Kepler dynamics of momentum* **p**=*m***v**

*Example of complete (***r***,***p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)



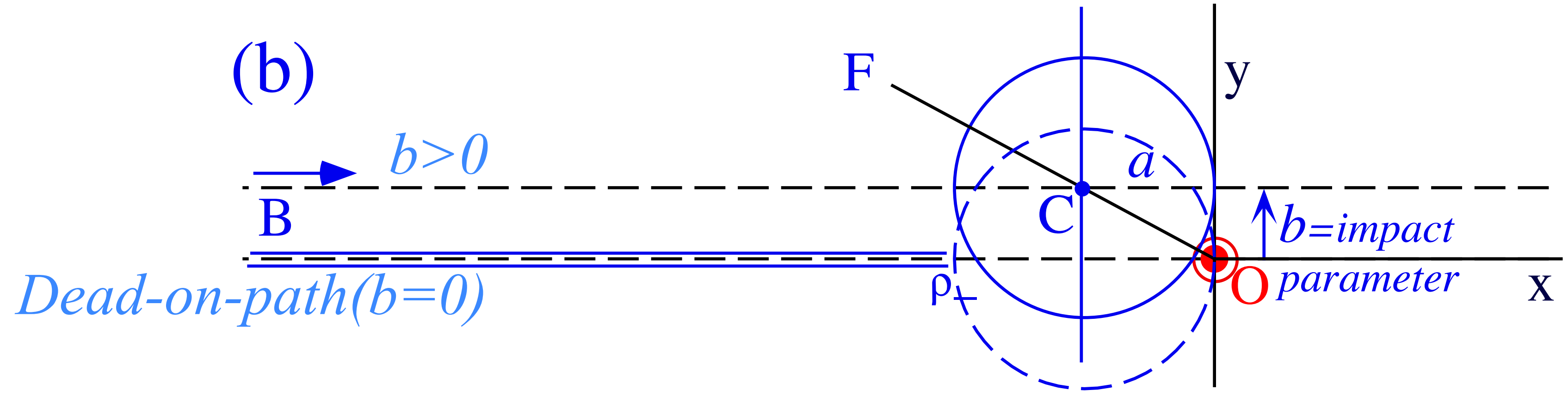
(a)
C''
C
C'
y
x
O
a
a
ρ
Dead-on-path(b=0)
Au radius~7.3·10-15m
(tiny)
Rutherford scattering of α+2
particles from Au+79 nucleus at O
Assume "Dead-On" closest approach 2a.
(E=k/2a)
a~10-11m >>7.3·10-15m
aε
a
b
-2
-1
-1.5
-0.5
0.5
-0.5
directrix
directrix
Rutherford scattering geometry...
Alpha-particle beam direction →
Gold nuclear target →
(Dead-on-path)



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach 2a.

(E=k/2a) $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$

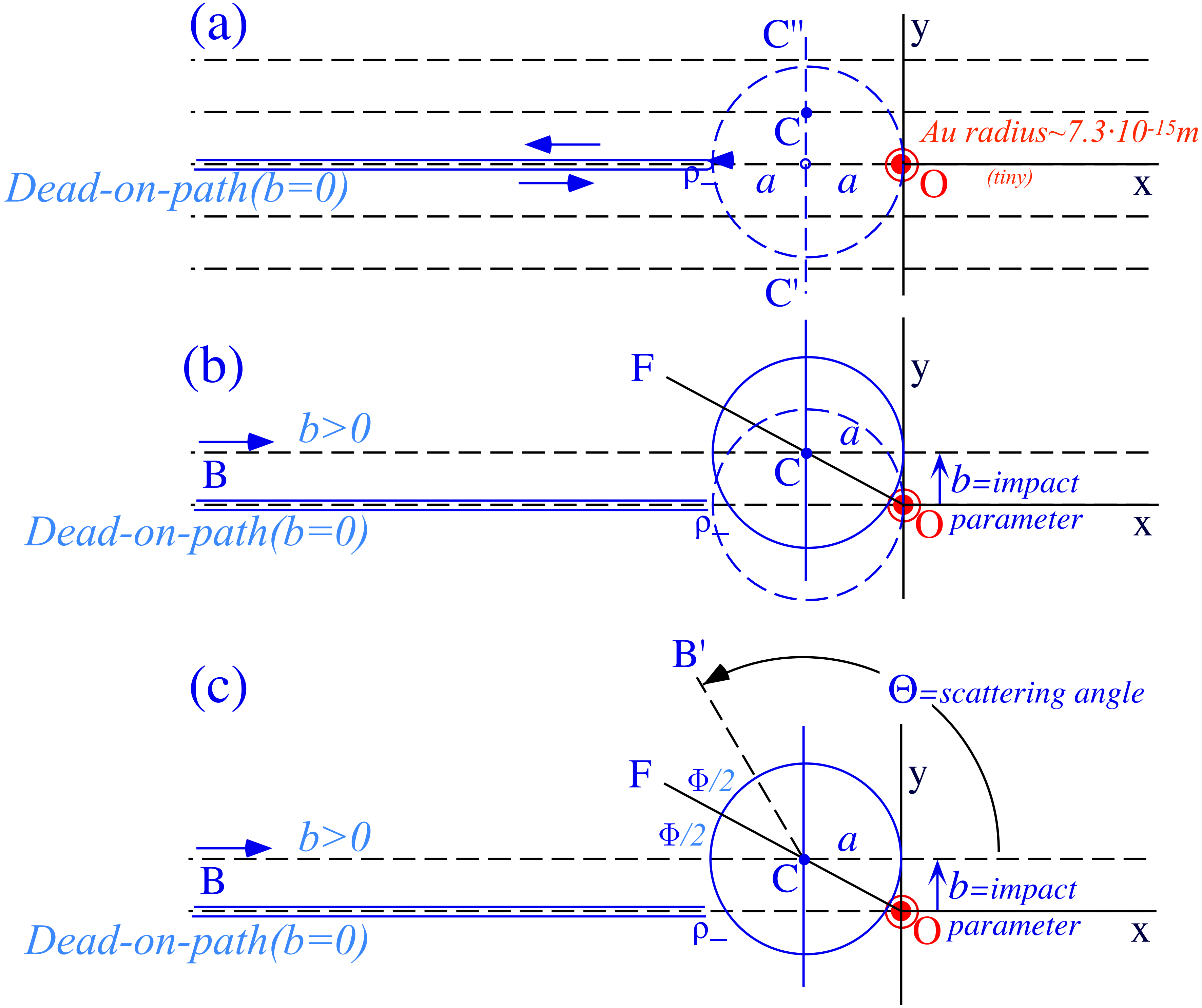


Pick an "impact parameter" line y =b.

Draw circle of radius ***a*** *around*

center point C=(-*a*,*b*) *tangent to* *y-axis.*

Draw "focus-locus" line OCF.



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach 2a.

(E=k/2a) $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$

Pick an "impact parameter" line y =b.

Draw circle of radius ***a*** *around*

center point C*=(-a,b) tangent to y-axis.*

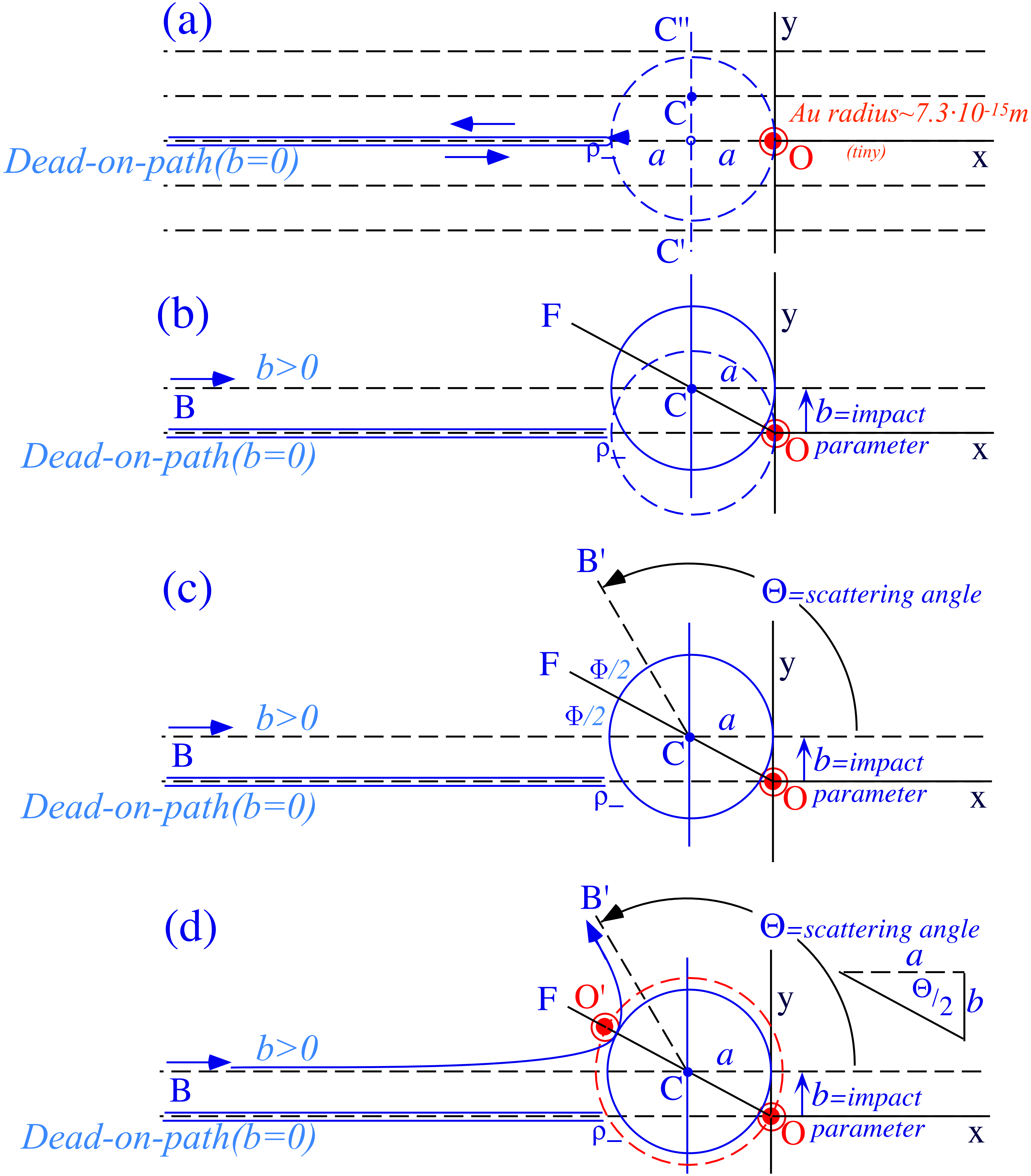
Draw "focus-locus" line OCF.

Copy angle ∠BCF *(equal to* Φ/2)

to make angle ∠FCB′ (*also equal to* Φ/2)

Resulting line CB′ *is outgoing asymptote*

at scattering angle Θ .



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach 2a.

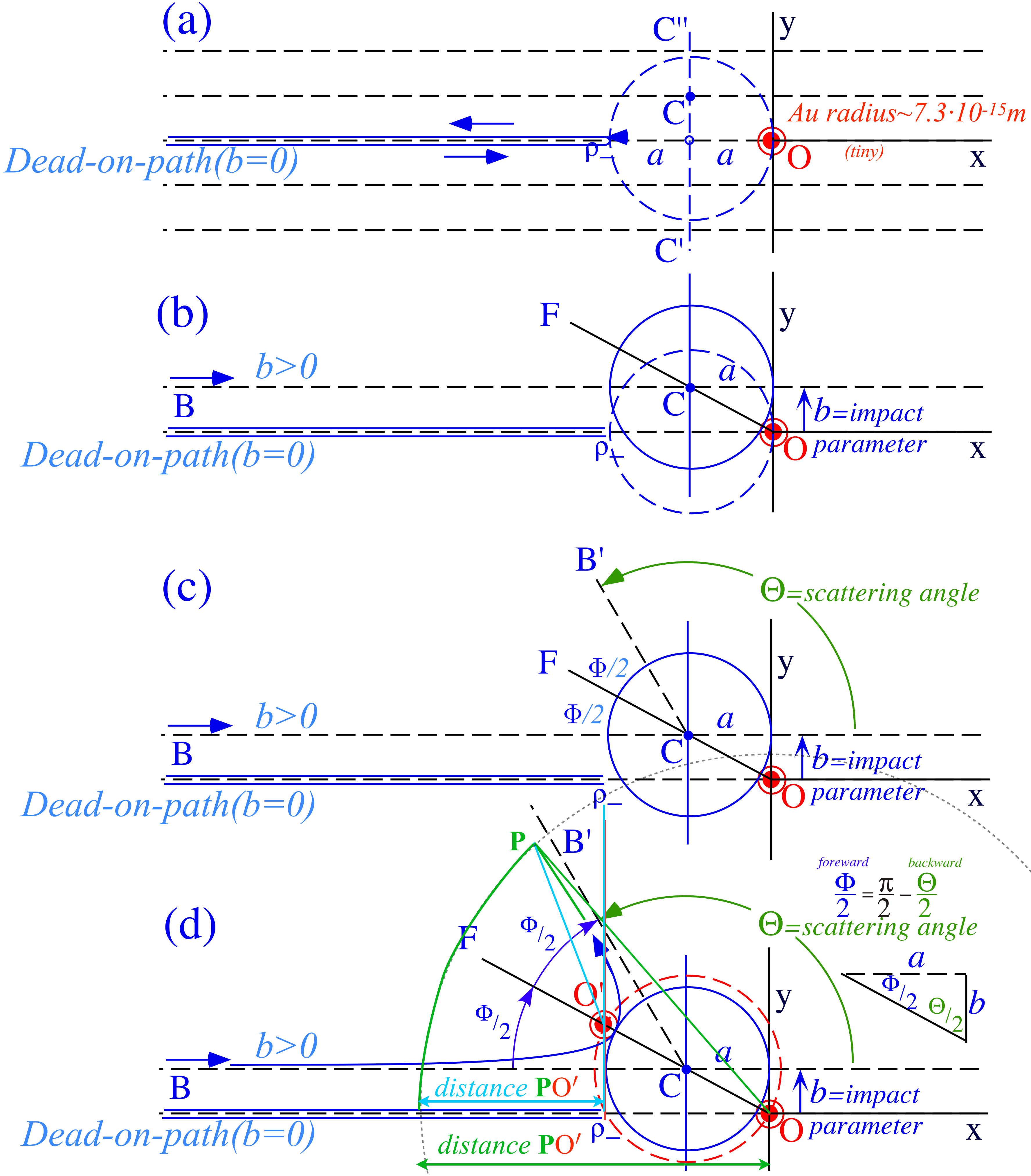
(E=k/2a) $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$

Pick an "impact parameter" line y =b.
Draw circle of radius ***a*** *around*
center point C=(-*a*,*b*) *tangent to* ***y****-axis.*
Draw "focus-locus" line OCF.

Copy angle ∠BCF (*equal to* Φ/2)
to make angle ∠FCB′ (*also equal to* Φ/2)
Resulting line CB′ *is outgoing asymptote*
at scattering angle Θ .

Locate secondary focus O′ *by drawing*
circle around point C *of diameter* CO
thru point O. *Diameter* O′CO *is 2aε.*
Hyperbolic orbit points **P** *now found*
using constant 2a=**P**O-**P**O′

Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O



Assume "Dead-On" closest approach 2a.

($E=k/2a$) $a \sim 10^{-11}m >> 7.3\cdot 10^{-15}m$

Pick an "impact parameter" line y = b.
Draw circle of radius a around center point C=(-a,b) tangent to y-axis.
Draw "focus-locus" line OCF.

Copy angle ∠BCF (equal to Φ/2) to make angle ∠FCB′ (also equal to Φ/2). Resulting line CB′ is outgoing asymptote at scattering angle Θ .

Locate secondary focus O′ by drawing circle around point C of diameter CO thru point O. Diameter O′CO is 2aε. Hyperbolic orbit points **P** *now found using constant 2a=***P***O-***P***O′*

Rutherford scattering and hyperbolic orbit geometry

➡ *Backward vs forward scattering angles and orbit construction example*

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Projection **ε•r** *geometry of* **ε**-*vector and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

Projection **ε•p** *geometry of* **ε**-*vector and momentum* **p**=*m***v**

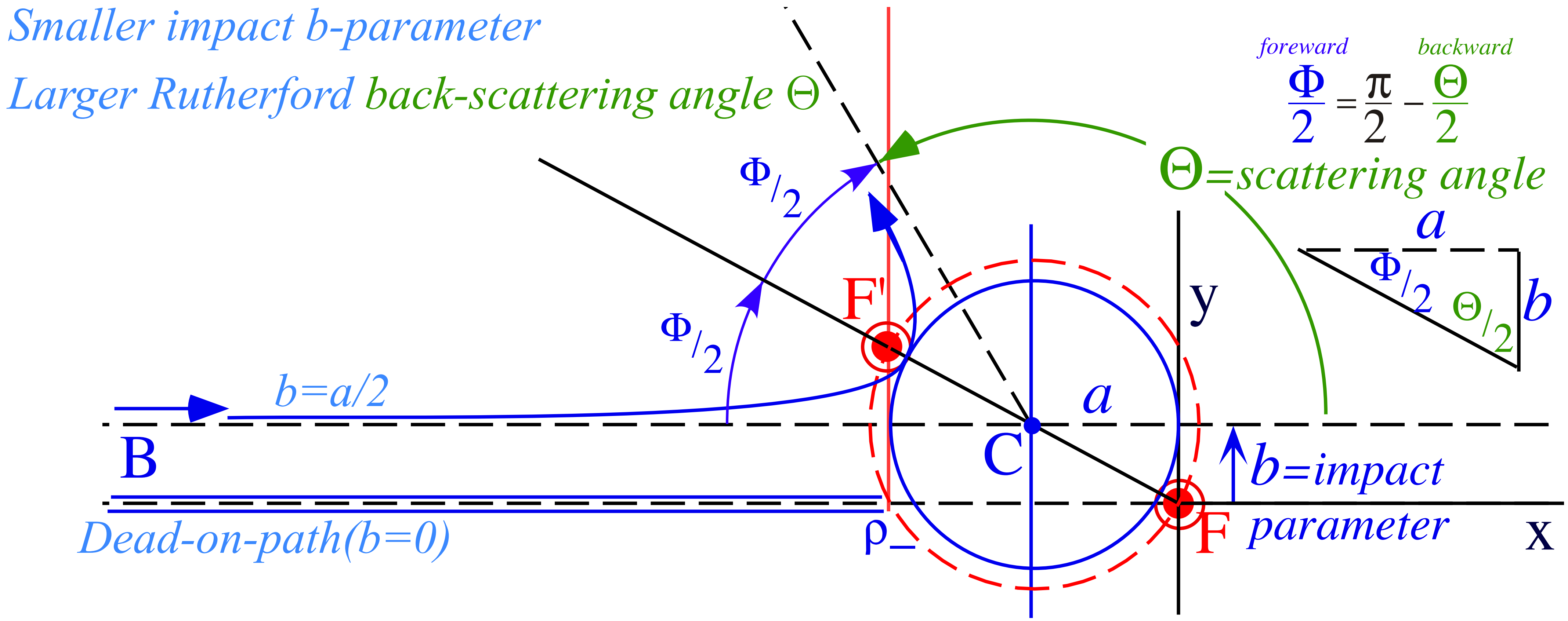
General geometric orbit construction using **ε**-*vector and (γ,R)-parameters*

Derivation of **ε**-*construction by analytic geometry*

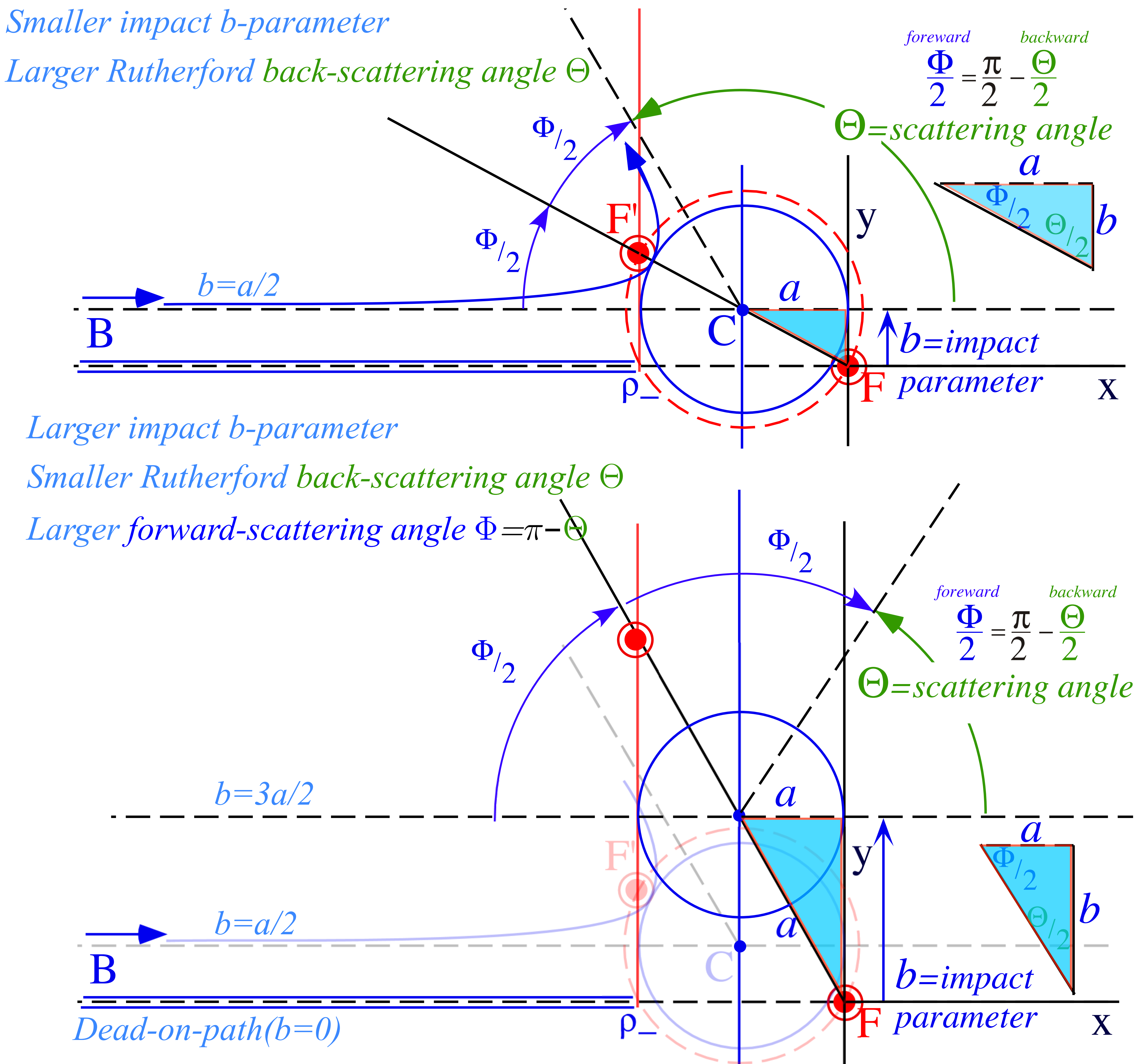
Coulomb orbit algebra of **ε**-*vector and Kepler dynamics of momentum* **p**=*m***v**

*Example of complete (***r***,***p***)-geometry of elliptical orbit*

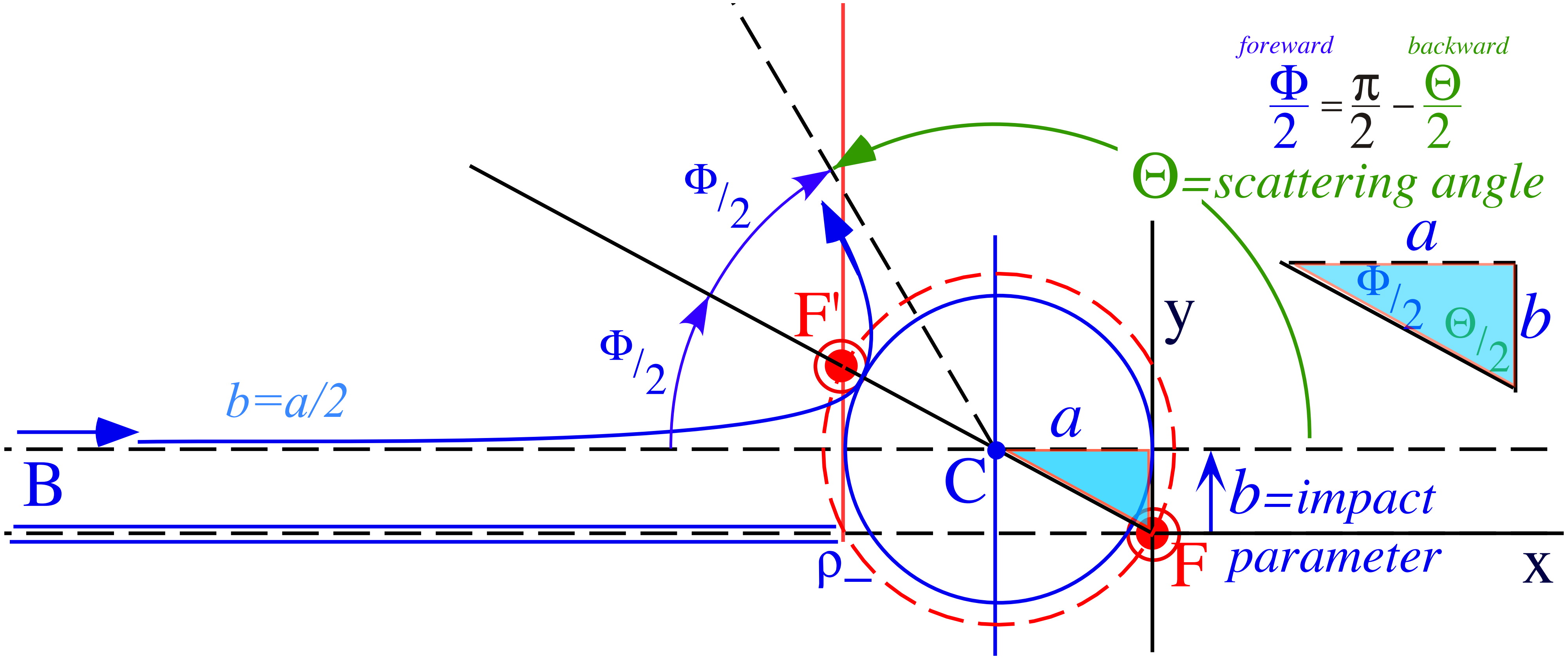
Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

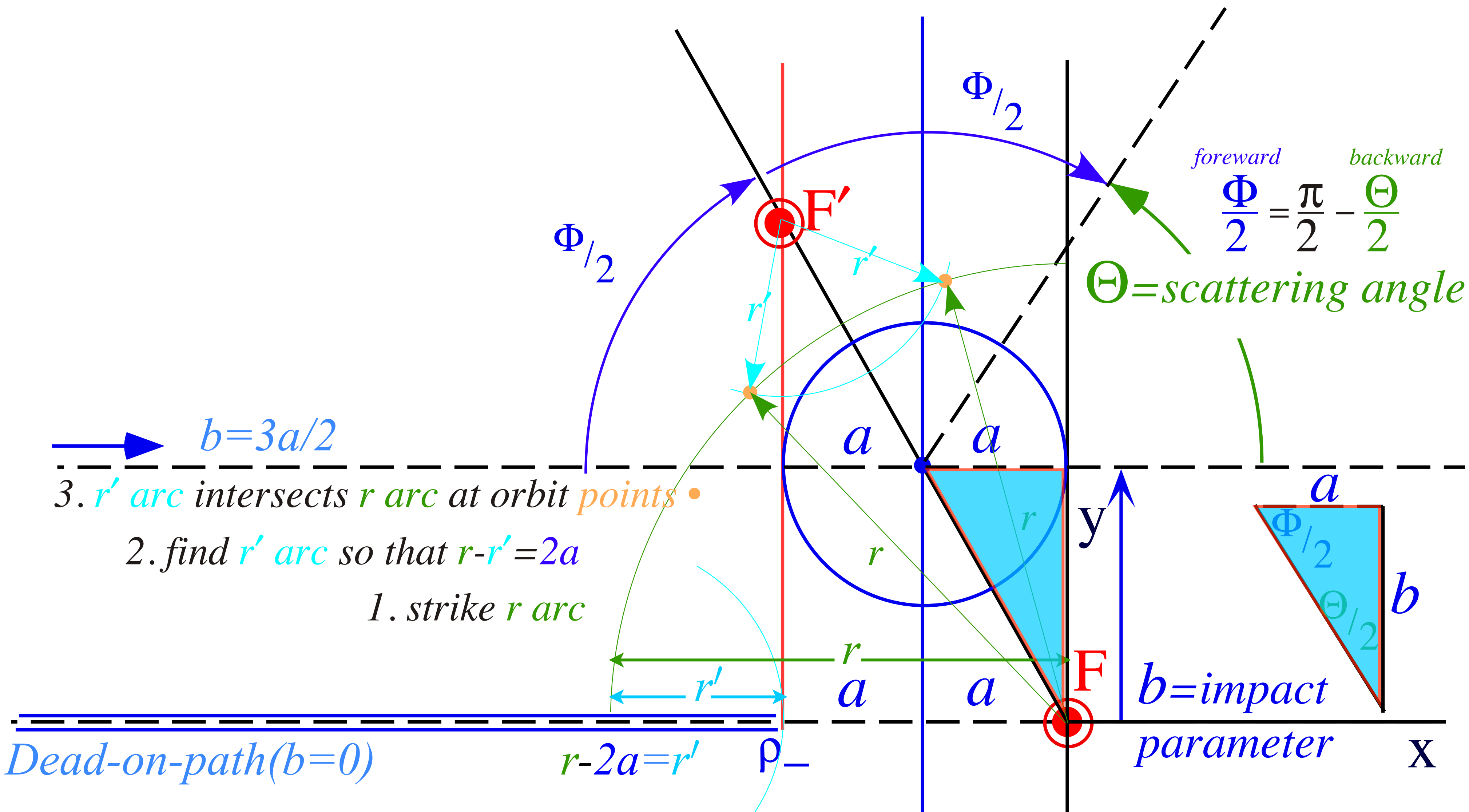


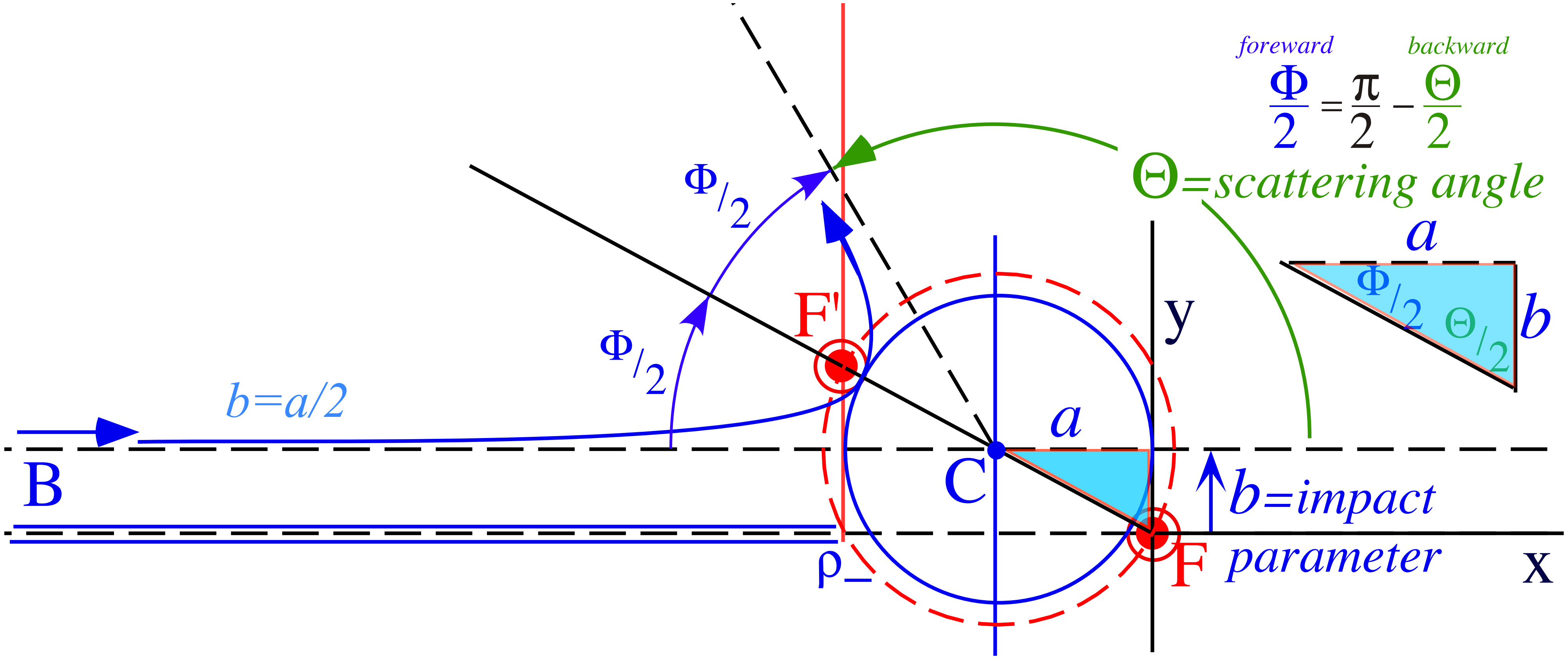
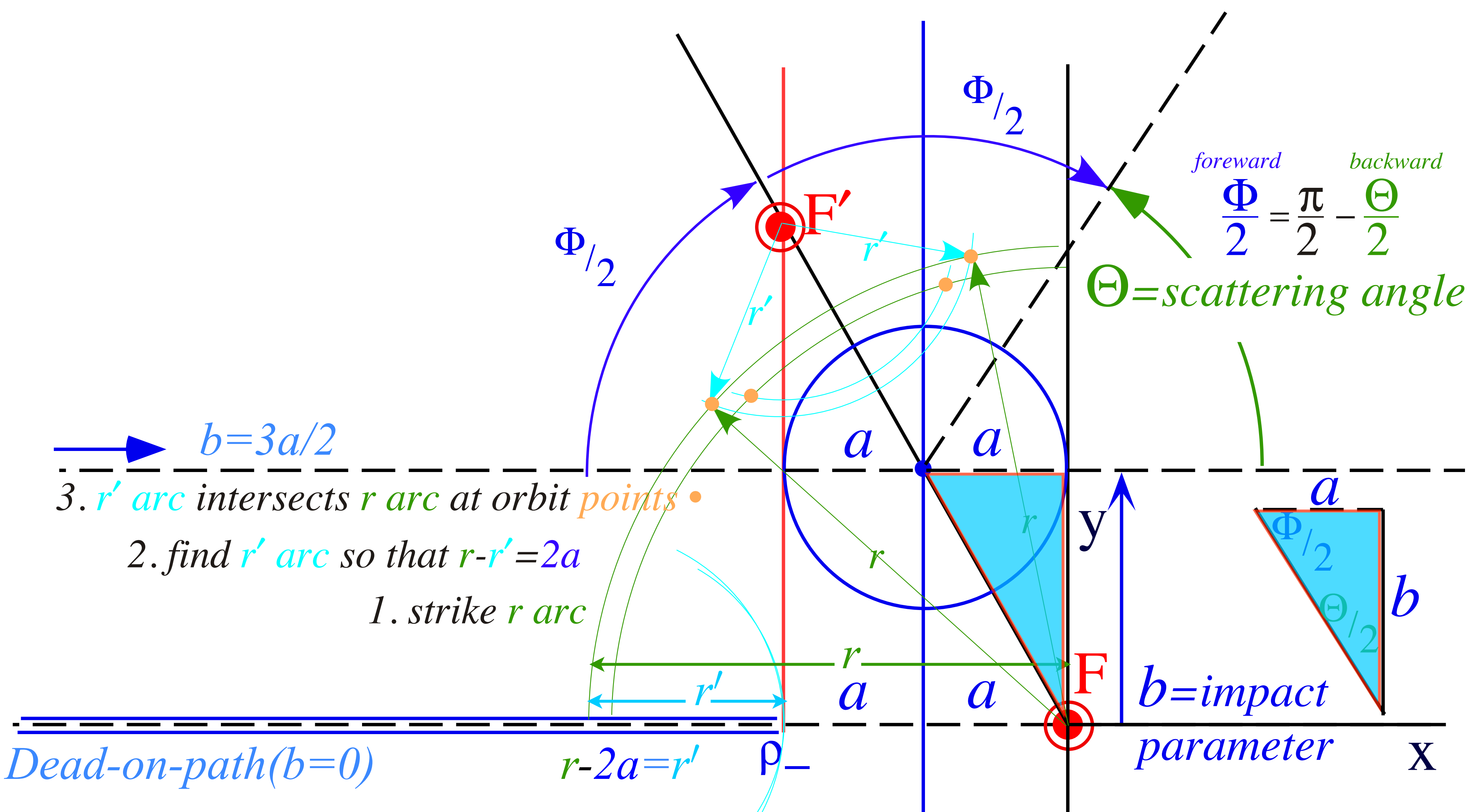
Smaller impact b-parameter
Larger Rutherford back-scattering angle Θ
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
Φ/2
Θ/2
b
y
b=a/2
a
B
C
b=impact
parameter
Dead-on-path(b=0)
ρ_
F
x

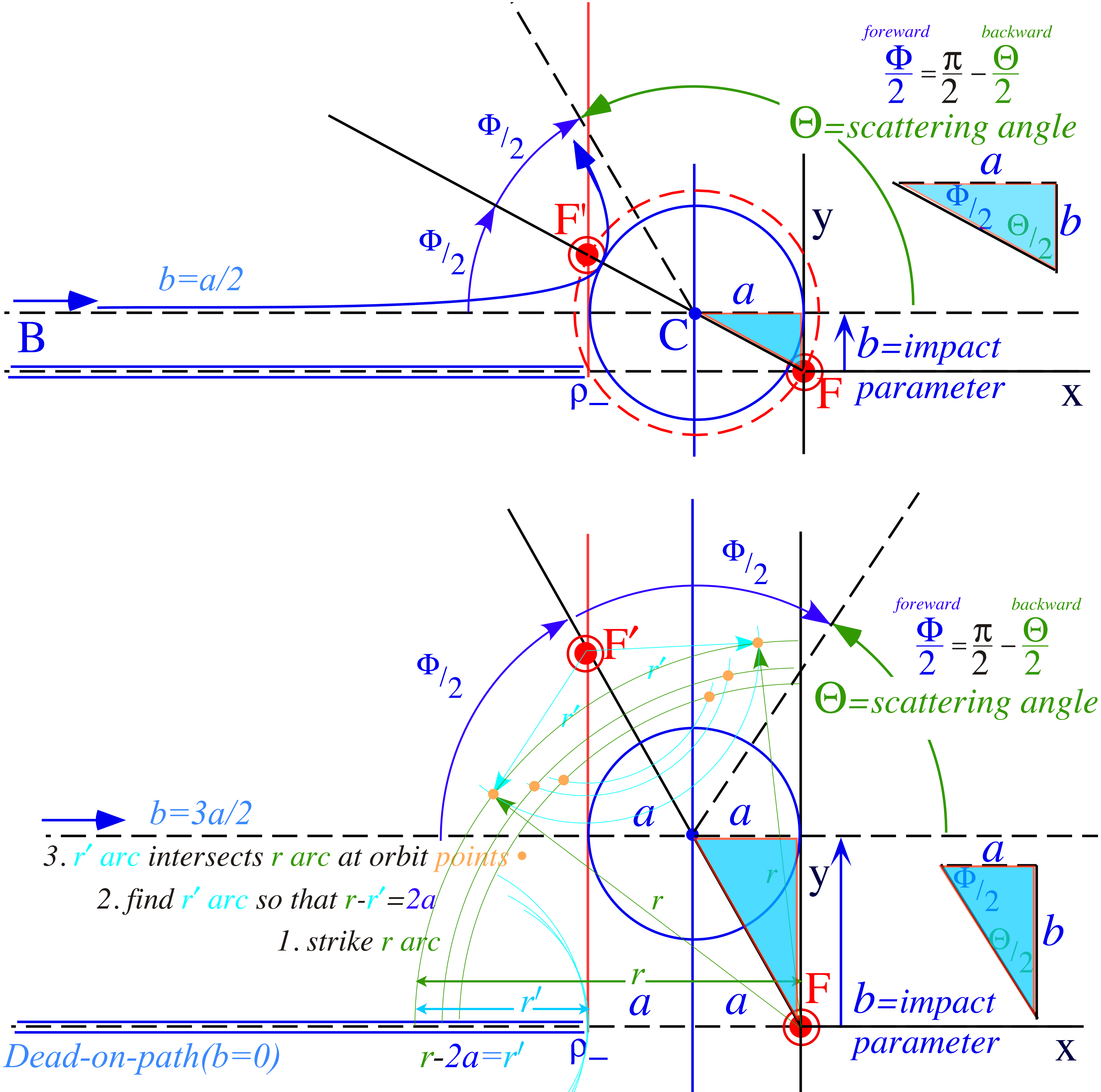


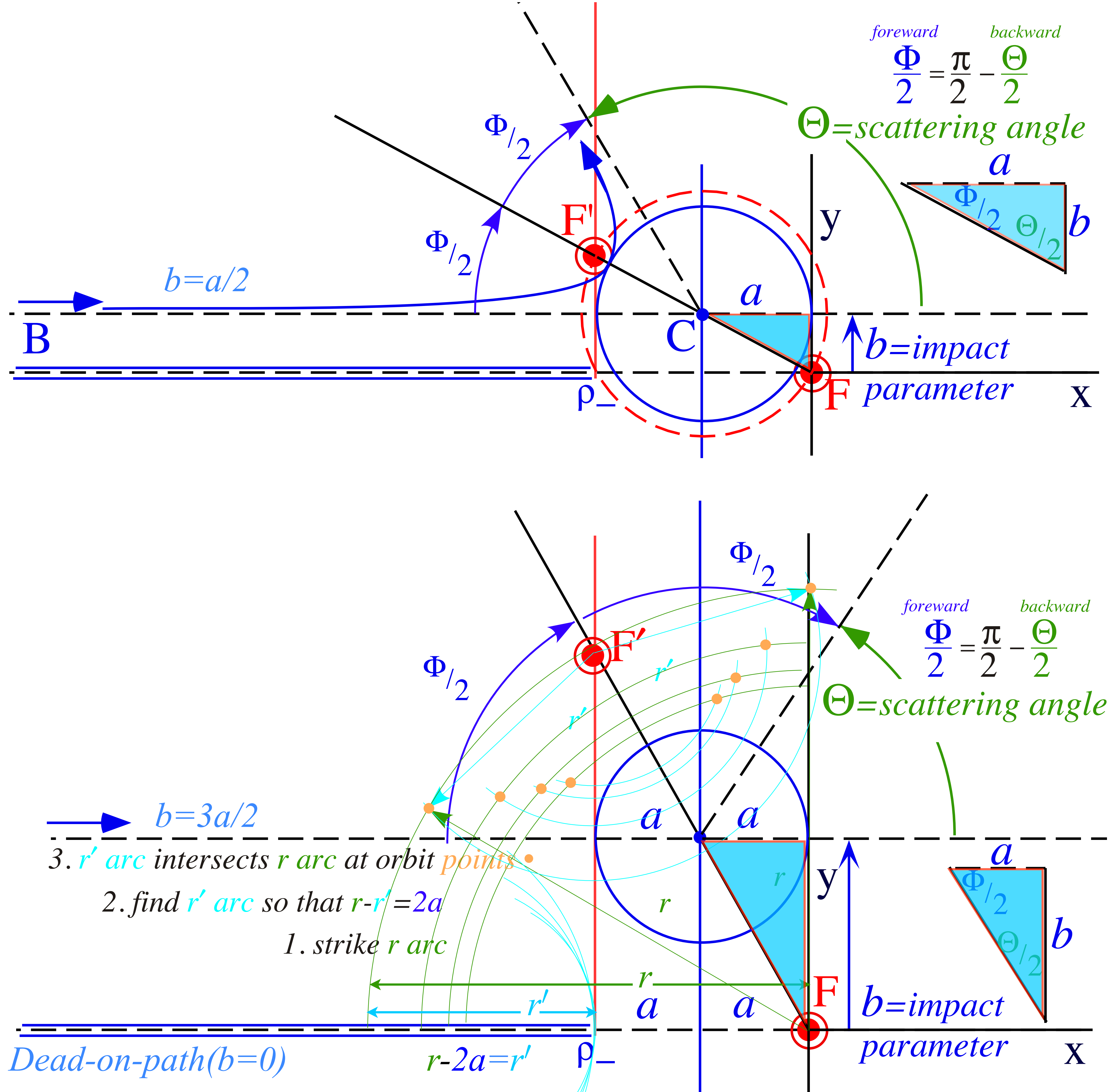
Smaller impact b-parameter
Larger Rutherford back-scattering angle Θ
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Φ/2
Θ/2
y
b=a/2
a
B
C
b=impact
parameter
ρ_
F
x
Larger impact b-parameter
Smaller Rutherford back-scattering angle Θ
Larger forward-scattering angle Φ=π−Θ
Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
Φ/2
Θ=scattering angle
b=3a/2
a
y
a
F'
b=a/2
a
Φ/2
Θ/2
b
B
C
b=impact
parameter
ρ_
F
x
Dead-on-path(b=0)


foreward backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
C
a
y
b
b=impact parameter
ρ_
F
x

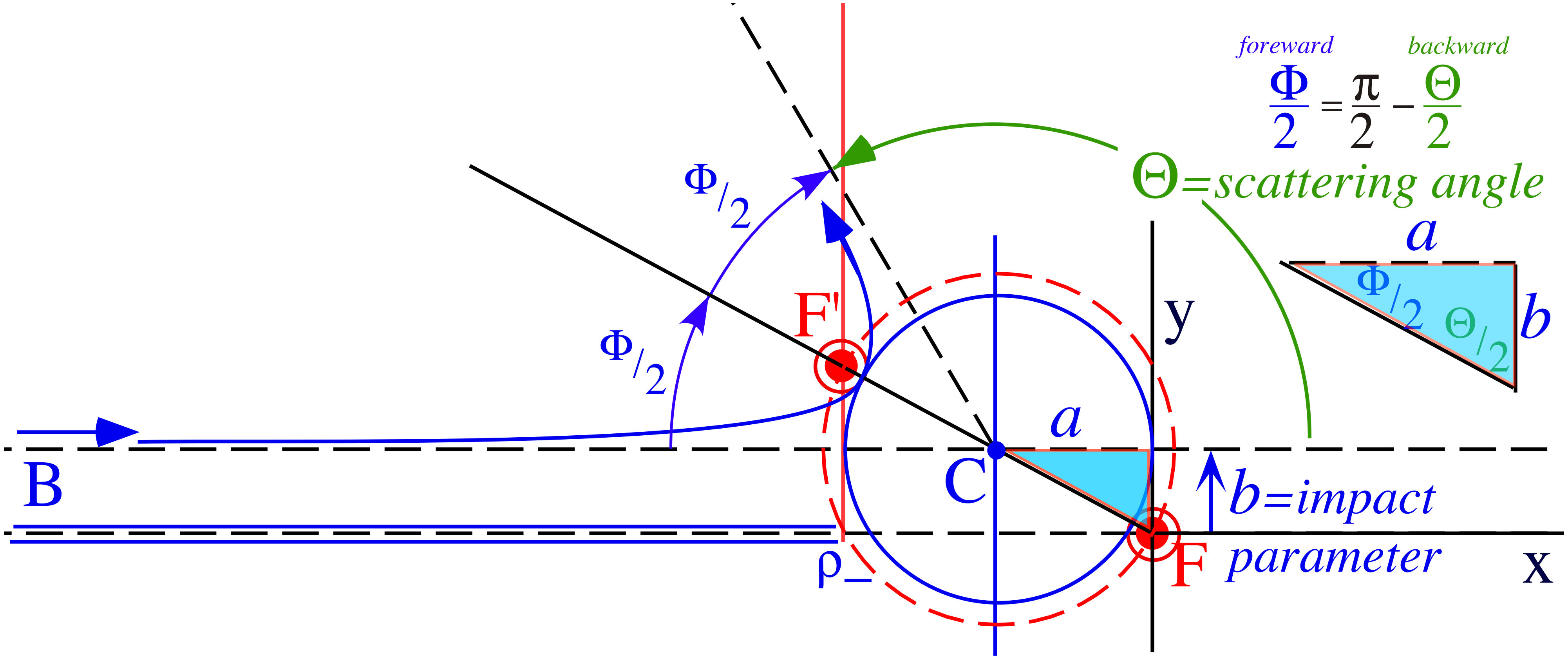

Φ/2
Φ/2
foreward backward
Φ/2 = π/2 − Θ/2
F′
Θ=scattering angle
r′
r′
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
y
b
F
b=impact parameter
x
Dead-on-path(b=0)
r-2a=r′
ρ_


foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Φ/2
Θ/2
y
b=a/2
a
B
C
b=impact
parameter
ρ_
F
x

Φ/2
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Φ/2
F′
r′
r′
Θ=scattering angle
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points
a
y
Φ/2
b
Θ/2
2. find r′ arc so that r-r′=2a
r
r
1. strike r arc
r
r′
F
a
a
b=impact
parameter
Dead-on-path(b=0)
r-2a=r′
ρ_
x

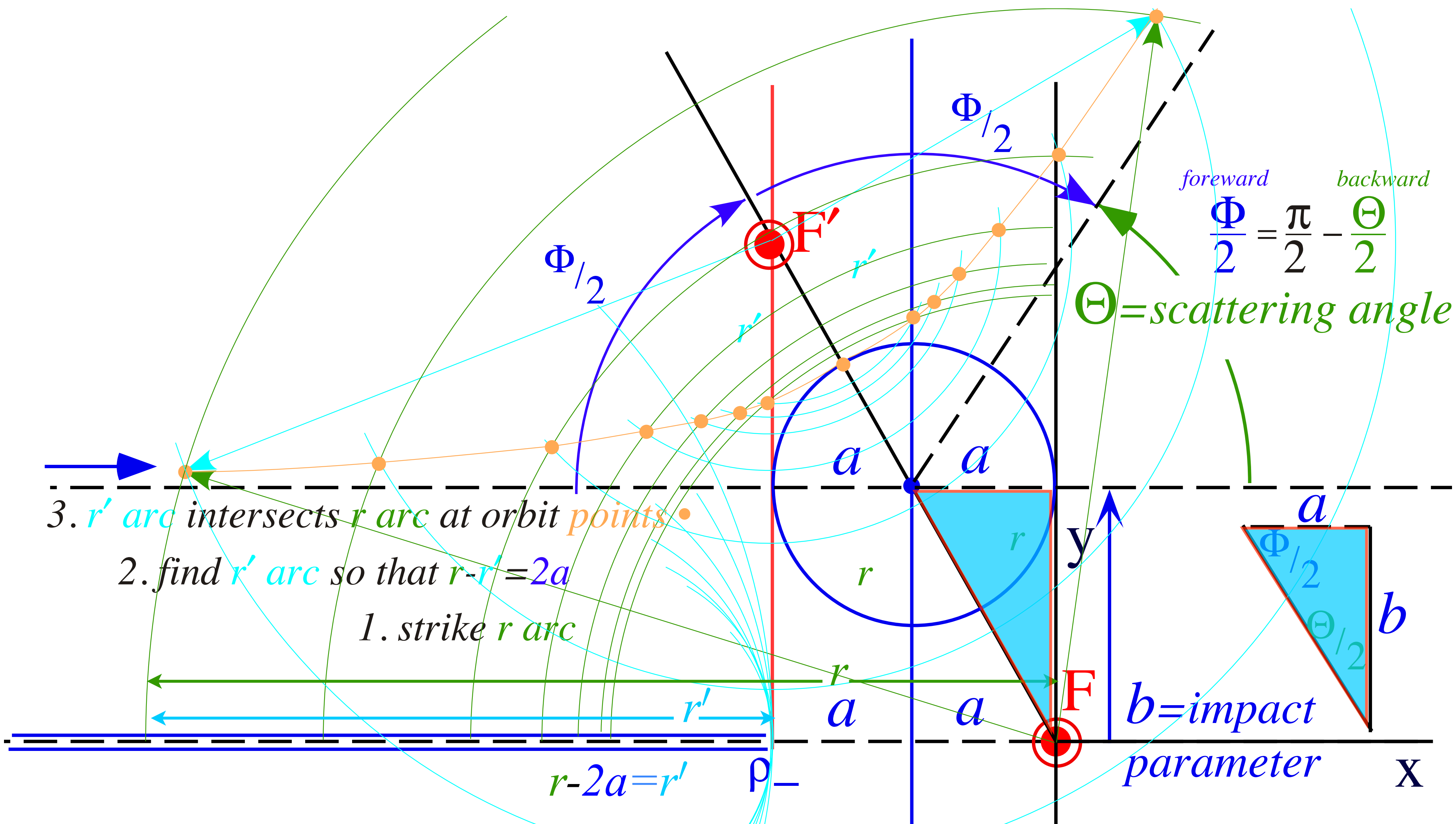

foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
a
b
Φ/2
Θ/2
y
C
B
b=impact
parameter
ρ_
F
x
Φ/2
Φ/2
F′
r′
r′
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
b=3a/2
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
a
Φ/2
Θ/2
b
r
r′
a
a
F
b=impact
parameter
Dead-on-path(b=0)
r-2a=r′
ρ_
x



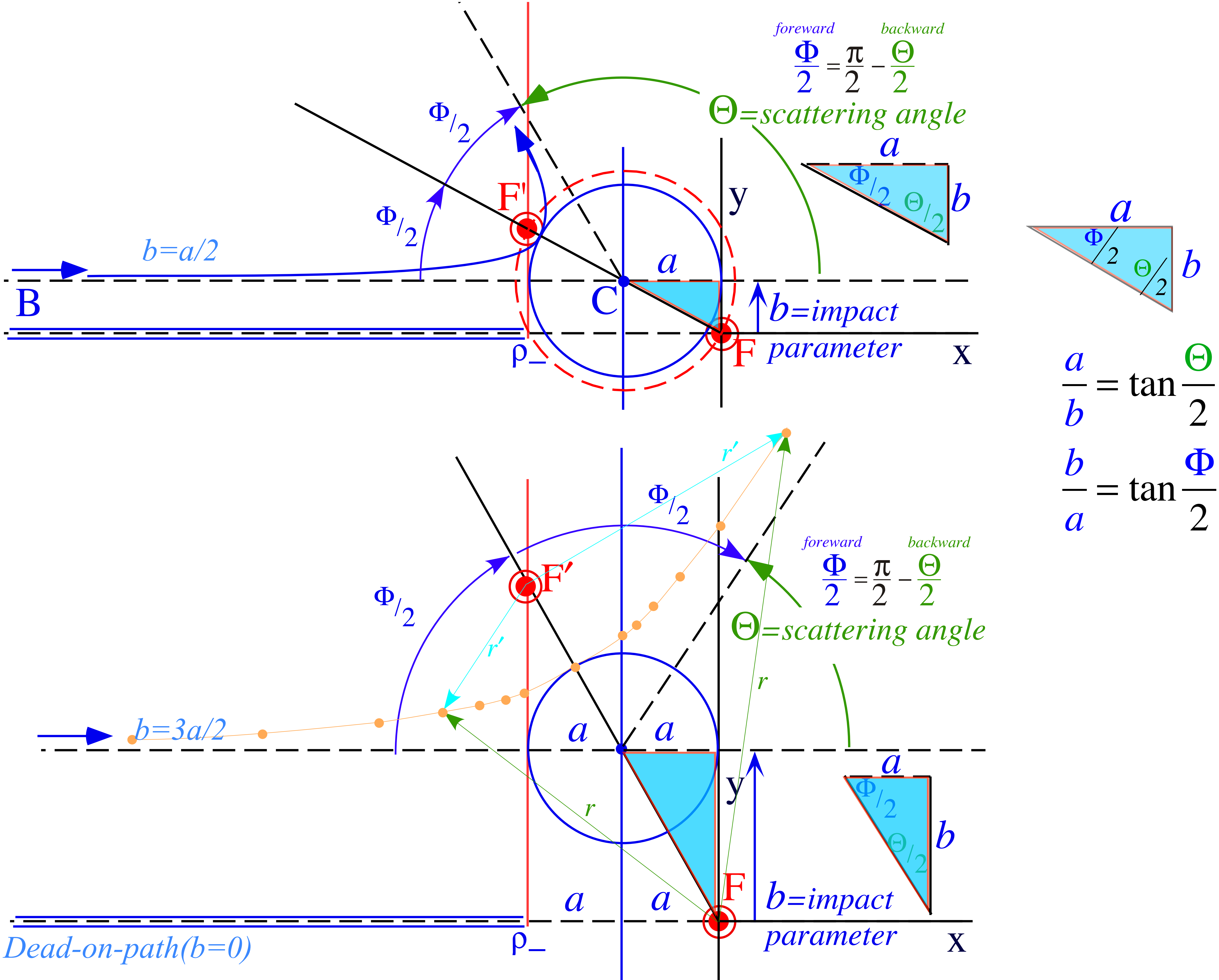
foreward backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
C
a
b
y
b=impact parameter
ρ_
F
x
Φ/2
F′
r′
Φ/2
b=3a/2
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r′
a
Dead-on-path(b=0)
r-2a=r′



foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Θ/2
y
a
B
C
b=impact
parameter
ρ_
F
x



Φ/2
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Φ/2
F′
r′
r′
Θ=scattering angle
a
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
Φ/2
Θ/2
b
r
r′
F
a
a
b=impact
parameter
r-2a=r′
ρ_
x


foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
b=a/2
B
C
a
y
b
b=impact
parameter
ρ_
F
x
a
b
Φ/2
Θ/2
a/b = tan Θ/2
b/a = tan Φ/2
r'
Φ/2
F'
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
r'
r
b=3a/2
a
a
r
y
a
Φ/2
Θ/2
b
F
a
a
b=impact
parameter
x
ρ_
Dead-on-path(b=0)

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

➡ *Parabolic "kite" and orbital envelope geometry*

Differential and total scattering cross-sections

Eccentricity vector **ε** *and* (ε,λ)*-geometry of orbital mechanics*

Projection **ε•r** *geometry of* **ε**-*vector and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

Projection **ε•p** *geometry of* **ε**-*vector and momentum* **p**=*m***v**

General geometric orbit construction using **ε**-*vector and* (γ,*R*)*-parameters*

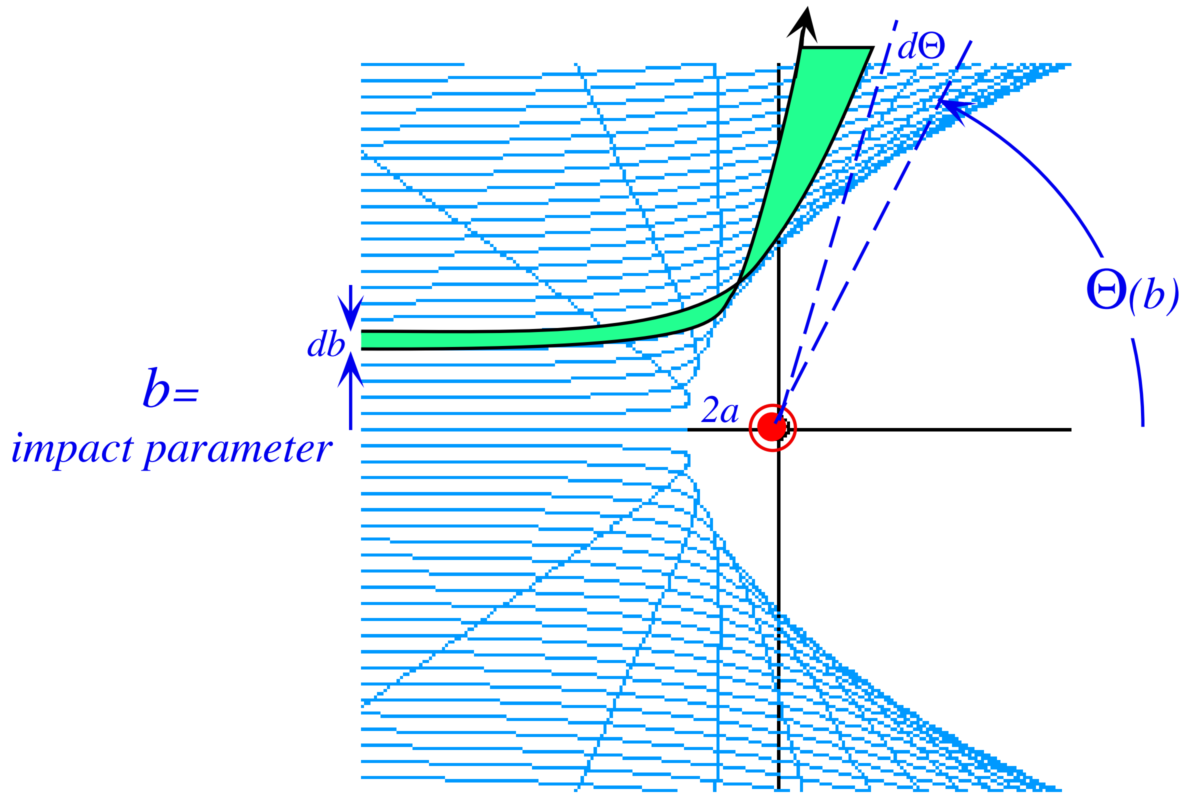
Derivation of **ε**-*construction by analytic geometry*

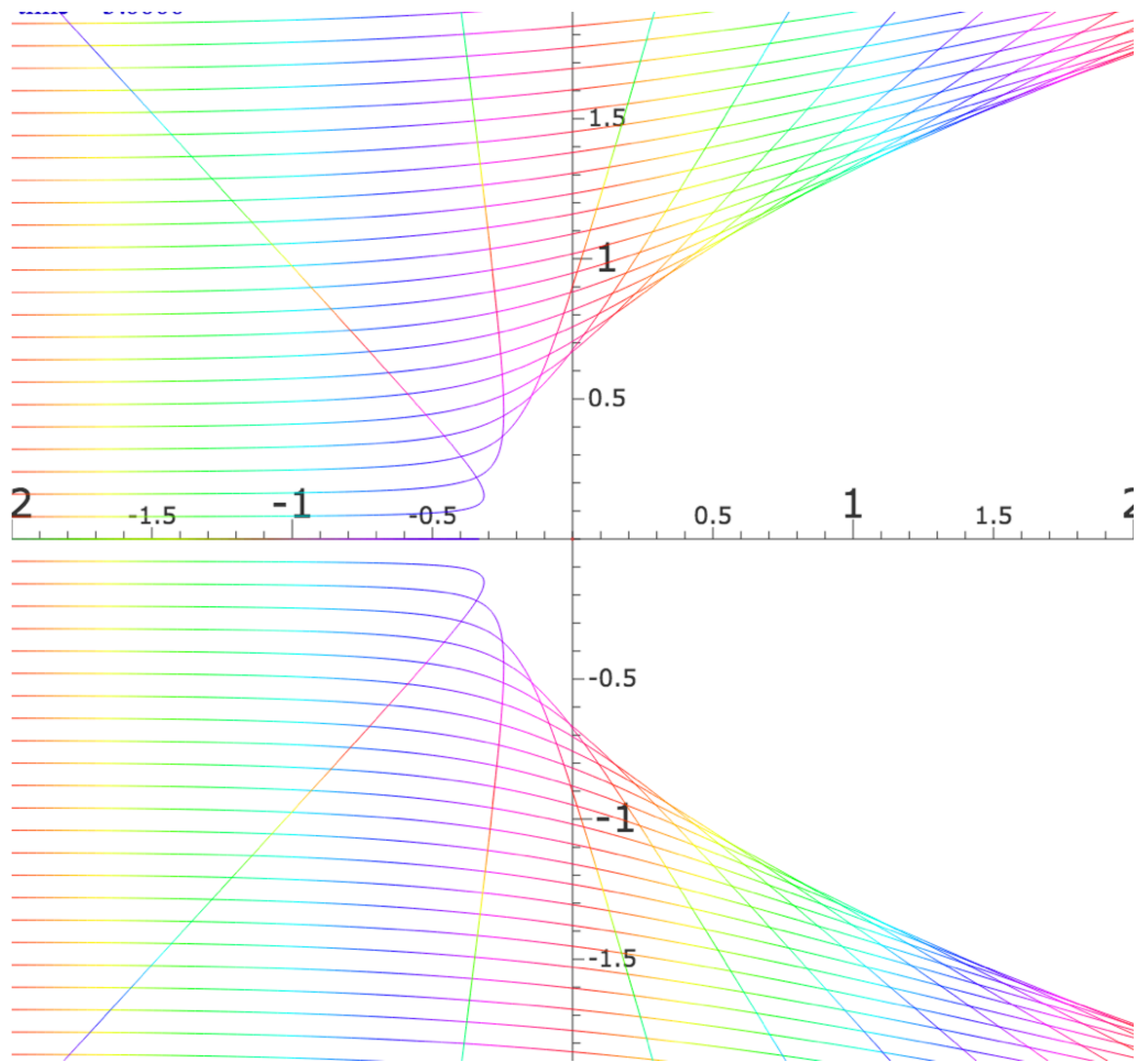
Coulomb orbit algebra of **ε**-*vector and Kepler dynamics of momentum* **p**=*m***v**

Example of complete (**r**,**p**)*-geometry of elliptical orbit*

Connection formulas for (γ,*R*)*-parameters with (a,b) and* (ε,λ)

Rutherford scattering geometry





http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=Rutherford

Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Volcanoes of Io (Paths=180, No color quant.) | Parabolic Fountain (Uniform) | Space Bomb (Coulomb) | Exploding Starlet (IHO) | Synchrotron Motion (Crossed E & B fields)

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Rutherford scattering | 2-Electron Orbits

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Atomic Orbits

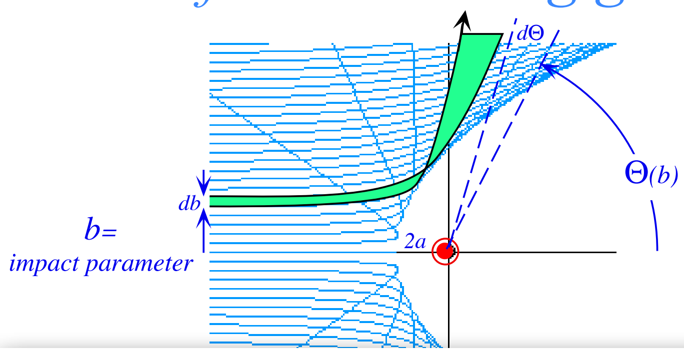
Chapter 4 Molecular Ion Orbits

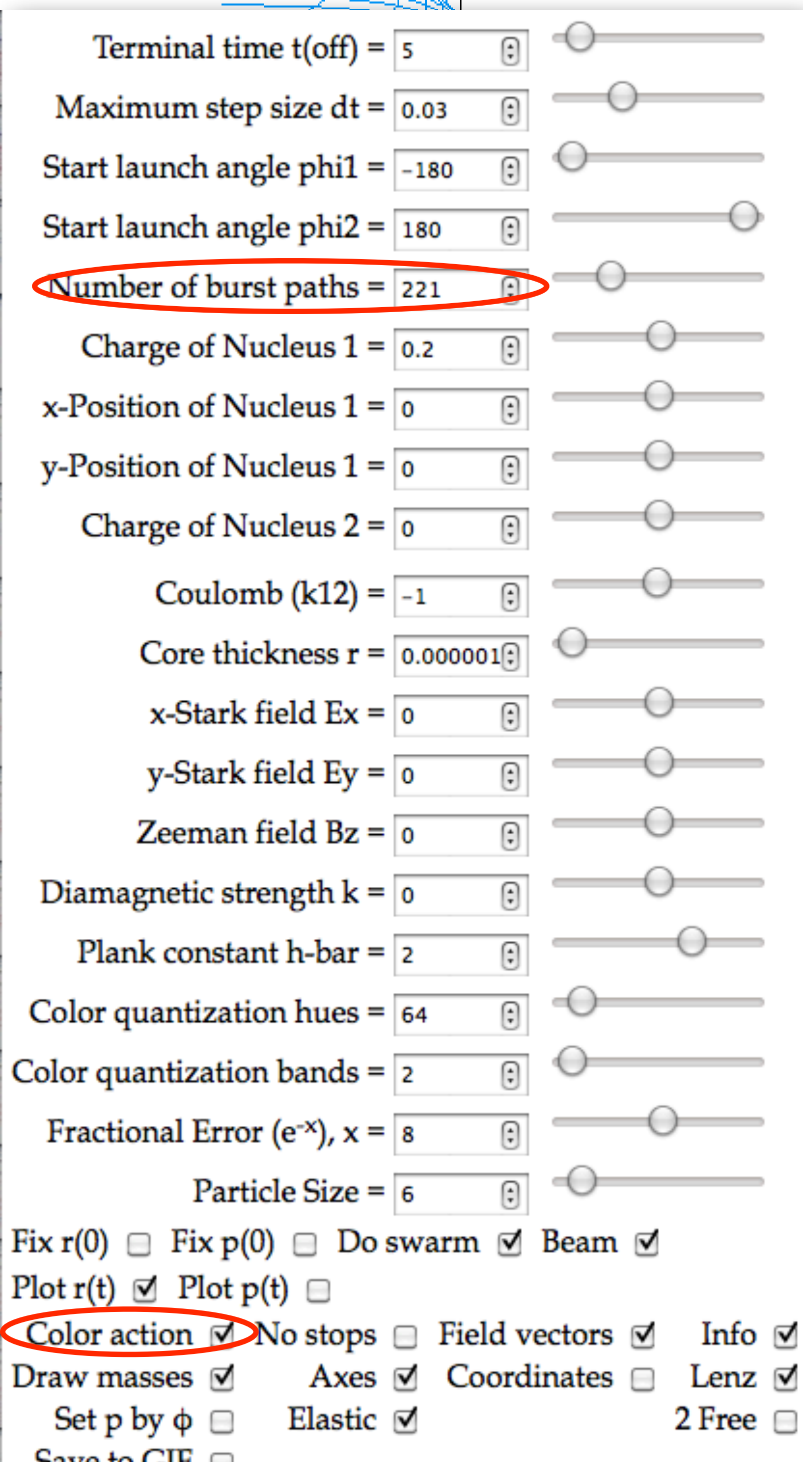
Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

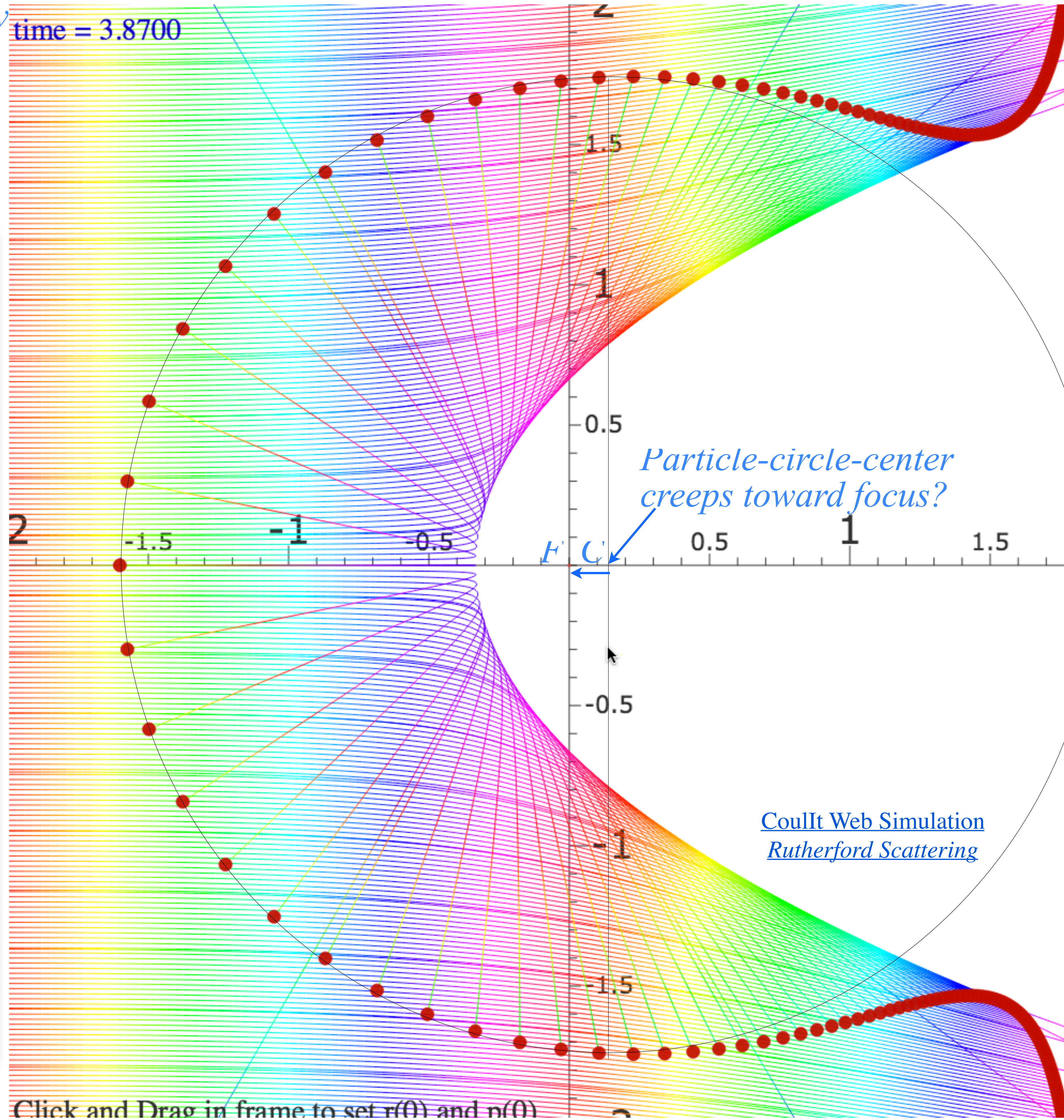
Molecular Ion Orbits

Oscillator Scattering | 2-Particle Orbits | 2-Particle Collision

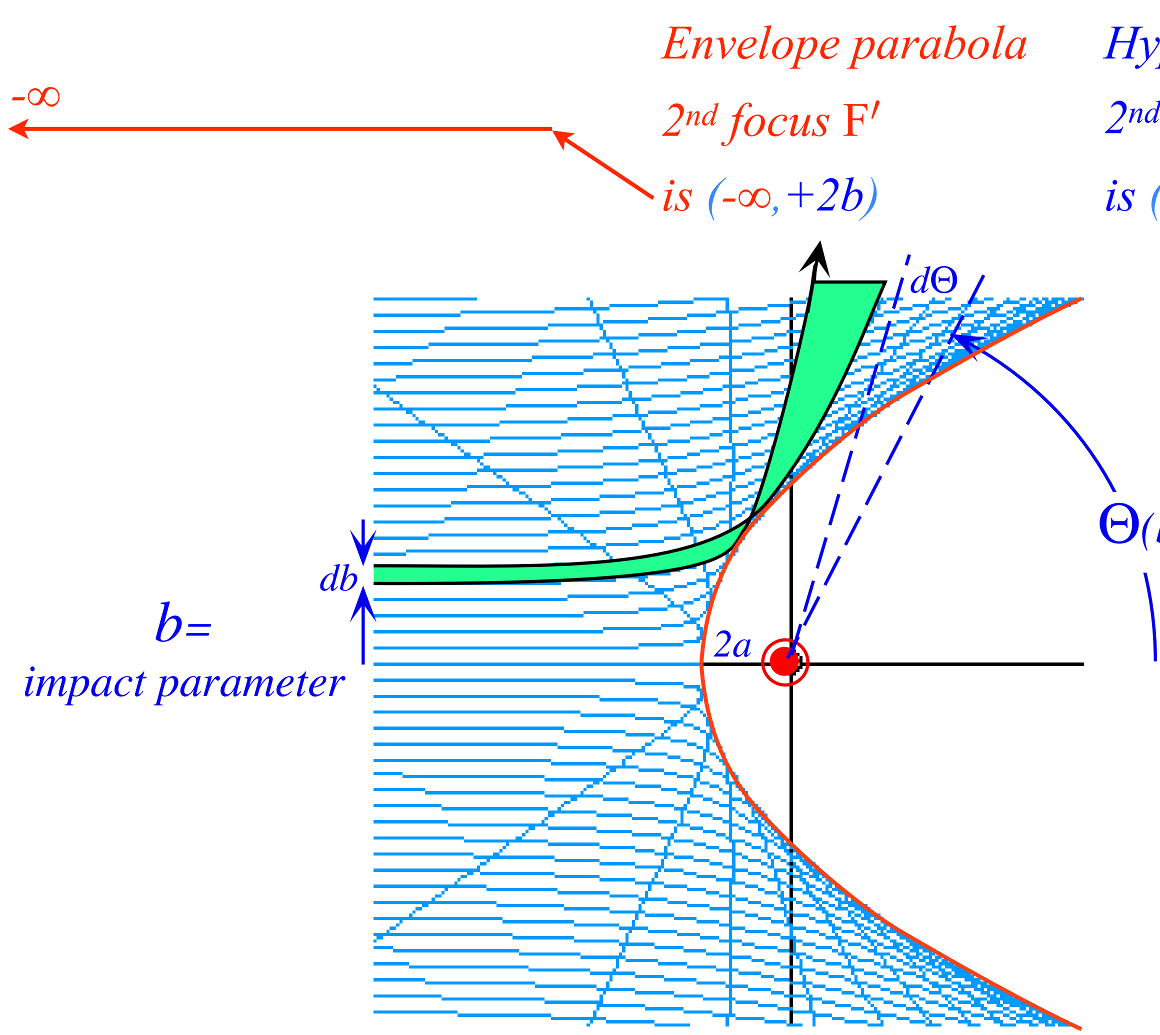
Rutherford scattering geometry



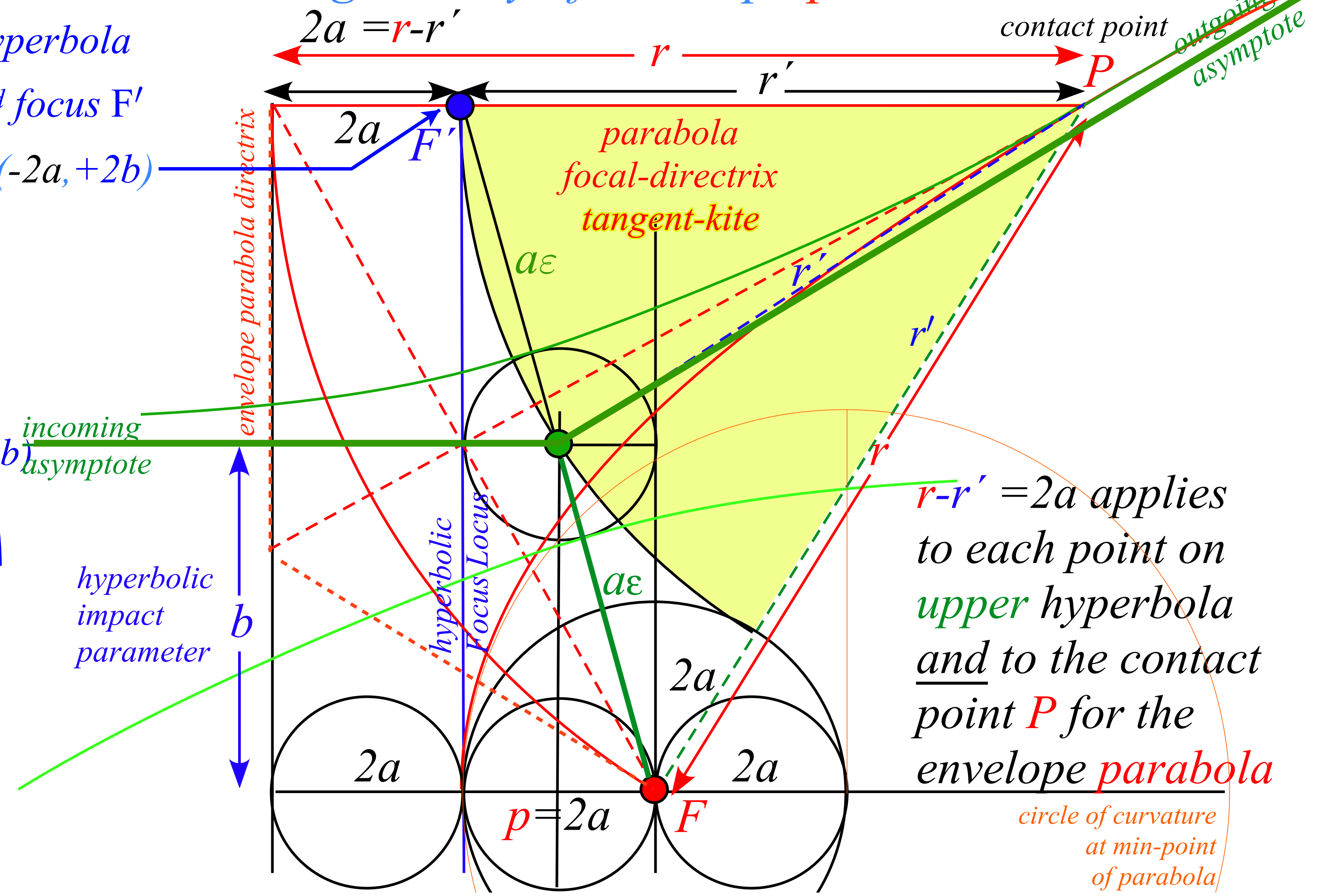




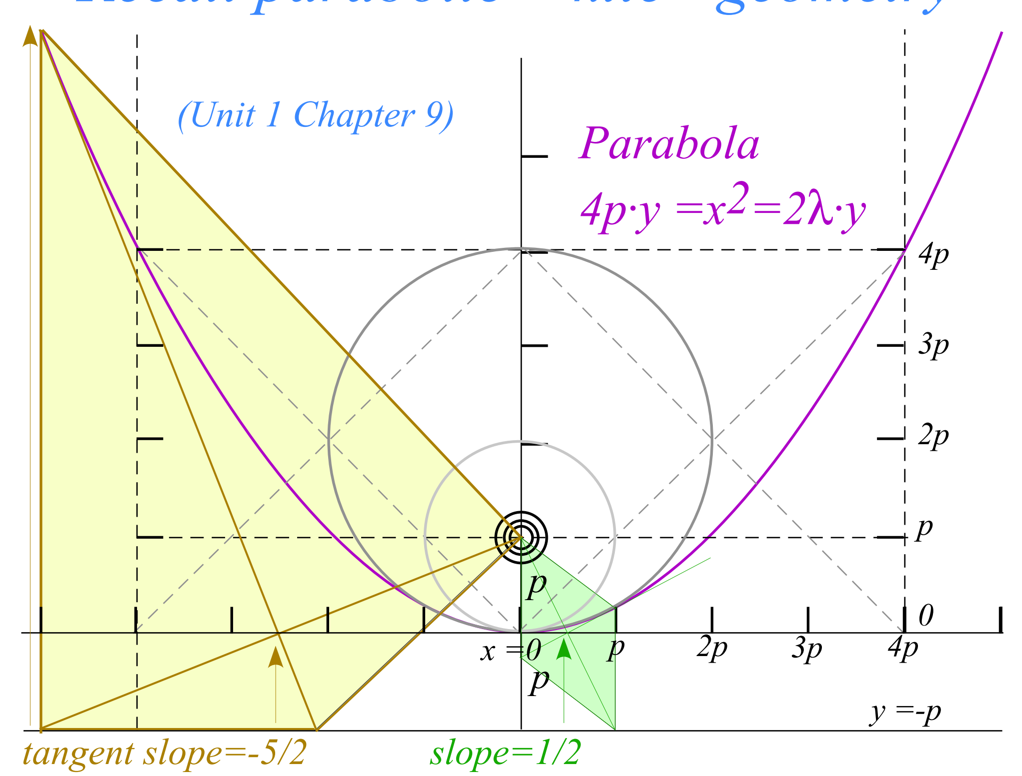
Rutherford scattering geometry



"Kite" geometry of envelope parabola

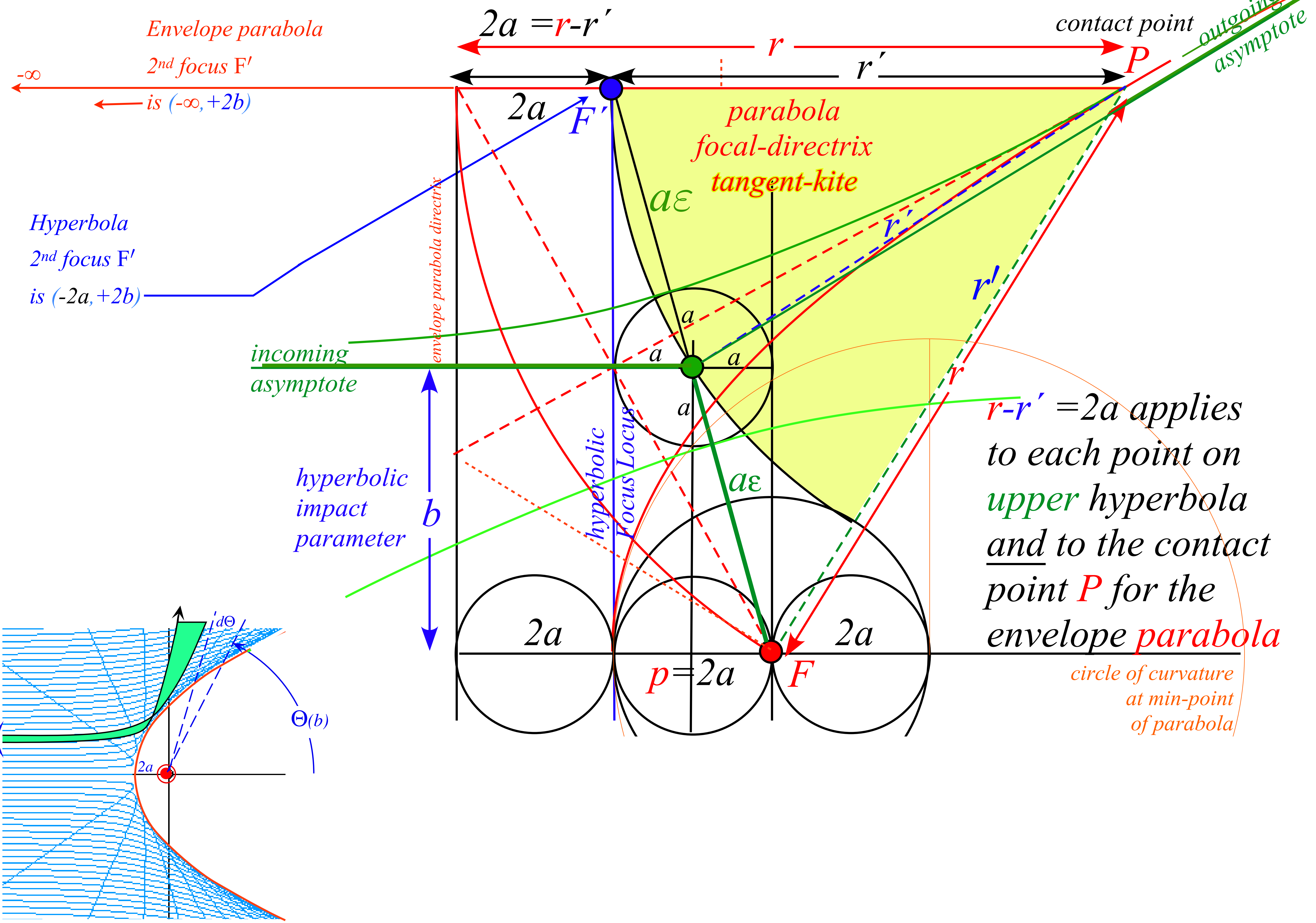


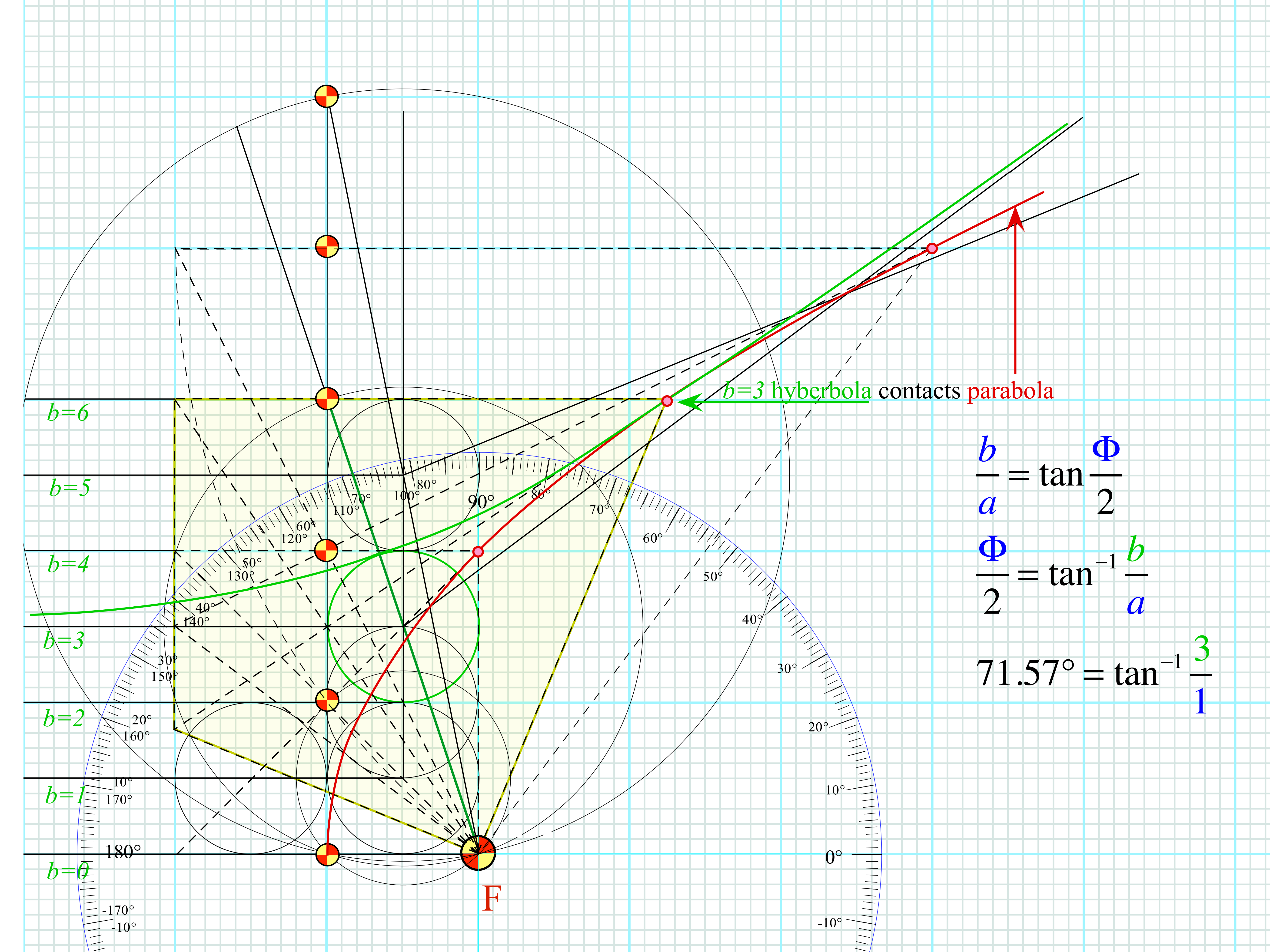
Recall parabolic " kite" geometry



Rutherford scattering geometry

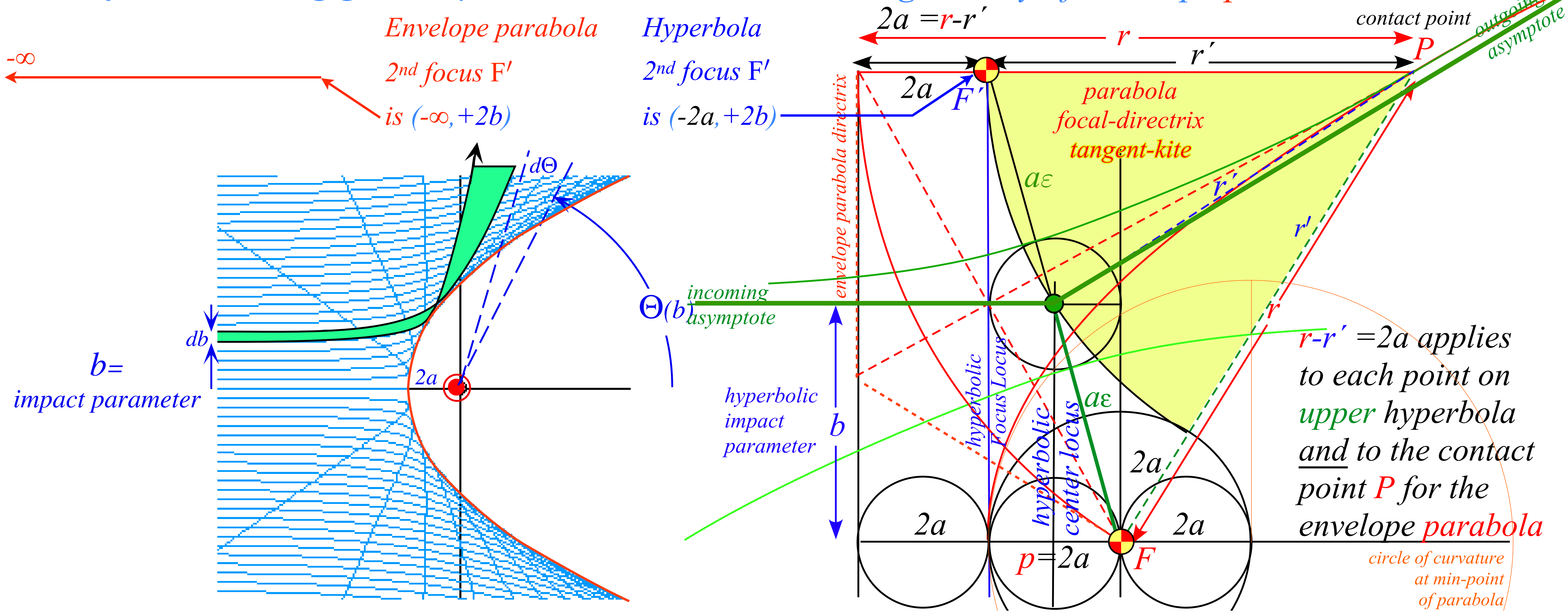
"Kite" geometry of envelope parabola



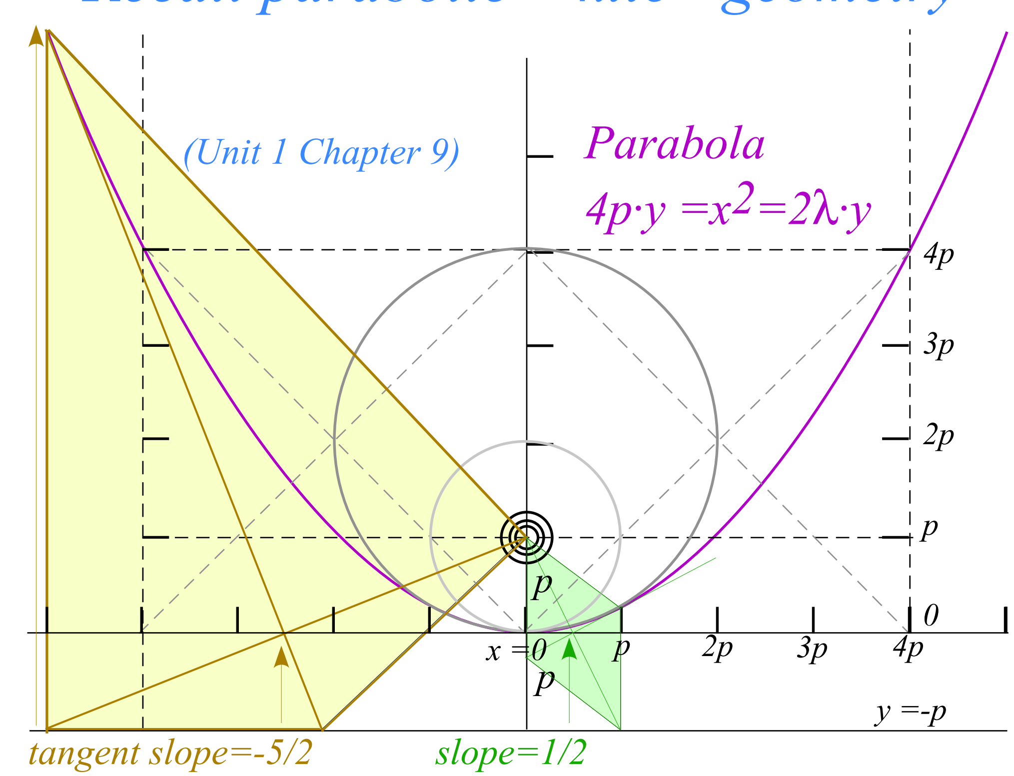

b=6
b=5
b=4
b=3
b=2
b=1
b=0
b=3 hyberbola contacts parabola
F
90°
180°
0°
b/a = tan Φ/2
Φ/2 = tan⁻¹ b/a
71.57° = tan⁻¹ 3/1

Rutherford scattering geometry

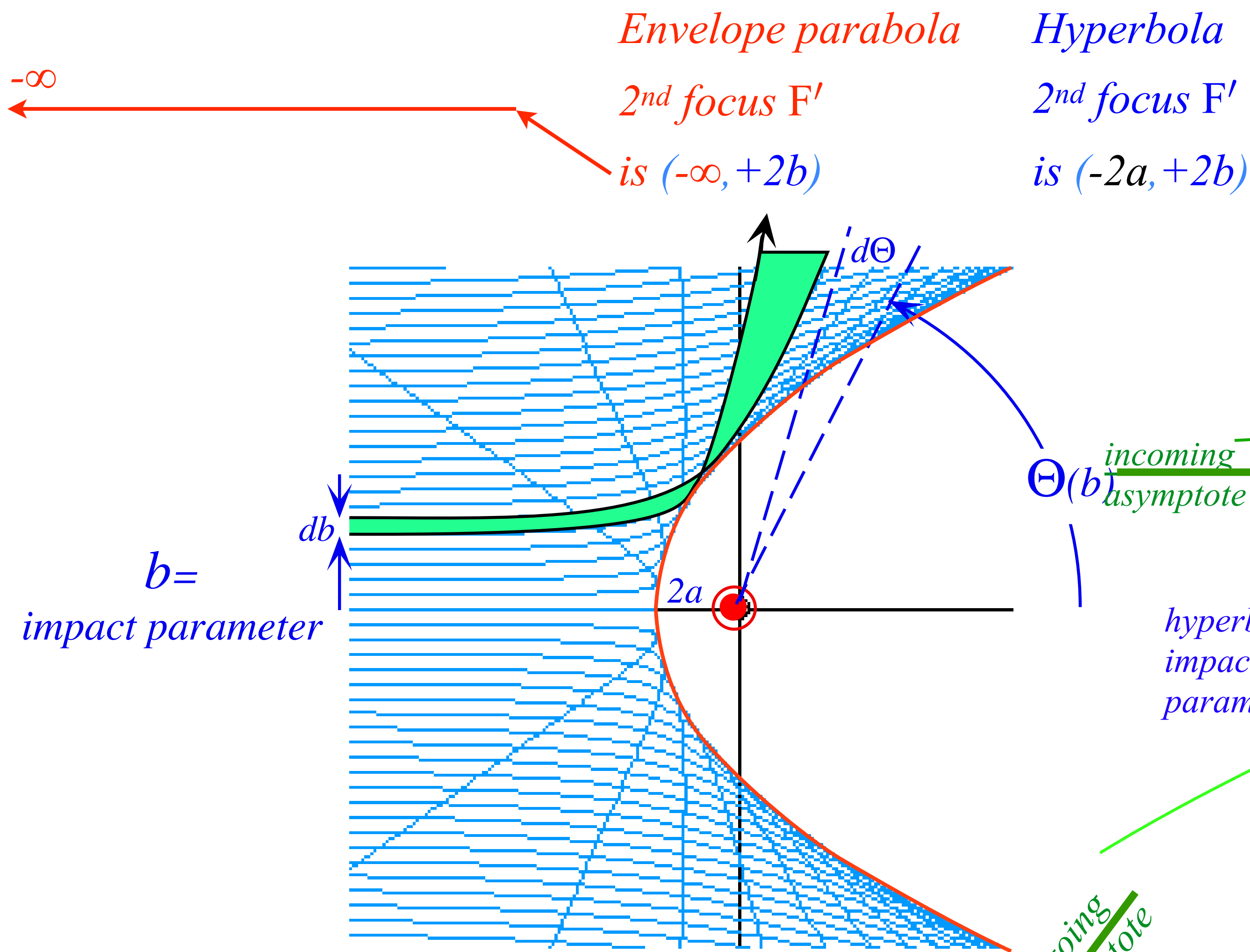
"Kite" geometry of envelope parabola



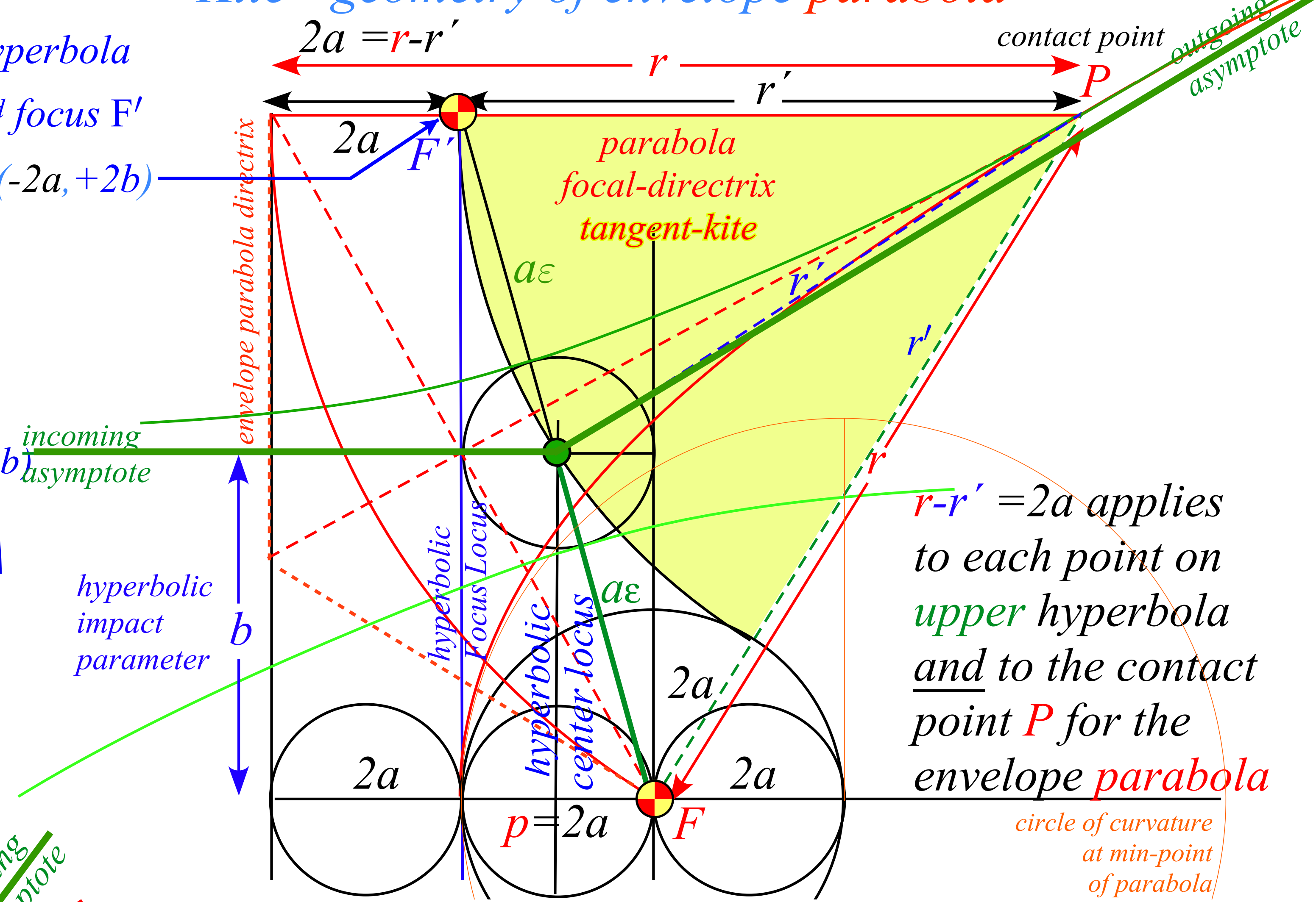
Recall parabolic " kite" geometry



Rutherford scattering geometry

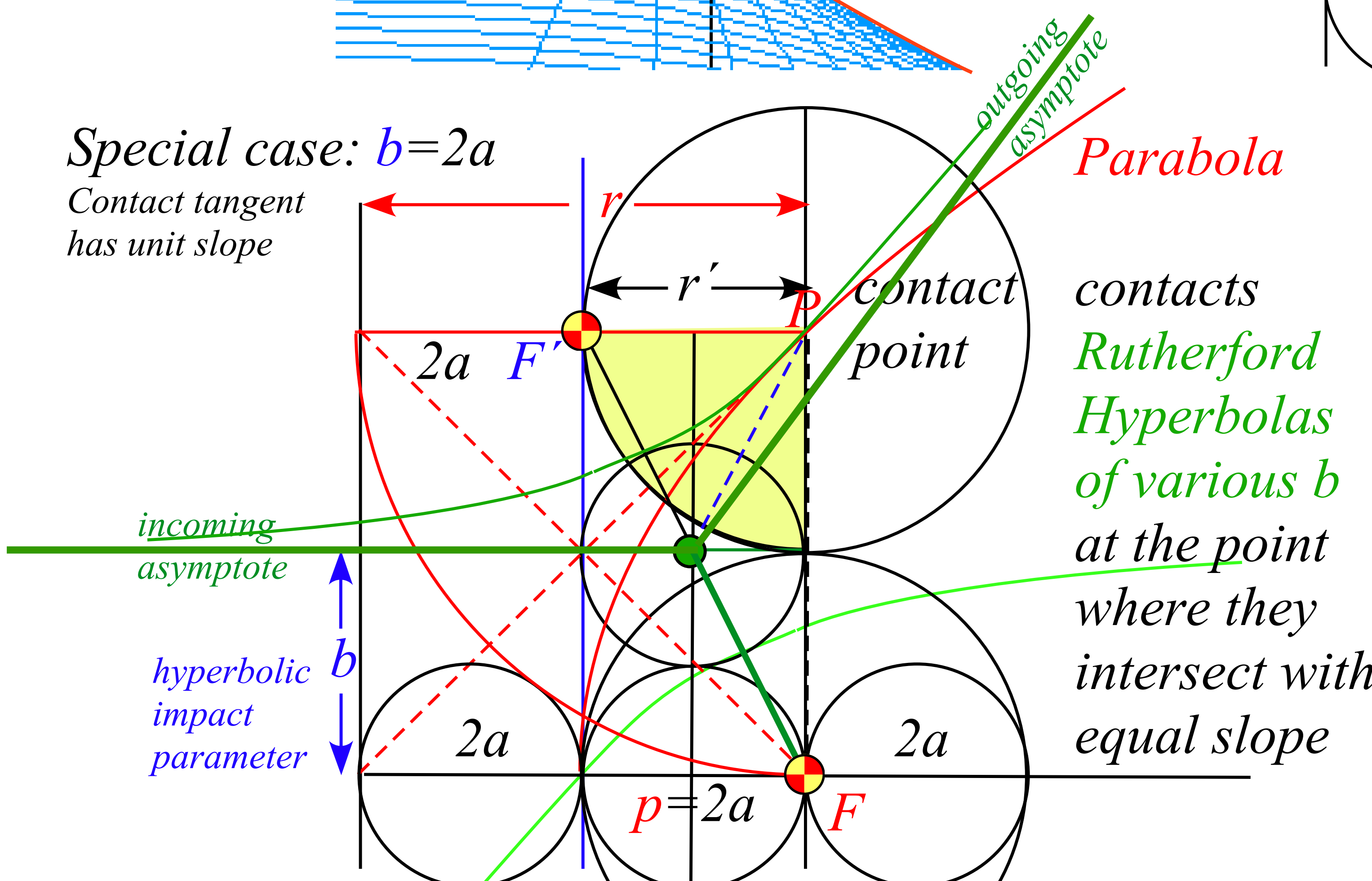


"Kite" geometry of envelope parabola

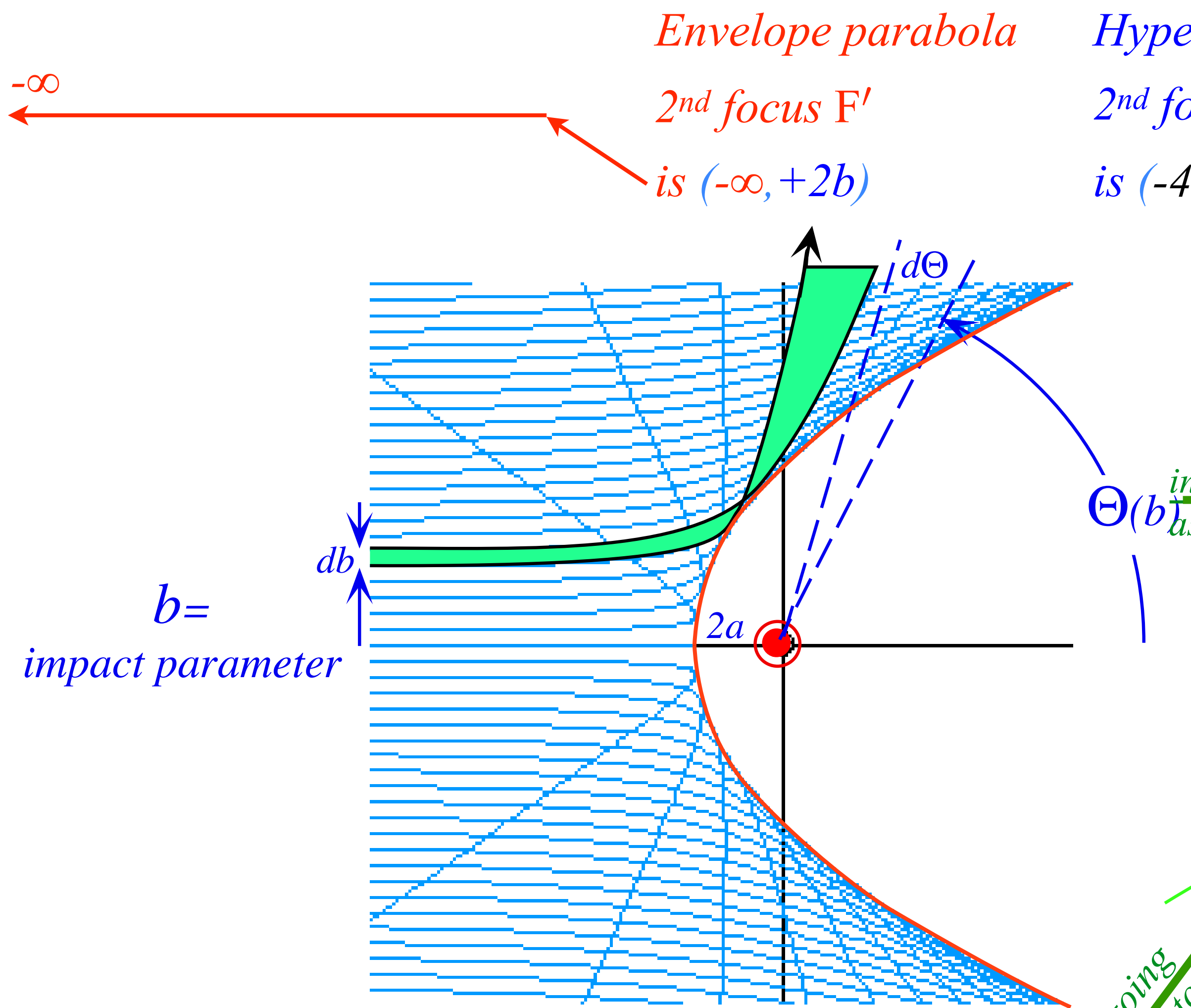


Special case: $b=2a$

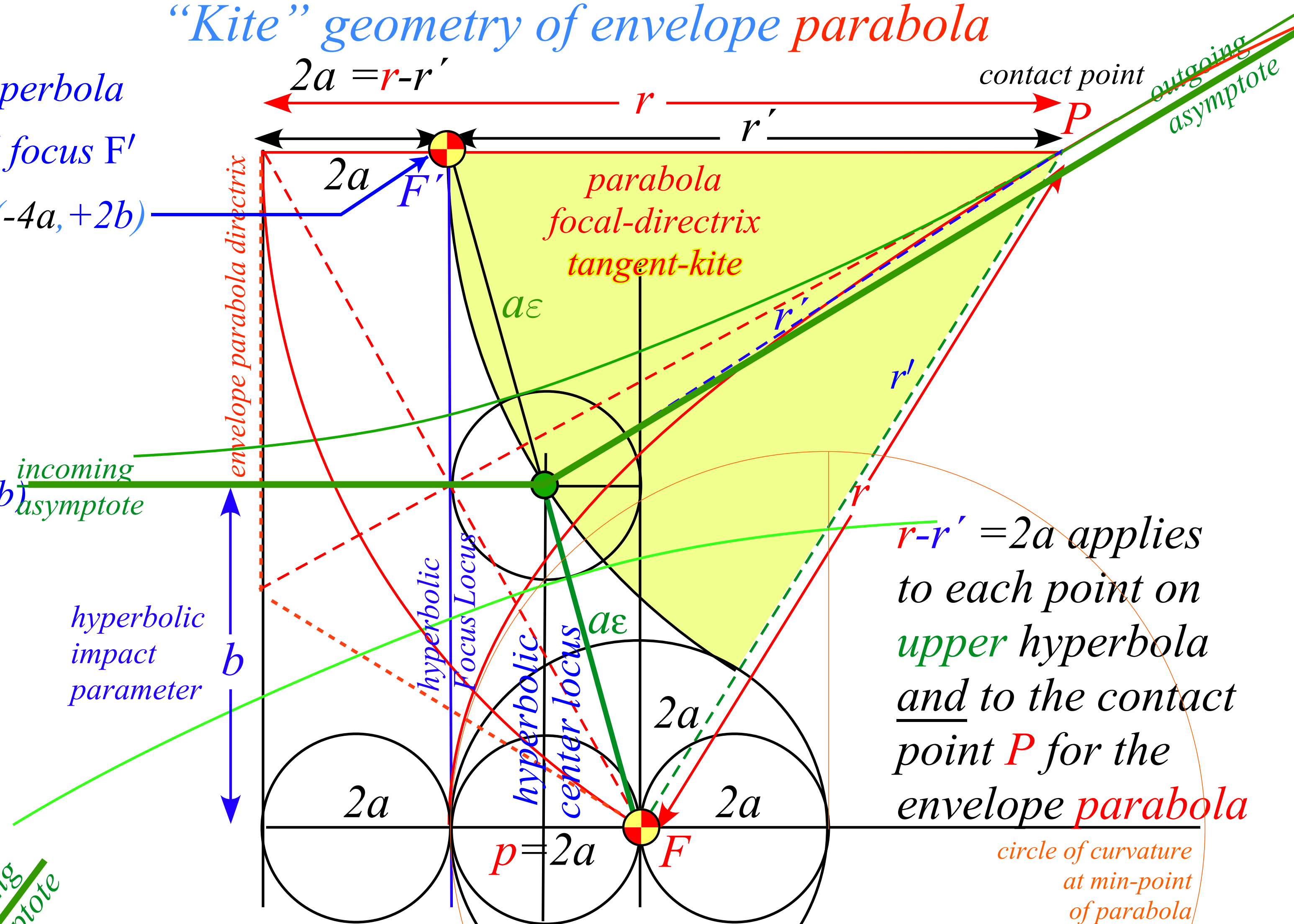
Contact tangent has unit slope



Rutherford scattering geometry



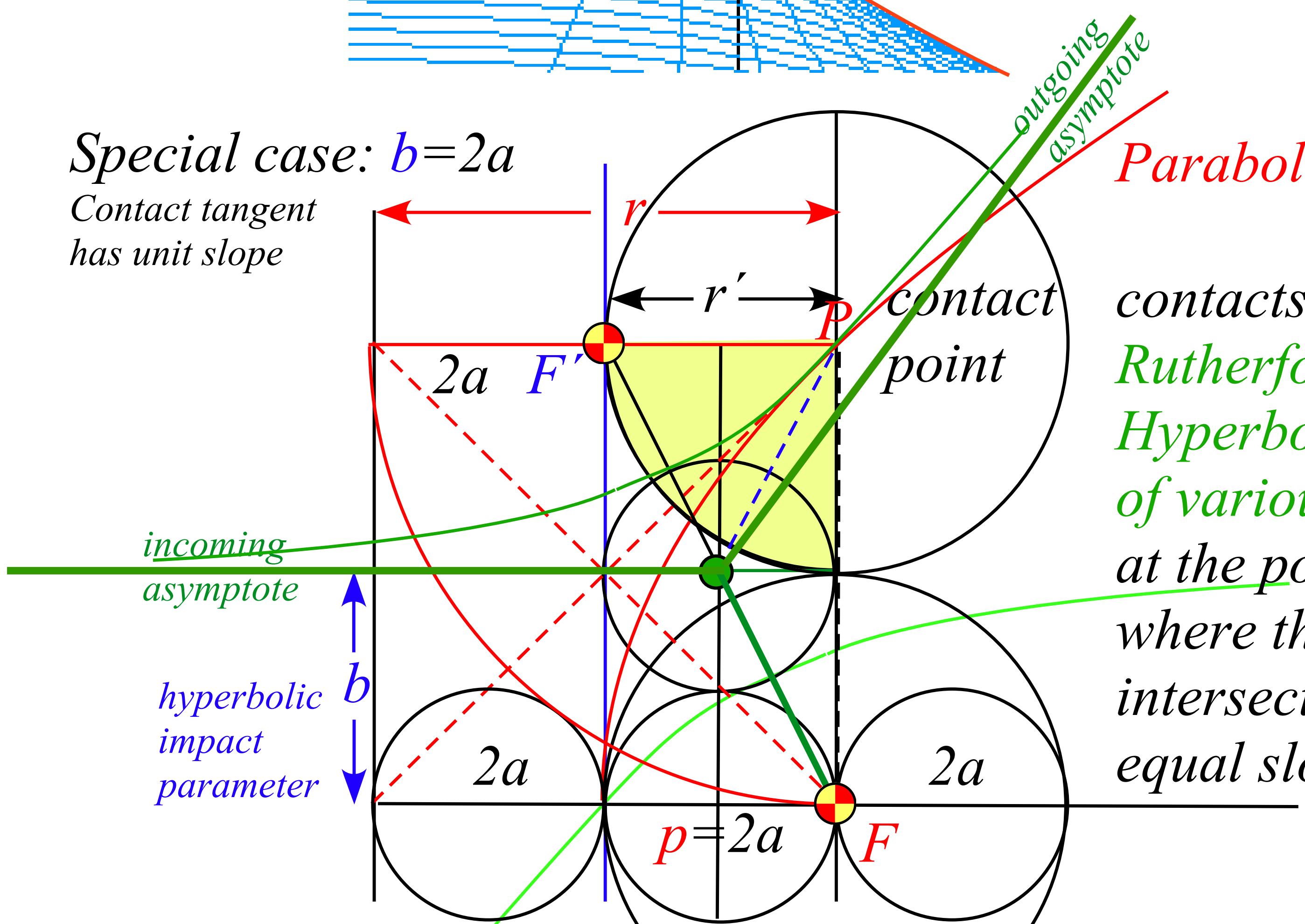
"Kite" geometry of envelope parabola



r-r' =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola

Special case: b=2a

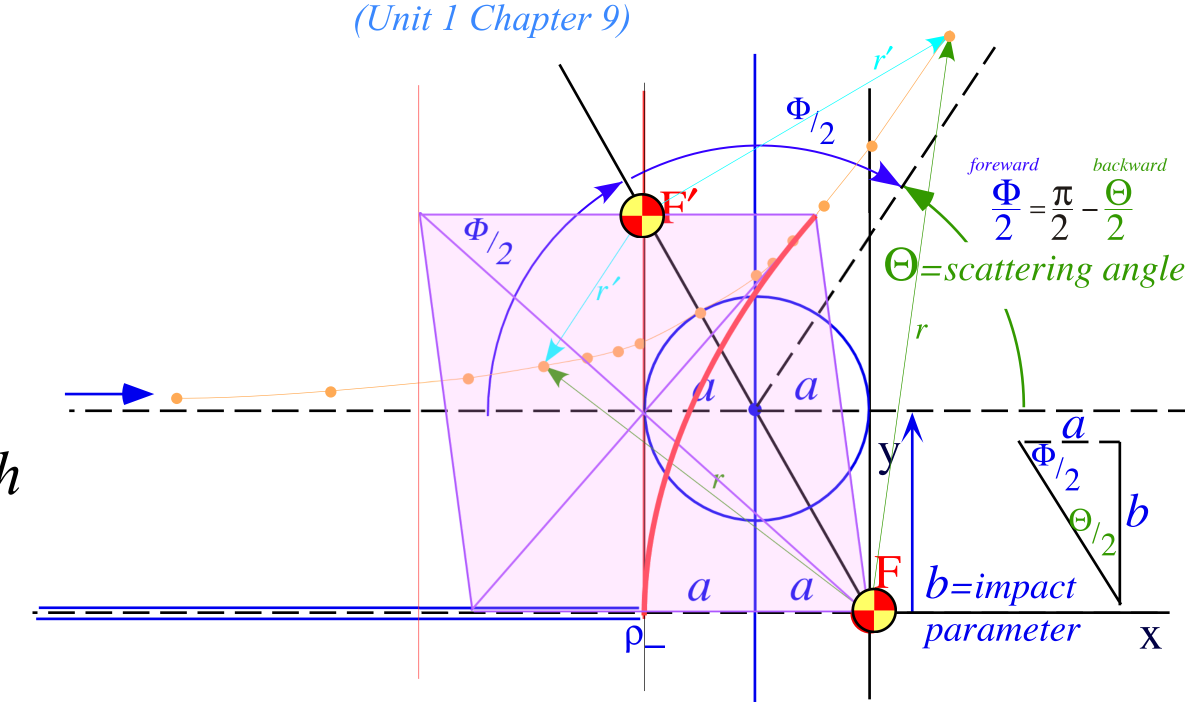
Contact tangent has unit slope



contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Recall parabolic " kite" geometry

(Unit 1 Chapter 9)



Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

➔ *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

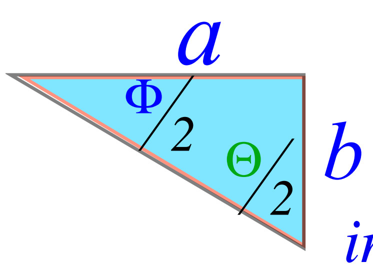
Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

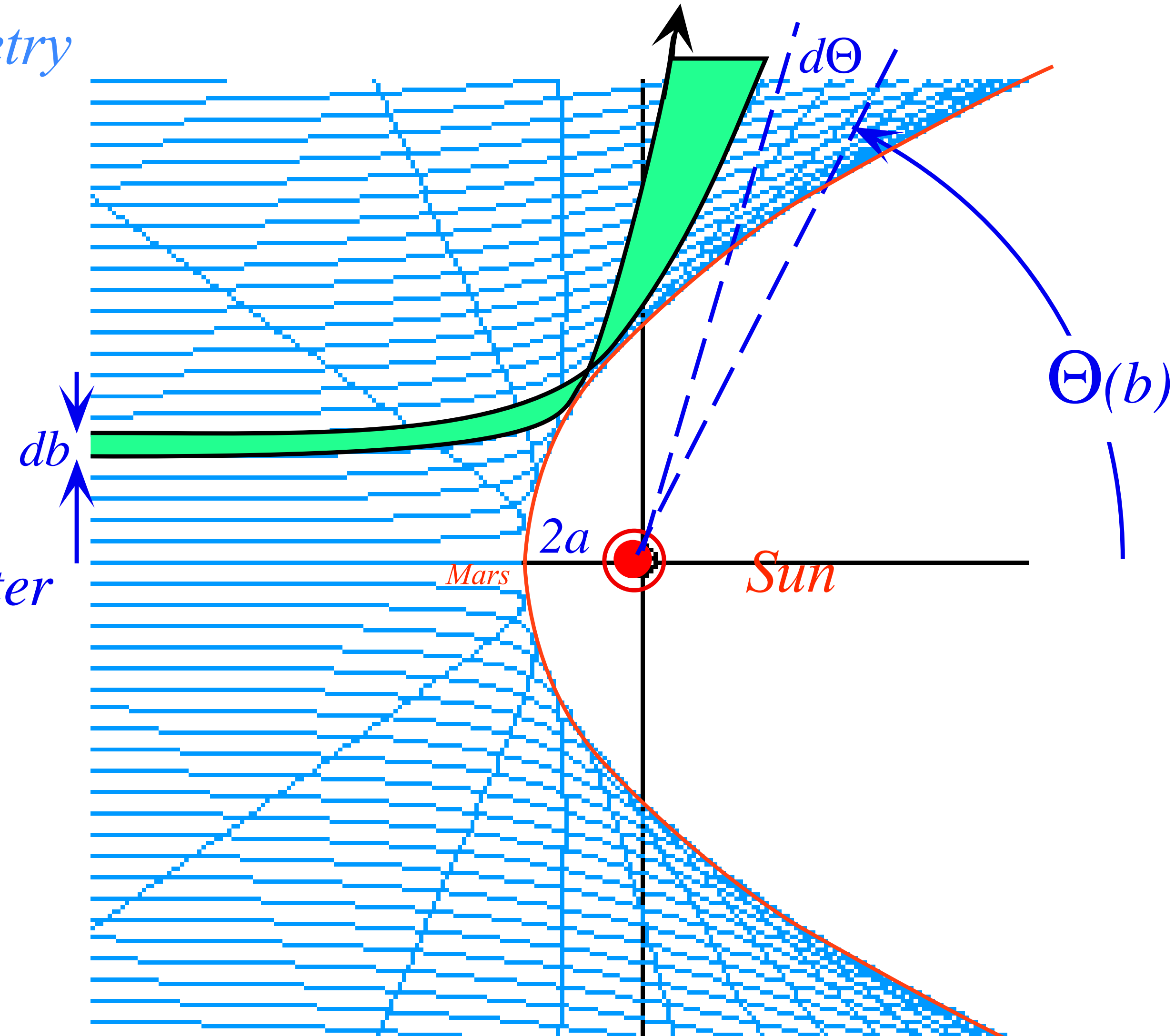
Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Rutherford scattering geometry



b= impact parameter

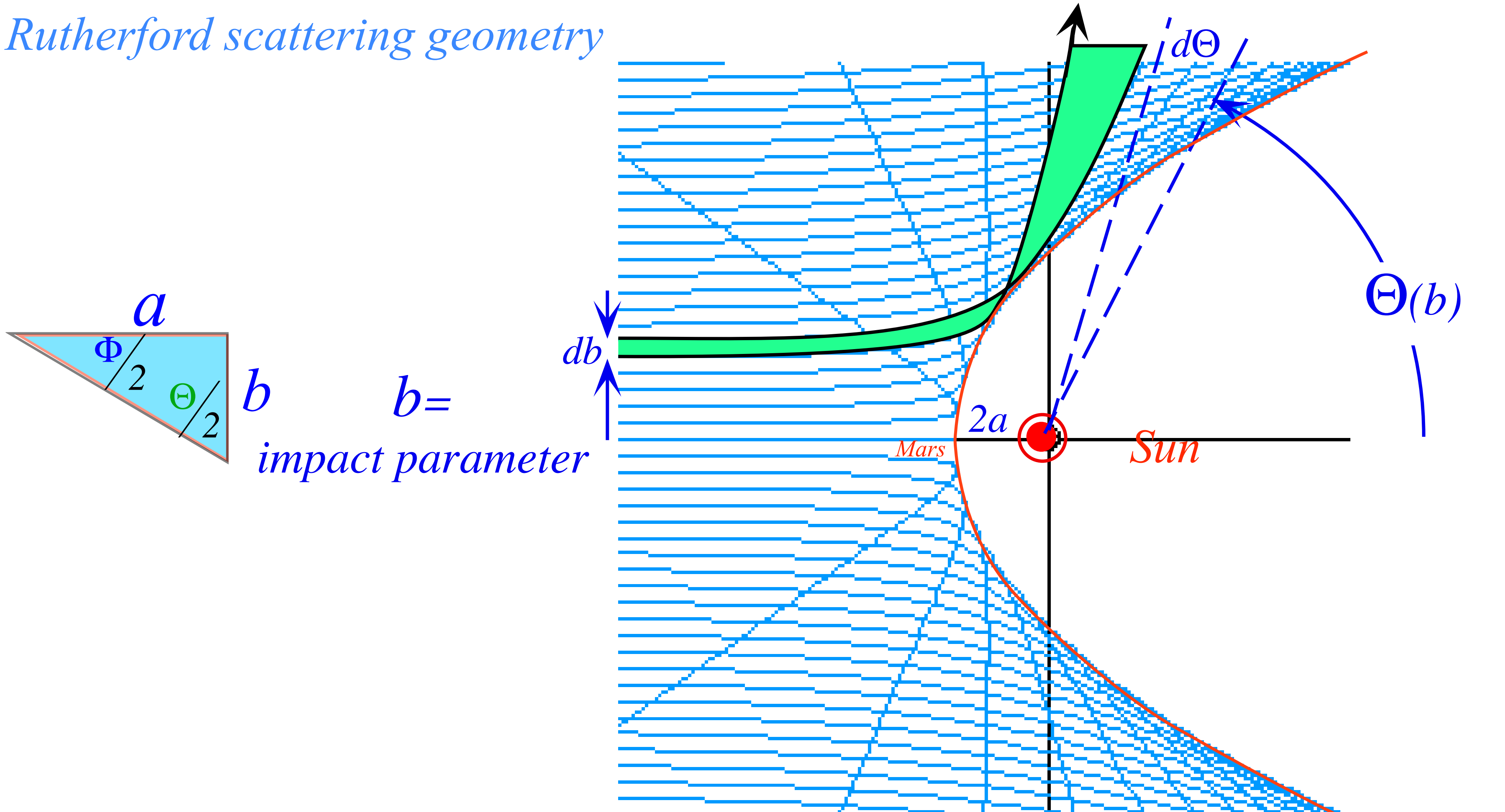


Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma = b \cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Rutherford scattering geometry



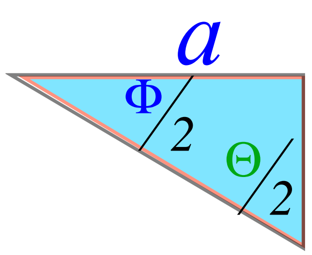
Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy.
Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma=b\cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

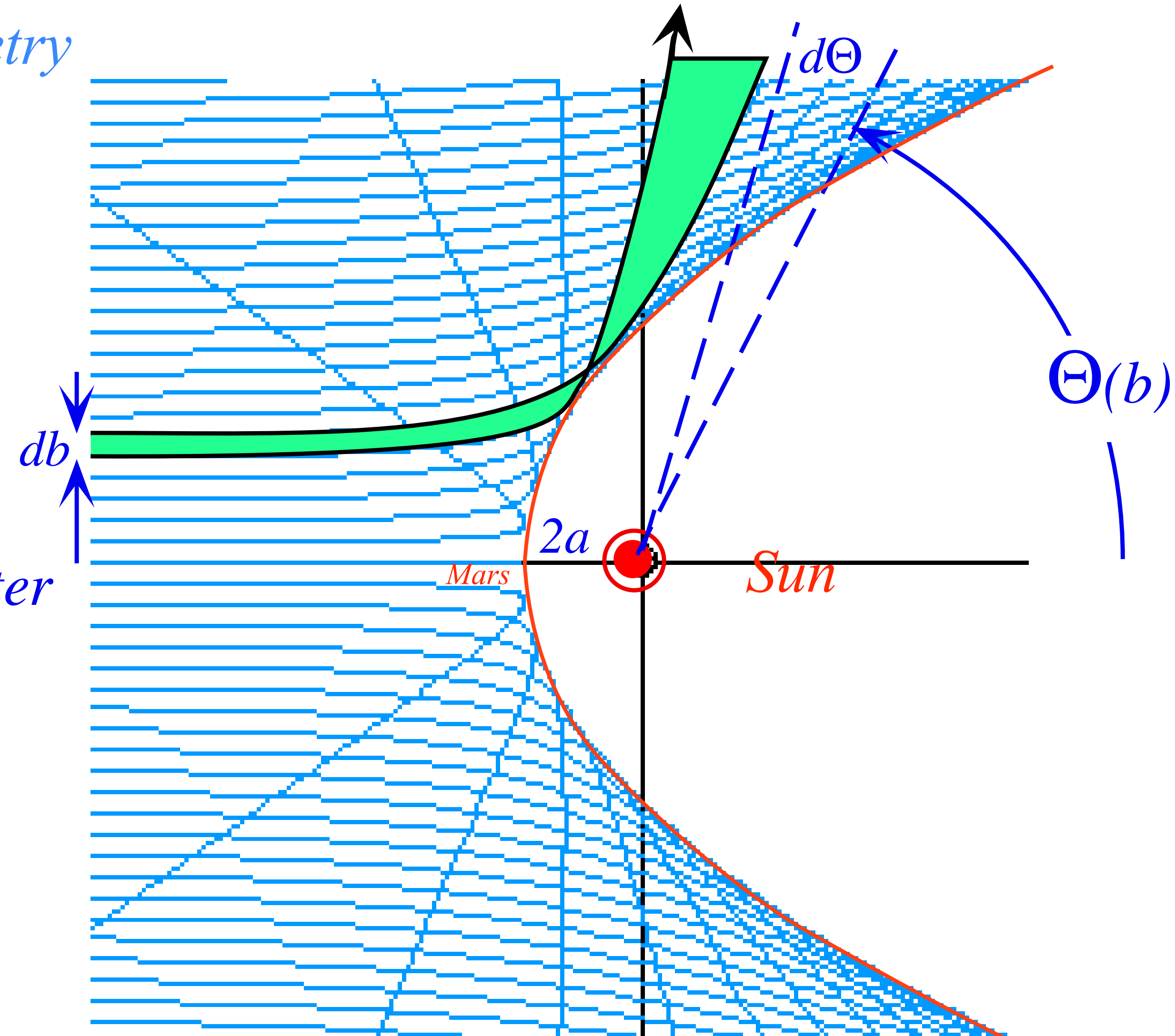
Ratio $\dfrac{d\sigma}{d\Omega} = \dfrac{b\, db\, d\varphi}{\sin\Theta d\Theta d\varphi} = \dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

Rutherford scattering geometry



b= *impact parameter*

$$\frac{b}{a} = \cot\frac{\Theta}{2}$$



Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

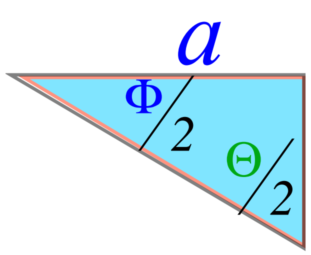
Incremental window $d\sigma = b\cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Ratio $$\frac{d\sigma}{d\Omega} = \frac{b\,db\,d\varphi}{\sin\Theta d\Theta d\varphi} = \frac{b}{\sin\Theta}\frac{db}{d\Theta}$$ is called the *differential scattering cross-section (DSC)*

Geometry: $b = a\cot\frac{\Theta}{2}$

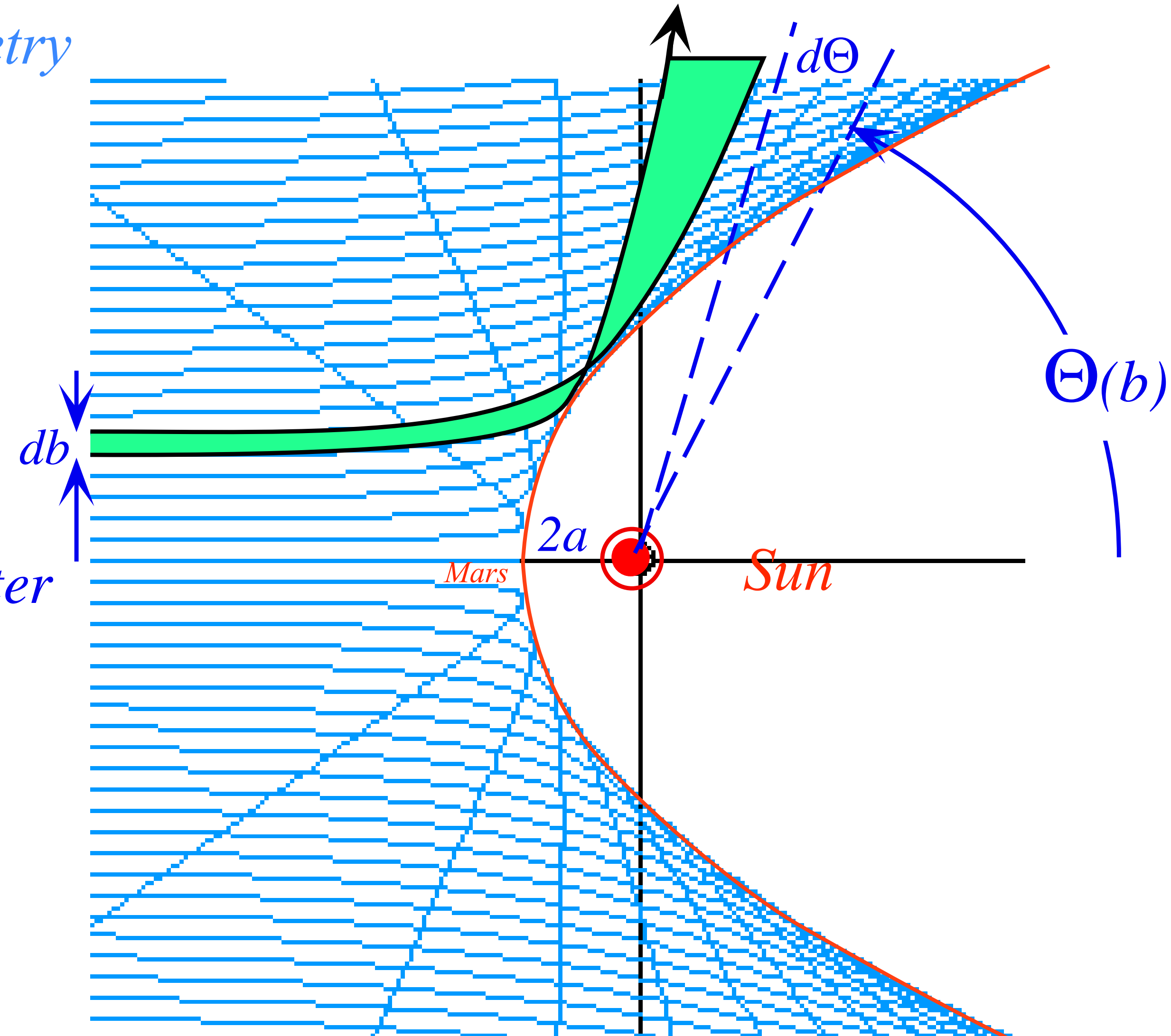
with: $$\frac{db}{d\Theta} = \frac{-a}{2}\csc^2\frac{\Theta}{2}$$

Rutherford scattering geometry



$b=$ *impact parameter*

$$\frac{b}{a}=\cot\frac{\Theta}{2}$$



Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma=b\cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA=R^2\sin\Theta d\Theta d\varphi=R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega=\sin\Theta d\Theta d\varphi$

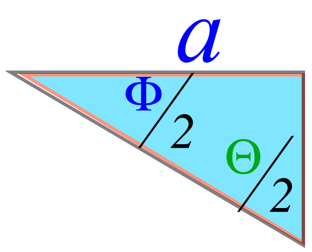
Ratio $\dfrac{d\sigma}{d\Omega}=\dfrac{b\,db\,d\varphi}{\sin\Theta d\Theta d\varphi}=\dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

Geometry: $b=a\cot\dfrac{\Theta}{2}=\dfrac{k}{2E}\cot\dfrac{\Theta}{2}$

with: $\dfrac{db}{d\Theta}=\dfrac{-a}{2}\csc^2\dfrac{\Theta}{2}=\dfrac{-a}{2\sin^2\dfrac{\Theta}{2}}$

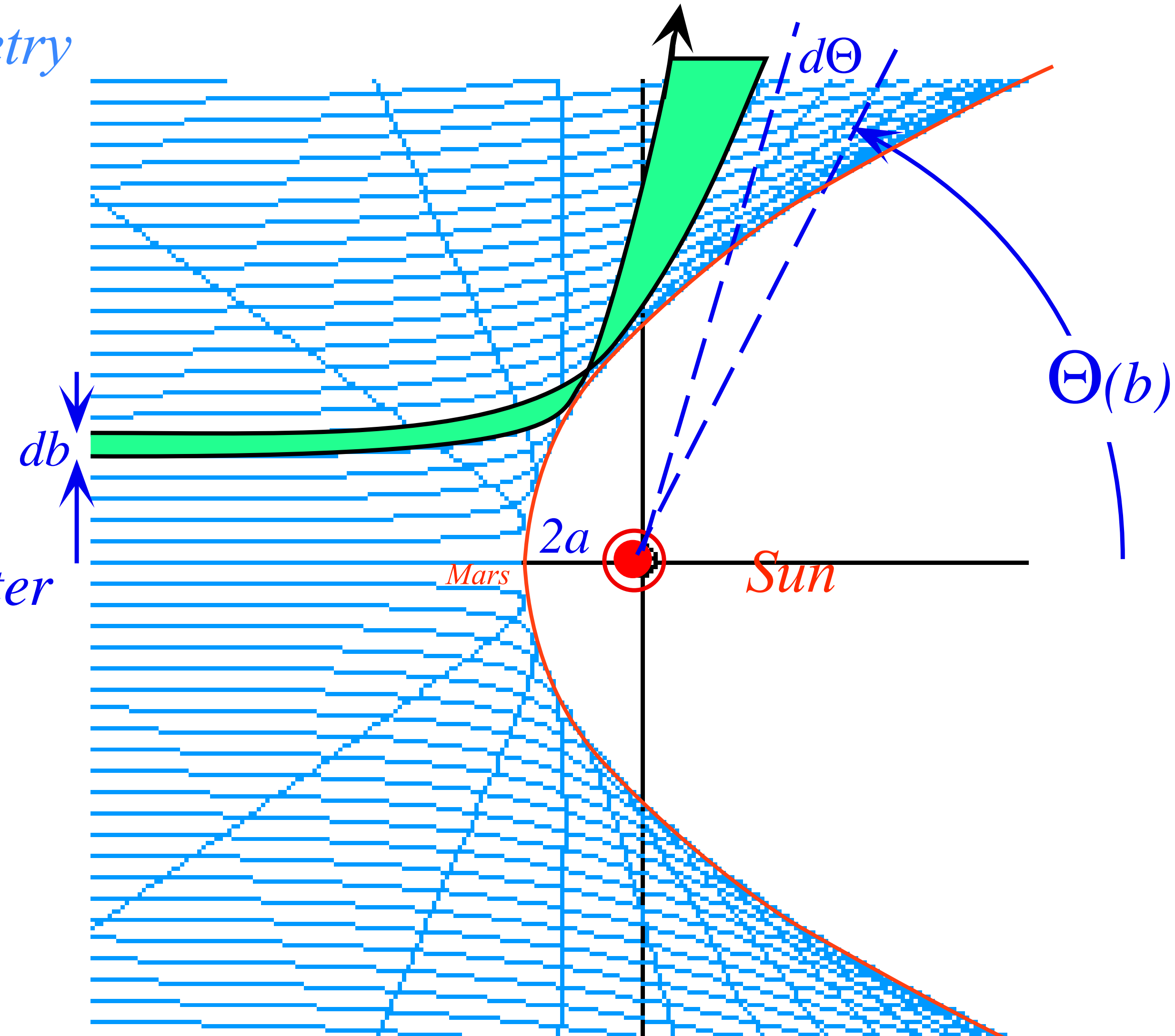
$(\textit{Never forget!}: a=\dfrac{-k}{2E})$

Rutherford scattering geometry



$b=$ *impact parameter*

$$\frac{b}{a}=\cot\frac{\Theta}{2}$$



Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Incremental window $d\sigma=b\cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA=R^2\sin\Theta d\Theta d\varphi=R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega=\sin\Theta d\Theta d\varphi$

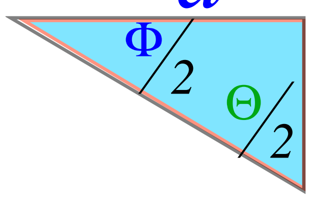
Ratio $\dfrac{d\sigma}{d\Omega}=\dfrac{b\,db\,d\varphi}{\sin\Theta d\Theta d\varphi}=\dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

Geometry: $b=a\cot\dfrac{\Theta}{2}=\dfrac{k}{2E}\cot\dfrac{\Theta}{2}$ gives the *Rutherford DSC*. $\dfrac{d\sigma}{d\Omega}=\dfrac{-a^2\cos\frac{\Theta}{2}}{2\sin\Theta\sin^3\frac{\Theta}{2}}$

with: $\dfrac{db}{d\Theta}=\dfrac{-a}{2}\csc^2\dfrac{\Theta}{2}=\dfrac{-a}{2\sin^2\frac{\Theta}{2}}$ and: $\sin\Theta=2\sin\dfrac{\Theta}{2}\cos\dfrac{\Theta}{2}$

(*Never forget*!: $a=\dfrac{-k}{2E}$)

Rutherford scattering geometry



b= *impact parameter*

$$\frac{b}{a} = \cot\frac{\Theta}{2}$$

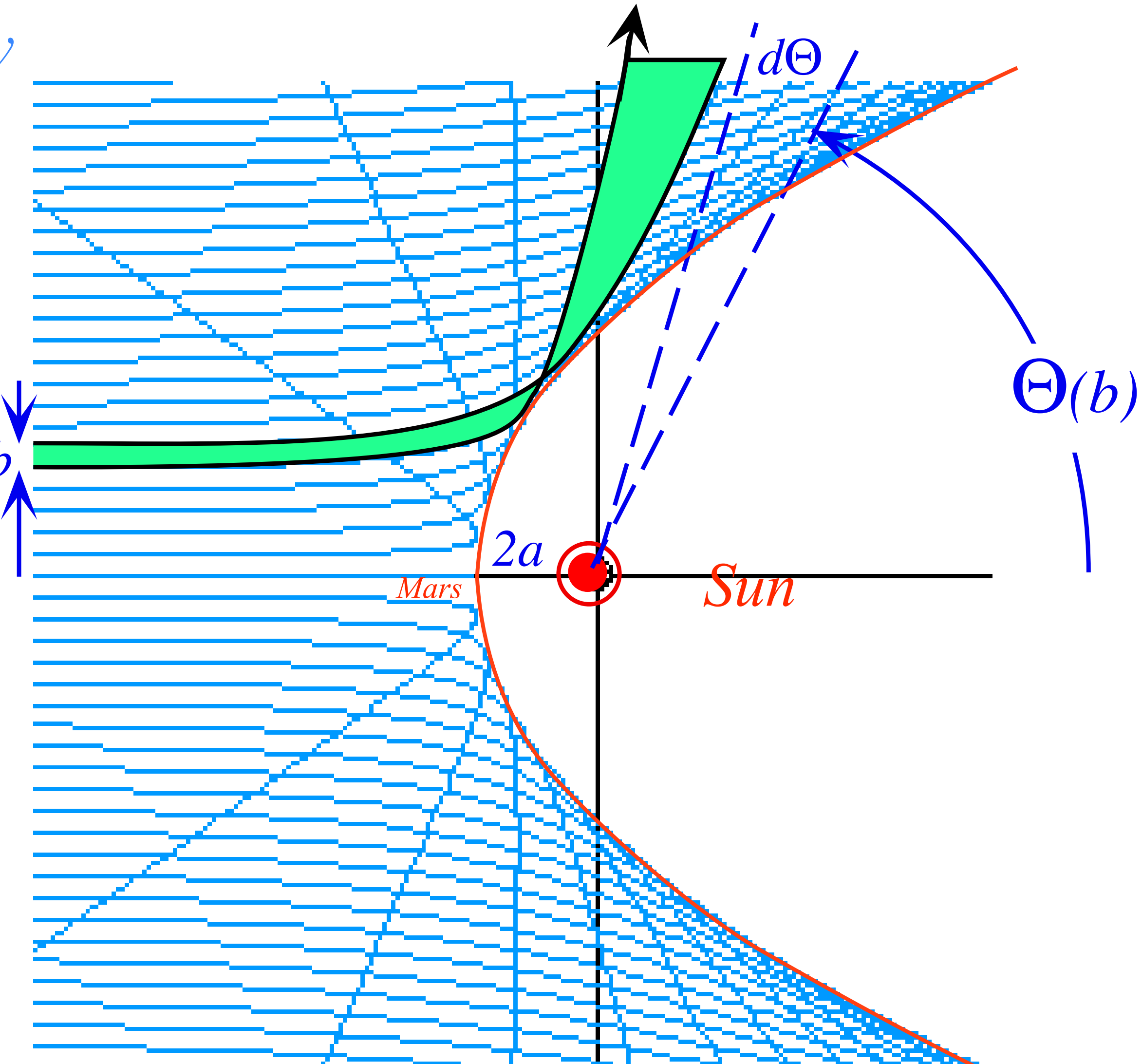


Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Incremental window $d\sigma = b \cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Ratio $\dfrac{d\sigma}{d\Omega} = \dfrac{b\, db\, d\varphi}{\sin\Theta d\Theta d\varphi} = \dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering cross-section (DSC)*

Geometry: $b = a\cot\frac{\Theta}{2} = \frac{k}{2E}\cot\frac{\Theta}{2}$ gives the *Rutherford DSC*.

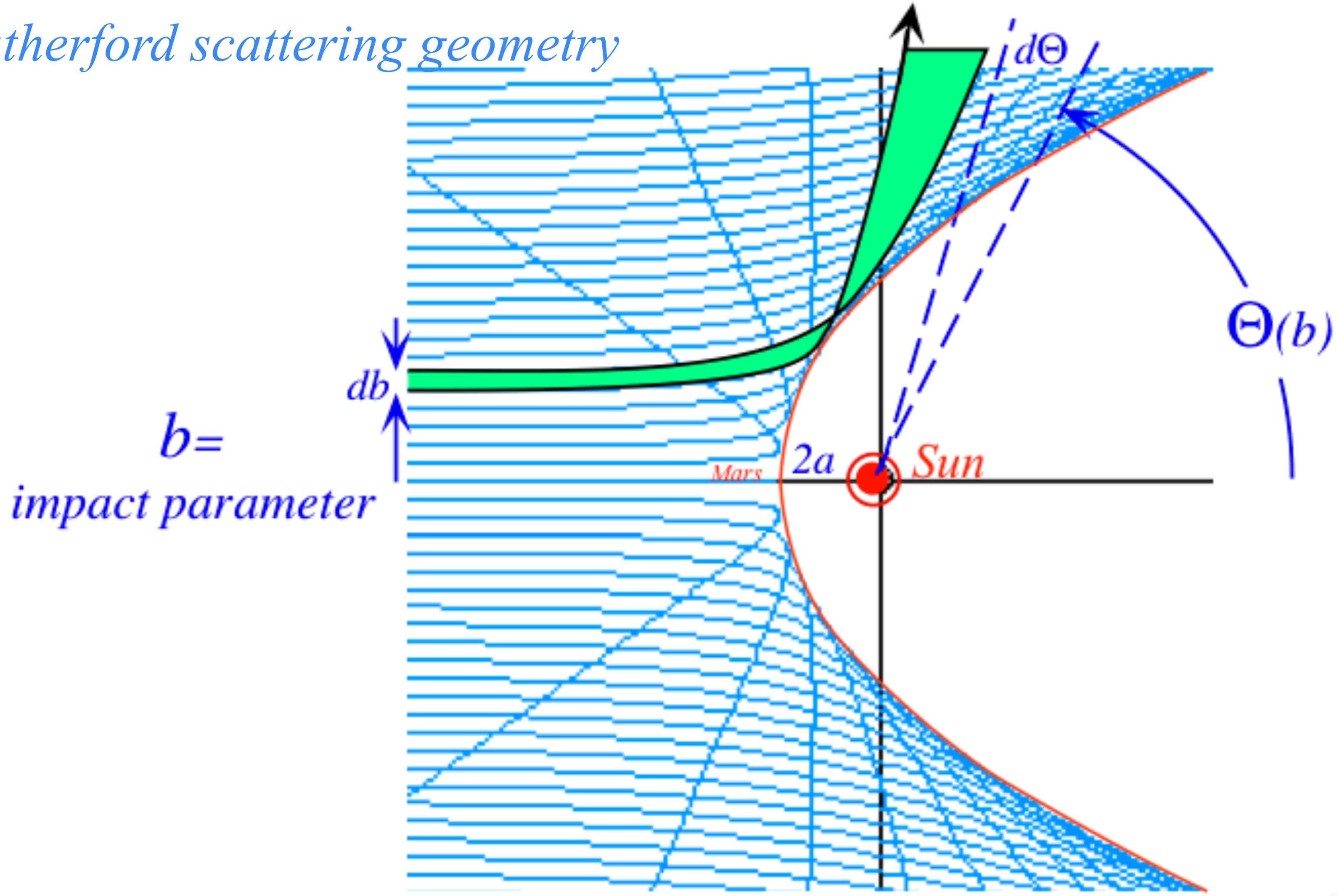
$$\frac{d\sigma}{d\Omega} = \frac{-a^2 \cos\frac{\Theta}{2}}{2\sin\Theta \sin^3\frac{\Theta}{2}} = \frac{-k^4}{16E^2}\sin^{-4}\frac{\Theta}{2}$$

with: $\dfrac{db}{d\Theta} = \dfrac{-a}{2}\csc^2\dfrac{\Theta}{2} = \dfrac{-a}{2\sin^2\frac{\Theta}{2}}$ and: $\sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}$

(*Never forget*!: $a = \dfrac{-k}{2E}$)

This classical result agrees exactly with 1st Born approximation to *quantum* Coulomb *DSC*!

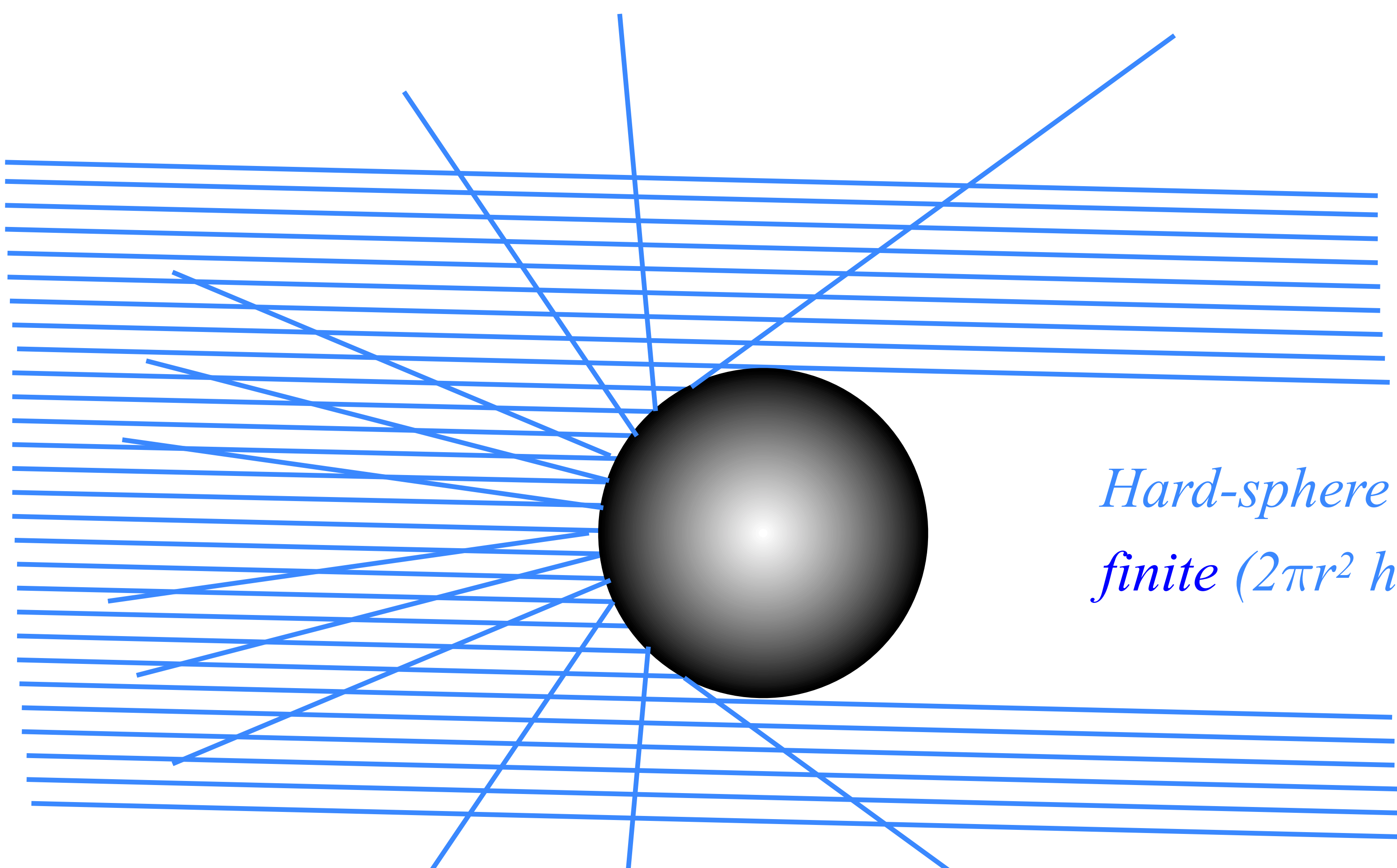
Rutherford scattering geometry



Two Extremes:

*Rutherford (Coulomb) scattering has **infinite** (∞) total cross section*

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \frac{k^4}{16E^2} \sin^{-4}\frac{\Theta}{2} = \infty$$



*Hard-sphere scattering has **finite** ($2\pi r^2$ here) total cross section*

CoulIt Web Simulation

Hard Sphere

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

➔ *Eccentricity vector* $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Eccentricity vector $\boldsymbol{\varepsilon}$ *and (ε,λ) geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Eccentricity vector **ε** *and (ε,λ) geometry of orbital mechanics*

Isotropic field *V=V(r)* guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb *V=-k/r* also conserves *eccentricity vector* **ε**

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1{}^2+p_1{}^2-x_2{}^2-p_2{}^2)$$
$$S_B=x_1p_1+x_2p_2$$
$$S_C=x_1p_2-x_2p_1$$

$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

Generates symmetry groups: $R(3) \subset R(3) \times R(3) \subset O(4)$

$\mathbf{A} = km \cdot \boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A = \tfrac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)$$

$$S_B = x_1 p_1 + x_2 p_2$$

$$S_C = x_1 p_2 - x_2 p_1$$

Generates symmetry groups: $U(1) \subset U(2)$

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

➤ *Projection* **ε•r** *geometry of* **ε***-vector and orbital radius* **r**

Review and connection to usual orbital algebra (previous lecture)

Projection **ε•p** *geometry of* **ε***-vector and momentum* **p**=m**v**

General geometric orbit construction using **ε***-vector and (γ,R)-parameters*

Derivation of **ε***-construction by analytic geometry*

Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* **p**=m**v**

*Example of complete (***r***,***p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

Eccentricity vector $\boldsymbol{\varepsilon}$ and (ε,λ) geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* $\mathbf{L}$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

Generates symmetry groups: *$R(3) \subset R(3)\times R(3) \subset O(4)$*

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A = \tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B = x_1p_1 + x_2p_2$$
$$S_C = x_1p_2 - x_2p_1$$

Generates symmetry groups: *$U(1) \subset U(2)$*

$\mathbf{A} = km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r} = \frac{\mathbf{r}\bullet\mathbf{r}}{r} - \frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km} = r - \frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km} = r - \frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

Eccentricity vector ε *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* ε

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B= x_1p_1 + x_2p_2$$
$$S_C= x_1p_2 - x_2p_1$$

A = *km*·ε is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of ε with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r} = \frac{\mathbf{r}\bullet\mathbf{r}}{r} - \frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km} = r - \frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km} = r - \frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km} = \mathbf{p}\bullet\hat{\mathbf{r}} = p_r$$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* ε

$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B=x_1p_1+x_2p_2$$
$$S_C=x_1p_2-x_2p_1$$

$\mathbf{A}=km\cdot\varepsilon$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of ε with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

Let angle ϕ be angle between ε and radial vector **r**

$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km}$$

...or of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

Eccentricity vector ε and (ε,λ) geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* ε

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B= x_1p_1 + x_2p_2$$
$$S_C= x_1p_2 - x_2p_1$$

A = *km*·ε is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of ε with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r} = \frac{\mathbf{r}\bullet\mathbf{r}}{r} - \frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km} = r - \frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km} = r - \frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

Let angle ϕ be angle between ε and radial vector **r**

$$\varepsilon\, r\cos\phi = r - \frac{L^2}{km} \quad \text{or:} \quad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

...or of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km} = \mathbf{p}\bullet\hat{\mathbf{r}} = p_r$$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* $\mathbf{L}$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A = \tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B = x_1p_1 + x_2p_2$$
$$S_C = x_1p_2 - x_2p_1$$

$\mathbf{A} = km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r} = \frac{\mathbf{r}\bullet\mathbf{r}}{r} - \frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km} = r - \frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km} = r - \frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector $\mathbf{r}$

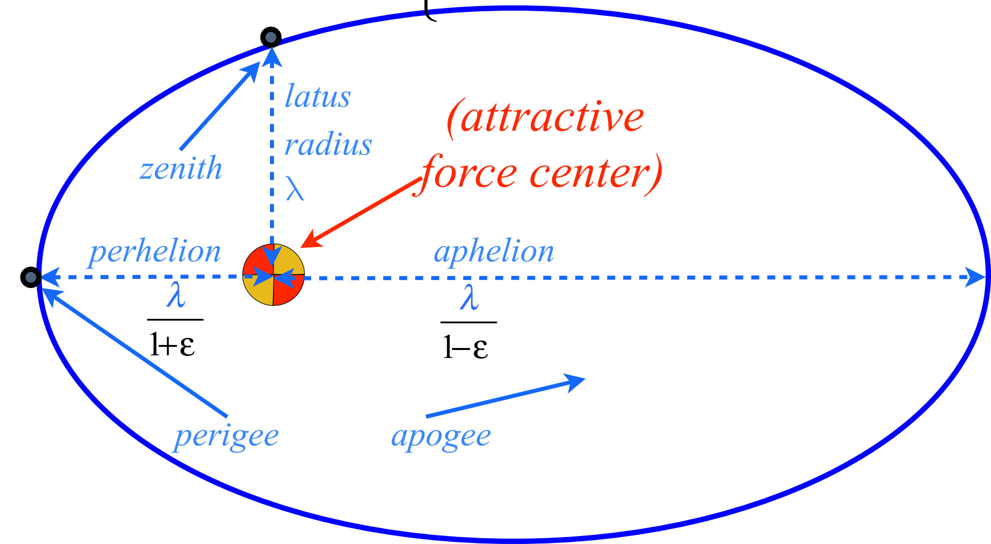
$$\varepsilon\, r\cos\phi = r - \frac{L^2}{km} \quad \text{or:} \quad r = \frac{L^2/km}{1-\varepsilon\cos\phi}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km} = \mathbf{p}\bullet\hat{\mathbf{r}} = p_r$$

For $\lambda=L^2/km$ that matches:

$$r = \frac{\lambda}{1-\varepsilon\cos\phi} = \begin{cases} \dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee} \\ \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith} \\ \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee} \end{cases}$$



Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B=x_1p_1+x_2p_2$$
$$S_C=x_1p_2-x_2p_1$$

$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

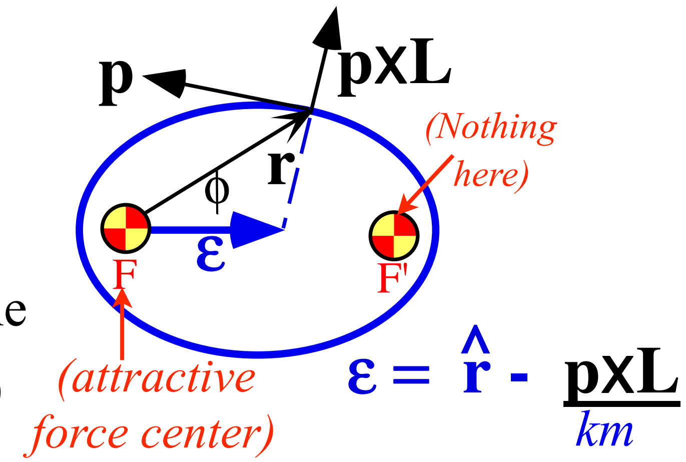
Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector **r**

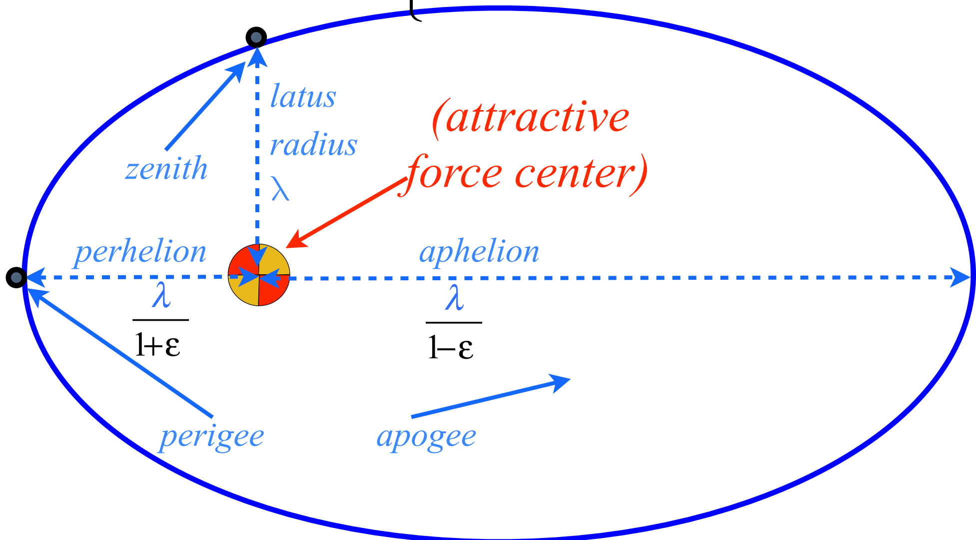
$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km}\quad\text{or:}\quad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

For $\lambda=L^2/km$ that matches:
$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\begin{cases}\dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee}\\ \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith}\\ \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee}\end{cases}$$

(a) Attractive (k>0)
Elliptic (E<0)

(Rotational momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)





Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B=x_1p_1+x_2p_2$$
$$S_C=x_1p_2-x_2p_1$$

$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector **r**

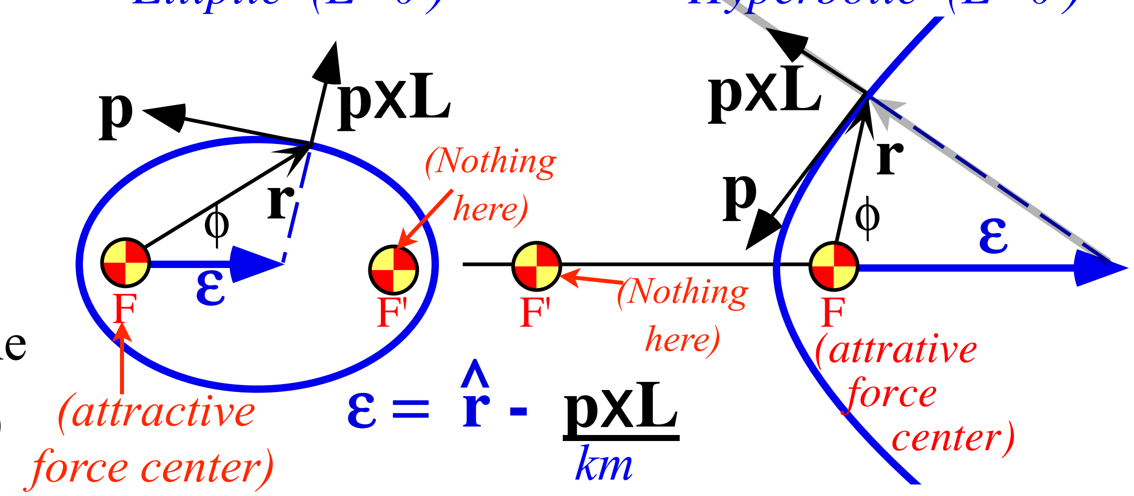
$$\varepsilon\,r\cos\phi=r-\frac{L^2}{km}\qquad\text{or:}\qquad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

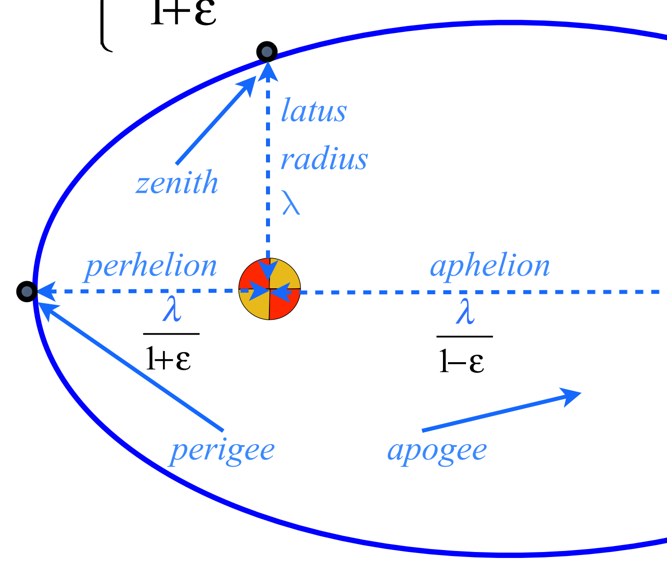
For $\lambda=L^2/km$ that matches: $r=\dfrac{\lambda}{1-\varepsilon\cos\phi}=\begin{cases}\dfrac{\lambda}{1-\varepsilon} & \text{if: }\phi=0 \quad \textit{apogee}\\ \lambda & \text{if: }\phi=\dfrac{\pi}{2} \quad \textit{zenith}\\ \dfrac{\lambda}{1+\varepsilon} & \text{if: }\phi=\pi \quad \textit{perigee}\end{cases}$

(a) Attractive (k>0)
Elliptic (E<0)

(b) Attractive (k>0)
Hyperbolic (E>0)

(Rotational momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)





Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* $\mathbf{L}$

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B=x_1p_1+x_2p_2$$
$$S_C=x_1p_2-x_2p_1$$

$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector $\mathbf{r}$

$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km}\qquad\text{or:}\qquad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

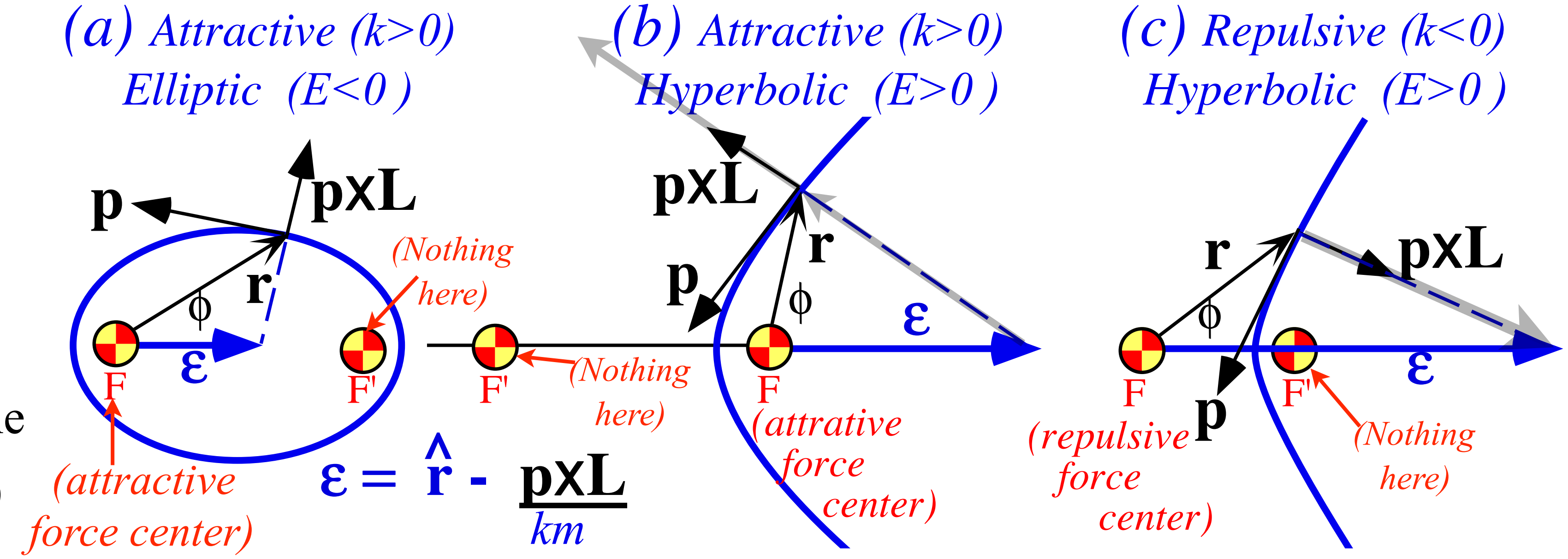
For $\lambda=L^2/km$ that matches: $r=\dfrac{\lambda}{1-\varepsilon\cos\phi}=\begin{cases}\dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee}\\ \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith}\\ \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee}\end{cases}$

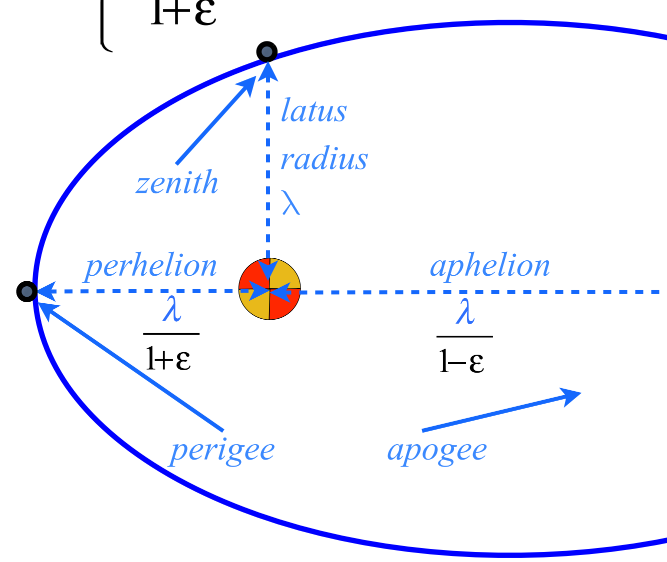
(Rotational momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)

(a) Attractive (k>0) Elliptic (E<0)

(b) Attractive (k>0) Hyperbolic (E>0)

(c) Repulsive (k<0) Hyperbolic (E>0)





Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

➔ *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

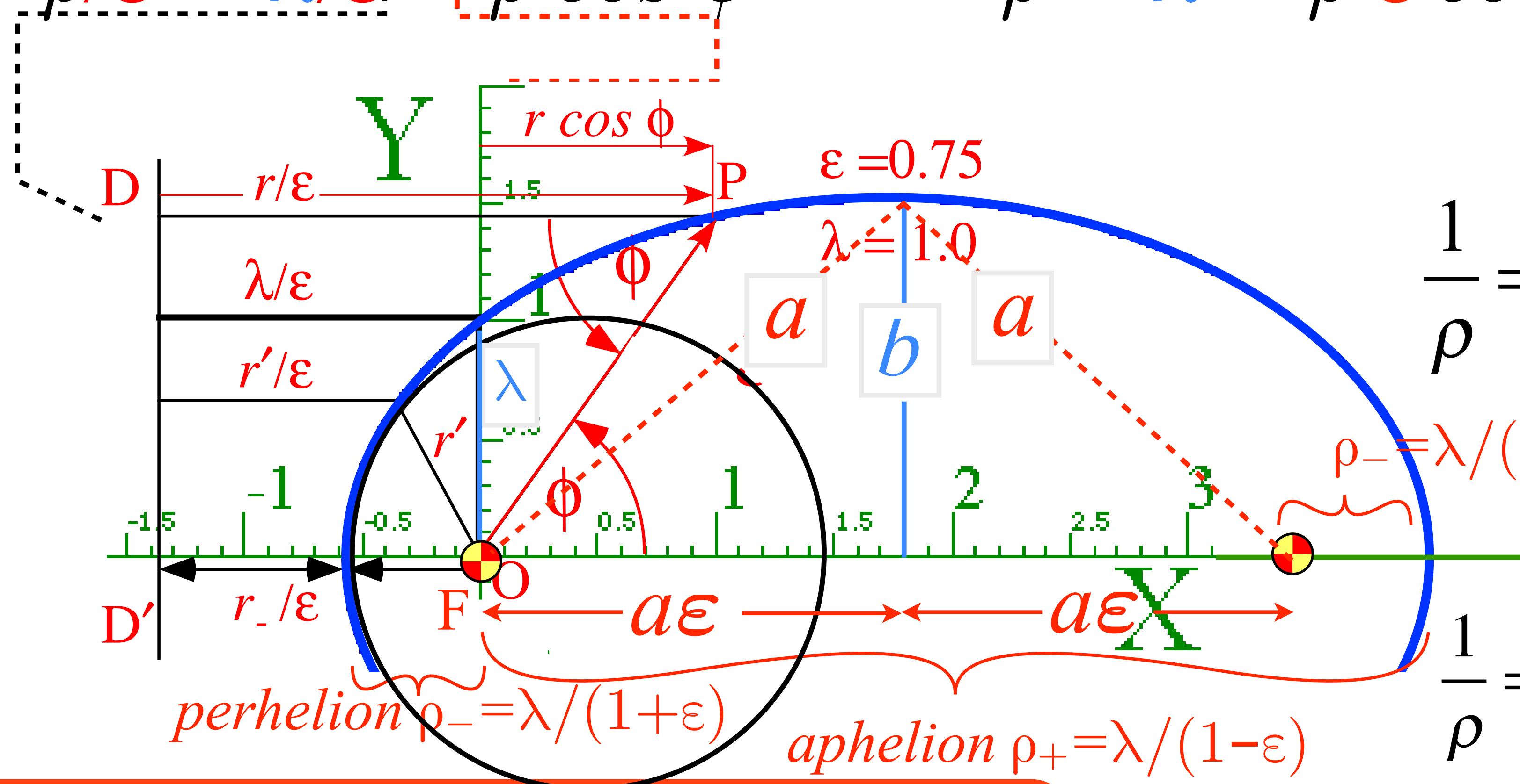
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

(From Lecture 25 p. 64-74) *Geometry of Coulomb orbits (Let: $r = \rho$ here)*

$$\rho/\varepsilon = \lambda/\varepsilon + \rho\cos\phi \qquad \rho = \lambda + \rho\,\varepsilon\cos\phi \qquad \rho = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$$\frac{1}{\rho} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by:

*Defining **eccentricity** ε*

Distance to F$_{ocal}$-point $= \varepsilon \cdot$ Distance to D$_{irectrix}$-line

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon<1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon>1$)*

(x,y) *parameters*	*physical constants*	(r,ϕ) *parameters*
major radius $a = \dfrac{k}{2E}$	*Energy* $E = \dfrac{k}{2a}$	$\varepsilon = \sqrt{\dfrac{k^2 m + 2\mu^2 E}{k^2 m}} = \sqrt{1 \pm \dfrac{b^2}{a^2}}$ *eccentricity*
minor radius $b = \dfrac{\mu}{\sqrt{2m\lvert E\rvert}}$	*Orbital Momentum* $\mu = \sqrt{km\lambda}$	*latus radius* $\lambda = \dfrac{\mu^2}{km} = \dfrac{b^2}{a}$

$$\varepsilon^2 = 1 - \frac{b^2}{a^2} \quad \text{(ellipse: } \varepsilon<1) \quad \frac{b}{a} = \sqrt{1-\varepsilon^2}$$

$$\varepsilon^2 = 1 + \frac{b^2}{a^2} \quad \text{(hyperbola: } \varepsilon>1) \quad \frac{b}{a} = \sqrt{\varepsilon^2 - 1}$$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon<1$)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb: $\varepsilon>1$)*

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

➡ *Projection* $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

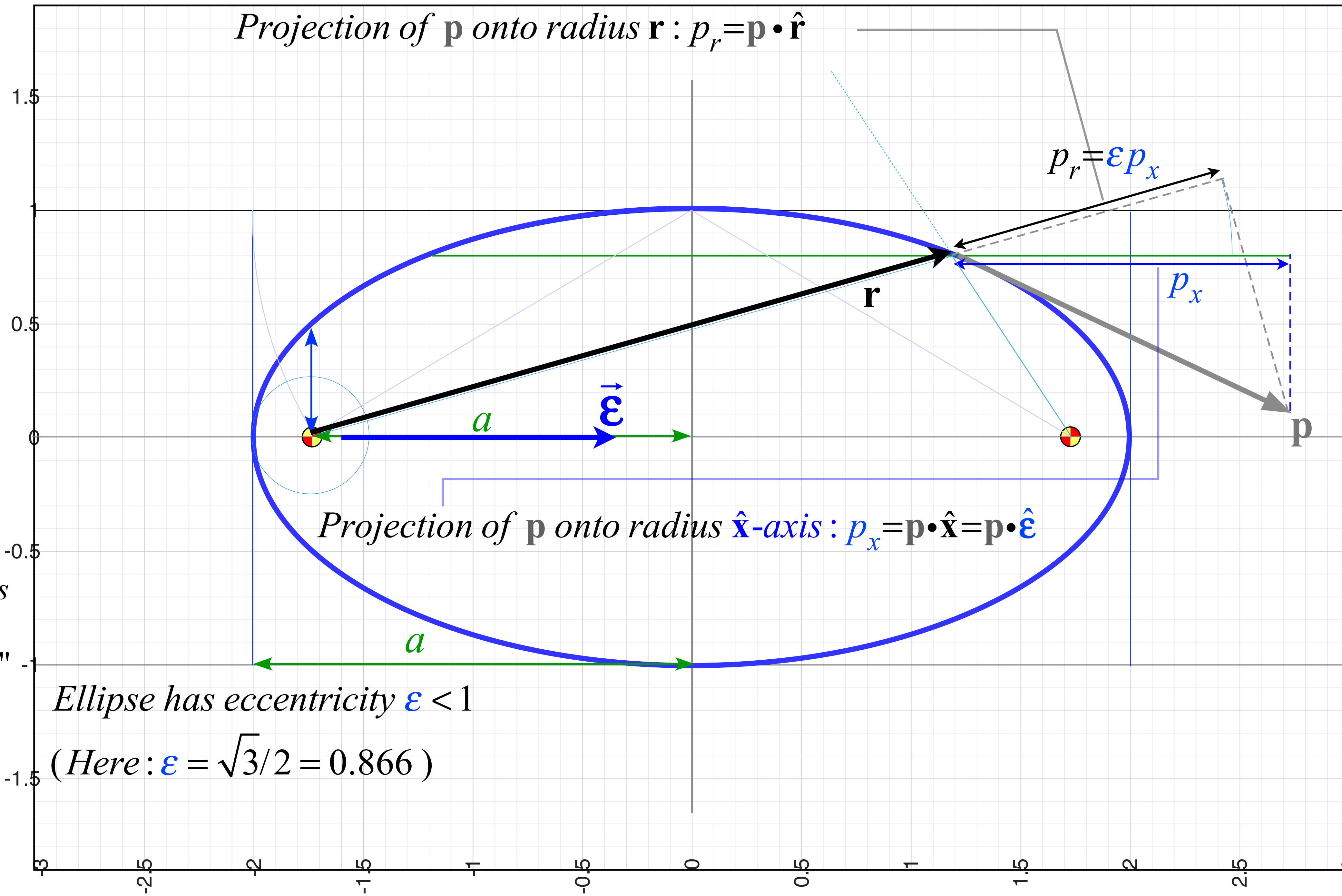
Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

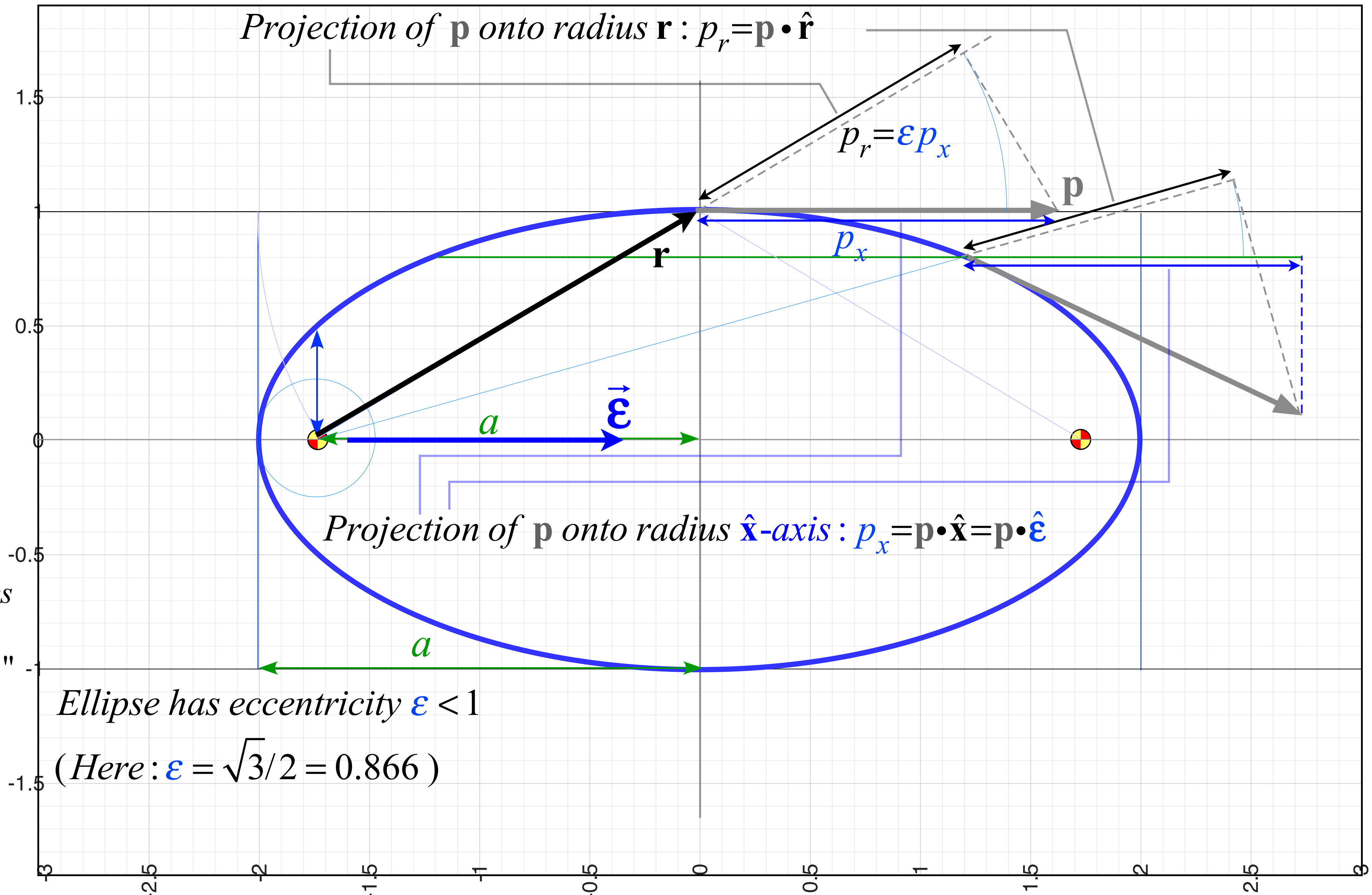
This says:
"*Projection p_r of $\mathbf{p}$ onto radial $\mathbf{r}$ or $\mathbf{r}'$ lines equals eccentricity ε times projection p_x of $\mathbf{p}$ onto orbit major axis:* $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "



Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}$$
$$= \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

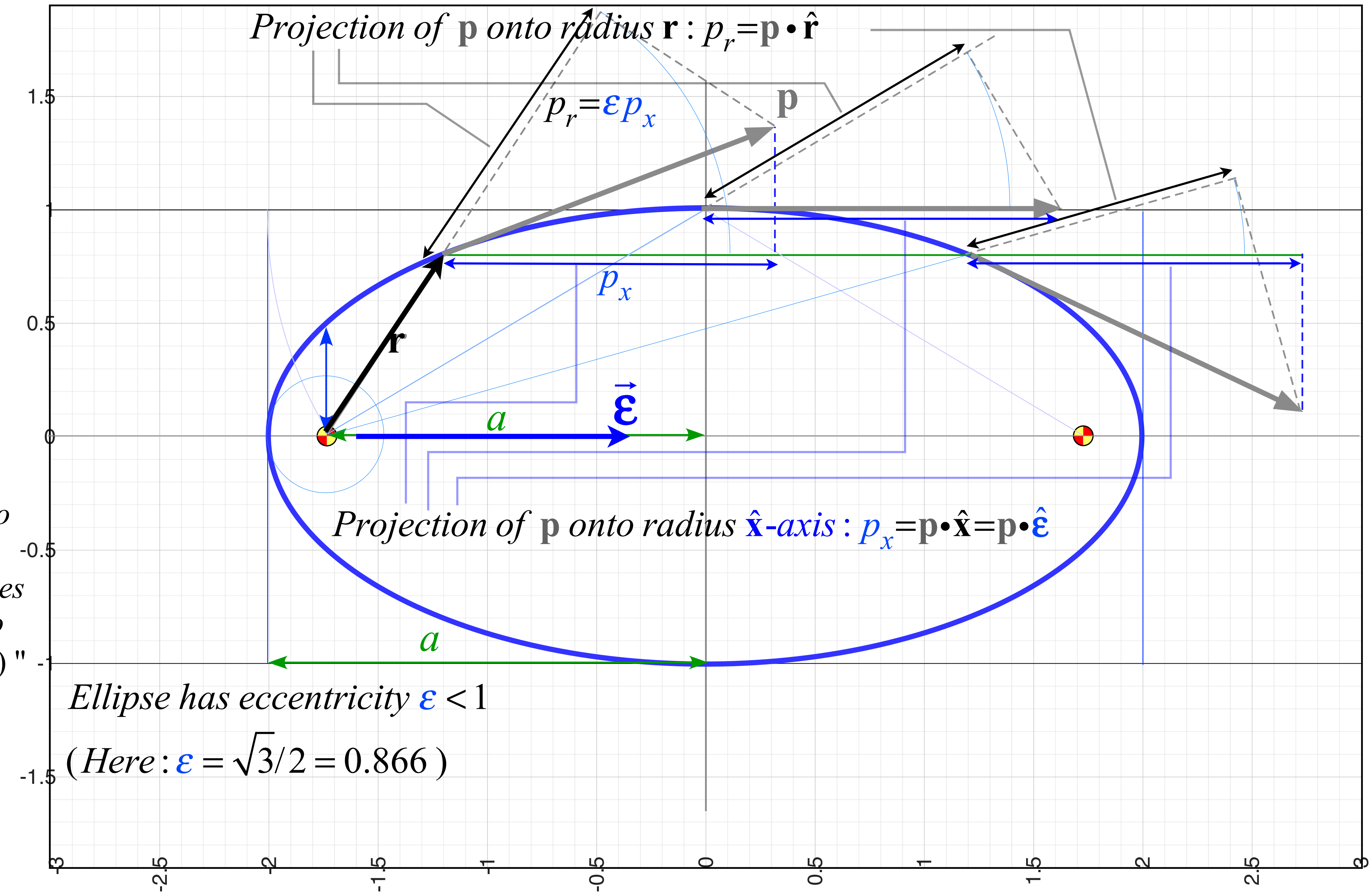


NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn to correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection* p_r *of* **p** *onto radial* **r** *or* **r′** *lines equals eccentricity* ε *times projection* p_x *of* **p** *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "



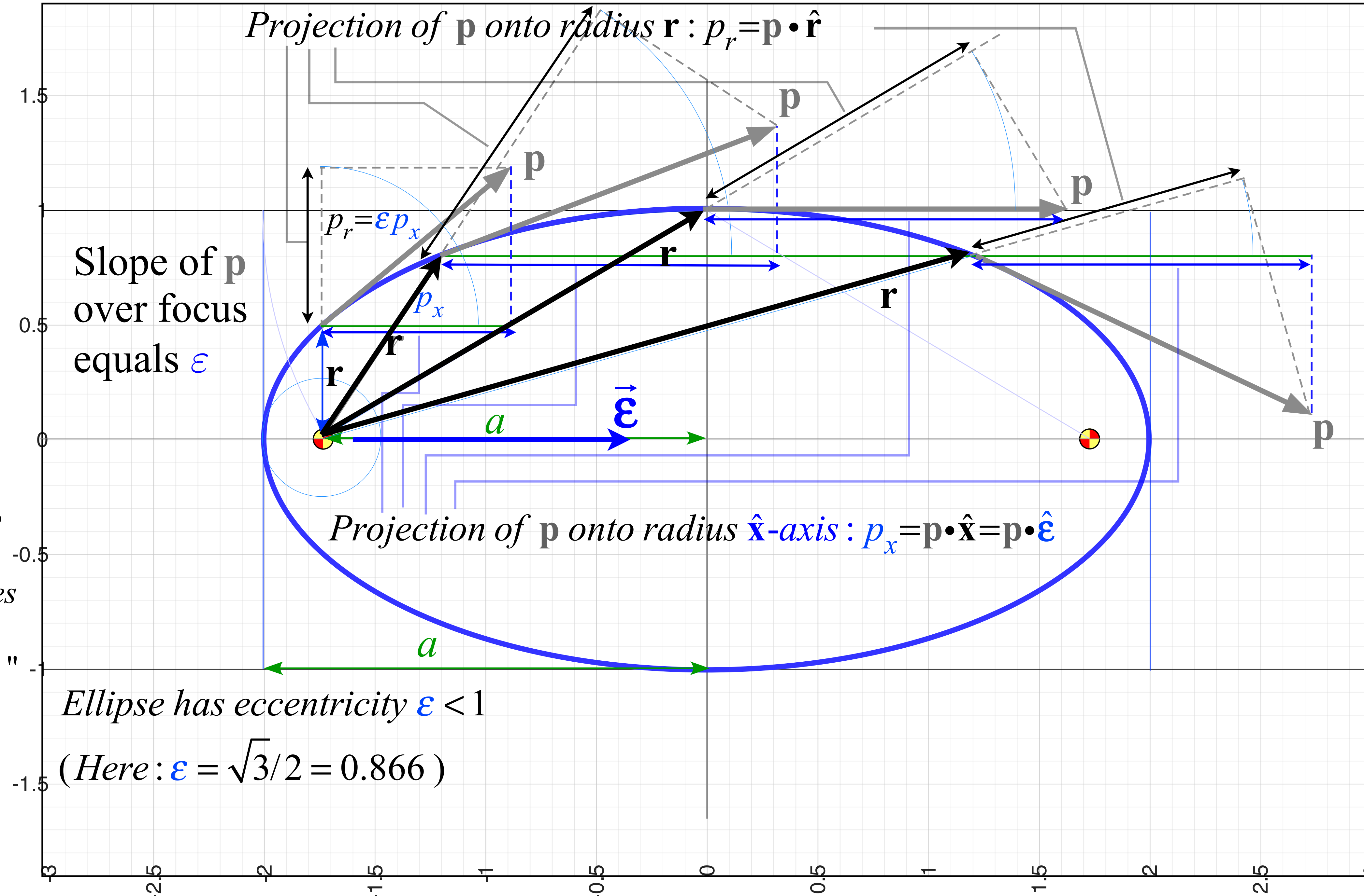
NOTE: Lengths of vectors **p** and -**p** are not drawn to correctly show that momentum **p**=m**v** grows as radial distance r=|**r**| falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:

"*Projection p_r of* **p** *onto radial* **r** *or* **r′** *lines equals eccentricity* ε *times projection p_x of* **p** *onto orbit major axis* : ($\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}}$) "

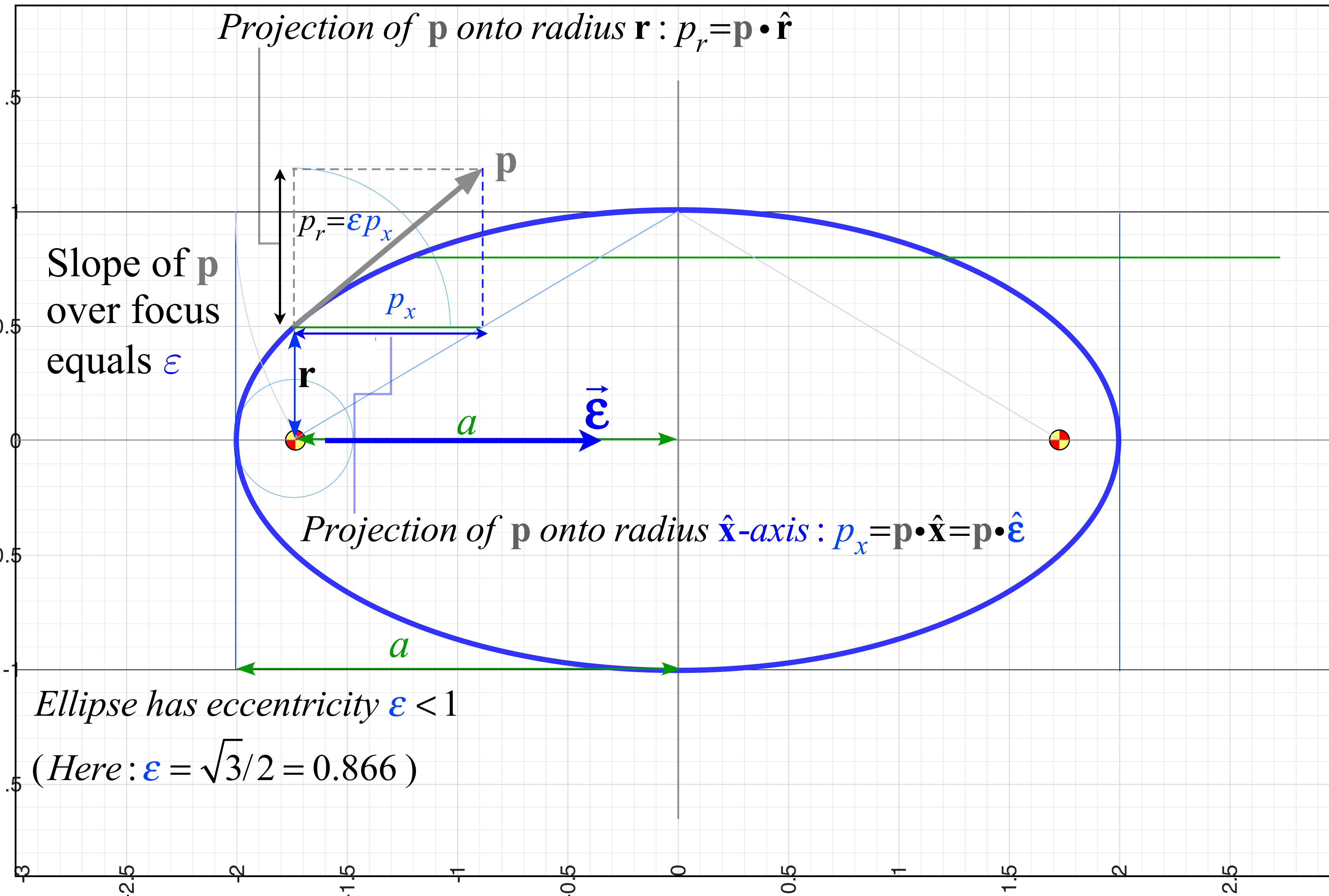


NOTE: Lengths of vectors **p** and -**p** are not drawn to correctly show that momentum **p**=m**v** grows as radial distance r=|**r**| falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection p_r of* **p** *onto radial* **r** *or* **r′** *lines equals eccentricity ε times projection p_x of* **p** *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "



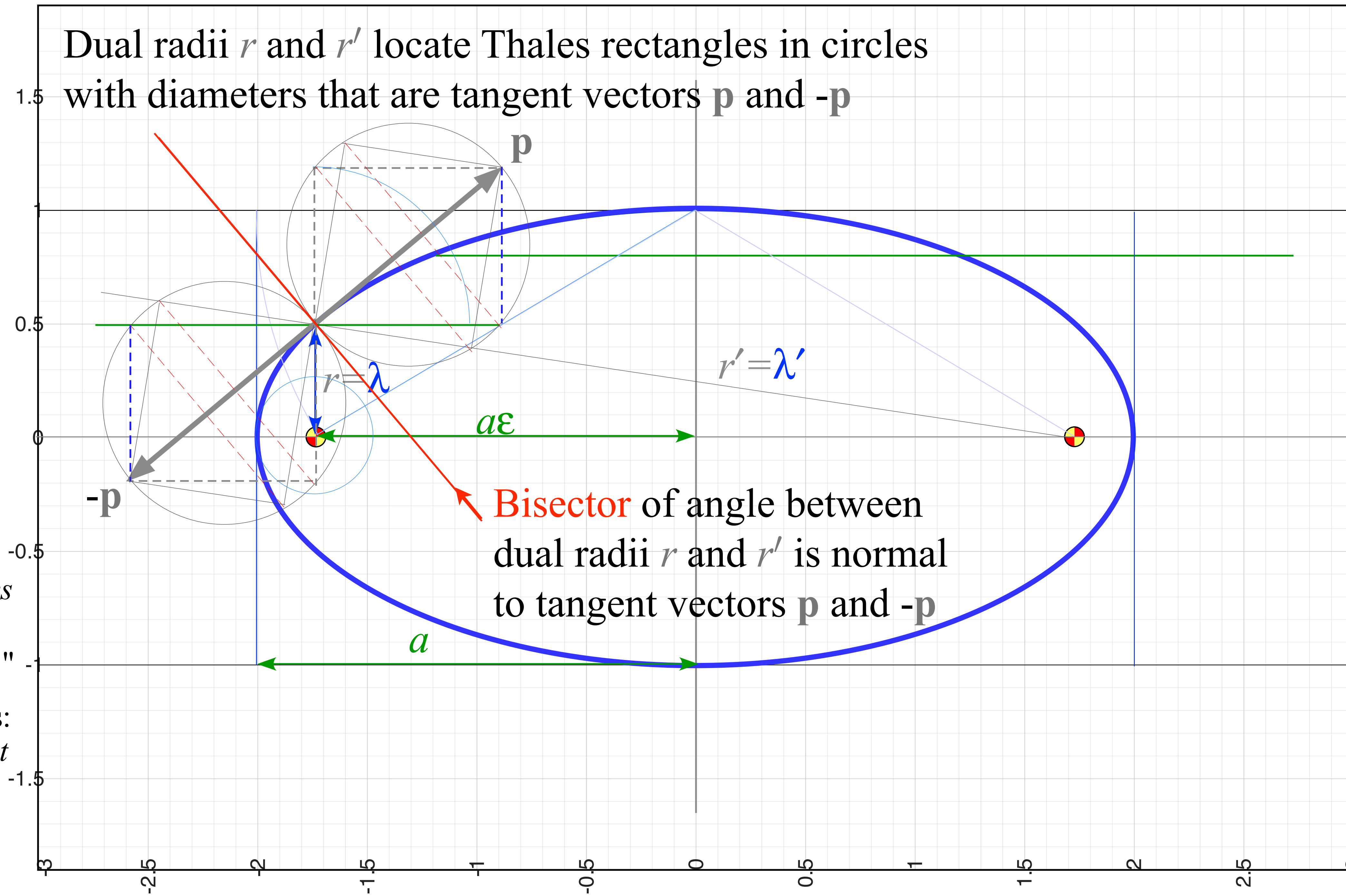
NOTE: Lengths of vectors **p** and -**p** are not drawn to correctly show that momentum **p**=m**v** grows as radial distance r=|**r**| falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection* p_r *of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis* : $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"



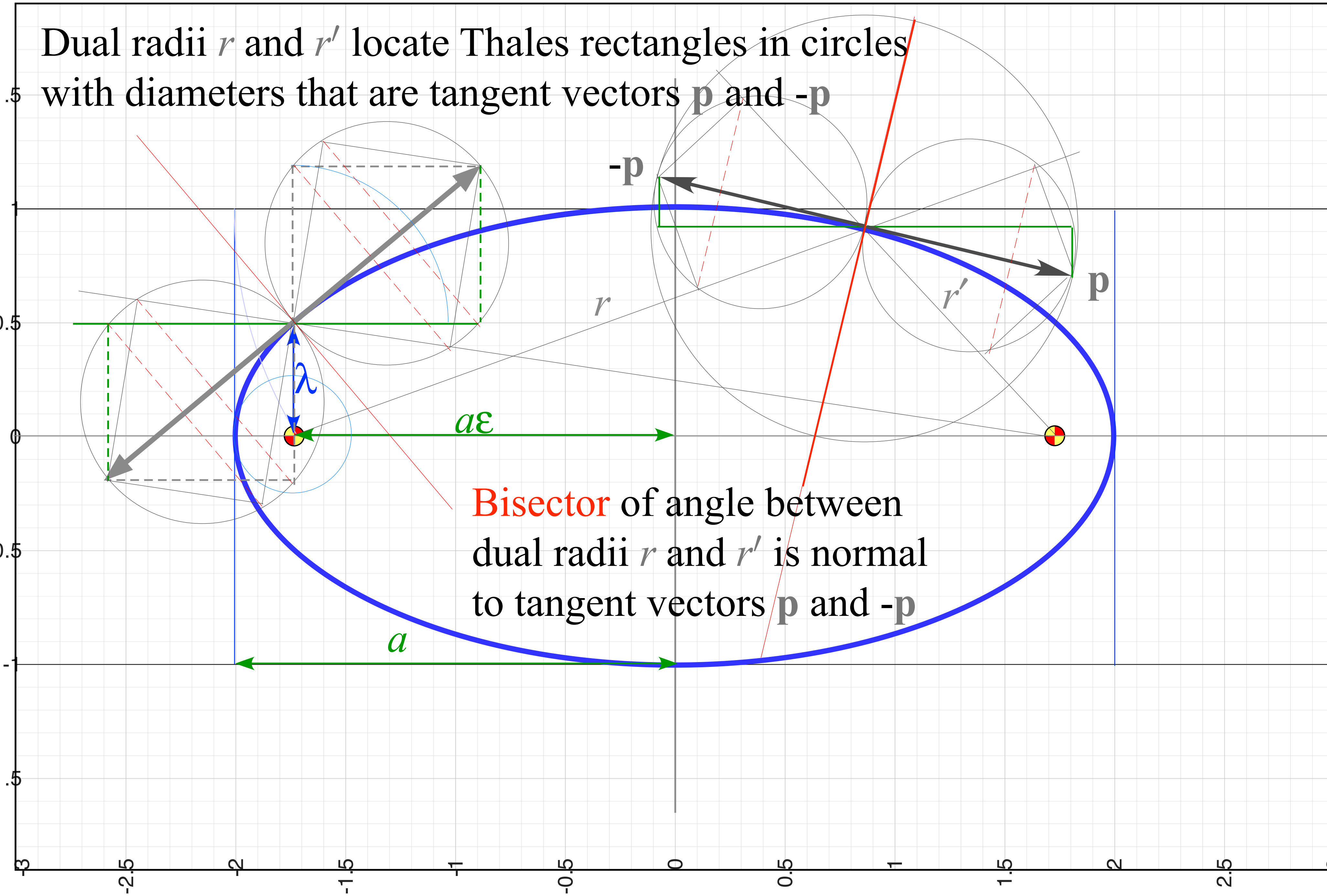
NOTE: Lengths of vectors $\mathbf{p}$ and $-\mathbf{p}$ are not drawn to correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km} = \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity ε times projection p_x of* $\mathbf{p}$ *onto orbit major axis*: $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$"

Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"



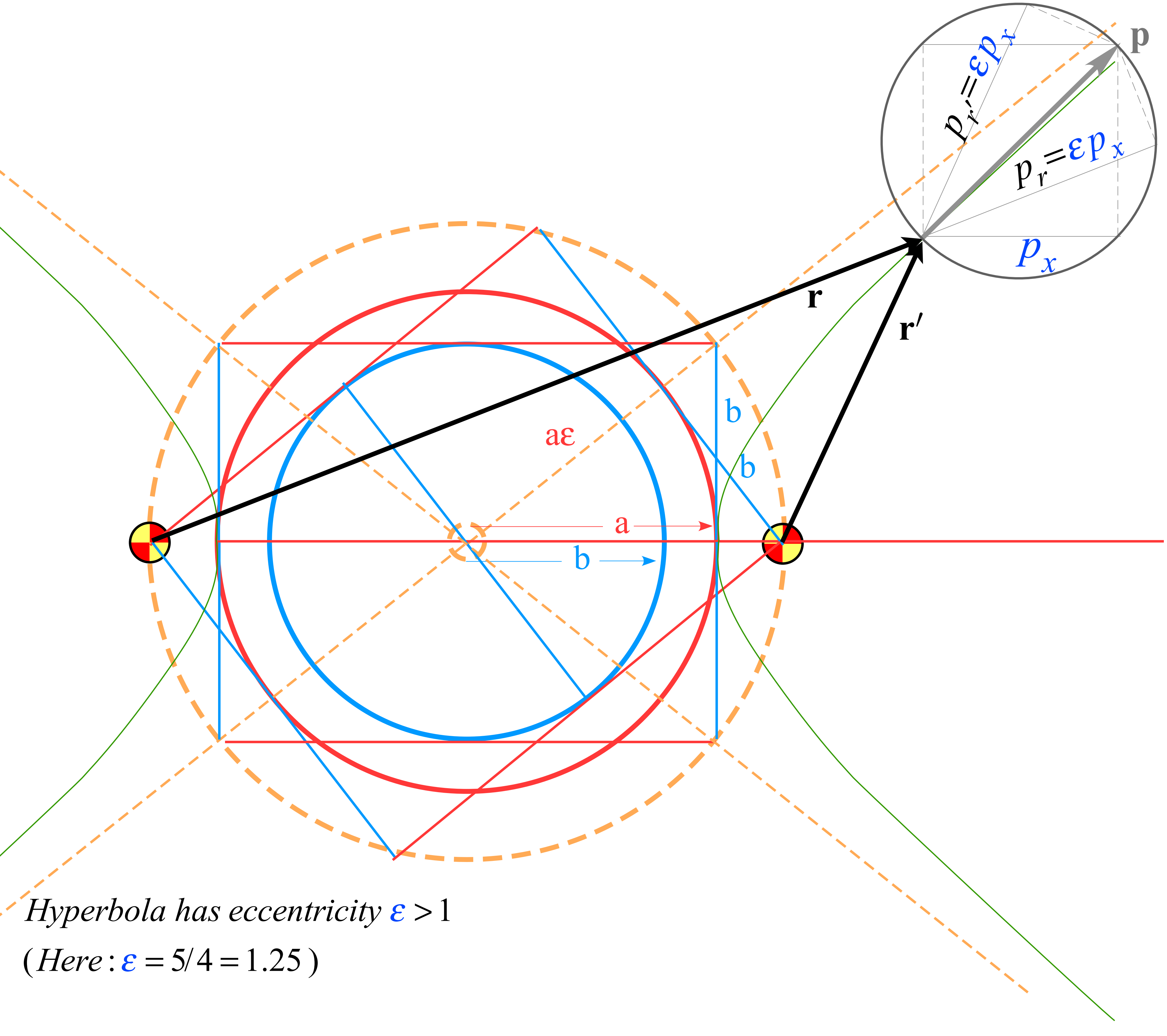
NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn to correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$
$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:
"*Projection* p_r *of* **p** *onto radial* **r** *or* **r′** *lines equals eccentricity* ε *times projection* p_x *of* **p** *onto orbit major axis*: $(\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}})$ "

Focal geometry demands:
"*Momentum* **p** *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* **r** *or* **r′** *lines*."



Hyperbola has eccentricity $\varepsilon > 1$

$(Here: \varepsilon = 5/4 = 1.25)$

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

➤ *General geometric orbit construction using* $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

*General geometric orbit construction using **ε-vector** and (γ,R)-parameters*

*Next several pages give step-by-step constructions of **ε-vector** and Coulomb orbit and trajectory physics*

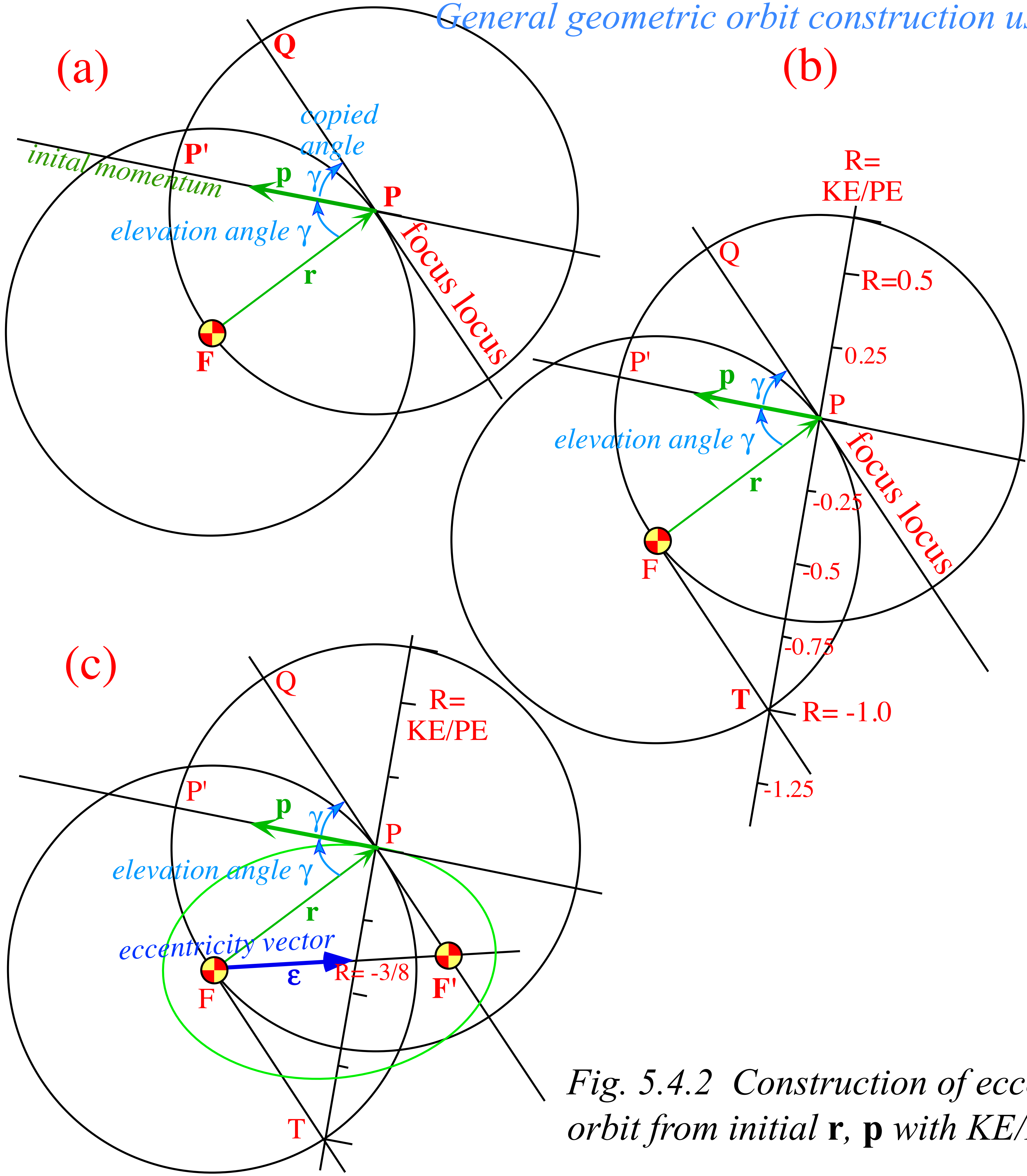


Fig. 5.4.2 Construction of eccentricity vector **ε** *and orbit from initial* **r**, **p** *with KE/PE=-3/8.*

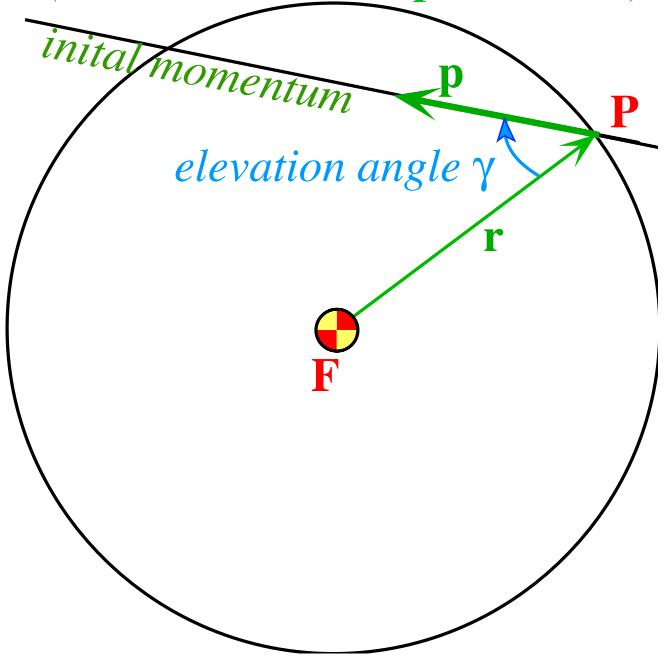
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

(momentum initial **p** *direction)*



*General geometric orbit construction using **ε**-vector and (γ,R)-parameters*

Pick launch point P

(radius vector **r** *)*

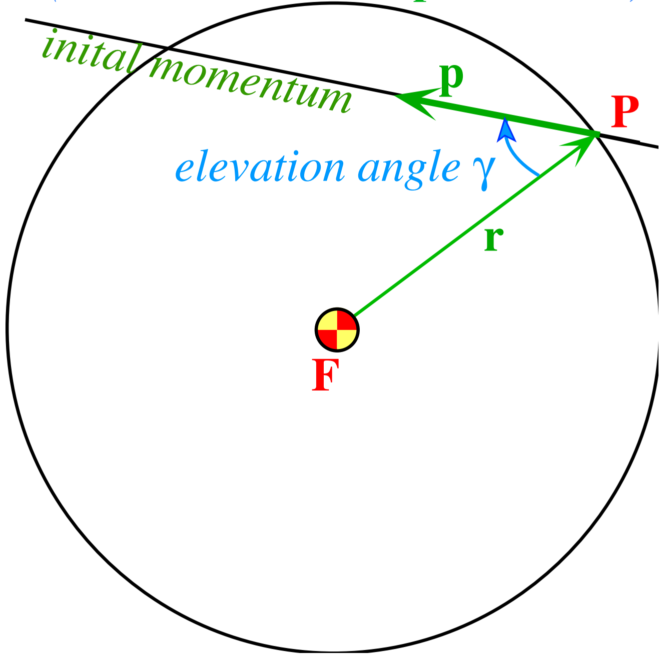
and elevation angle γ from radius

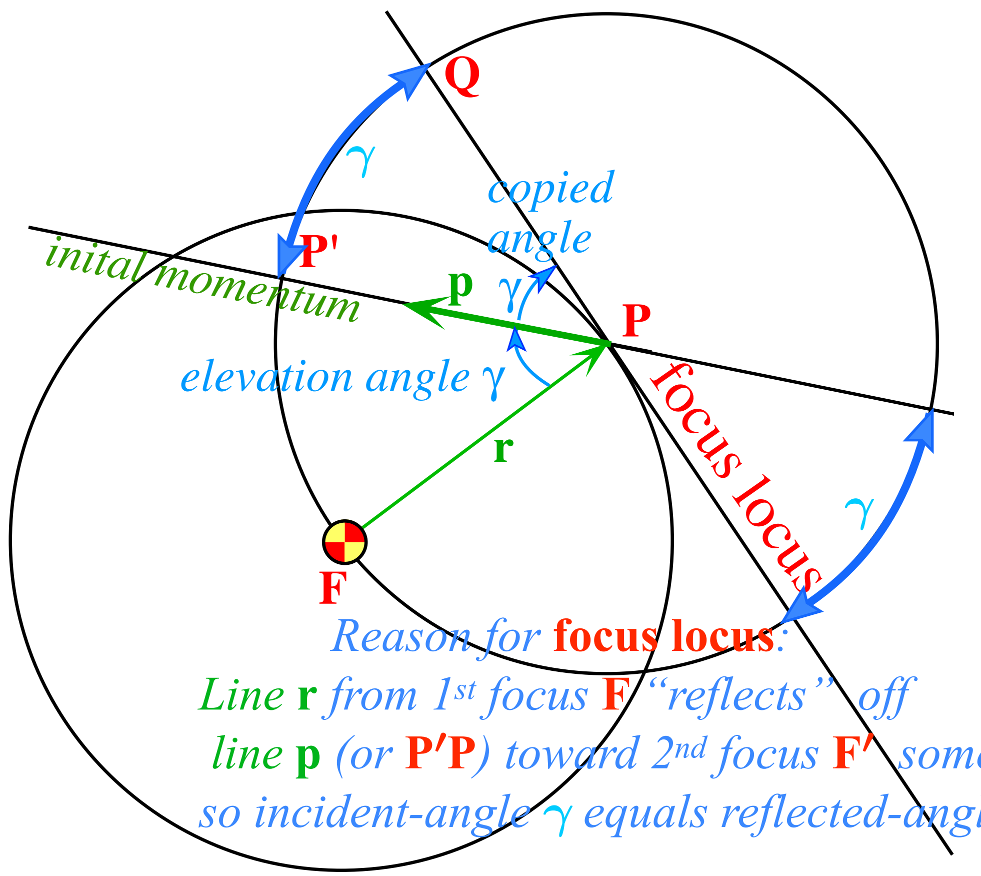
(momentum initial **p** *direction)*

Copy F-center circle around launch point P

Copy elevation angle γ (∠FPP′) onto ∠P′PQ

Extend resulting line QPQ′ *to make* **focus locus**





Reason for **focus locus***:*

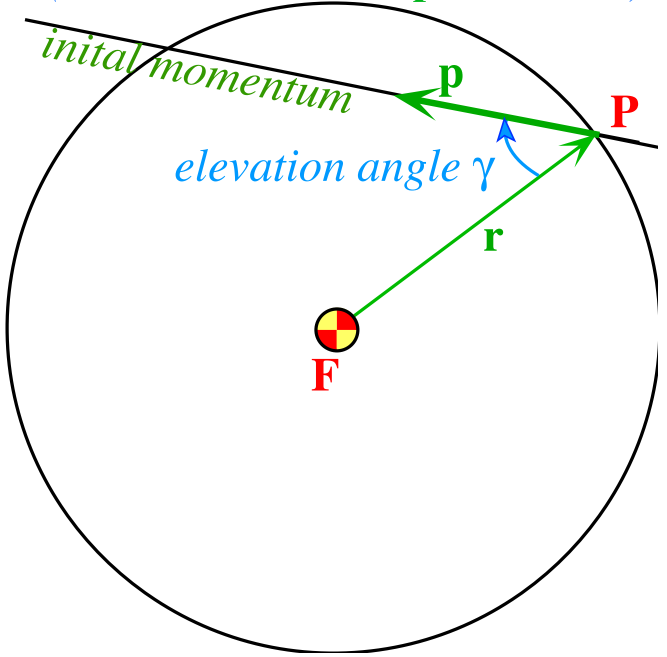
Line **r** *from 1st focus* **F** *"reflects" off*

line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*

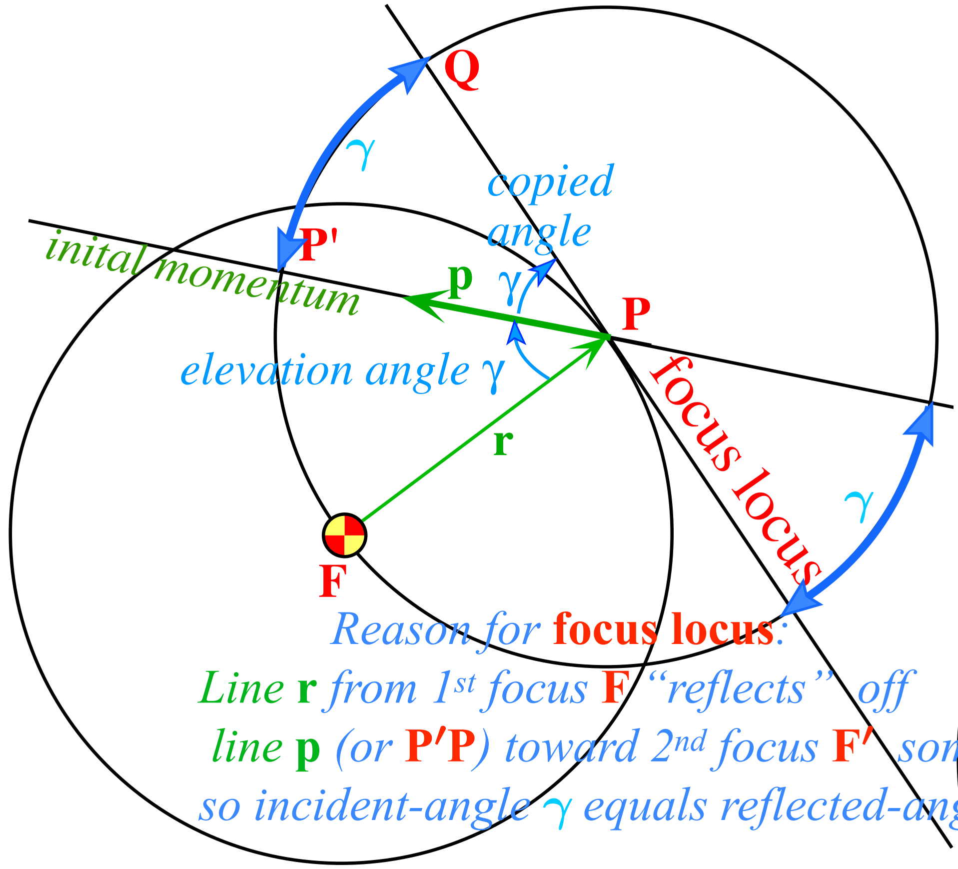
so incident-angle γ equals reflected-angle γ

General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

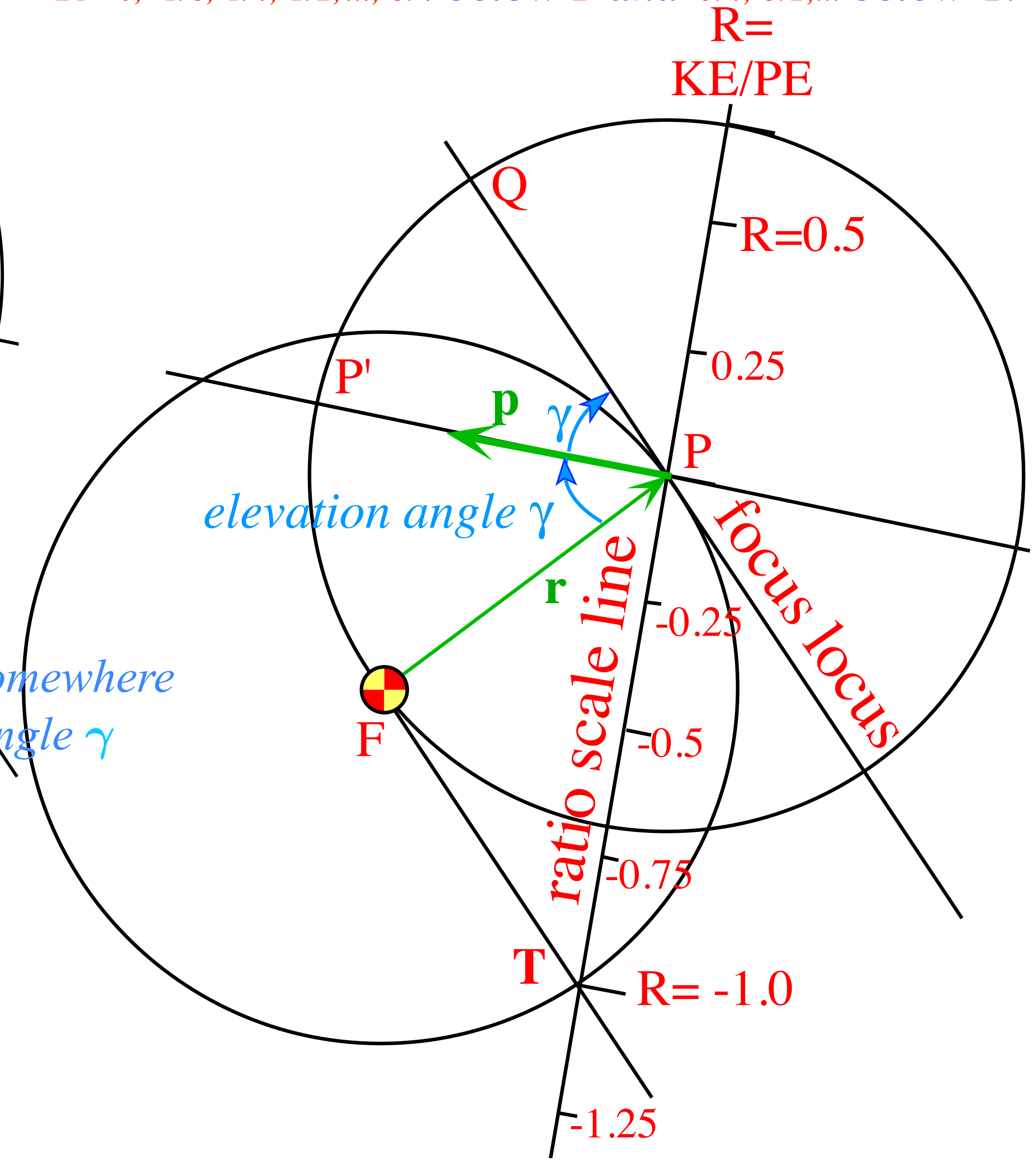


Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**



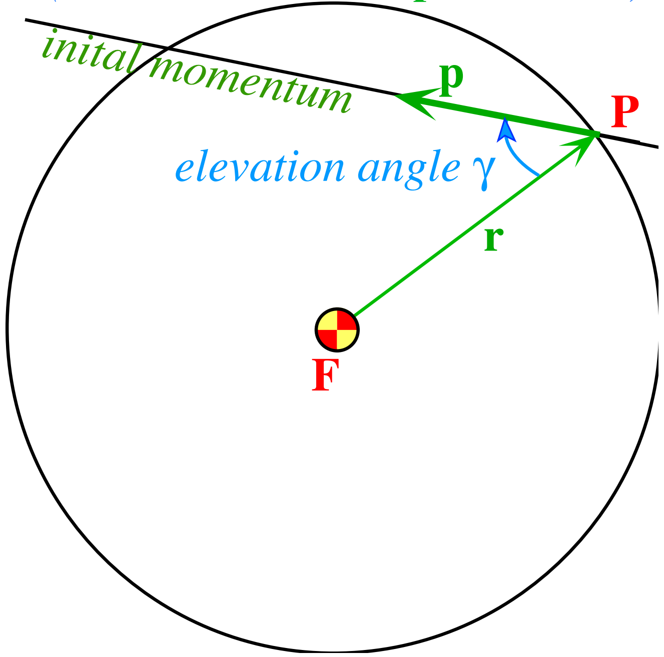
Reason for **focus locus***:*
Line **r** *from 1st focus* **F** *"reflects" off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ *equals reflected-angle* γ

*Copy double angle 2*γ *(∠FPQ) onto ∠PFT*
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.

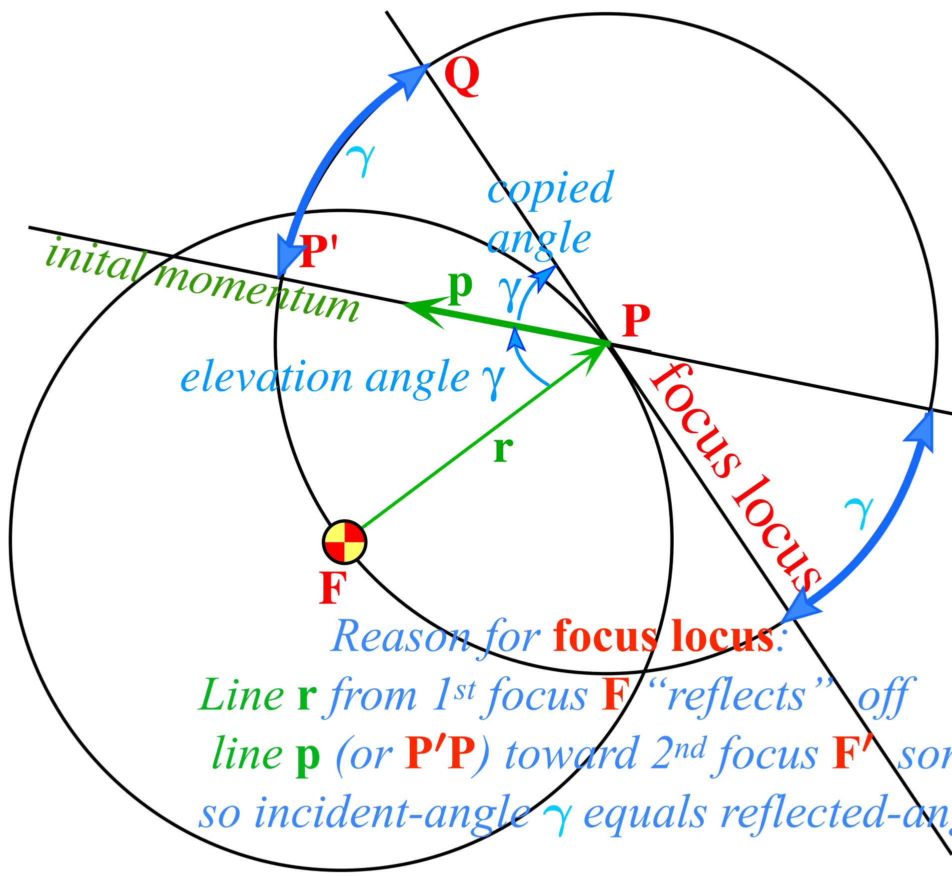


General geometric orbit construction using *ε-vector* and *(γ,R)-parameters*

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

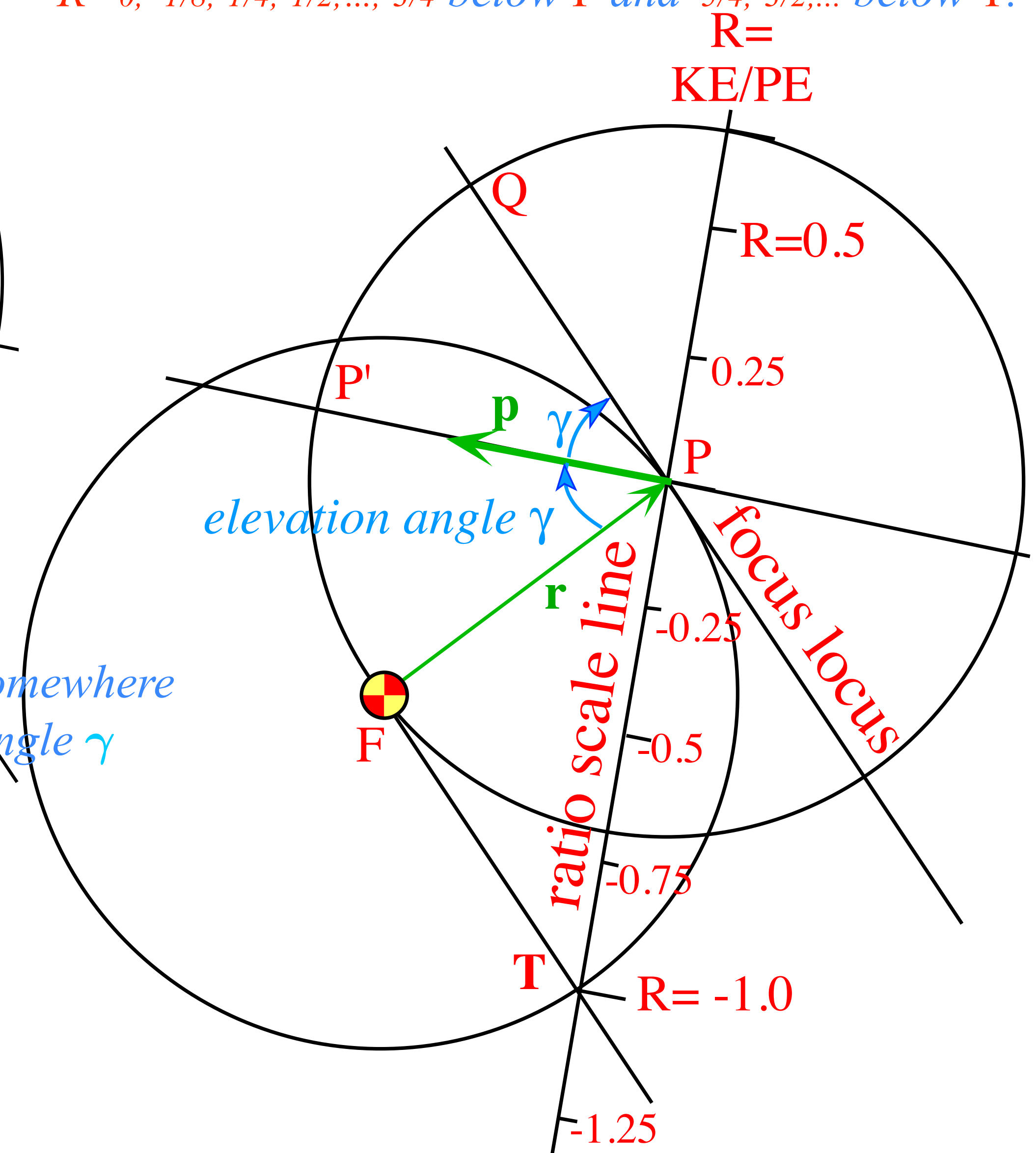


Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus**:
Line **r** *from 1st focus* **F** *"reflects" off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ *equals reflected-angle* γ

Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.

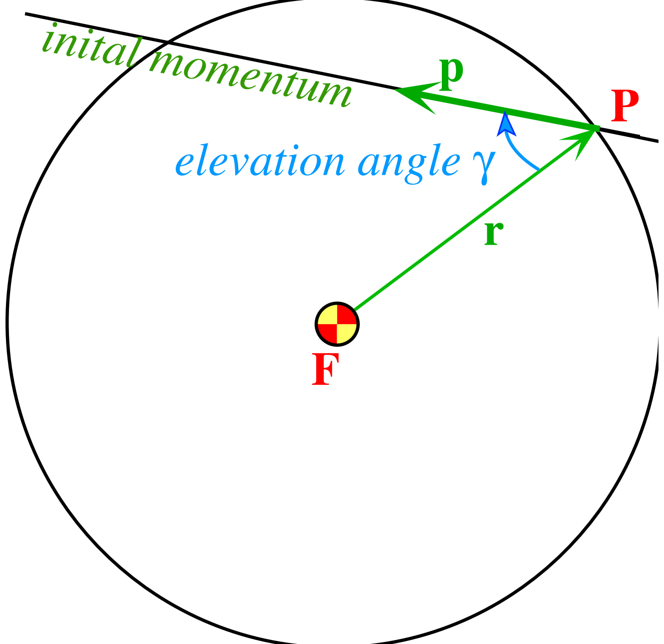


$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

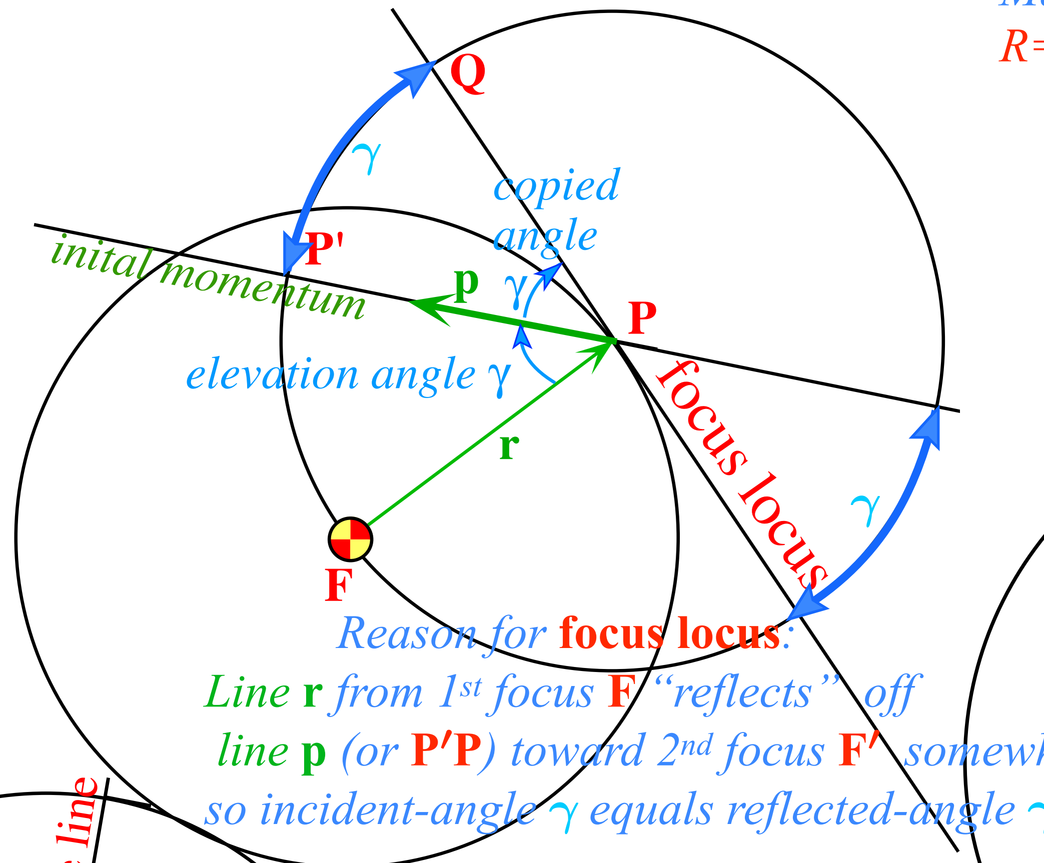
$$= \pm \left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$

General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

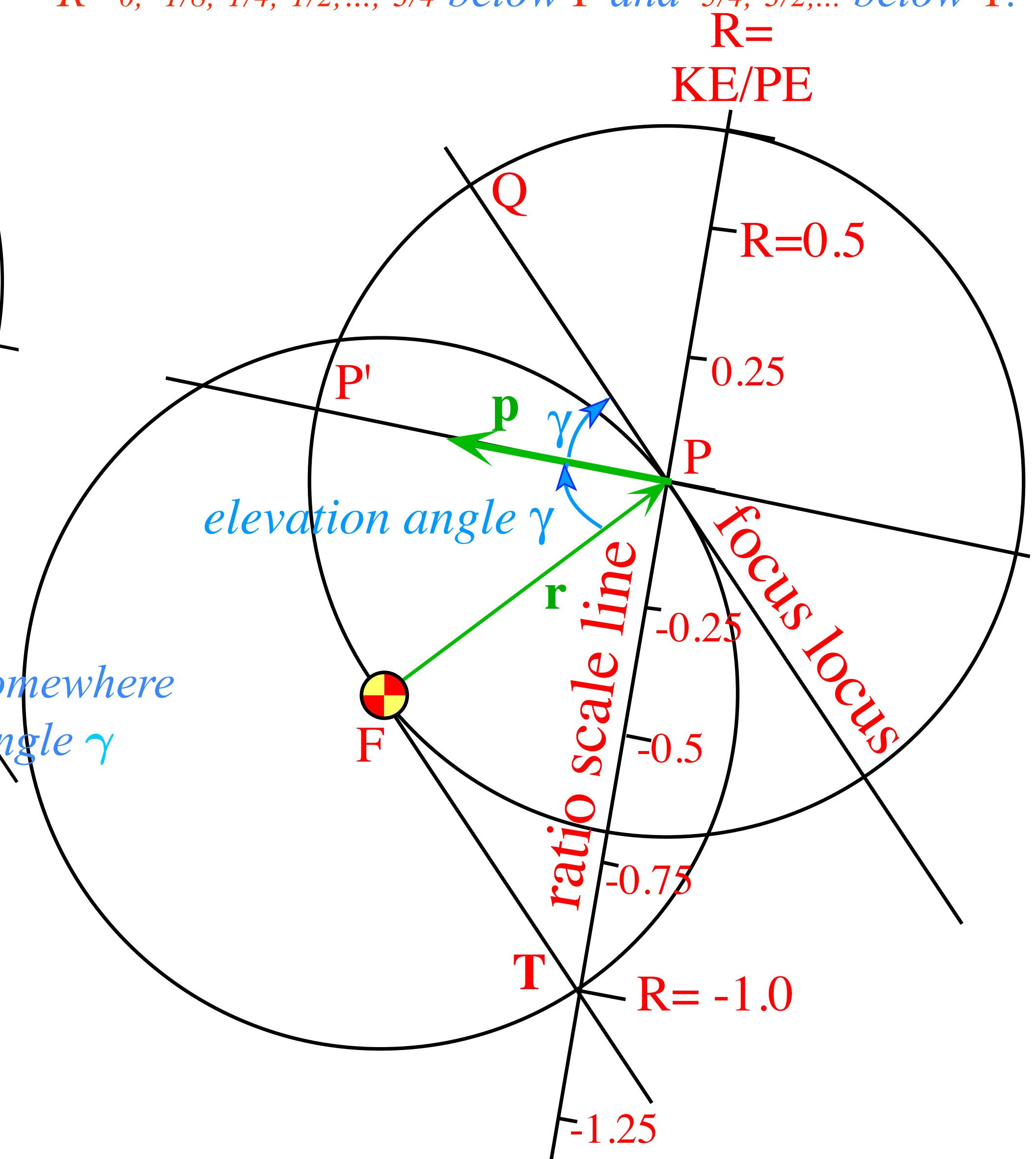


Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**

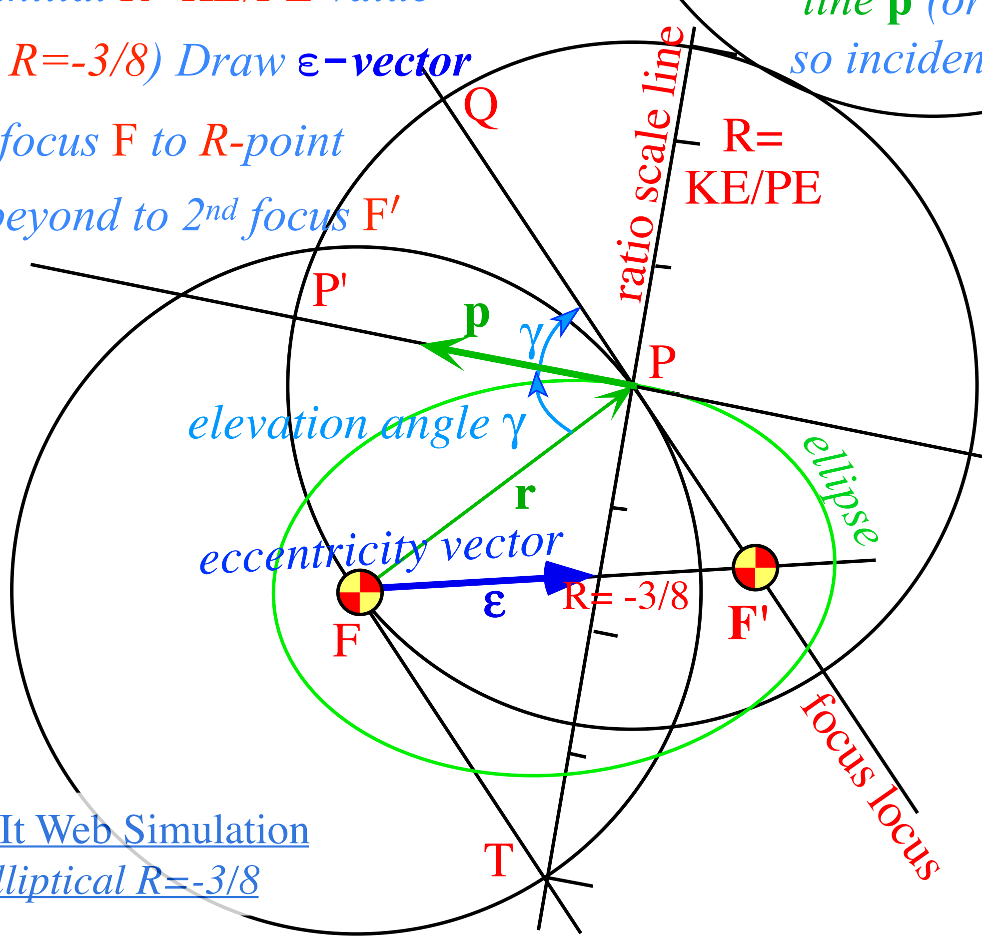


Reason for **focus locus***:*
Line **r** *from 1st focus* **F** *"reflects" off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ *equals reflected-angle* γ

Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and R=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.



Pick initial R=KE/PE value
(here R=-3/8) Draw ***ε−vector***
from focus F *to R-point*
and beyond to 2nd focus F′



[CoulIt Web Simulation](#)
[Elliptical R=-3/8](#)

focus F *and 2nd focus* F′ *allow final construction of orbital trajectory. Here it is an R=-3/8 ellipse.*

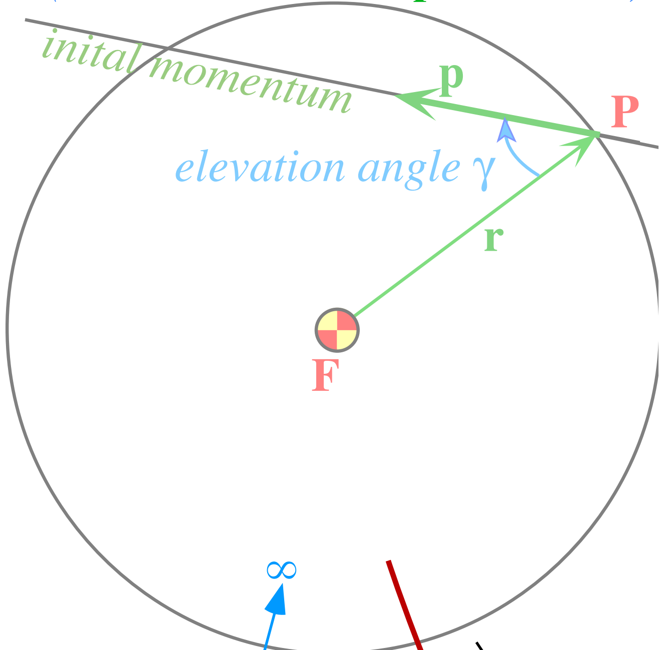
$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

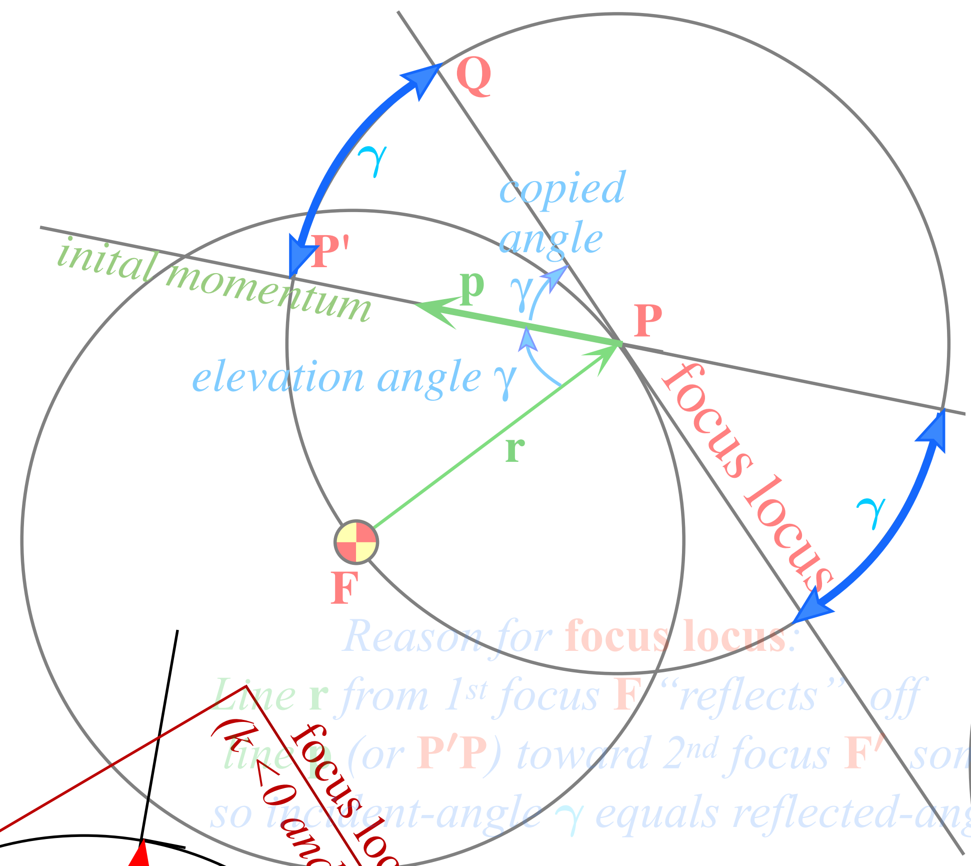
(Detailed Analytic geometry of ε-vector follows.)

General geometric orbit construction using ε-vector and (γ,R)-parameters

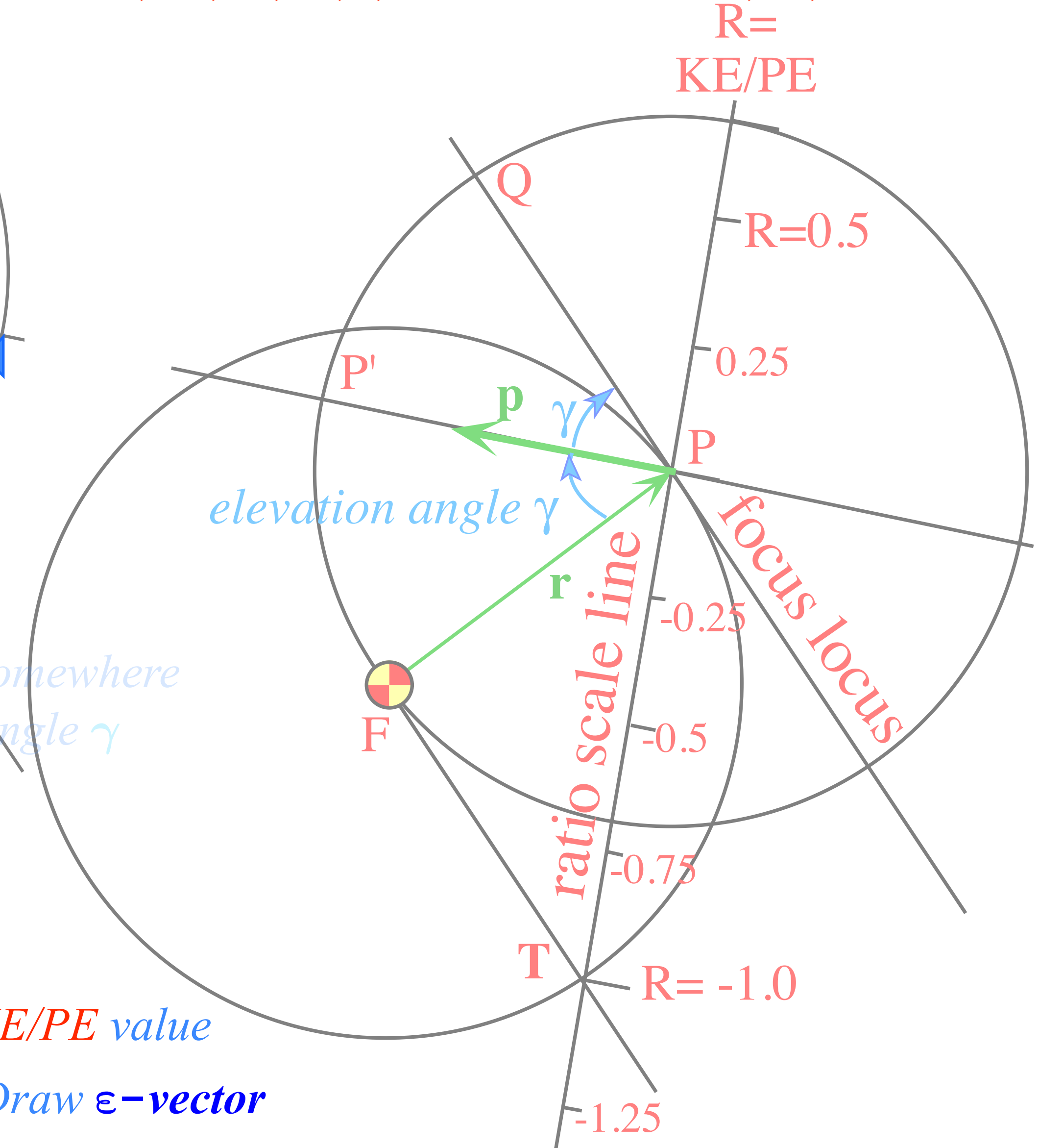
Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

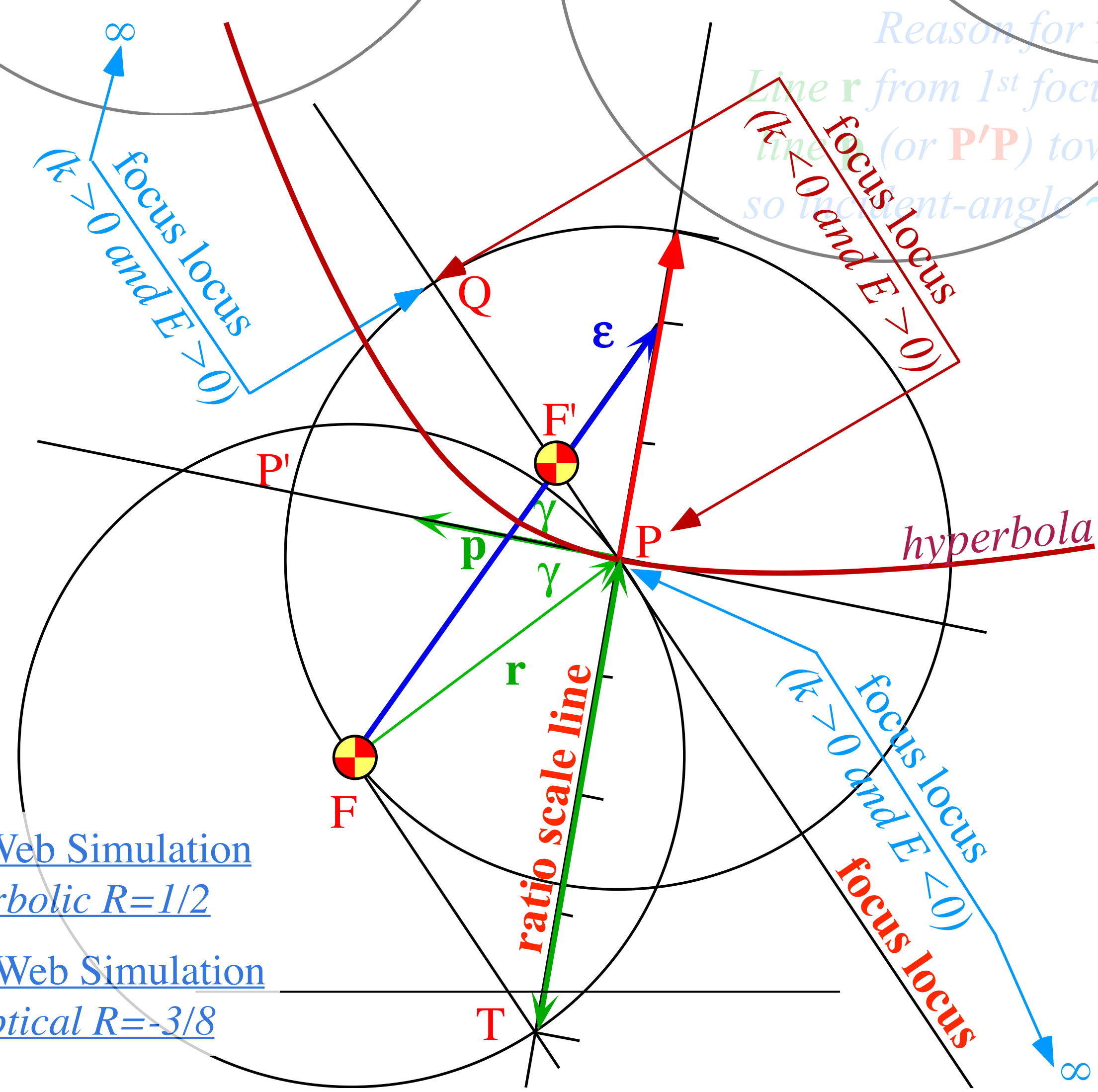


Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**



Copy double angle 2γ (∠FPQ) onto ∠PFT
Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.





Pick initial R=KE/PE value
(here R=+1/2) Draw ***ε−vector***
from focus F *to R-point*
(Here it intersects 2nd focus F′

focus F *and 2nd focus* F′ *allow final*
construction of orbital trajectory.
Here it is an R=+1/2 hyperbola.

$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)} = \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

CoulIt Web Simulation
Hyperbolic R=1/2

CoulIt Web Simulation
Elliptical R=-3/8

(Detailed Analytic geometry of ***ε-vector*** *follows.)*

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

➡ *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Derivation of ε-construction by analytic geometry

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

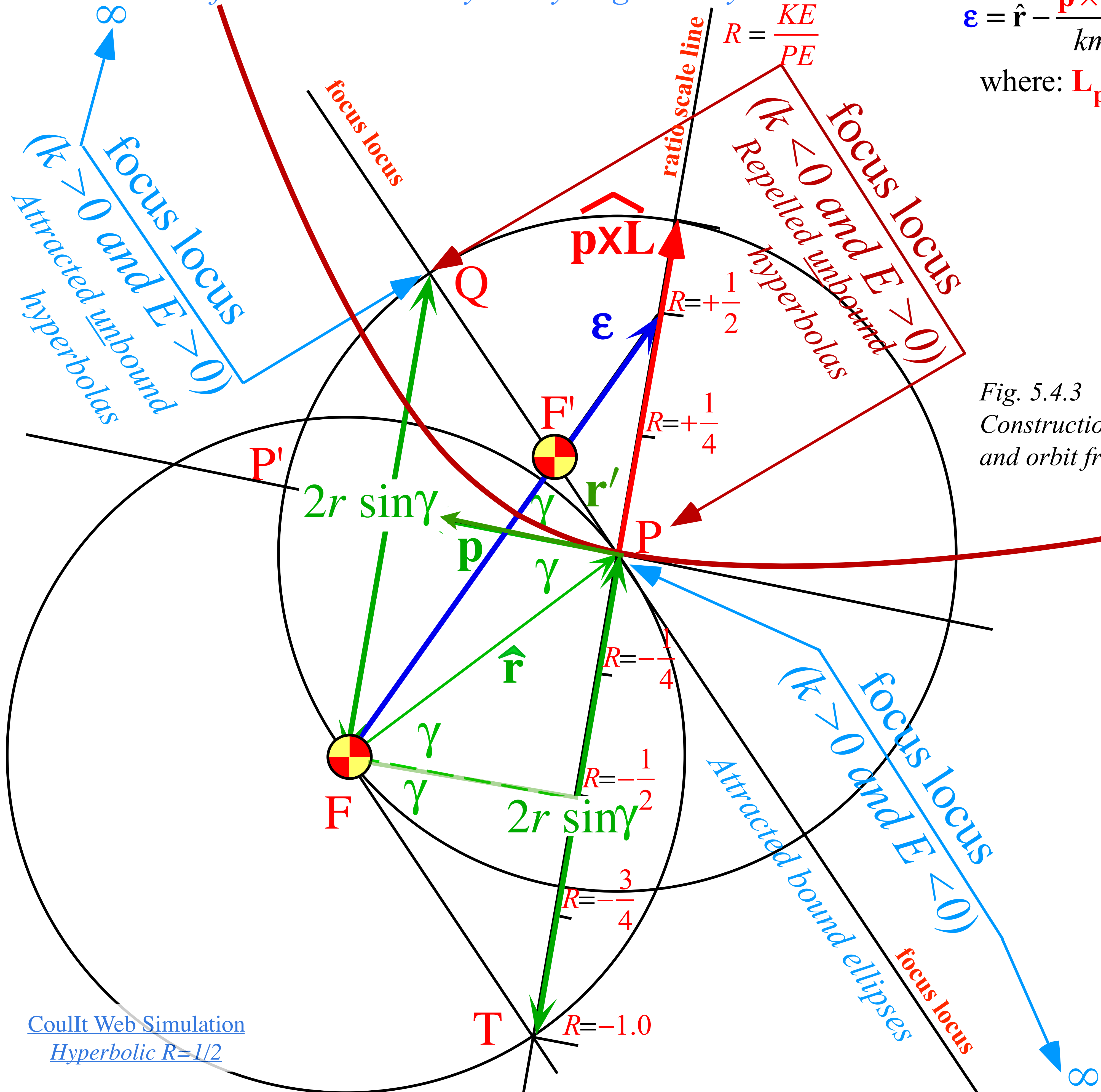


Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma \frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma \frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

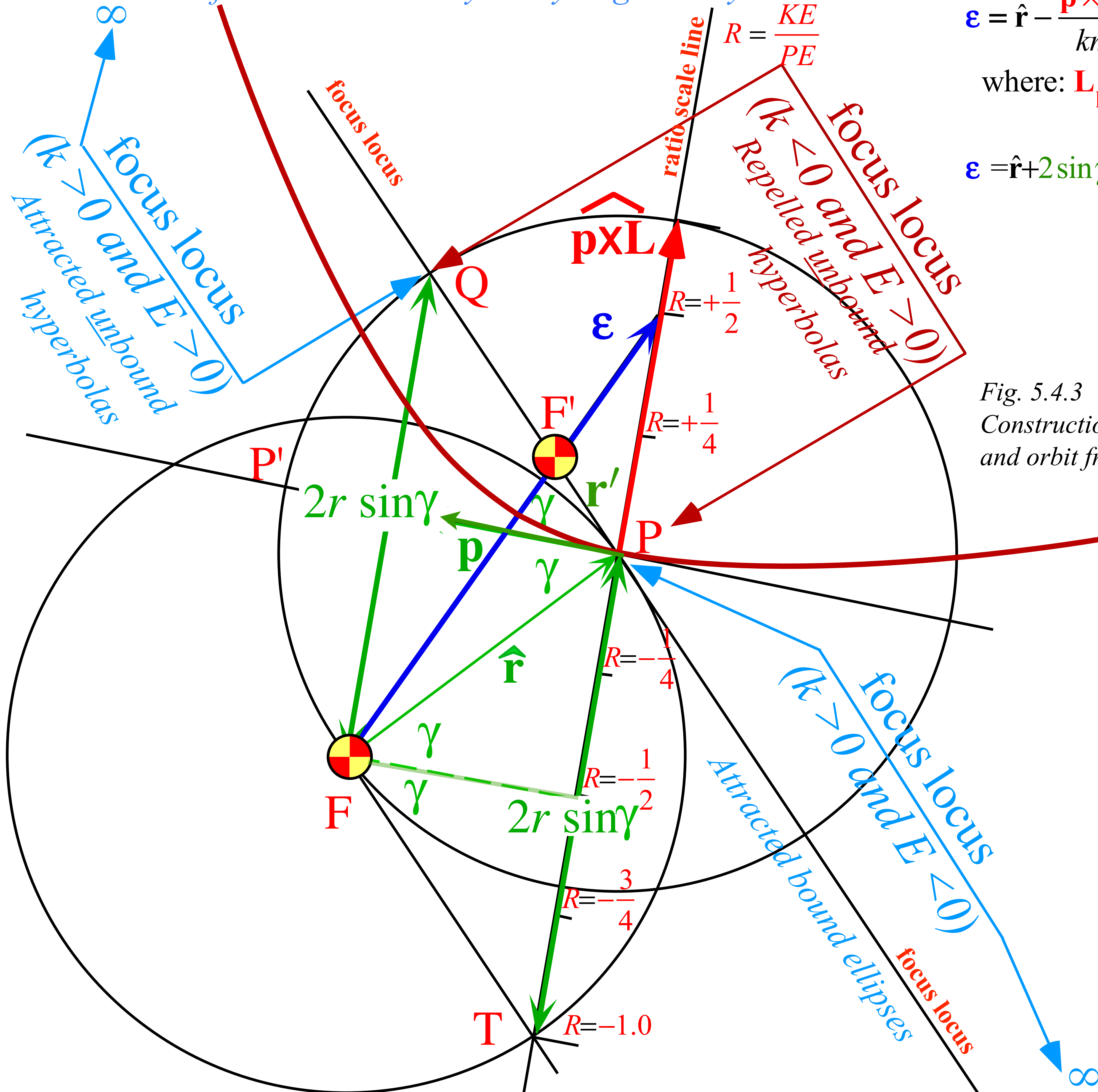
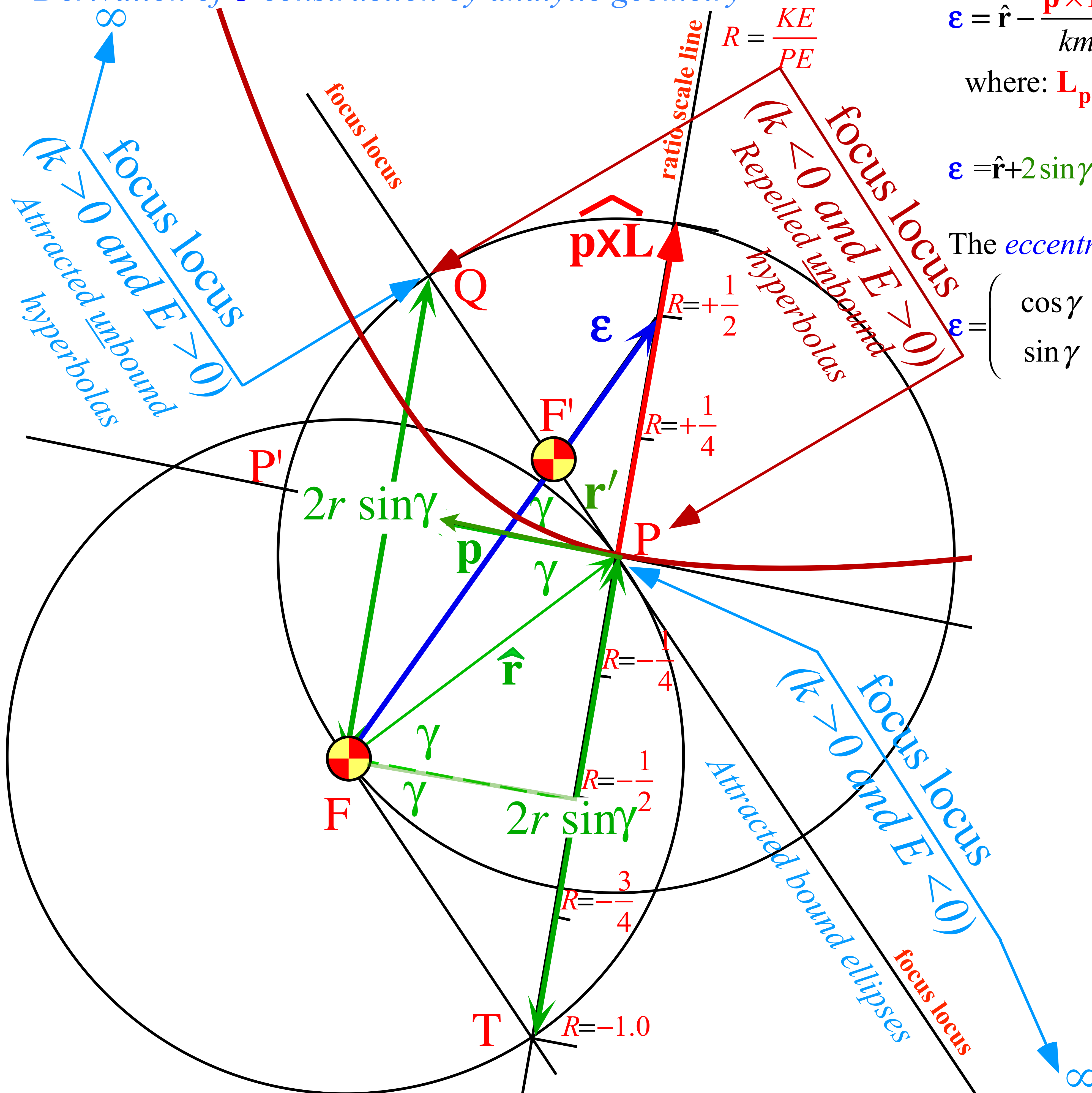


Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of $\boldsymbol{\varepsilon}$-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

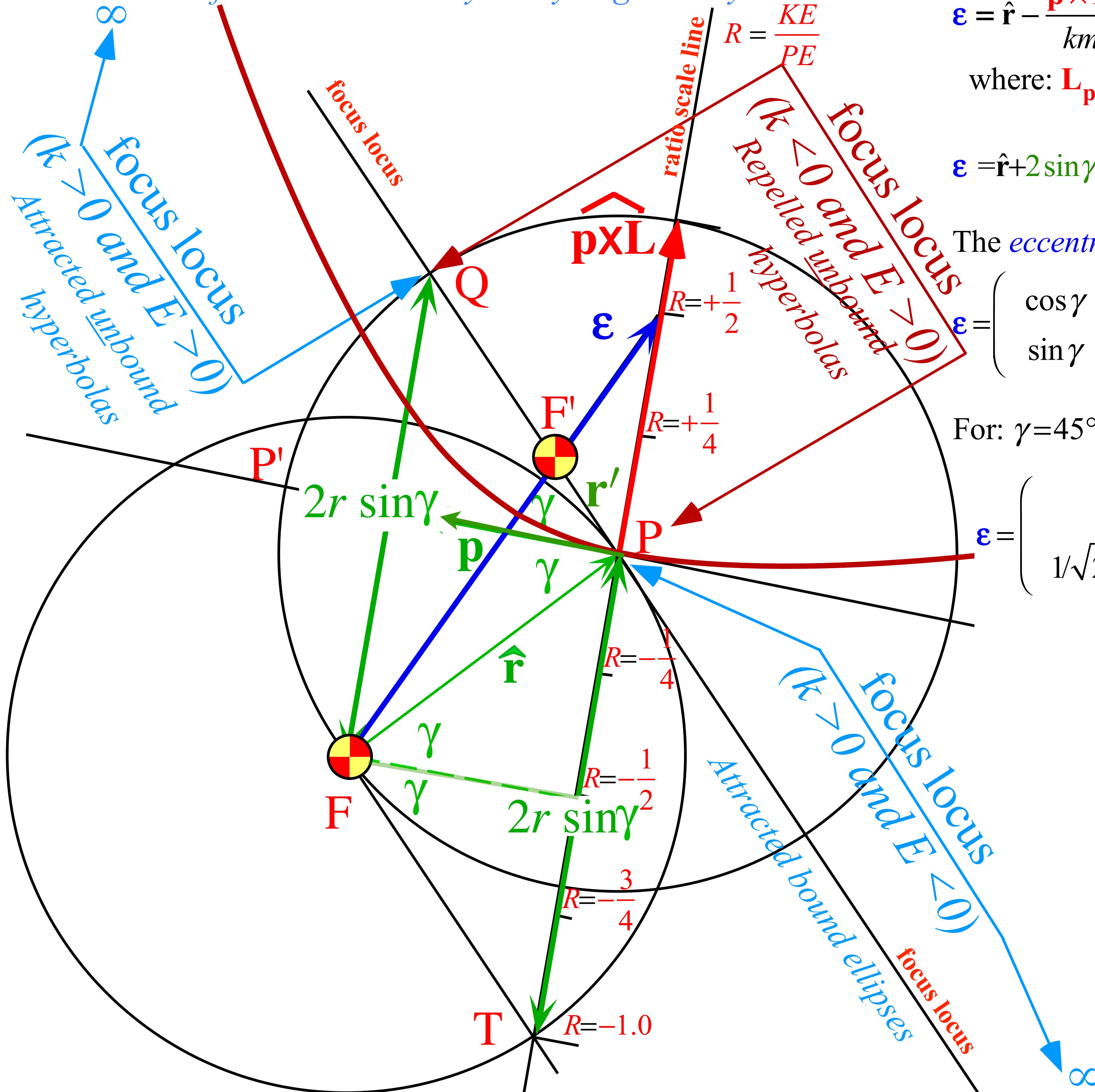
The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

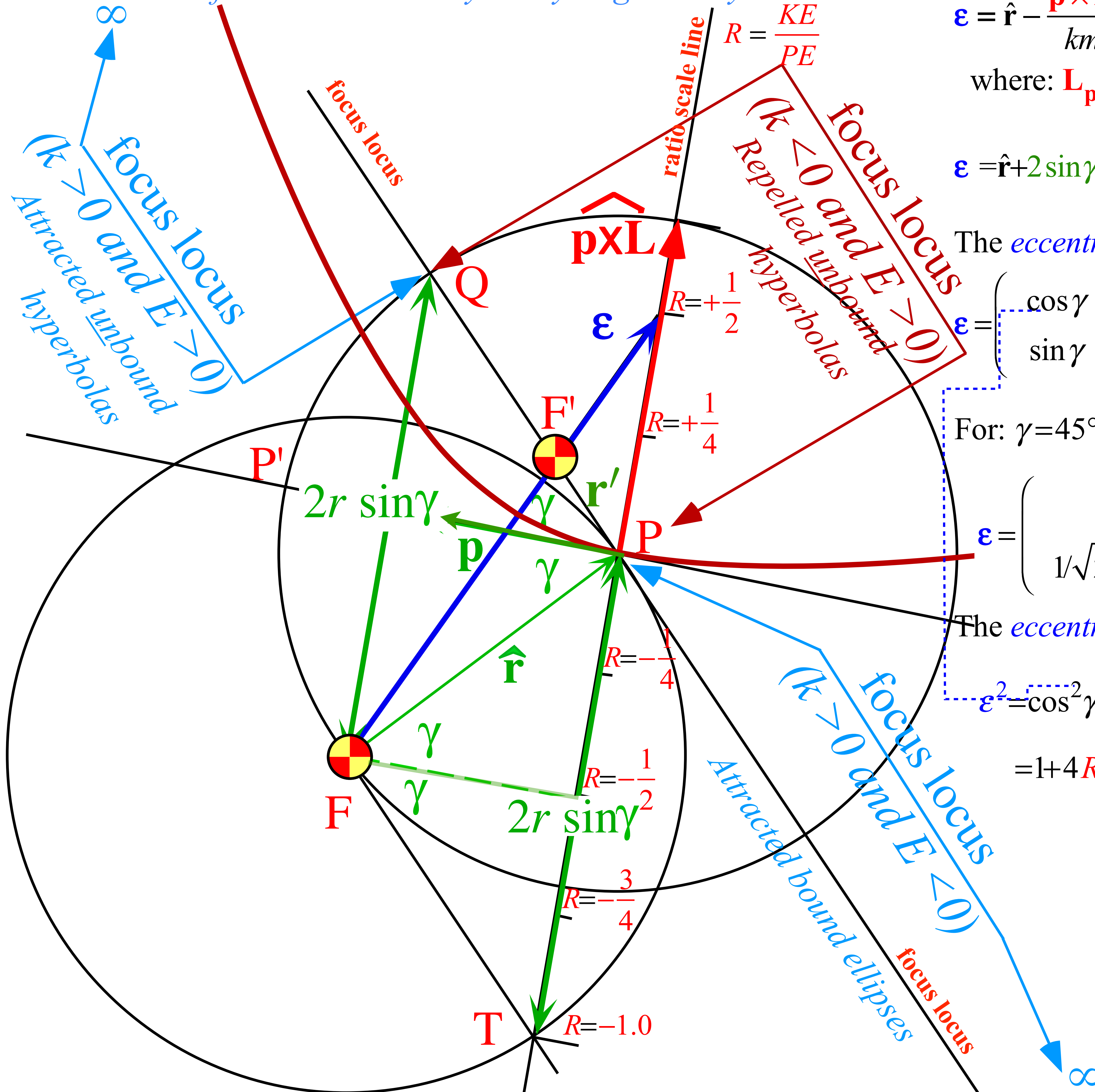
For: $\gamma = 45°$ and: $R = +\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

For: $\gamma = 45°$ and: $R = +\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

The *eccentricity* parameter defined by:

$$\varepsilon^2 = \cos^2\gamma + (2R+1)^2\sin^2\gamma = 1 \pm \frac{a^2}{b^2}$$

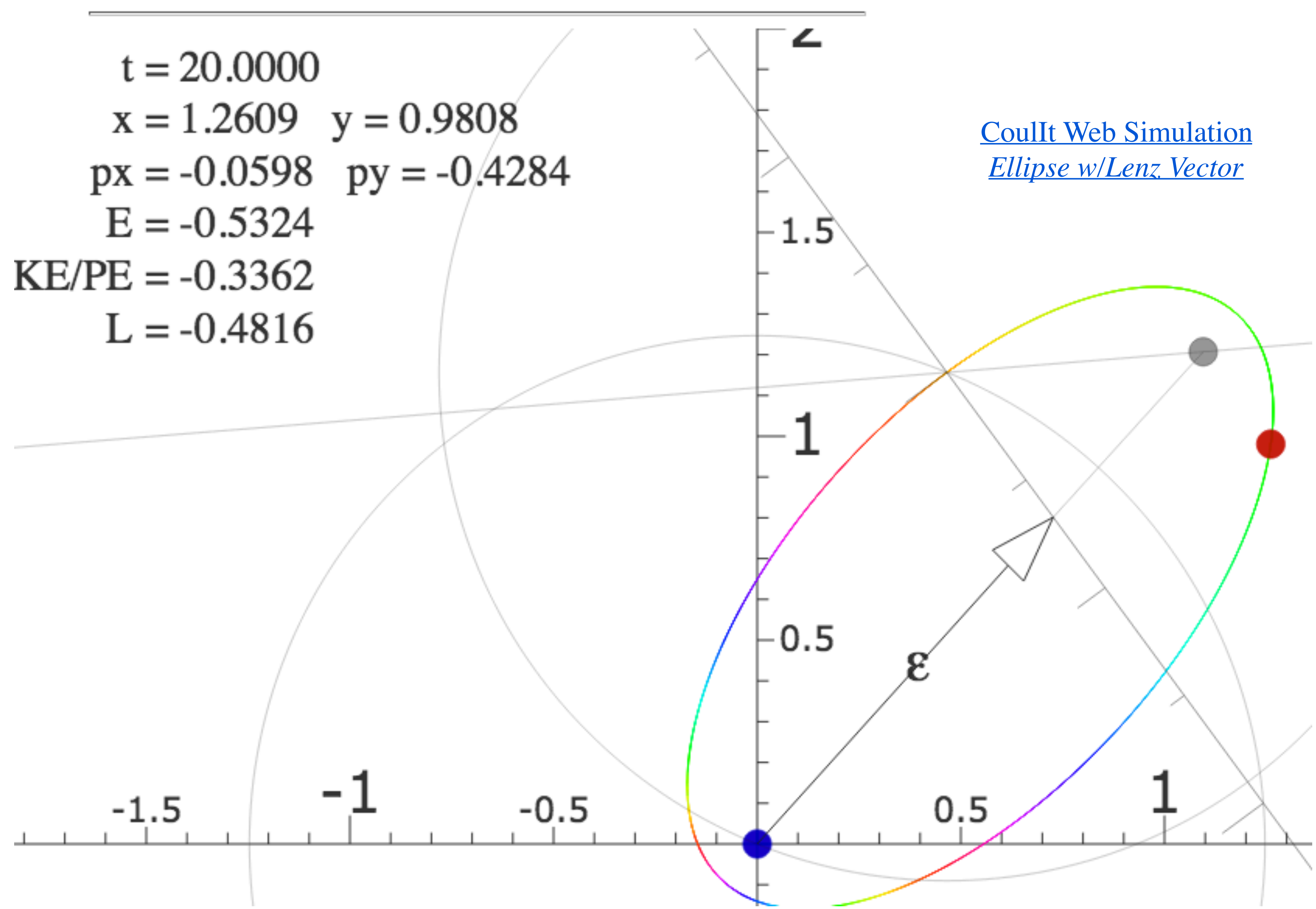
$$= 1 + 4R(R+1)\sin^2\gamma = \frac{5}{2} \quad \text{where: } \sin^2\gamma = \frac{1}{2}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Initial position x(0) =	0.465648
Initial position y(0) =	1.156488
Initial momentum px(0) =	0.591603
Initial momentum py(0) =	0.435114
Terminal time t(off) =	20
Maximum step size dt =	0.01
Charge of Nucleus 1 =	-1
x-Position of Nucleus 1 =	0
y-Position of Nucleus 1 =	0
Charge of Nucleus 2 =	0
Coulomb (k12) =	-1
Core thickness r =	0.000001
x-Stark field Ex =	0
y-Stark field Ey =	0
Zeeman field Bz =	0
Diamagnetic strength k =	0
Plank constant h-bar =	2
Color quantization hues =	64
Color quantization bands =	2
Fractional Error (e^{-x}), x =	8
Particle Size =	9

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☑ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by φ ☐ Elastic ☑ 2 Free ☐
Save to GIF ☐



Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Chapter 4 Molecular Ion Orbits

Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

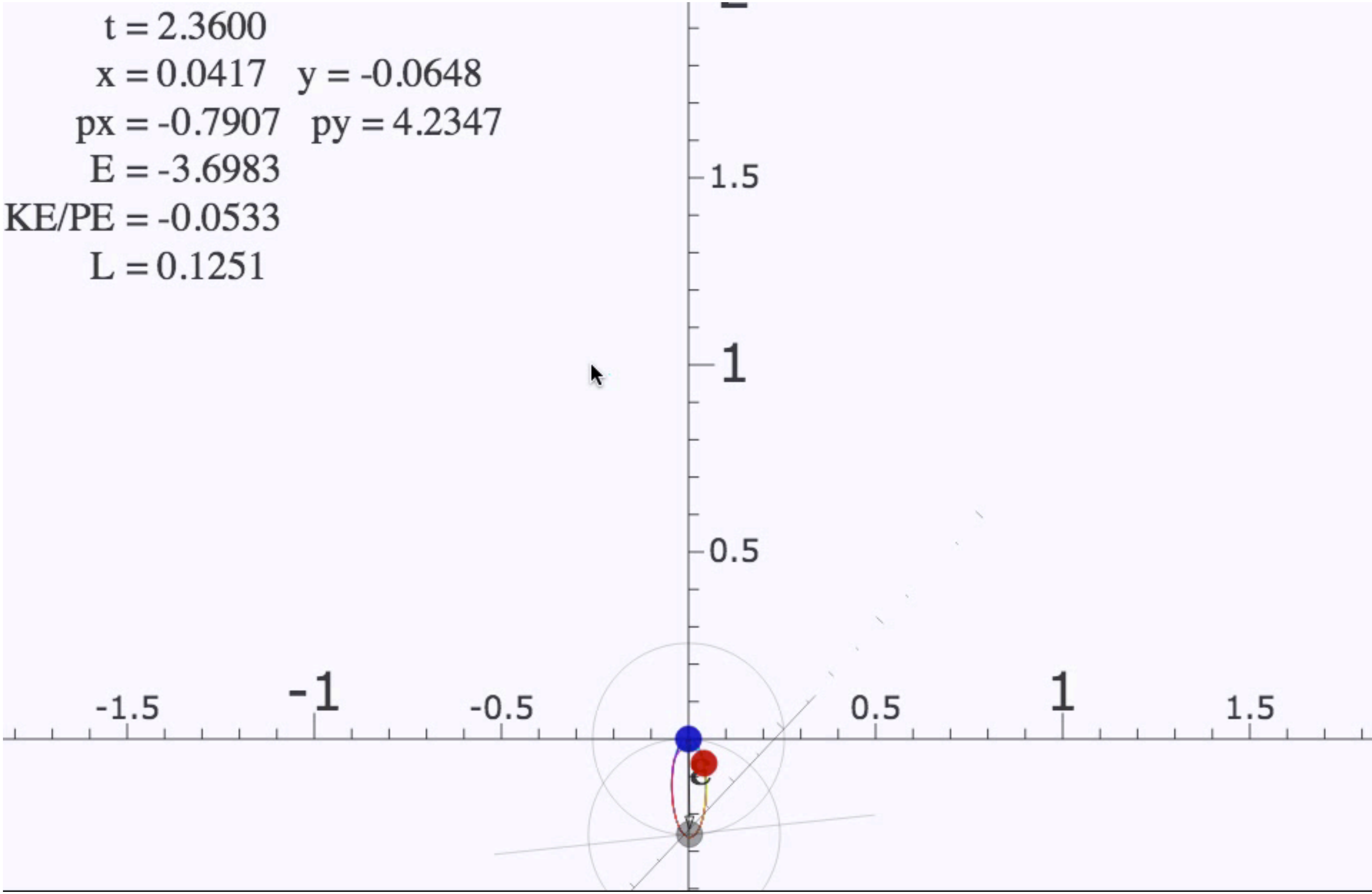
Volcanoes of Io (Paths=180, No color quant.) | Parabolic Fountain (Uniform)
Space Bomb (Coulomb) | Exploding Starlet (IHO)
Synchrotron Motion (Crossed E & B fields)

Rutherford scattering | 2-Electron Orbits

Atomic Orbits

Molecular Ion Orbits

Oscillator Scattering | 2-Particle Orbits | 2-Particle Collision



Play this movie of ε-construction by CoulItWeb

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

➤ *Coulomb orbit algebra of* $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Coulomb orbit algebra of **ε***-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=*m***v**

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r = \frac{\lambda}{1-\varepsilon\cos\phi} = \frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle φ using: $L = mr^2\frac{d\phi}{dt} = mr^2\dot{\phi}$

$$\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m}\frac{1}{r^2}$$

Coulomb orbit algebra of ***ε*-vector** *and Kepler dynamics of momentum* **p**=*m***v**

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ***ε*-*vector*** *and Kepler dynamics of momentum* **p**=*m***v**

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r = \frac{\lambda}{1-\varepsilon\cos\phi} = \frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r} = \frac{dr}{dt} = \frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

Polar angle φ using: $L = mr^2\frac{d\phi}{dt} = mr^2\dot{\phi}$

$$\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m}\frac{1}{r^2} = \frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

using: $\frac{1}{r^2} = \left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of **ε-vector** *and Kepler dynamics of momentum* **p**=*m***v**

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum p=mv

*Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** or momentum **p**=m**v***

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r}=\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum p=mv

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi$$

$$\dot{r}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

again using: $L=mr^2\dot{\phi}$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=m**v**

*Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** or momentum **p**=m**v***

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi \qquad \text{using: } \frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$$

$$\dot{r}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi \qquad \text{again using: } L=mr^2\dot{\phi}$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot{\phi}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

$$\text{using: } \frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

Cartesian x = r cos φ:

$$\dot{x}=\frac{dx}{dt}=\qquad \dot{r}\cos\phi-\sin\phi\; r\dot{\phi}$$

Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\qquad \dot{r}\sin\phi+\cos\phi\; r\dot{\phi}$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity $\mathbf{v}$ *or momentum* $\mathbf{p}=m\mathbf{v}$

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi$$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

$$\boxed{\dot{r}}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=\boxed{-\frac{k}{L}\varepsilon\sin\phi}$$

again using: $L=mr^2\dot{\phi}$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$\boxed{r\dot{\phi}}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Cartesian x = r cos φ:

$$\dot{x}=\frac{dx}{dt}=\boxed{\dot{r}}\cos\phi-\sin\phi\,\boxed{r\dot{\phi}}$$

$$=\boxed{-\frac{k}{L}\varepsilon\sin\phi}\cos\phi-\sin\phi\,\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\boxed{\dot{r}}\sin\phi+\cos\phi\,\boxed{r\dot{\phi}}$$

$$=\boxed{-\frac{k}{L}\varepsilon\sin\phi}\sin\phi+\cos\phi\,\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum p=mv

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot r=\frac{dr}{dt}=\frac{L^2}{km}\,\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot r=\frac{L^2}{km}\,\frac{-\varepsilon\sin\phi\,\dot\phi}{(1-\varepsilon\cos\phi)^2}$$

$$\dot r=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot\phi\,\varepsilon\sin\phi \qquad \textit{using: } \frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$$

$$\dot r=-\frac{k}{L^2}mr^2\dot\phi\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi \qquad \textit{again using: } L=mr^2\dot\phi$$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot\phi$

$$\dot\phi=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$r\dot\phi=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\frac{k}{L}(1-\varepsilon\cos\phi)$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Cartesian x = r cos φ:

$$\dot x=\frac{dx}{dt}=\dot r\cos\phi-\sin\phi\, r\dot\phi$$

$$=-\frac{k}{L}\varepsilon\sin\phi\cos\phi-\sin\phi\,\frac{k}{L}(1-\varepsilon\cos\phi)$$

$$=-\frac{k}{L}\sin\phi$$

Cartesian y = r sin φ:

$$\dot y=\frac{dy}{dt}=\dot r\sin\phi+\cos\phi\, r\dot\phi$$

$$=-\frac{k}{L}\varepsilon\sin\phi\sin\phi+\cos\phi\,\frac{k}{L}(1-\varepsilon\cos\phi)$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=m**v**

*Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** or momentum **p**=m**v***

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

$$\dot{r}=\frac{dr}{dt}=\frac{L^2}{km}\frac{-\frac{d}{dt}(-\varepsilon\cos\phi)}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=\frac{L^2}{km}\frac{-\varepsilon\sin\phi\,\dot{\phi}}{(1-\varepsilon\cos\phi)^2}$$

$$\dot{r}=-\frac{L^2}{km}\left(\frac{km}{L^2}\right)^2 r^2\dot{\phi}\,\varepsilon\sin\phi$$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

$$\boxed{\dot{r}}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=\boxed{-\frac{k}{L}\varepsilon\sin\phi}$$

again using: $L=mr^2\dot{\phi}$

Polar angle φ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

$$\boxed{r\dot{\phi}}=\frac{L}{mr}=\frac{L}{m}\frac{1}{r}=\frac{L}{m}\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)=\boxed{\frac{k}{L}(1-\varepsilon\cos\phi)}$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Cartesian x = r cos φ:

$$\dot{x}=\frac{dx}{dt}=\boxed{\dot{r}}\cos\phi-\sin\phi\,\boxed{r\dot{\phi}}$$

$$=-\frac{k}{L}\sin\phi$$

Cartesian y = r sin φ:

$$\dot{y}=\frac{dy}{dt}=\boxed{\dot{r}}\sin\phi+\cos\phi\,\boxed{r\dot{\phi}}$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

Velocity:

Momentum:

$$\boxed{p_x=m\dot{x}=-\frac{mk}{L}\sin\phi}$$

$$\boxed{p_y=m\dot{y}=\frac{mk}{L}(\cos\phi-\varepsilon)}$$

***p** traces an off-center circle!*

Rutherford scattering and hyperbolic orbit geometry

- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

- *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

- *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

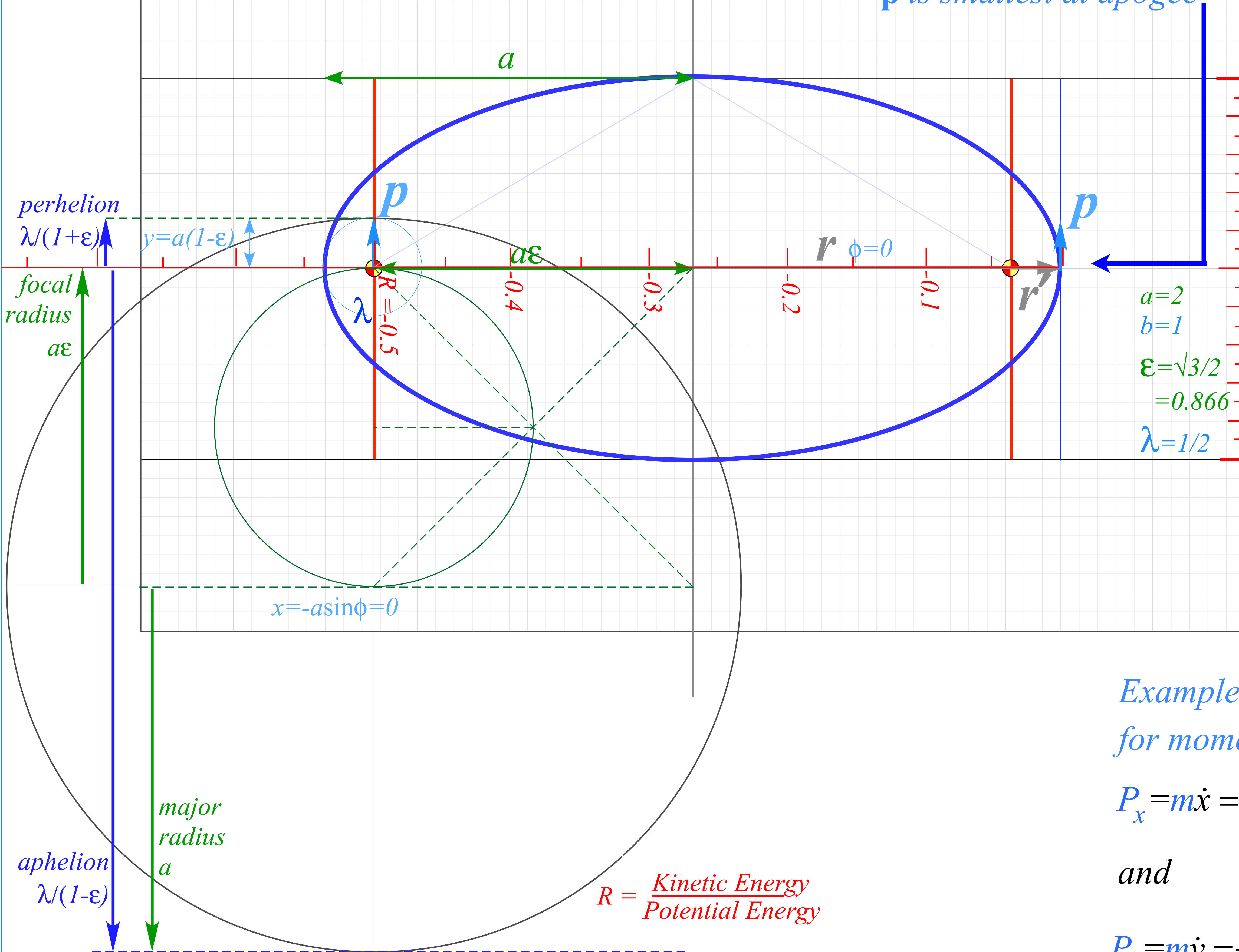
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

- ➔ *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Coulomb **p**=*m***v** *geometry* (ϕ=0)

p *is smallest at apogee*

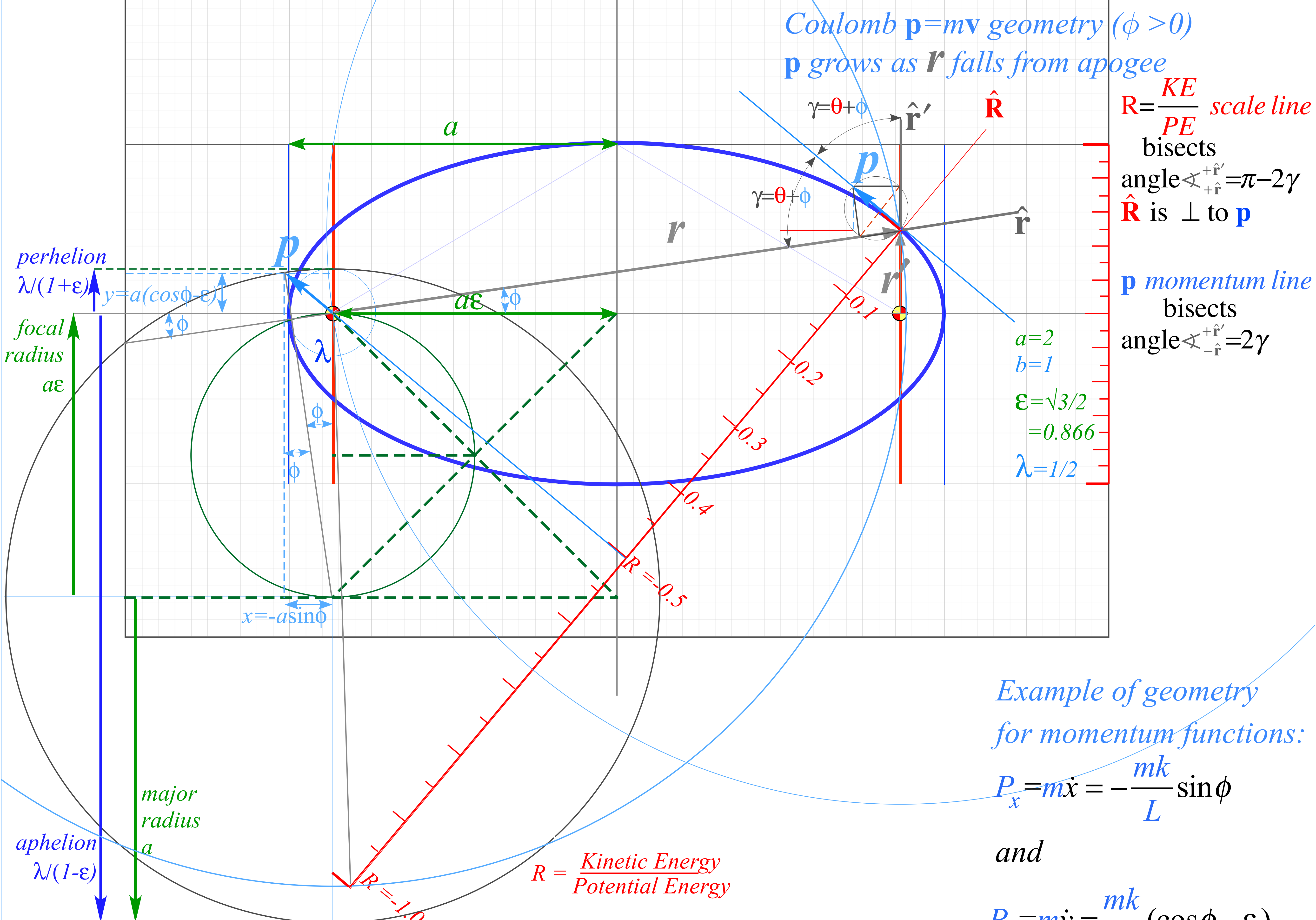


Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$



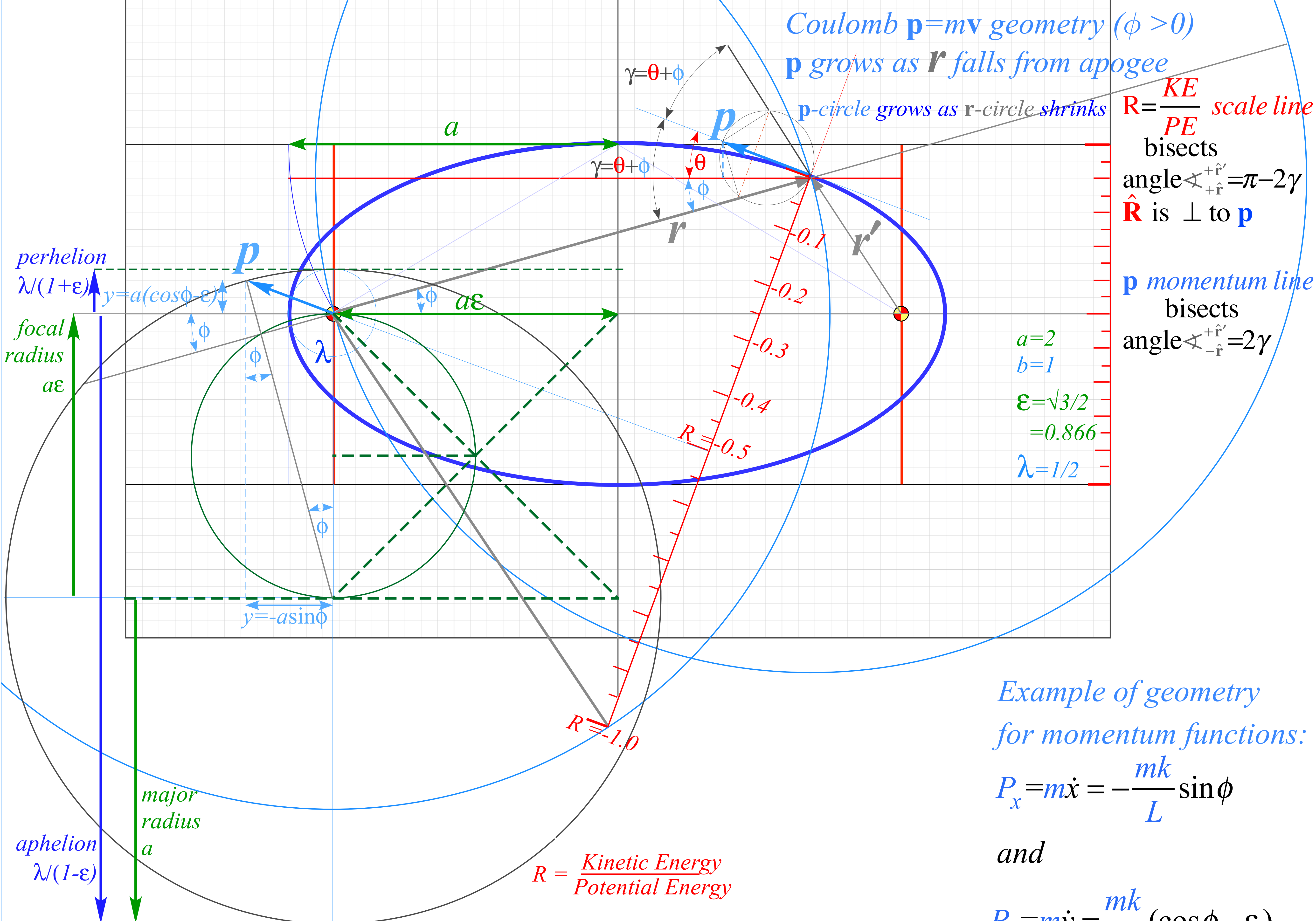
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 67-71.*

*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



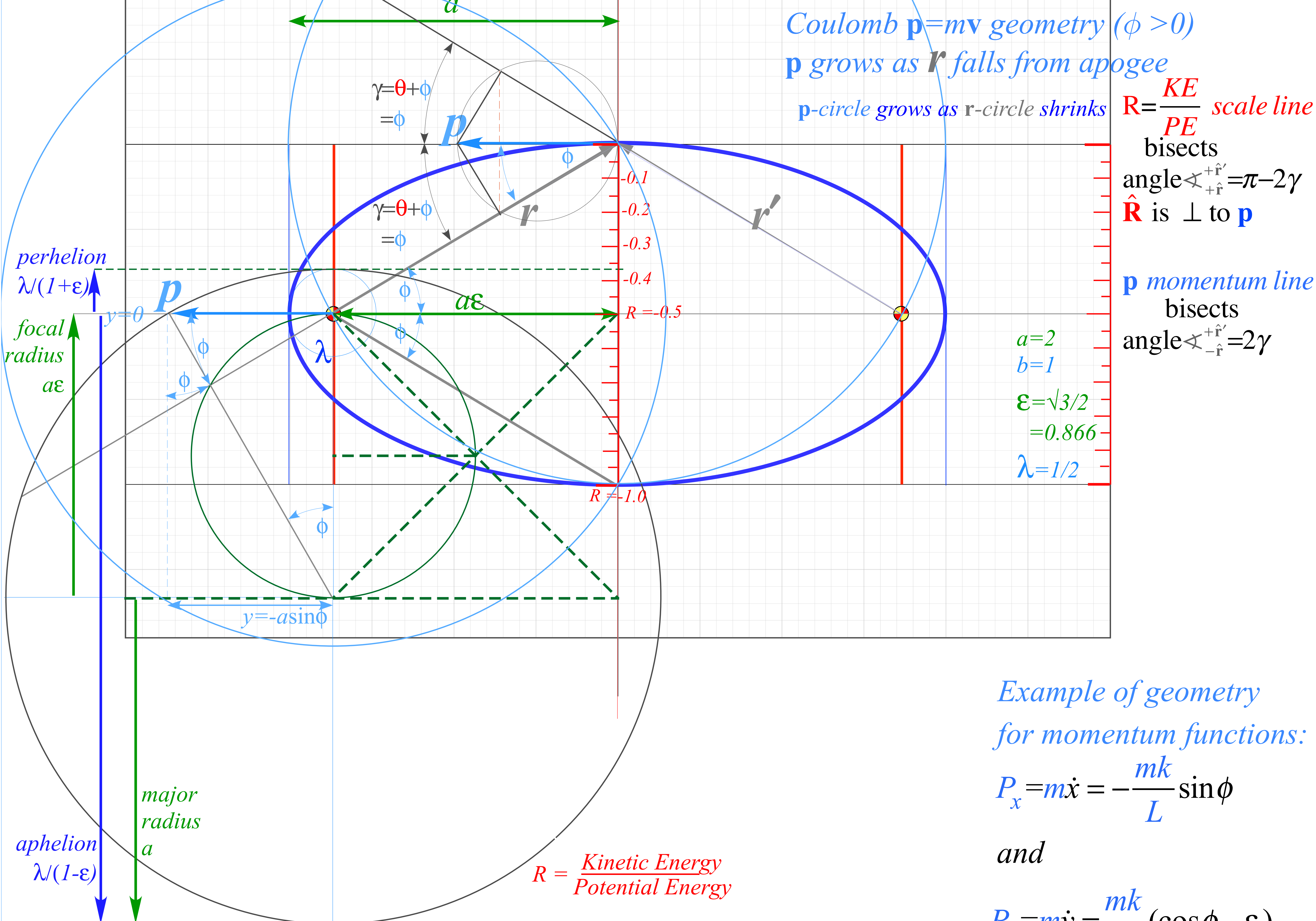
Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ as is derived before on p. 66-70.
*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



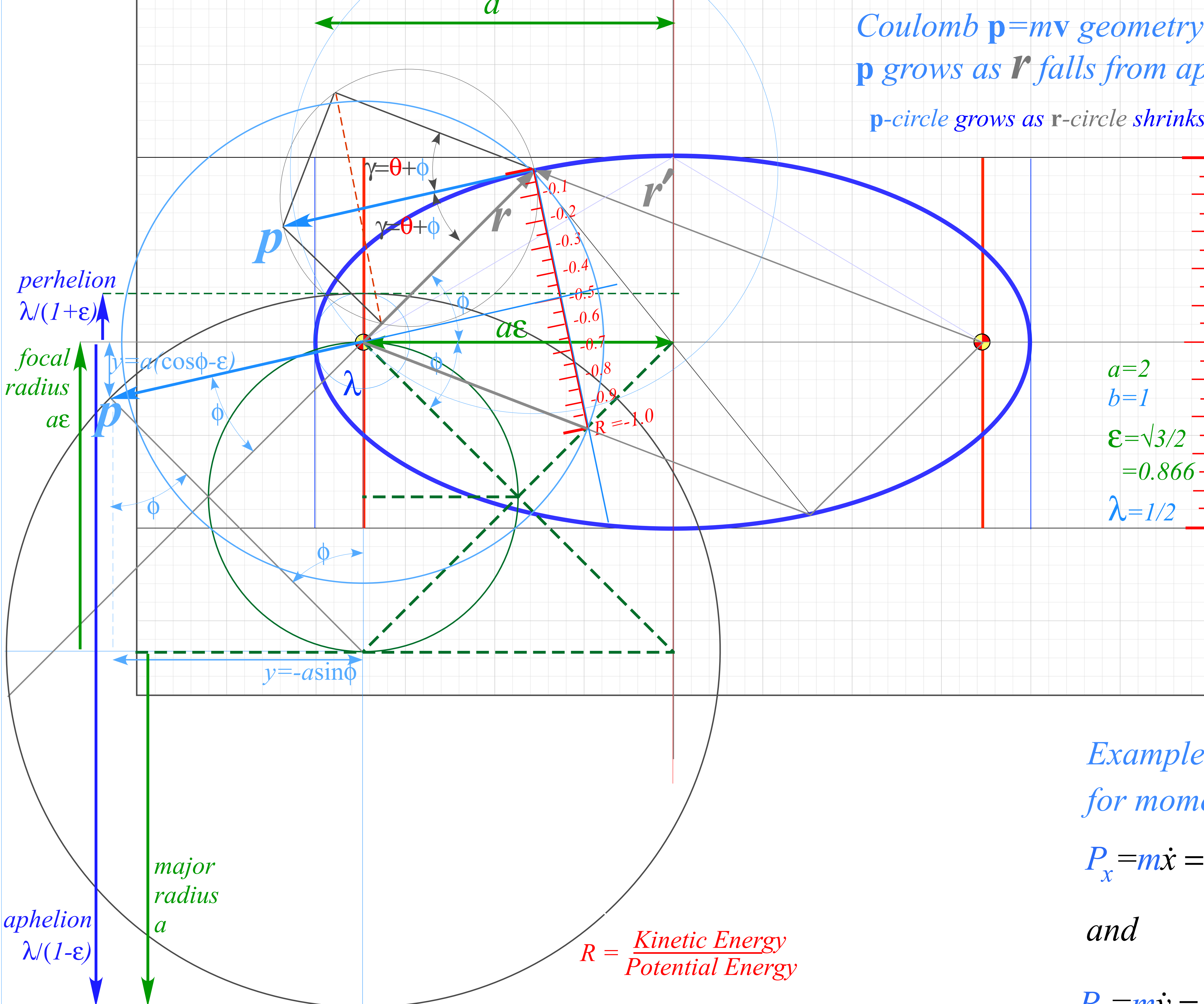
Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 66-70.*
*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



Example of geometry for momentum functions:

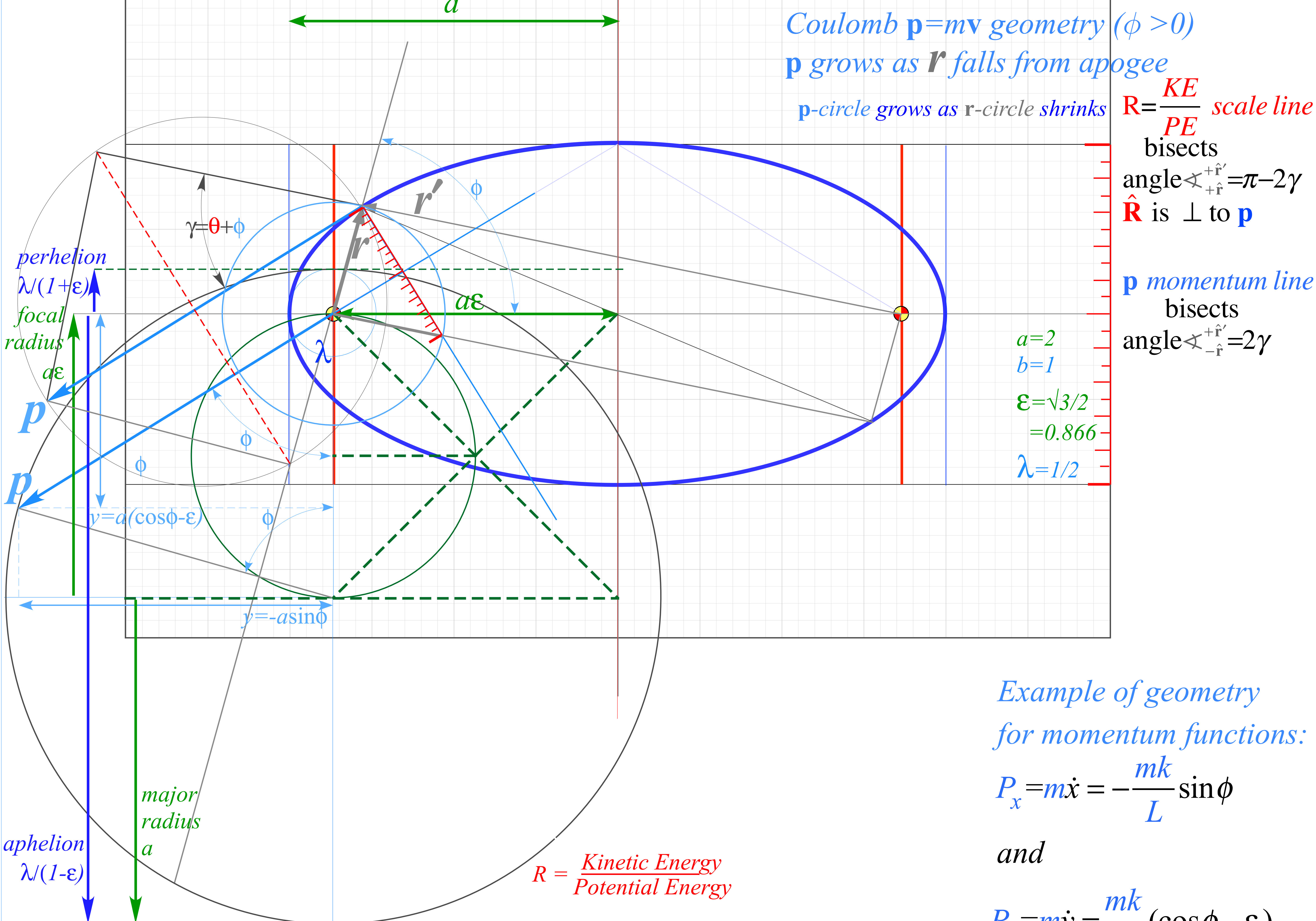
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 66-70.*

*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



Example of geometry for momentum functions:

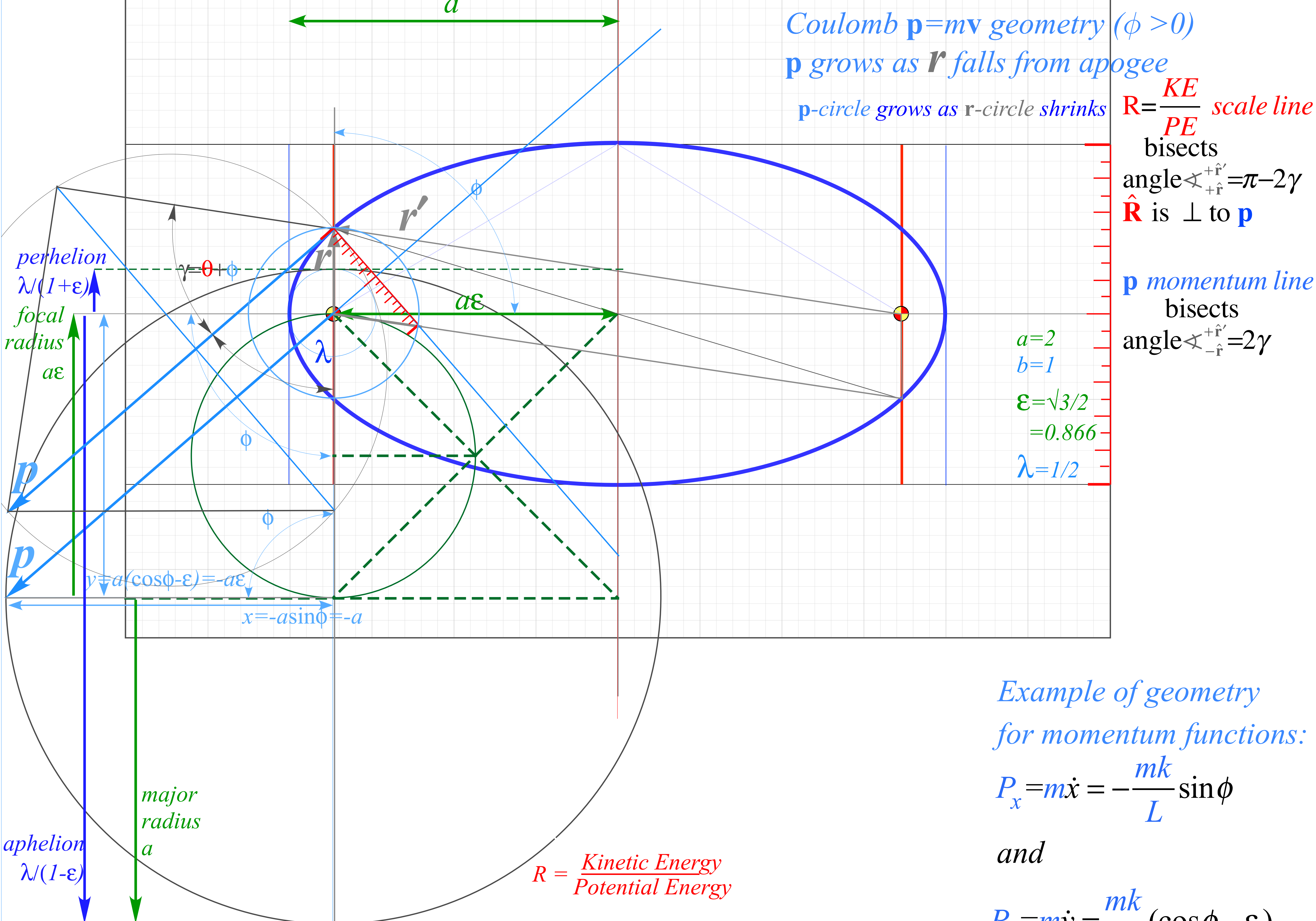
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 66-70.*

*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*



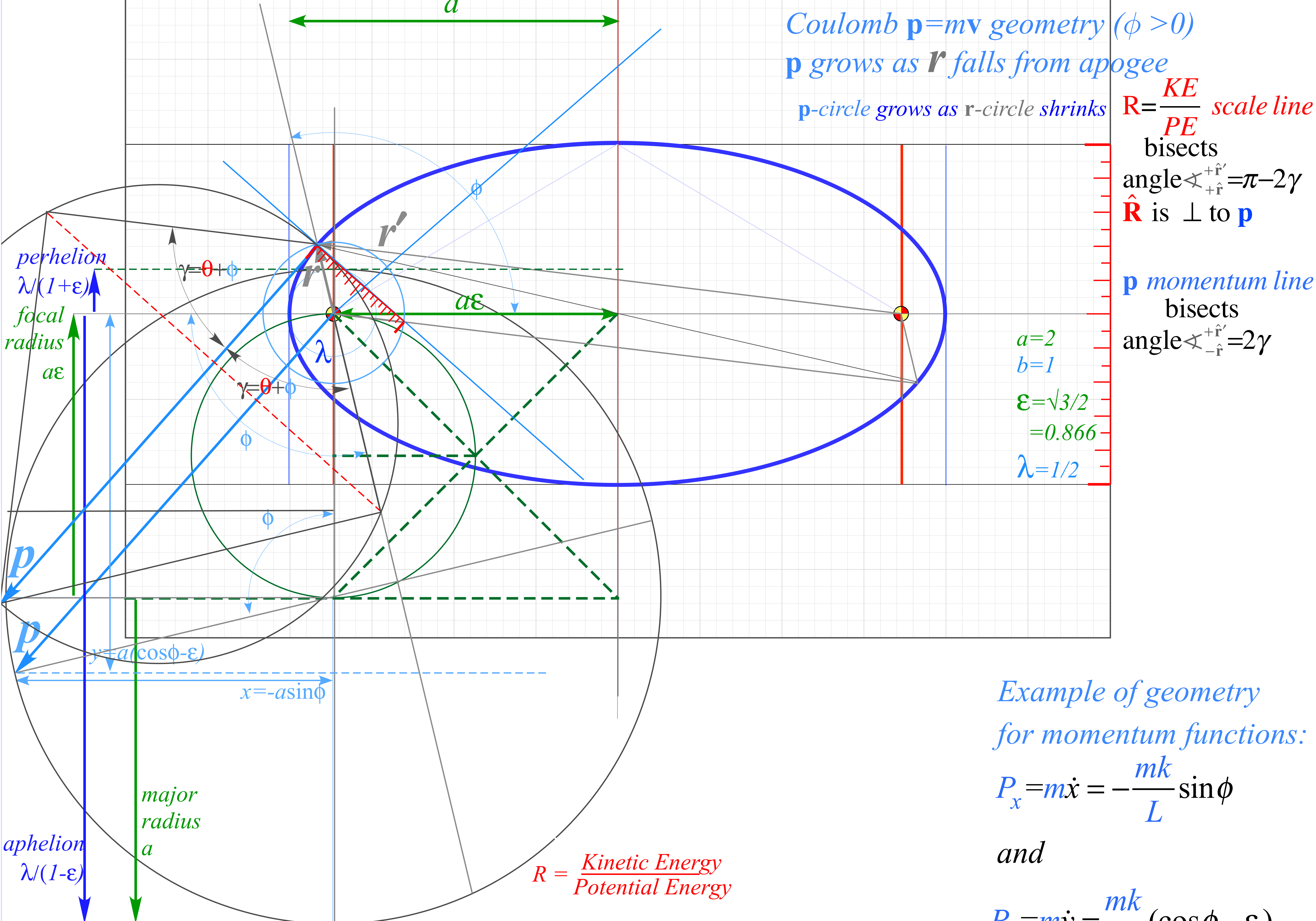
0>*R*=*KE/PE*>-1 *scale subtends angle* 2γ *with length* 2*r* sinγ *as is derived before on p. 66-70.*
Note similarity of (**R**,**r**)*-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$



Example of geometry for momentum functions:

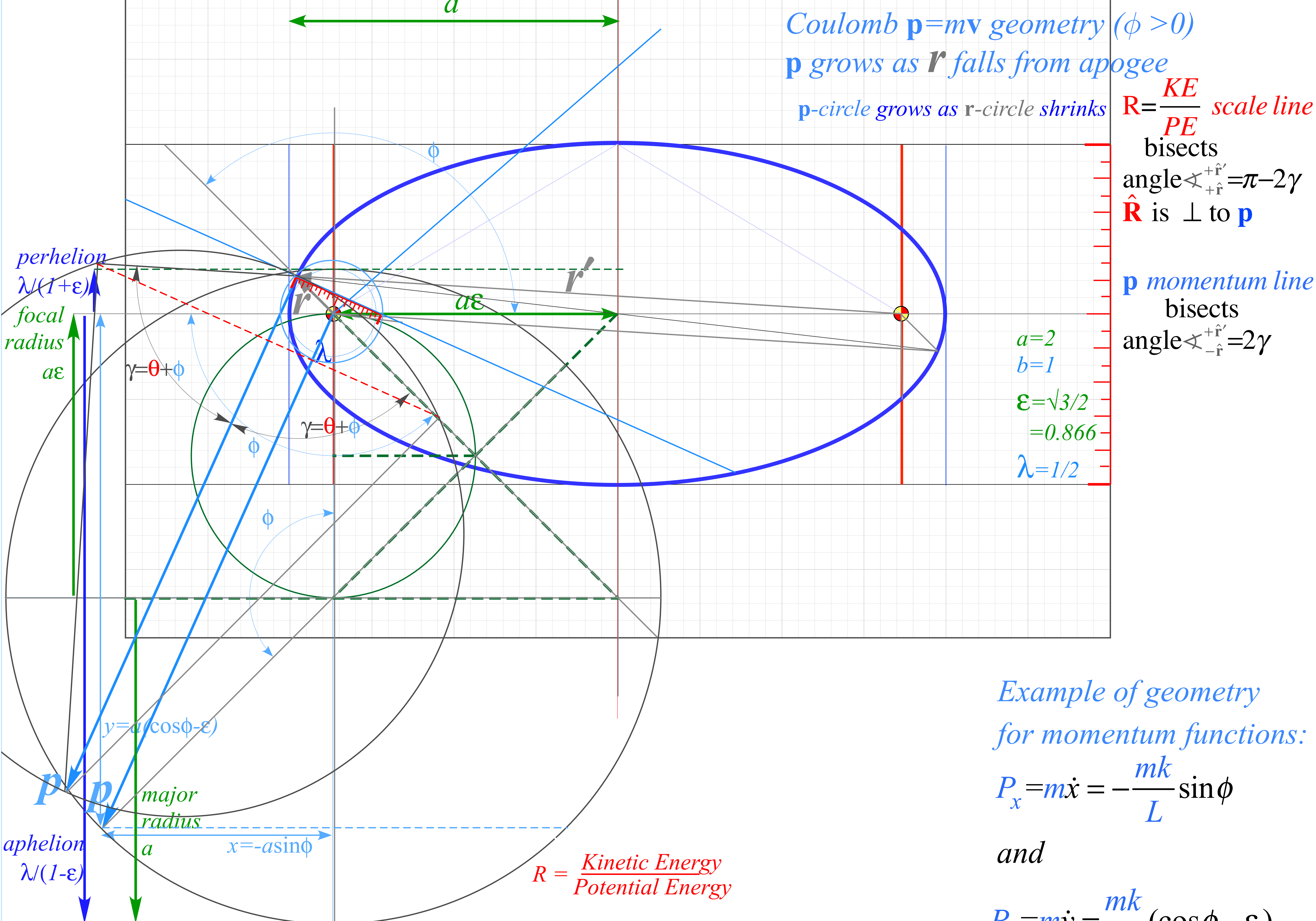
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ as is derived before on p. 66-70.

Note similarity of (R,r)-triangle in r-circle of radius r to that in p-circle of diameter p above.



Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

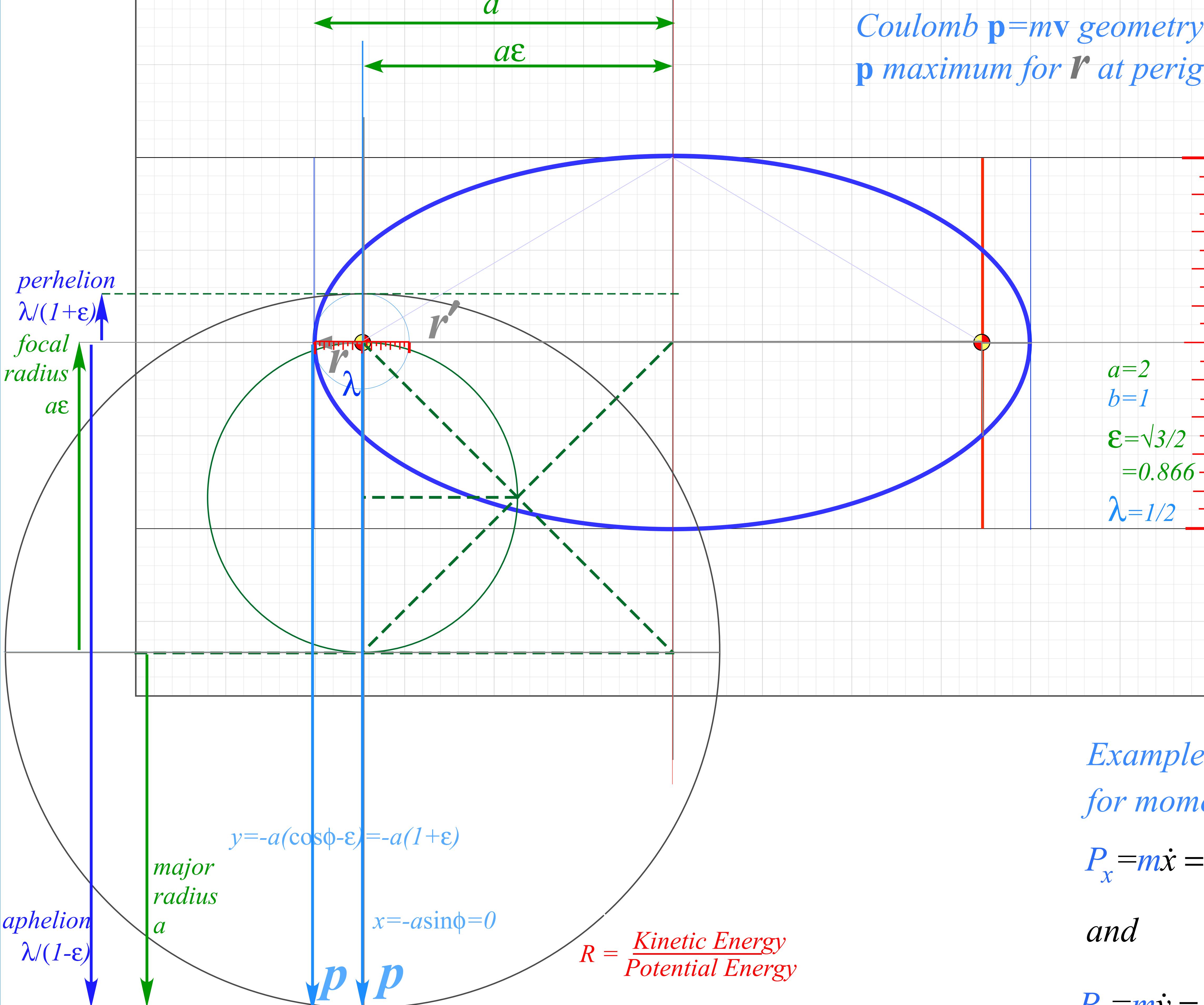
and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ as is derived before on p. 66-70.

Note similarity of (R,r)-triangle in r-circle of radius r to that in p-circle of diameter p above.

Coulomb **p**=*m***v** *geometry (ϕ >0)*
p *maximum for* ***r*** *at perigee*



[CoulIt Web Simulation *Elliptical with Momentum*]

Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 66-70.*

*Note similarity of (***R**,**r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

Review and connection to usual orbital algebra (previous lecture)

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

➔ *Connection formulas for* (γ,R)*-parameters with (a,b) and* (ε,λ)

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:

1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)

Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1)$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1)$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$= 1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$$

$$= 1+\frac{b^2}{a^2} \quad \text{for hyperbola } (\varepsilon > 1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1) \text{ where: } \quad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ implying: } R(R+1) < 0$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola } (\varepsilon > 1) \text{ where: } \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ implying: } R(R+1) > 0$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1 \quad \text{implying:} \quad R(R+1) < 0$$

(or: $-R^2 > R$)
(or: $0 > R > -1$)

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1 \quad \text{implying:} \quad R(R+1) > 0$$

(or: $-R^2 < R$)
(or:$-1 > R > 0$)

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$=1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon<1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=-\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)<0$$

(or: $-R^2>R$)
(or: $0>R>-1$)

$$=1+\frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon>1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)>0$$

(or: $-R^2<R$)
(or: $-1>R>0$)

Total $\frac{-k}{2a}=E=energy=KE+PE$ relates ratio $R=\frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a}=E=KE+PE=R\cdot PE+PE=(R+1)PE=(R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a}=(R+1)\frac{1}{r}=(R+1)$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$=1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon<1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=-\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)<0$$

(or: $-R^2>R$)
(or: $0>R>-1$)

$$=1+\frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon>1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)>0$$

(or: $-R^2<R$)
(or: $-1>R>0$)

Total $\frac{-k}{2a}=E=energy=KE+PE$ relates ratio $R=\frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a}=E=KE+PE=R\cdot PE+PE=(R+1)PE=(R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a}=(R+1)\frac{1}{r}=(R+1)$$

$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)} \text{ assuming } \textbf{unit} \text{ initial radius } (r\equiv 1).\right)$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$=1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon<1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=-\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)<0$$

(or: $-R^2>R$)
(or: $0>R>-1$)

$$=1+\frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon>1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)>0$$

(or: $-R^2<R$)
(or: $-1>R>0$)

Total $\frac{-k}{2a}=E=energy=KE+PE$ relates ratio $R=\frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a}=E=KE+PE=R\cdot PE+PE=(R+1)PE=(R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a}=(R+1)\frac{1}{r}=(R+1)$$

$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)} \text{ assuming } \textbf{unit} \text{ initial radius } (r\equiv 1).\right)$$

$$4R(R+1)\sin^2\gamma=\mp\frac{b^2}{a^2} \quad \text{implies:} \quad 2\sqrt{\mp R(R+1)}\sin\gamma=\frac{b}{a} \quad \text{or:} \quad b=2a\sqrt{\mp R(R+1)}\sin\gamma$$

$$b=r\sqrt{\frac{\mp R}{R+1}}\sin\gamma\left(=\sqrt{\frac{\mp R}{R+1}}\sin\gamma \text{ assuming } \textbf{unit} \text{ initial radius } (r\equiv 1)\right)$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$=1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon<1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=-\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)<0$$

(or: $-R^2>R$)
(or: $0>R>-1$)

$$=1+\frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon>1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)>0$$

(or: $-R^2<R$)
(or: $-1>R>0$)

Total $\frac{-k}{2a}=E=energy=KE+PE$ relates ratio $R=\frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a}=E=KE+PE=R\cdot PE+PE=(R+1)PE=(R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a}=(R+1)\frac{1}{r}=(R+1)$$

$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)} \text{ assuming \textbf{unit} initial radius } (r\equiv 1).\right)$$

$$4R(R+1)\sin^2\gamma=\mp\frac{b^2}{a^2} \quad \text{implies:} \quad 2\sqrt{\mp R(R+1)}\sin\gamma=\frac{b}{a} \quad \text{or:} \quad b=2a\sqrt{\mp R(R+1)}\sin\gamma$$

$$b=r\sqrt{\frac{\mp R}{R+1}}\sin\gamma\left(=\sqrt{\frac{\mp R}{R+1}}\sin\gamma \text{ assuming \textbf{unit} initial radius } (r\equiv 1)\right)$$

Latus radius is similarly related:

$$\lambda=\frac{b^2}{a}=\mp 2rR\sin^2\gamma$$

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:

1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)

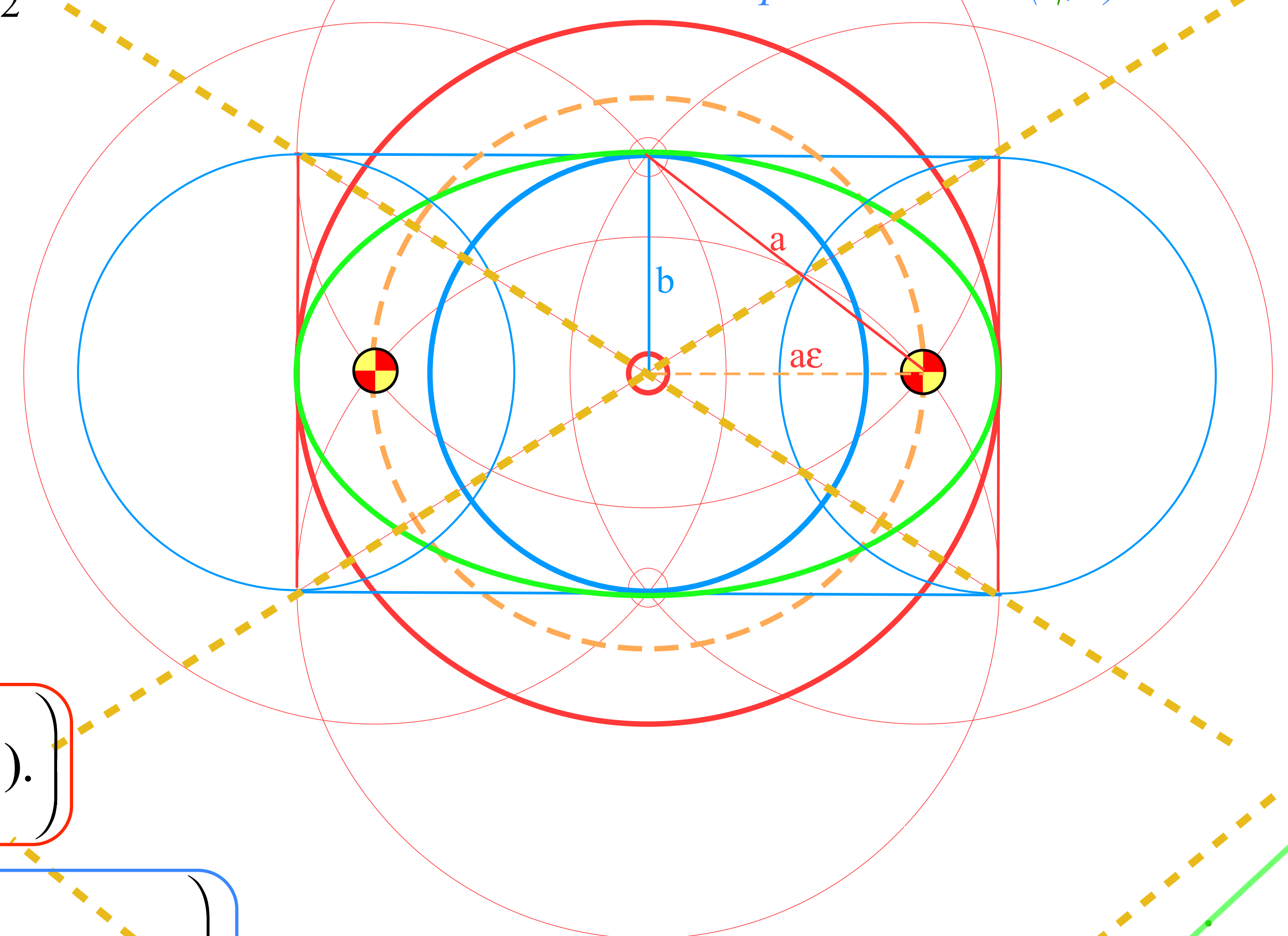
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \text{ ellipse}(\varepsilon < 1) \qquad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2}$$

$$= 1 + \frac{b^2}{a^2} \text{ hyperbola } (\varepsilon > 1) \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2}$$



$$a = \frac{r}{2(R+1)} = \left(\frac{1}{2(R+1)} \text{ assuming } \textbf{unit} \text{ initial radius } (r \equiv 1).\right)$$

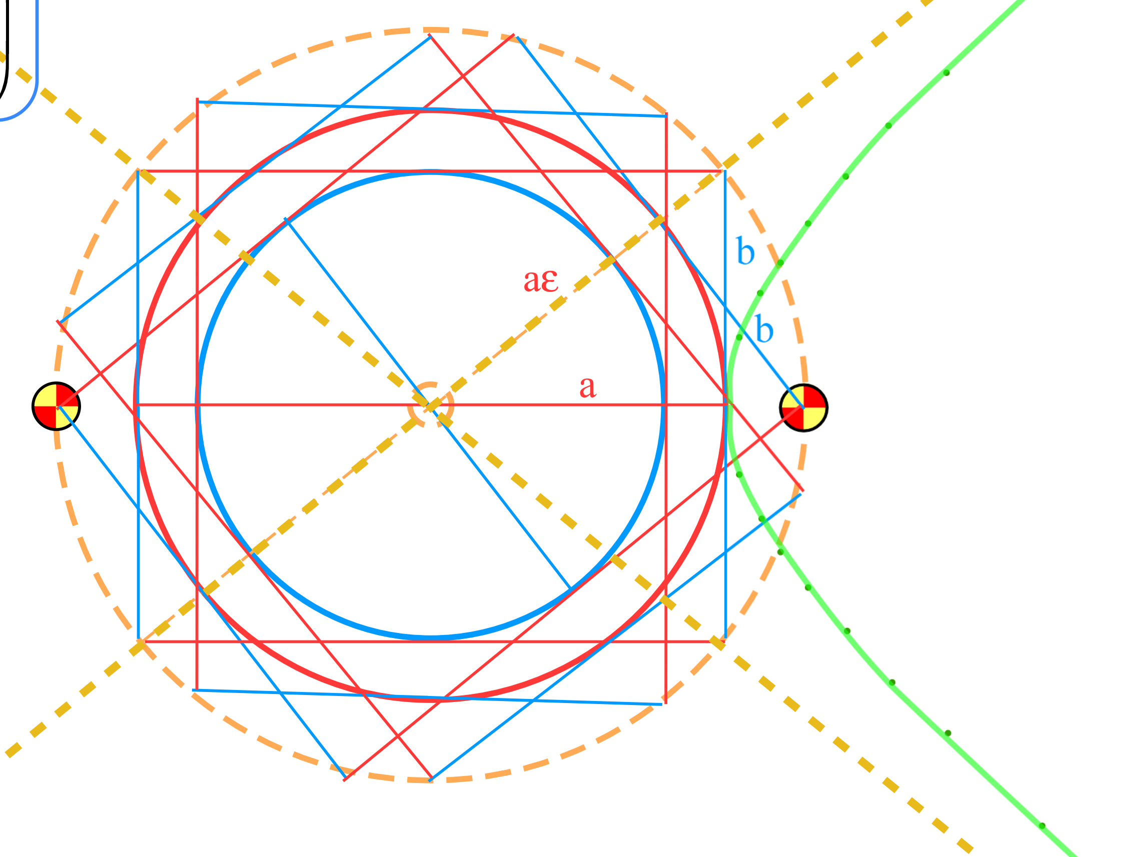
$$b = r\sqrt{\frac{\mp R}{R+1}}\sin\gamma \left(= \sqrt{\frac{\mp R}{R+1}}\sin\gamma \text{ assuming } \textbf{unit} \text{ initial radius } (r \equiv 1)\right)$$

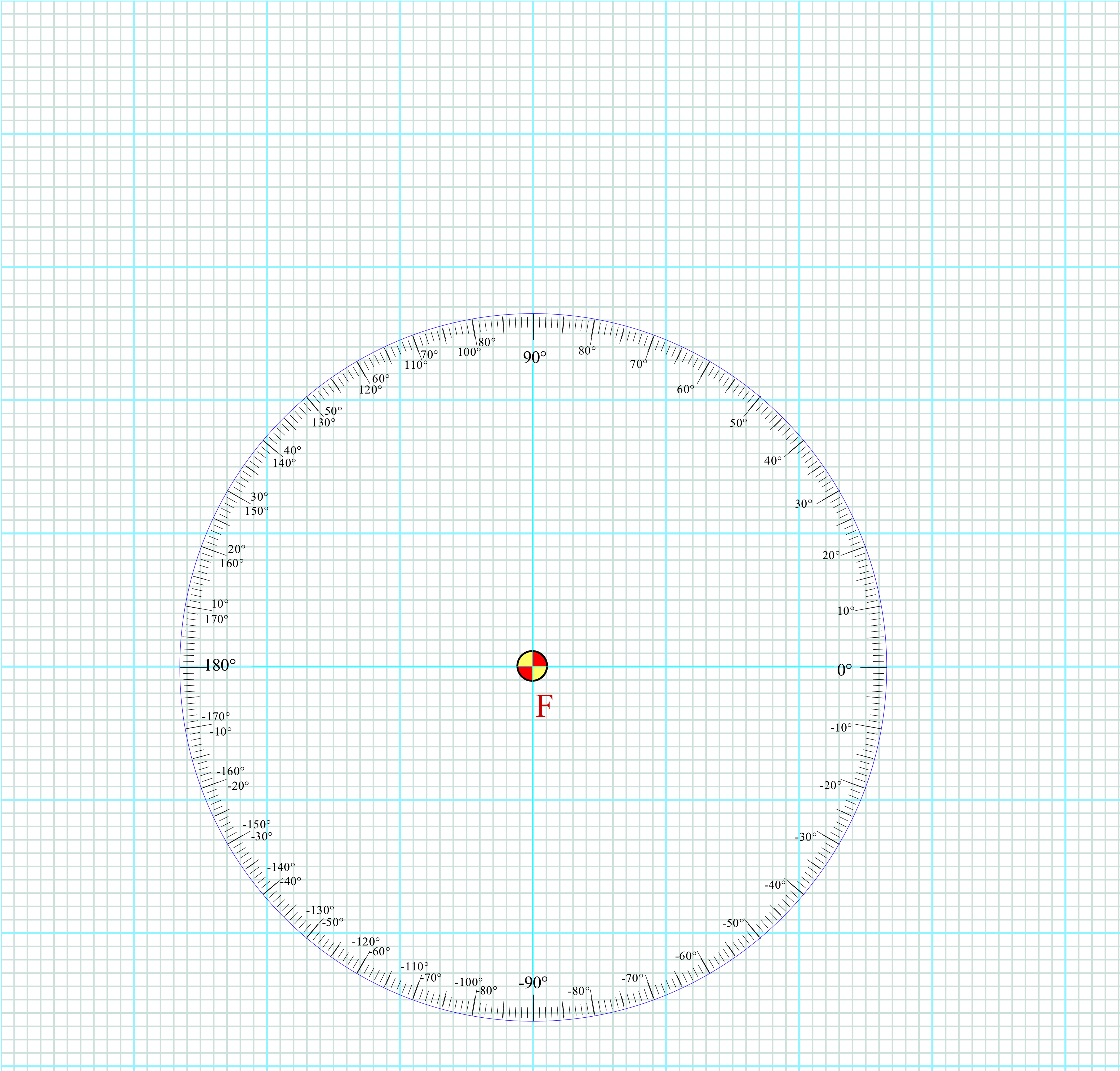
Latus radius is similarly related:

$$\lambda = \frac{b^2}{a} = \mp 2rR\sin^2\gamma$$

From ε^2 result (at top):

$$\frac{b}{a} = 2\sqrt{\mp R(R+1)}\sin\gamma = \sqrt{\pm(1-\varepsilon^2)}$$




90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
-10°
-20°
-30°
-40°
-50°
-60°
-70°
-80°
-90°
-80°
-100°
-70°
-110°
-60°
-120°
-50°
-130°
-40°
-140°
-30°
-150°
-20°
-160°
-10°
-170°
180°
170°
10°
160°
20°
150°
30°
140°
40°
130°
50°
120°
60°
110°
70°
100°
80°
F