Lecture 23
Thur. 11.16.2017

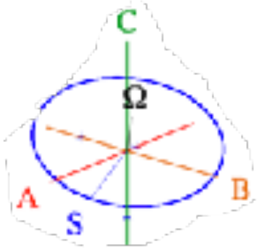
U(2)~R(3) algebra/geometry in classical or quantum theory

(Classical Mechanics with a BANG! Units 4-6, Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5 and Ch. 7)

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$



The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

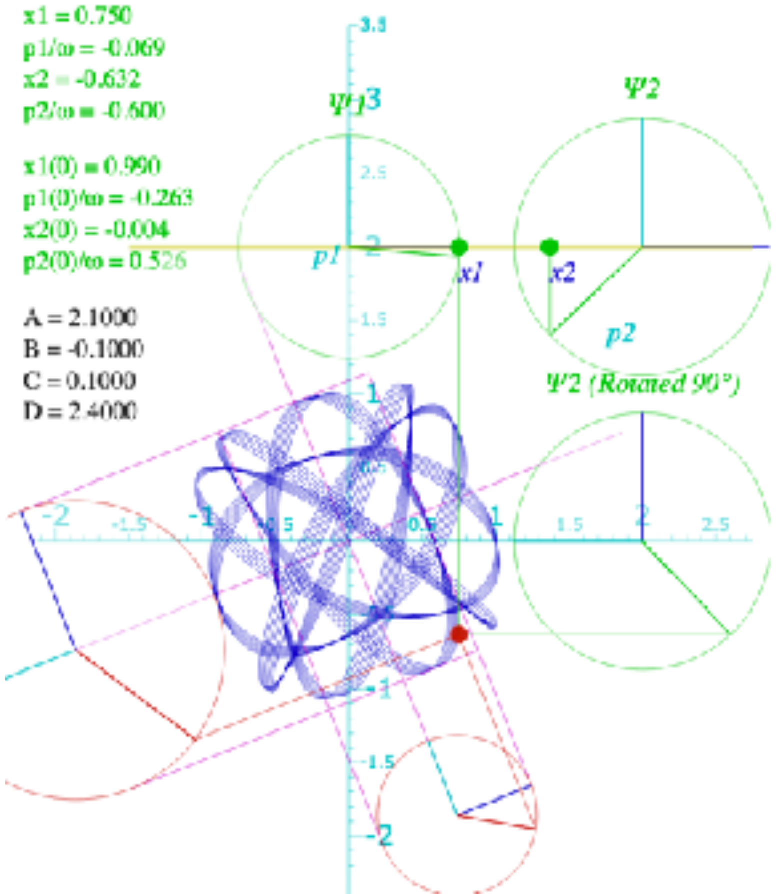
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

- *Bloch equation for density operator*



Reviewing fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler-defined state |αβγ⟩ described by Stoke's **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian **H**[φϑΘ]=*exp(-i***Ω•S***)·t and angular velocity* **Ω**(φϑ)*·t*=**Θ***-vector*

Euler-defined operator **R***(αβγ) derived from Darboux-defined* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* **Θ**=0−4π*-sequence* [φϑ] *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

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ABC-Type elliptical polarized motion

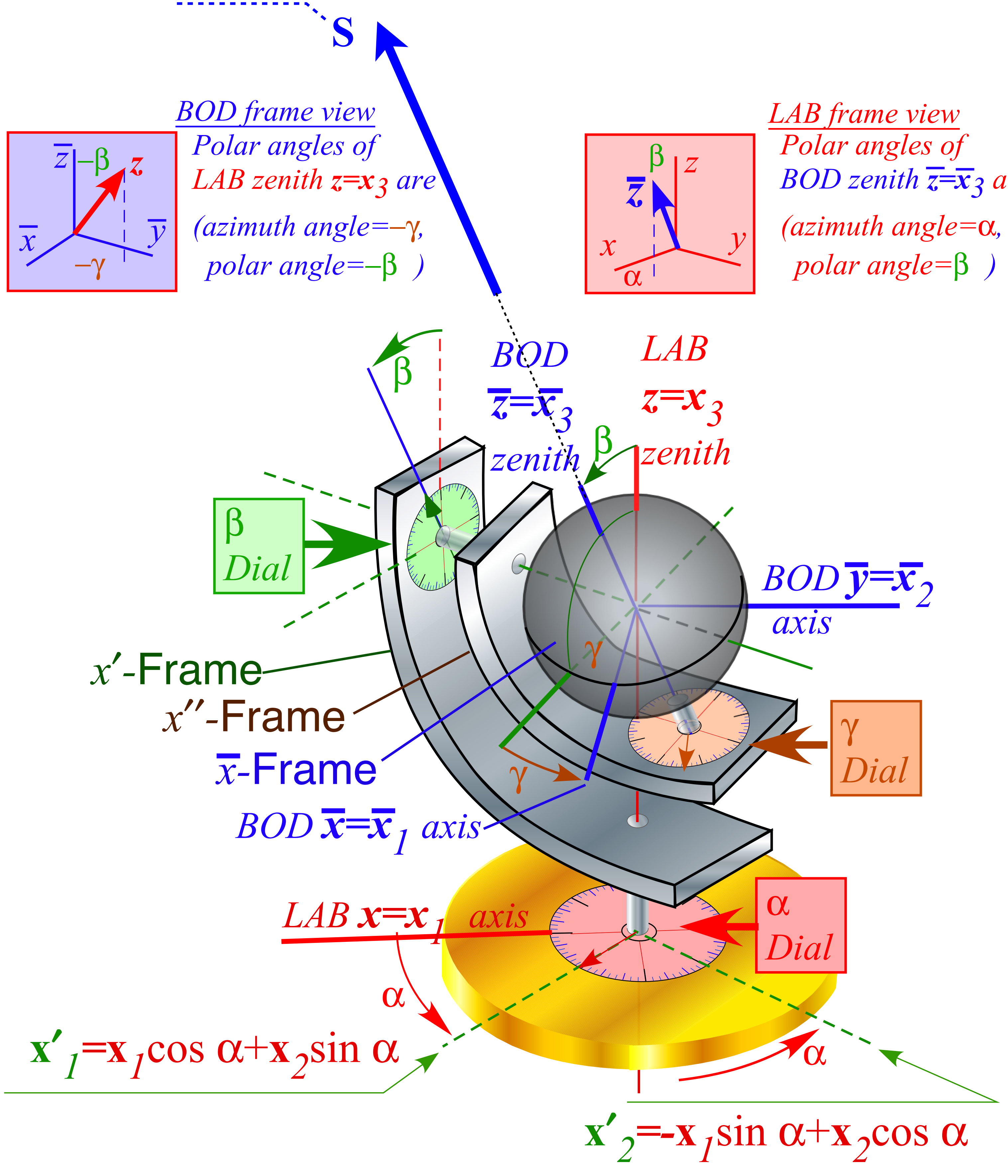
Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$,*and* **R**$(0,0,\gamma)$

3D-real **S**-*vector represents state* $|\alpha,\beta,\gamma\rangle$ *of U(2) oscillator*

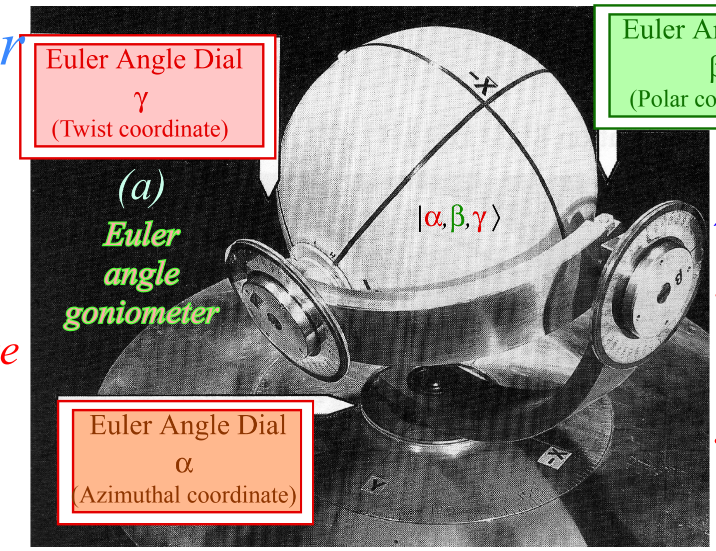


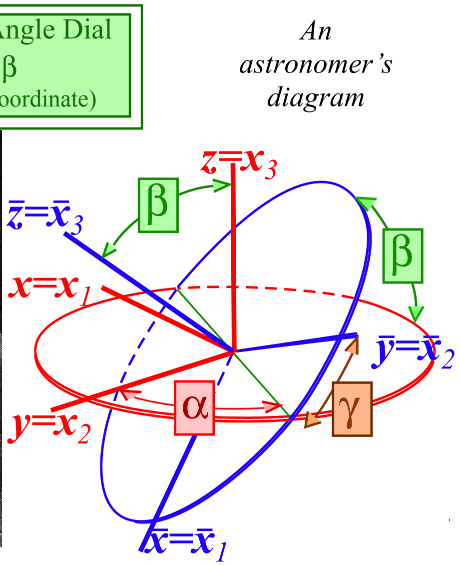
BOD frame view
Polar angles of LAB zenith $z=x_3$ are (azimuth angle=−γ, polar angle=−β)

LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are (azimuth angle=α, polar angle=β)

$$\mathbf{x}'_1=\mathbf{x}_1\cos\alpha+\mathbf{x}_2\sin\alpha$$

$$\mathbf{x}'_2=-\mathbf{x}_1\sin\alpha+\mathbf{x}_2\cos\alpha$$





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Euler angles

Under Construction!
Web based U(2) Calculator - Euler State

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}\left(\alpha\beta\gamma\right)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha\,\text{about}\,Z]\cdot\mathbf{R}[\beta\,\text{about}\,Y]\cdot\mathbf{R}[\gamma\,\text{about}\,Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

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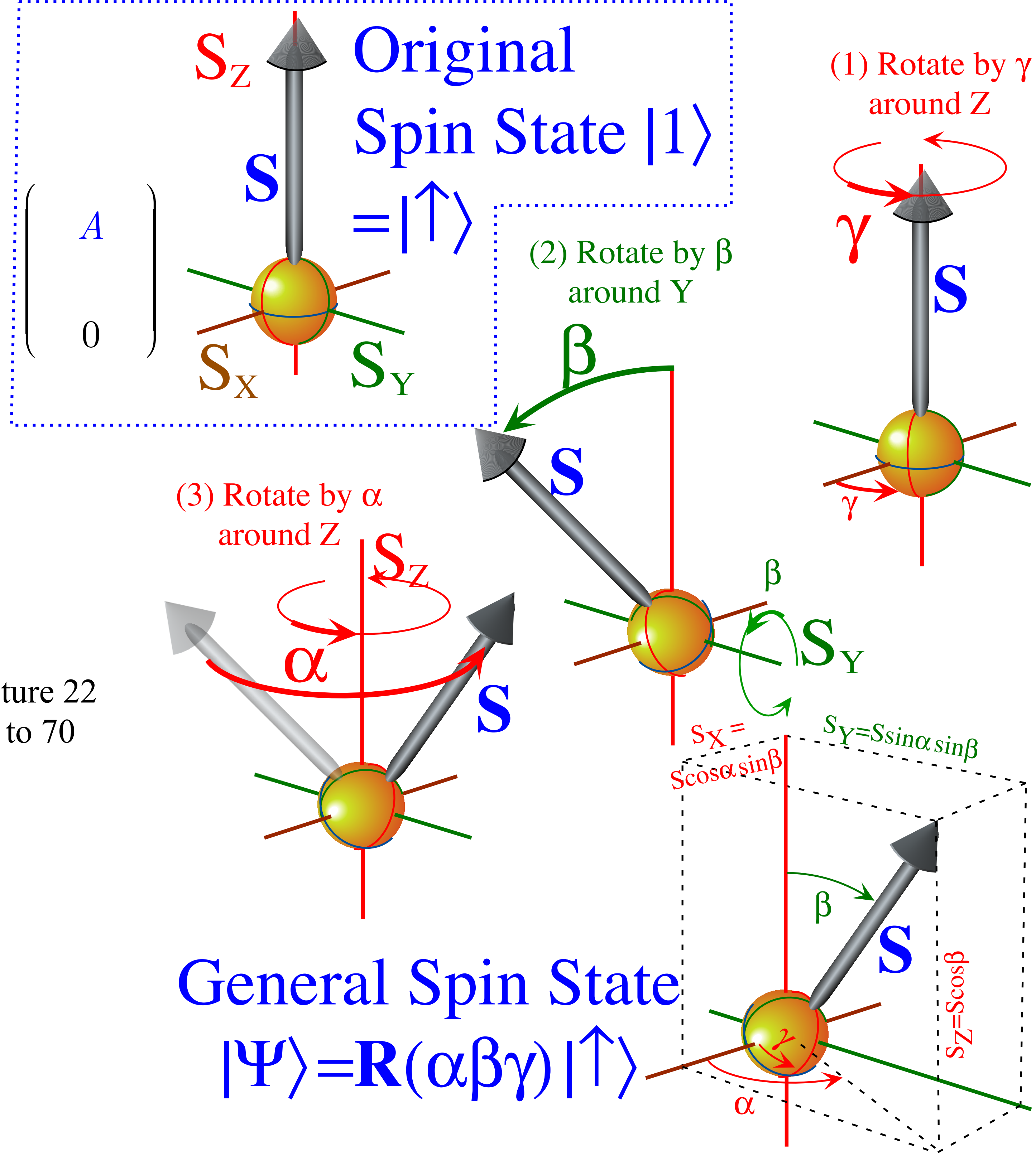

Original
Spin State |1⟩
=|↑⟩

General Spin State
|Ψ⟩=R(αβγ)|↑⟩

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

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Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\text{Asymmetry } S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}] \quad = \frac{I}{2}\cos\beta$$

$$\text{Balance } \quad S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2} \cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

$$\text{Chirality } \quad S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2} \cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$

Three ways to picture U(2) spin or pseudo-spin states

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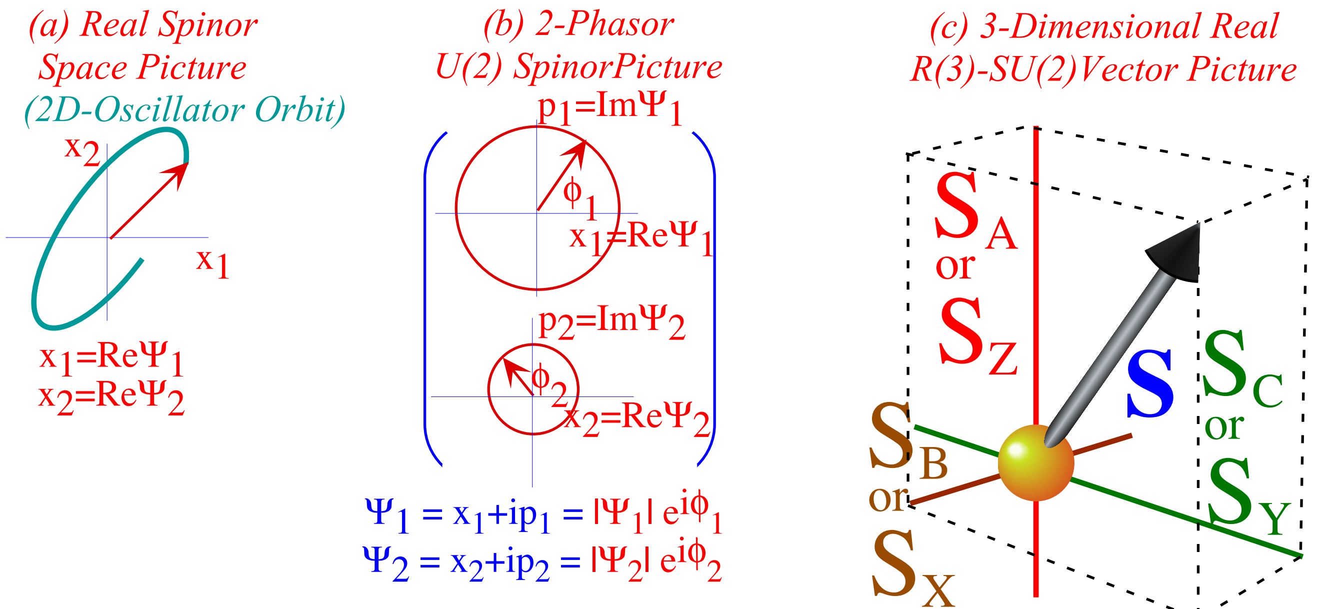

$S_A = (\Psi_1{}^* \Psi_1 \ - \ \Psi_2{}^* \Psi_2)/2$
$S_B = (\Psi_1{}^* \Psi_2 \ + \ \Psi_2{}^* \Psi_1)/2$
$S_C = (\Psi_1{}^* \Psi_2 \ - \ \Psi_2{}^* \Psi_1)/2i$

(a)
Ellipsometry

(b)
U(2) phasors

(c)
3D real R(3) vectors

General Spin State
$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\uparrow\rangle$

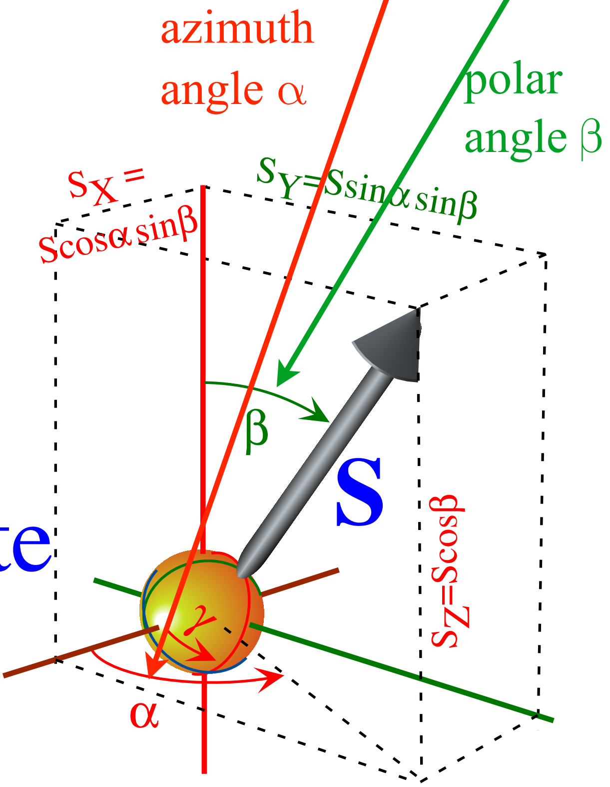

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Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

$$\text{Asymmetry } S_A = \frac{1}{2}\left(a\middle|\sigma_A\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \boxed{\frac{I}{2}\cos\beta}$$

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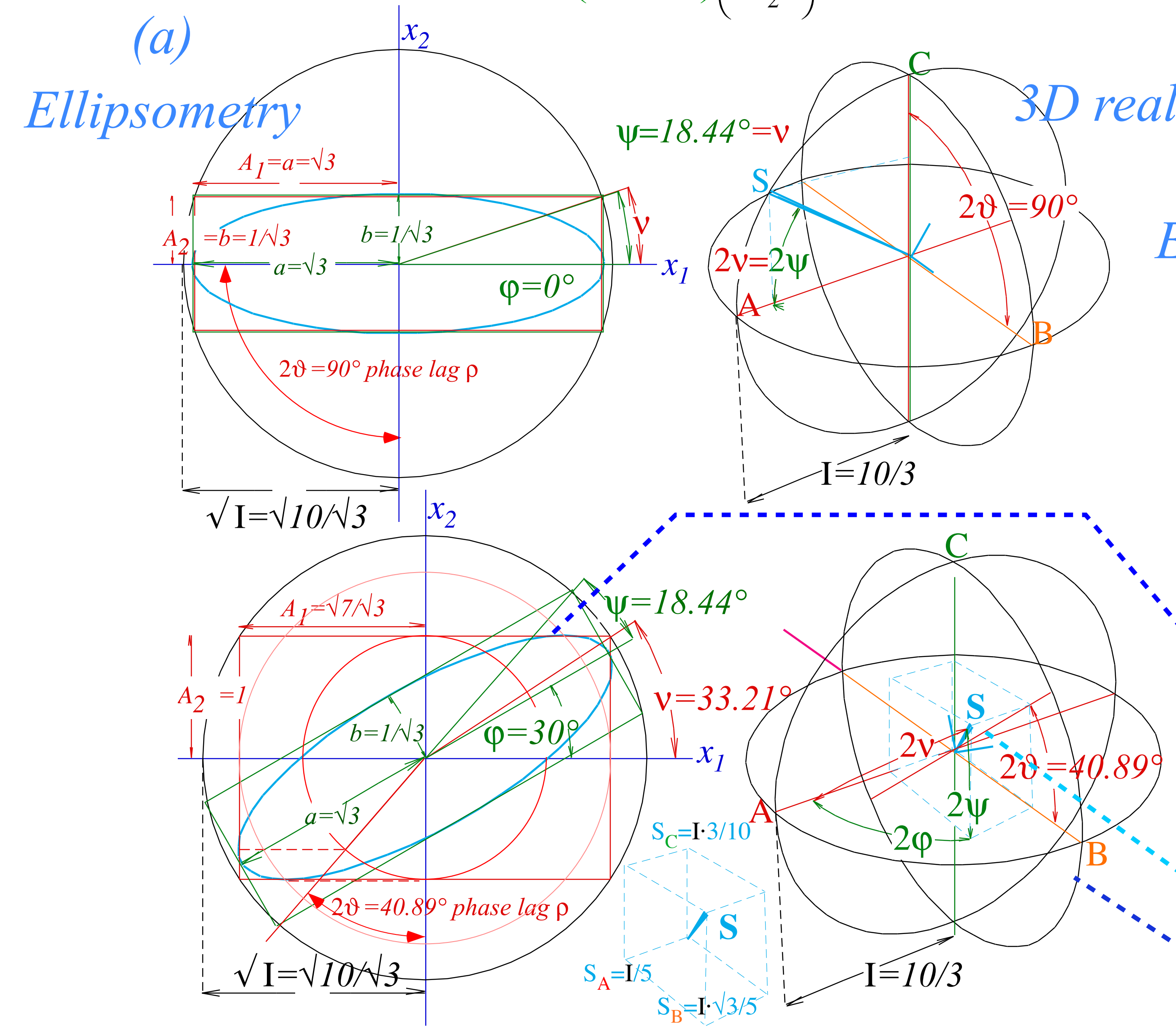
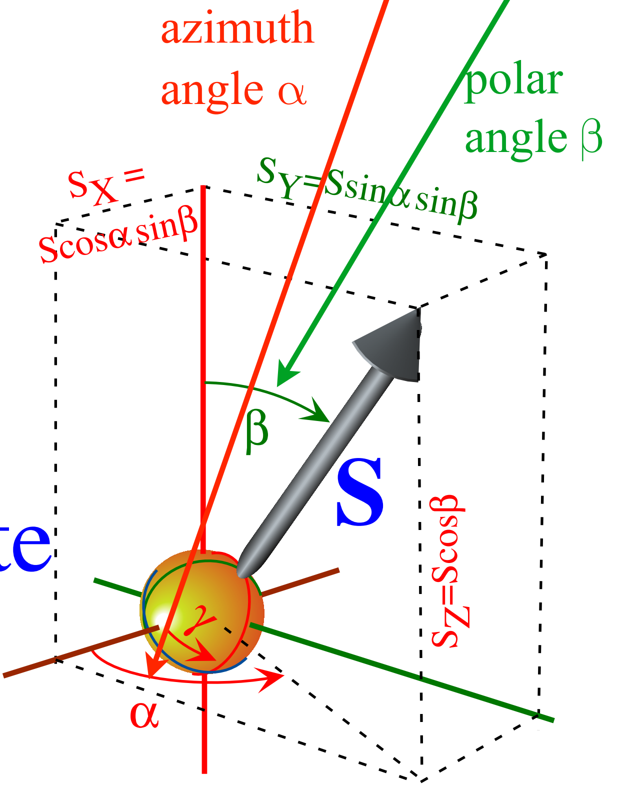


Ellipsometry of U(2) states detailed at end of this Lecture

General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$

Complex U(2) ellipse of any state corresponds to a single point **S** *in R(3) on the Stoke's sphere*



Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).

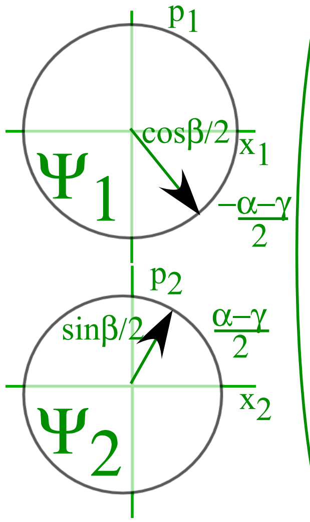
U(2) World : Complex 2D Spinors

Ellipsometry of U(2) states described by Two "Worlds" U(2) or R(3)

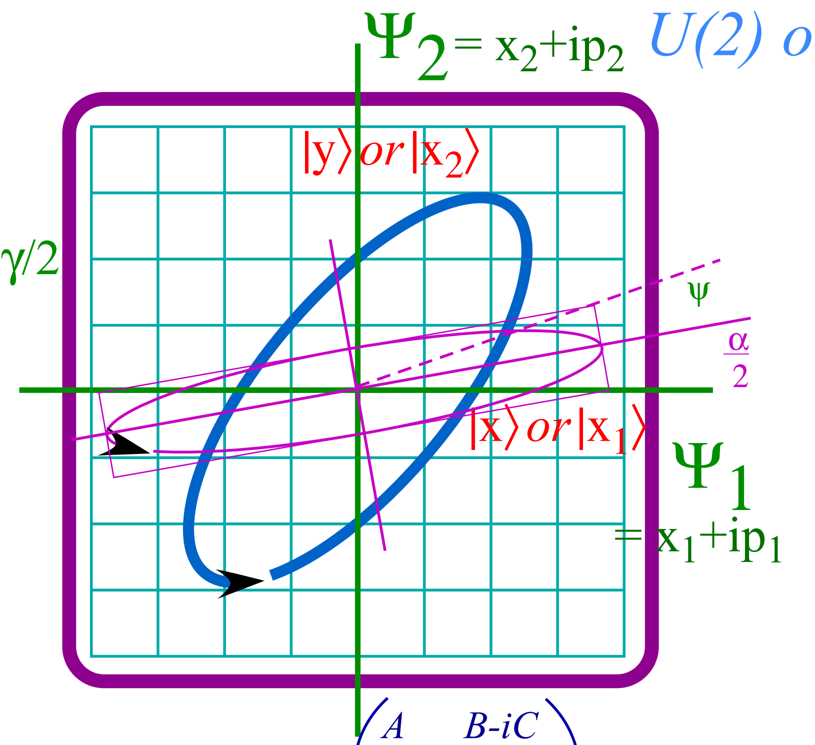
U(2) World labeled by two complex phasors and driven by complex operator

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

2-State ket $|\Psi\rangle=$



$$\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2 \\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

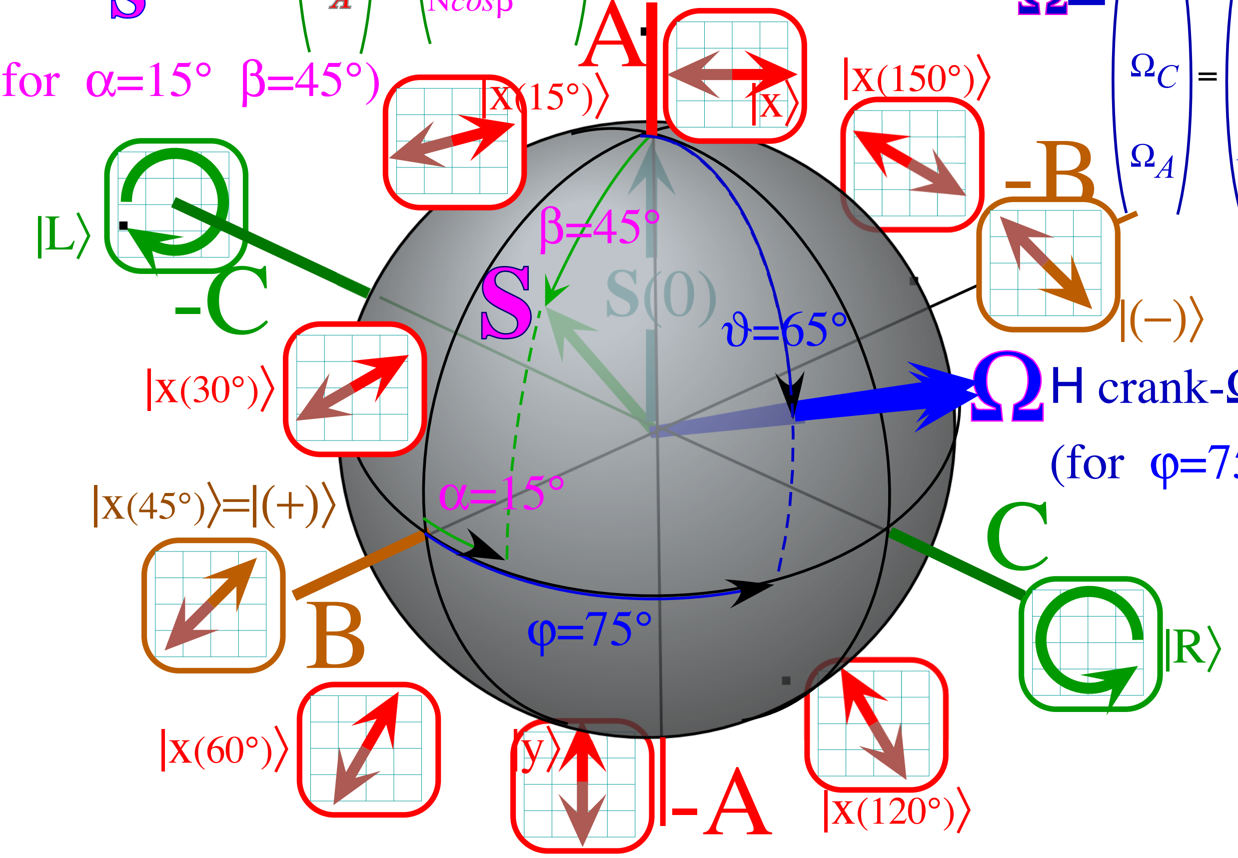
$$\begin{pmatrix} S_B \\ S_C \\ S_A \end{pmatrix} = \begin{pmatrix} N\sin\beta\cos\alpha \\ N\sin\beta\sin\alpha \\ N\cos\beta \end{pmatrix} \frac{1}{2}$$

(for $\alpha=15°$ $\beta=45°$)

H-Operator $\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

Angular velocity

$$\mathbf{\Omega} = \begin{pmatrix} \Omega_B \\ \Omega_C \\ \Omega_A \end{pmatrix} = \begin{pmatrix} 2B \\ 2C \\ A-D \end{pmatrix} = \begin{pmatrix} \Omega\sin\vartheta\cos\varphi \\ \Omega\sin\vartheta\sin\varphi \\ \Omega\cos\vartheta \end{pmatrix}$$



H crank-Ω vector

(for $\varphi=75°$ $\vartheta=65°$)

*R(3) World labeled by real 3-D "spin" vector **S** of angular momentum and driven by real 3-D "spin" vector **Ω** of angular velocity*

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- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
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Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Here spin-rotor **S***-polar coordinates are Euler angles*

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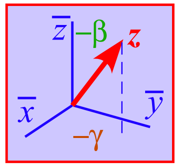
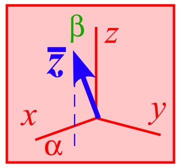
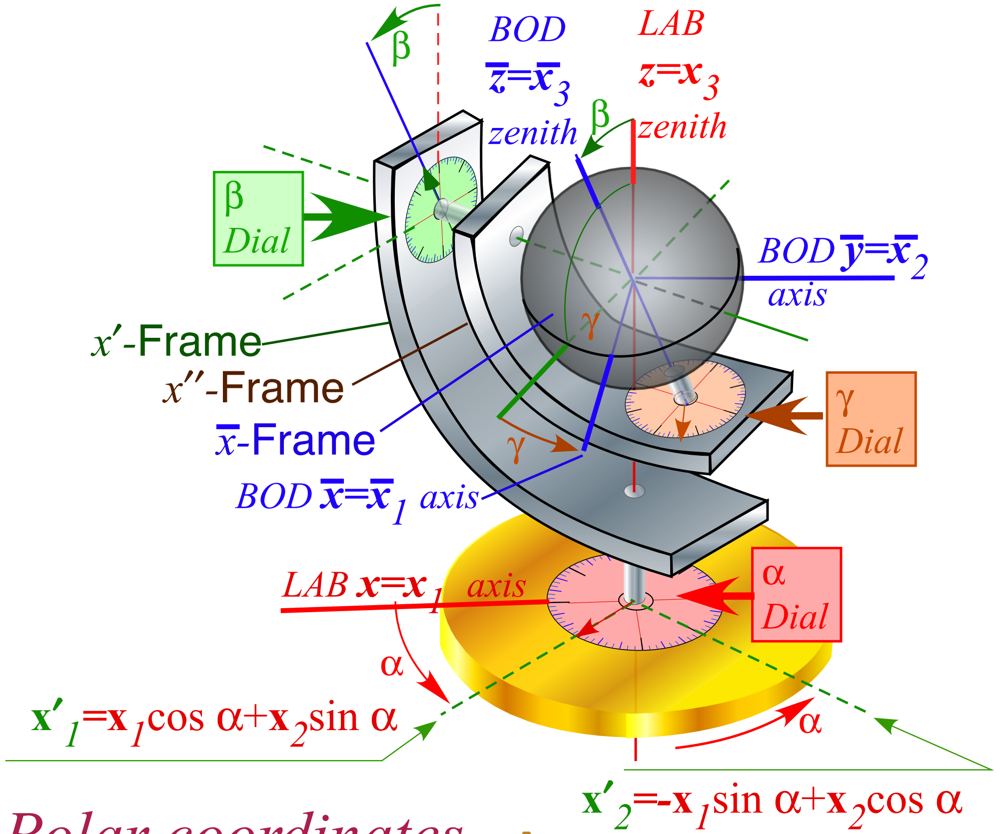


BOD frame view
Polar angles of
LAB zenith z=x₃ are
(azimuth angle=−γ,
polar angle=−β)



LAB frame view
Polar angles of
BOD zenith $\bar{z}=\bar{x}_3$ are
(azimuth angle=α,
polar angle=β)



Polar coordinates for unit Spin vector $\hat{\mathbf{S}}$

$$\hat{S}_X = \cos\alpha \ \sin\beta$$
$$\hat{S}_Y = \sin\alpha \ \sin\beta$$
$$\hat{S}_Z = \cos\beta$$

Spin State&Operator

$$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$$

by Euler angles

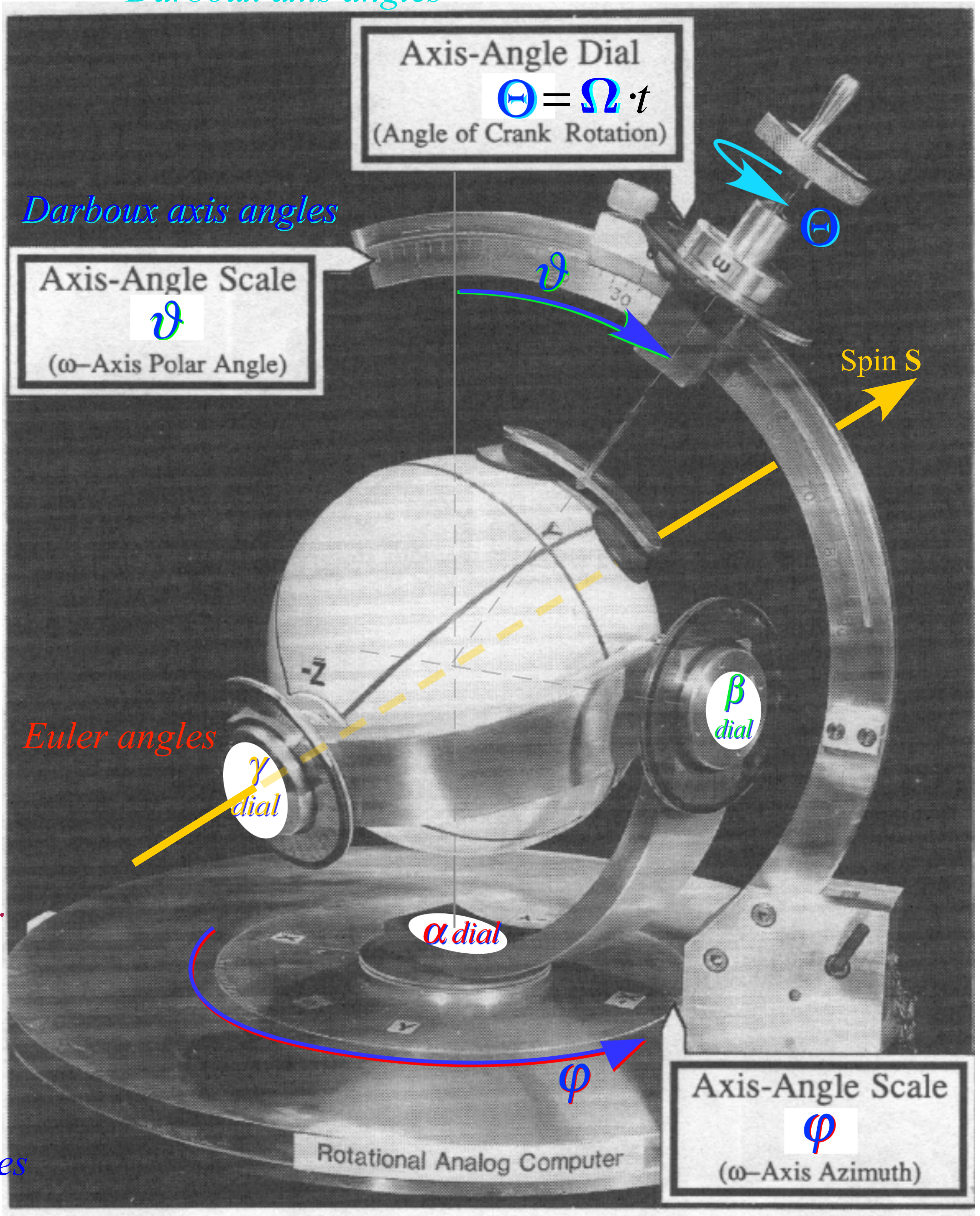
Polar coordinates for unit axis vector $\hat{\boldsymbol{\Theta}}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

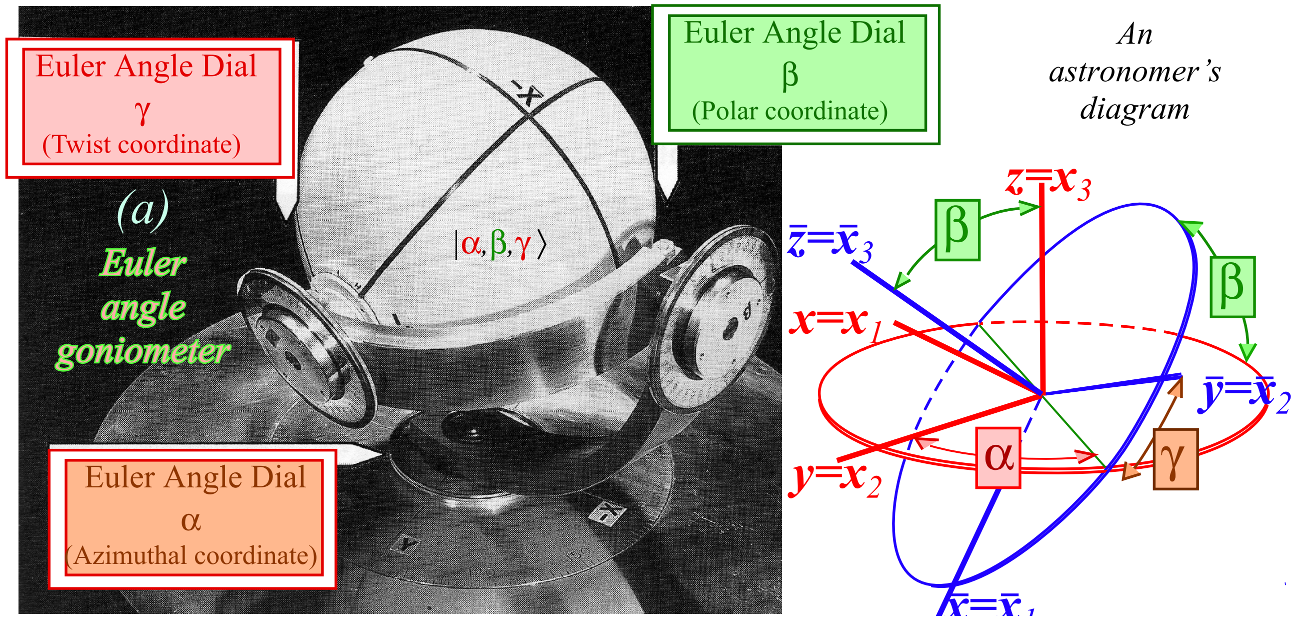
State&Operator

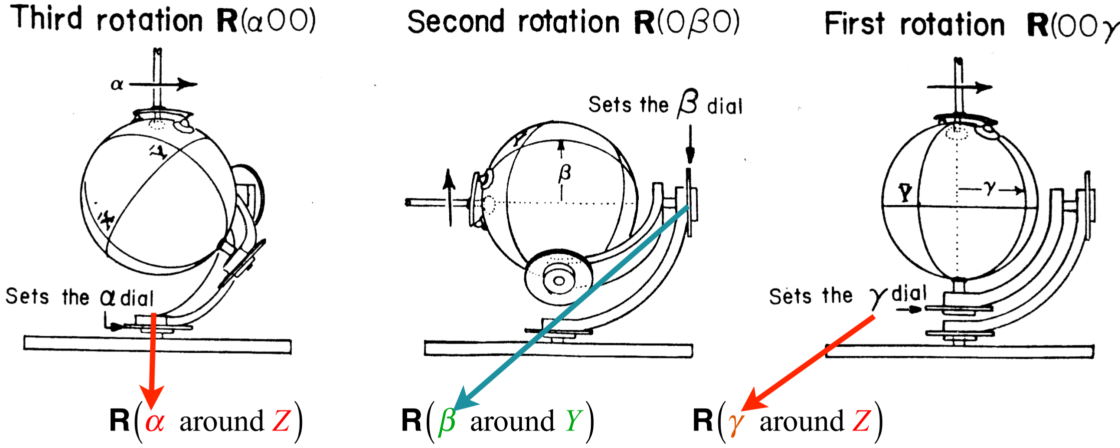
$$|[\varphi\vartheta\Theta]\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\uparrow\rangle$$

by Darboux axis angles



Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$





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$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

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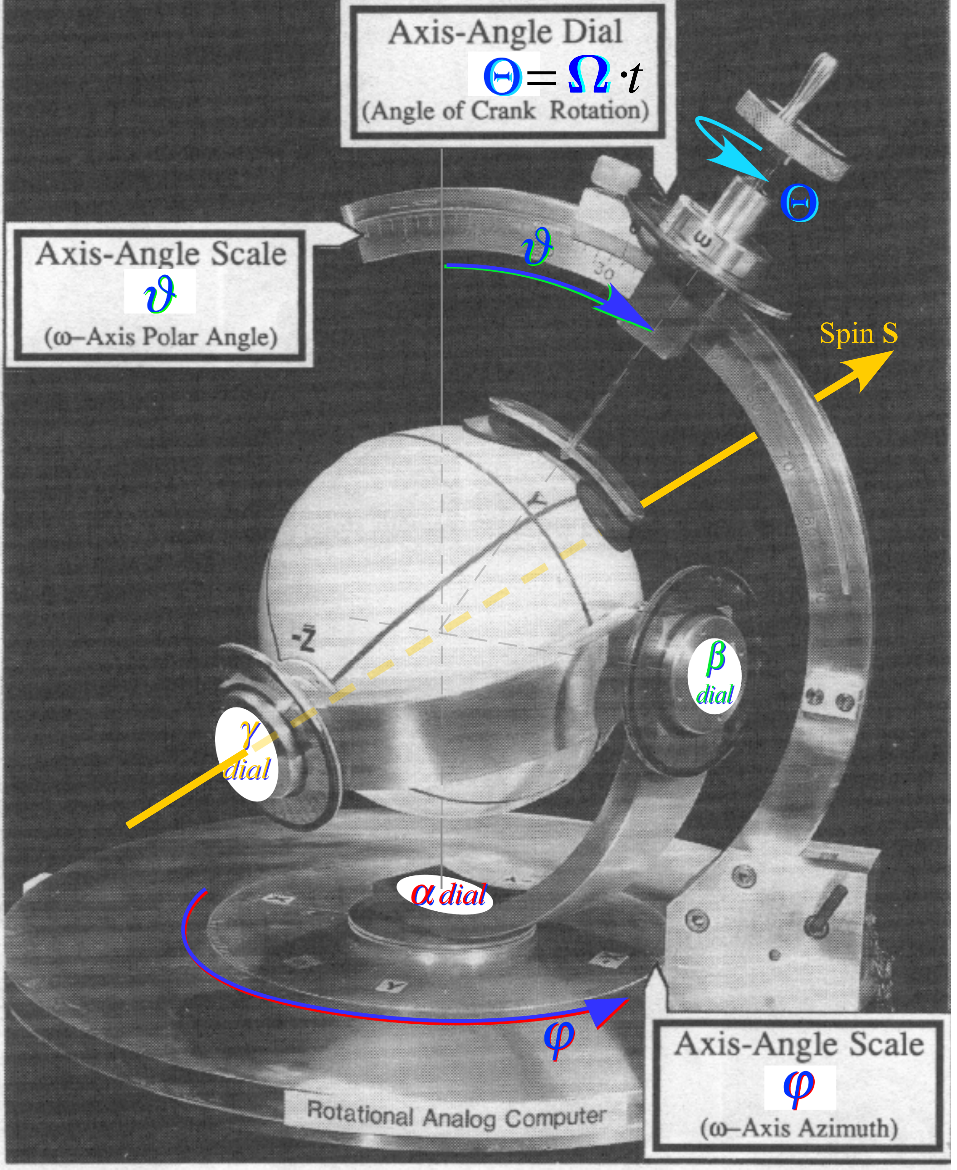
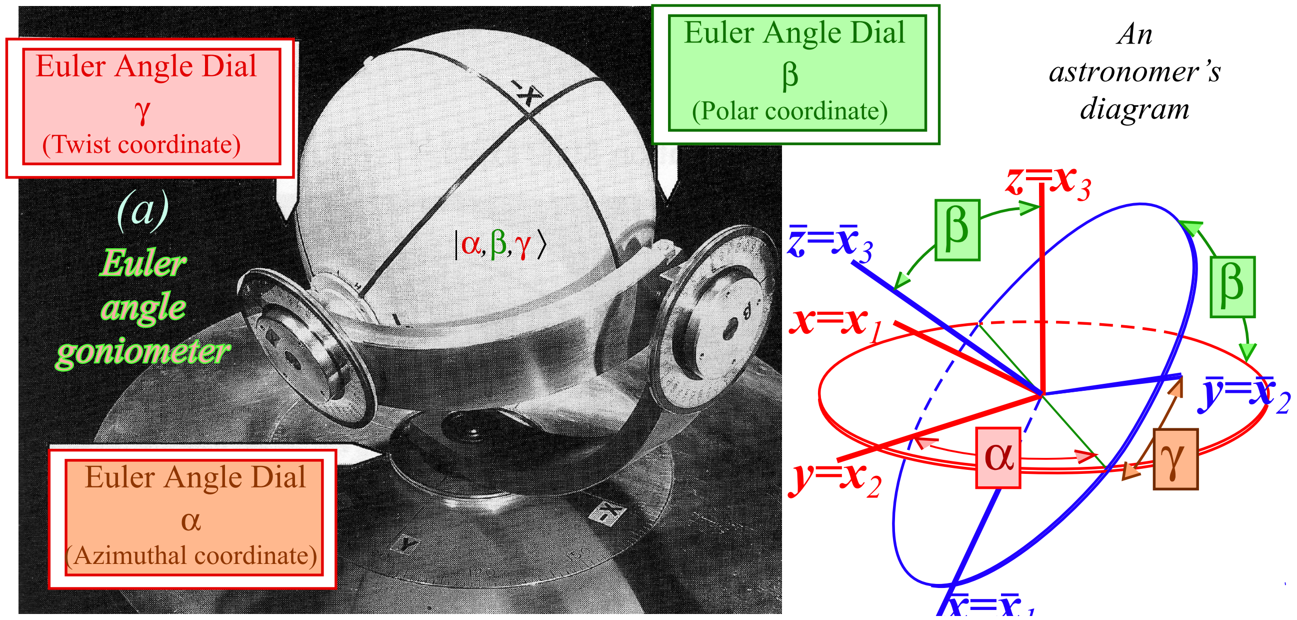


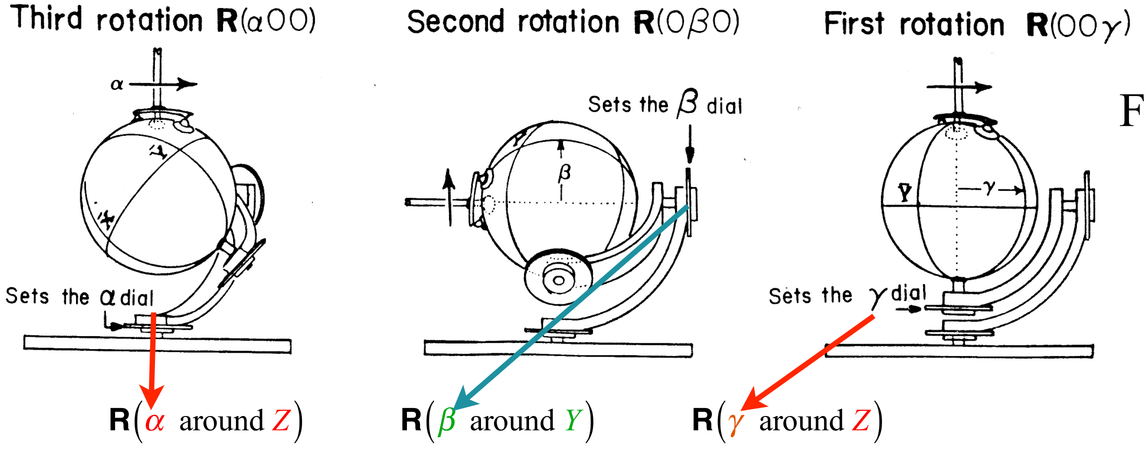
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right)\\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler **R**(αβγ) versus *Darboux* **R**[φϑΘ]





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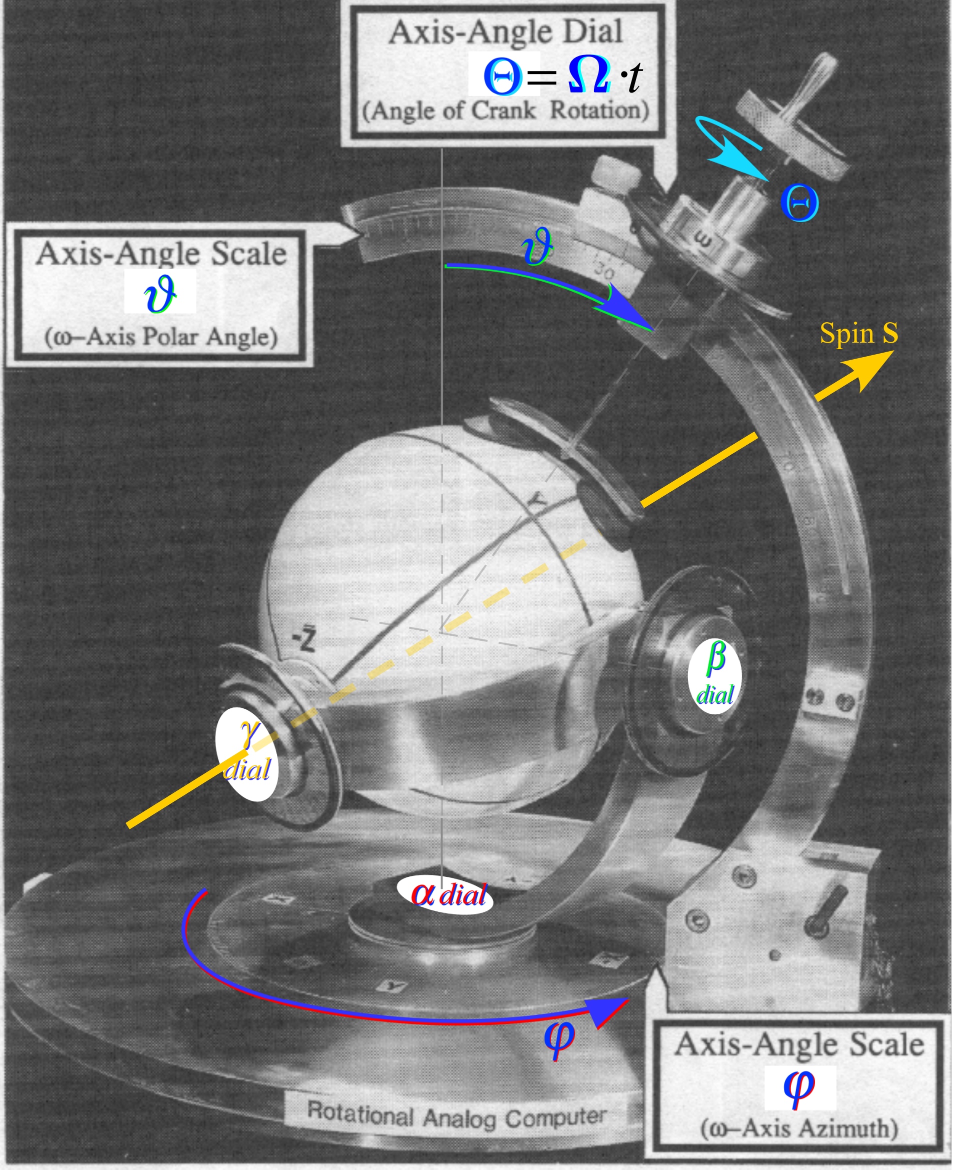


$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition*:

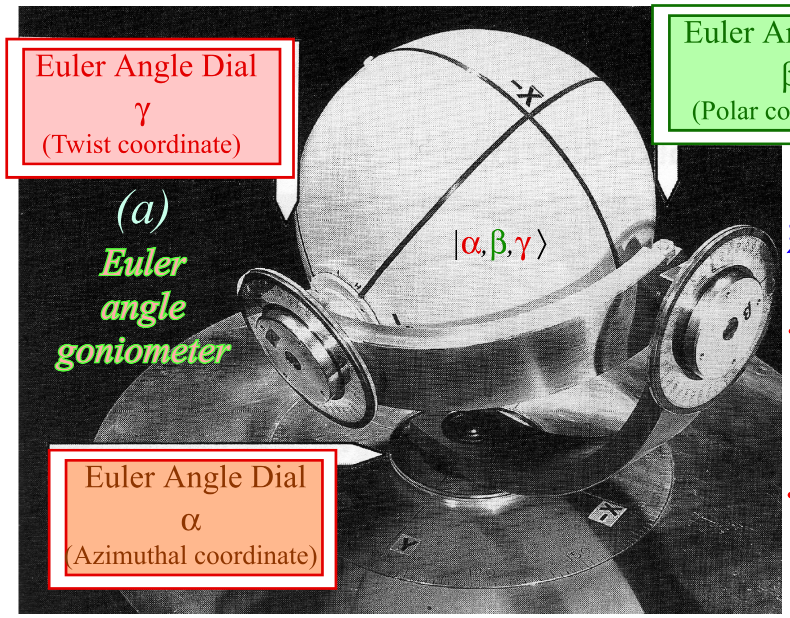
$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

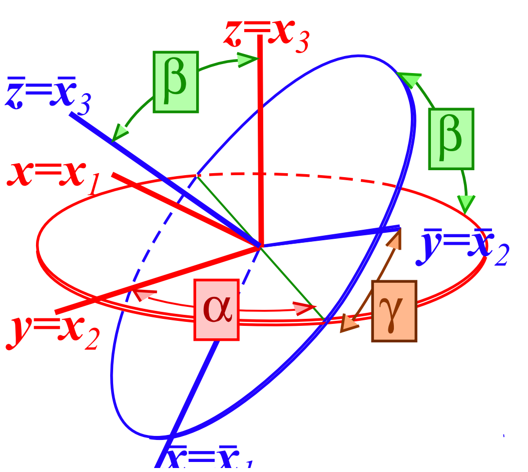
$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

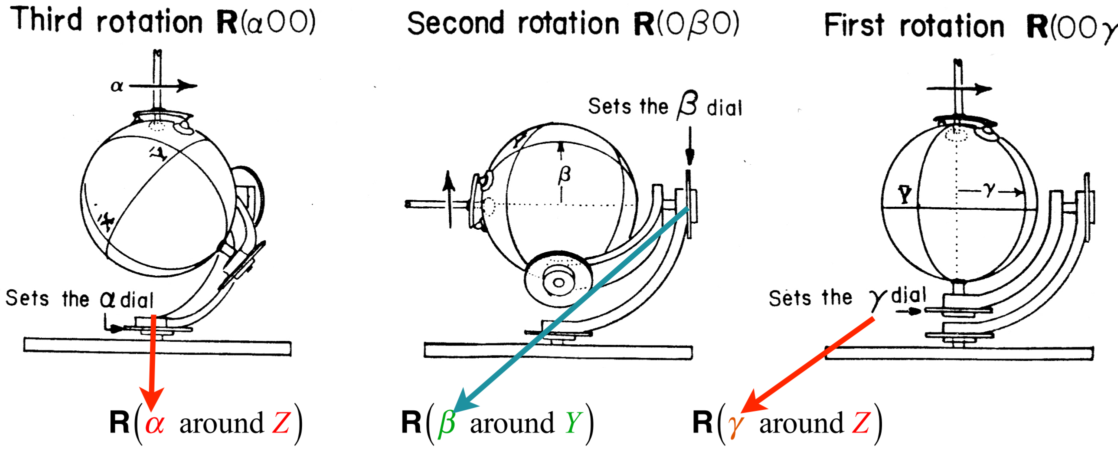
$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler **R**(αβγ) versus *Darboux* **R**[φϑΘ]



An astronomer's diagram





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$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

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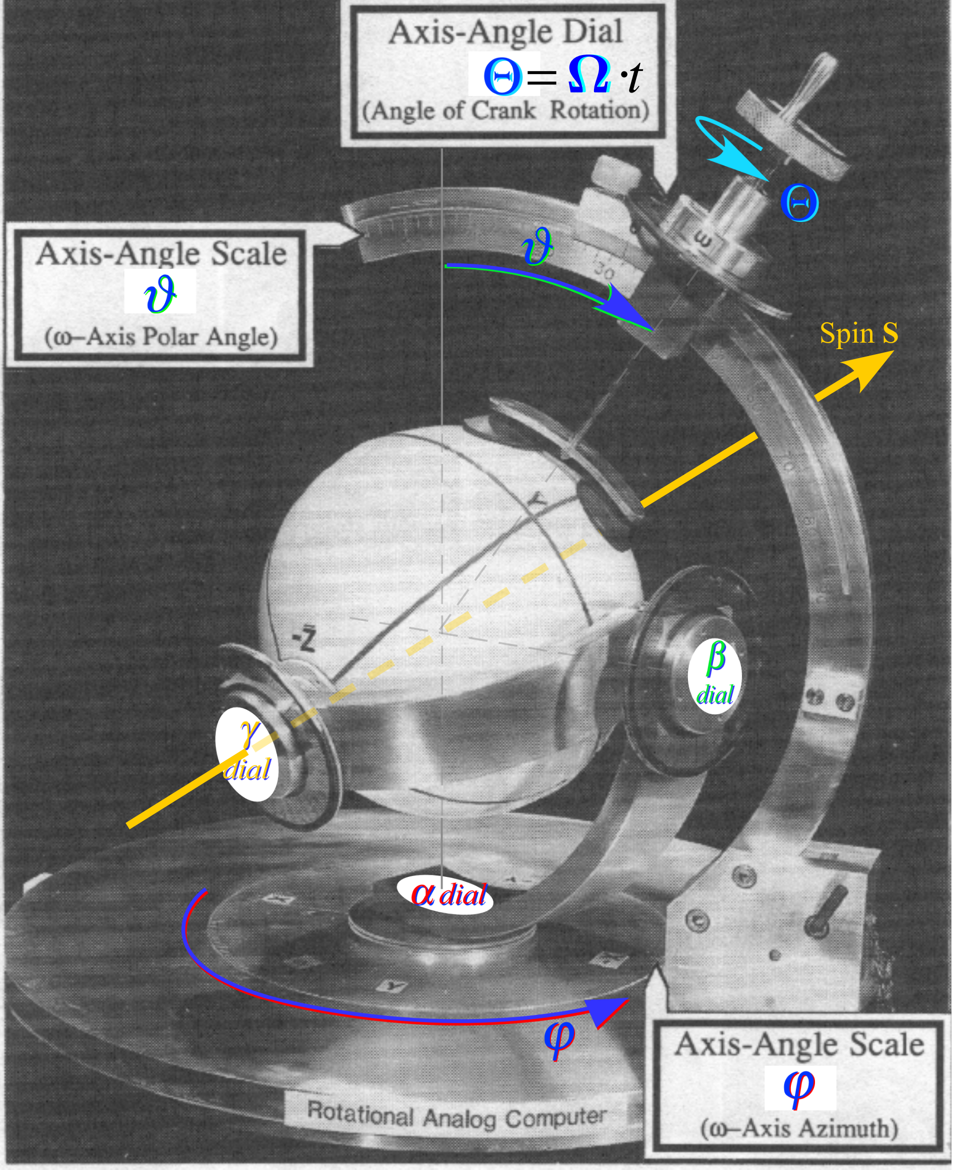
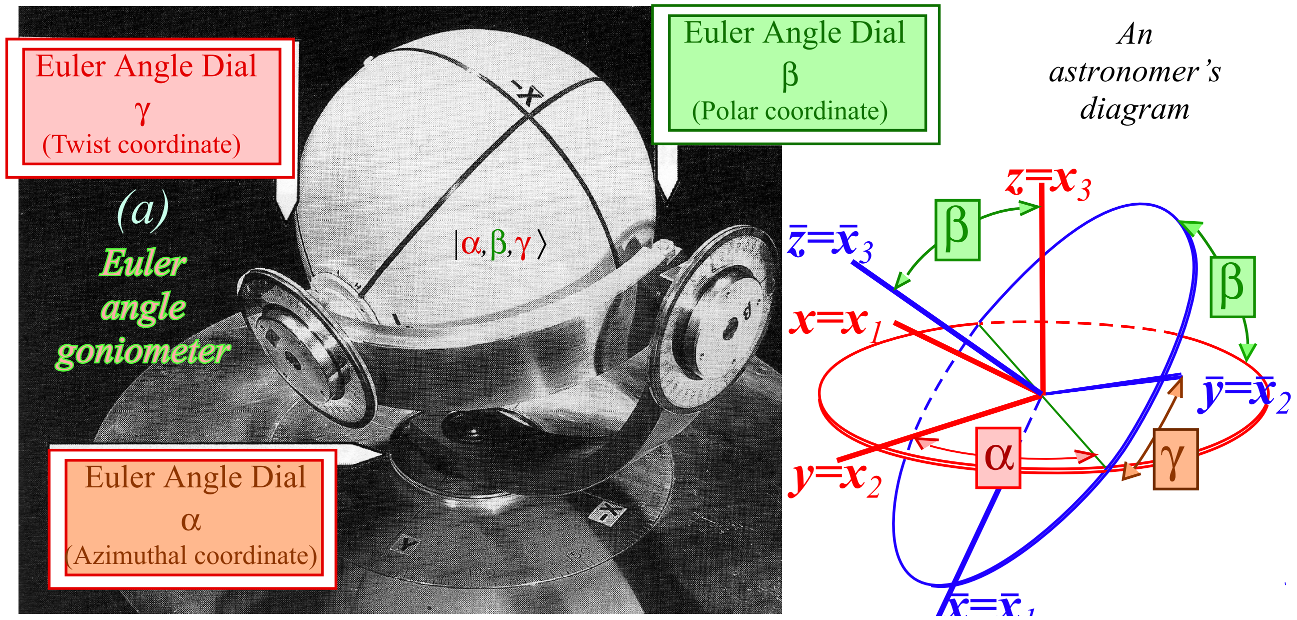


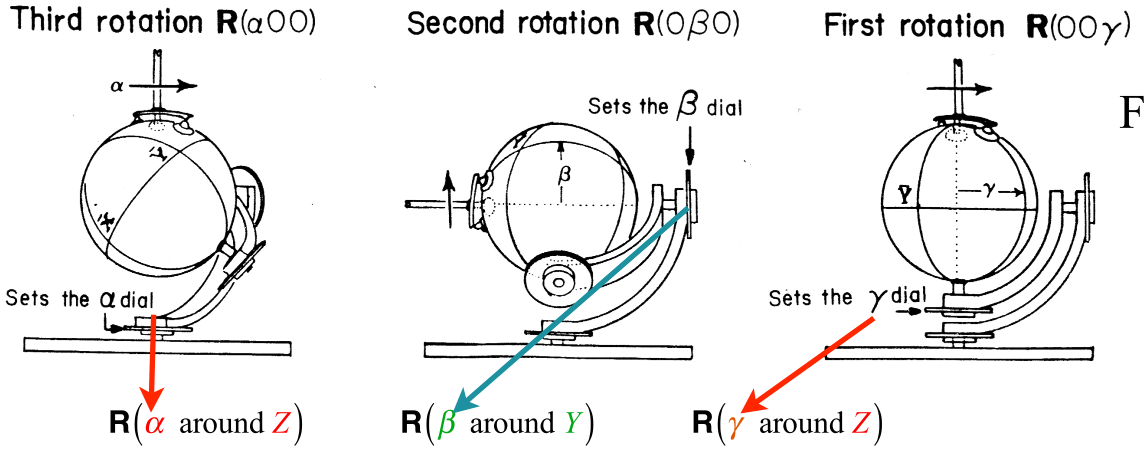
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler **R**(αβγ) versus *Darboux* **R**[φϑΘ]





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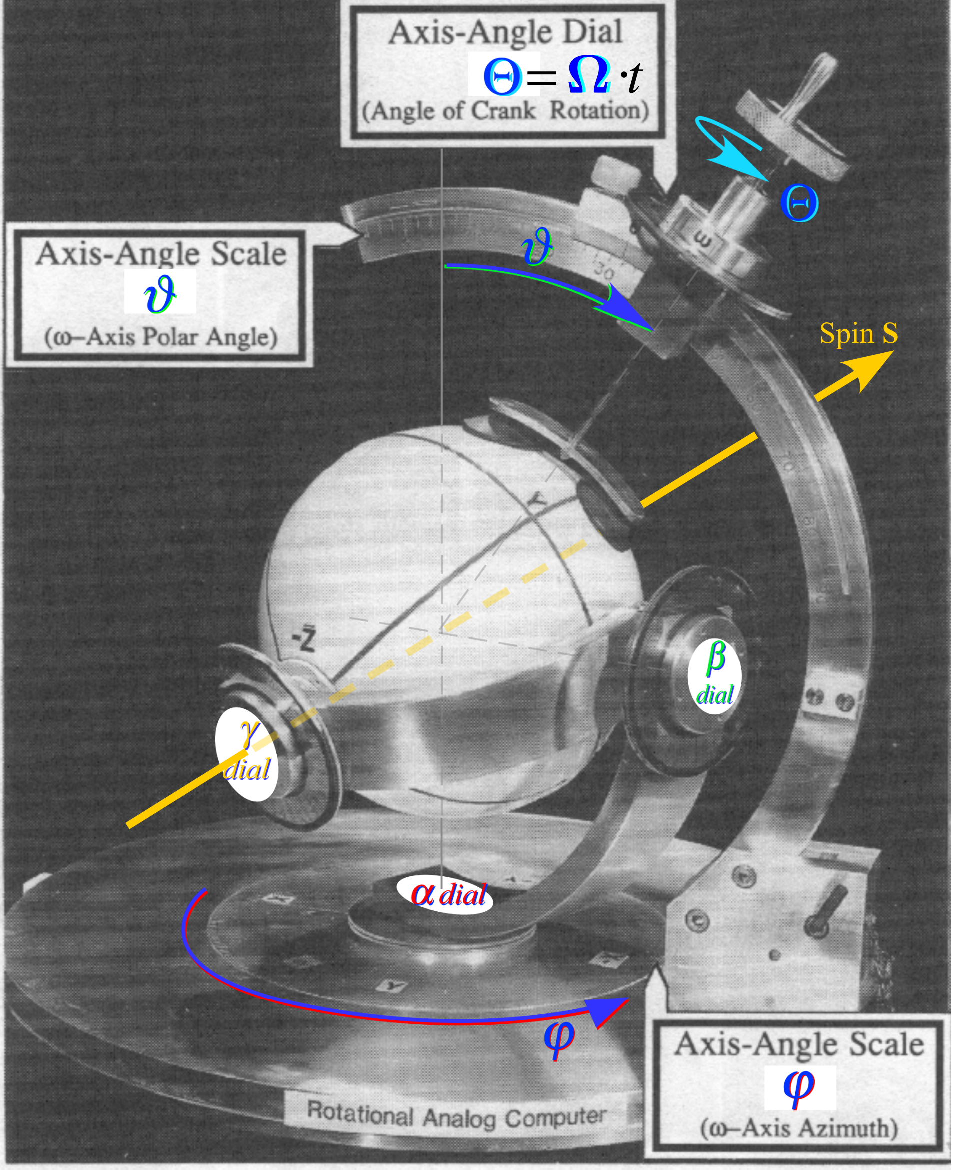


$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

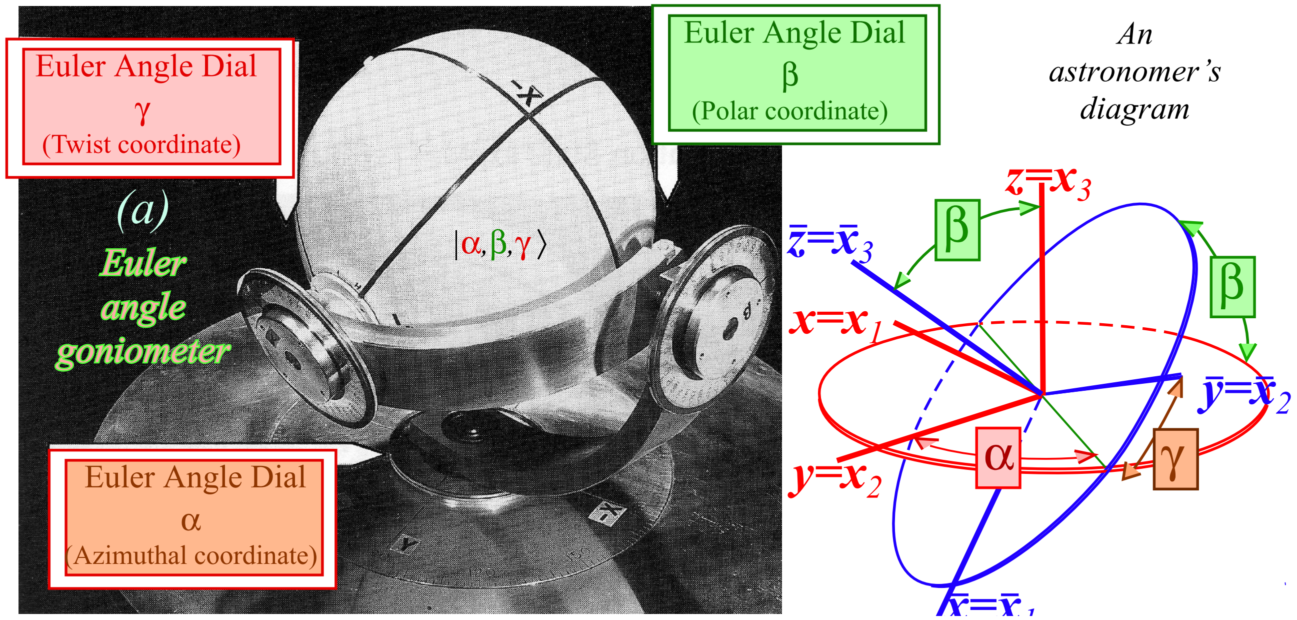
Euler **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition* lets us relate **R**(αβγ) = **R**[φϑΘ] ...

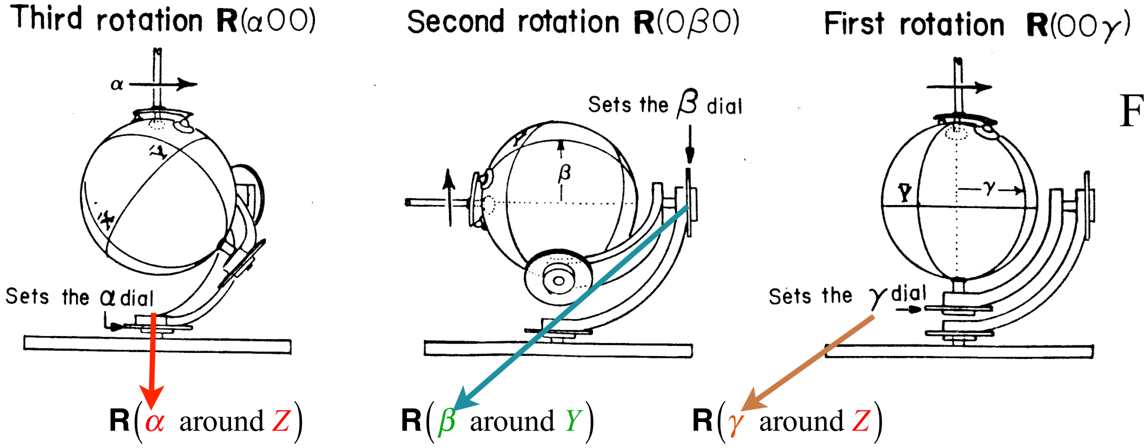
$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Euler **R**(αβγ) versus *Darboux* **R**[φϑΘ]



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}}$$
$$=\mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



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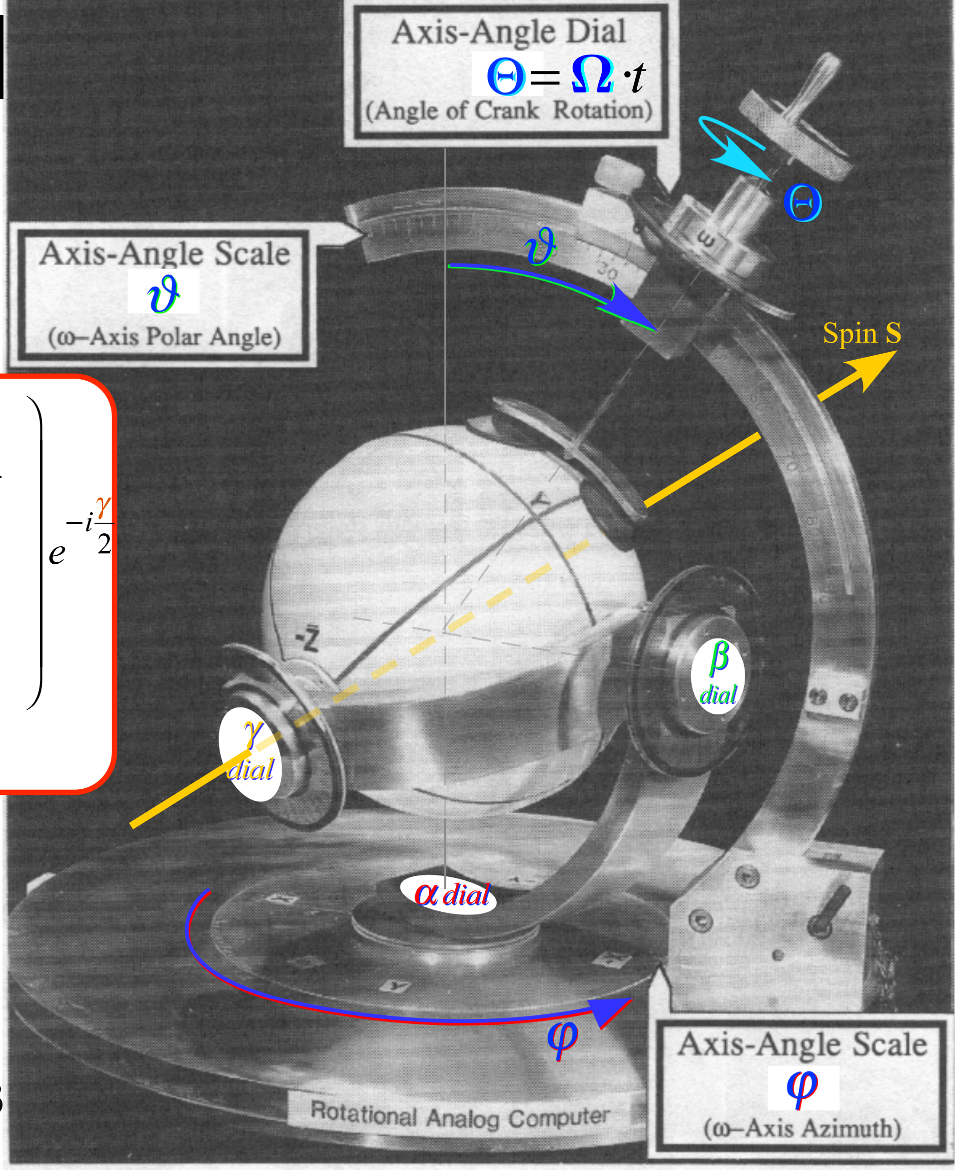

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$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

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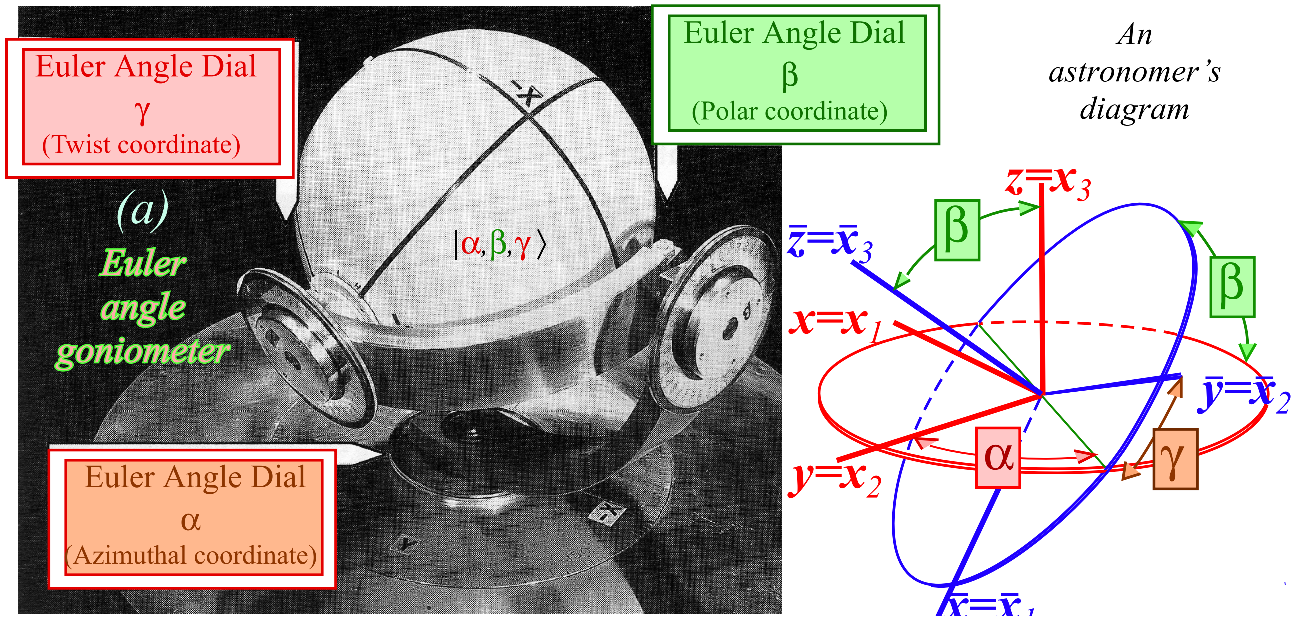
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}}$$

Phase coherence angle

Population inversion angle

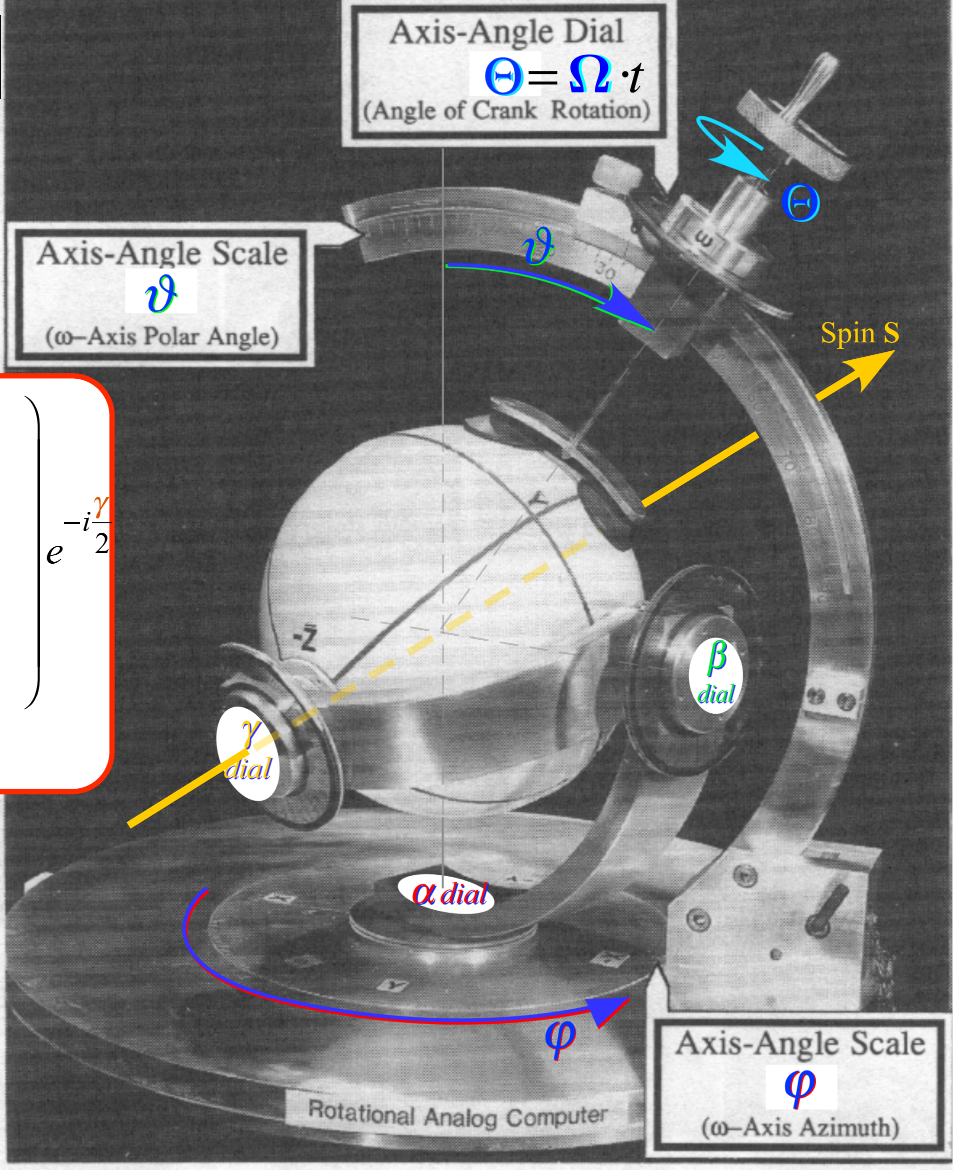
Overall phase angle

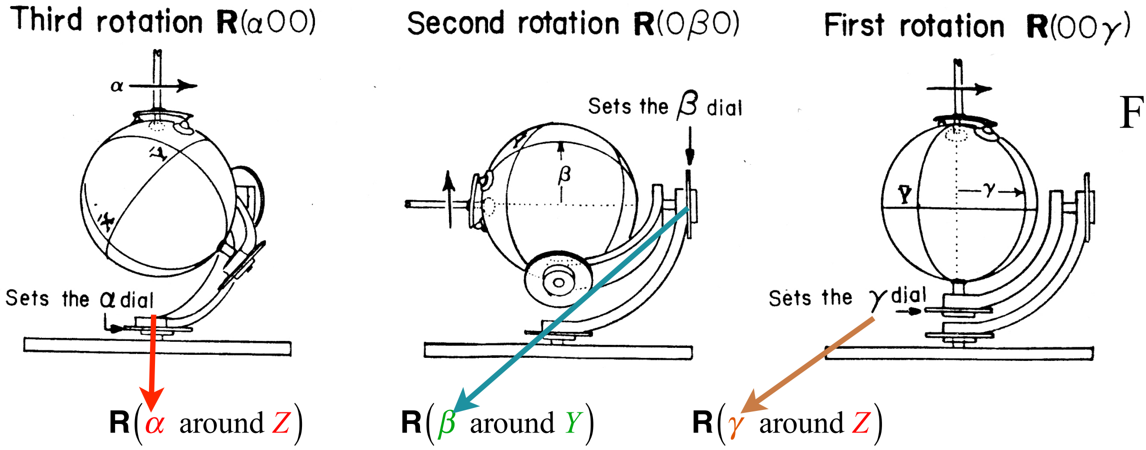
Euler **R**(αβγ) versus *Darboux* **R**[φϑΘ]



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\frac{\gamma}{2}}$$

$$=\mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$





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$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

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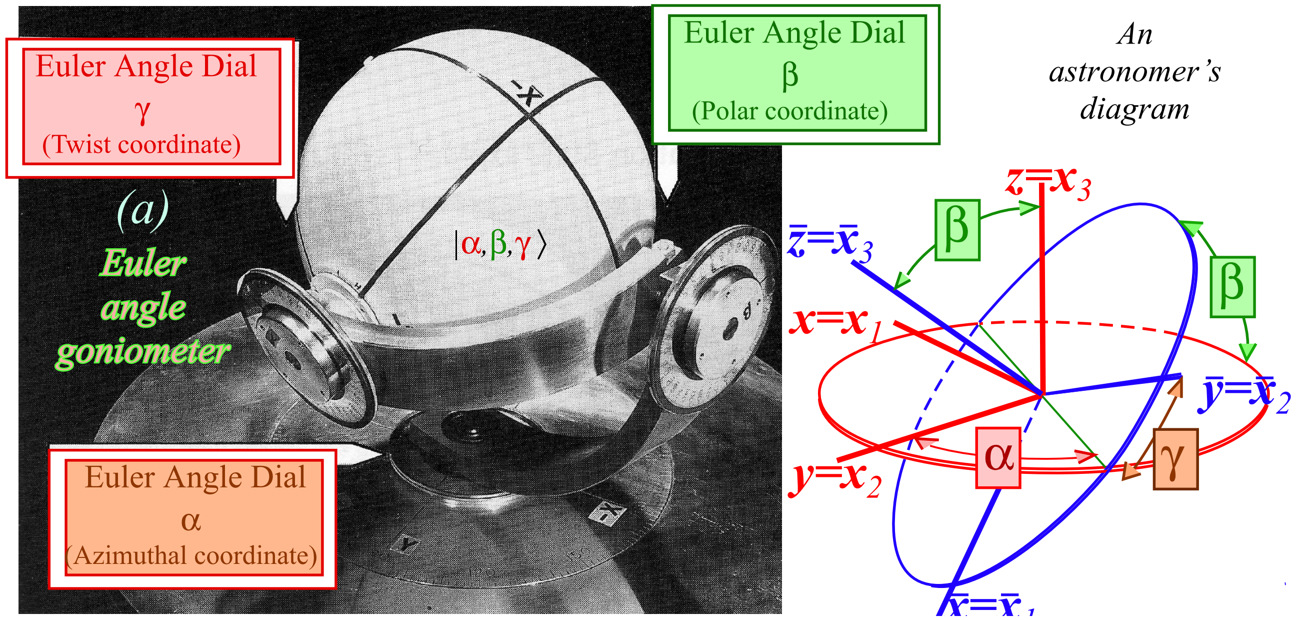
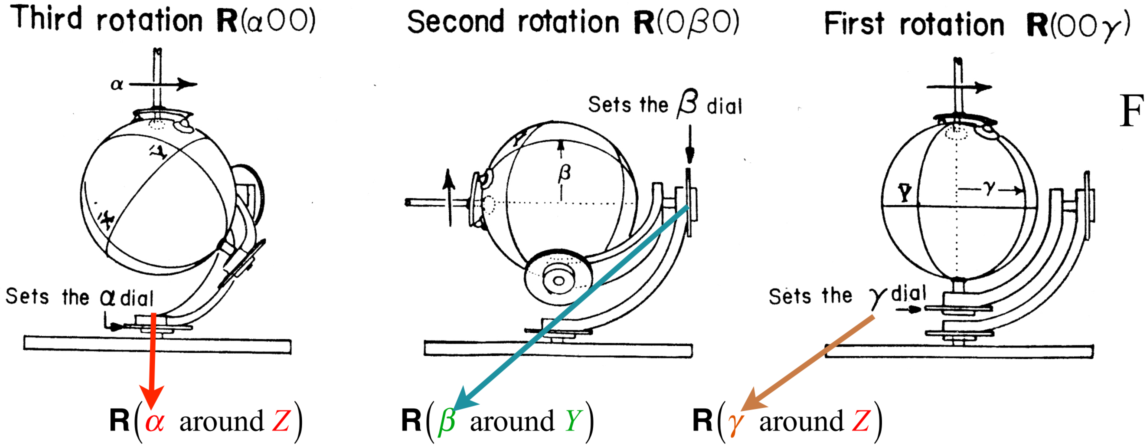
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2=\cos\Theta/2$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]





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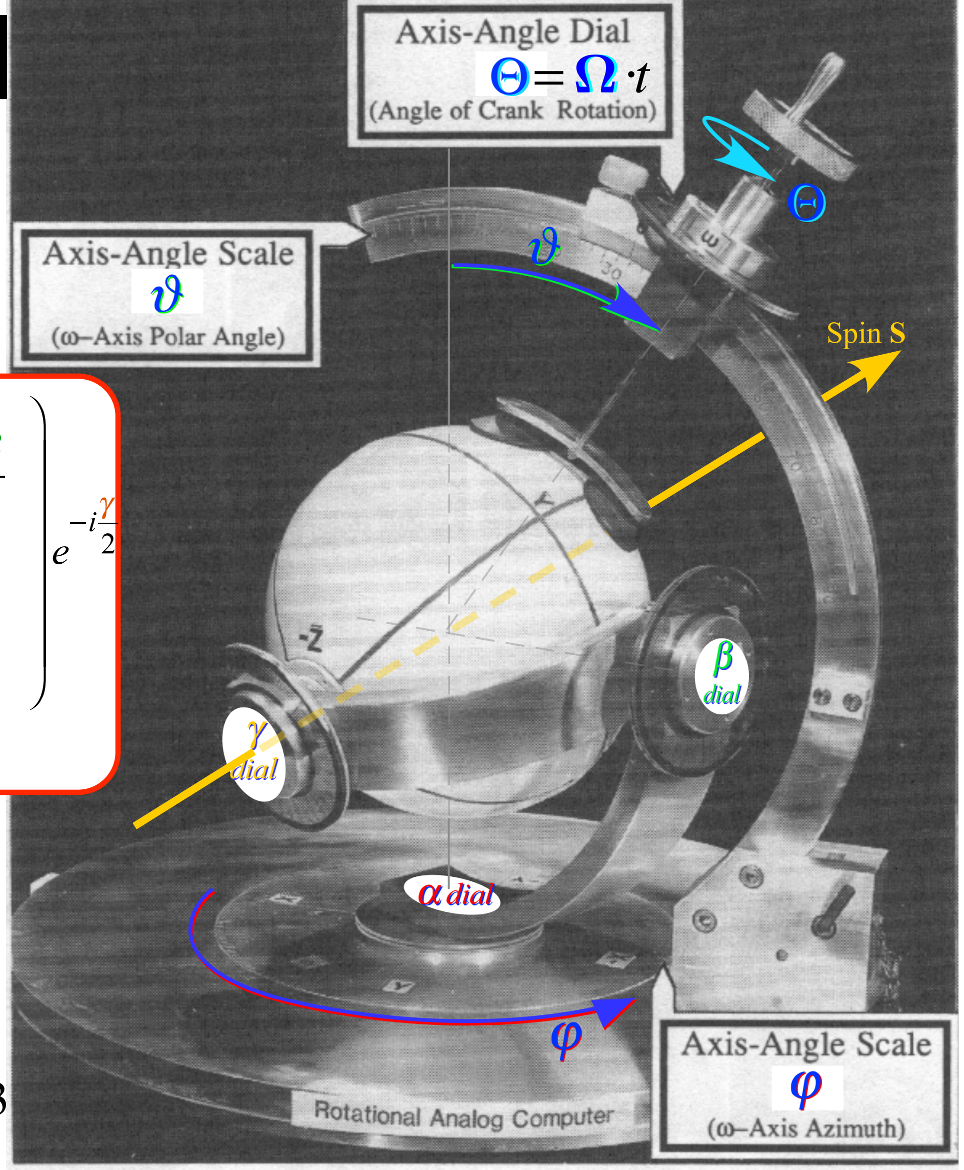
$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



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$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}\left[\varphi\vartheta\Theta\right] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition* lets us relate **R**(αβγ) = **R**[φϑΘ] ...

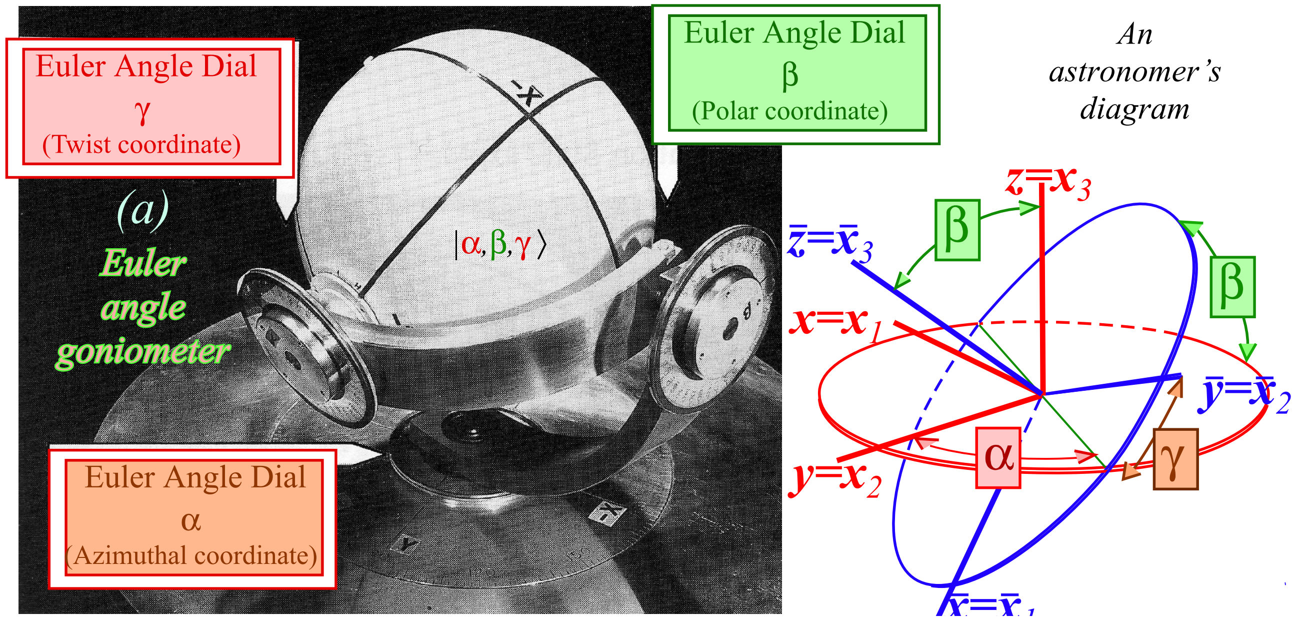
$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

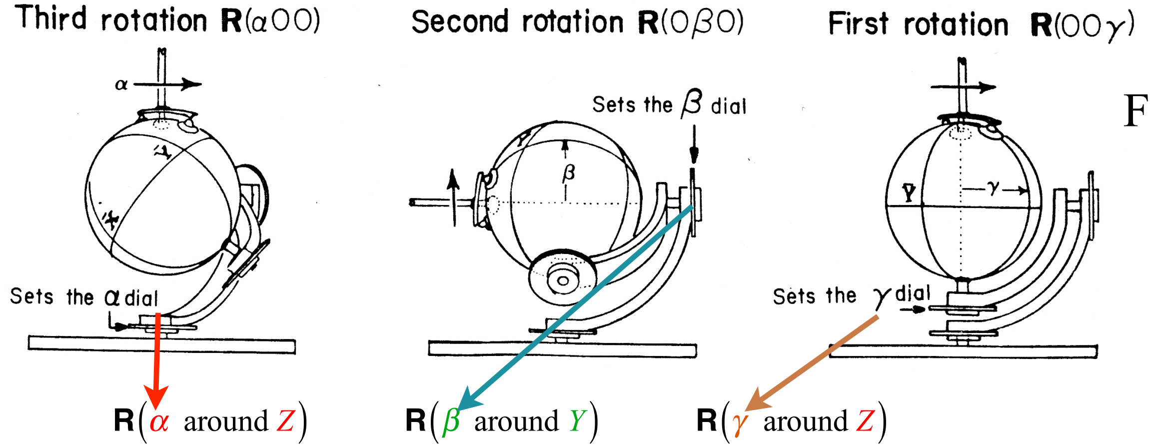
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \ \sin\Theta/2$$

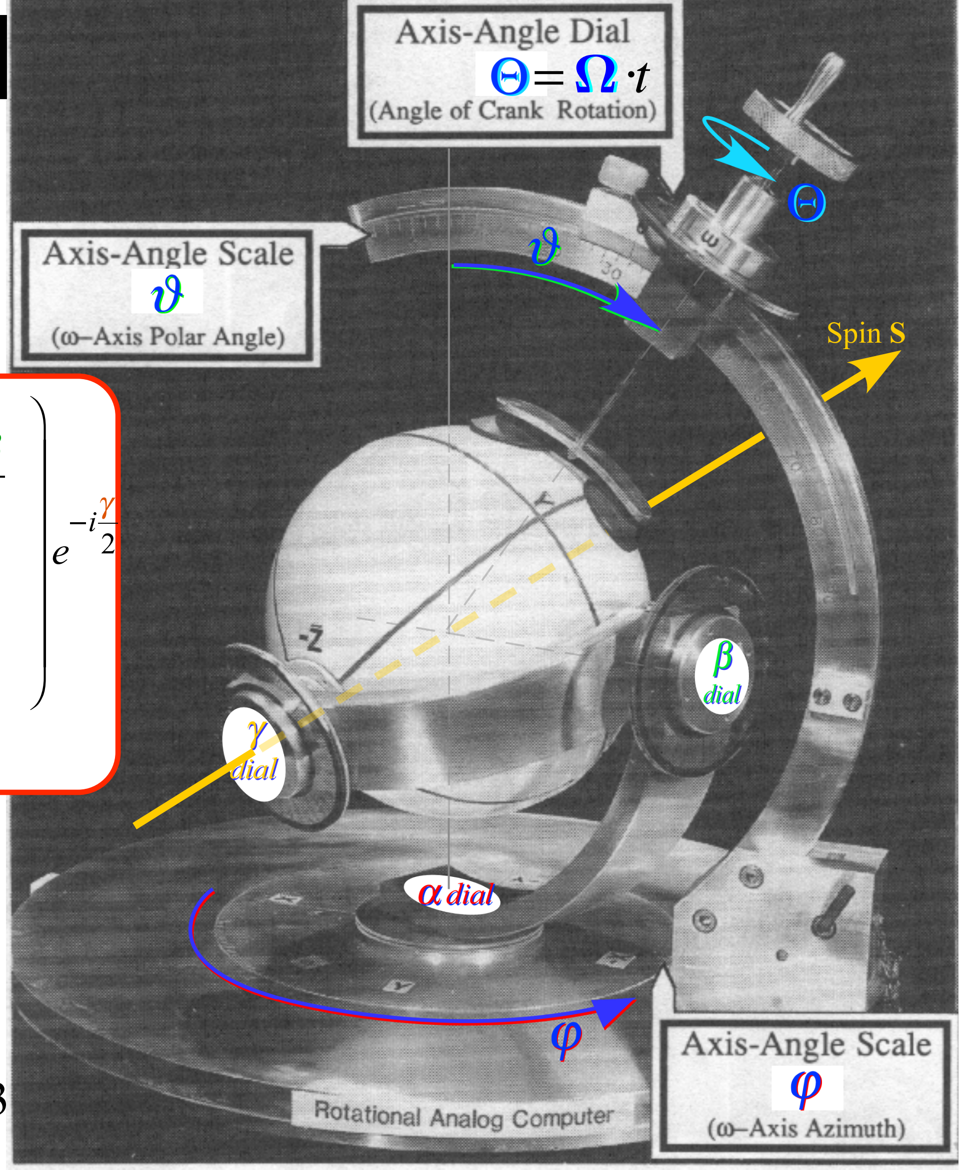
Euler R(αβγ) versus *Darboux* R[φϑΘ]





From Lecture 22
page 62 to 70

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Lecture 22
page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) = R[φϑΘ] ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

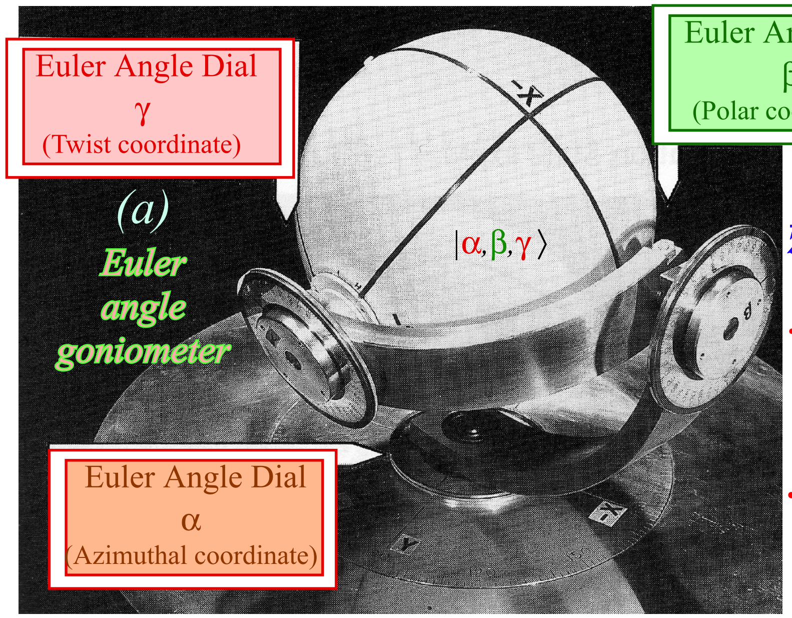
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\ \cos\beta/2 = \cos\Theta/2$

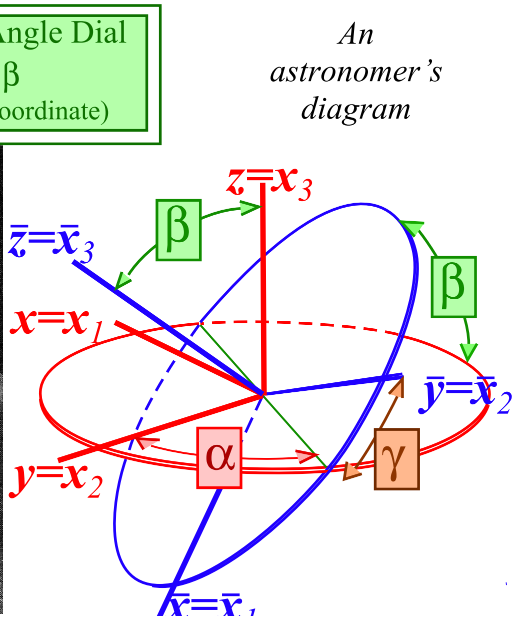
$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\ \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_Y\ \sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

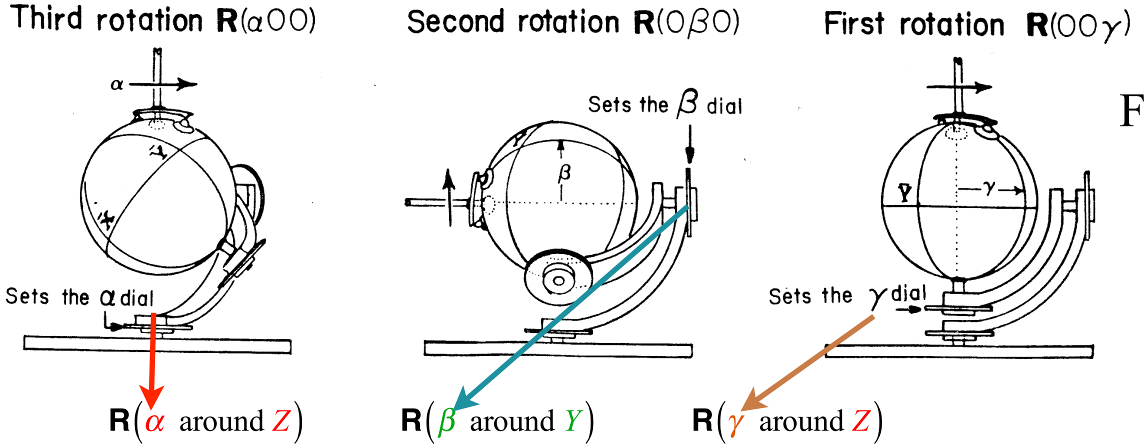
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$



Euler Angle Dial
β
(Polar coordinate)



An astronomer's diagram

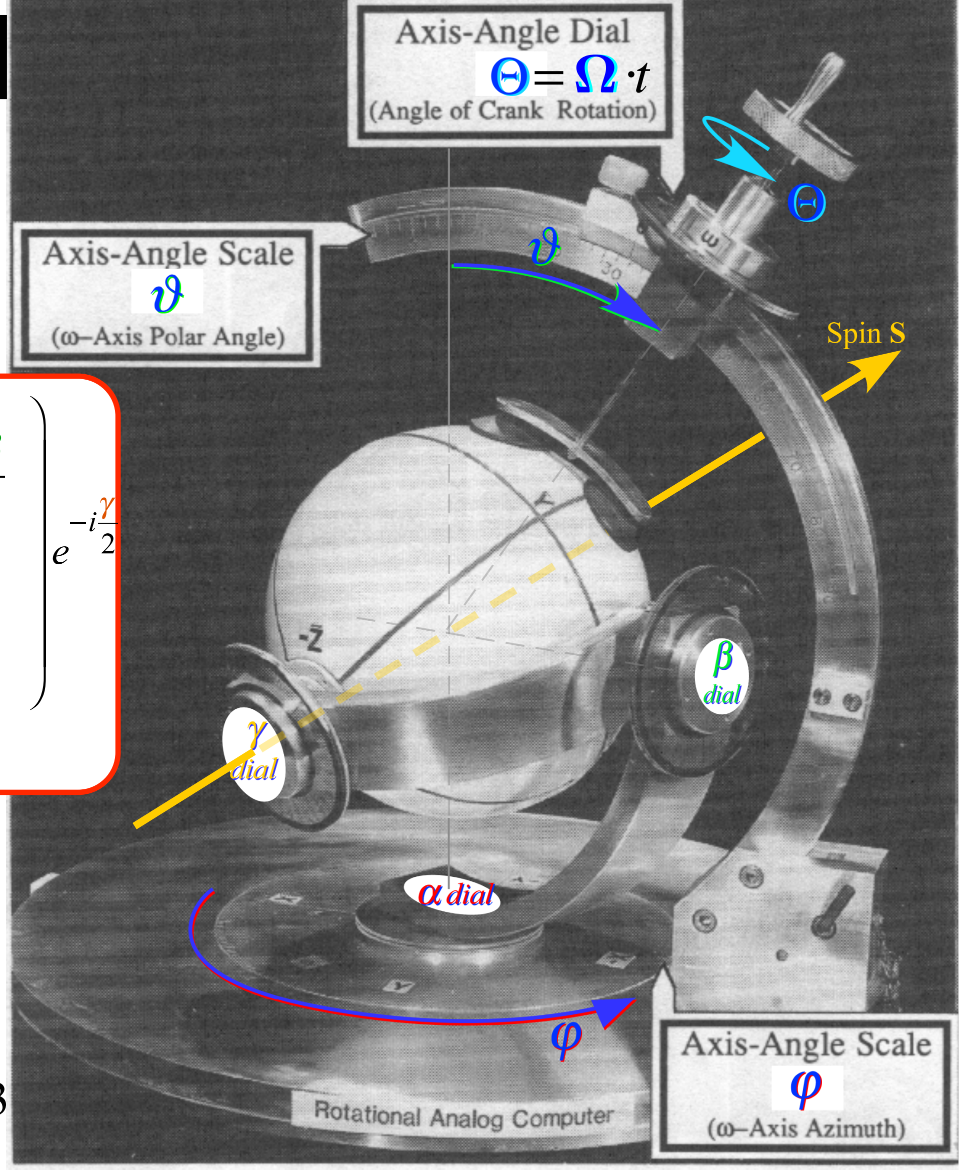


Third rotation $\mathbf{R}(\alpha 00)$ Second rotation $\mathbf{R}(0\beta 0)$ First rotation $\mathbf{R}(00\gamma)$

$\mathbf{R}(\alpha$ around $Z)$ $\mathbf{R}(\beta$ around $Y)$ $\mathbf{R}(\gamma$ around $Z)$

From Lecture 22 page 62 to 70

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma) = \mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\ \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_Y\ \sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\ \cos\beta/2 = \hat{\Theta}_Z\ \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$

U(2) operator vs (x, p)

$$\begin{pmatrix} x_1+ip_1 & -x_2+ip_2 \\ x_2+ip_2 & x_1-ip_1 \end{pmatrix}$$

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ]*(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩   αβγ *make better coordinates but:* **R**(αβγ)|000⟩ = **R**(αβγ) |**1**⟩ = **R**[φϑΘ]|**1**⟩

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \overset{\frown}{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \overset{\frown}{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \overset{\frown}{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

Euler **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$ *(So:* **R**$(\alpha\beta\gamma)$=**R**$[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate **R**$(\alpha\beta\gamma)$ to **R**$[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩   αβγ *make better coordinates but:* **R**(αβγ)|000⟩ = **R**(αβγ) |**1**⟩ = **R**[φϑΘ]|**1**⟩

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

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$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ](*So:* **R**(αβγ)=**R**[φϑΘ])

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩ αβγ *make better coordinates but:* **R**(αβγ)|000⟩ = **R**(αβγ) |**1**⟩ = **R**[φϑΘ]|**1**⟩

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:***R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩ αβγ *make better coordinates but:* **R**(αβγ)|000⟩ = **R**(αβγ) |**1**⟩ = **R**[φϑΘ]|**1**⟩

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta \ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta \ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

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$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ]*(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩ αβγ *make better coordinates but:* **R**(αβγ)|000⟩ = **R**(αβγ) |**1**⟩ = **R**[φϑΘ]|**1**⟩

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

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$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

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This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Euler **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$ *(So:* **R**$(\alpha\beta\gamma)$=**R**$[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate **R**$(\alpha\beta\gamma)$ to **R**$[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

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This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

Euler **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$ *(So:*$\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ](*So:* **R**(αβγ)=**R**[φϑΘ])

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *αβγ make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}x_1+ip_1\\ \\ x_2+ip_2\end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \qquad \cos\vartheta\ \sin\Theta/2$

$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$

$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$

$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$

$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$

$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$

$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$

$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$

$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$\varphi = (\alpha - \gamma + \pi)/2$

$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$

$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$

$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ (So: $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$)

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$

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$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$

$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$

$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$

$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$

$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$

$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$

$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

Example: *Euler angles* $(\alpha=50^\circ\ \beta=60^\circ\ \gamma=70^\circ)$

$\varphi = (50^\circ-70^\circ+180^\circ)/2 = 80^\circ$

$\vartheta = \tan^{-1}[\tan 60^\circ/2/\sin(50^\circ+\gamma)/2] = 33.7^\circ$

$\Theta = 2\cos^{-1}[\cos 60^\circ/2\cos(50^\circ+\gamma)/2] = 128.7^\circ$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] (*So:* **R**(αβγ)=**R**[φϑΘ])

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

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$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

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$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha - \gamma + \pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* (α=50° β=60° γ=70°)

$$\varphi = (50° - 70° + 180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\ \sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\ \cos(50°+\gamma)/2] = 128.7°$$

Reverse check: (αβγ) in terms of [φϑΘ]

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$

$$\beta = 2\sin^{-1}(\sin 128.7°/2\ \sin 33.7°) = 60.022°$$

$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2) = 70.007°$$

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- ➡ *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

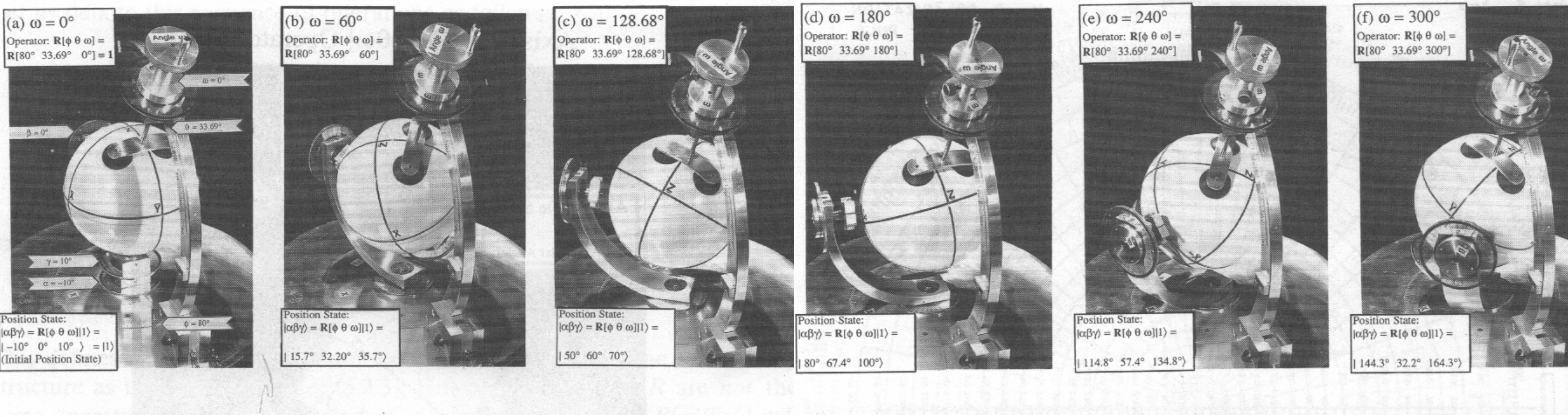
The ABC's of U(2) dynamics-Archetypes
- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates
- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

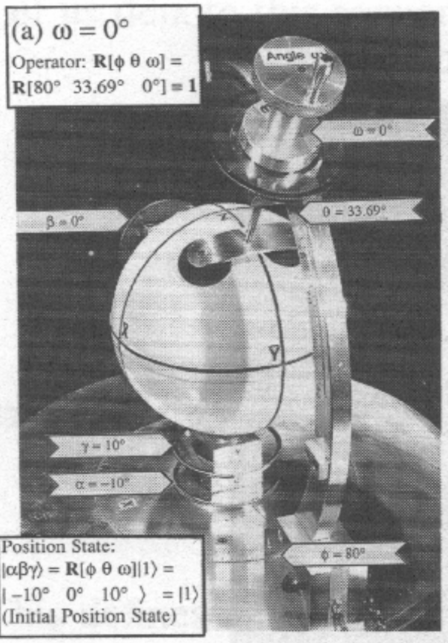
$\Theta=0°$ $\Theta=60°$ $\Theta=128.7°$ $\Theta=180°$ $\Theta=240°$ $\Theta=300°$



Under Construction: Web based U(2) Calculator - Euler & Darboux Angles

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

$\Theta=0^\circ$

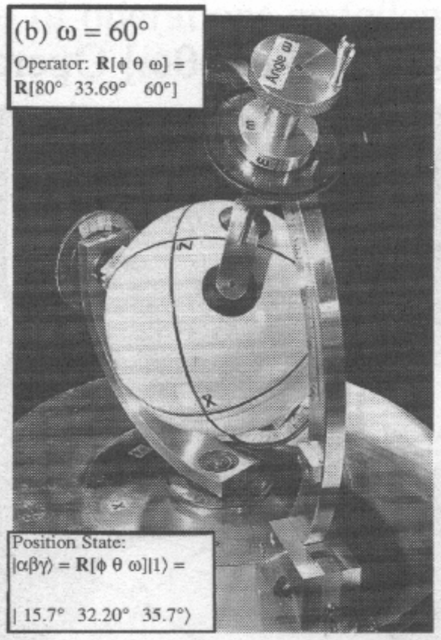


(a) $\omega = 0^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 0^\circ] = \mathbf{1}$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|-10^\circ\ \ 0^\circ\ \ 10^\circ\rangle = |1\rangle$ (Initial Position State)

$\Theta=60^\circ$

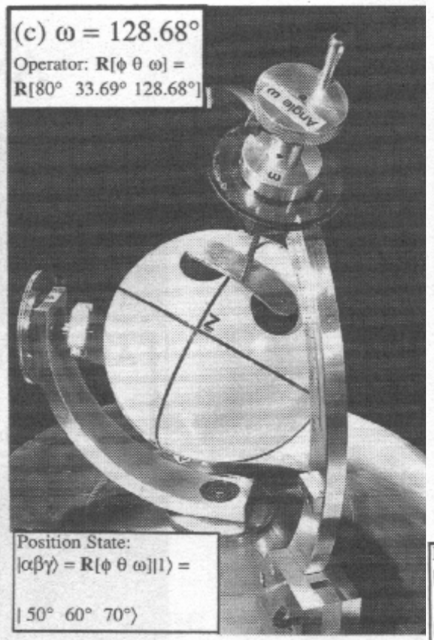


(b) $\omega = 60^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 60^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 15.7^\circ\ \ 32.20^\circ\ \ 35.7^\circ\rangle$

$\Theta=128.7^\circ$

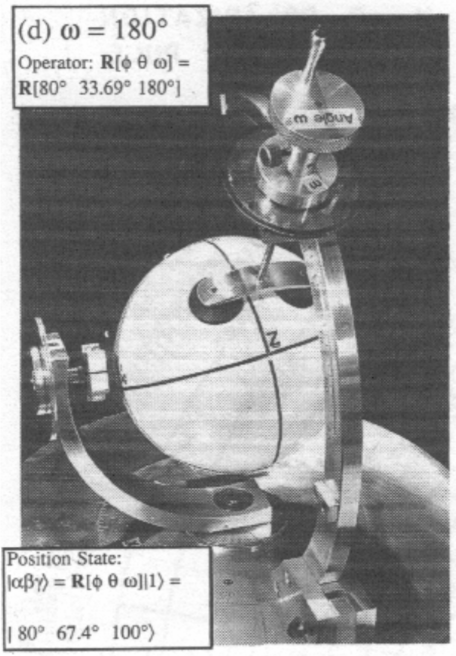


(c) $\omega = 128.68^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 128.68^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 50^\circ\ \ 60^\circ\ \ 70^\circ\rangle$

$\Theta=180^\circ$

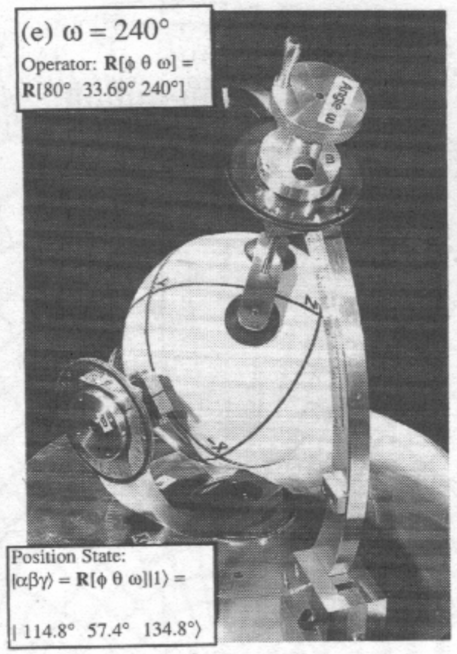


(d) $\omega = 180^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 180^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 80^\circ\ \ 67.4^\circ\ \ 100^\circ\rangle$

$\Theta=240^\circ$

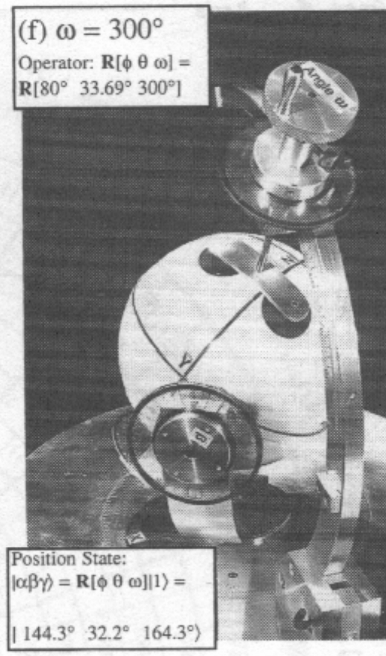


(e) $\omega = 240^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 240^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 114.8^\circ\ \ 57.4^\circ\ \ 134.8^\circ\rangle$

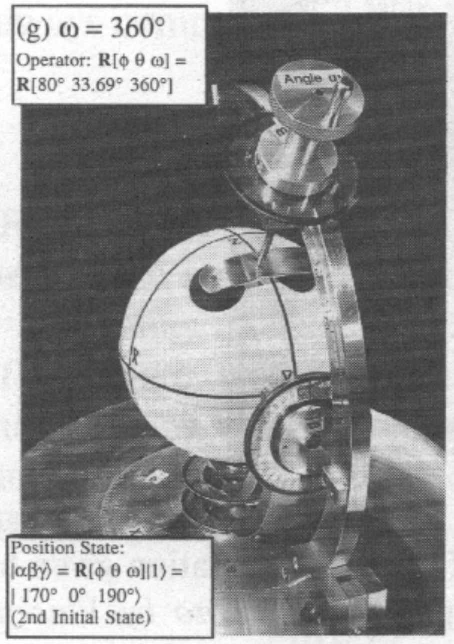
$\Theta=300^\circ$



(f) $\omega = 300^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 300^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 144.3^\circ\ \ 32.2^\circ\ \ 164.3^\circ\rangle$



(g) $\omega = 360^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ 33.69^\circ\ 360^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 170^\circ\ \ 0^\circ\ \ 190^\circ\rangle$ (2nd Initial State)

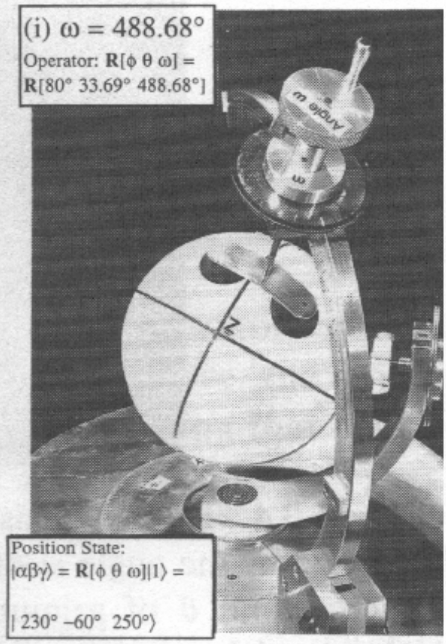
$\Theta=360^\circ$

(h) $\omega = 420^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ 33.69^\circ\ 420^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 195.7^\circ\ -32.2^\circ\ \ 215.7^\circ\rangle$

$\Theta=420^\circ$

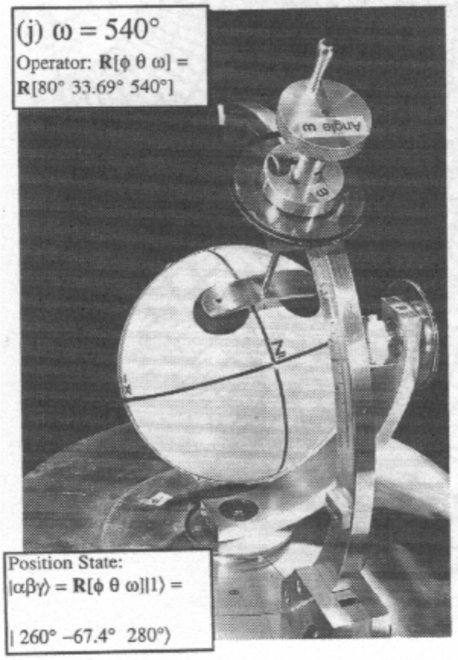


(i) $\omega = 488.68^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ 33.69^\circ\ 488.68^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 230^\circ\ -60^\circ\ \ 250^\circ\rangle$

$\Theta=488.7^\circ$

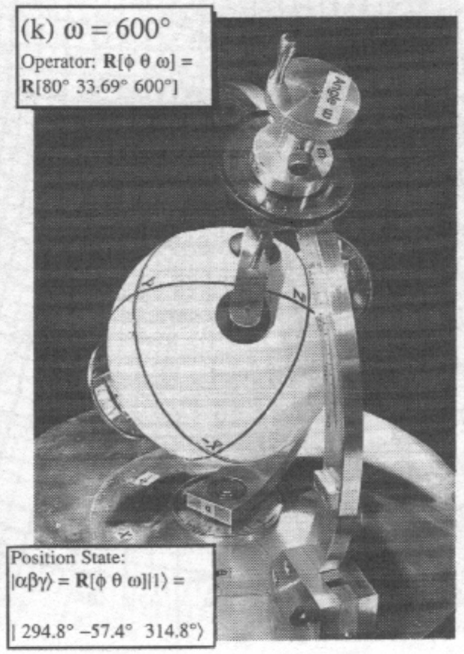


(j) $\omega = 540^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ 33.69^\circ\ 540^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 260^\circ\ -67.4^\circ\ \ 280^\circ\rangle$

$\Theta=540^\circ$



(k) $\omega = 600^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ 33.69^\circ\ 600^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 294.8^\circ\ -57.4^\circ\ \ 314.8^\circ\rangle$

$\Theta=600^\circ$

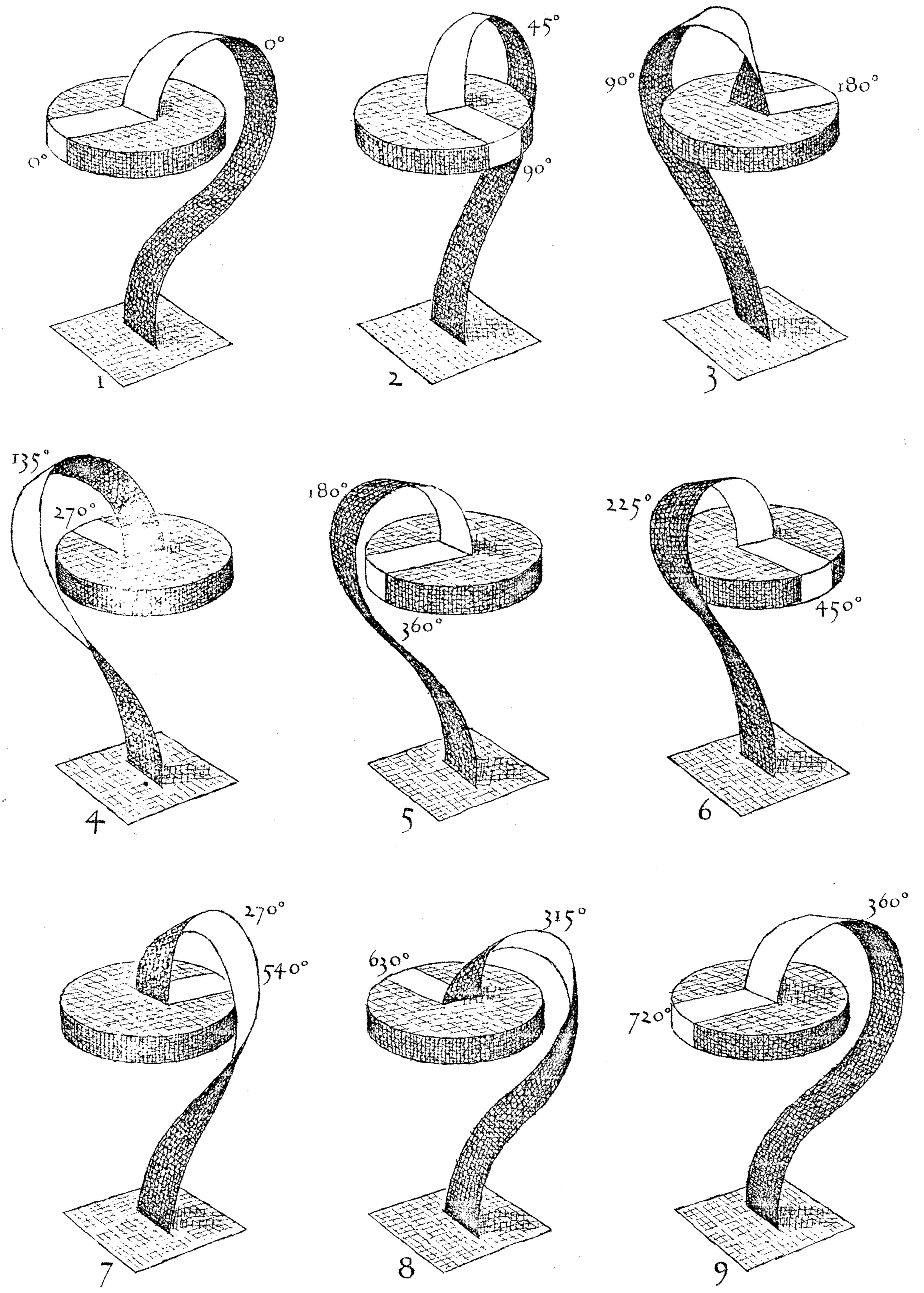
(l) $\omega = 660^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ 33.69^\circ\ 660^\circ]$

Position State: $|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$ $|\ 324.3^\circ\ -32.2^\circ\ \ 344.3^\circ\rangle$

$\Theta=660^\circ$

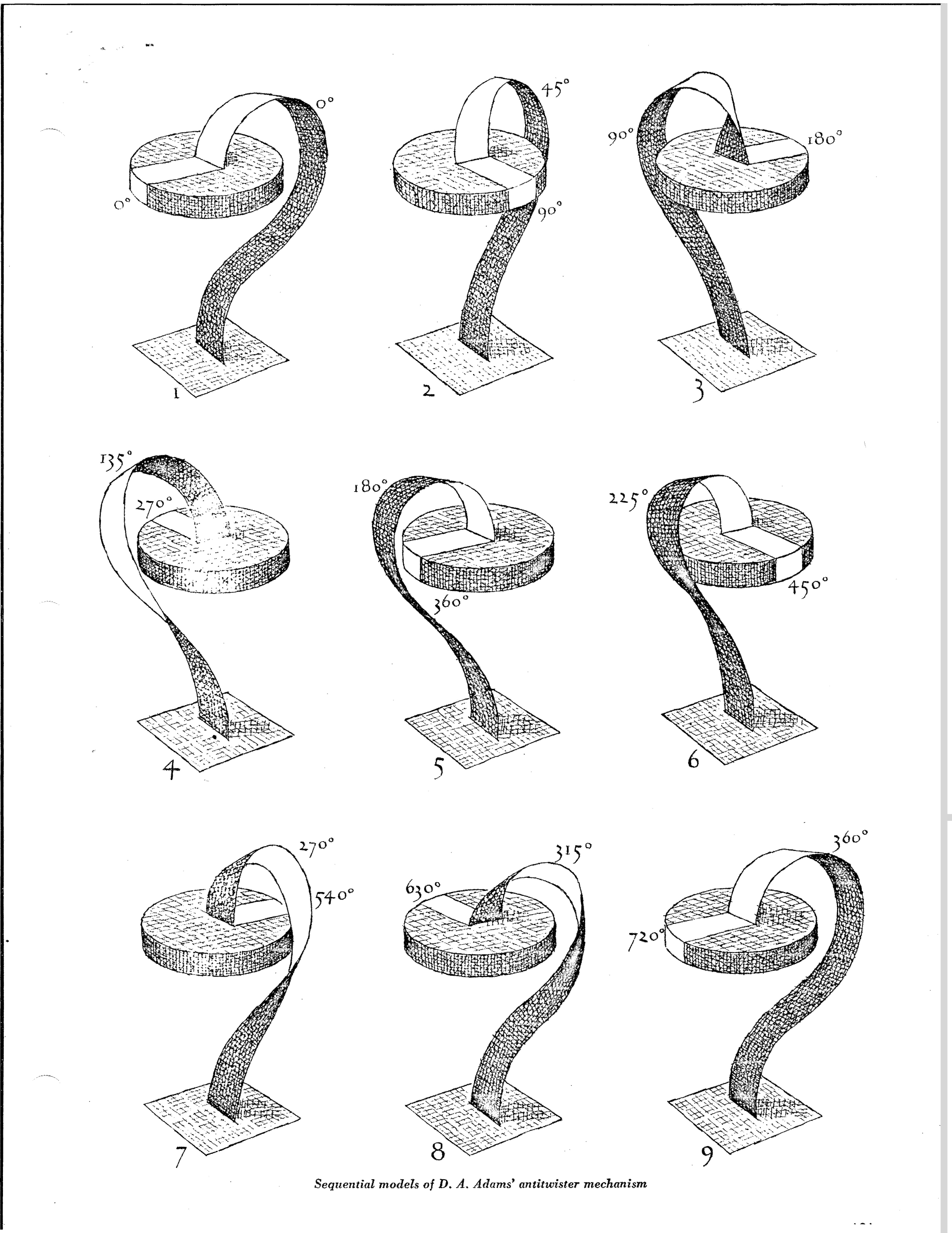
Some "real-world" applications of the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

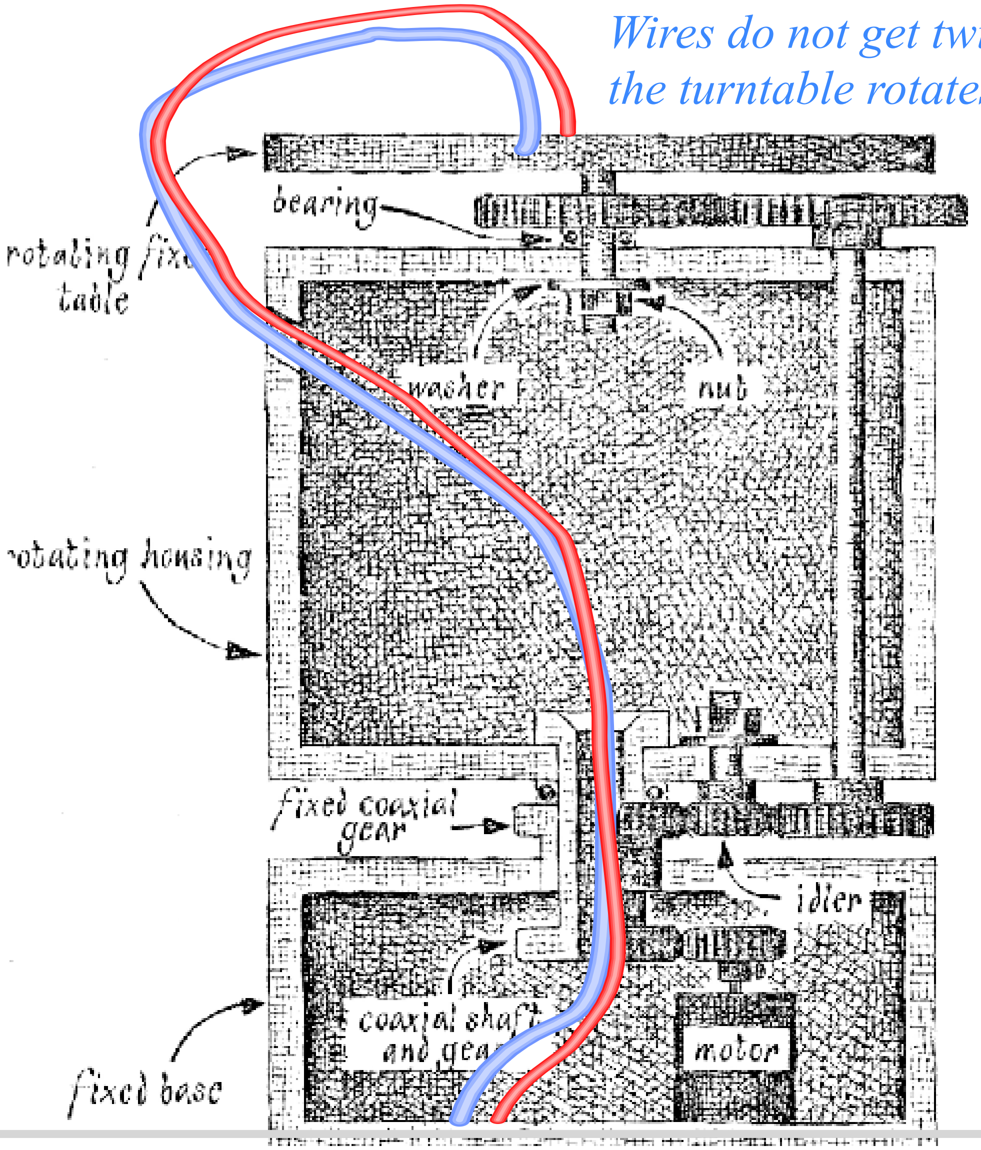
From Scientific American
December 1975-p.120-125

Some "real-world" applications of the U(2)-R(3) spinor-vector topology



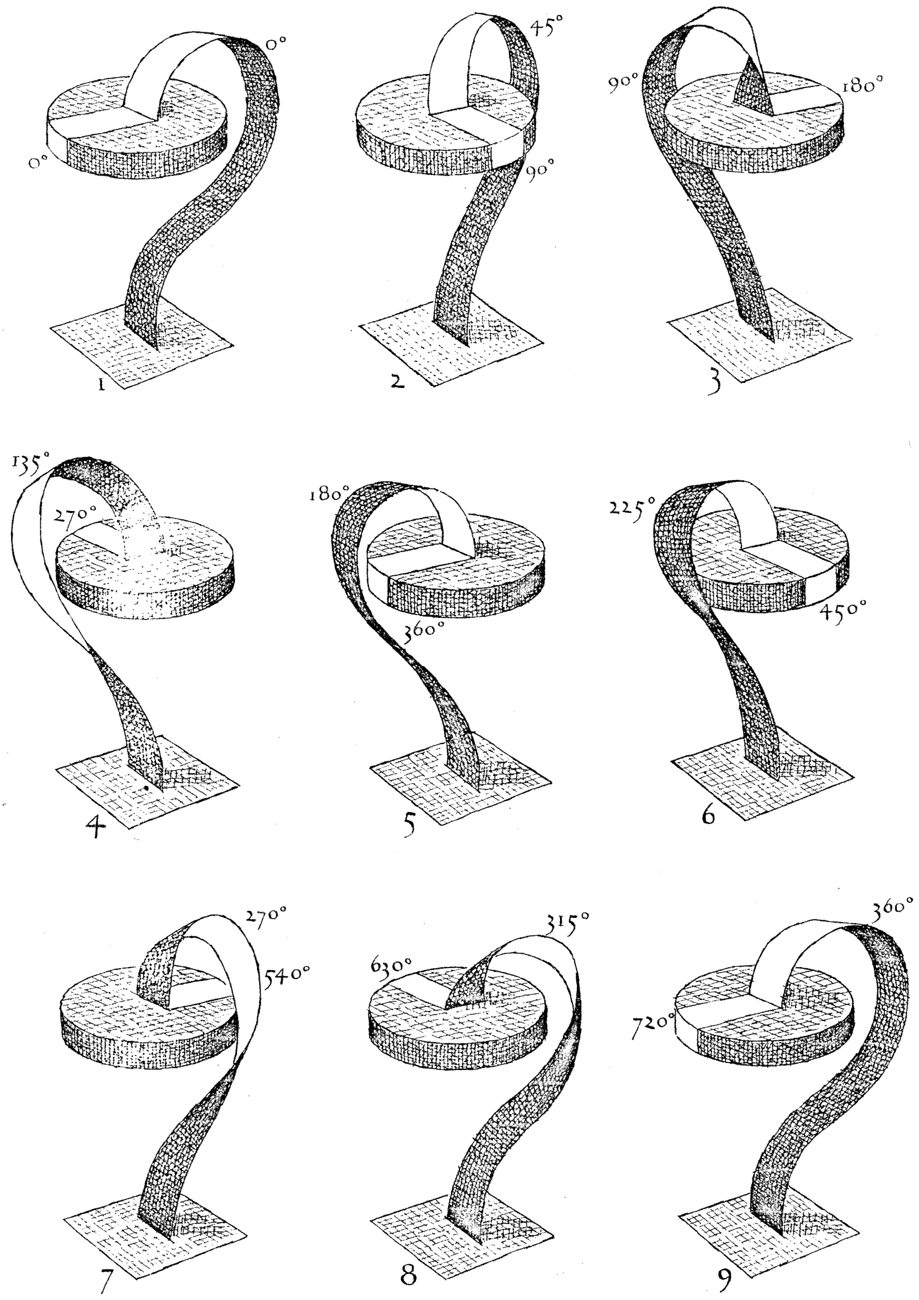
Sequential models of D. A. Adams' antitwister mechanism

From Scientific American December 1975-p.120-125



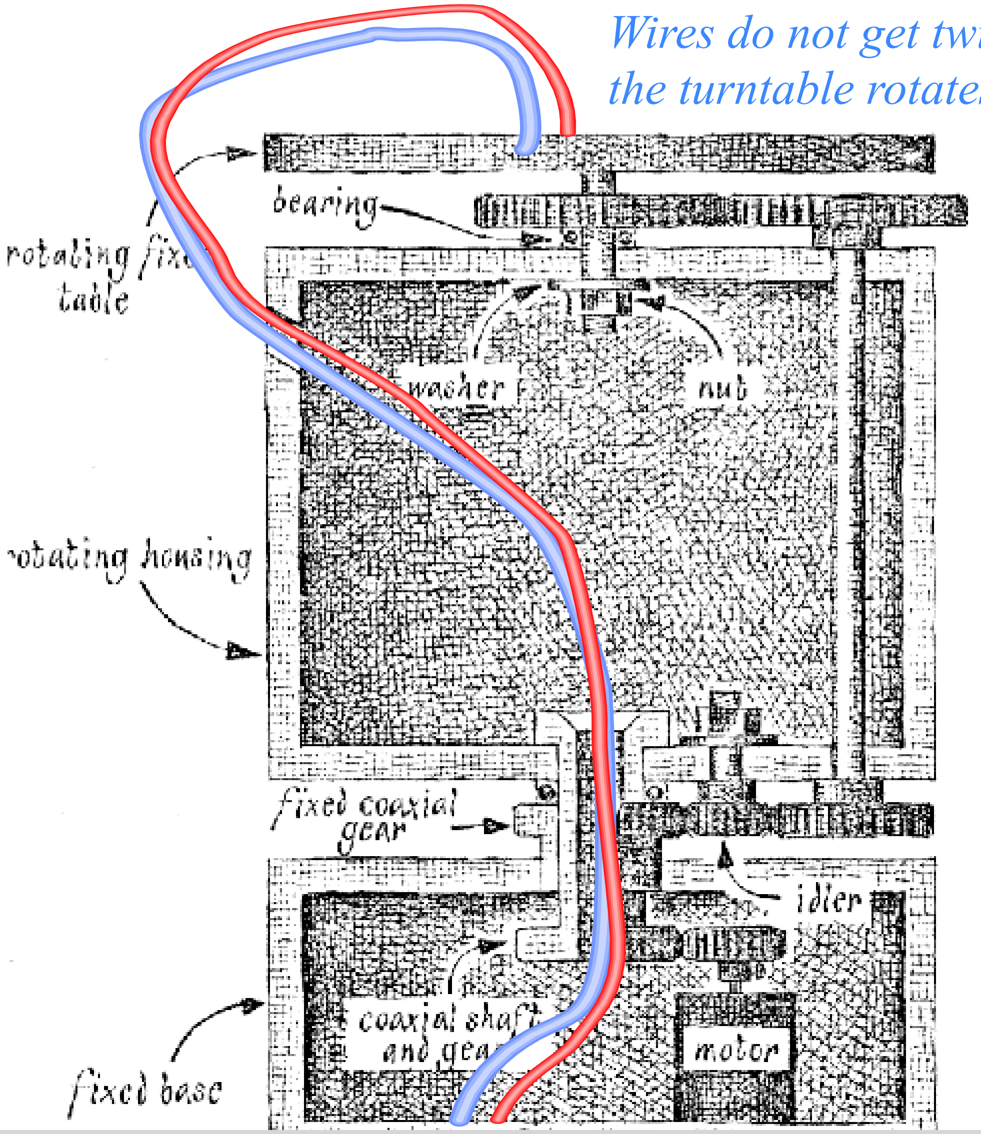
Wires do not get twisted up as the turntable rotates

Some "real-world" applications of the U(2)-R(3) spinor-vector topology

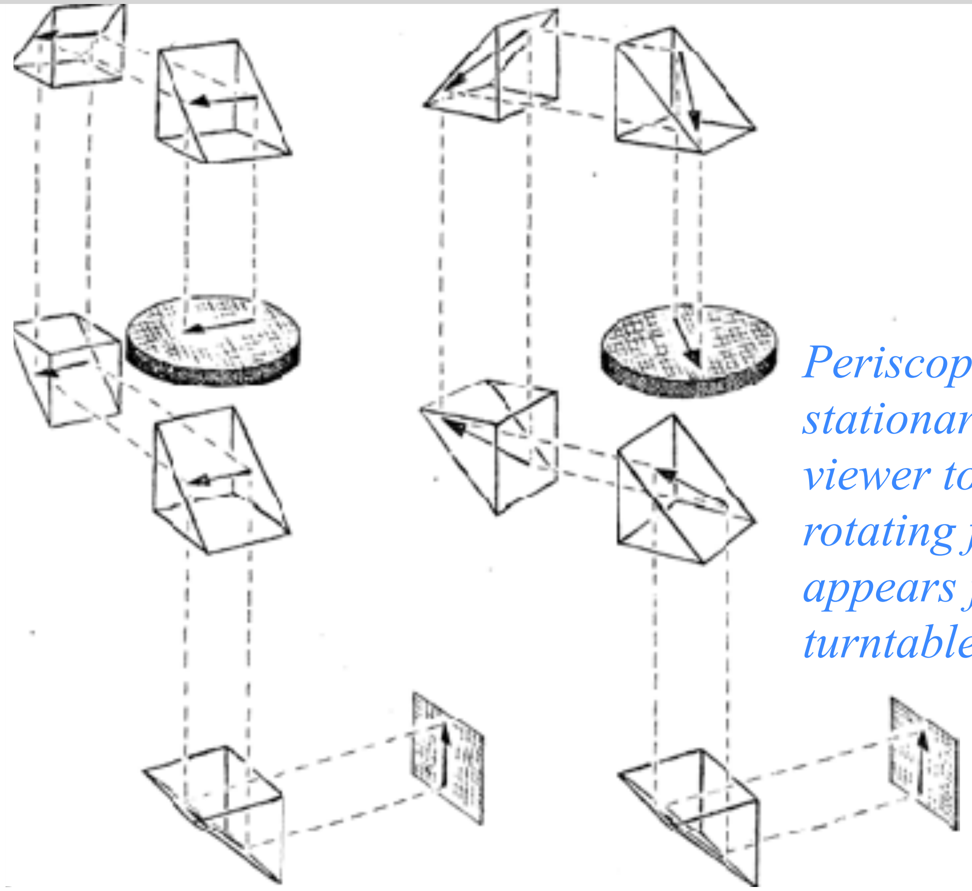


Sequential models of D. A. Adams' antitwister mechanism

From Scientific American December 1975-p.120-125



Wires do not get twisted up as the turntable rotates



Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Reviewing fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

- *Euler-defined state |αβγ⟩ described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**[φϑΘ]=*exp(-i***Ω•S***)·t and angular velocity* **Ω**(φϑ)*·t*=**Θ***-vector*
- *Euler-defined operator* **R***(αβγ) derived from Darboux-defined* **R**[φϑΘ] *and vice versa*
- *Euler* **R***(αβγ) rotation* **Θ**=0−4π*-sequence* [φϑ] *fixed (and "real-world" applications)*

➤ *Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian* $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$ ◄

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle (αβγ) ellipse coordinates*

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \mathbf{\Omega}_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\mathbf{\Omega}=\mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \quad \Omega_0 \quad \mathbf{1} \quad + \quad \mathbf{\Omega}_A \quad \mathbf{S}_A \quad + \quad \mathbf{\Omega}_B \quad \mathbf{S}_B \quad + \quad \mathbf{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\mathbf{\Omega}_A{}^2 + \mathbf{\Omega}_B{}^2 + \mathbf{\Omega}_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm\sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

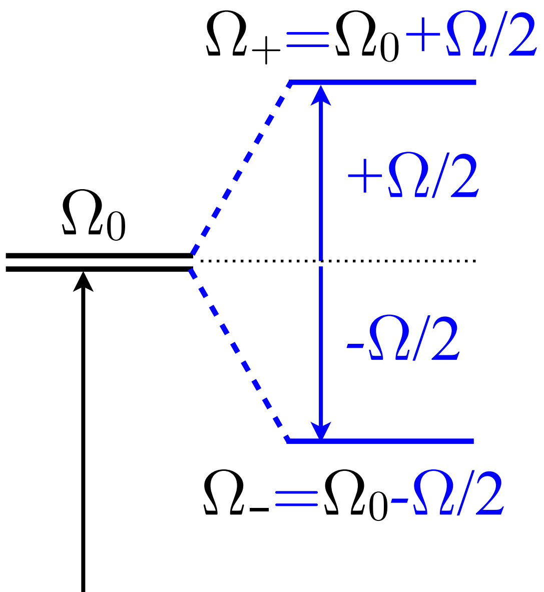
$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

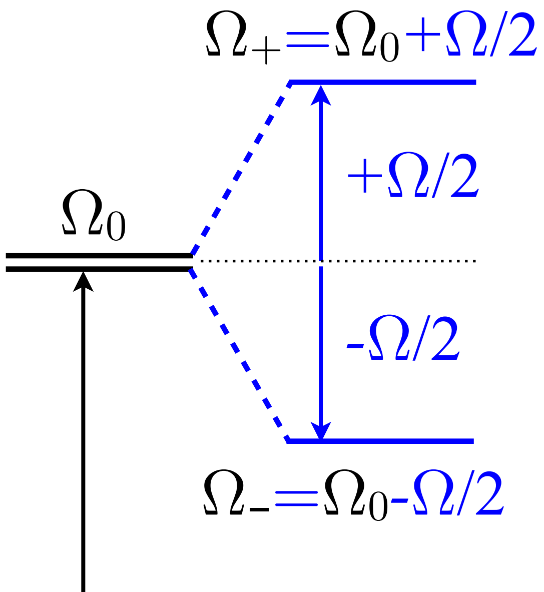
$$\mathbf{H} = \Omega_0 \mathbf{1} + \mathbf{\Omega}_A \mathbf{S}_A + \mathbf{\Omega}_B \mathbf{S}_B + \mathbf{\Omega}_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$
$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\,]$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

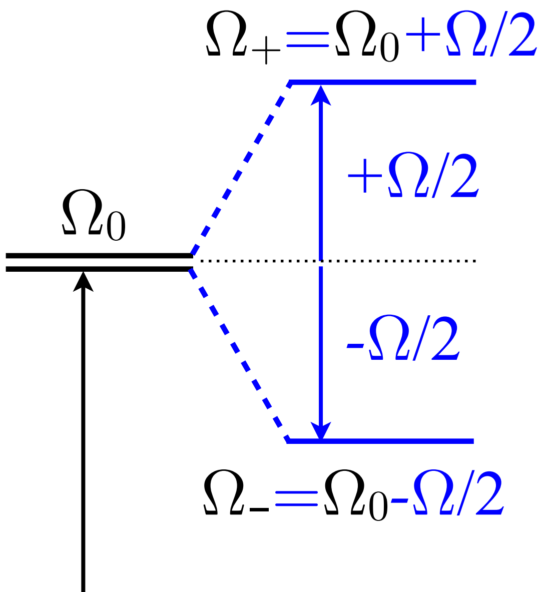
Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

$$\boxed{\begin{array}{l} \textit{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 \\ = \dfrac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2} \end{array}}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\;]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)\,/\,\sqrt{(A-D)^2 + 4B^2 + 4C^2}\;]$, $\qquad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\;]$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** matrix:

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ of crank vector $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ and: $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

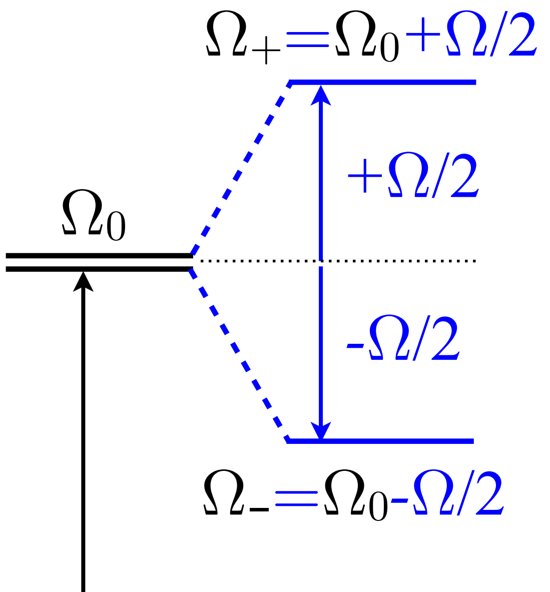
$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, and: $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\ ]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)\,/\,\sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\qquad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

Step 3.To find eigenvectors replace Euler angles (azimuth α, polar β) of Euler-state

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$\Omega_+ = \Omega_0 + \Omega/2$

$+\Omega/2$

Ω_0

$-\Omega/2$

$\Omega_- = \Omega_0 - \Omega/2$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\ \sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\ \sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega \sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\ ]$

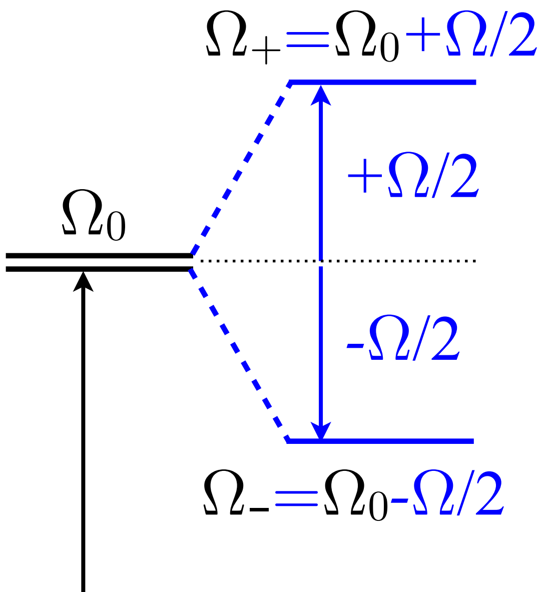
or: $\vartheta = \cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\qquad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

with the Darboux axis polar angles (azimuth φ *, polar* ϑ *) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

$$= \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\boldsymbol{\Omega}=\boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \quad \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\ \sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\ \sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\ \sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\ ]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)\ /\ \sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\quad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

$$\boxed{\textit{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 = \frac{A + D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}}$$

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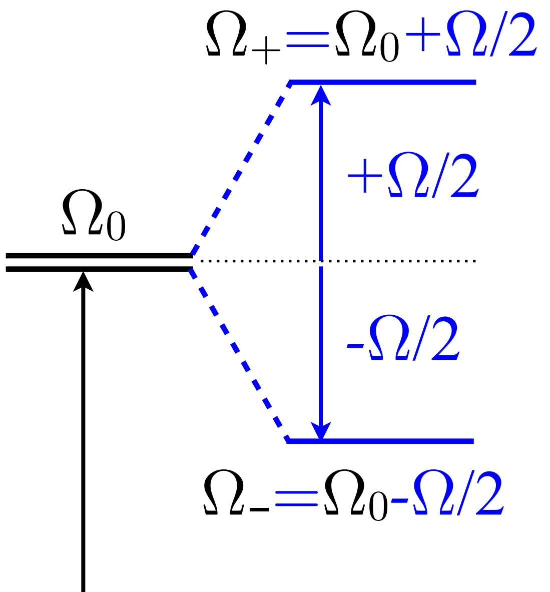
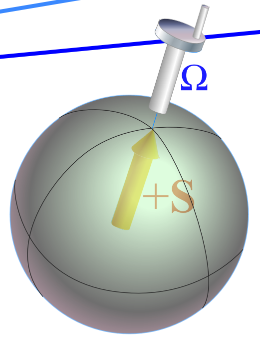
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$



Spin **+S**

Up-Crank

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \quad \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\,]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)\,/\,\sqrt{(A-D)^2 + 4B^2 + 4C^2}\,]$, $\quad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\,]$

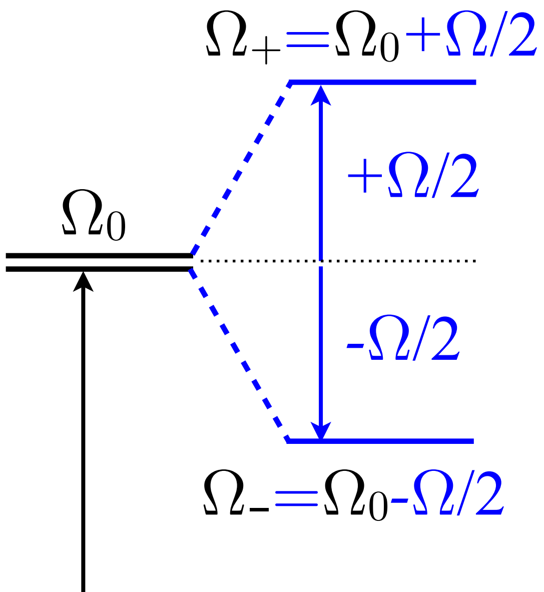
Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

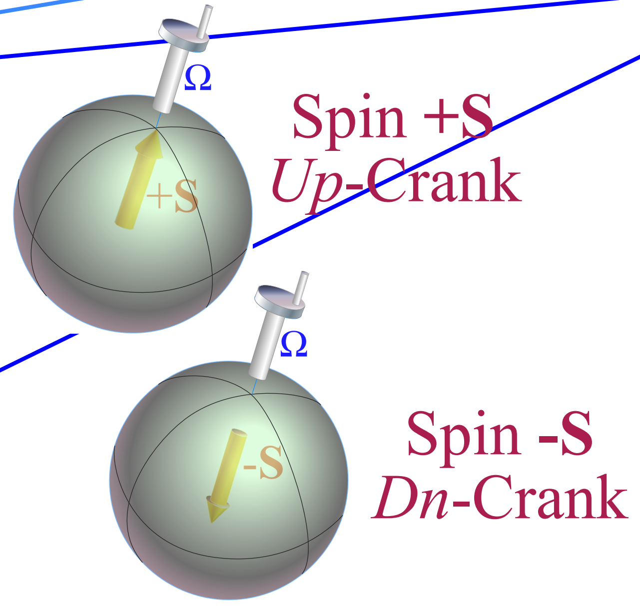
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta \pm \pi$*) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$\left|\Omega_-\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\boldsymbol{\Omega}=\boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \quad \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2+\Omega_C{}^2}\ ]$

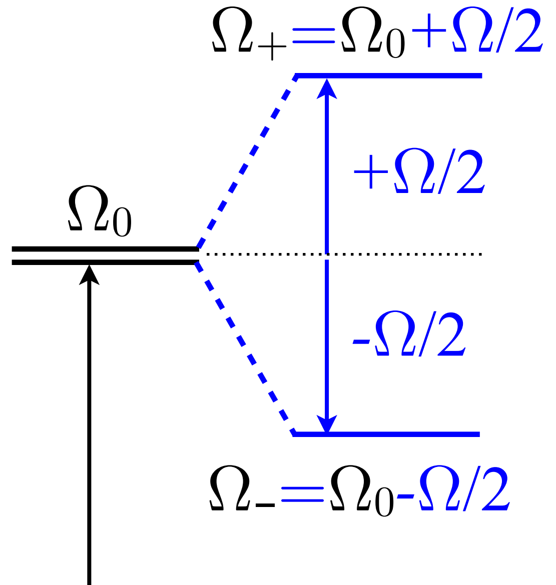
or: $\vartheta = \cos^{-1}[(A\text{-}D)\,/\,\sqrt{(A-D)^2+4B^2+4C^2}\ ]$, $\qquad \varphi = \cos^{-1}[B/\sqrt{B^2+C^2}\ ]$

$$\boxed{\begin{aligned}&\textit{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 \\ &= \frac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}\end{aligned}}$$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

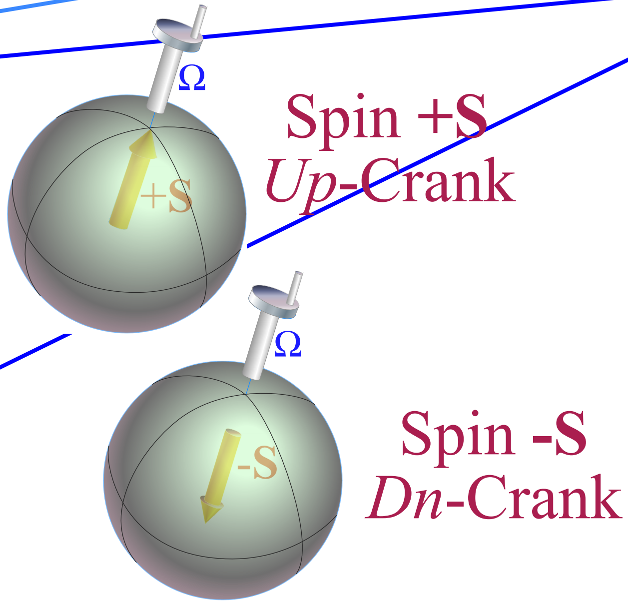
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta\pm\pi$*) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$\left|\Omega_-\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



More reliable computation:

$\varphi = atan2(C, B)$

$[\tan^{-1}(C\,/\,B)\ \textit{is unreliable}]$

$\vartheta = atan2(2\sqrt{B^2+C^2}, A-D)$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\left(\begin{array}{cc} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{array}\right)=\left(\begin{array}{cc} A & B-iC \\ B+iC & D \end{array}\right)$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

$$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

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$$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$$

Step 2.Convert Cartesian to polar form: $(\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta)$

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega\ =\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}\ =\sqrt{64}=8$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

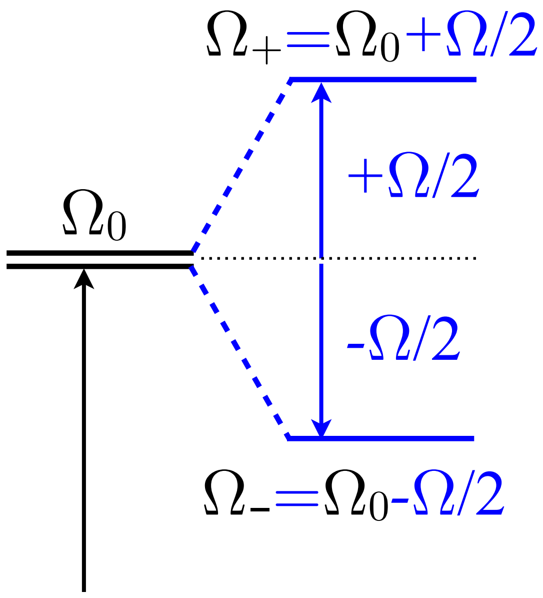
$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

$$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$$

Step 2.Convert Cartesian to polar form: ($\Omega_A = \Omega\cos\vartheta$, $\Omega_B = \Omega\cos\varphi\,\sin\vartheta$, $\Omega_C = \Omega\sin\varphi\,\sin\vartheta$)

$$\Omega_0 = \frac{A+D}{2} = 10$$

and: $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2} = \sqrt{(4)^2 + 4\sqrt{6}^2 + 4\sqrt{6}^2} = \sqrt{16+24+24} = \sqrt{64} = 8$



$\boxed{eigenvalue - 1}$

$$\omega_\uparrow = 10 + \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 + 4 = 14$$

$\boxed{eigenvalue - 2}$

$$\omega_\downarrow = 10 - \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 - 4 = 6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix}12 & \sqrt{6}(1-i)\\ \sqrt{6}(1+i) & 8\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\begin{pmatrix}10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3}\\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3}\end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

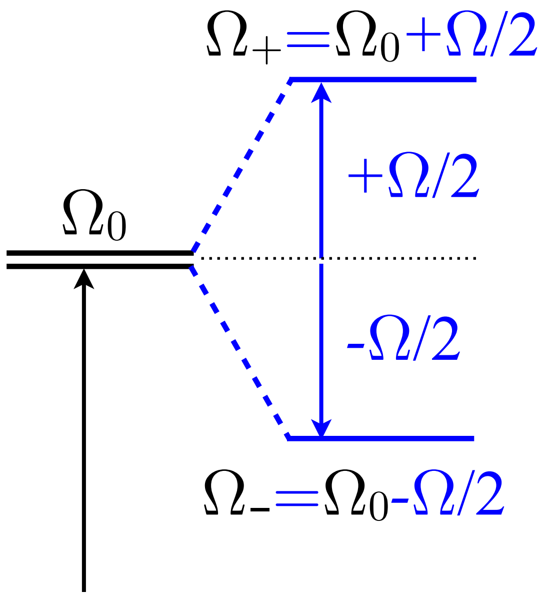
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega=\sqrt{{\Omega_A}^2+{\Omega_B}^2+{\Omega_C}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



$\underline{|eigenvalue-1}$

$$\omega_{\uparrow}=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\underline{|eigenvalue-2}$

$$\omega_{\downarrow}=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$$

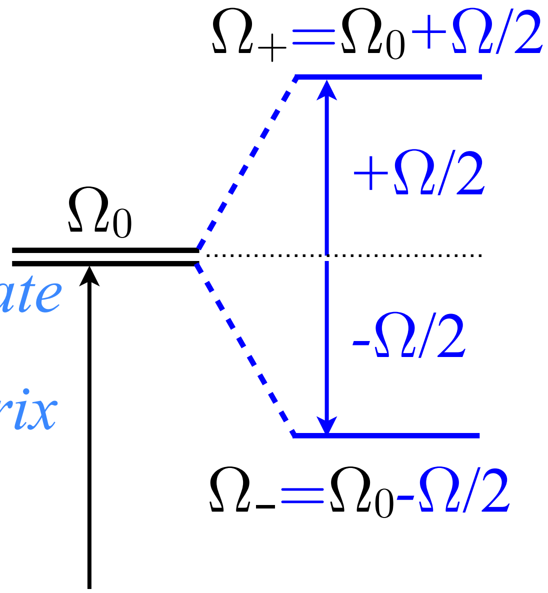
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta$, $\Omega_B=\Omega\cos\varphi\,\sin\vartheta$, $\Omega_C=\Omega\sin\varphi\,\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega\ =\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}\ =\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}\ ]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\ ]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



Step 3.To find eigenvectors replace Euler angles (azimuth α, polar β) of Euler-state

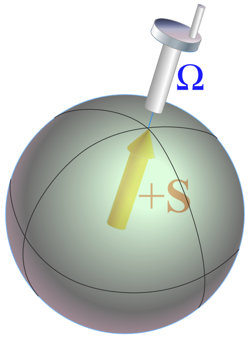
with the Darboux axis polar angles (azimuth φ , polar ϑ or $\vartheta\pm\pi$) of **H***-matrix*

eigenvalue -1

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

eigenvector -1

$$|\uparrow\rangle=\begin{pmatrix} e^{-i\frac{\pi}{8}}\cos\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\sin\frac{\pi}{6} \end{pmatrix}=\begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

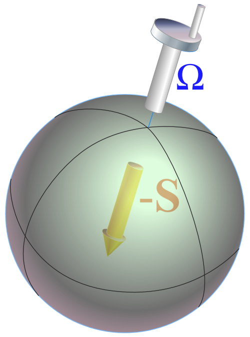


eigenvalue -2

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

eigenvector -2

$$|\downarrow\rangle=\begin{pmatrix} -e^{-i\frac{\pi}{8}}\sin\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\cos\frac{\pi}{6} \end{pmatrix}=\begin{pmatrix} -e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \\ 1 \end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$



Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

The *ABC*'s of *U(2)* dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)* $\longrightarrow$
$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\boldsymbol{\sigma}_{\mathrm{B}} + C\,\boldsymbol{\sigma}_{\mathrm{C}} + \frac{A-D}{2}\,\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\mathbf{H} = \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\boldsymbol{\sigma}_{\mathrm{B}} + \frac{\Omega_{\mathrm{C}}}{2}\,\boldsymbol{\sigma}_{\mathrm{C}} + \frac{\Omega_{\mathrm{A}}}{2}\,\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_{\mathrm{A}} \\ \Omega_B \\ \Omega_{\mathrm{C}} \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)* →

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_{\mathrm{B}} + C\,\boldsymbol{\sigma}_{\mathrm{C}} + \frac{A-D}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_{\mathrm{B}} + \frac{\Omega_{\mathrm{C}}}{2}\boldsymbol{\sigma}_{\mathrm{C}} + \frac{\Omega_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\vec{\Omega}=\begin{pmatrix}\Omega_{\mathrm{A}}\\ \Omega_B\\ \Omega_{\mathrm{C}}\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\textit{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_{\mathrm{A}}\\ \Omega_B\\ \Omega_{\mathrm{C}}\end{pmatrix}=\begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix}\qquad \textit{Eigen}-\textit{Spin}:\vec{\mathbf{S}}=\begin{pmatrix}S_{\mathrm{A}}\\ S_B\\ S_{\mathrm{C}}\end{pmatrix}=\begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)* $\longrightarrow$

$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

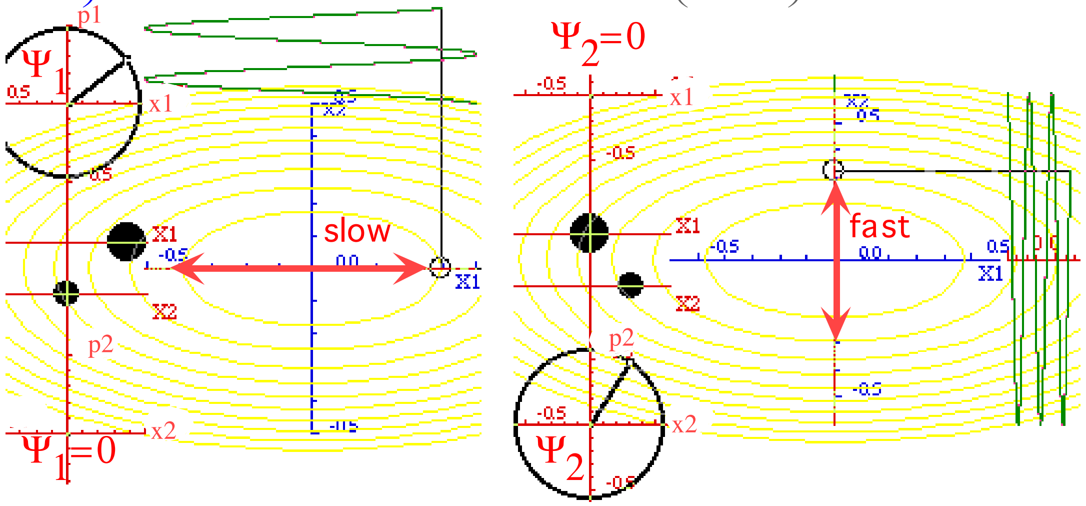
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

$$\textit{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad \textit{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\;\sigma_B + C\;\sigma_C + \frac{A-D}{2}\sigma_A$$

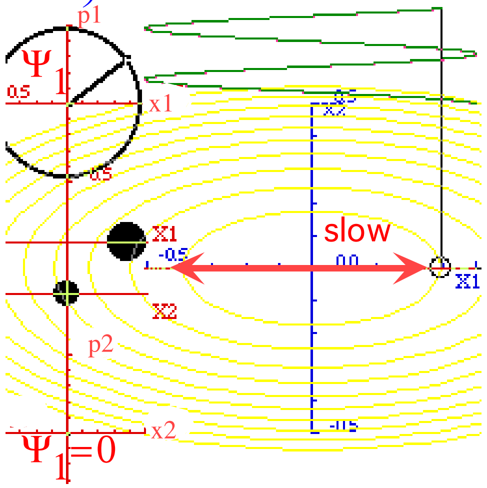
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

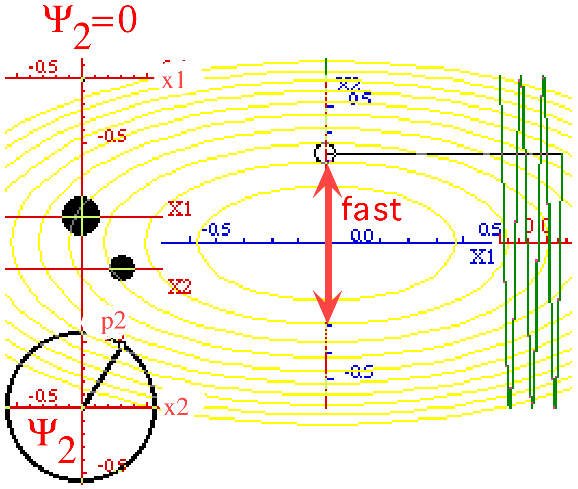
$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

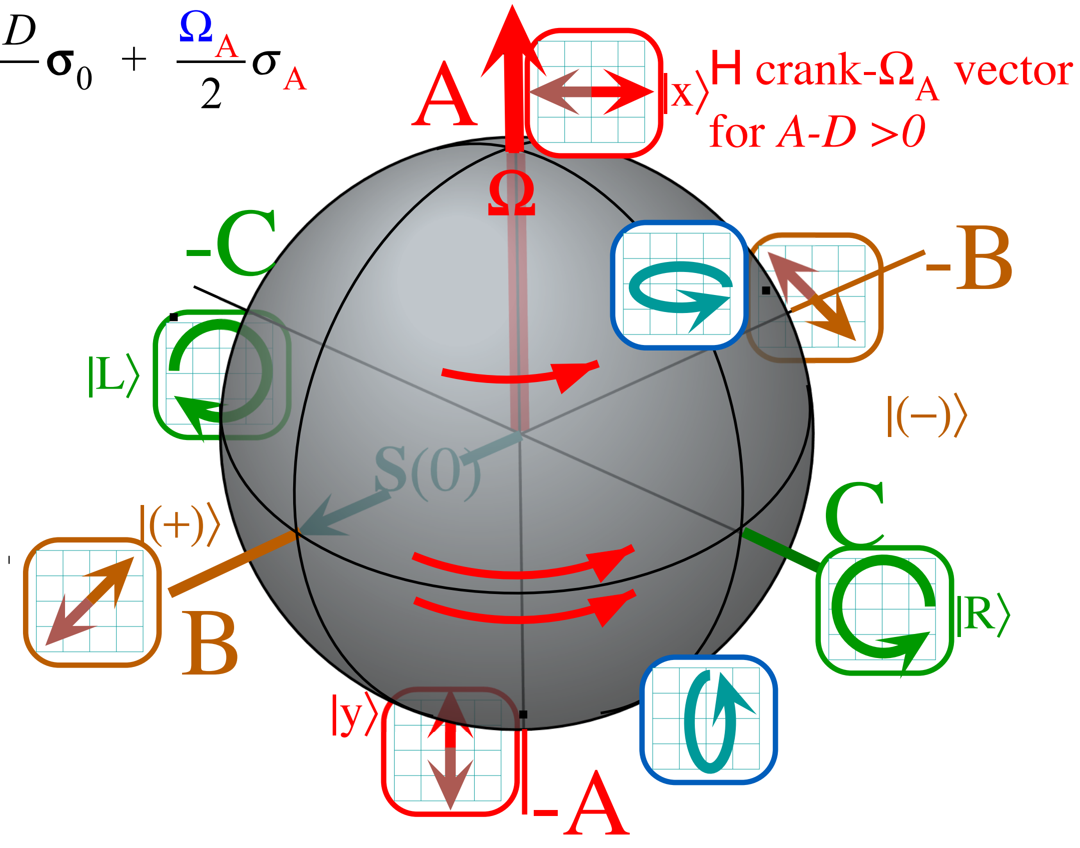
Asymmetric Diagonal A-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$Crank:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix}\qquad Eigen-Spin:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$







The ABC's of U(2) dynamics

Density operator **ρ** (see p.128-147) →

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H}=\frac{A+D}{2}\mathbf{1}+B\,\sigma_B+C\,\sigma_C+\frac{A-D}{2}\sigma_A$$

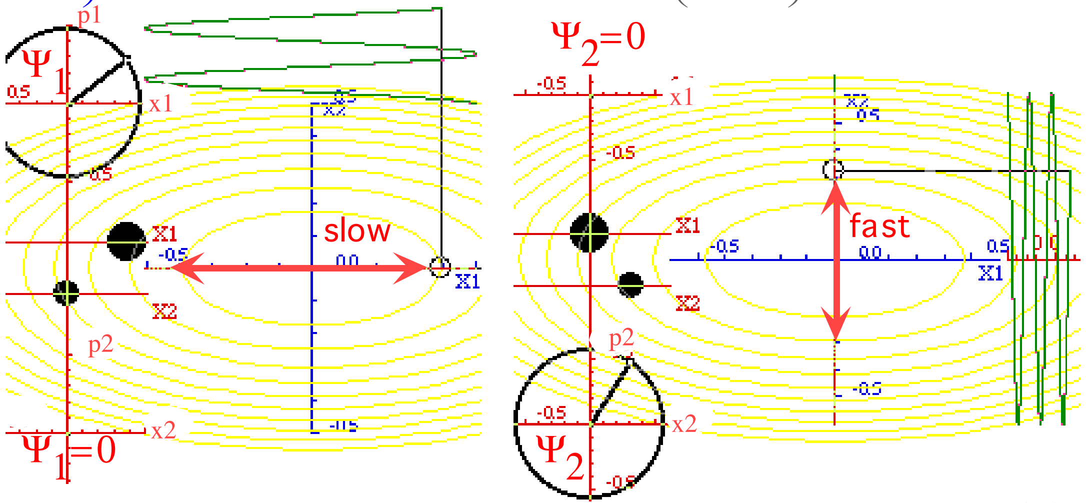
$$\mathbf{H}=\frac{A+D}{2}\sigma_0+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C+\frac{\Omega_A}{2}\sigma_A$$

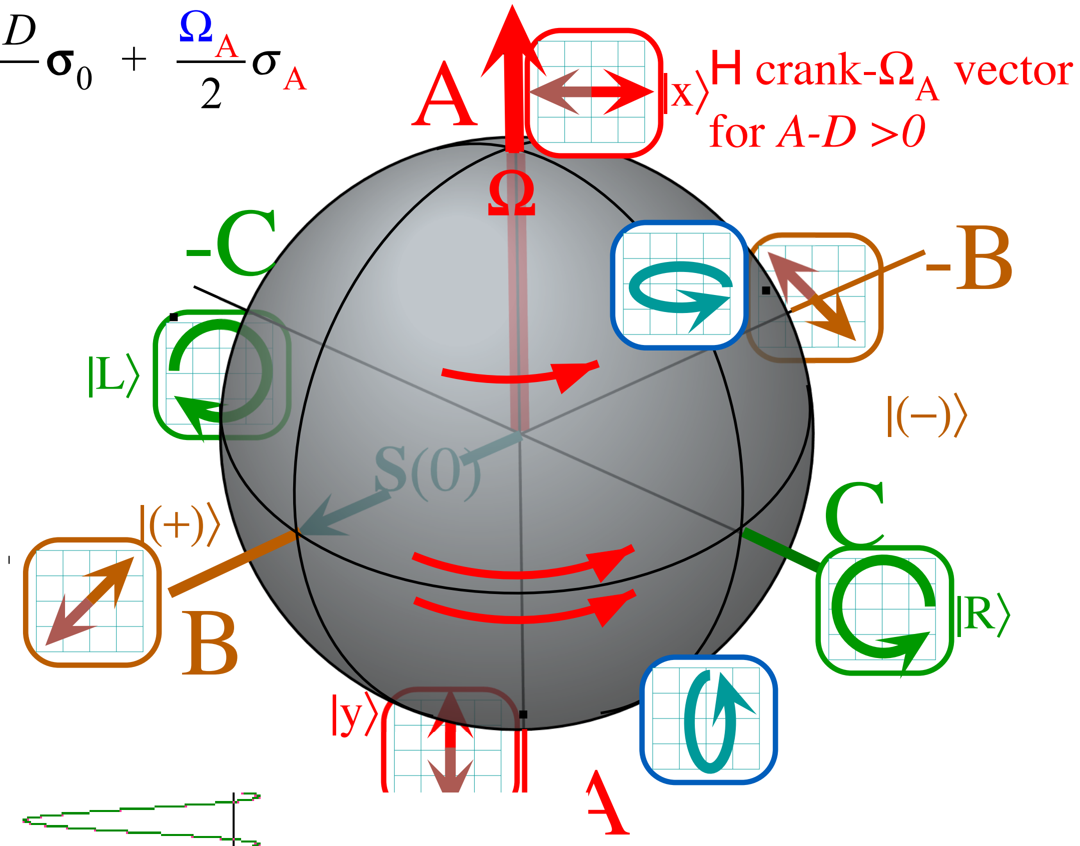
$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Asymmetric Diagonal A-Type motion

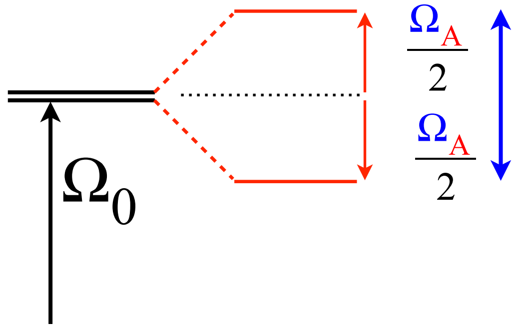
$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{A+D}{2}\sigma_0+\frac{\Omega_A}{2}\sigma_A$$

$$Crank:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix}\qquad Eigen-Spin:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$

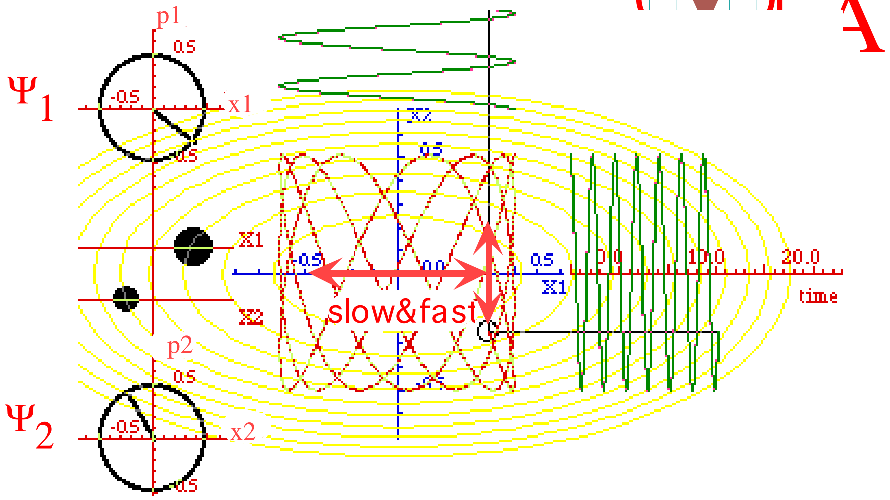




Beat dynamics:

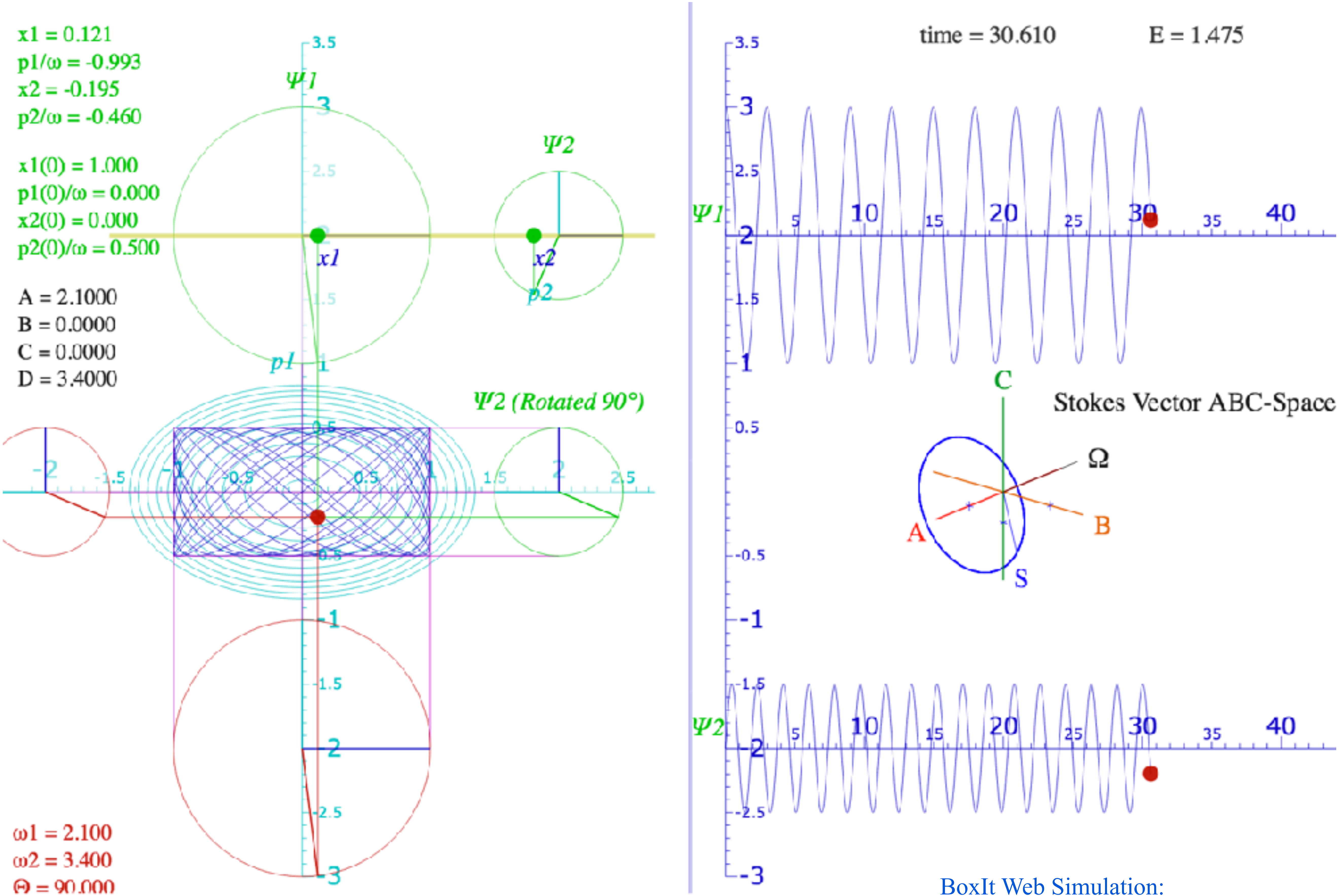


Total beat frequency Ω_A



BoxIt (A-Type) Web Simulation

A-Type elliptical polarized motion



x1 = 0.121
p1/ω = -0.993
x2 = -0.195
p2/ω = -0.460

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = 0.0000
C = 0.0000
D = 3.4000

ω1 = 2.100
ω2 = 3.400
Θ = 90.000

time = 30.610
E = 1.475

BoxIt Web Simulation:
A-Type with A=2.1, D=3.4

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\longrightarrow$

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_{\mathrm{B}} + C\,\boldsymbol{\sigma}_{\mathrm{C}} + \frac{A-D}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_{\mathrm{B}} + \frac{\Omega_{\mathrm{C}}}{2}\boldsymbol{\sigma}_{\mathrm{C}} + \frac{\Omega_{\mathrm{A}}}{2}\boldsymbol{\sigma}_{\mathrm{A}}$$

$$\vec{\boldsymbol{\Omega}}=\begin{pmatrix}\Omega_{\mathrm{A}}\\ \Omega_B\\ \Omega_{\mathrm{C}}\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\boldsymbol{\sigma}_{\mathrm{B}}$$

The ABC's of U(2) dynamics

Density operator **ρ** (see p.128-147) →

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

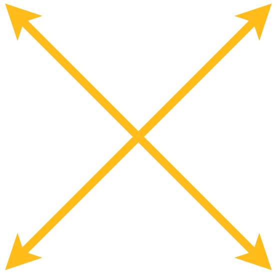
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

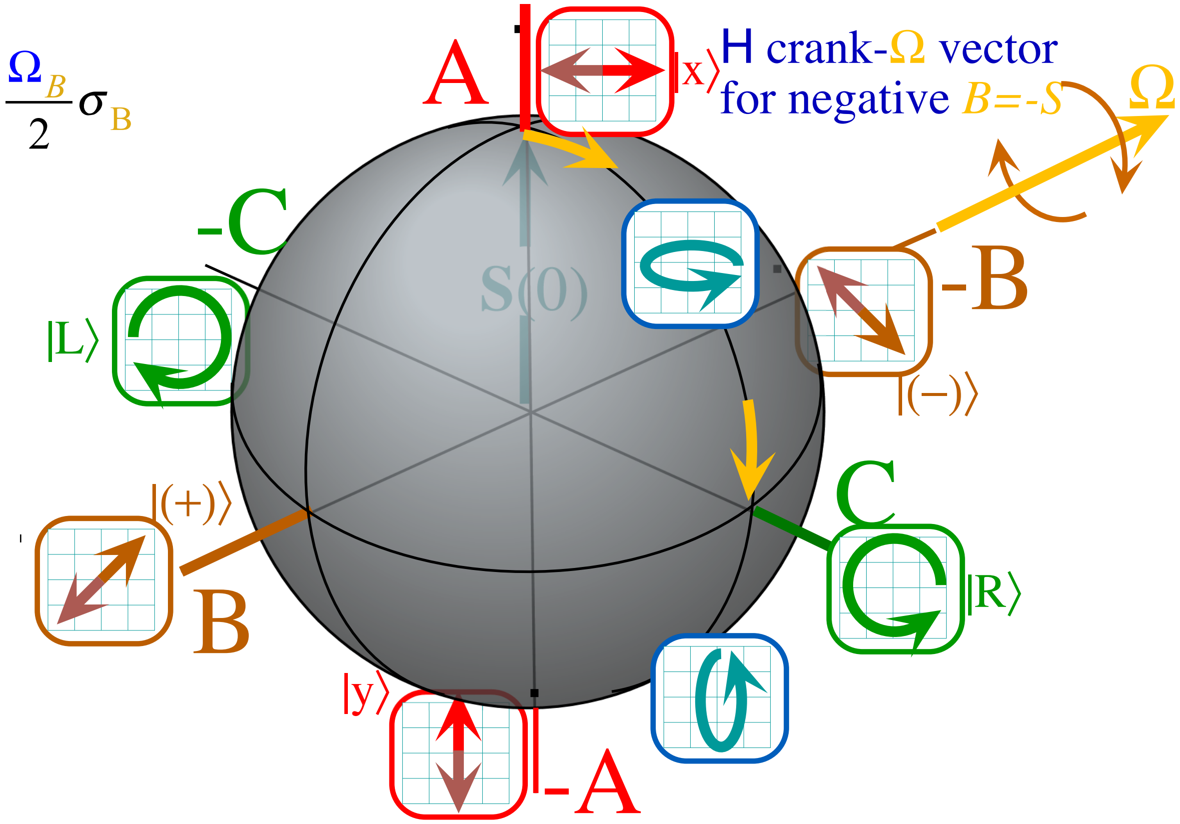
$$\vec{\boldsymbol{\Omega}}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\boldsymbol{\sigma}_B$$

$$Crank:\ \vec{\boldsymbol{\Omega}}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}0\\ 2B\\ 0\end{pmatrix}\qquad Eigen-Spin:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$





The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

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In general:

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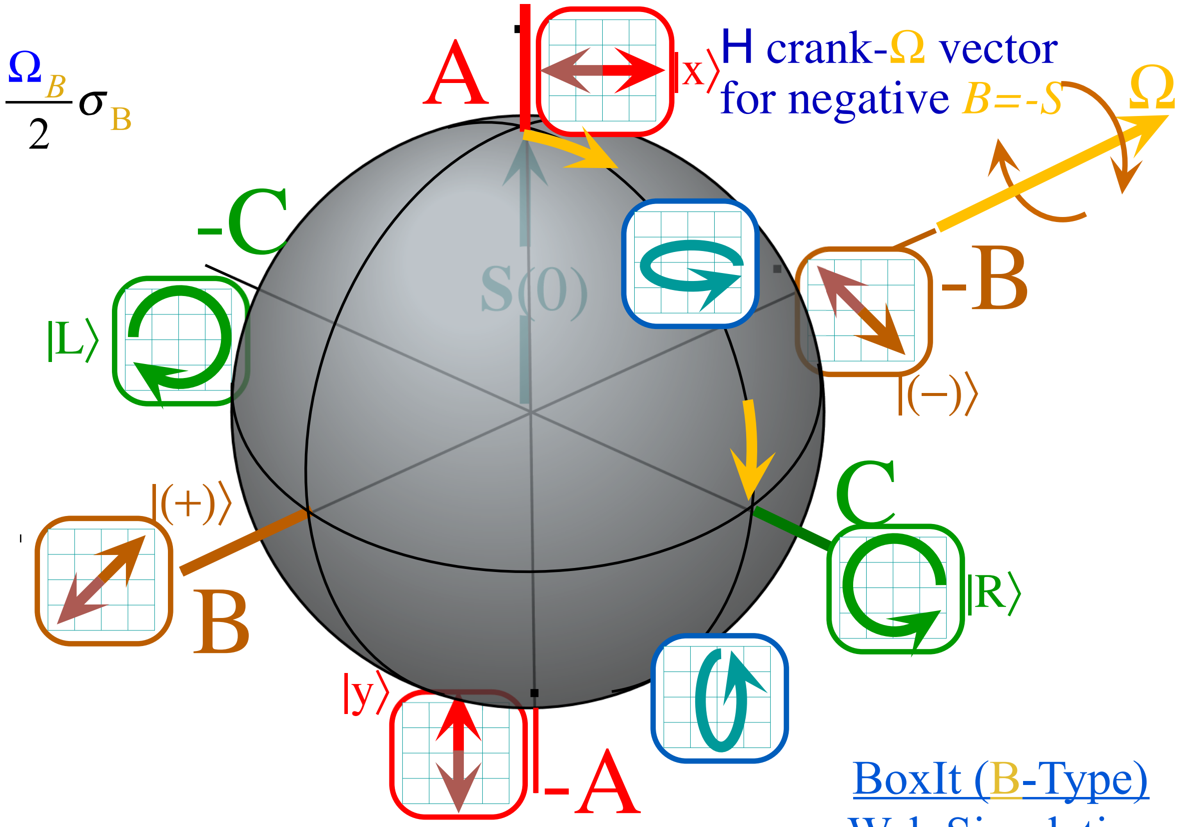
$$\mathbf{H} = \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Bilateral-Balanced B-Type motion

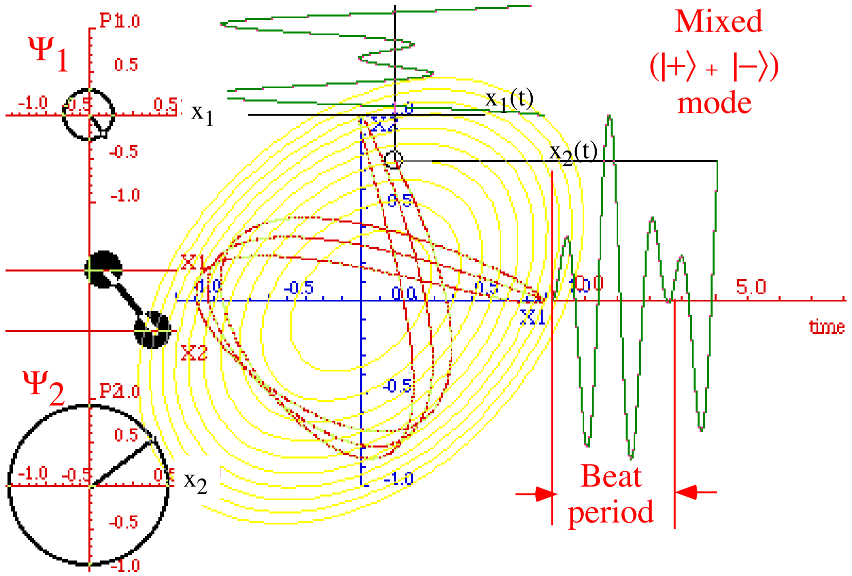
$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}=\ \Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\sigma_B$$

$$Crank:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}0\\ 2B\\ 0\end{pmatrix}\qquad Eigen-Spin:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$



BoxIt (B-Type)
Web Simulation

Beat dynamics:



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$

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$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

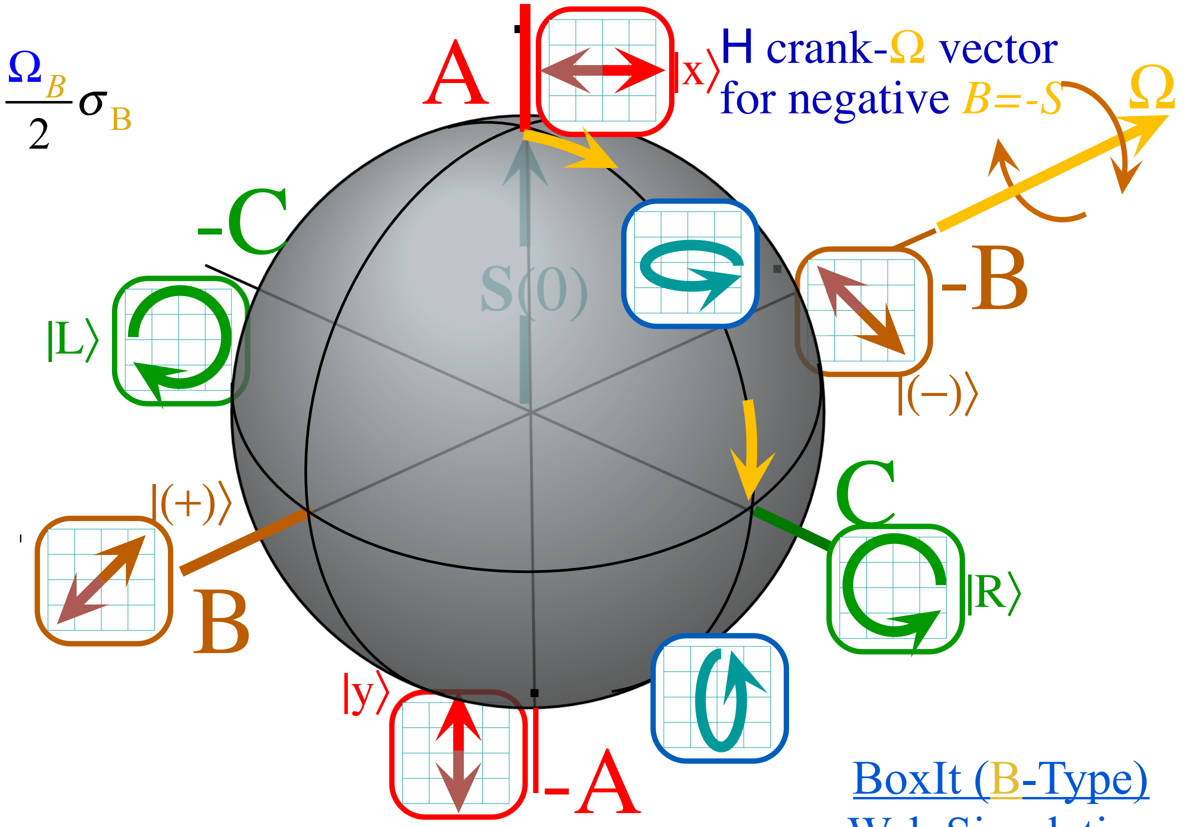
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

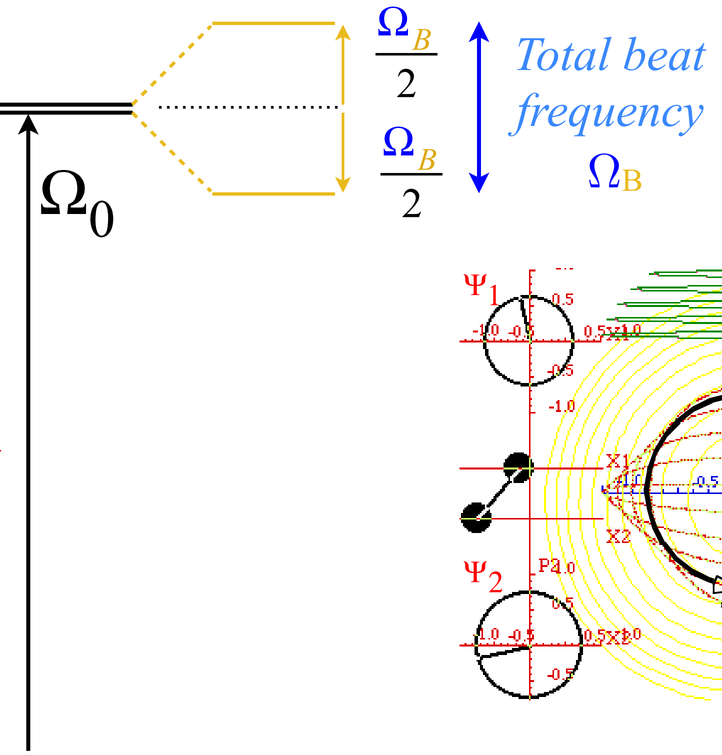
Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_B}{2}\sigma_B$$

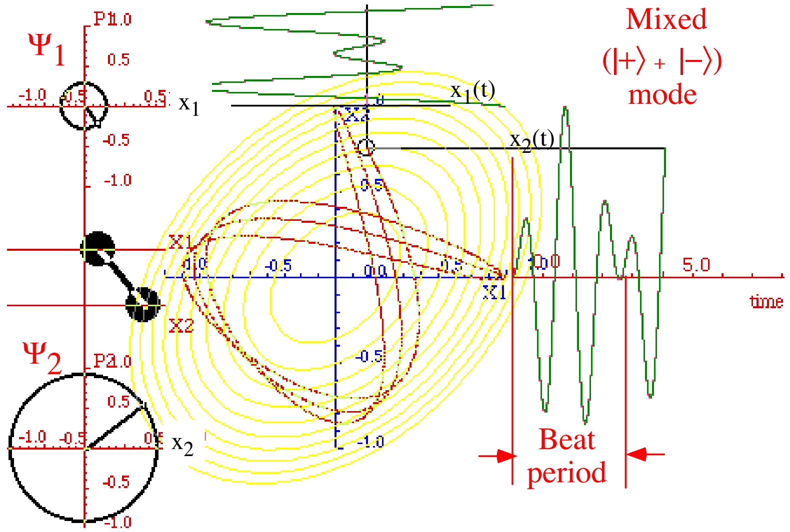
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$

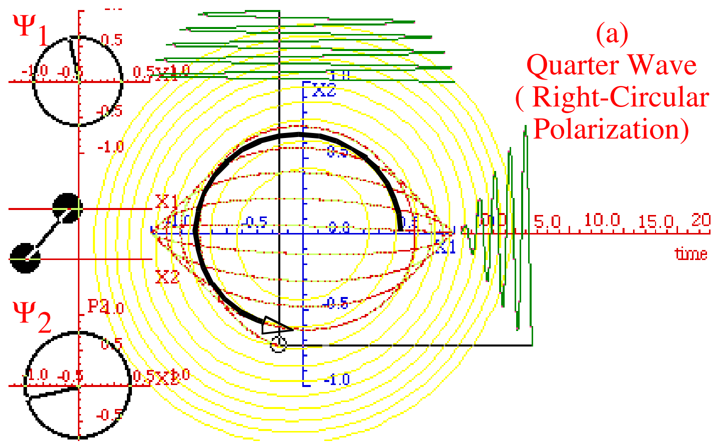


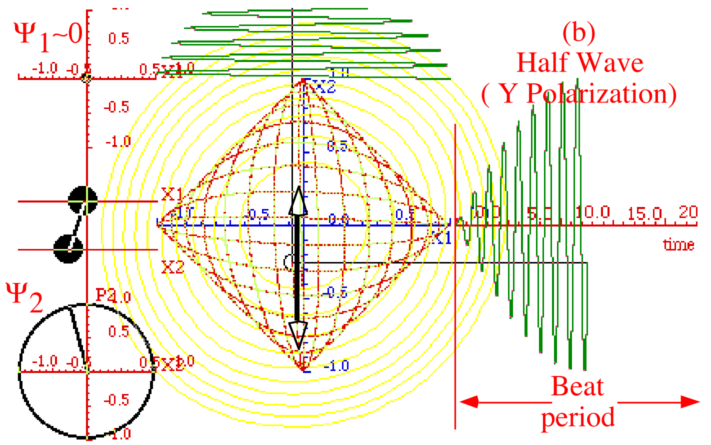
Beat dynamics:



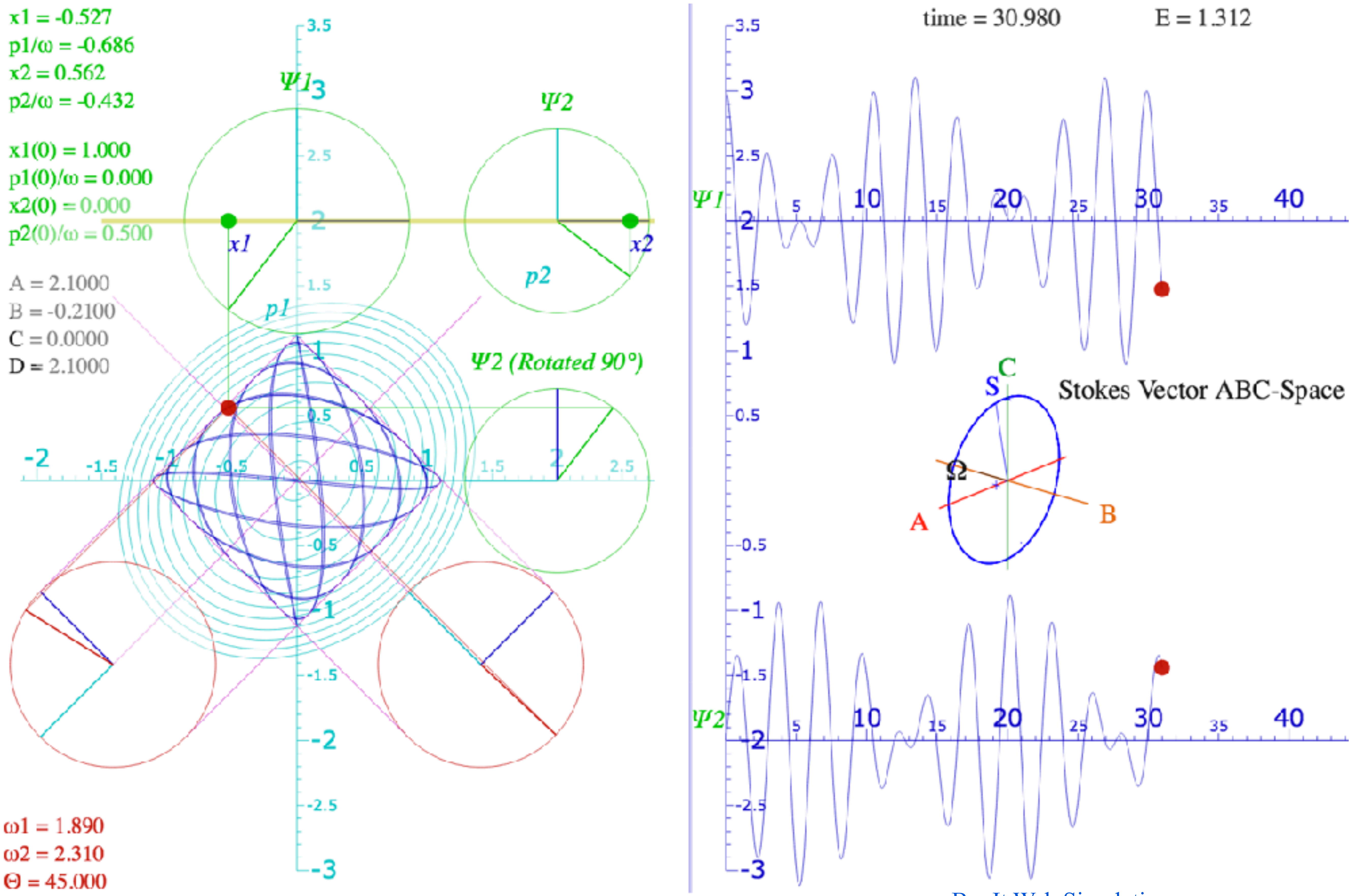
BoxIt (B-Type) Web Simulation







B-Type elliptical polarized motion



x1 = -0.527
p1/ω = -0.686
x2 = 0.562
p2/ω = -0.432

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

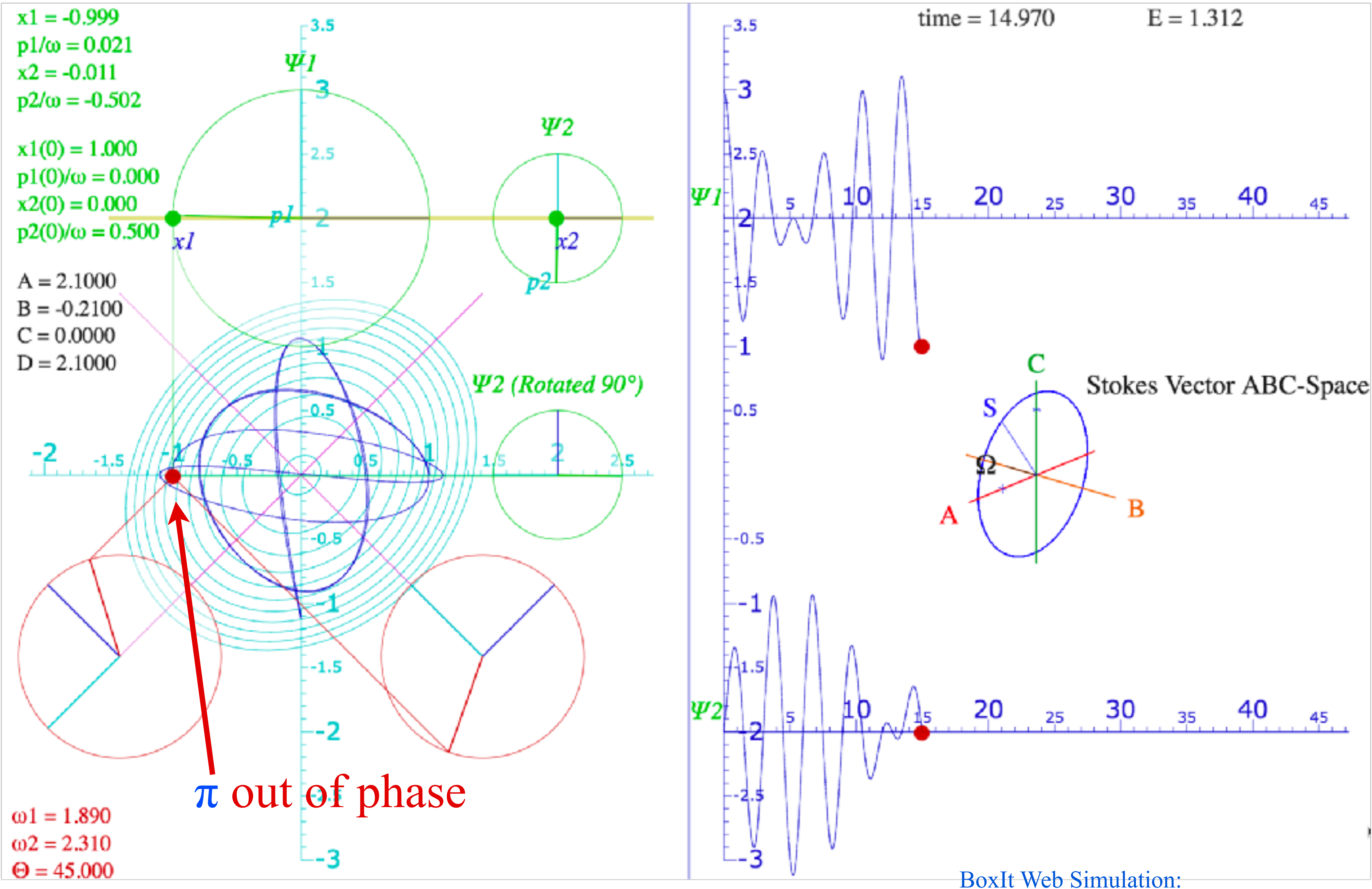
A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000

ω1 = 1.890
ω2 = 2.310
Θ = 45.000

time = 30.980
E = 1.312

BoxIt Web Simulation:
B-Type with A, D=2.1; B=-0.21

B-Type elliptical polarized motion Note that one 360°=2π rotation of **S** leaves (x_1,x_2) at $-(x_1,x_2)$



x1 = -0.999
p1/ω = 0.021
x2 = -0.011
p2/ω = -0.502

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000

ω1 = 1.890
ω2 = 2.310
Θ = 45.000

time = 14.970
E = 1.312

π out of phase

BoxIt Web Simulation:
B-Type with A, D=2.1; B=-0.21

To assess the rationality of any number we approximate it using successive levels of *continued fractions*.

$$\alpha = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \ddots}}}}$$

Example 1: the number π = 3.1415926..., and recipe for getting n_k

$$A_0 = \alpha = 3.14159265...$$

$$A_1 = \frac{1}{A_0 - n_0} = 7.06...$$

$$A_2 = \frac{1}{A_1 - n_1} = 15.99...$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.003...$$

$$n_0 = INT(A_0) = 3$$

$$n_1 = INT(A_1) = 7$$

$$n_2 = INT(A_2) = 15$$

$$n_3 = INT(A_3) = 1$$

$$\pi \cong \quad = 3.000..$$

$$\pi \cong 3 + \frac{1}{7} = \frac{22}{7} = 3.1428$$

$$\pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.141509$$

$$\pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15 + 1}} = \frac{355}{113} = 3.14159292$$

Example 2: the *Golden Mean G*=(1+√5)/2=1.618033989…

$$A_0 = G = 1.618033989...$$

$$A_1 = \frac{1}{A_0 - n_0} = 1.6180...$$

$$A_2 = \frac{1}{A_1 - n_1} = 1.6180...$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.6180...$$

$$n_0 = INT(A_0) = 1$$

$$n_1 = INT(A_1) = 1$$

$$n_2 = INT(A_2) = 1$$

$$n_3 = INT(A_3) = 1$$

$$G \cong \quad = 1.000..$$

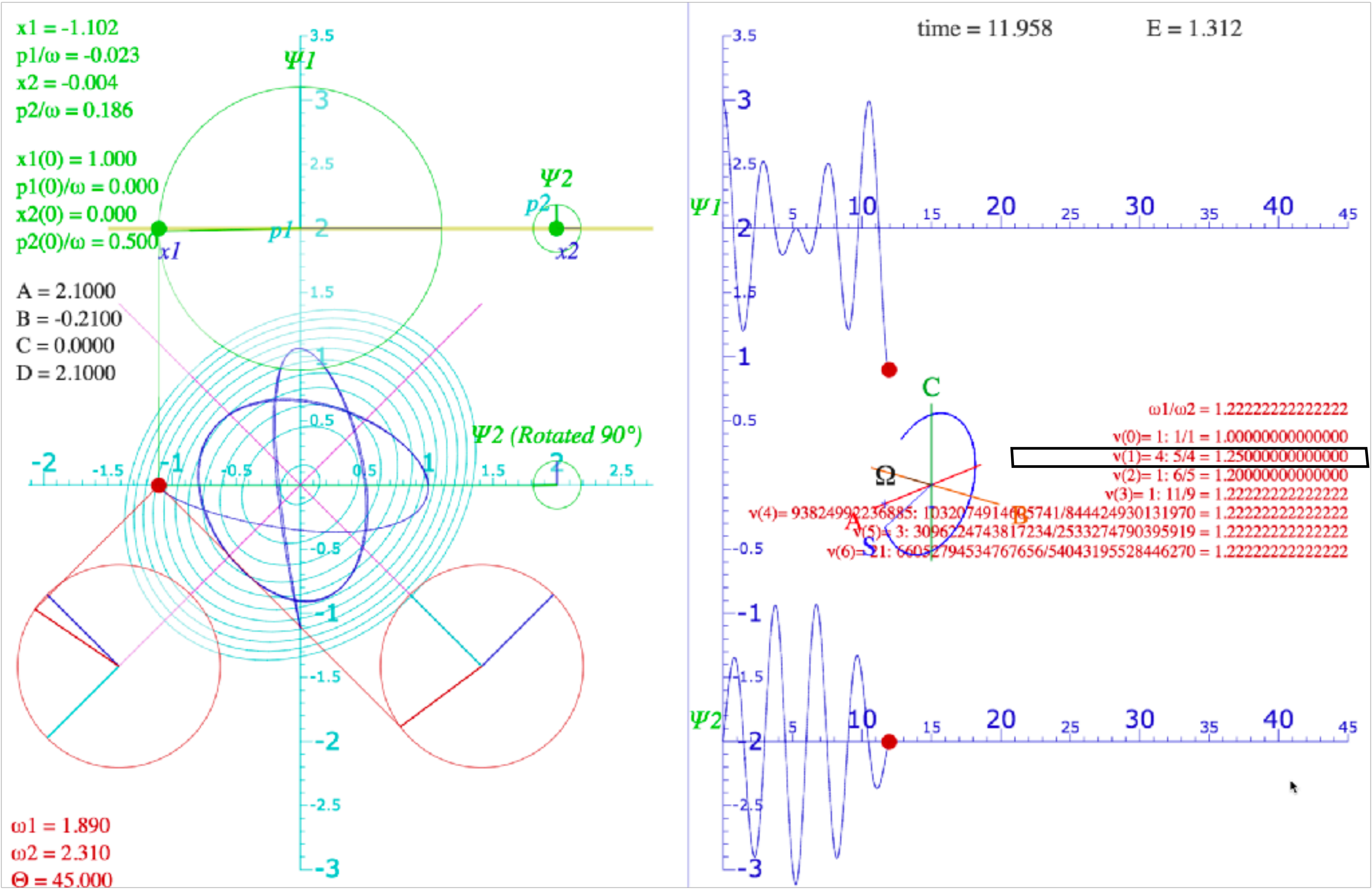
$$G \cong 1 + \frac{1}{1} = \frac{2}{1} = 2.000$$

$$G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = 1.500$$

$$G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}} = \frac{5}{3} = 1.666...$$

Note:
Fibonacci
numbers:
1,2,3,5,8,…

The most irrational number is closest to being rational!



x1 = -1.102
p1/ω = -0.023
x2 = -0.004
p2/ω = 0.186

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000

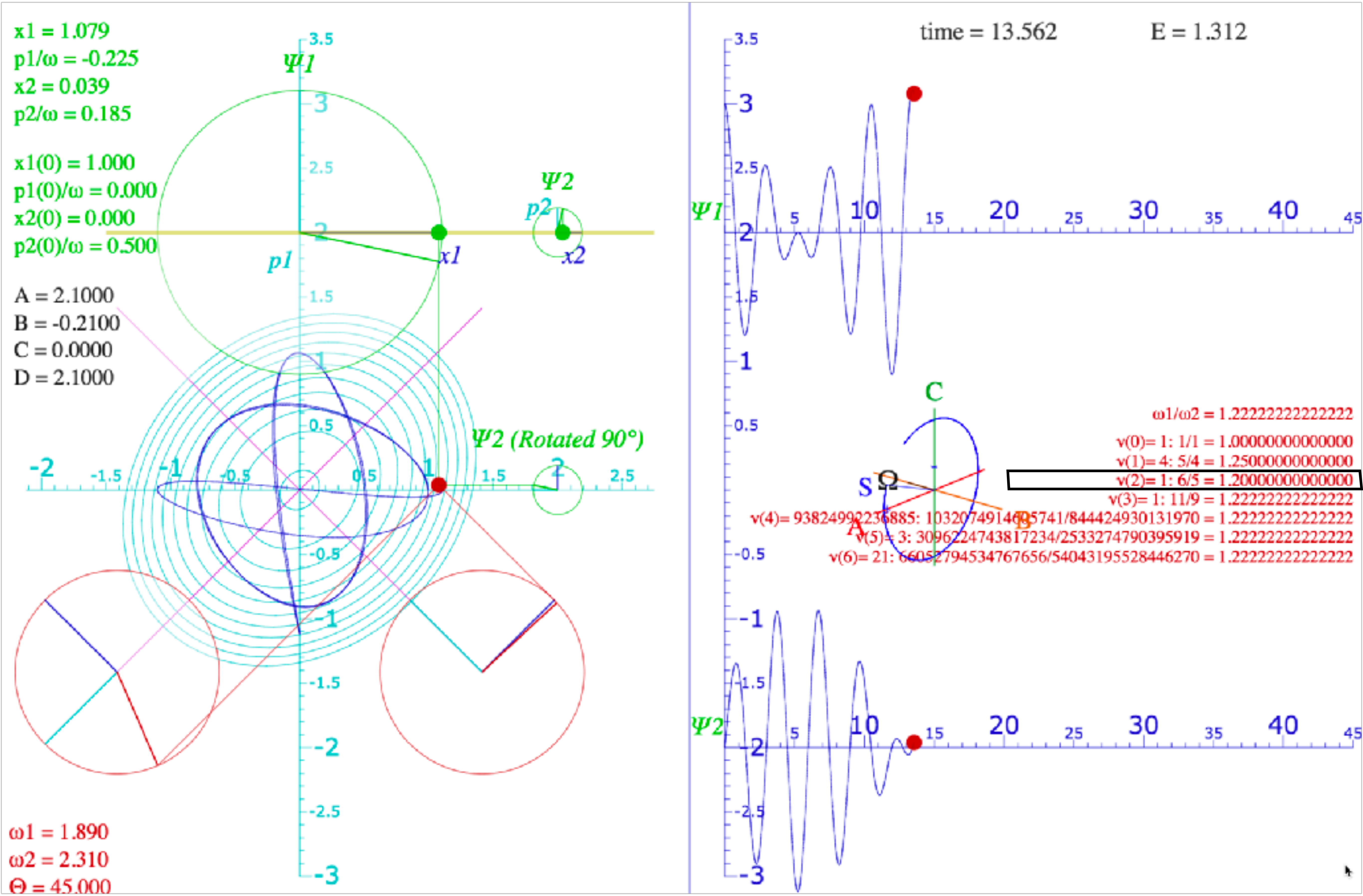
ω1 = 1.890
ω2 = 2.310
Θ = 45.000

time = 11.958
E = 1.312

```
ω1/ω2 = 1.22222222222222

v(0)= 1: 1/1 = 1.00000000000000
v(1)= 4: 5/4 = 1.25000000000000
v(2)= 1: 6/5 = 1.20000000000000
v(3)= 1: 11/9 = 1.22222222222222
v(4)= 93824992236885: 1032074914605741/844424930131970 = 1.22222222222222
v(5)= 3: 3096224743817234/2533274790395919 = 1.22222222222222
v(6)= 21: 66052794534767656/54043195528446270 = 1.22222222222222
```

BoxIt Web Simulation: B-Type with
A, D=2.1; B=-0.21



x1 = 1.079
p1/ω = -0.225
x2 = 0.039
p2/ω = 0.185

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000

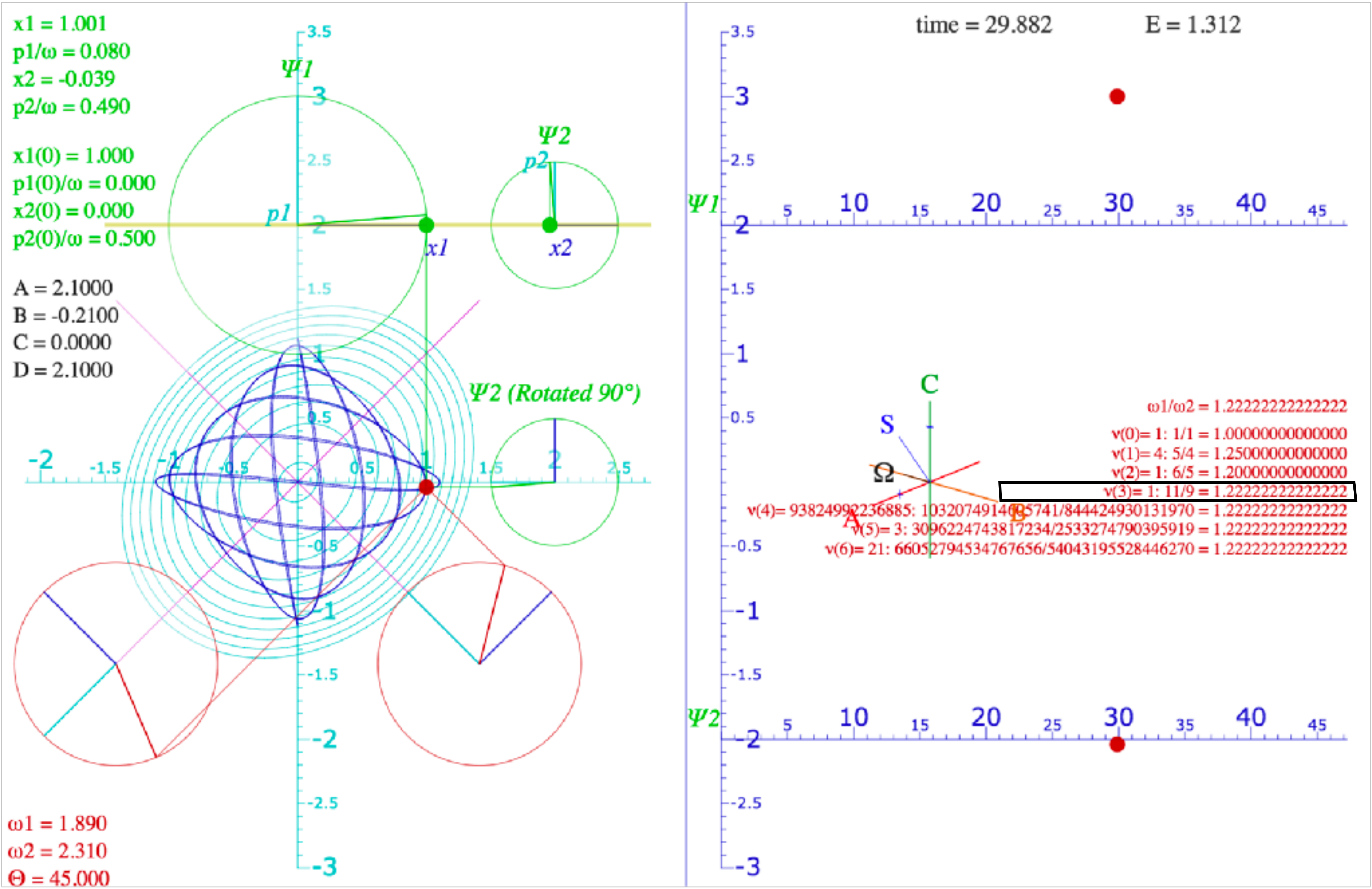
ω1 = 1.890
ω2 = 2.310
Θ = 45.000

time = 13.562
E = 1.312

ω1/ω2 = 1.22222222222222

ν(0)= 1: 1/1 = 1.00000000000000
ν(1)= 4: 5/4 = 1.25000000000000
ν(2)= 1: 6/5 = 1.20000000000000
ν(3)= 1: 11/9 = 1.22222222222222
ν(4)= 938249922236885: 1032074914605741/844424930131970 = 1.22222222222222
ν(5)= 3: 3096224743817234/2533274790395919 = 1.22222222222222
ν(6)= 21: 66052794534767656/54043195528446270 = 1.22222222222222

BoxIt Web Simulation: B-Type with A, D=2.1; B=-0.21



x1 = 1.001
p1/ω = 0.080
x2 = -0.039
p2/ω = 0.490

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000

ω1 = 1.890
ω2 = 2.310
Θ = 45.000

time = 29.882
E = 1.312

ω1/ω2 = 1.2222222222222

ν(0)= 1: 1/1 = 1.0000000000000
ν(1)= 4: 5/4 = 1.2500000000000
ν(2)= 1: 6/5 = 1.2000000000000
ν(3)= 1: 11/9 = 1.2222222222222
ν(4)= 93824992236885: 1032074914605741/844424930131970 = 1.2222222222222
ν(5)= 3: 3096224743817234/2533274790395919 = 1.2222222222222
ν(6)= 21: 6605279453476765 6/5404319552844627 0 = 1.2222222222222

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S**-*vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$-*vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$-*sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle $(\alpha\beta\gamma)$ ellipse coordinates*

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) →

$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

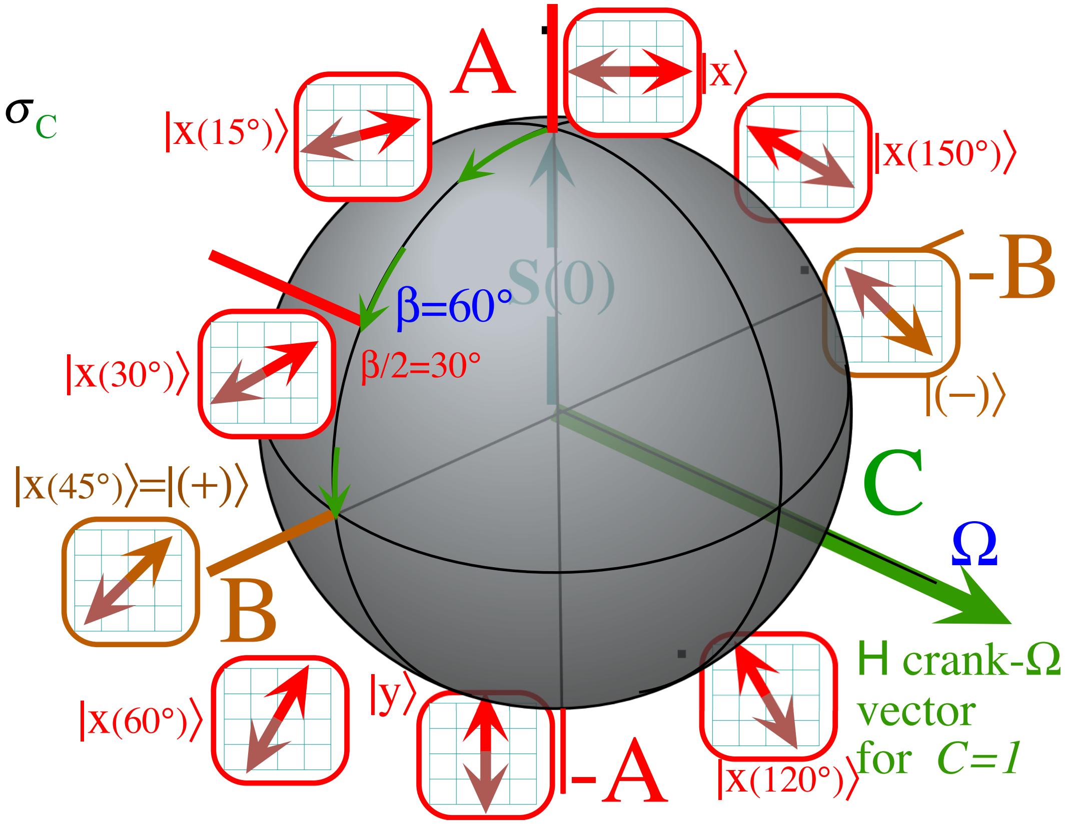
$$\mathbf{H} = \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Circular-Coriolis... C-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle\\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 2C\end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \pm S\end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator **ρ** *(see p.128-147)* →

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

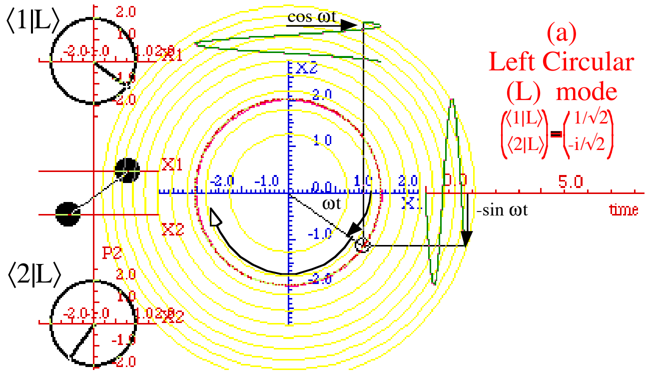
$$\mathbf{H} = \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Circular-Coriolis... C-Type motion

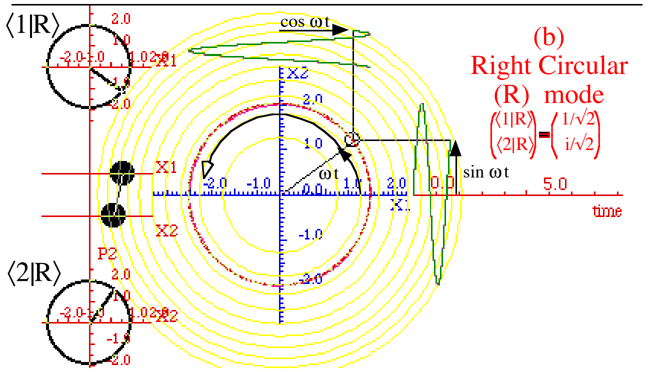
$$\begin{pmatrix}\langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle\\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

$$\textit{Crank}:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 2C\end{pmatrix} \qquad \textit{Eigen-Spin}:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \pm S\end{pmatrix}$$



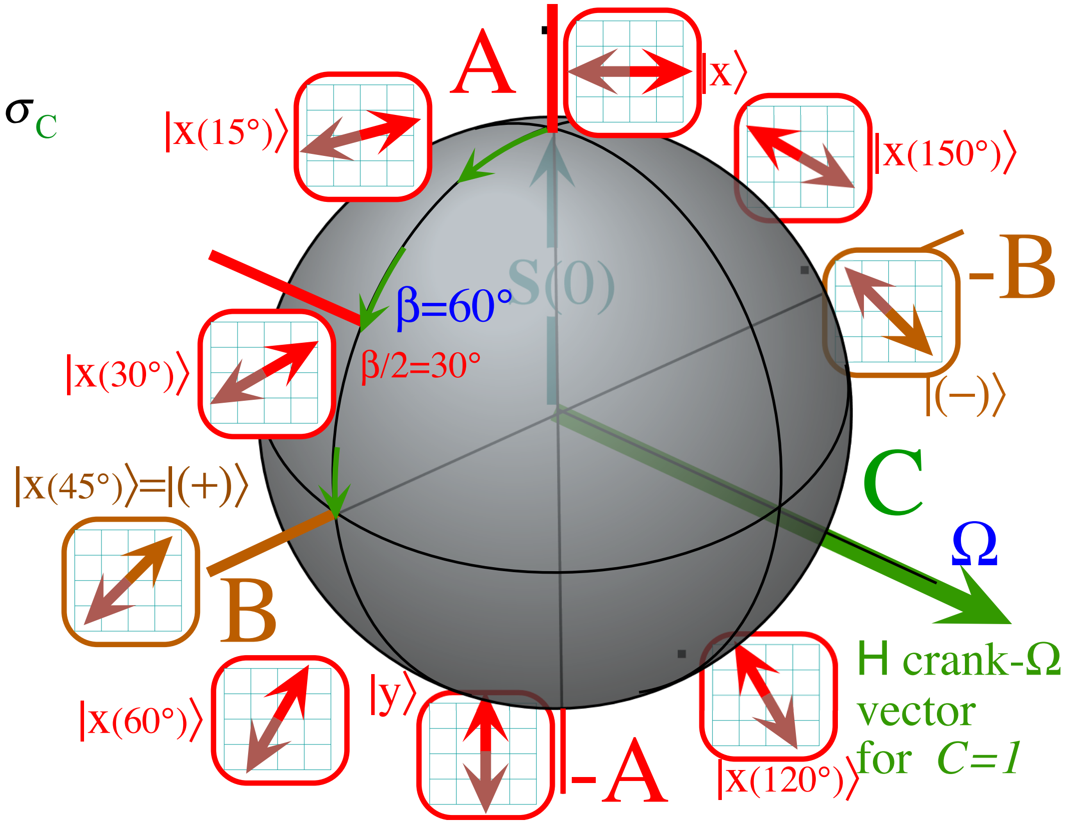
(a)
Left Circular
(L) mode

$$\begin{pmatrix}\langle 1|L\rangle\\ \langle 2|L\rangle\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$$



(b)
Right Circular
(R) mode

$$\begin{pmatrix}\langle 1|R\rangle\\ \langle 2|R\rangle\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}$$



H crank-Ω vector for *C=1*

The ABC's of U(2) dynamics

Density operator **ρ** (see p.128-147) →

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle1|\mathbf{H}|1\rangle & \langle1|\mathbf{H}|2\rangle\\ \langle2|\mathbf{H}|1\rangle & \langle2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

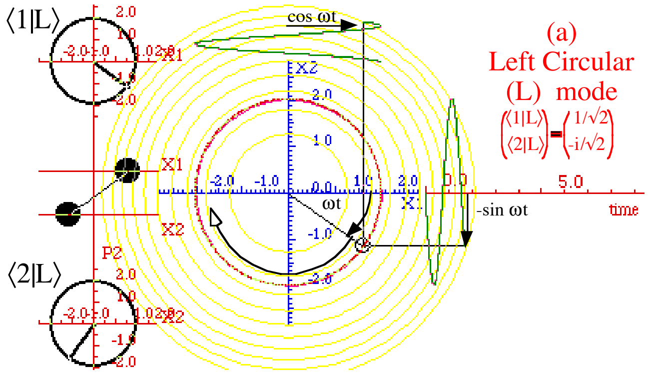
$$\mathbf{H} = \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Circular-Coriolis... C-Type motion

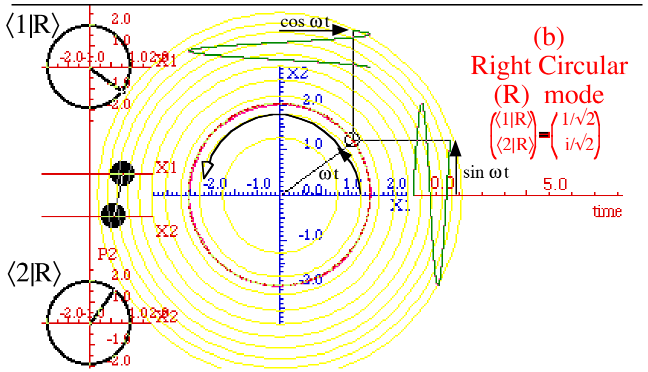
$$\begin{pmatrix}\langle1|\mathbf{H}^C|1\rangle & \langle1|\mathbf{H}^C|2\rangle\\ \langle2|\mathbf{H}^C|1\rangle & \langle2|\mathbf{H}^C|2\rangle\end{pmatrix} = \begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix} = \Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_C}{2}\sigma_C$$

$$\text{Crank}: \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 2C\end{pmatrix} \qquad \text{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \pm S\end{pmatrix}$$



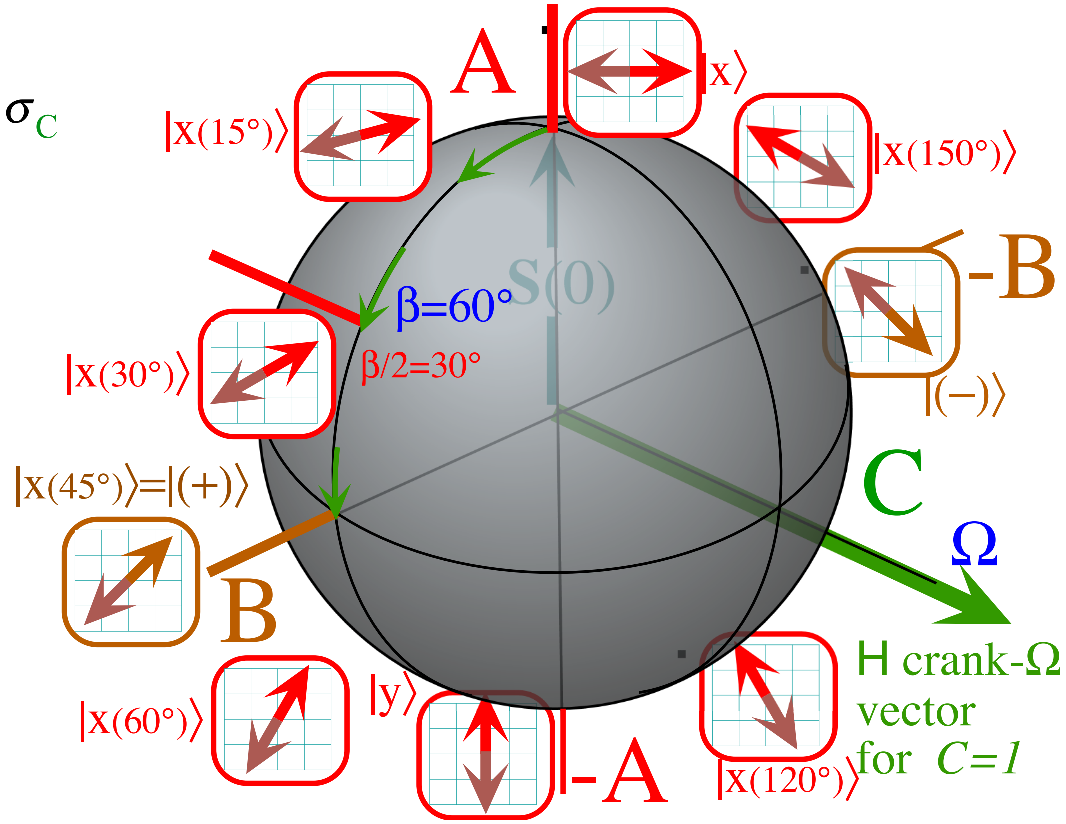
(a) Left Circular (L) mode

$\begin{pmatrix}\langle1|L\rangle\\ \langle2|L\rangle\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$



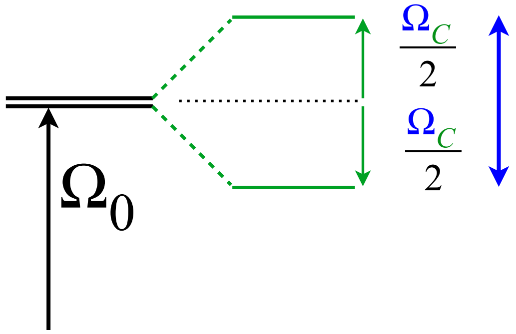
(b) Right Circular (R) mode

$\begin{pmatrix}\langle1|R\rangle\\ \langle2|R\rangle\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}$

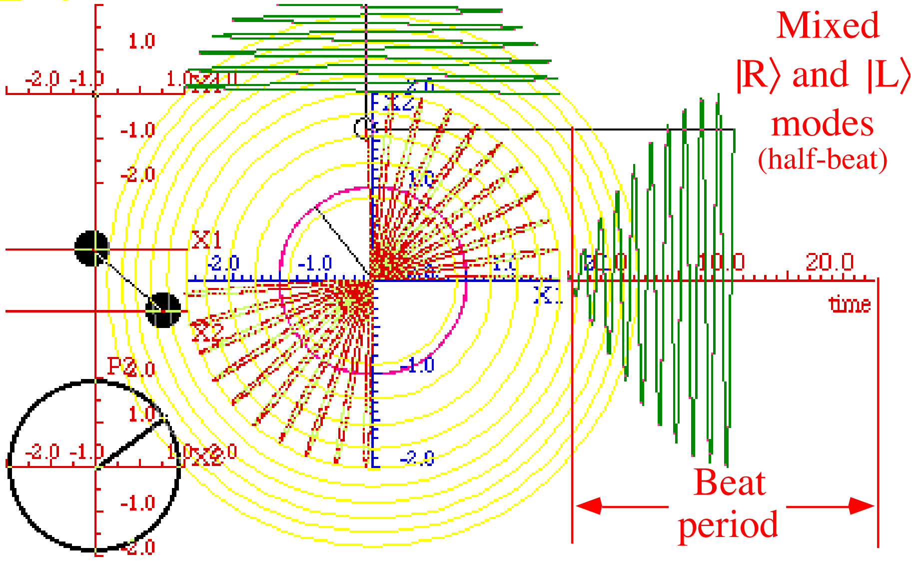


H crank-Ω vector for C=1

Beat dynamics:



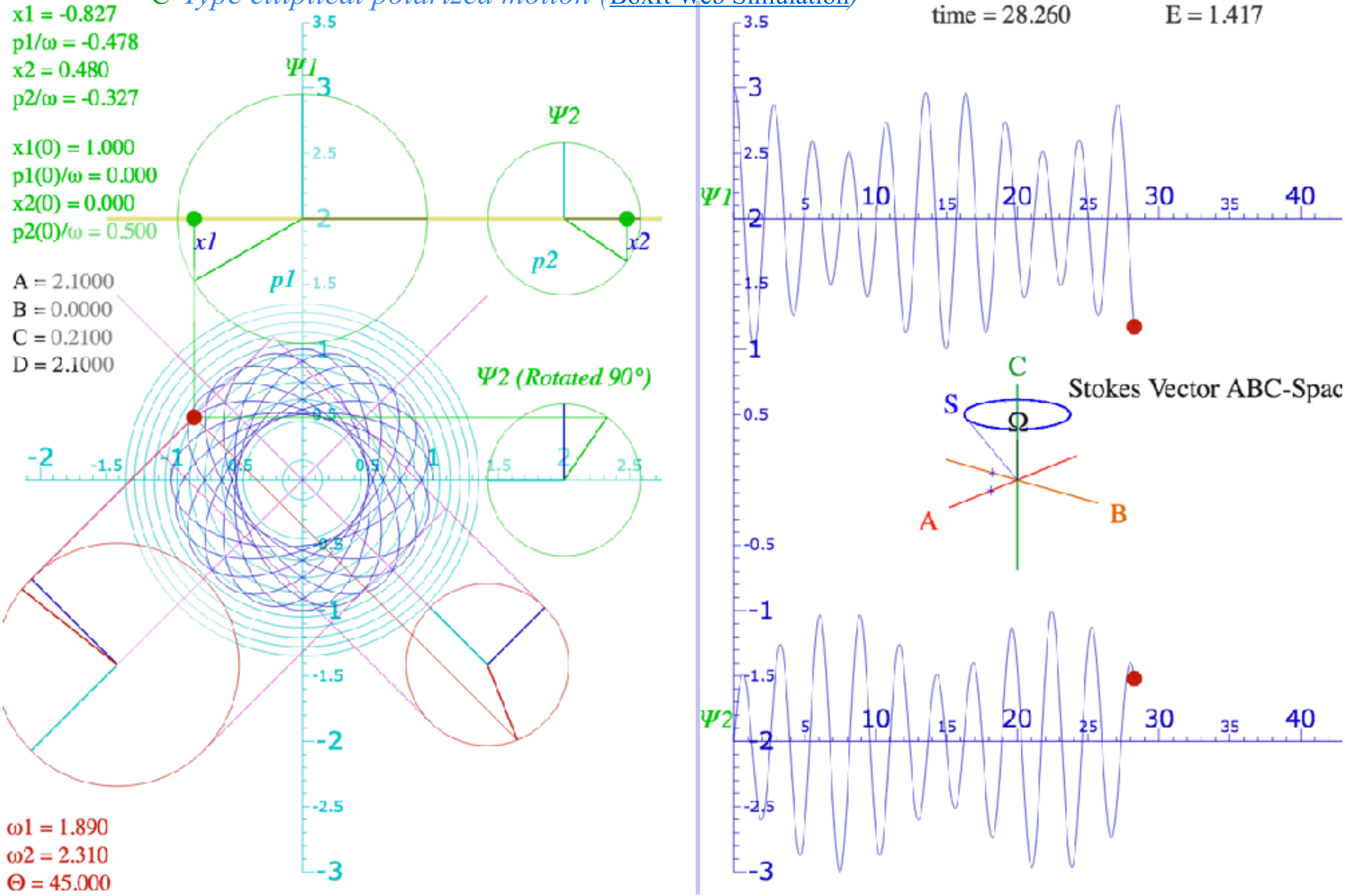
Total beat frequency Ω_C



Mixed |R⟩ and |L⟩ modes (half-beat)

BoxIt (C-Type) Web Simulation

C-Type elliptical polarized motion (BoxIt Web Simulation)



x1 = -0.827
p1/ω = -0.478
x2 = 0.480
p2/ω = -0.327

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = 0.0000
C = 0.2100
D = 2.1000

ω1 = 1.890
ω2 = 2.310
Θ = 45.000

time = 28.260
E = 1.417

BoxIt Web Simulation:
C-Type with A, D=2.1; C=-0.21

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes
Asymmetric-Diagonal A-Type motion
Bilateral-Balanced B-Type motion
Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes
AB-Type motion and Wigner's Avoided-Symmetry-Crossings
ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates
Conventional amp-phase ellipse coordinates
Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics-Mixed modes

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\sigma_B + C\sigma_C + \frac{A-D}{2}\sigma_A$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

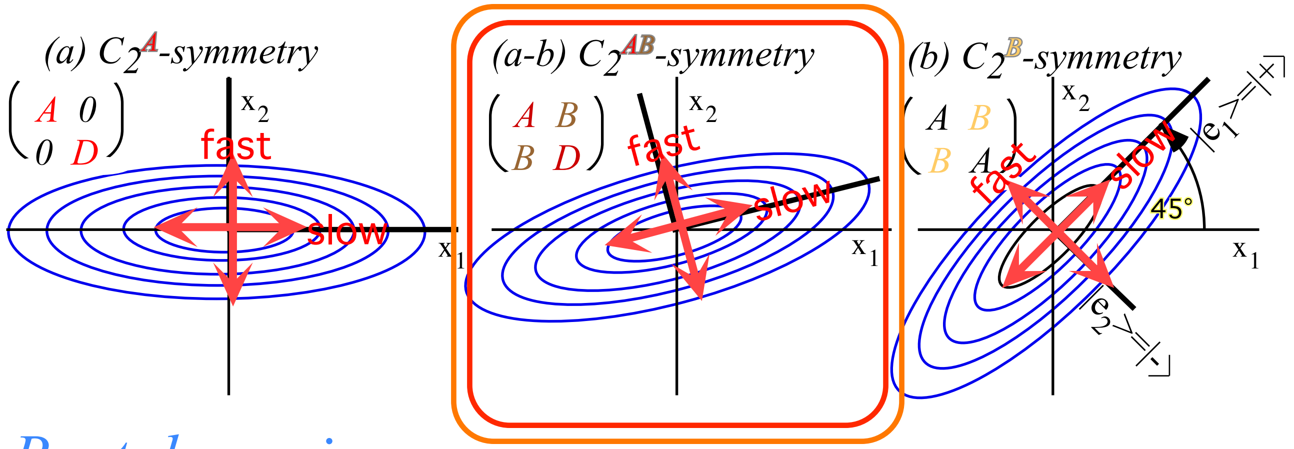
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Tilted-plane polarization AB-Type motion

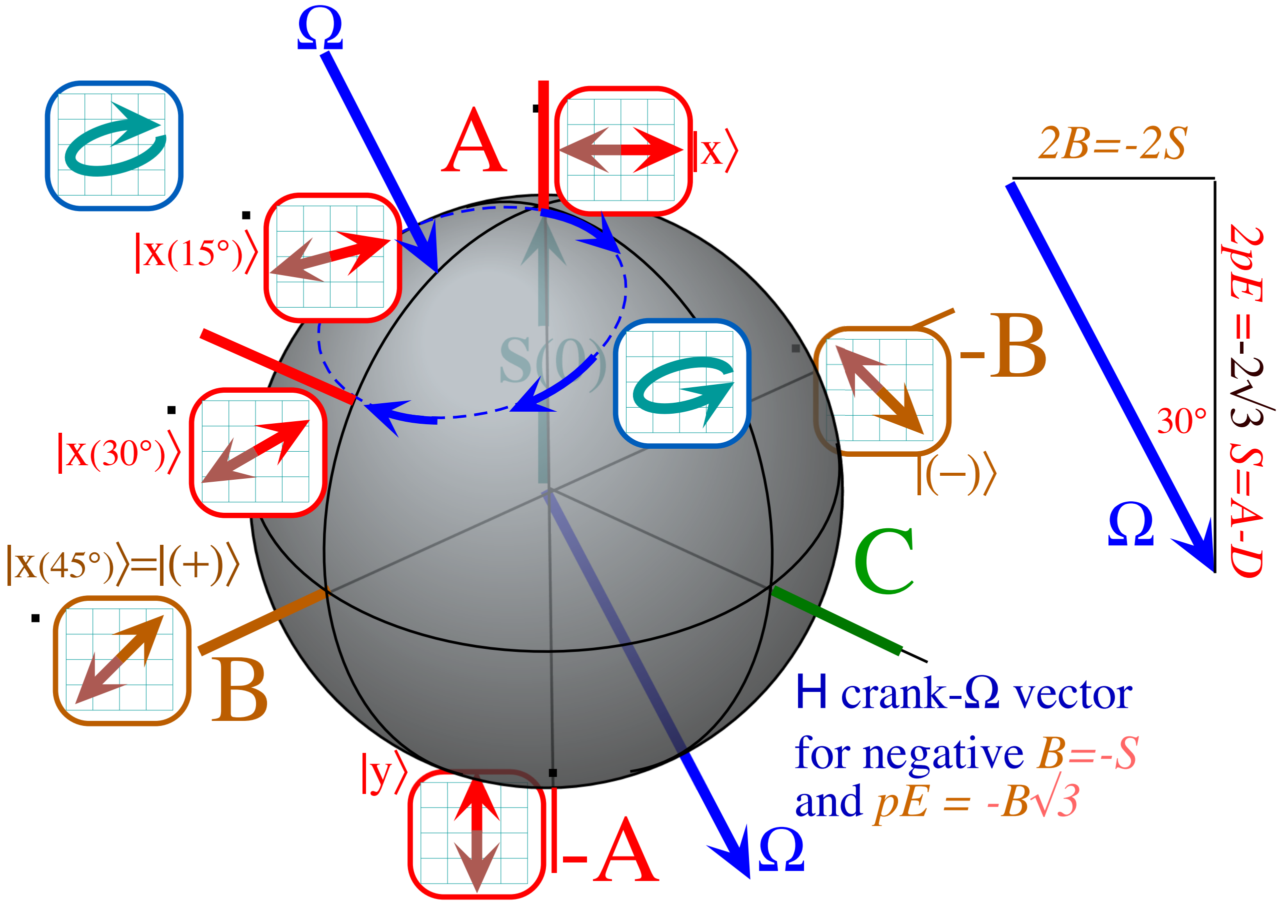
$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix}$$

$$Eigen-Spin: \vec{\mathbf{S}} = \pm S\sigma\boldsymbol{\Omega}$$



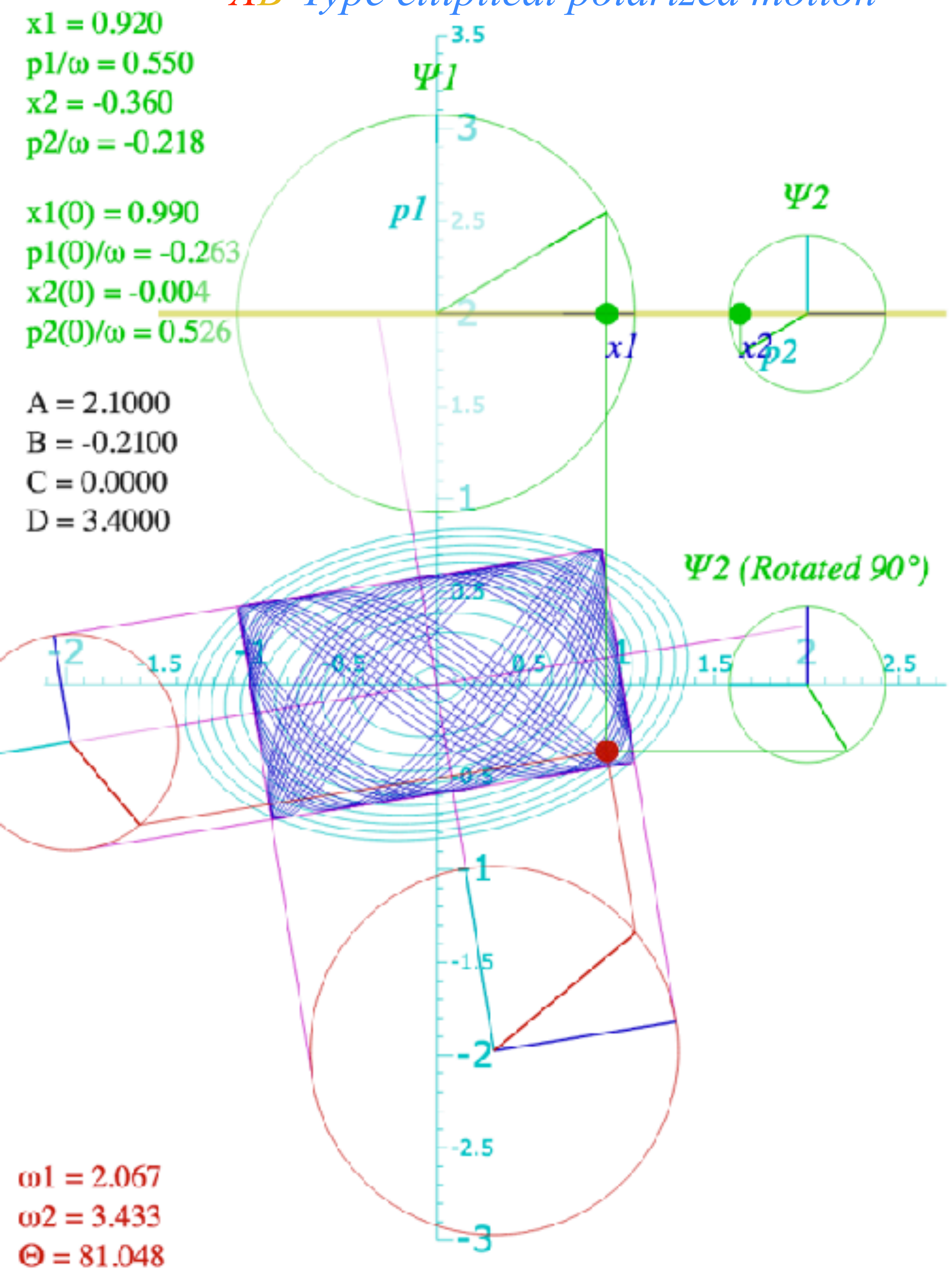
Beat dynamics:



H crank-Ω vector
for negative B=-S
and pE = -$B\sqrt{3}$

BoxIt (AB-Type Motion)
Web Simulation

AB-Type elliptical polarized motion

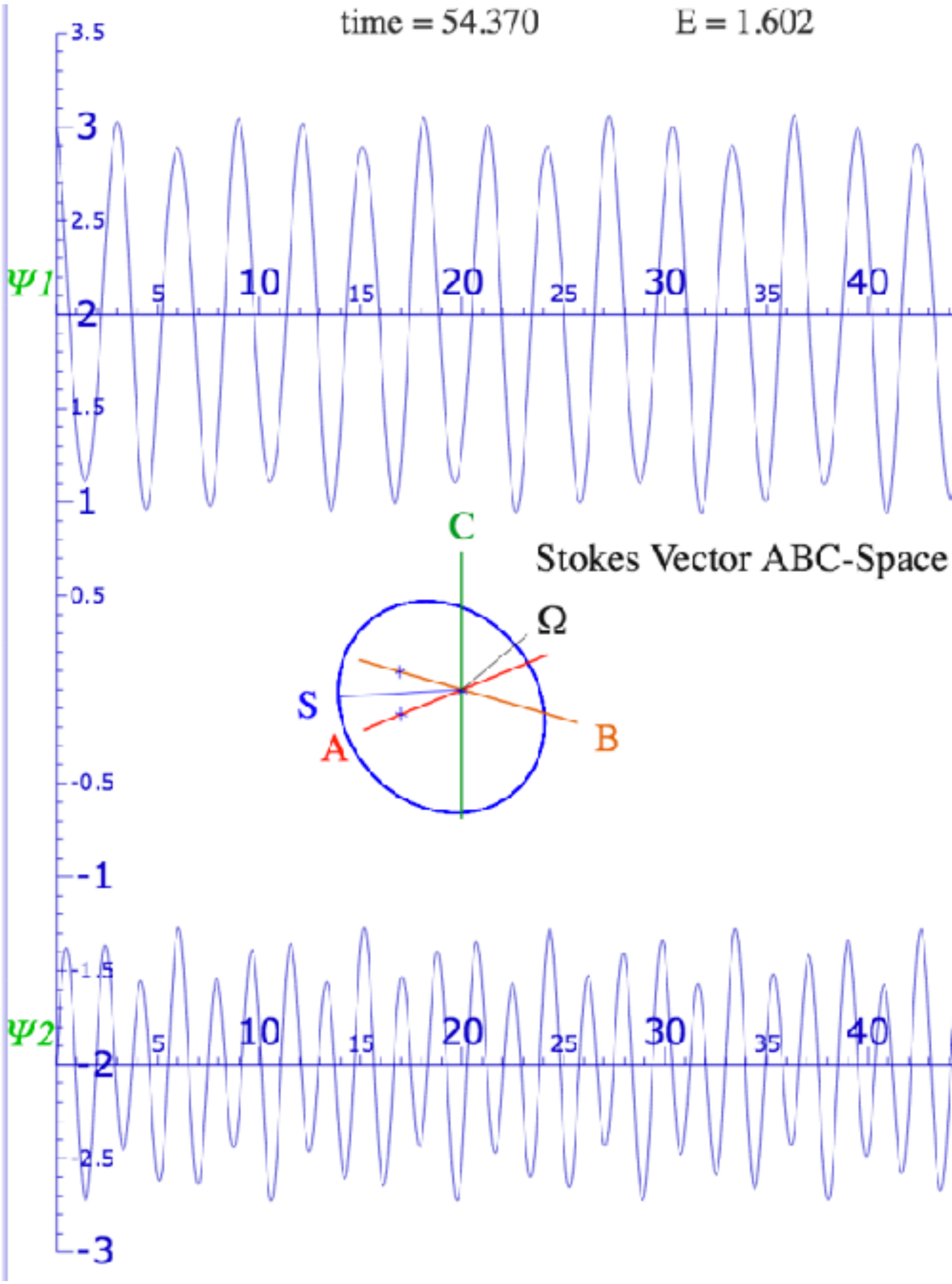


x1 = 0.920
p1/ω = 0.550
x2 = -0.360
p2/ω = -0.218

x1(0) = 0.990
p1(0)/ω = -0.263
x2(0) = -0.004
p2(0)/ω = 0.526

A = 2.1000
B = -0.2100
C = 0.0000
D = 3.4000

ω1 = 2.067
ω2 = 3.433
Θ = 81.048



time = 54.370
E = 1.602

BoxIt Web Simulation:
AB-Type with A=2.1; B=-0.21; D=3.4

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A+B\boldsymbol{\sigma}_B=\mathbf{H}=\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}$

$\mathbf{H}=\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}$ Secular equation: $\varepsilon^2-0\cdot\varepsilon-(A^2+B^2)$ gives *hyperbolic* energy levels: $\varepsilon=\pm\sqrt{A^2+B^2}$

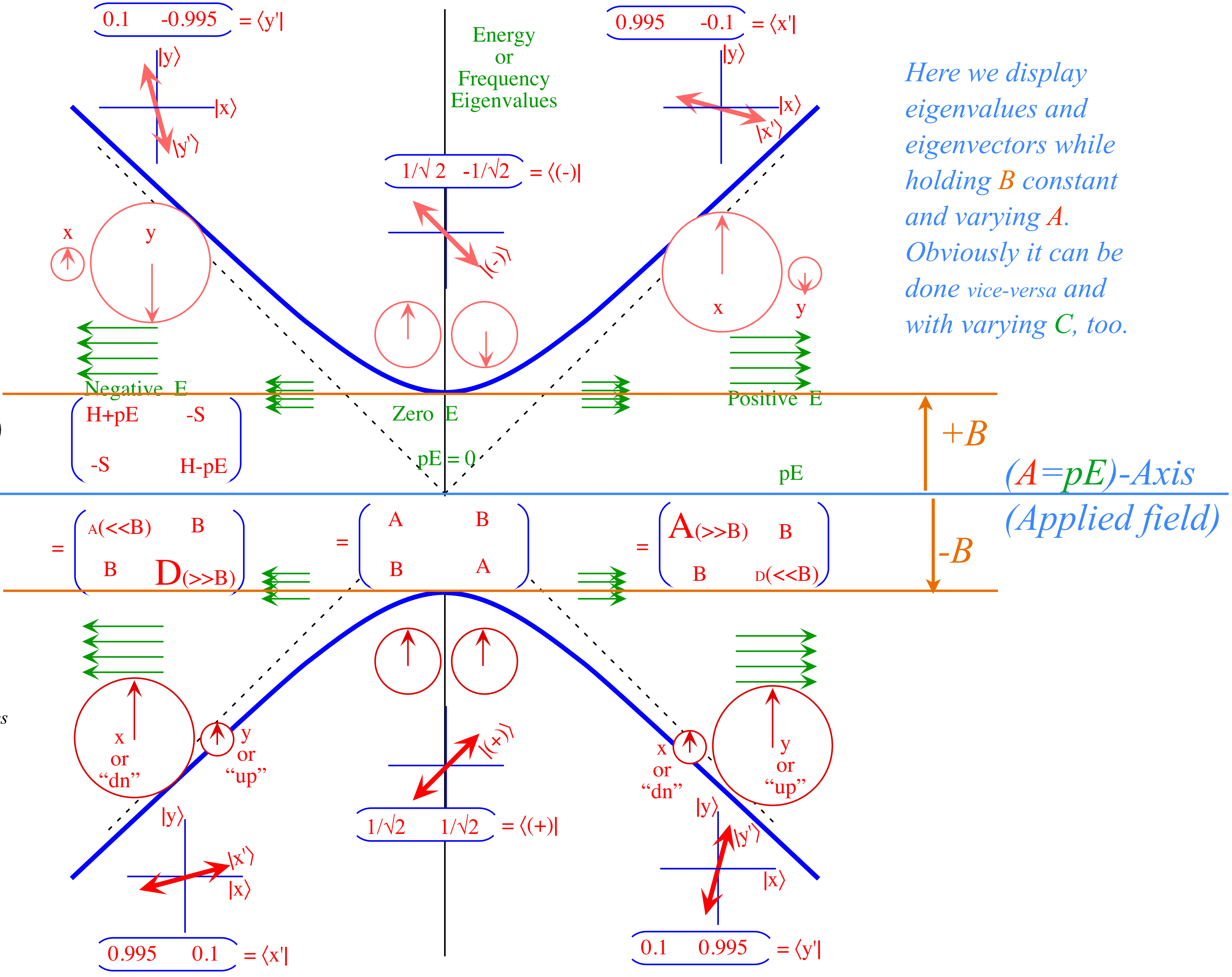


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

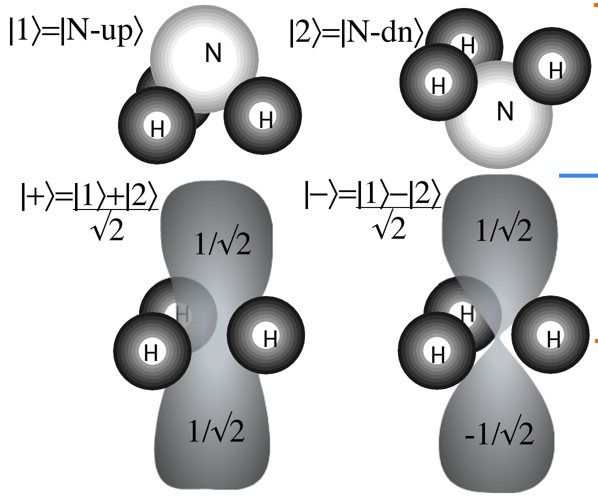


Fig. 10.3.2 Ammonia (NH$_3$) inversion states
(a) Base states (b) C$_2$-Eigenstates

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

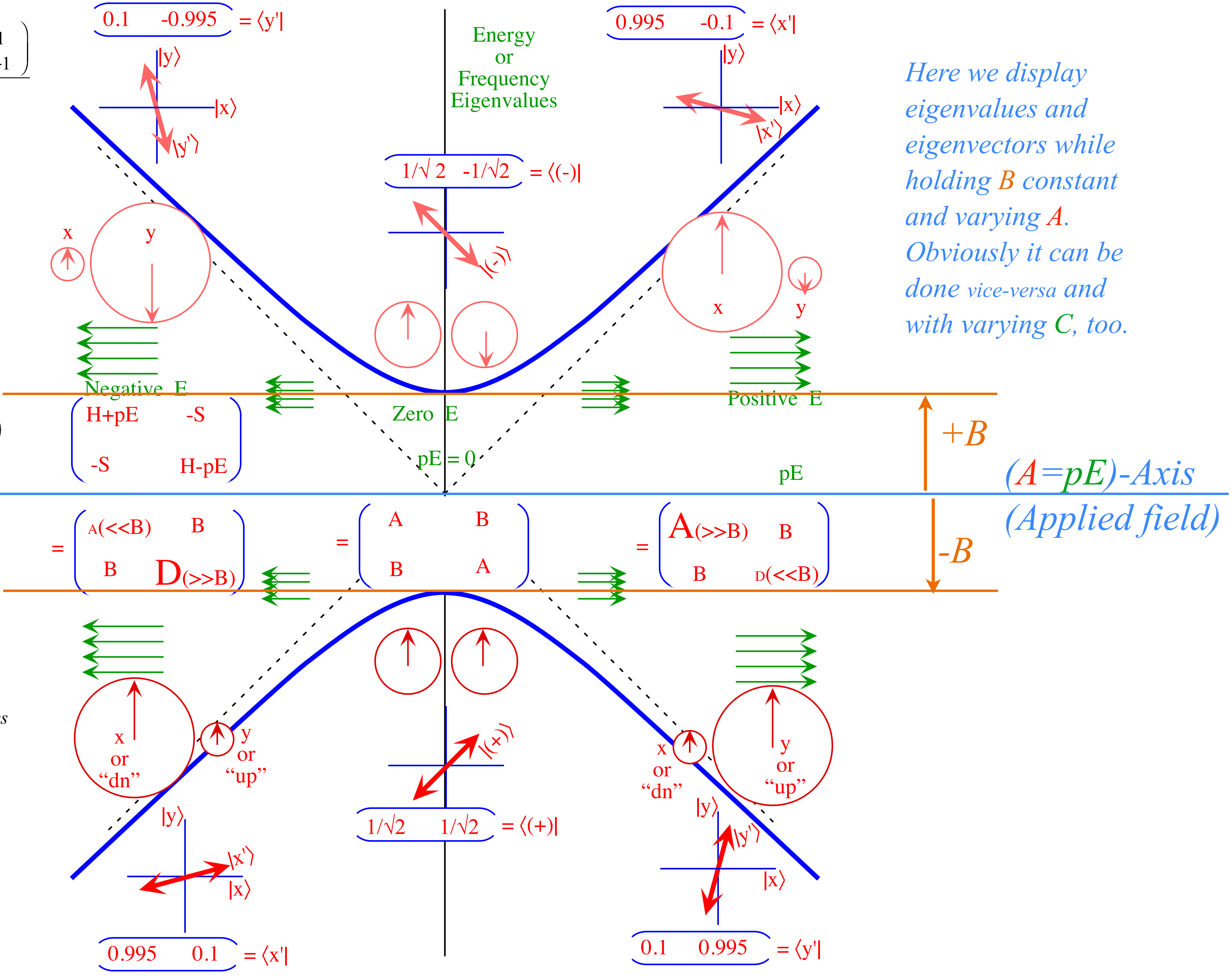


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

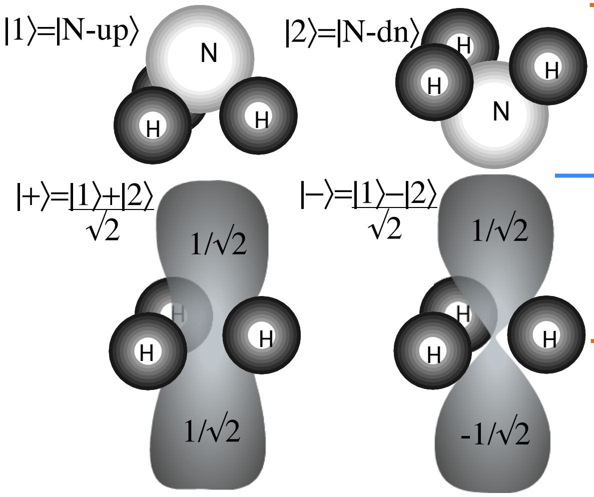


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A+B\boldsymbol{\sigma}_B=\mathbf{H}=\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}$

$\mathbf{H}=\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}$ Secular equation: $\varepsilon^2-0\cdot\varepsilon-(A^2+B^2)$ gives *hyperbolic* energy levels: $\varepsilon=\pm\sqrt{A^2+B^2}$

$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}=\frac{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}{\sqrt{2}}\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}\frac{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}{\sqrt{2}}$$

$$=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}+A+B & B-A\\ +A-B & B+A\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

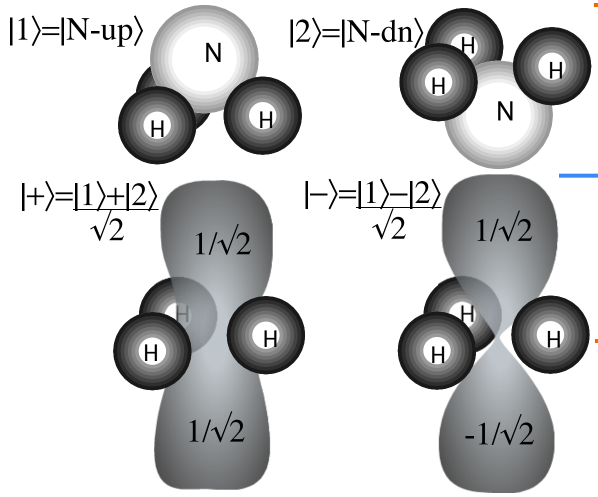
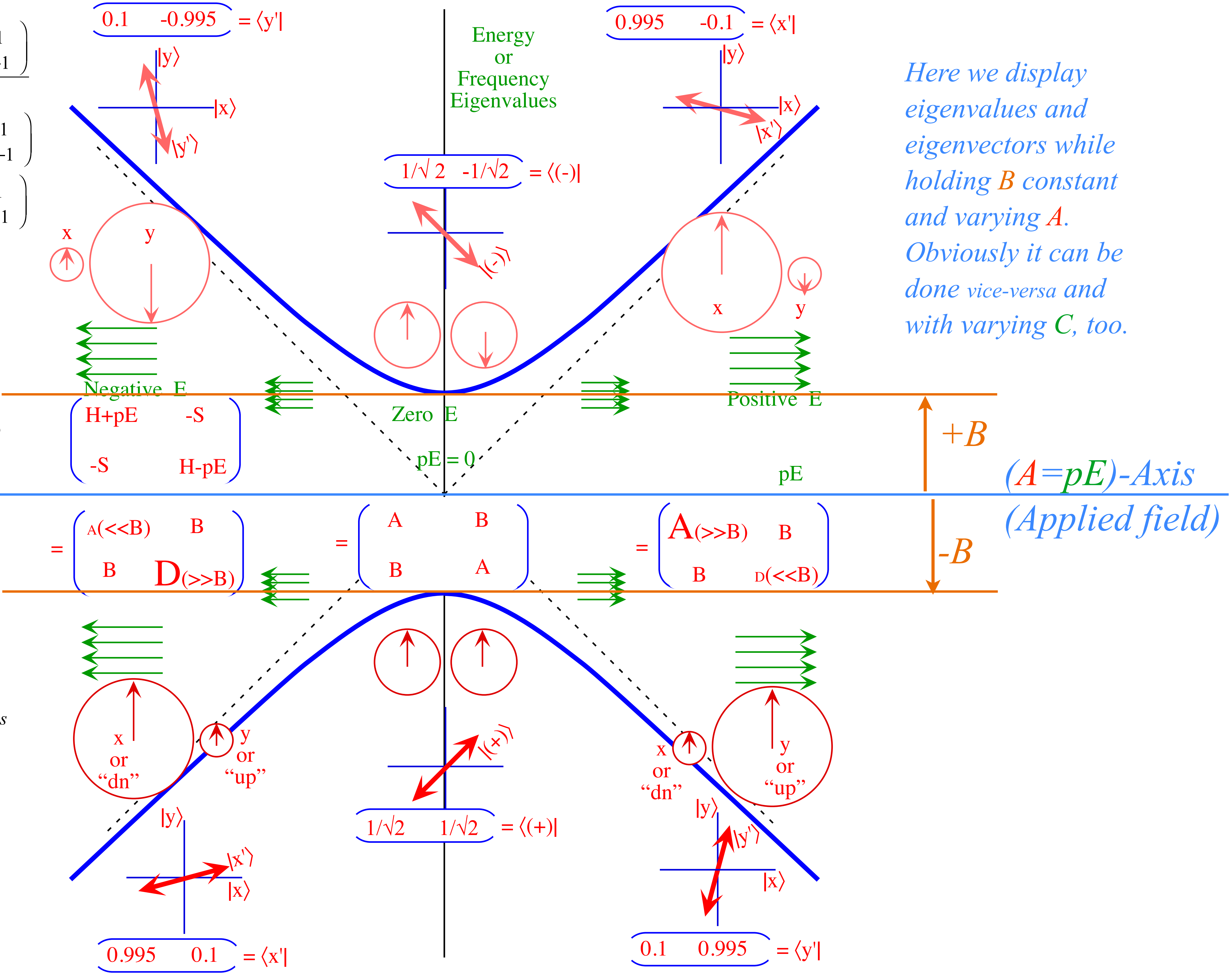


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



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A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

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$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

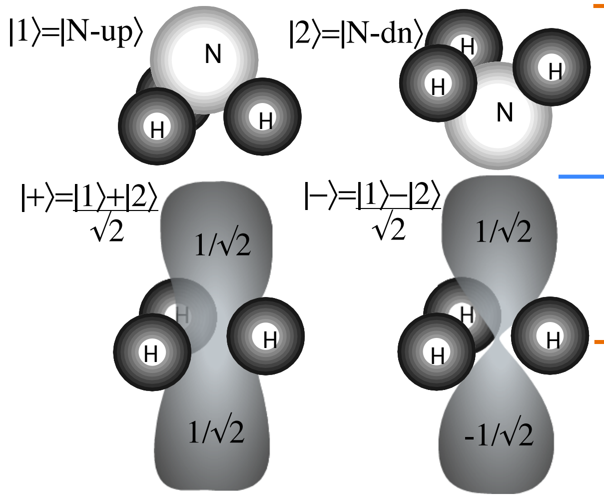


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

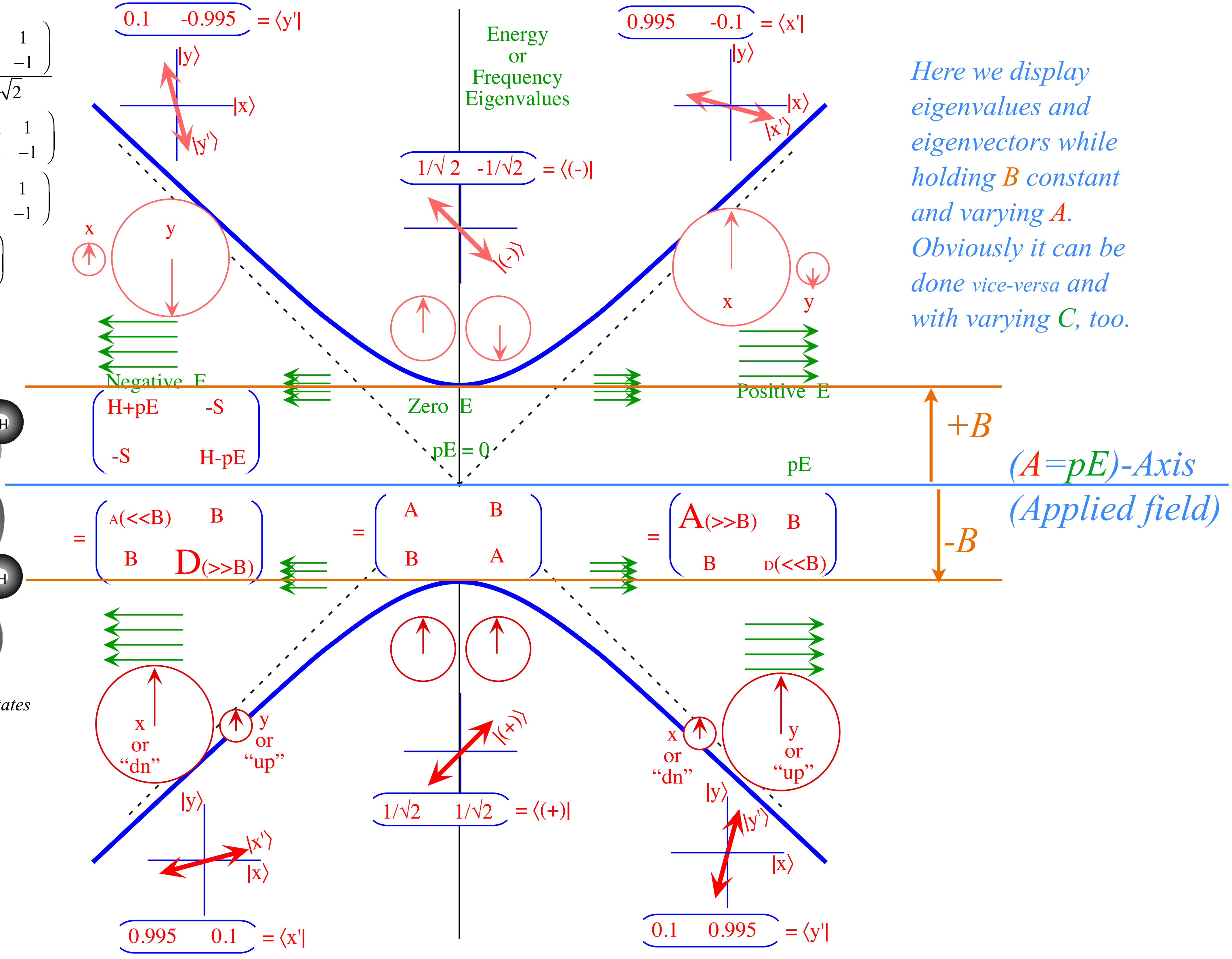


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

$$= \begin{pmatrix} +B & A \\ A & -B \end{pmatrix}$$

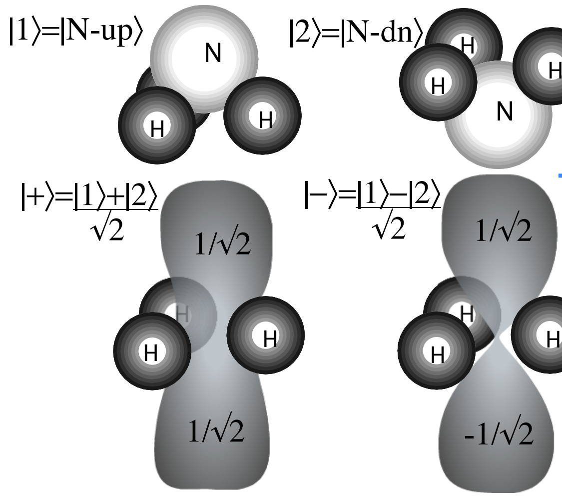


Fig. 10.3.2 Ammonia (NH$_3$*) inversion states*
(a) Base states (b) C$_2$*-Eigenstates*

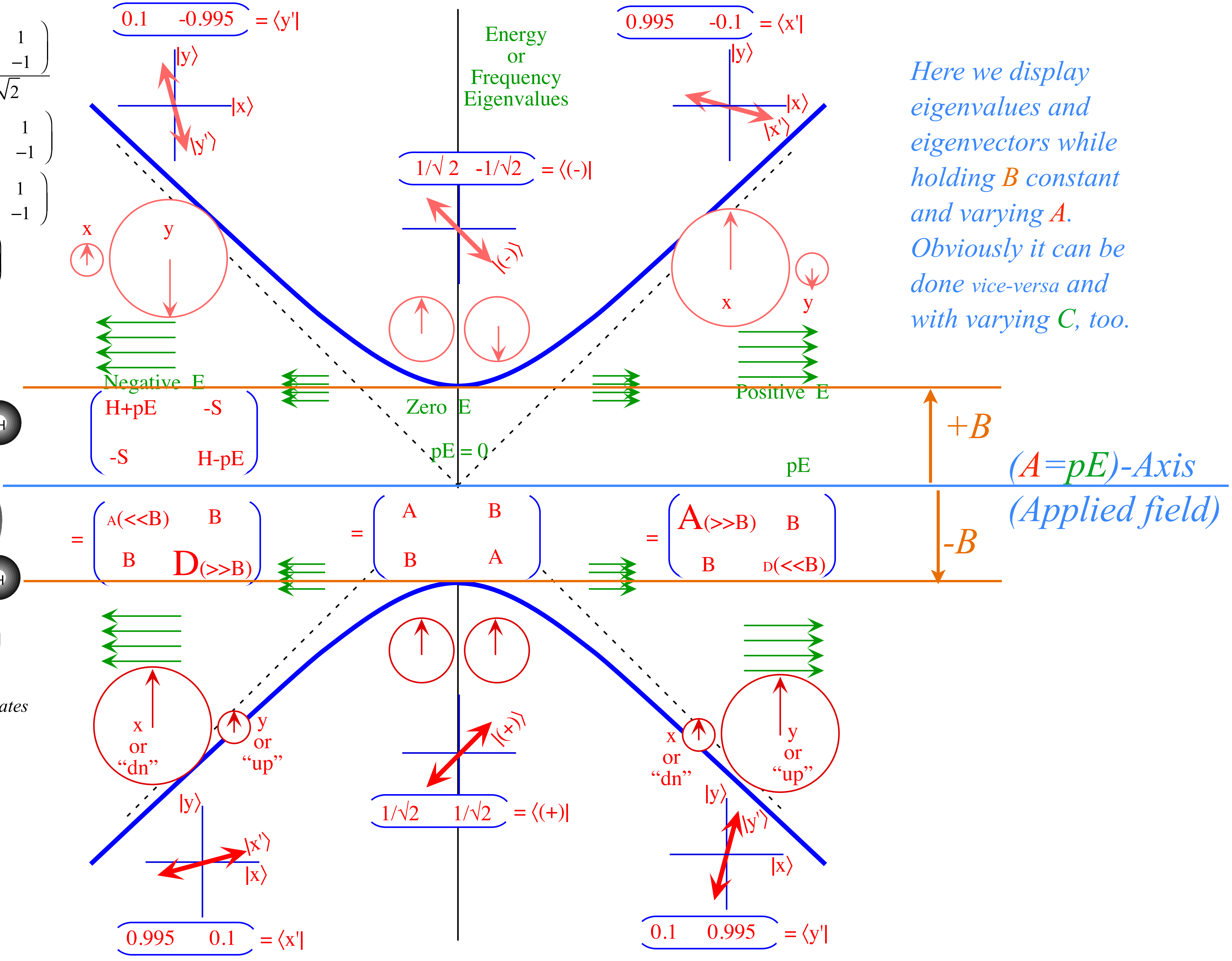


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

The failure of perturbation methods to get exact hyperbolic eigenvalues

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} \ ,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} \ .$$

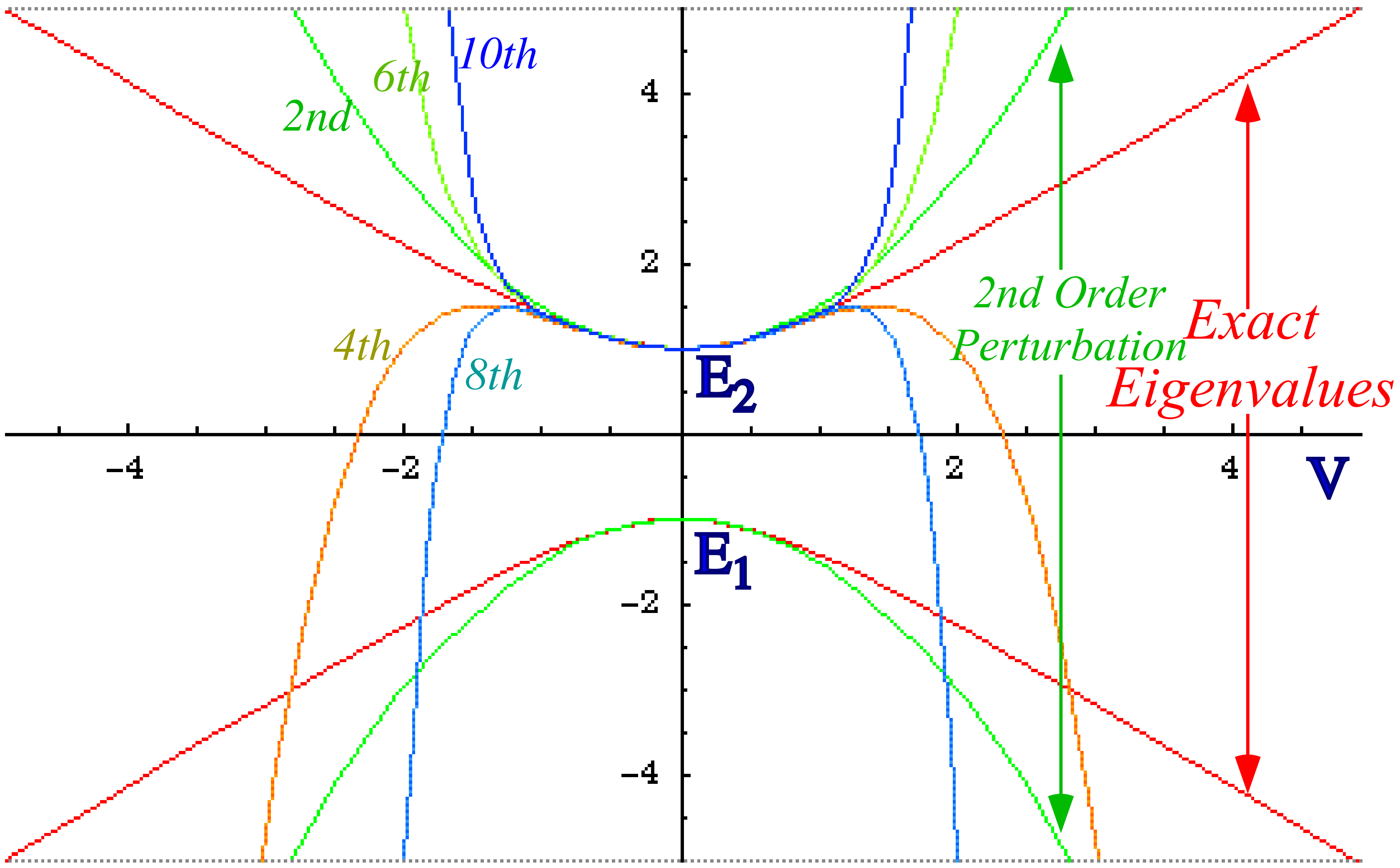


Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations

$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1 E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 + E_2)^2 - 4E_1E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4V^2}}{2} \ ,$$

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \ , \text{ where: } \Delta = |E_1 - E_2|$$

A view of a conical intersection:

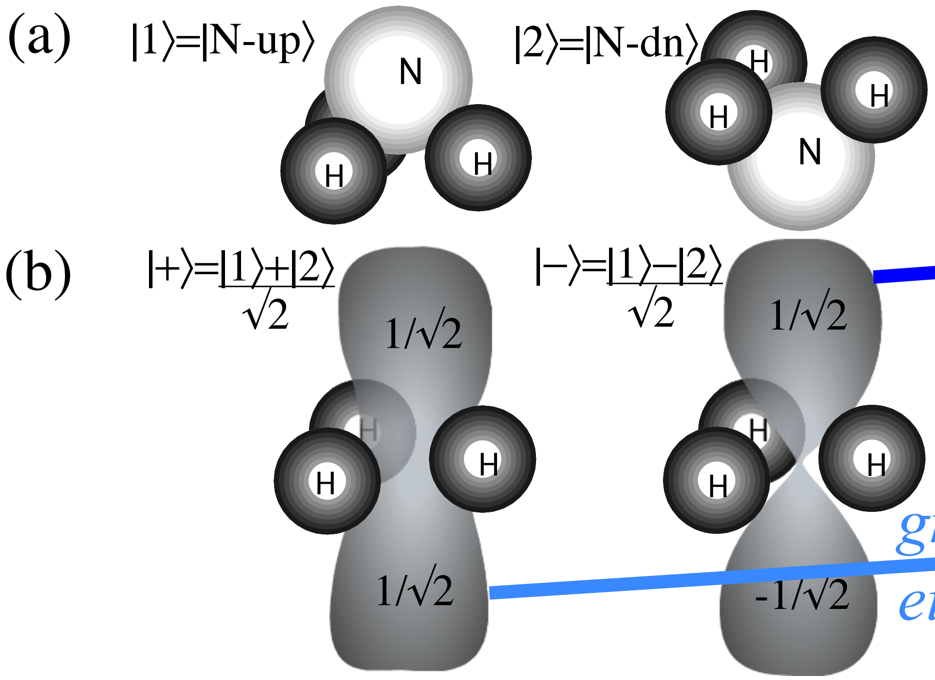
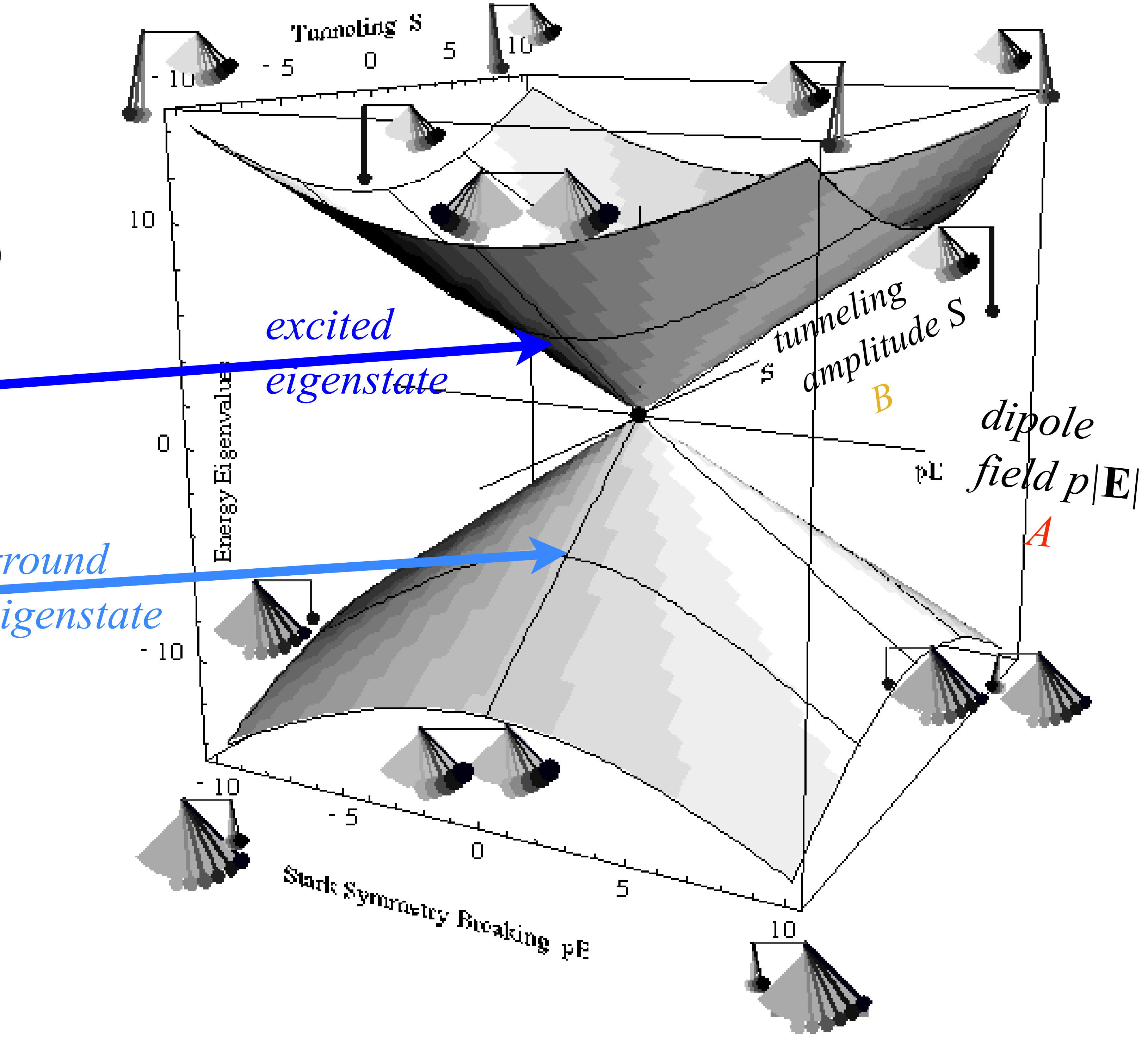


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

(Also known as a “Dirac-point”)

A view of a conical intersection: Any vertical cross-section is hyperbolic avoided-crossing

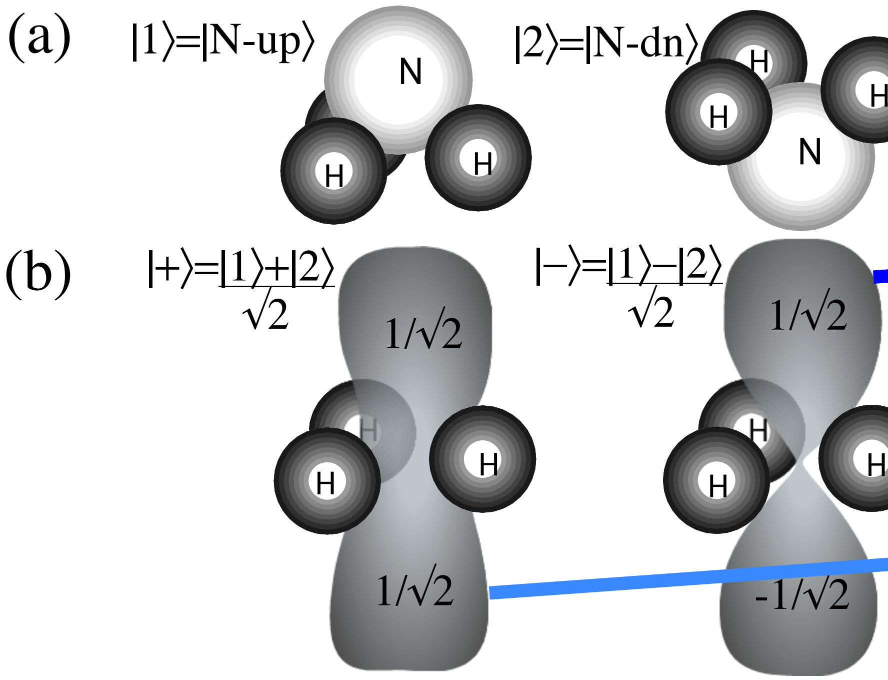
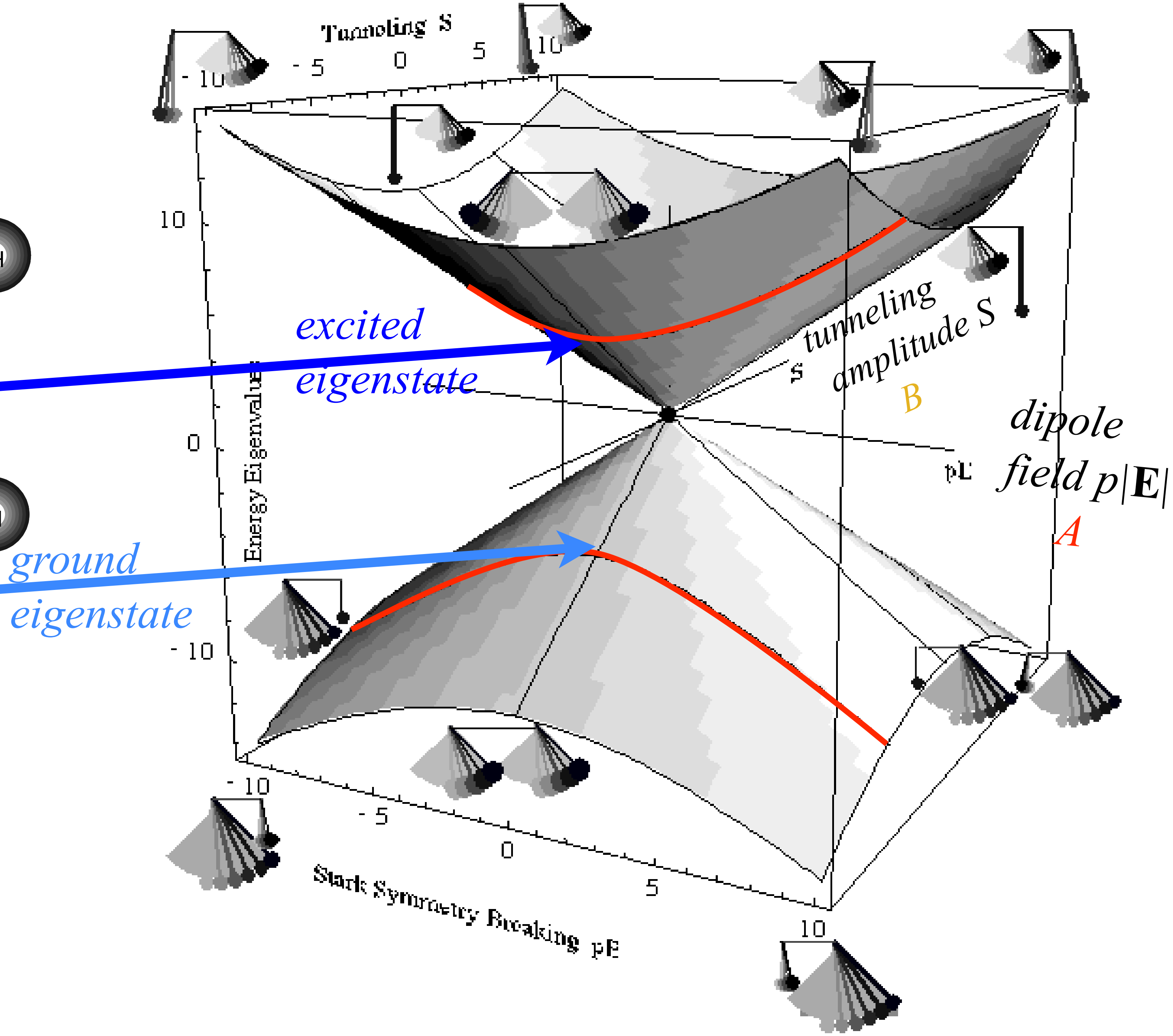


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

(Also known as a "Dirac-point")

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

ABC-Type elliptical polarized motion

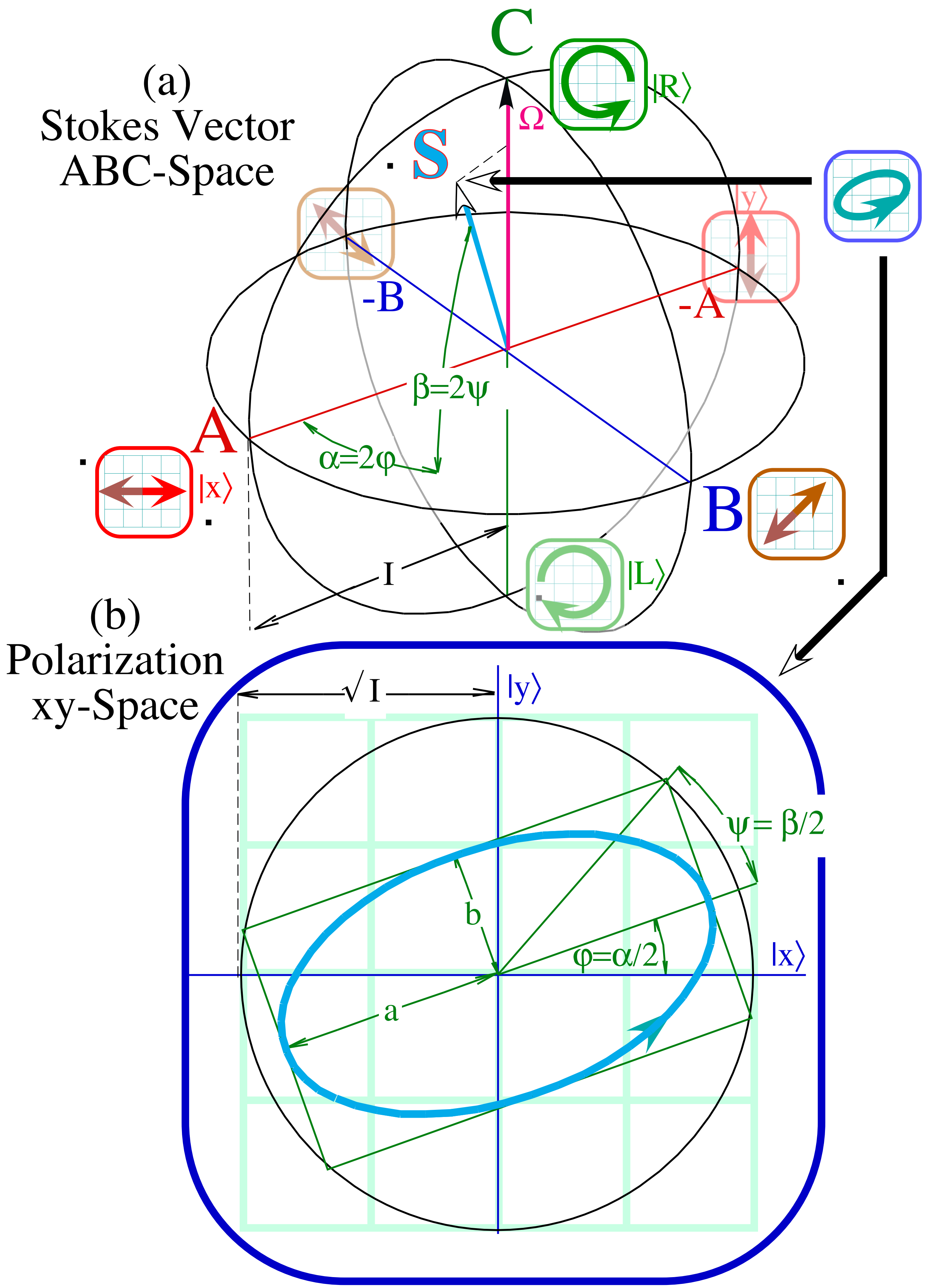


Fig. 10.B.3

Euler-like coordinates for
(a) R(3) spin vector
(b) U(2) polarization ellipse

ABC-Type elliptical polarized motion

(from Principles of Symmetry, Dynamics, and Spectroscopy)

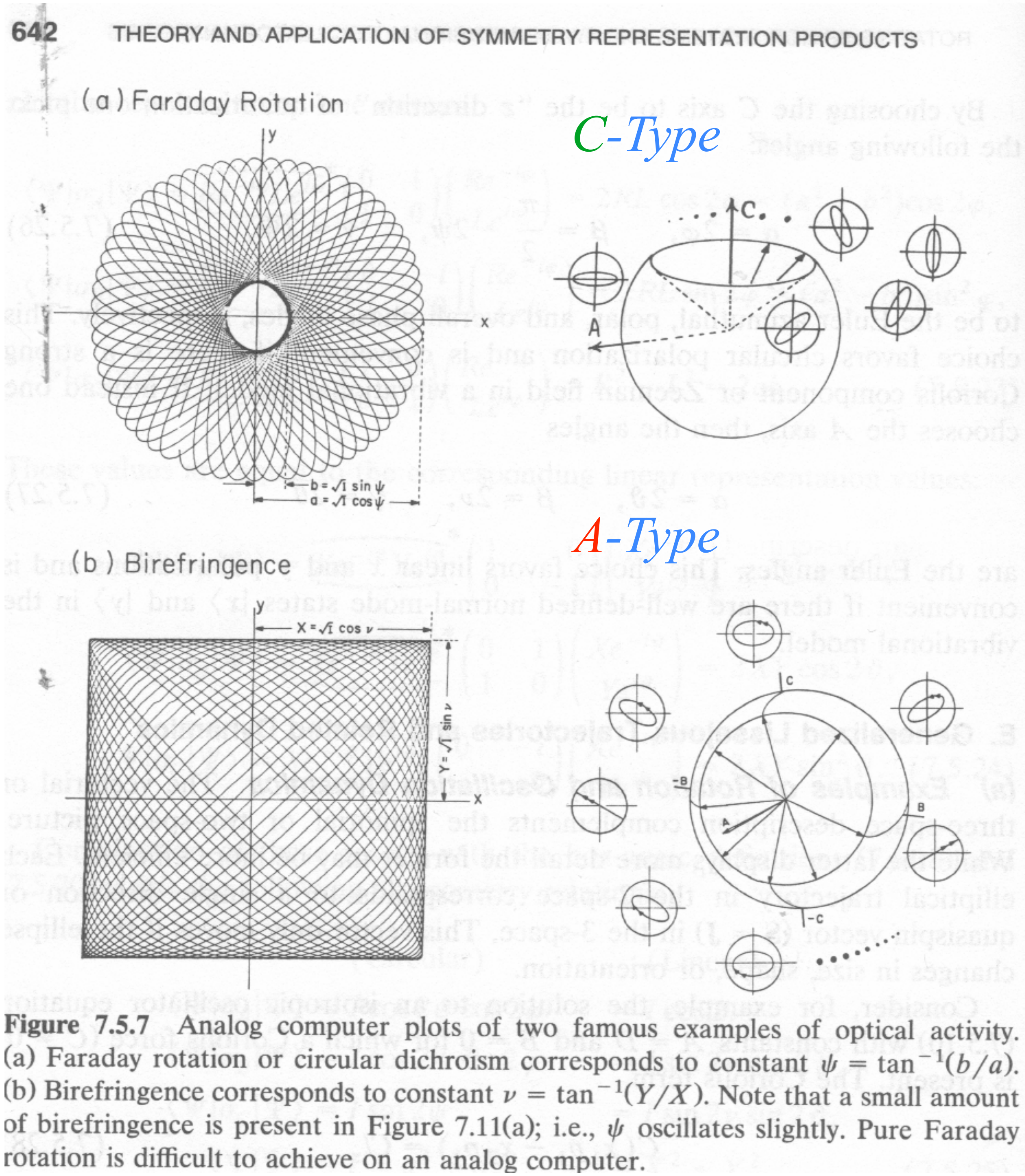
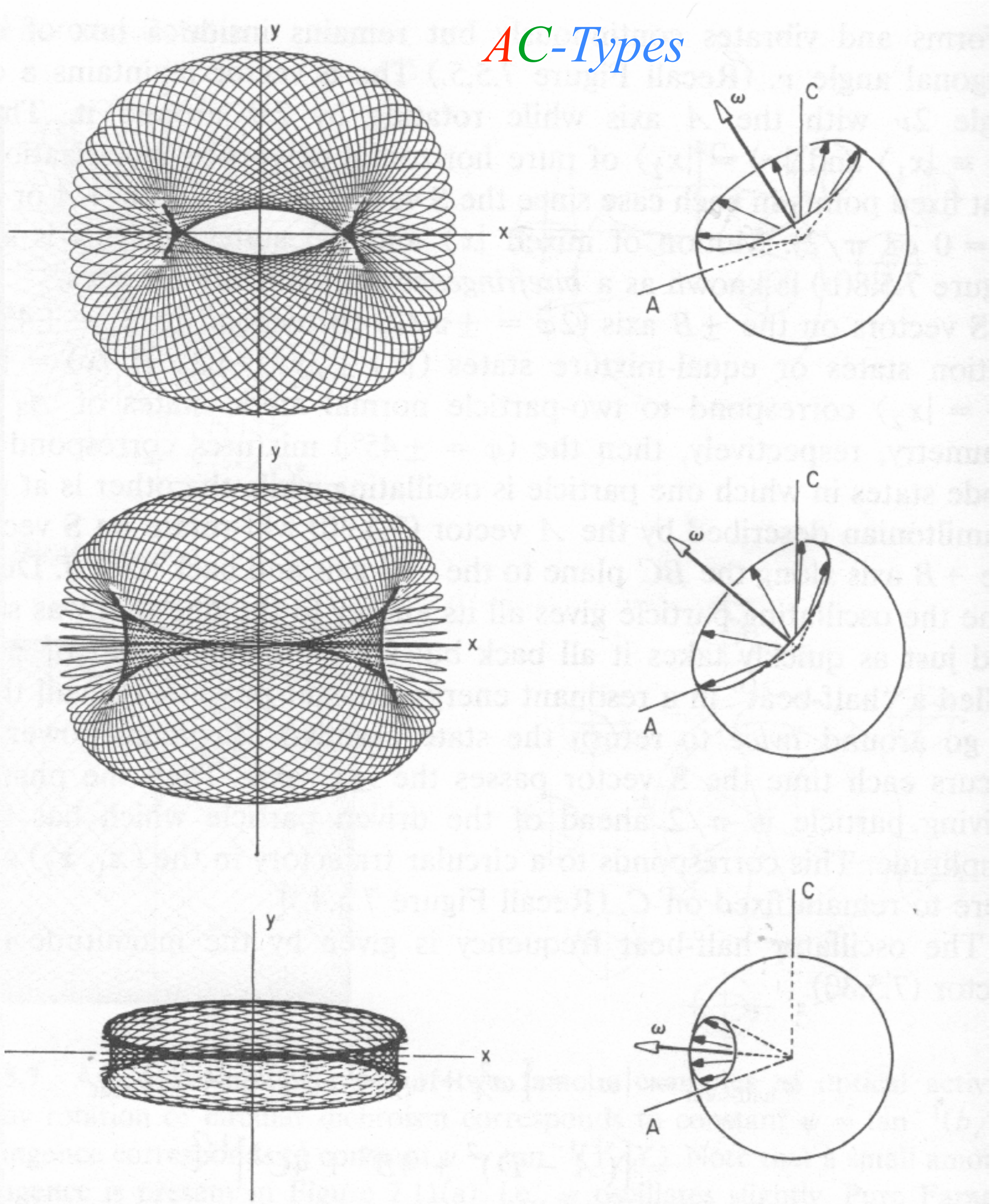
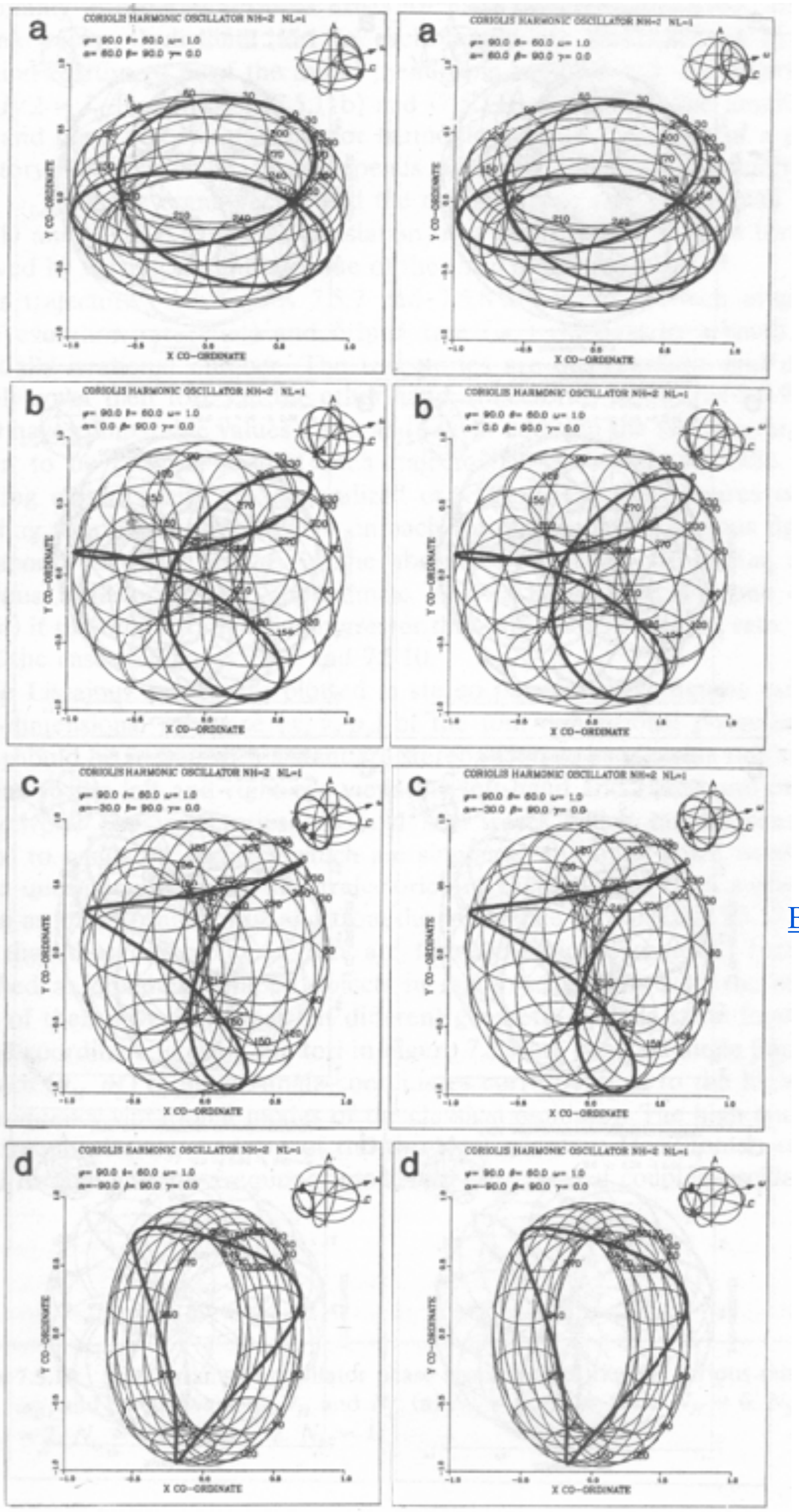


Figure 7.5.7 Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., ψ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.

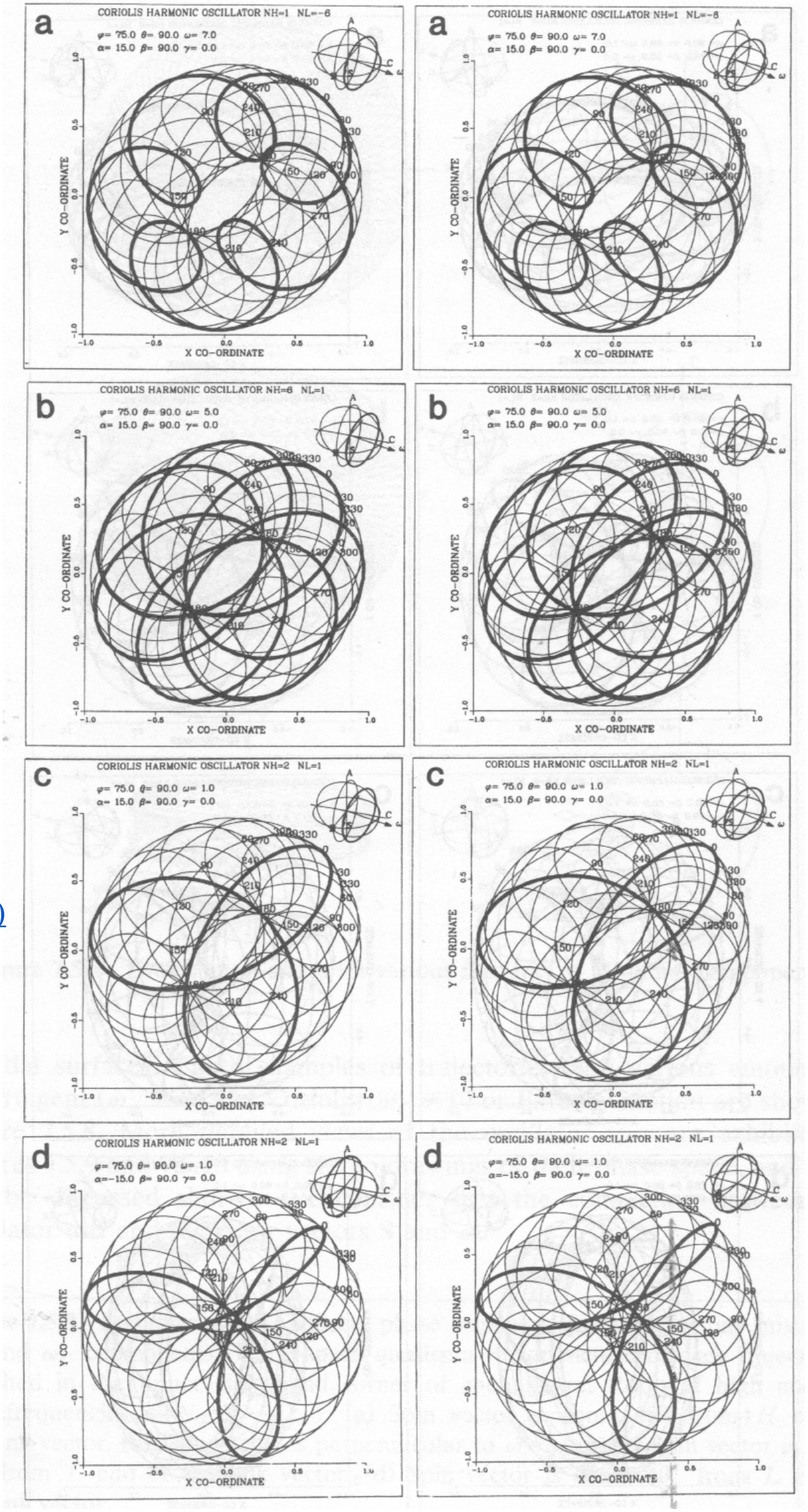


7.5.8 Evolution of states for various mixtures of A and C components.

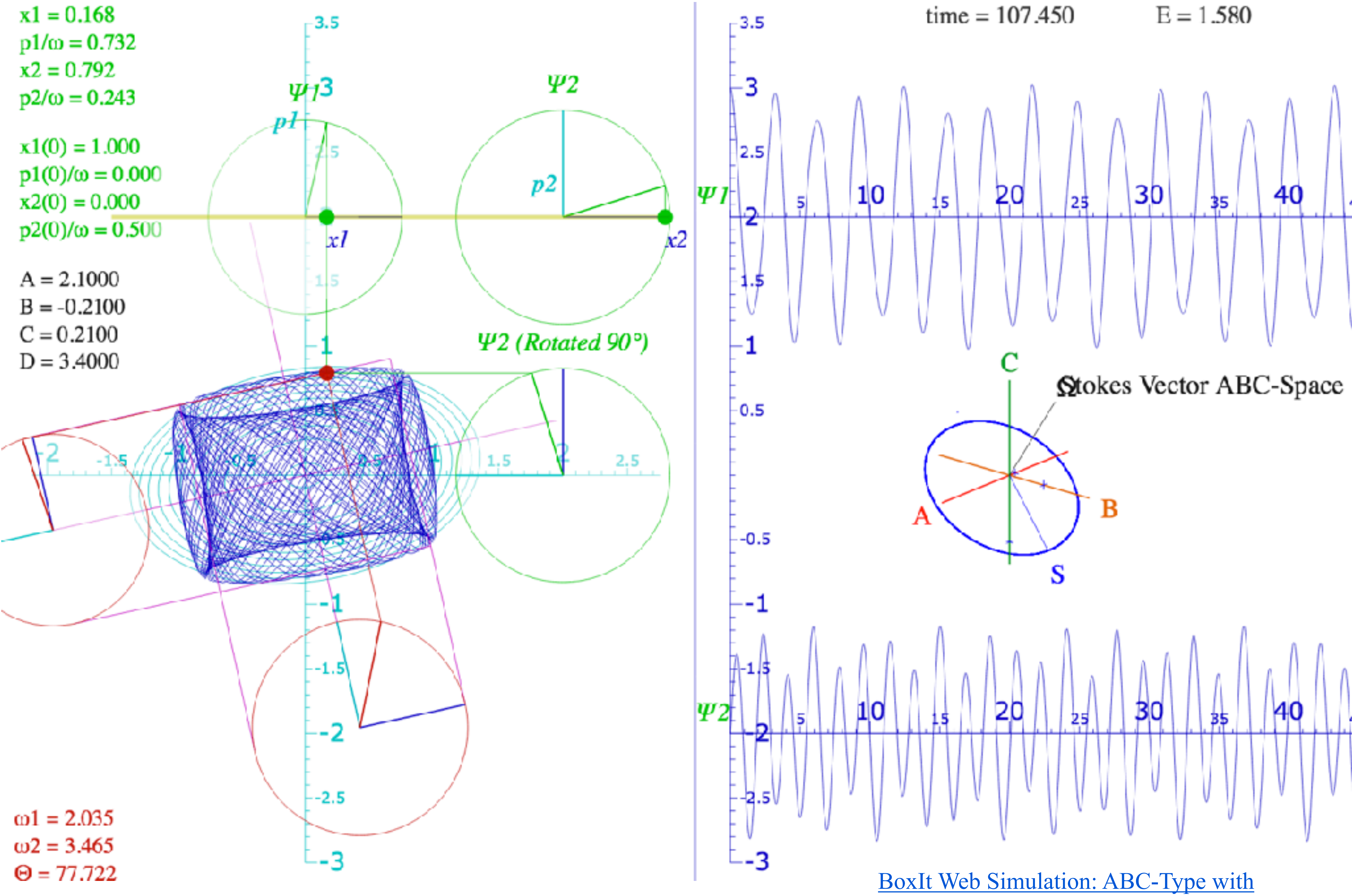


ABC-Type elliptical polarized dynamics

BoxIt (ABC-Motion)
Web Simulation



ABC-Type elliptical polarized motion



x1 = 0.168
p1/ω = 0.732
x2 = 0.792
p2/ω = 0.243

x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500

A = 2.1000
B = -0.2100
C = 0.2100
D = 3.4000

ω1 = 2.035
ω2 = 3.465
Θ = 77.722

time = 107.450
E = 1.580

BoxIt Web Simulation: ABC-Type with A=2.1; B=-0.21; C=0.21; D=3.4

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ and Darboux **R**$[\varphi\vartheta\Theta]$ representations of $U(2)$ and $R(3)$

- Euler-defined state $|\alpha\beta\gamma\rangle$ described by Stoke's **S**-vector, phasors, or ellipsometry
- Darboux defined Hamiltonian **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ and angular velocity $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$-vector
- Euler-defined operator **R**$(\alpha\beta\gamma)$ derived from Darboux-defined **R**$[\varphi\vartheta\Theta]$ and vice versa
- Euler **R**$(\alpha\beta\gamma)$ rotation $\mathbf{\Theta}=0-4\pi$-sequence $[\varphi\vartheta]$ fixed (and “real-world” applications)

Quick $U(2)$ way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The *ABC*'s of $U(2)$ dynamics-Archetypes

- *A*symmetric-*D*iagonal *A*-Type motion
- *B*ilateral-*B*alanced *B*-Type motion
- *C*ircular-*C*oriolis… *C*-Type motion

The *ABC*'s of $U(2)$ dynamics-Mixed modes

- *AB*-Type motion and Wigner's Avoided-Symmetry-Crossings
- *ABC*-Type elliptical polarized motion

Ellipsometry using $U(2)$ symmetry and related coordinates

- Conventional amp-phase ellipse coordinates
- Euler Angle $(\alpha\beta\gamma)$ ellipse coordinates

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates and related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes

A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$x_1= A_1 cos(\omega t+\rho_1)$$

$$-p_1=A_1 sin(\omega t+\rho_1)$$

$$x_2= A_2 cos(\omega t-\rho_1)$$

$$-p_2=A_2 sin(\omega t-\rho_1)$$

Amp-phase parameters $(A_1,A_2, \omega t,\rho_1)$

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ \\ x_2+ip_2 \end{pmatrix}$$

$p_1 = -A_1 \sin(\omega t + \rho_1)$

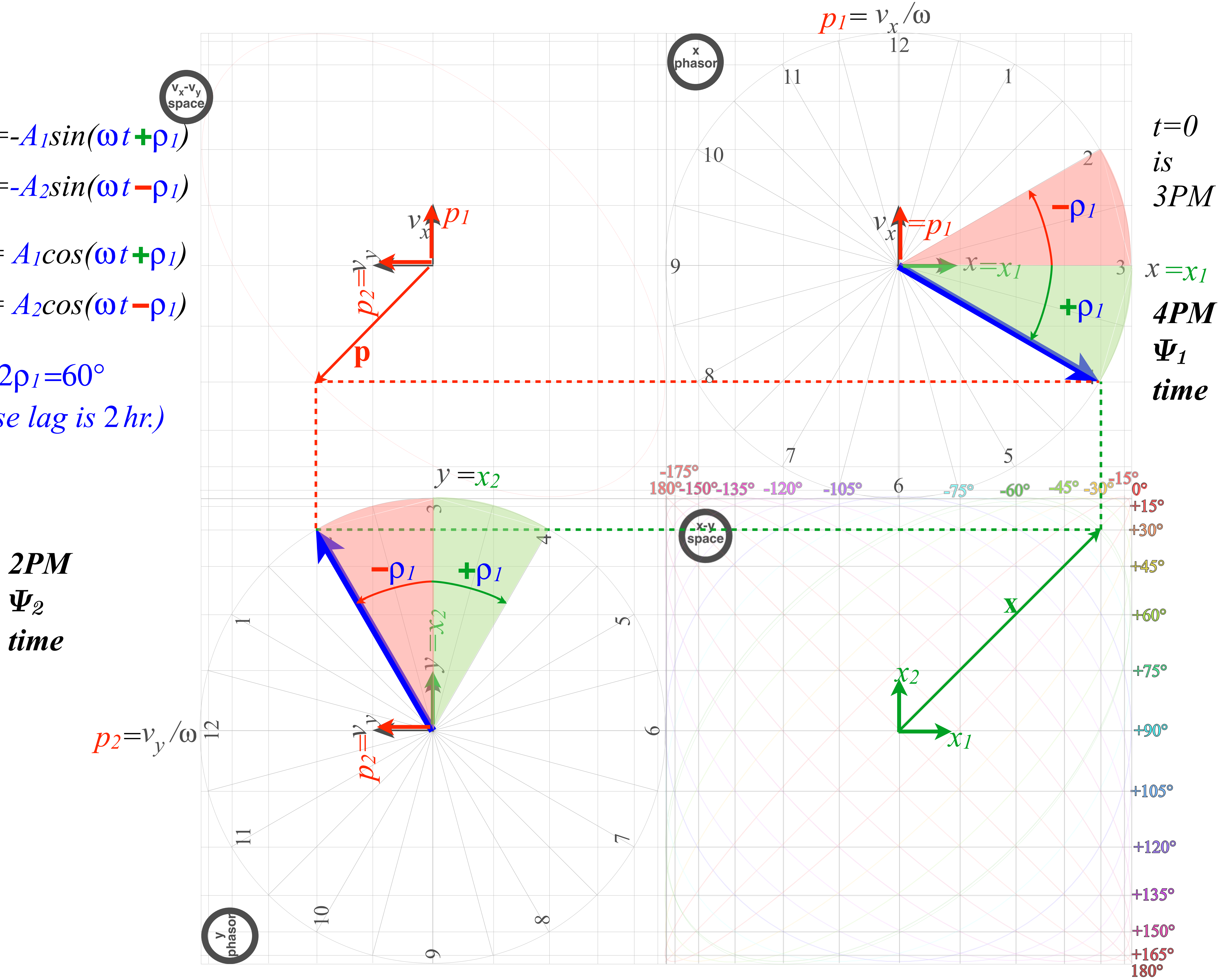
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2 hr.)



$p_1=-A_1\sin(\omega t+\rho_1)$

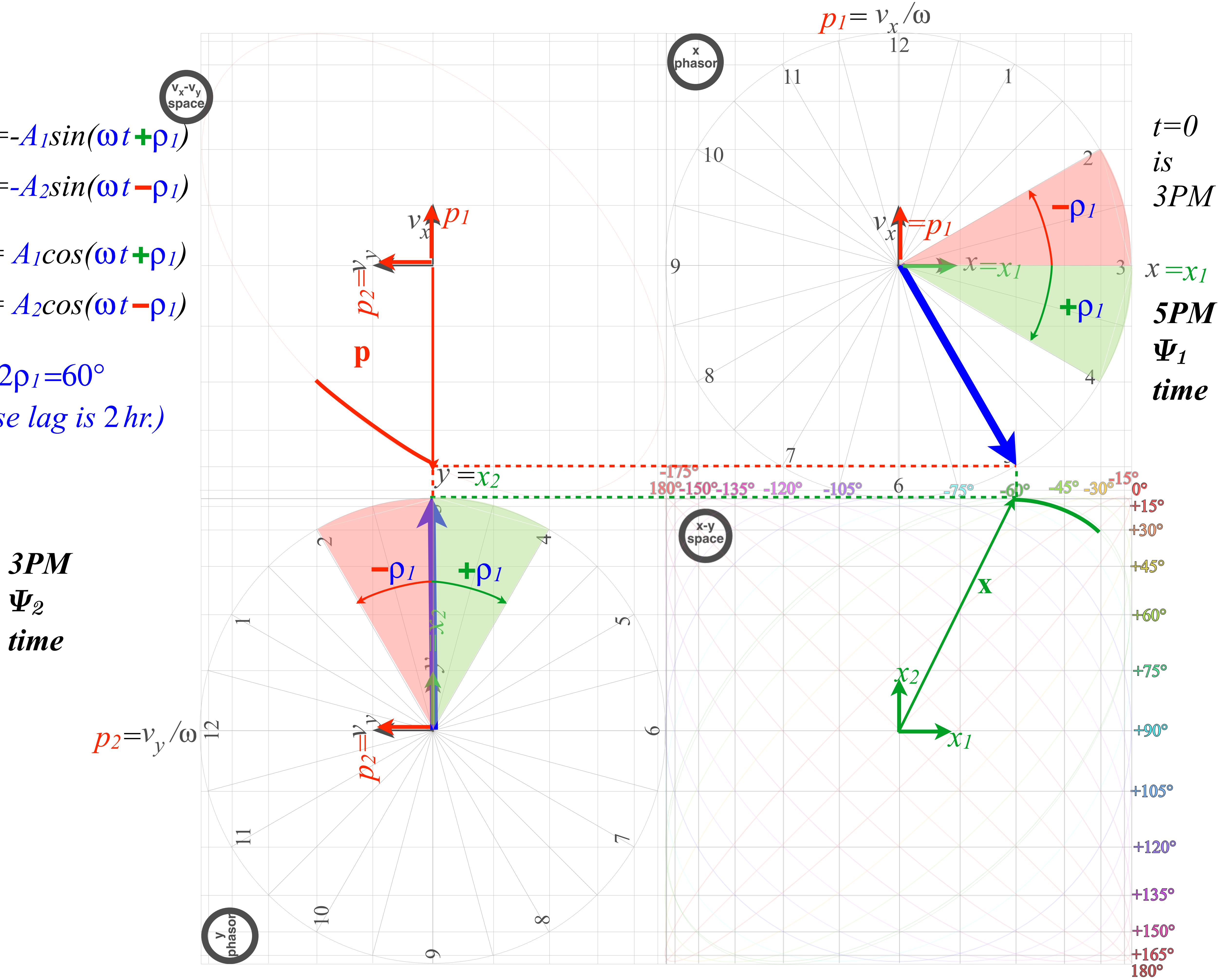
$p_2=-A_2\sin(\omega t-\rho_1)$

$x_1=A_1\cos(\omega t+\rho_1)$

$x_2=A_2\cos(\omega t-\rho_1)$

$2\rho_1=60°$

(phase lag is 2 hr.)



$p_1 = -A_1 \sin(\omega t + \rho_1)$

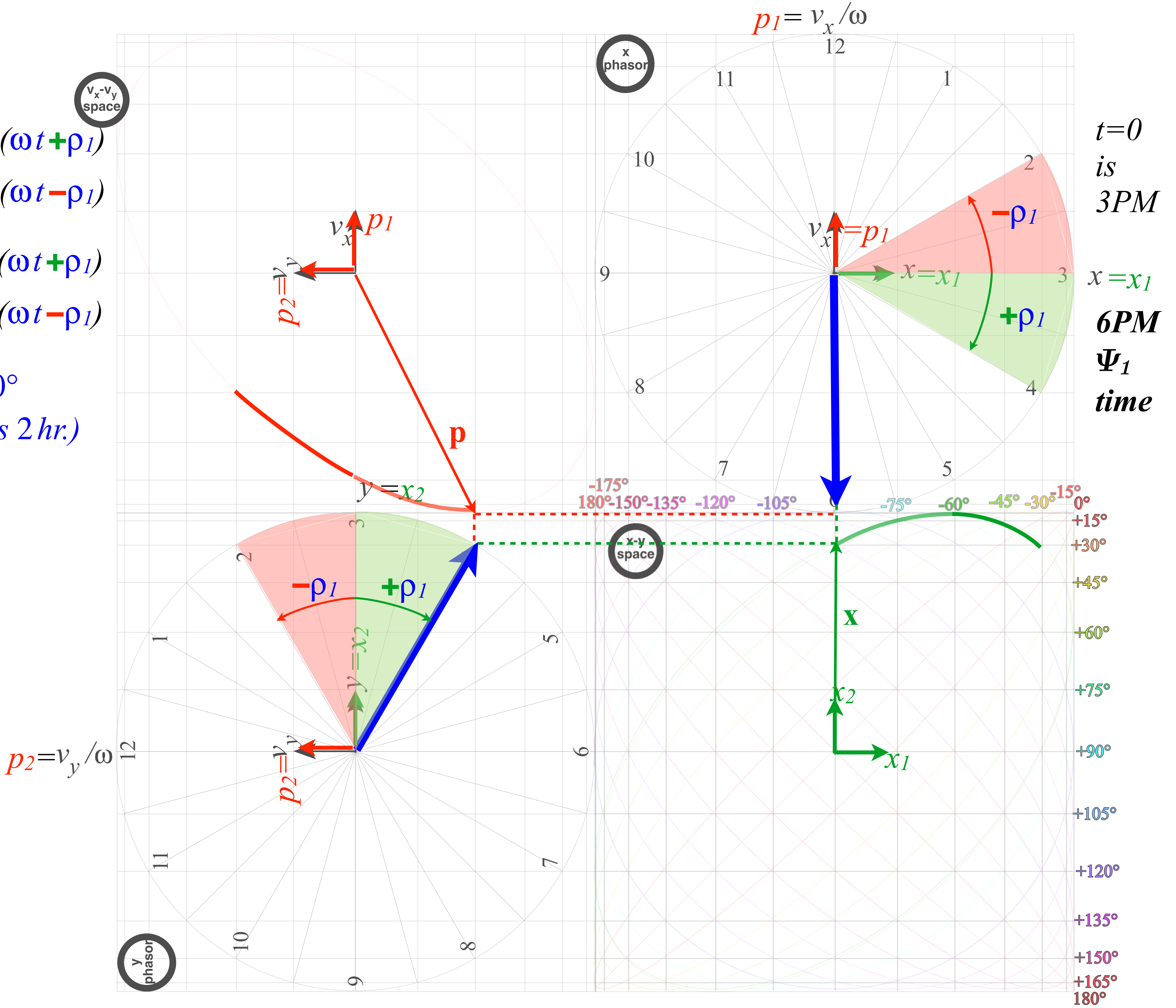
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

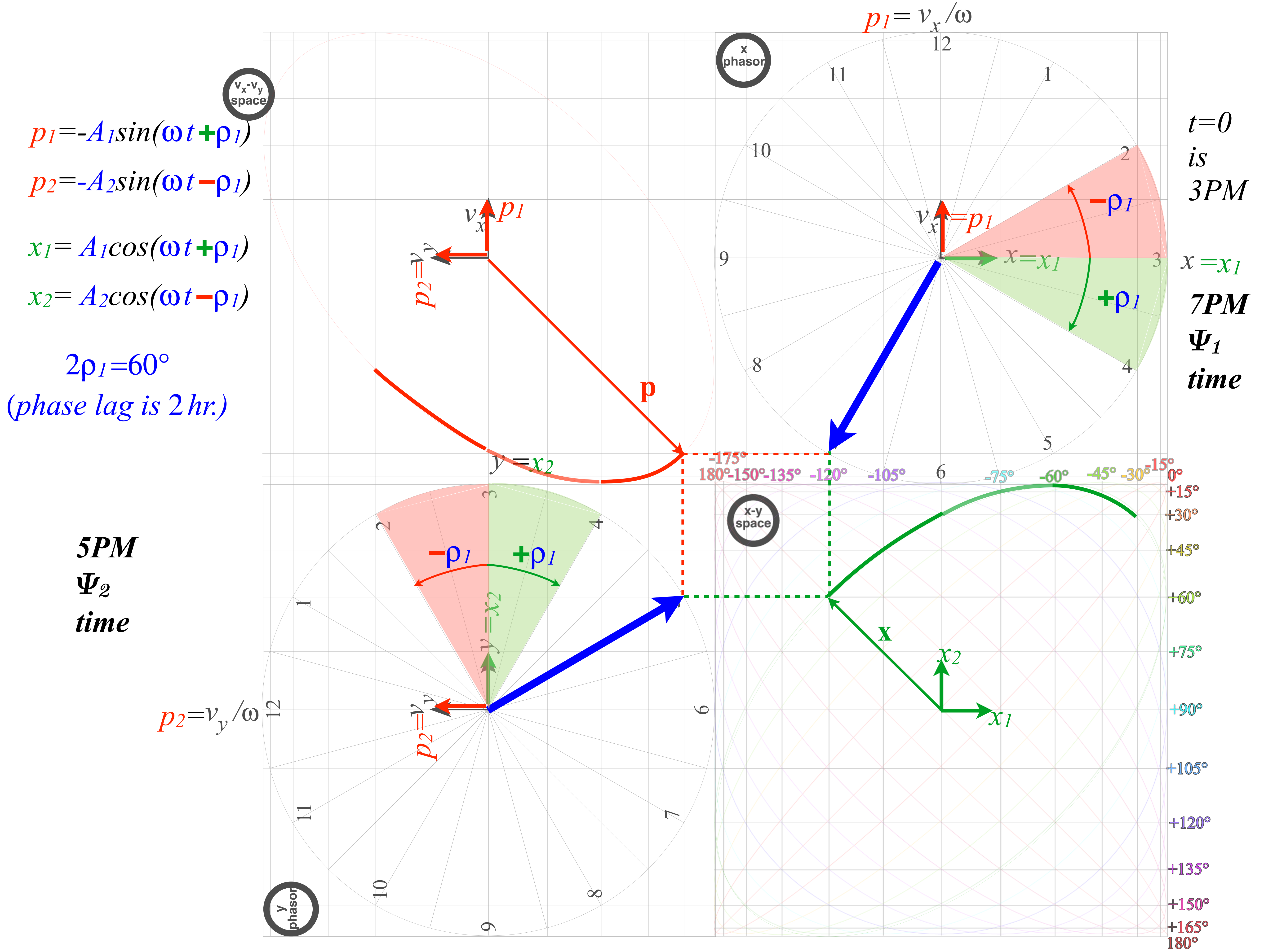
(phase lag is 2 hr.)



t=0
is
3PM

6PM
Ψ_1
time

4PM
Ψ_2
time



$p_1=-A_1\sin(\omega t+\rho_1)$

$p_2=-A_2\sin(\omega t-\rho_1)$

$x_1=A_1\cos(\omega t+\rho_1)$

$x_2=A_2\cos(\omega t-\rho_1)$

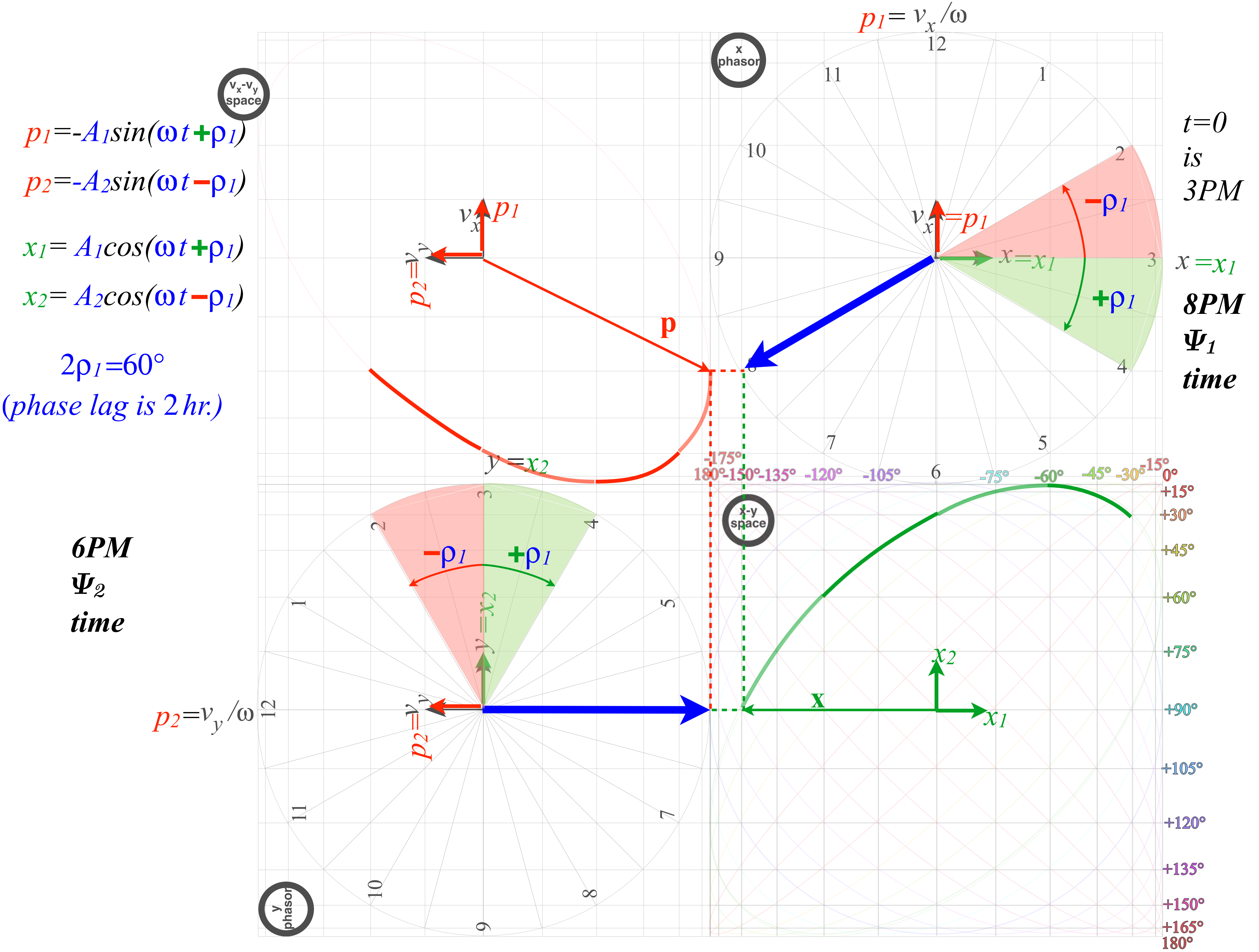
$2\rho_1=60°$

(phase lag is 2 hr.)

t=0
is
3PM

7PM
Ψ_1
time

5PM
Ψ_2
time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2 hr.)

$t=0$ is 3PM

8PM

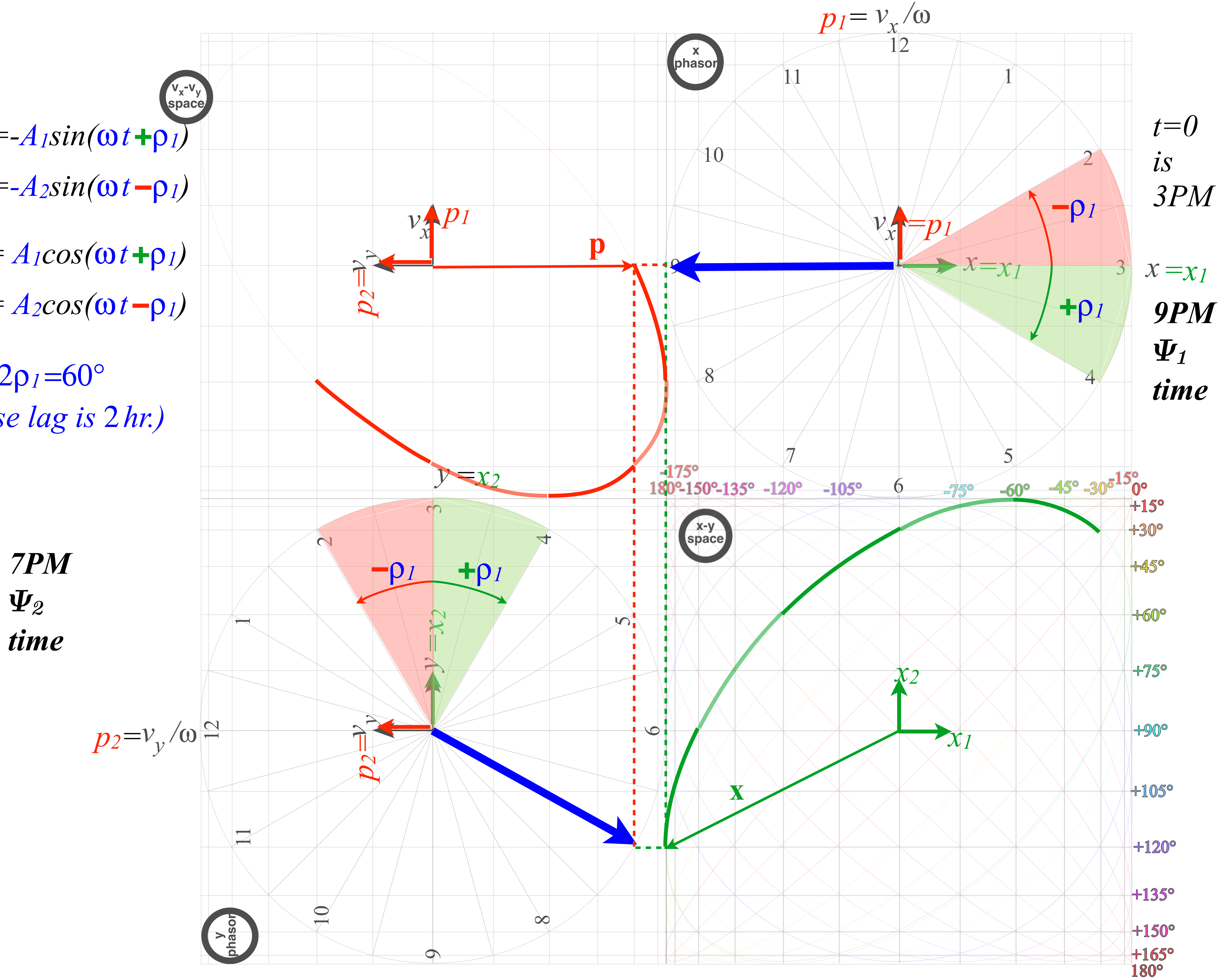
Ψ_1

time

6PM

Ψ_2

time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2 hr.)

t=0
is
3PM

9PM
Ψ₁
time

7PM
Ψ₂
time

$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

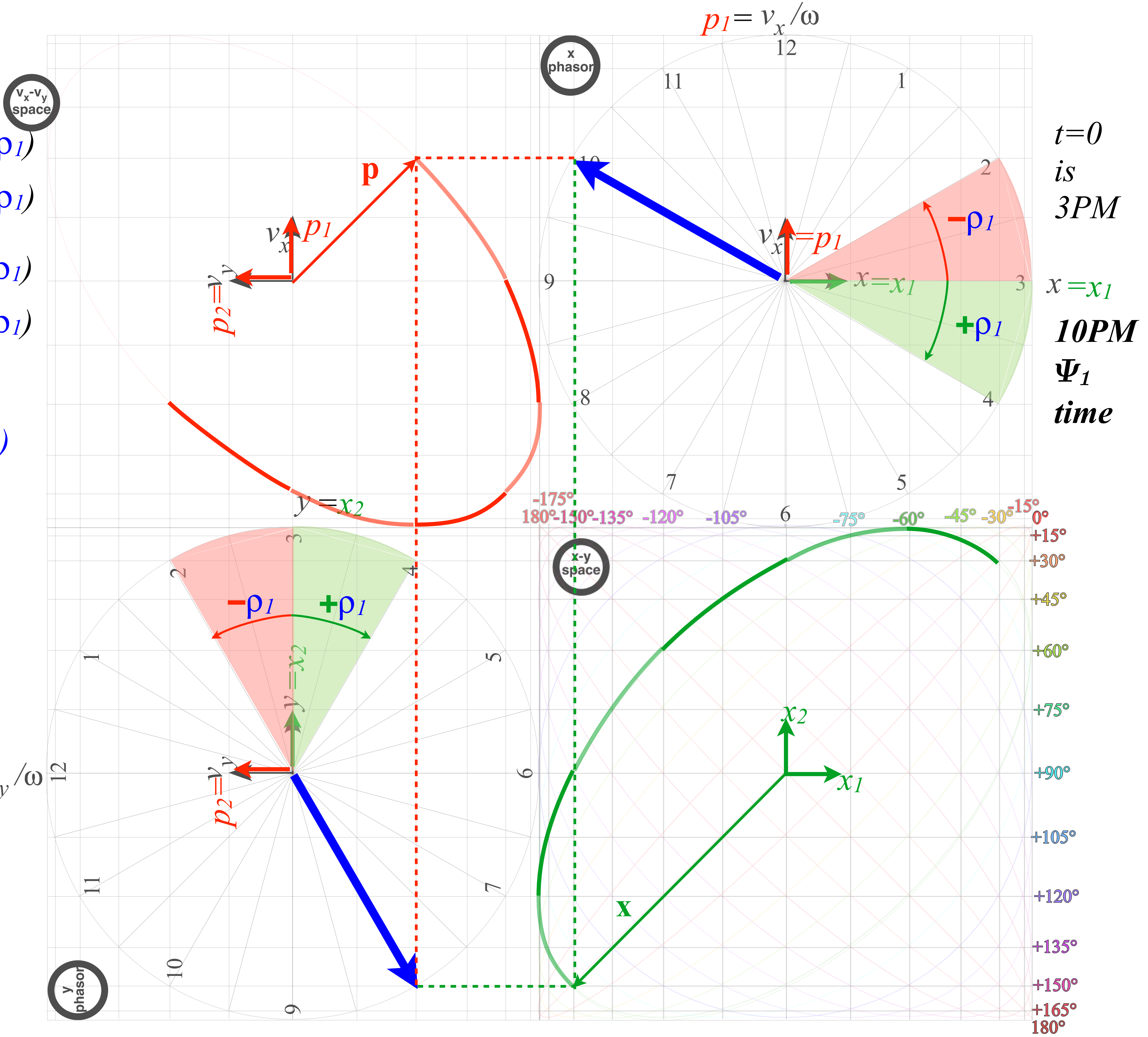
$2\rho_1 = 60°$

(phase lag is 2 hr.)

t=0
is
3PM

10PM
Ψ_1
time

8PM
Ψ_2
time



p_1=$-A_1\sin(\omega t+\rho_1)$

p_2=$-A_2\sin(\omega t-\rho_1)$

x_1= $A_1\cos(\omega t+\rho_1)$

x_2= $A_2\cos(\omega t-\rho_1)$

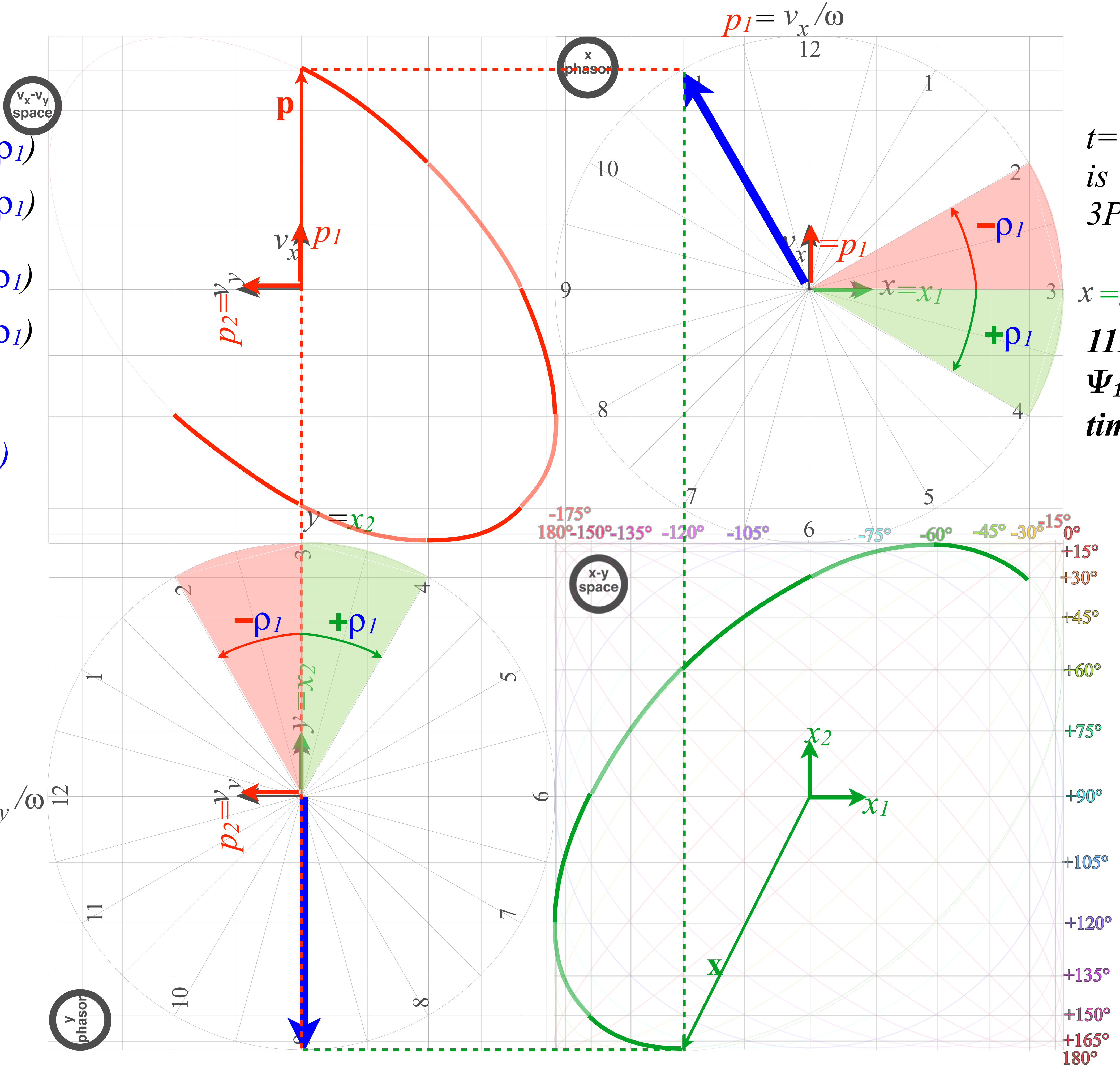
$2\rho_1=60°$

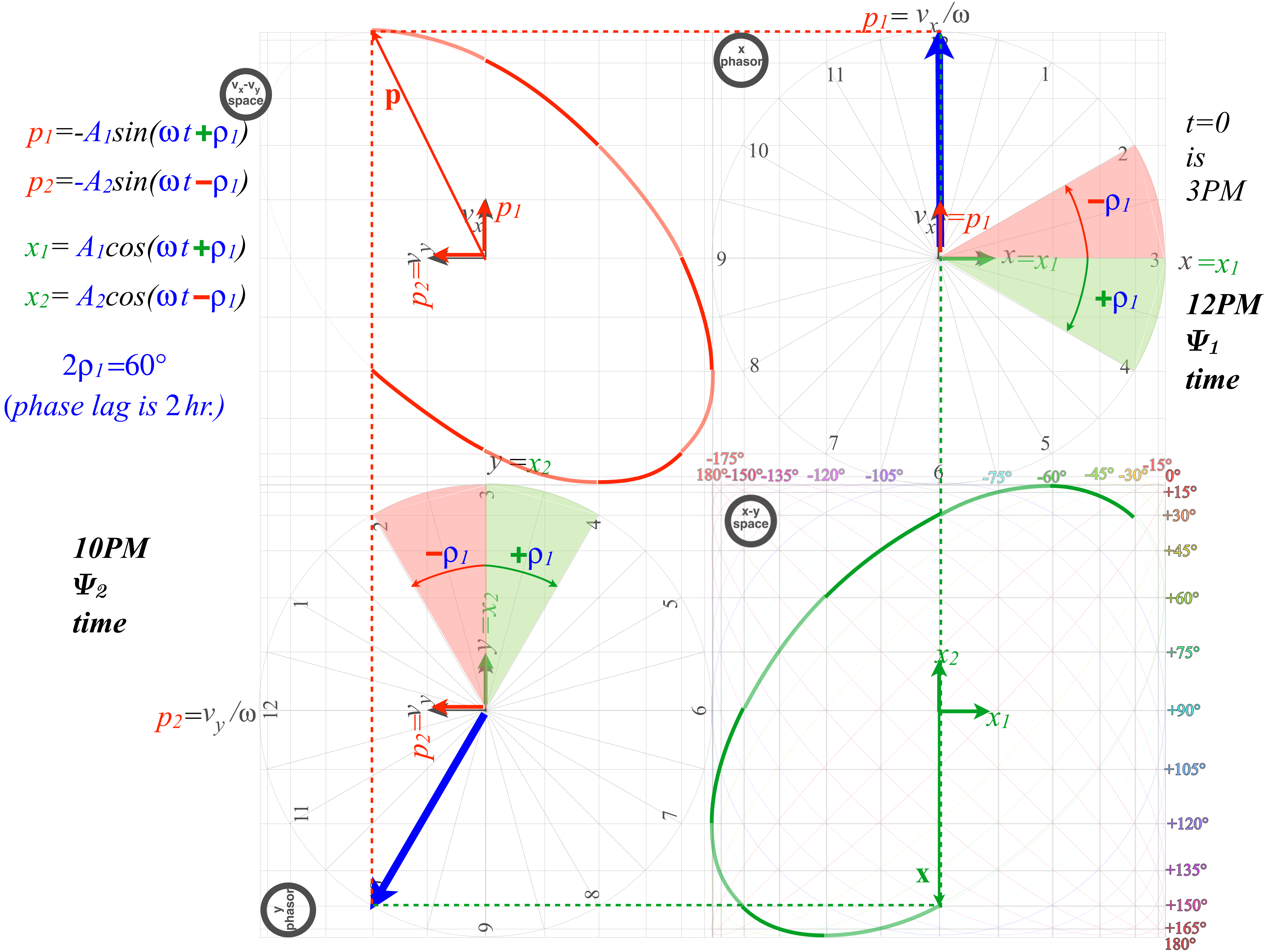
(phase lag is 2 hr.)

t=0
is
3PM

11PM
Ψ_1
time

9PM
Ψ_2
time





$$p_1=-A_1\sin(\omega t+\rho_1)$$

$$p_2=-A_2\sin(\omega t-\rho_1)$$

$$x_1=A_1\cos(\omega t+\rho_1)$$

$$x_2=A_2\cos(\omega t-\rho_1)$$

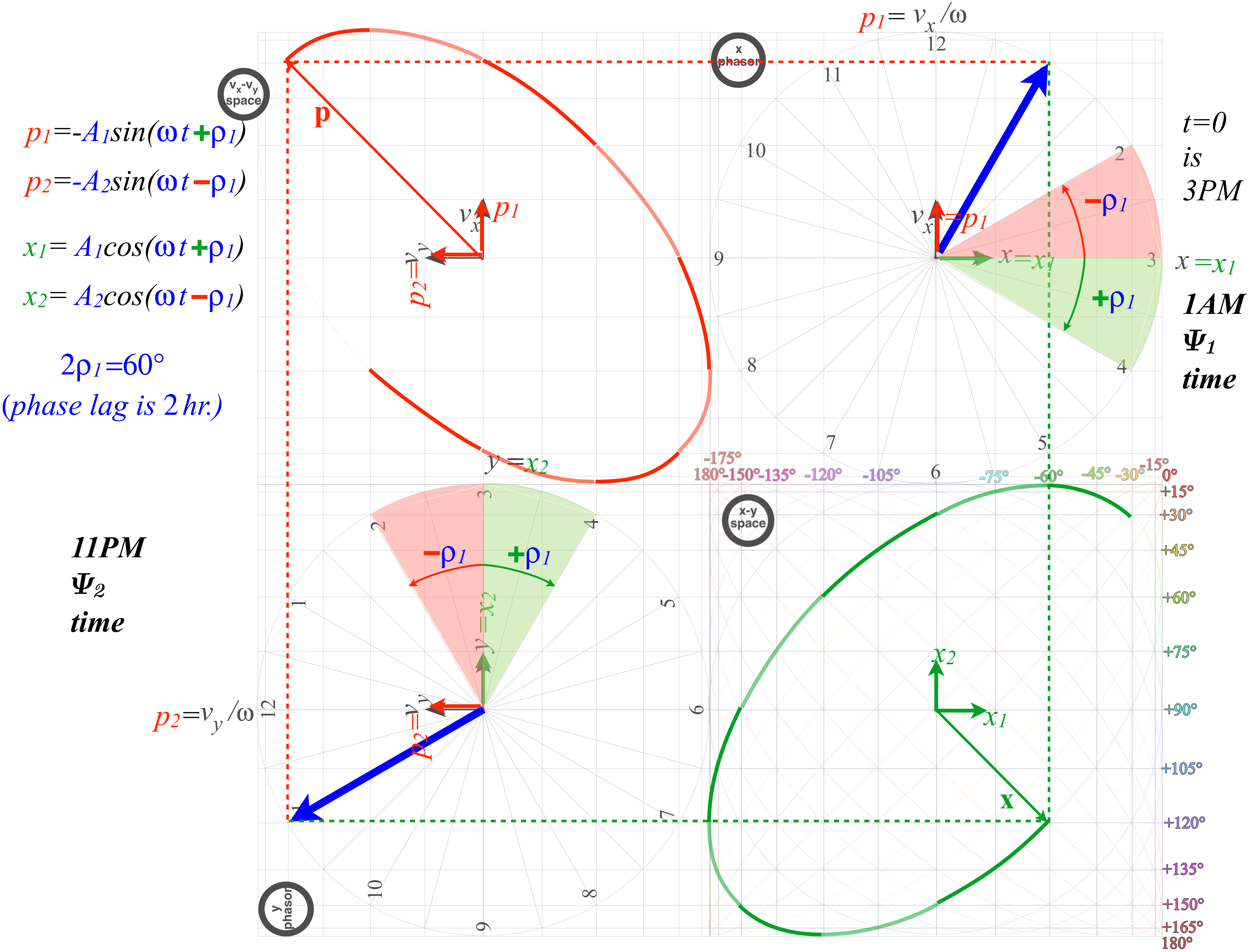
$2\rho_1=60°$

(phase lag is 2 hr.)

t=0 is 3PM

12PM* Ψ_1 *time

10PM* Ψ_2 *time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2 hr.)

t=0 is 3PM

1AM ***Ψ_1*** ***time***

11PM ***Ψ_2*** ***time***

p_1=-A_1sin(ωt+ρ_1)

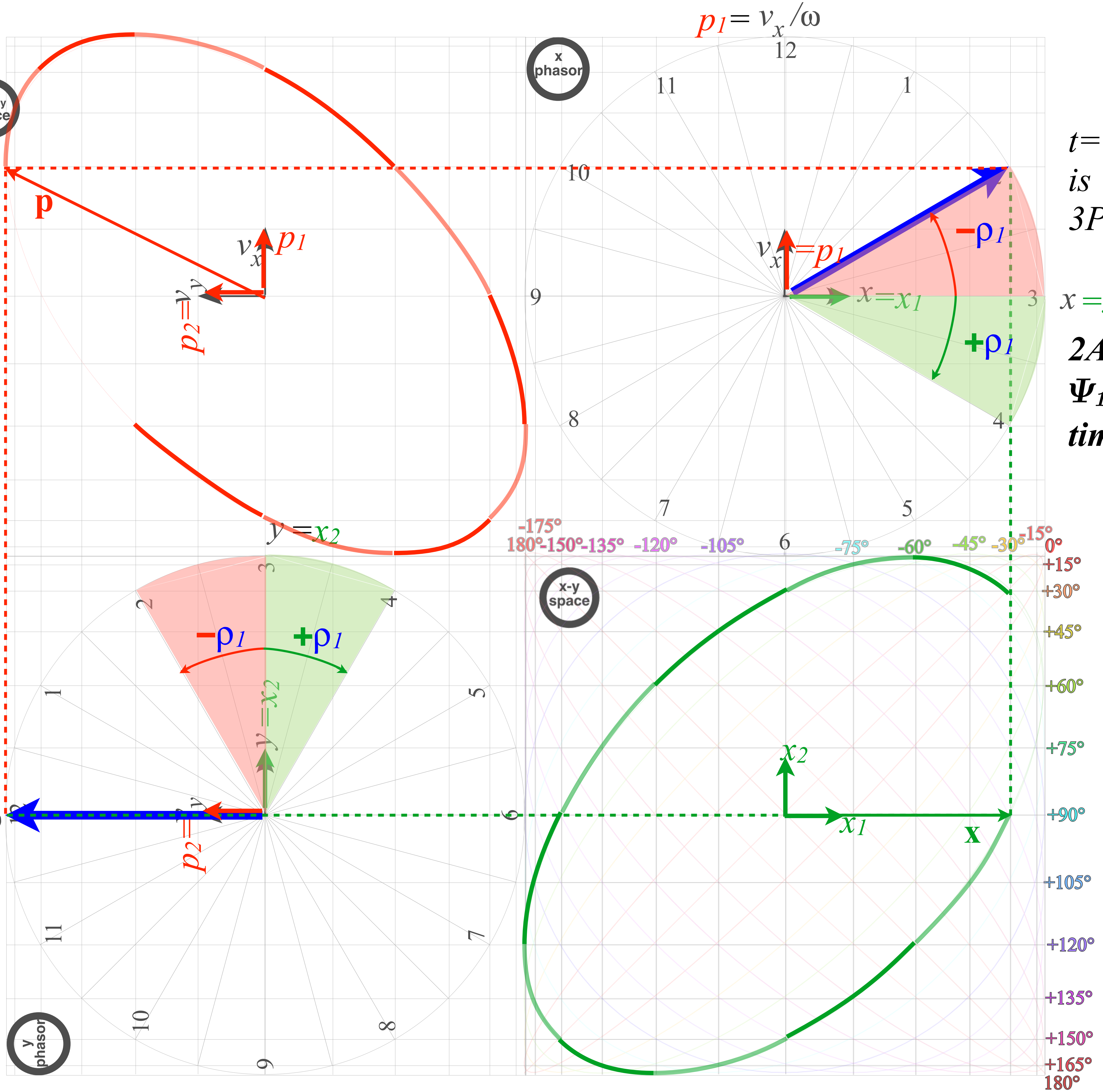
p_2=-A_2sin(ωt−ρ_1)

x_1= A_1cos(ωt+ρ_1)

x_2= A_2cos(ωt−ρ_1)

2ρ_1=60°

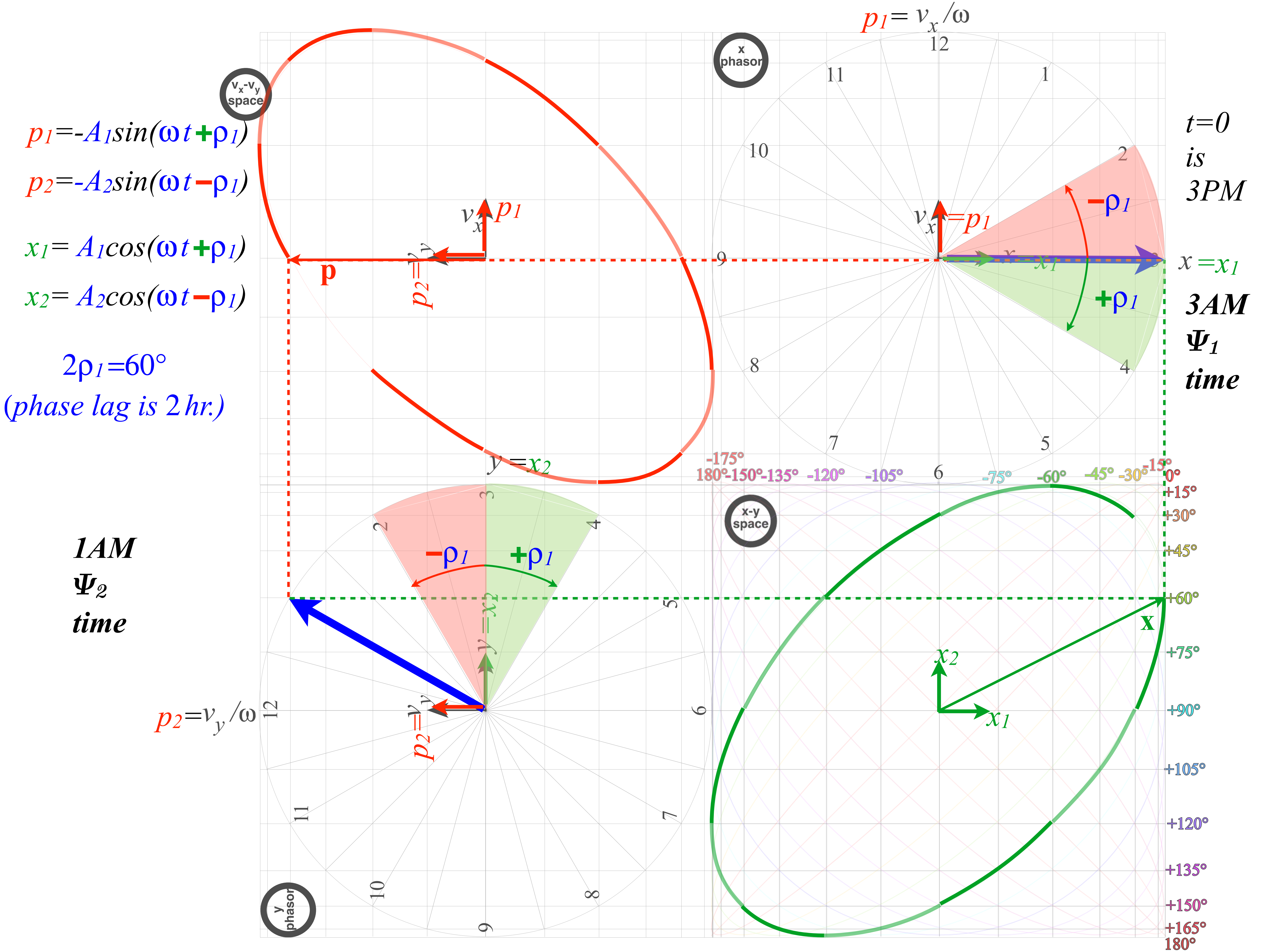
(phase lag is 2 hr.)



t=0
is
3PM

2AM
Ψ_1
time

12PM
Ψ_2
time



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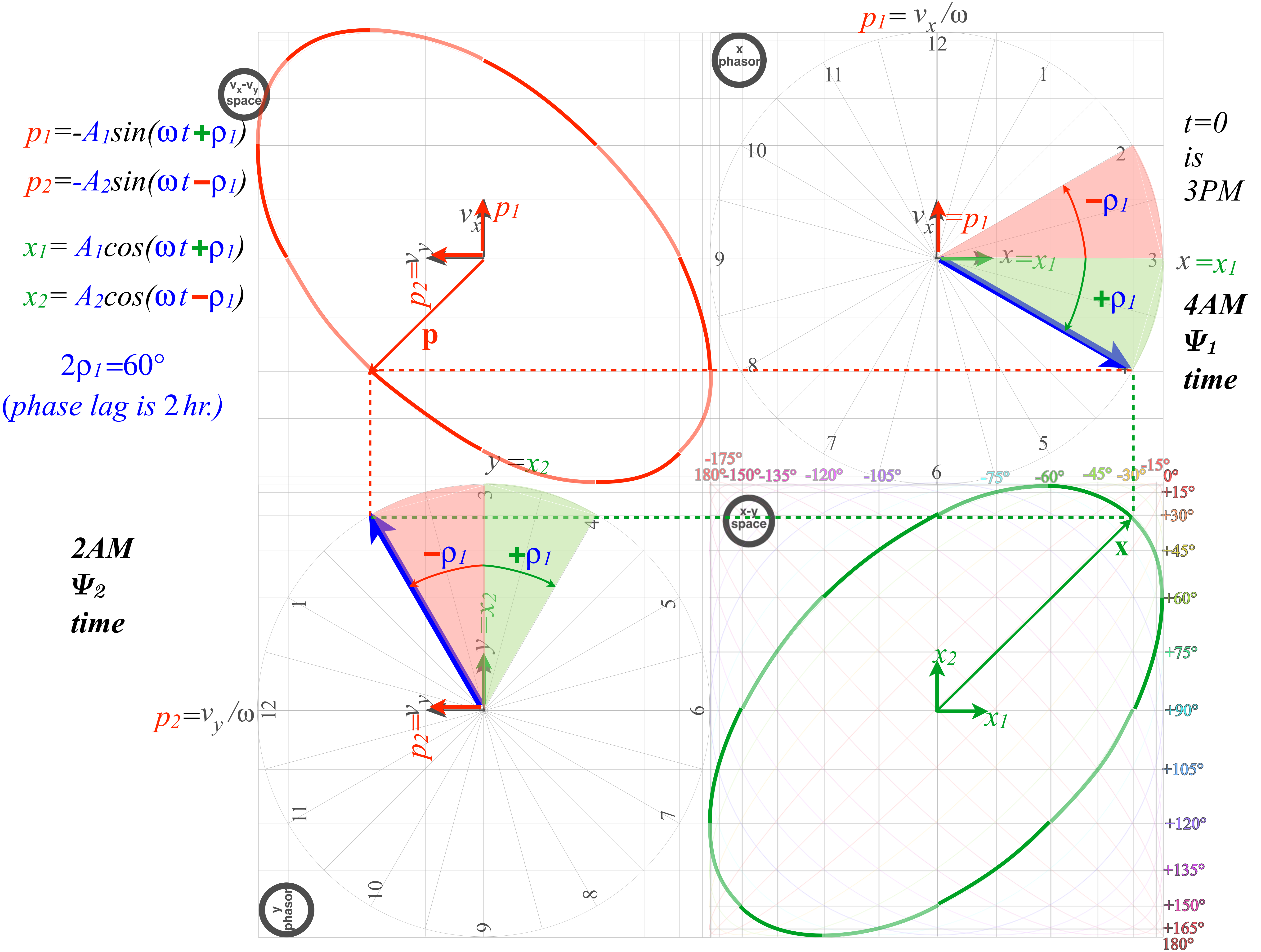
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2 hr.)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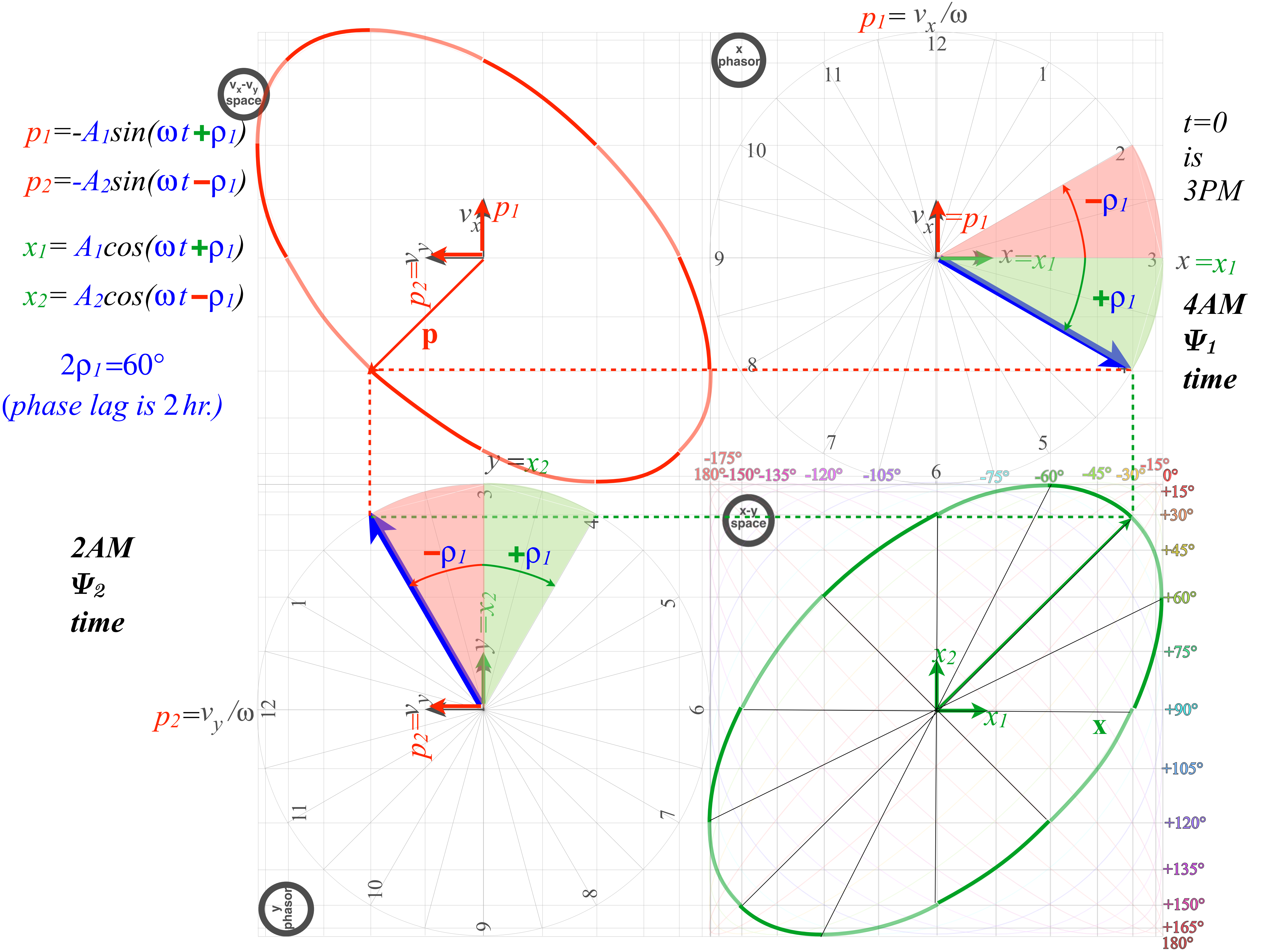
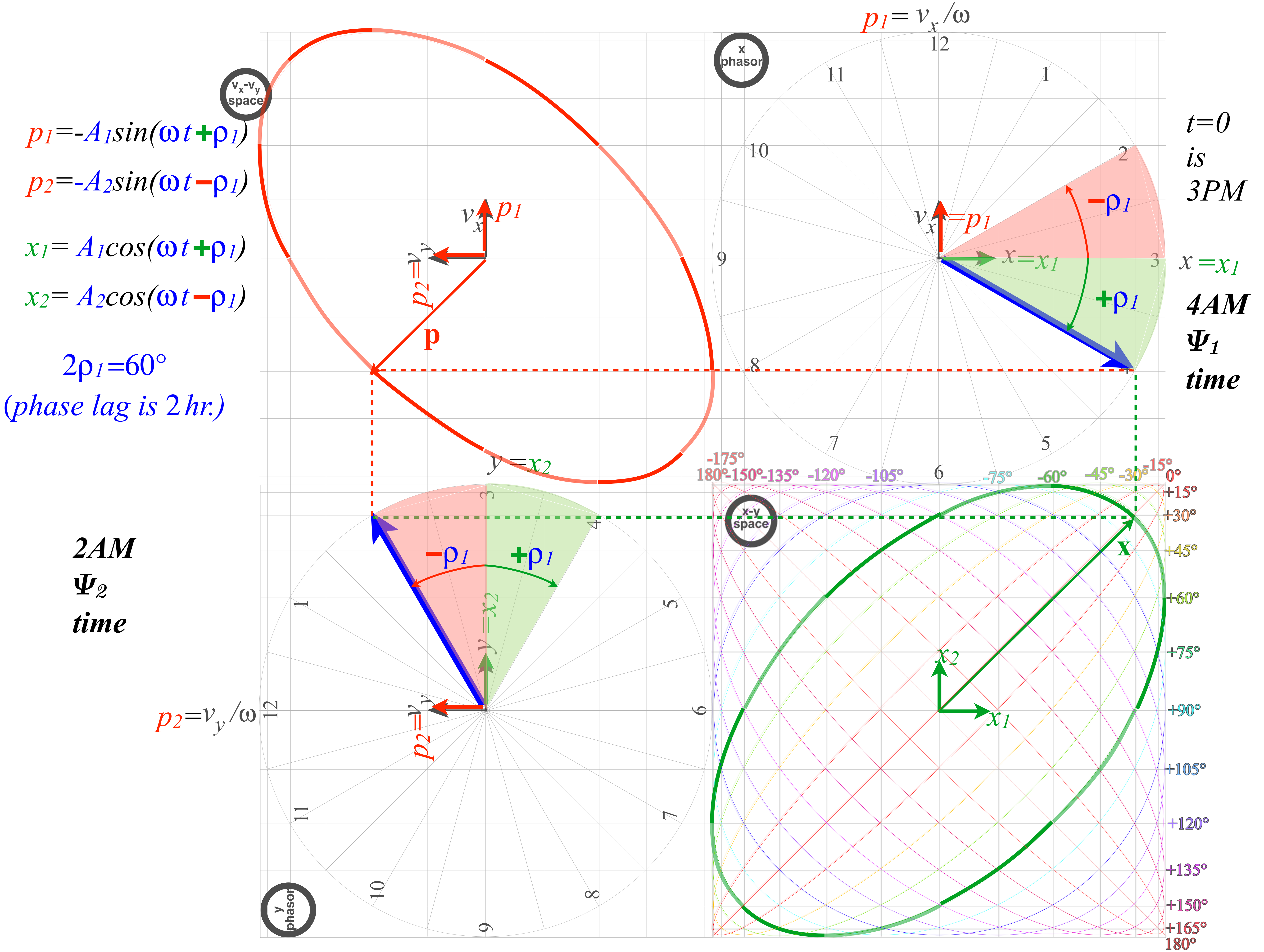
$x_1= A_1\cos(\omega t+\rho_1)$
$x_2= A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2 hr.)
v_x-v_y space
x phasor
y phasor
x-y space
$p_1= v_x/\omega$
$p_2=v_y/\omega$
v_x
p_1
$v_x=p_1$
$p_2=v_y$
p
$y=x_2$
$x=x_1$
$-\rho_1$
$+\rho_1$
t=0
is
3PM
4AM
Ψ_1
time
2AM
Ψ_2
time
x
x_1
x_2
12
1
2
3
4
5
6
7
8
9
10
11
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



$p_1=-A_1\sin(\omega t+\rho_1)$
$p_2=-A_2\sin(\omega t-\rho_1)$
$x_1= A_1\cos(\omega t+\rho_1)$
$x_2= A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2 hr.)
v_x-v_y space
x phasor
y phasor
x-y space
$p_1= v_x/\omega$
$p_2=v_y/\omega$
t=0 is 3PM
4AM
Ψ_1
time
2AM
Ψ_2
time
v_x p_1
$p_2=v_y$
p
$y=x_2$
$v_x=p_1$
$x=x_1$
$x=x_1$
$-\rho_1$
$+\rho_1$
$-\rho_1$
$+\rho_1$
$y=x_2$
x_2
x_1
x
12
11
1
10
2
9
3
8
7
5
6
4
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2 hr.)

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles ($\alpha\beta\gamma$) and A.

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$

$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$

$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$

$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles (αβγ)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles (αβγ) and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

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$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

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$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

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$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$\tan\beta/2 = A_2/A_1$ $\quad A^2 = A_1^2+A_2^2$ $\quad \alpha = 2\rho_1$ $\quad \gamma = 2\omega\cdot t$

Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* $(A_1,A_2, \omega t,\rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

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The ABC's of U(2) dynamics-Archetypes

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- *ABC-Type elliptical polarized motion*

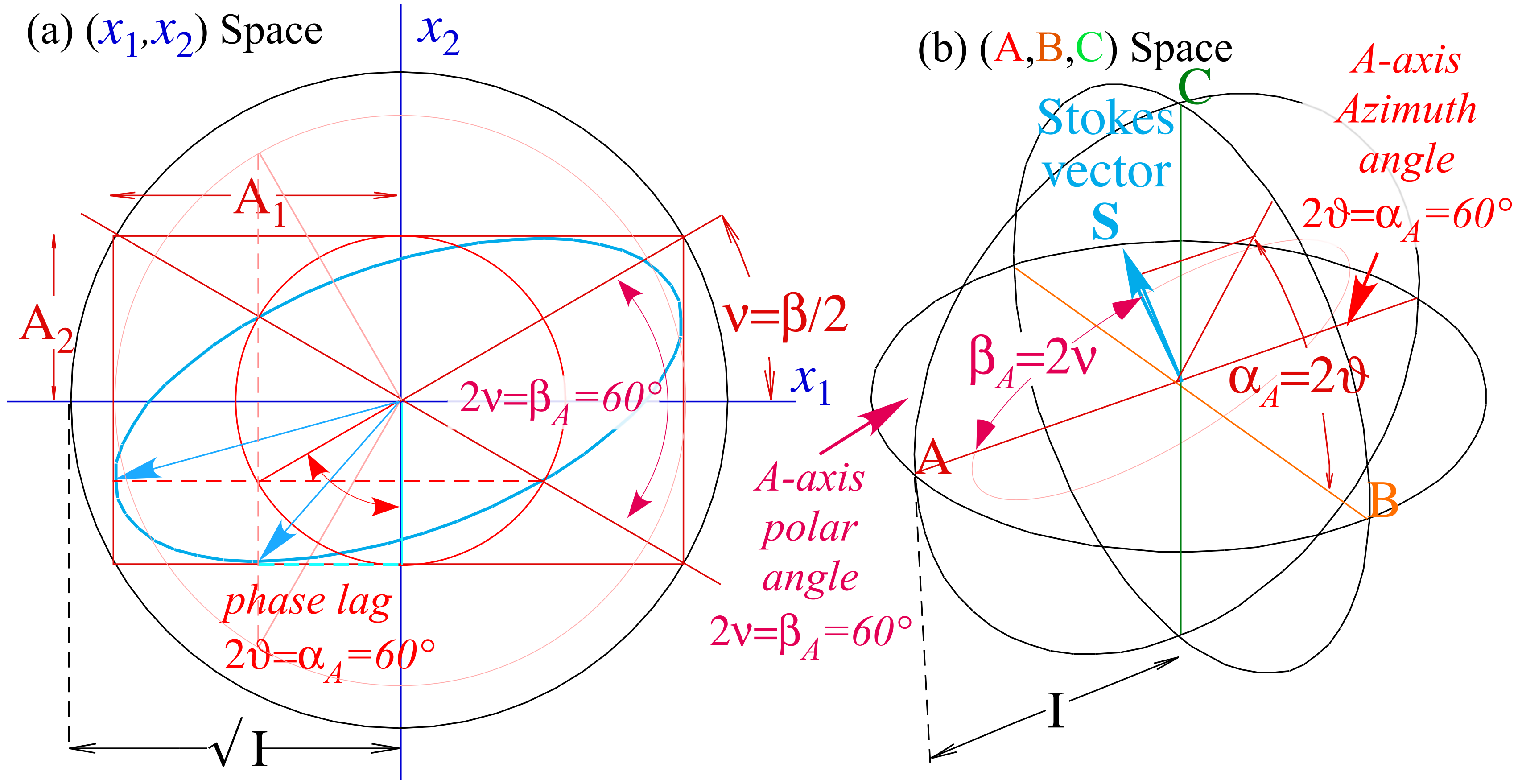
Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

The A-view in {x1,x2}-basis

Angles $\alpha_A=\rho_1-\rho_2=2\rho_1$, $\beta_A=2\tan^{-1}A_2/A_1$, $\gamma_A=2\omega\cdot t$ define ellipses with intensity $I=A^2=A_1^2+A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$



A or Z-axis Euler angles

$\alpha=\alpha_A=\rho_1-\rho_2=2\rho_1=60°$

$\beta=\beta_A=2\tan^{-1}A_2/A_1=60°$

$\gamma_A=2\omega\cdot t$

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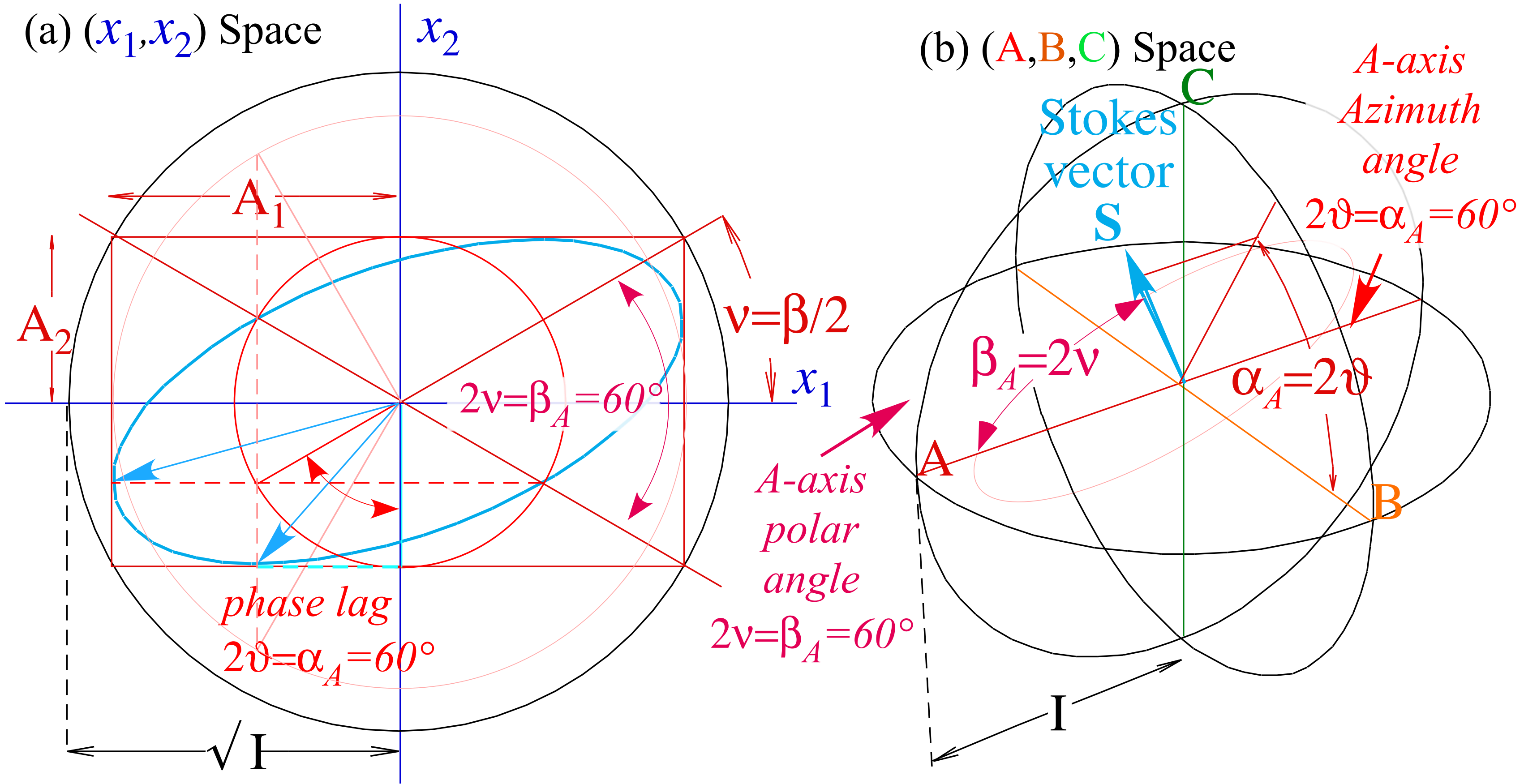

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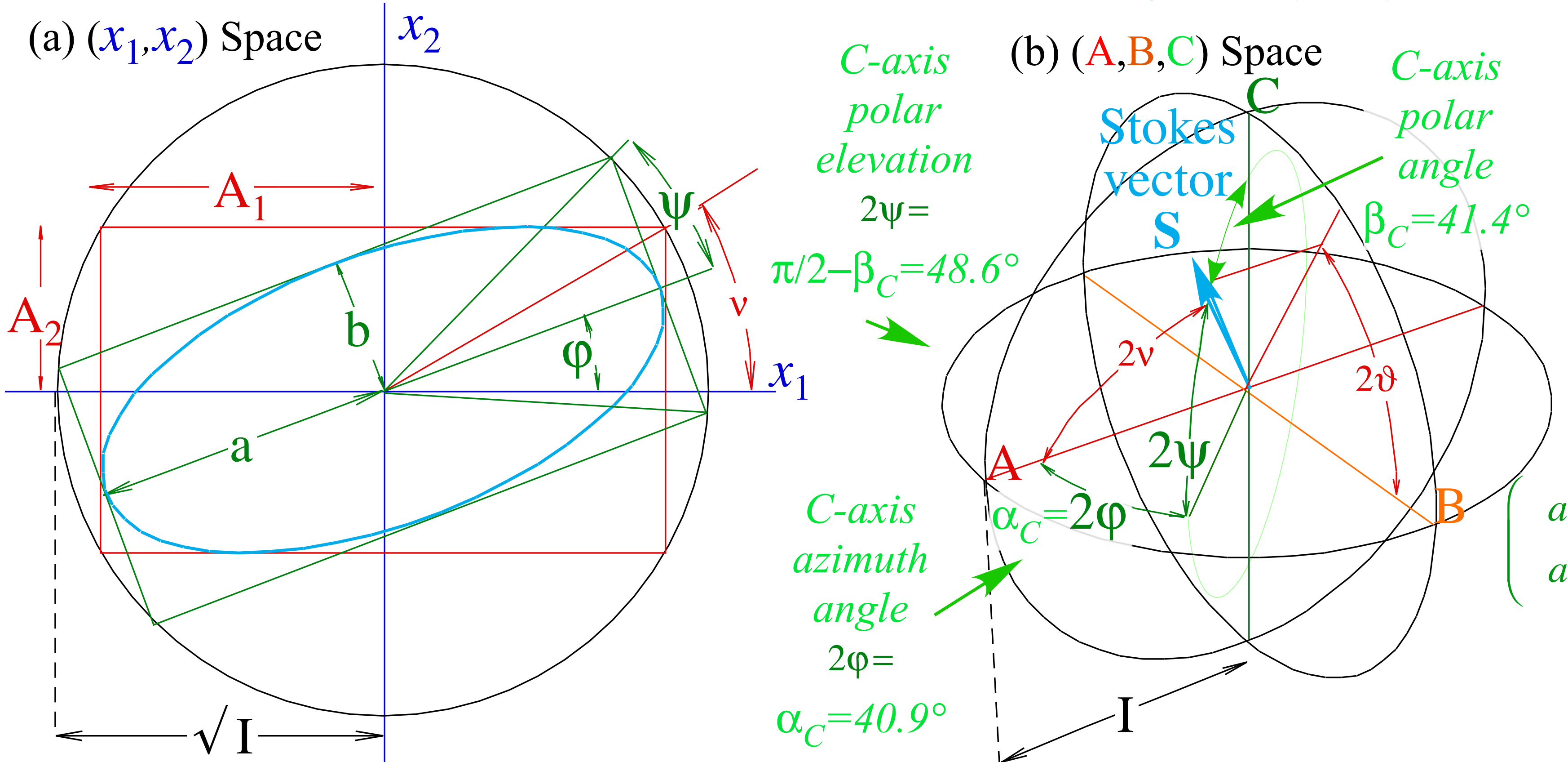
$\gamma_A=2\omega\cdot t$

The C-view in {xR,xL}-basis

The same orbit viewed in right-left $\{x_R,x_L\}$-basis of circular polarization with angles ($\alpha_C,\beta_C,\gamma_C$).



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R+ip_R \\ x_R+ip_R \end{pmatrix}$$

Converting an *A*-based set of Stokes parameters into a *C*-based set or a *B*-based set involves cyclic permutation of *A*, *B*, and *C* polar formulas

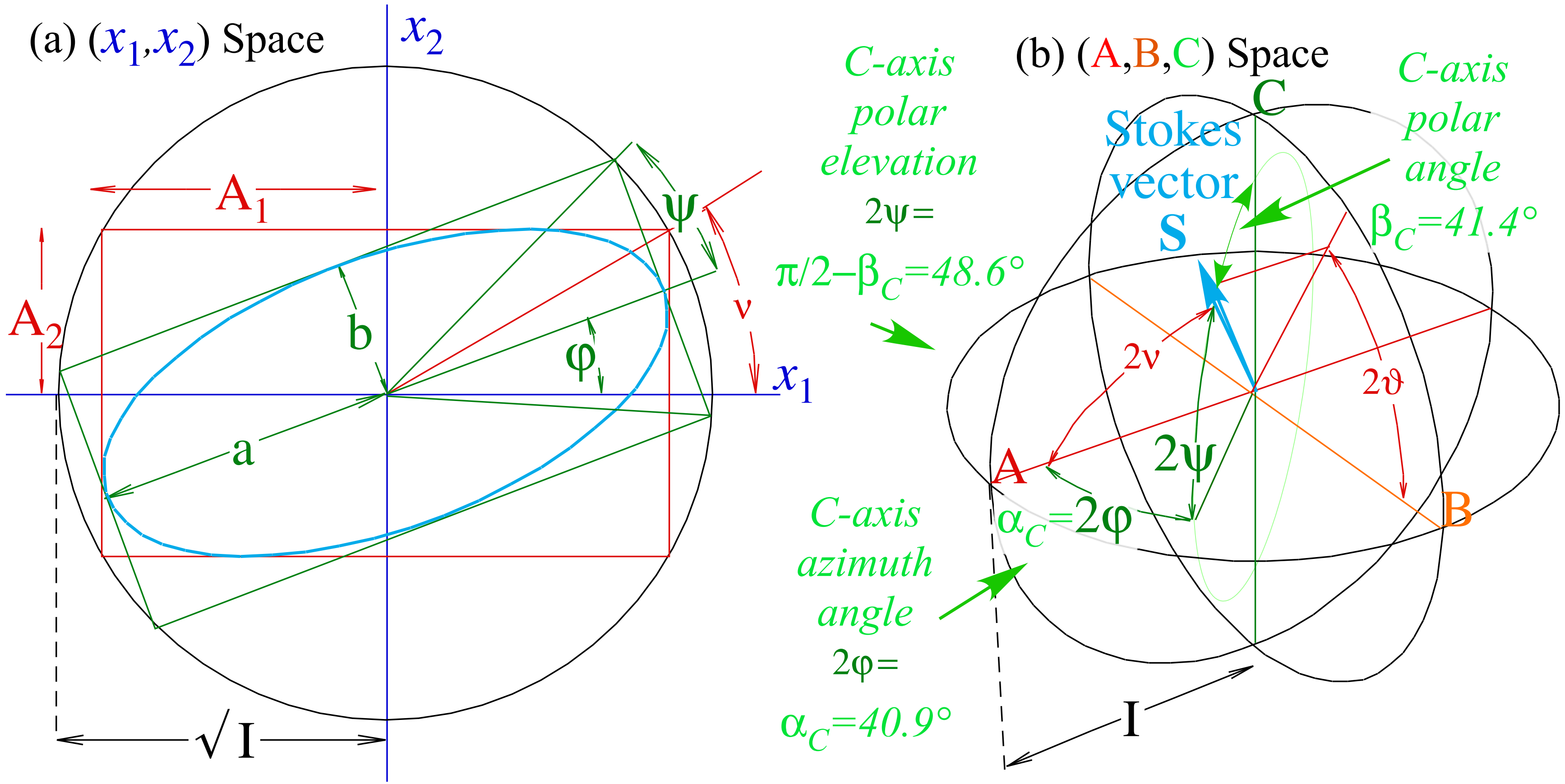
$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R, x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.



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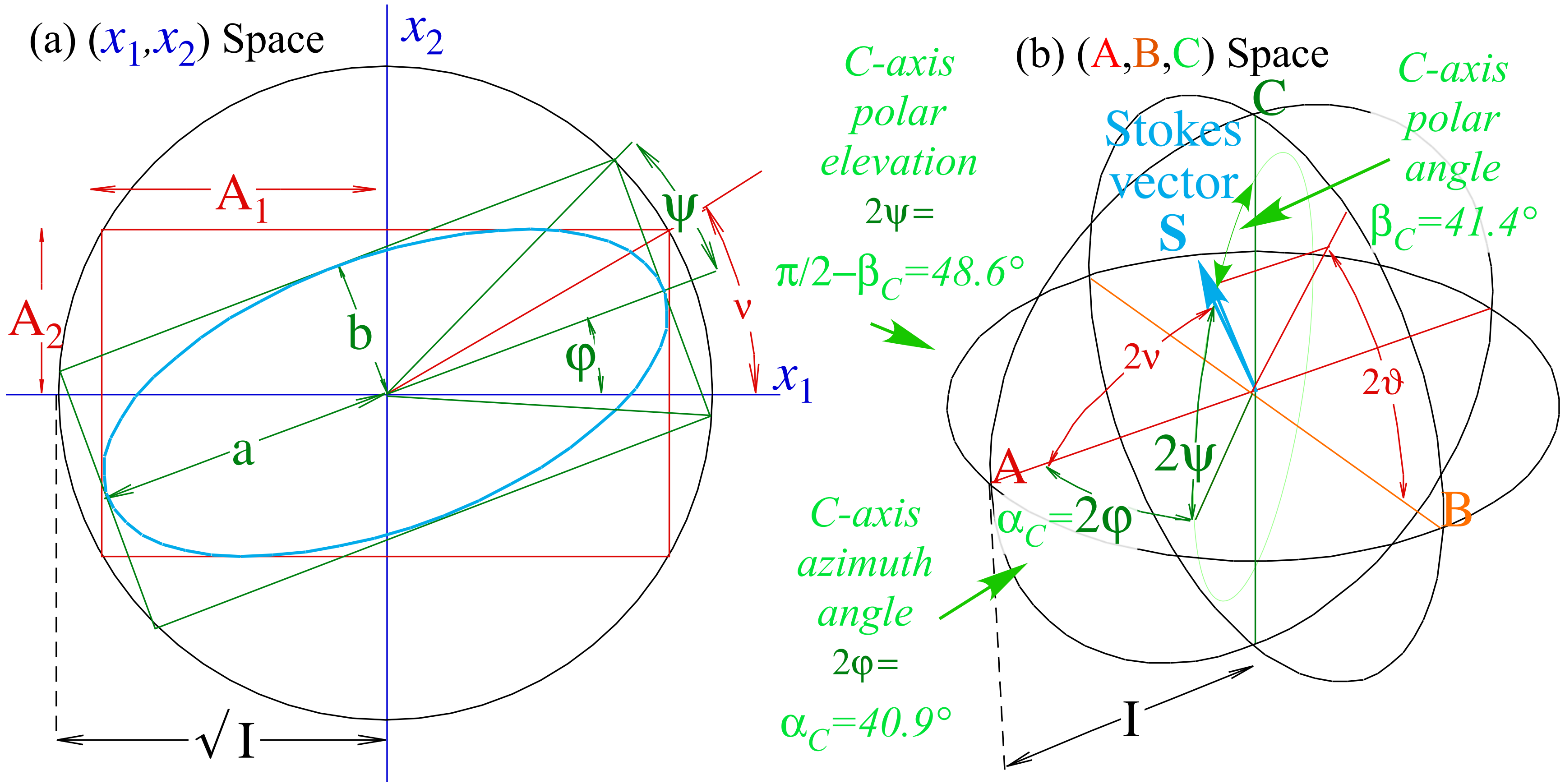
Chirality $$S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C*-view in* $\{x_R, x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

Angles (α_C, β_C): *C*-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$



Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

Asymmetry $$S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$$

Balance $$S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$$

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The C*-view in* $\{x_R, x_L\}$*-basis*

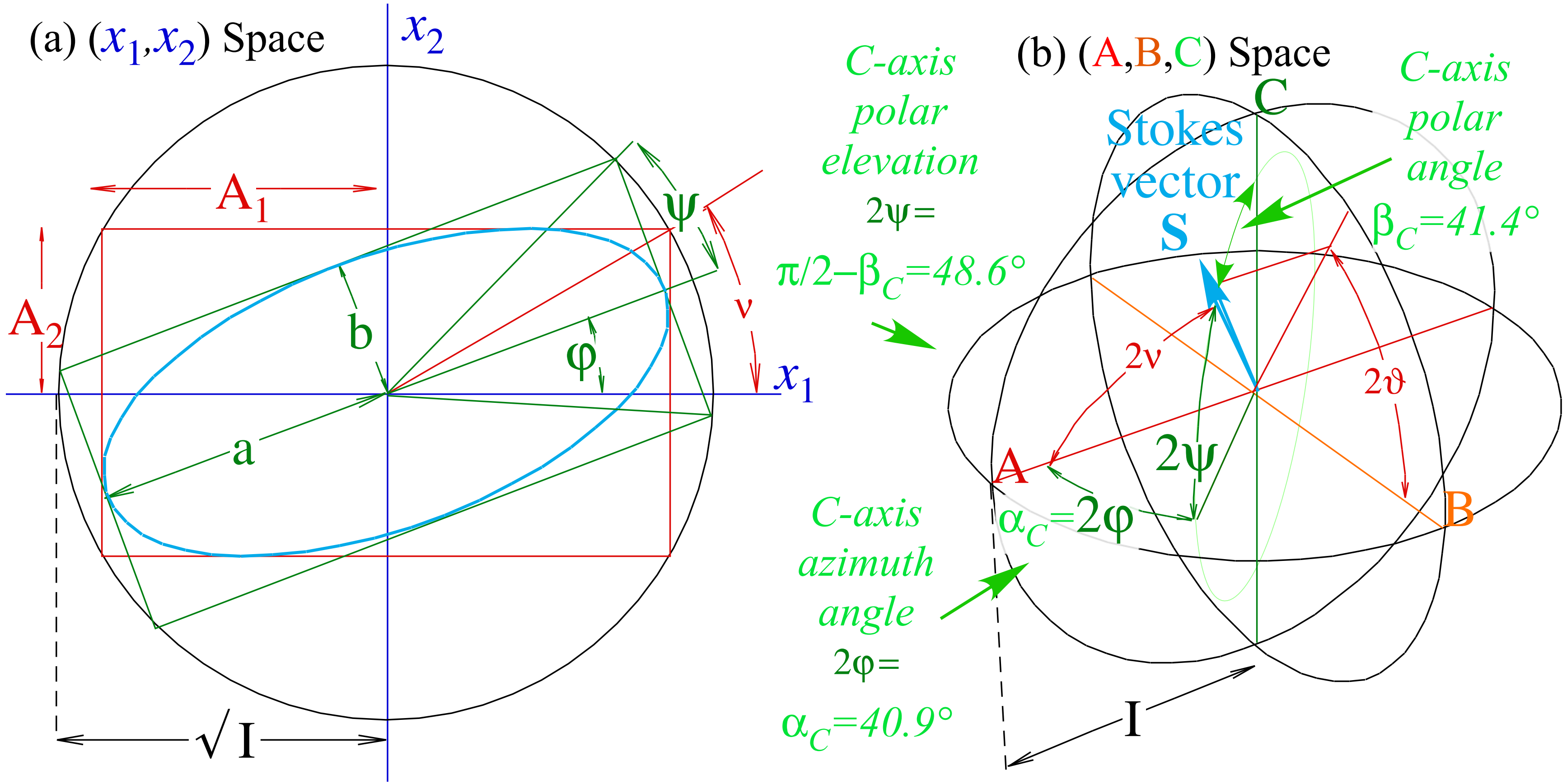
The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

Angles (α_C, β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$

C-axis azimuth angle α_C relates to A-axis angles α_A and β_A . See $\alpha_C = 2\varphi$ below.

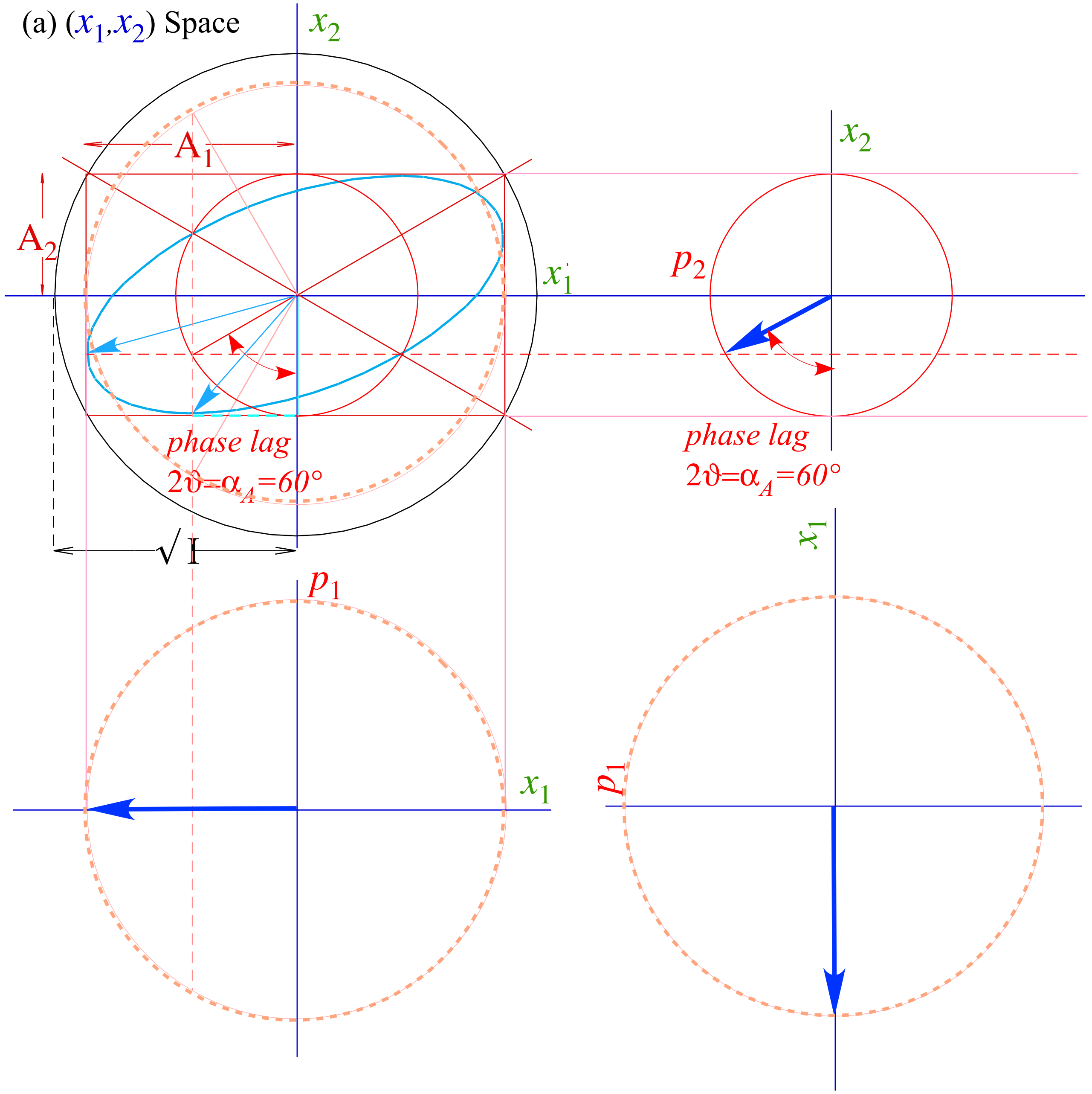
$$\frac{\cos\alpha_A \sin\beta_A}{\cos\beta_A} = \tan\alpha_C \qquad \text{or:} \quad \alpha_C = ATN2(\cos\alpha_A \sin\beta_A / \cos\beta_A) = ATN2\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2} / \frac{1}{2}\right) = 40.9°$$

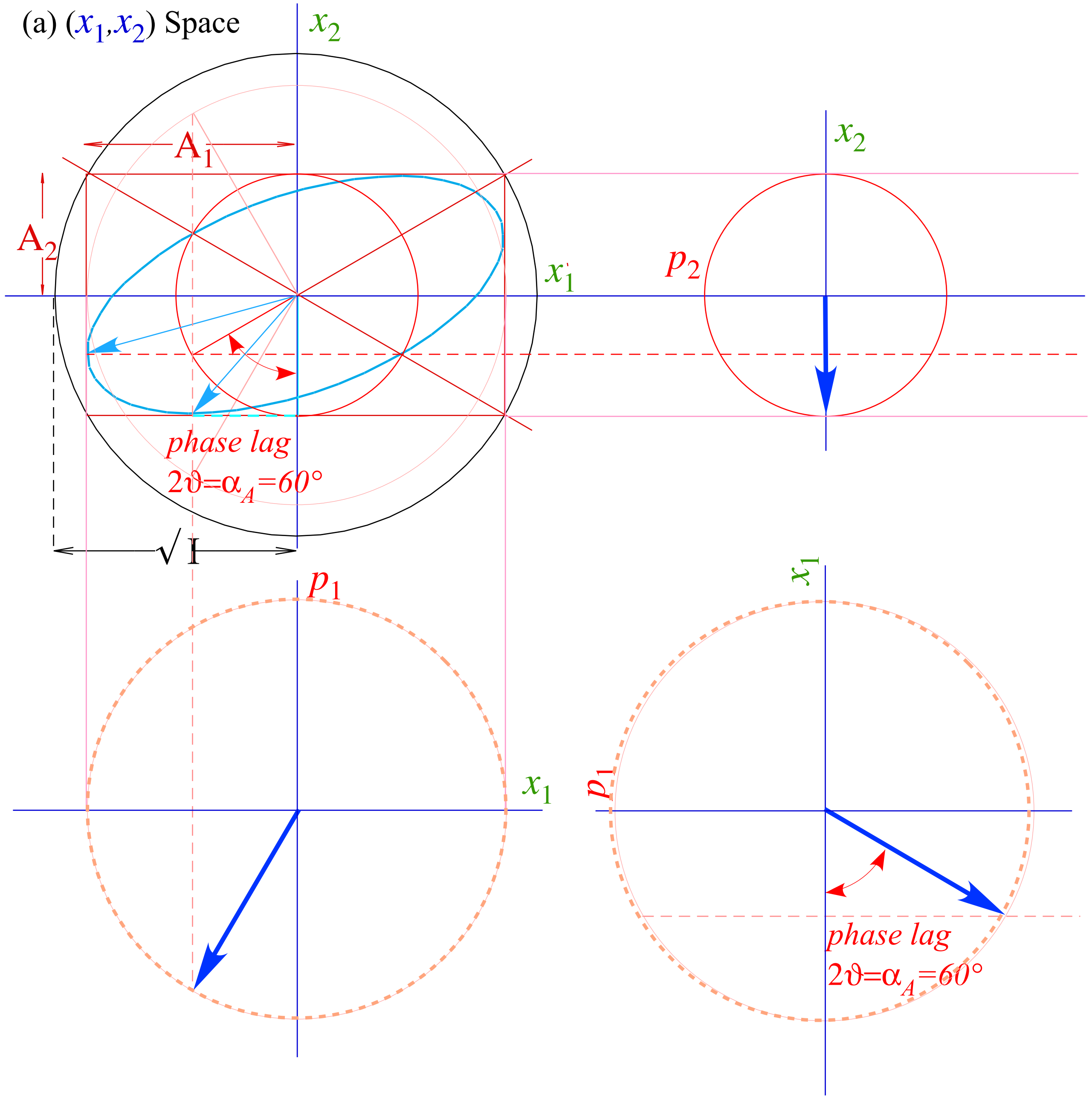


C-axis polar elevation 2ψ= π/2−β_C=48.6°

C-axis polar angle β_C=41.4°

C-axis azimuth angle 2φ= α_C=40.9°


(a) (x_1,x_2) Space
x_2
A_1
A_2
x_1
x_2
p_2
phase lag
$2\vartheta=\alpha_A=60°$
phase lag
$2\vartheta=\alpha_A=60°$
x_1
$\sqrt{I}$
p_1
x_1
p_1


(a) (x_1,x_2) Space
x_2
A_1
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x_1
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x_1
p_1
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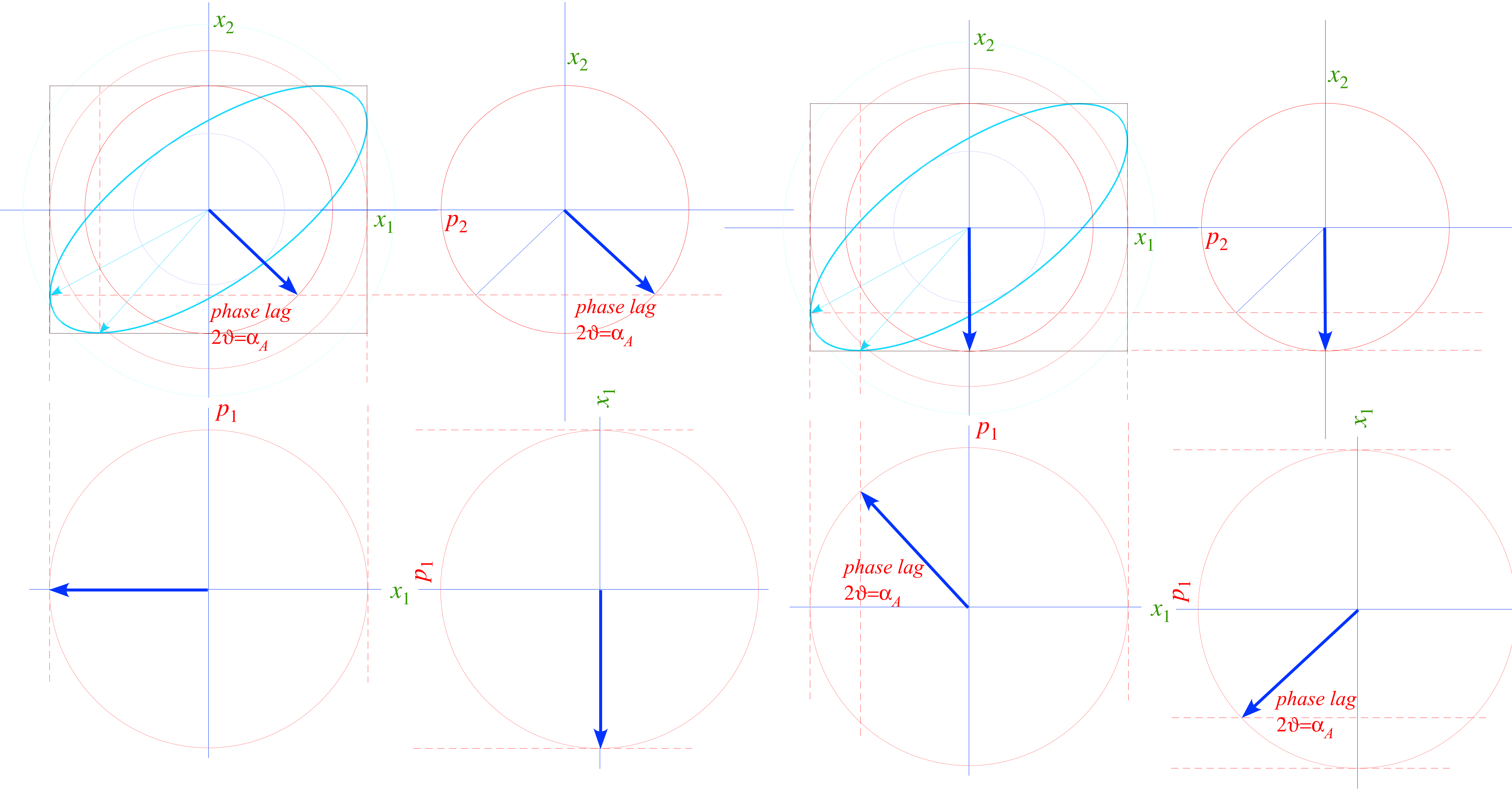
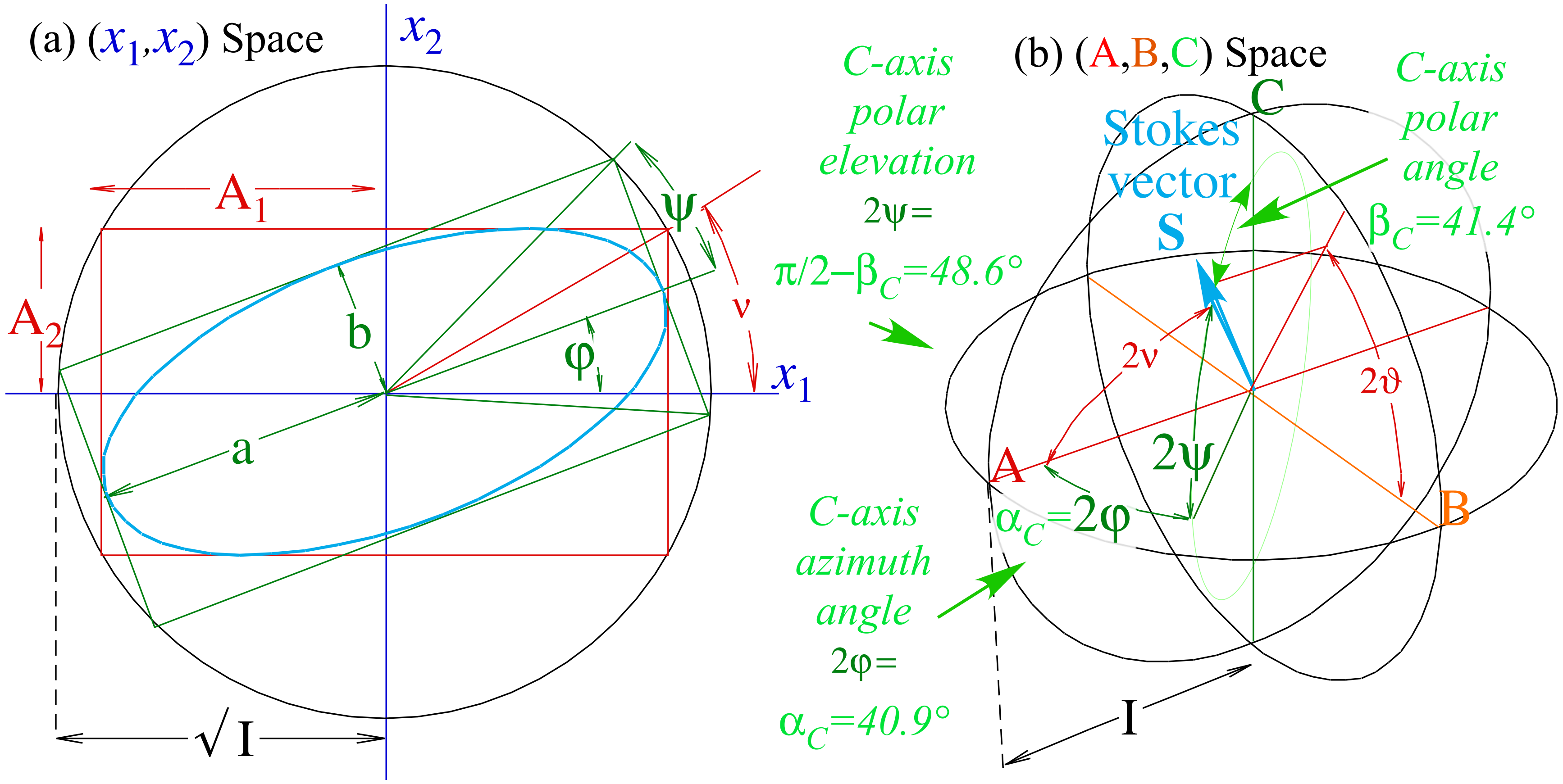

x_2
x_1
p_2
x_2
phase lag
$2\vartheta=\alpha_A$
phase lag
$2\vartheta=\alpha_A$
p_1
x_1
x_1
p_1
x_2
x_1
p_2
x_2
p_1
phase lag
$2\vartheta=\alpha_A$
x_1
x_1
p_1
phase lag
$2\vartheta=\alpha_A$

The C-view in $\{x_R, x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



A 90° *B* –rotation $\mathbf{R}(\pi/4)\left|x_1\right\rangle = \left|x_R\right\rangle$ of axis *A* into *C* gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_L + ip_L \end{pmatrix}$$

Polarization ellipse and spinor state dynamics

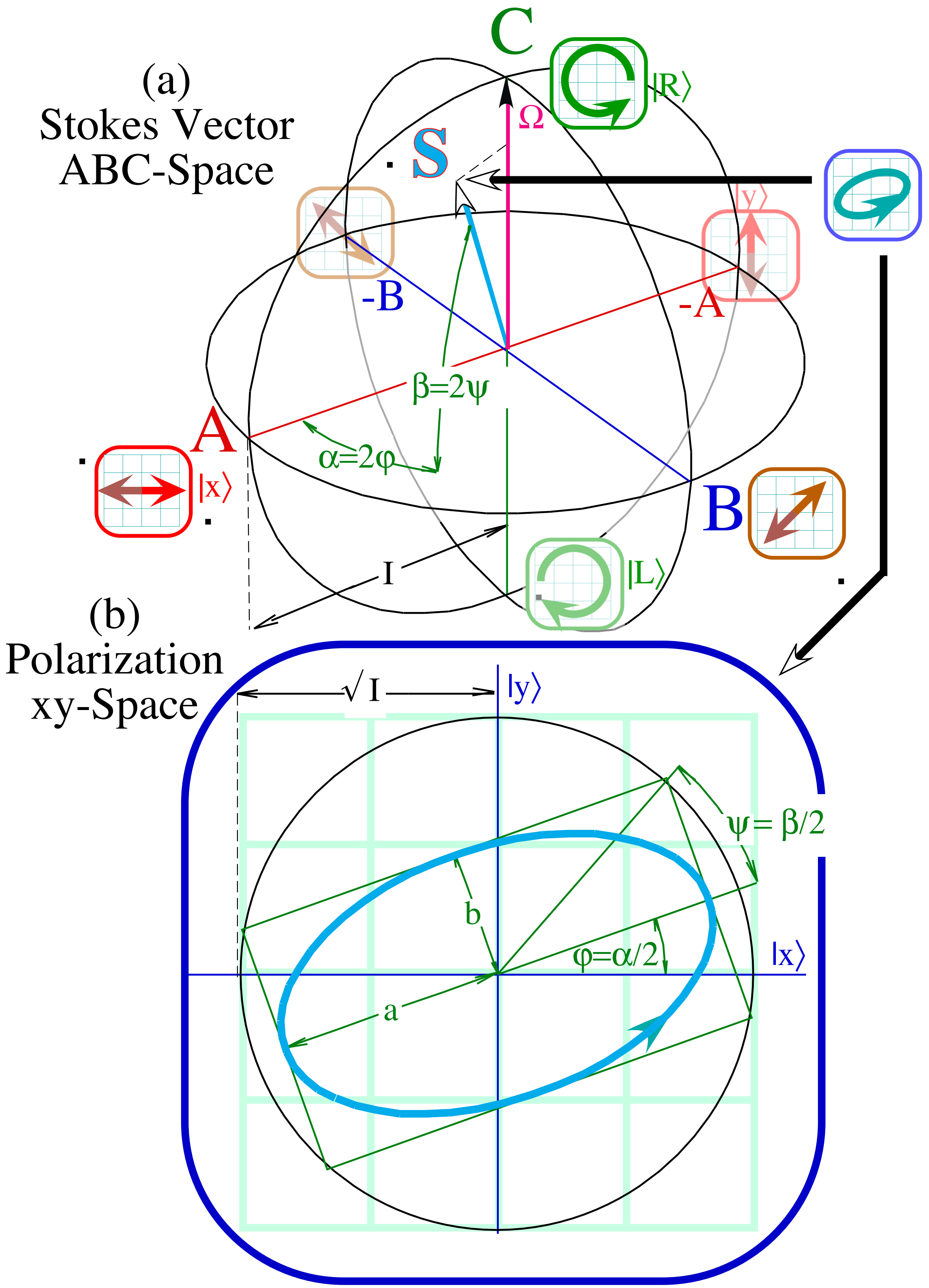


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

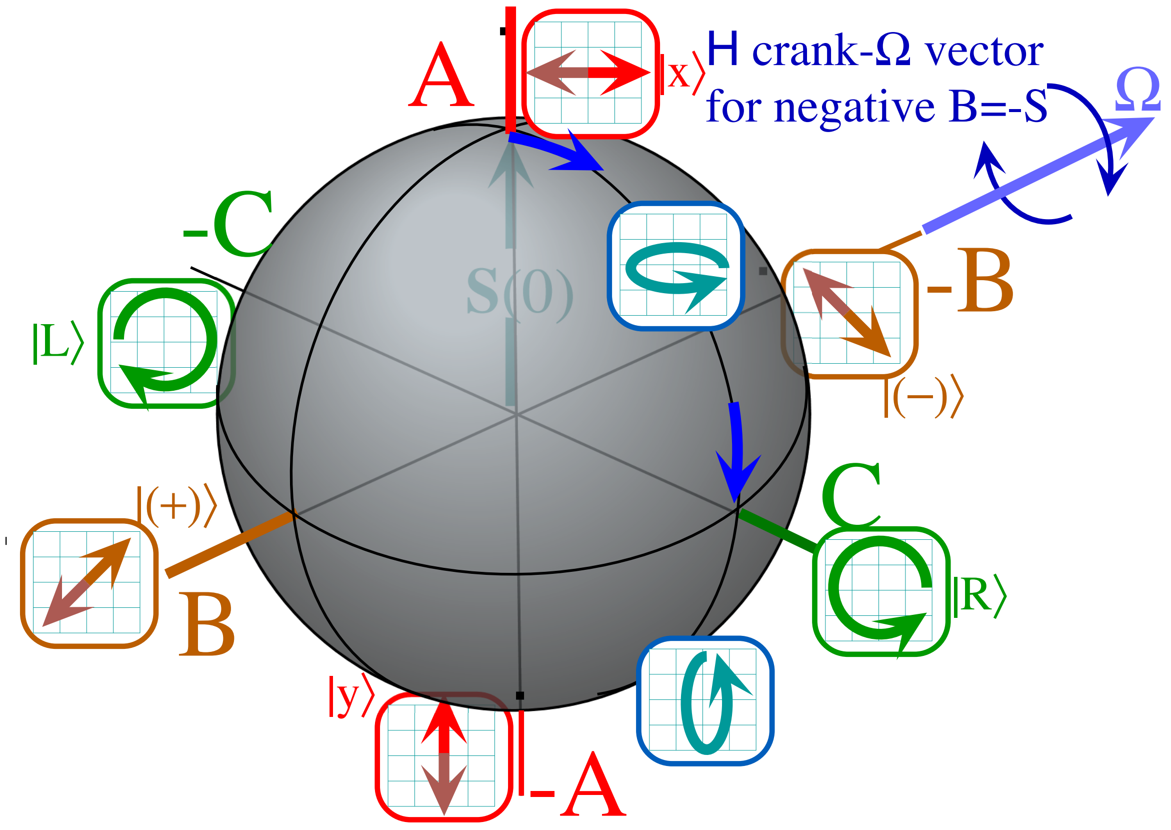
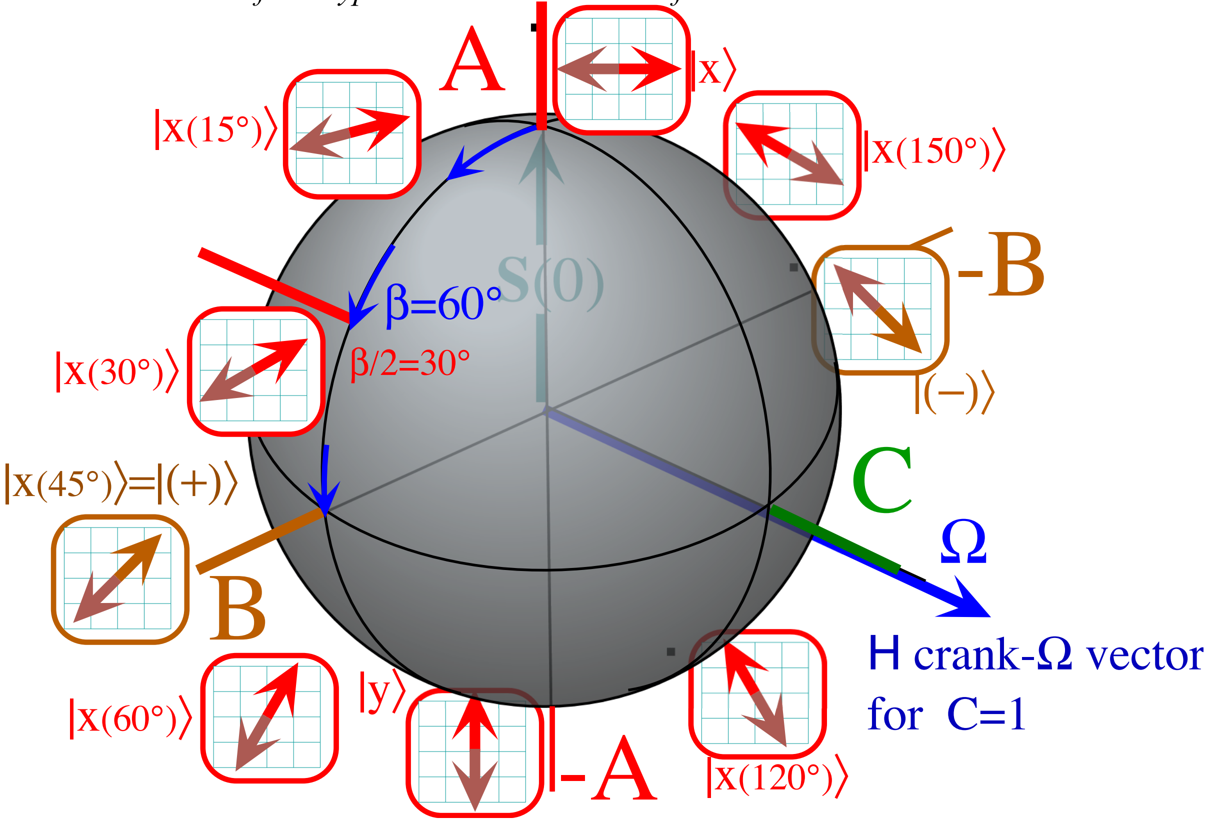


Fig. 10.5.5 Time evolution of a B-type beat. **S**-*vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S**-*vector rotates from A to B to -A to -B and back to A.*



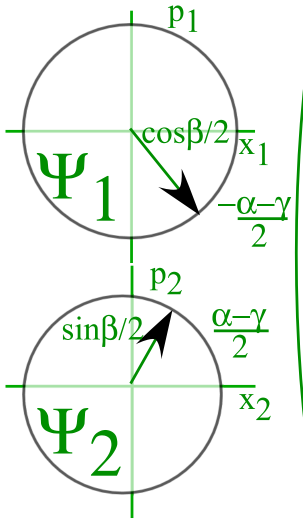
U(2) World : Complex 2D Spinors

Ellipsometry of U(2) states described by Two "Worlds" U(2) or R(3)

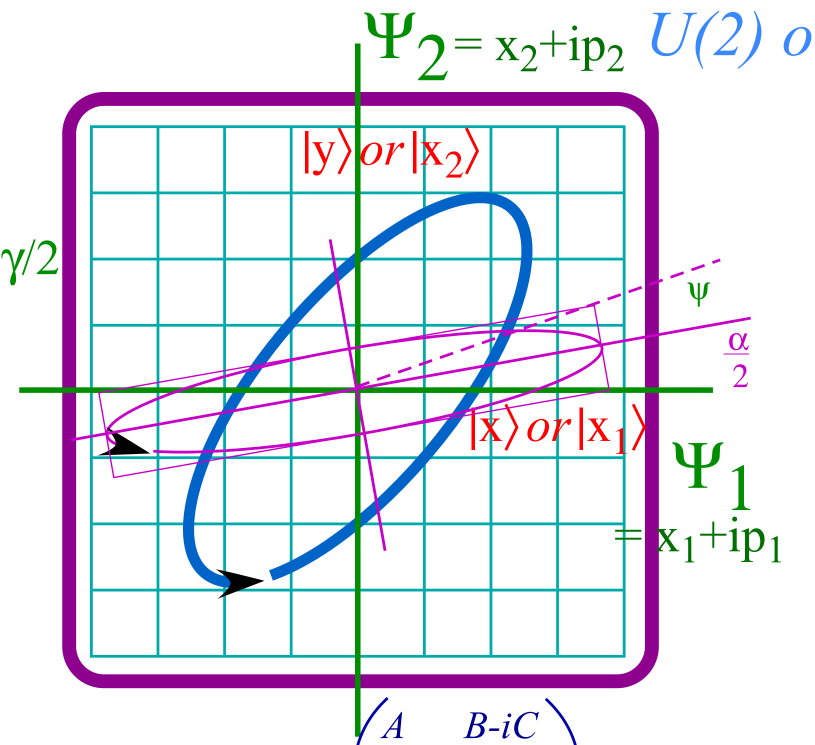
U(2) World labeled by two complex phasors and driven by complex operator

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

2-State ket $|\Psi\rangle$=



$$\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2 \\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State Spin Vector S

$$\begin{pmatrix}S_B \\ S_C \\ S_A\end{pmatrix} = \begin{pmatrix}N\sin\beta\cos\alpha \\ N\sin\beta\sin\alpha \\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

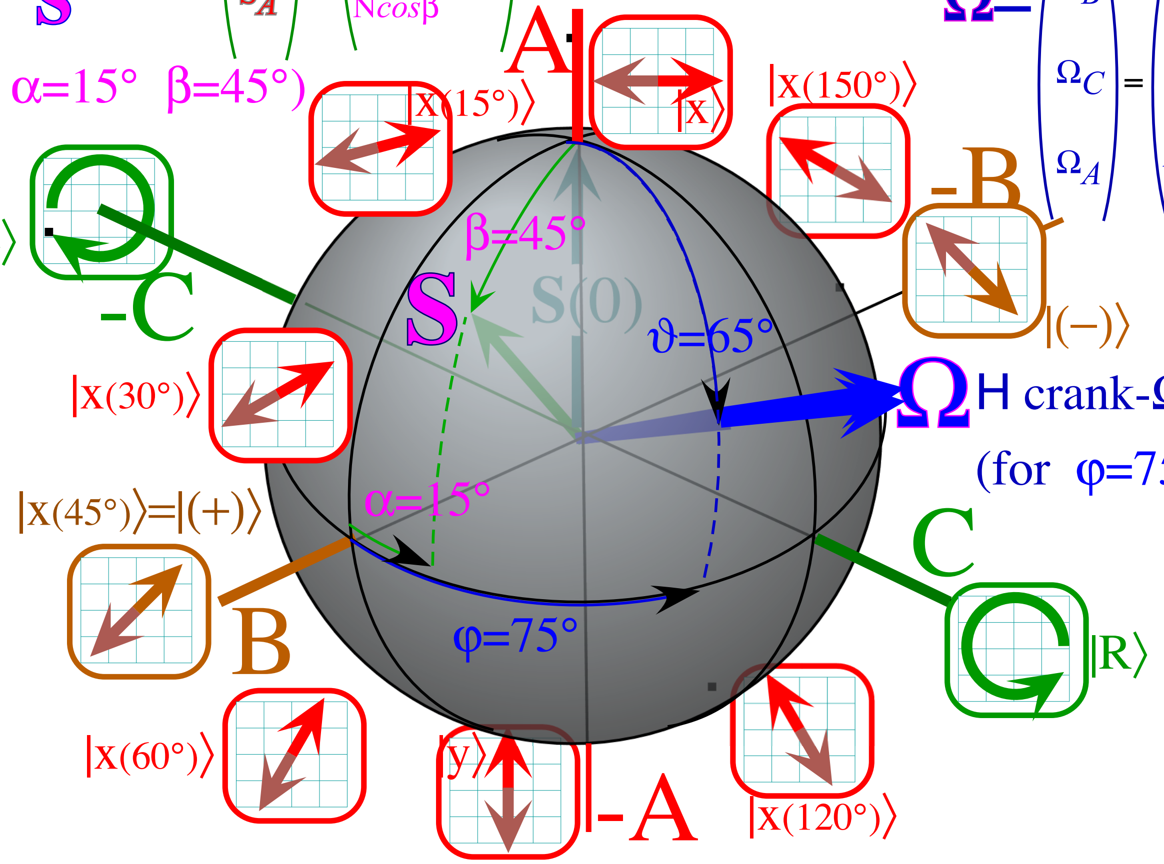
(for α=15° β=45°)

H-Operator $\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

Angular velocity

$$\mathbf{\Omega}=\begin{pmatrix}\Omega_B \\ \Omega_C \\ \Omega_A\end{pmatrix} = \begin{pmatrix}2B \\ 2C \\ A-D\end{pmatrix} = \begin{pmatrix}\Omega\sin\vartheta\cos\varphi \\ \Omega\sin\vartheta\sin\varphi \\ \Omega\cos\vartheta\end{pmatrix}$$

R(3) World labeled by real 3-D "spin" vector **S** *of angular momentum and driven by real 3-D "spin" vector* **Ω** *of angular velocity*



H crank-Ω vector

(for φ=75° ϑ=65°)

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

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- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

- *Bloch equation for density operator*

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

Euler phase-angle coordinates (α,β,γ)
and ***norm*** *N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

U(2) density operator approach to symmetry dynamics

x_1=cos[(γ+α)/2]cosβ/2
p_1=-sin[(γ+α)/2]cosβ/2

x_2=cos[(γ–α)/2]sinβ/2
p_2=-sin[(γ–α)/2]sinβ/2

Euler phase-angle coordinates (α, β, γ)
and ***norm*** *N of quantum state* |Ψ⟩

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

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scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\,\sin\beta$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

Euler phase-angle coordinates (α,β,γ)
and norm N of quantum state $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2\quad=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$:

$$S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2\quad=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\sin\alpha\sin\beta$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ)

and norm N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$

$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$

$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ***-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$:

$$S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$

$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$

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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad =N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2 \quad =N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$ $\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}$$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

...2-by-2 *density operator* ρ

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$

$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac12$: $\quad\frac12\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac12$: $\quad S_Z=S_A=\frac12\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac12$: $\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad =N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac12$: $\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2 \quad =N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac12 N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac12 N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\underset{\uparrow\ \rho}{\begin{pmatrix}\frac12 N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac12 N-S_Z\end{pmatrix}}=\frac12 N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\begin{pmatrix}0&1\\1&0\end{pmatrix}+S_Y\begin{pmatrix}0&-i\\i&0\end{pmatrix}+S_Z\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$\leftarrow$ ...so state *density operator* $\boldsymbol{\rho}$ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac12$: $\frac12\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac12$: $S_Z=S_A=\frac12\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac12$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac12$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2 \quad = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$
$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac12 N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac12 N-S_Z$

Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

$$=\begin{pmatrix}\frac12 N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac12 N-S_Z\end{pmatrix}=\frac12 N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\uparrow\ \rho \quad =\frac12 N\ \mathbf{1}\ +S_X\ \boldsymbol{\sigma}_X\ +S_Y\ \boldsymbol{\sigma}_Y\ +S_Z\ \boldsymbol{\sigma}_Z\ =\frac12 N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ)
and norm N of quantum state $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}\quad \text{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\rho\qquad=\frac{1}{2}N\quad\mathbf{1}\quad+S_X\quad\sigma_X\quad+S_Y\quad\sigma_Y\quad+S_Z\quad\sigma_Z\quad=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

...so state *density operator* $\boldsymbol{\rho}$ has $\boldsymbol{\sigma}$-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$\mathbf{H}=\quad\omega_0\quad\sigma_0\quad+\frac{\Omega_A}{2}\quad\sigma_A\quad+\frac{\Omega_B}{2}\quad\sigma_B\quad+\frac{\Omega_C}{2}\quad\sigma_C\quad=\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$

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$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ***-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\quad\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* ρ = |Ψ⟩⟨Ψ|=

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{}$$

$$\uparrow\ \boldsymbol{\rho}\ =\frac{1}{2}N\ \mathbf{1}\ +S_X\ \boldsymbol{\sigma}_X\ +S_Y\ \boldsymbol{\sigma}_Y\ +S_Z\ \boldsymbol{\sigma}_Z\ =\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* **ρ** has **σ**-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{}$$

$$\mathbf{H}=\ \omega_0\ \sigma_0\ +\frac{\Omega_A}{2}\ \sigma_A\ +\frac{\Omega_B}{2}\ \sigma_B\ +\frac{\Omega_C}{2}\sigma_C\ =\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H}=\ \Omega_0\ \mathbf{1}\ +\Omega_A\ \mathbf{S}_A\ +\Omega_B\ \mathbf{S}_B\ +\Omega_C\mathbf{S}_C\ =\Omega_0\mathbf{1}+\vec{\Omega}\bullet\mathbf{S}$$

$$\boxed{\begin{aligned}\boldsymbol{\rho}&=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H}&=\Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S**-*vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$-*vector*
- *Euler-defined operator* **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$-*sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

- *Bloch equation for density operator*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \mathit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2} N \mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

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Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

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$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^\dagger = \mathbf{H}$.
$\boldsymbol{\rho}^\dagger = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \mathit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

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Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

Given ρ and **H** in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}}+\cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}}+\cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}}+\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}}+\cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}}+\cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}}+\frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* **S** and *crank* **Ω** *point in different directions*.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^\dagger = \mathbf{H}$.
$\rho^\dagger = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]}$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

This cancels *This remains*

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H}=\Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad \text{-}i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]}$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta}+i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$

$$= \mathbf{A}\bullet\mathbf{B} + i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\boldsymbol{\sigma}$$

This cancels

This remains

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}}\right)\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}\left(\vec{\mathbf{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle=\mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right|=\left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger}=\mathbf{H}$.
$\boldsymbol{\rho}^{\dagger}=\boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho}=|\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho}=i\hbar\dot{\boldsymbol{\rho}}=i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right|+i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right|=\mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right|-\left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho}=i\hbar\dot{\boldsymbol{\rho}}=\mathbf{H}\boldsymbol{\rho}-\boldsymbol{\rho}\mathbf{H}=\left[\mathbf{H},\boldsymbol{\rho}\right]}$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma})=A_\alpha B_\beta\sigma_\alpha\sigma_\beta=A_\alpha B_\beta\left(\delta_{\alpha\beta}+i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
$$=A_\alpha B_\alpha+i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$=\mathbf{A}\bullet\mathbf{B}+i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\boldsymbol{\sigma}$$

This cancels

This remains

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho}=\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)=\hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\boldsymbol{\rho}\mathbf{H}=\left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)=\hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho}-\boldsymbol{\rho}\mathbf{H}=\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)-\frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho}=i\hbar\dot{\boldsymbol{\rho}}=\frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}}\right)\bullet\boldsymbol{\sigma}-\frac{i\hbar}{2}\left(\vec{\mathbf{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)=i\hbar\dot{\vec{\mathbf{S}}}\bullet\boldsymbol{\sigma}=\ i\hbar\left(\vec{\boldsymbol{\Omega}}\times\mathbf{S}\right)\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boxed{\begin{aligned}\rho &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\sigma\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\sigma\end{aligned}}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle=\mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right|=\left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger}=\mathbf{H}$.
$\rho^{\dagger}=\rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho=i\hbar\dot{\rho}=i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right|+i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right|=\mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right|-\left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\rho=i\hbar\dot{\rho}=\mathbf{H}\rho-\rho\mathbf{H}=\left[\mathbf{H},\rho\right]}$$

$$\left(\mathbf{A}\bullet\sigma\right)\left(\mathbf{B}\bullet\sigma\right)=A_\alpha B_\beta\sigma_\alpha\sigma_\beta=A_\alpha B_\beta\left(\delta_{\alpha\beta}+i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
$$=A_\alpha B_\alpha+i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$=\mathbf{A}\bullet\mathbf{B}+i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\sigma$$

Given ρ and **H** in terms *spin* S-vector and *crank* Ω-vector:

$$\mathbf{H}\rho=\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1}+\vec{S}\bullet\sigma\right)=\hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\Omega}\bullet\sigma+\hbar\Omega_0\vec{S}\bullet\sigma+\frac{\hbar}{2}\left(\vec{\Omega}\bullet\sigma\right)\left(\vec{S}\bullet\sigma\right)$$

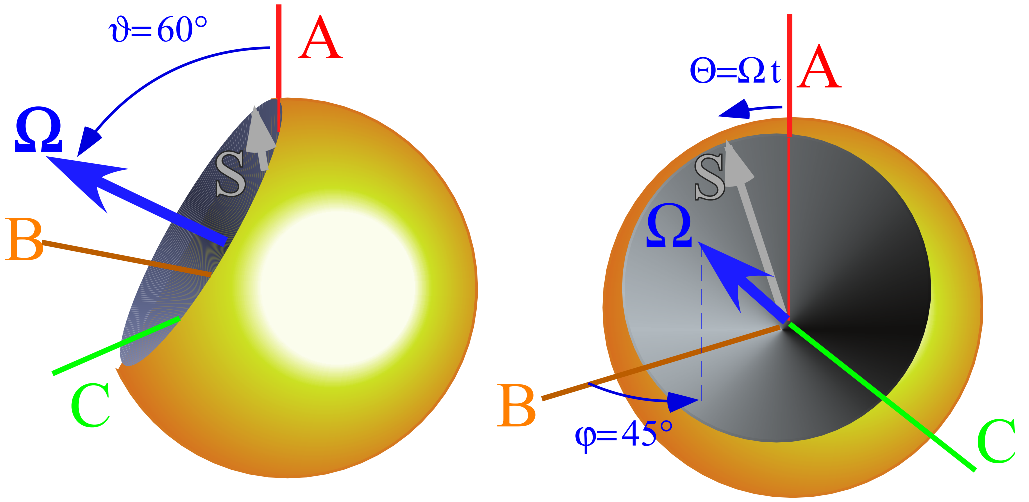
$$\rho\mathbf{H}=\left(\frac{N}{2}\mathbf{1}+\vec{S}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)=\hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\Omega}\bullet\sigma+\hbar\Omega_0\vec{S}\bullet\sigma+\frac{\hbar}{2}\left(\vec{S}\bullet\sigma\right)\left(\vec{\Omega}\bullet\sigma\right)$$

Last terms don't cancel if the *spin* S and *crank* Ω *point in different directions*.

$$\mathbf{H}\rho-\rho\mathbf{H}=\frac{\hbar}{2}\left(\vec{\Omega}\bullet\sigma\right)\left(\vec{S}\bullet\sigma\right)-\frac{\hbar}{2}\left(\vec{S}\bullet\sigma\right)\left(\vec{\Omega}\bullet\sigma\right)$$

$$i\hbar\frac{\partial}{\partial t}\rho=i\hbar\dot{\rho}=\frac{i\hbar}{2}\left(\vec{\Omega}\times\vec{S}\right)\bullet\sigma-\frac{i\hbar}{2}\left(\vec{S}\times\vec{\Omega}\right)\bullet\sigma$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1}+\vec{S}\bullet\sigma\right)=i\hbar\dot{\vec{S}}\bullet\sigma=\ i\hbar\left(\vec{\Omega}\times\mathbf{S}\right)\bullet\sigma$$



Factoring out •σ gives a classical/quantum *gyro-precession equation*.

$$\boxed{\frac{\partial\vec{S}}{\partial t}=\dot{\vec{S}}=\vec{\Omega}\times\vec{S}}$$