Lecture 19
Tues. 10.31.2017

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

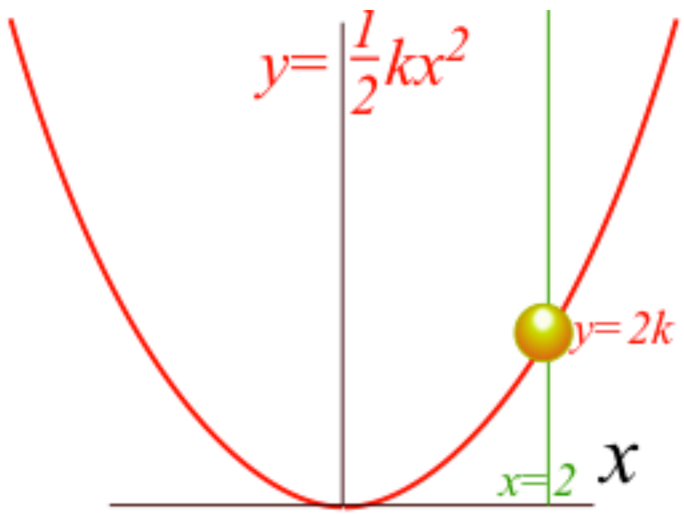
Some Ways to do constraint analysis

- *Way 1. Simple constraint insertion*
- *Way 2. GCC constraint webs*
  - *Find covariant force equations*
  - *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

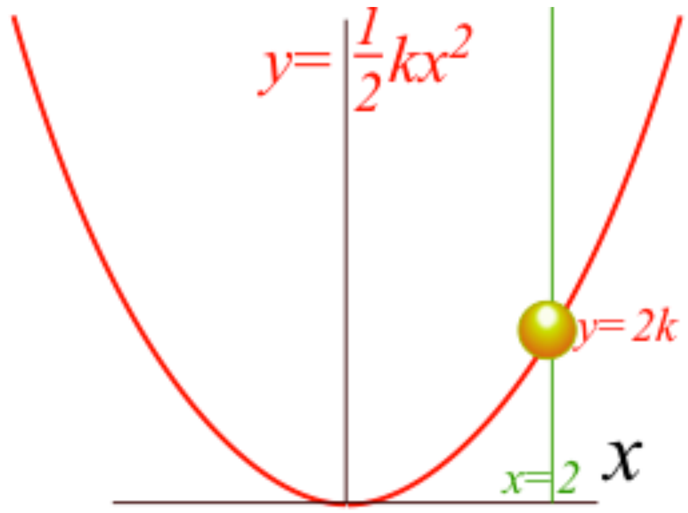
- *Way 3. OCC constraint webs*
  - *Preview of atomic-Stark orbits*
    - *Classical Hamiltonian separability*
- *Way 4. Lagrange multipliers*
  - *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *"Non-Holonomic" multipliers*

Simple constrained problem...



...and a variety of solutions

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

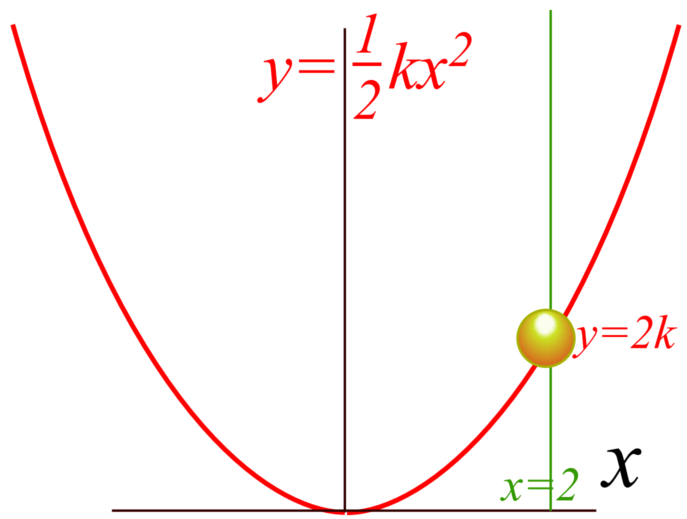
Way 2. GCC constraint webs

Find covariant force equations

Compare covariant vs. contravariant forces

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



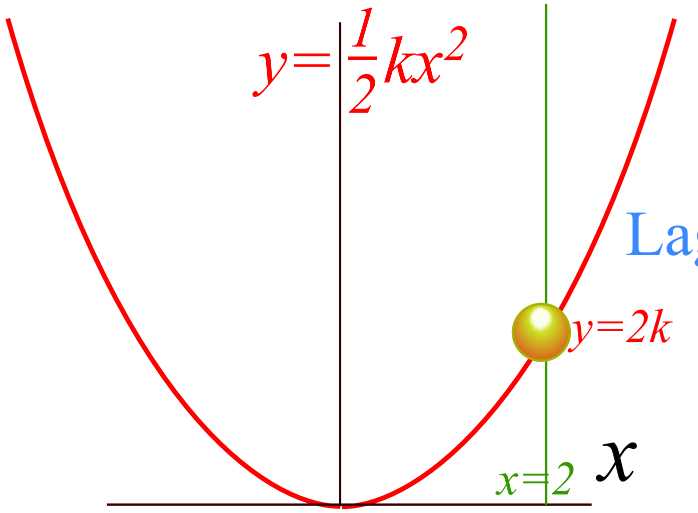
Way 1. Lagrangian has the constraint(s) simply inserted.

$$L = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2) - mgy$$

Let: $y = \frac{1}{2}kx^2$ and: $\dot{y} = kx\dot{x}$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

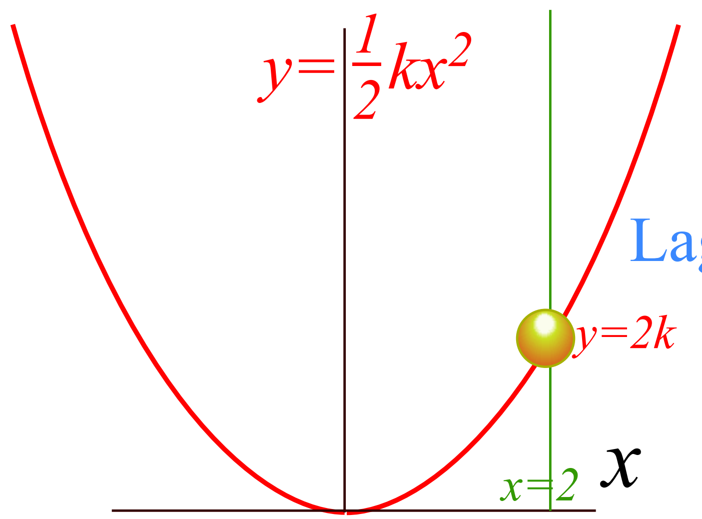
Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial \dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$

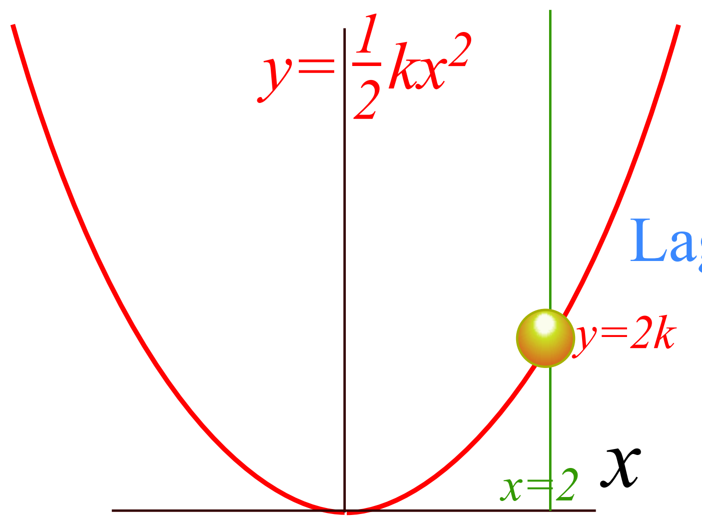
$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}$$

$$f_x=\frac{\partial L}{\partial x}$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

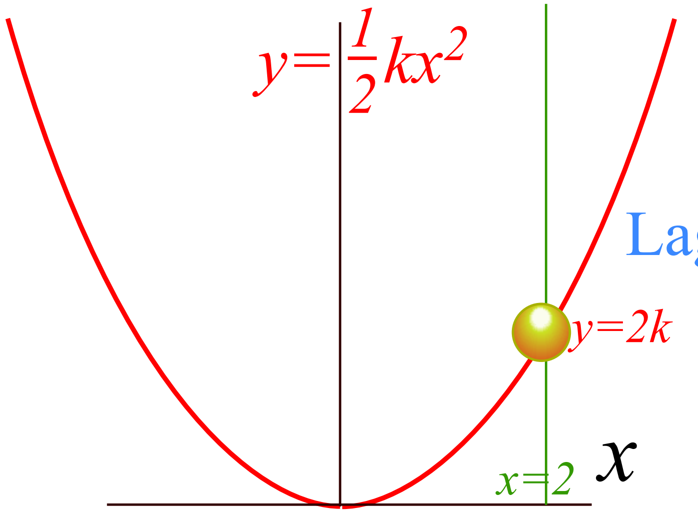
$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$
$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}$$
$$=m(\dot{x}+k^2x^2\dot{x})$$

$$f_x=\frac{\partial L}{\partial x}$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

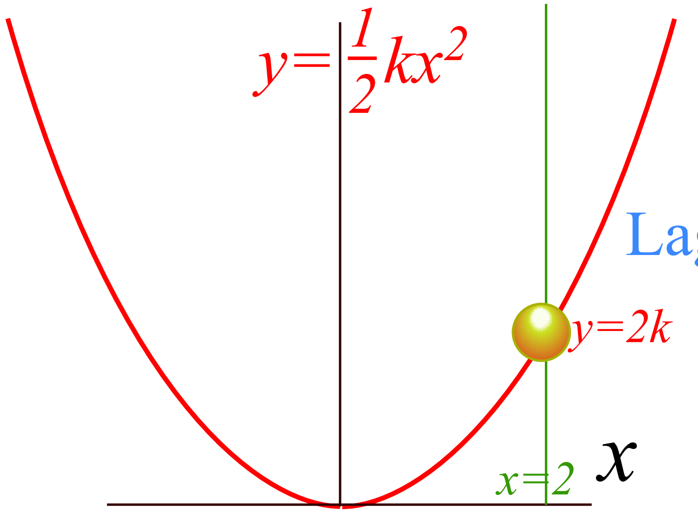
$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$
$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

$$f_x=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$
$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

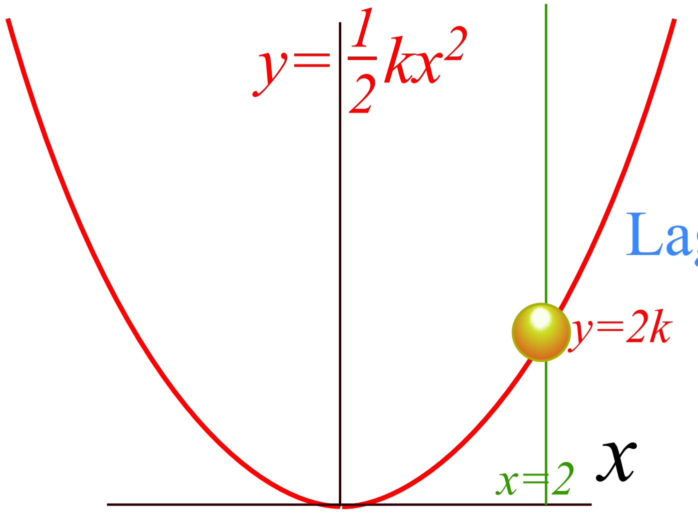
$$f_x=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$
$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

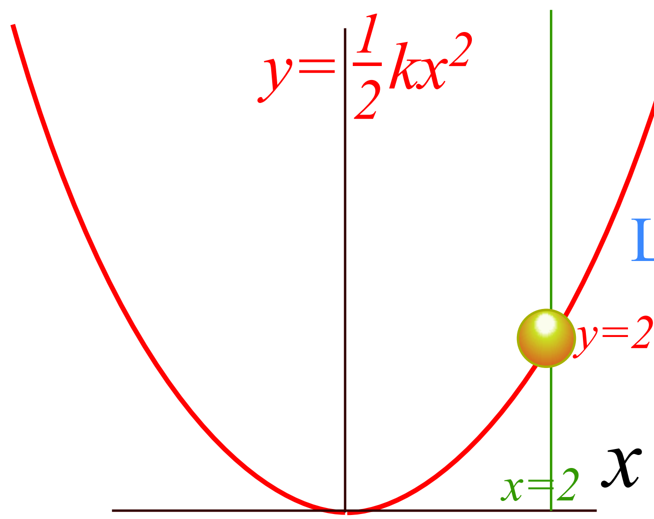
$$f_x=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$

$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

$$f_x=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

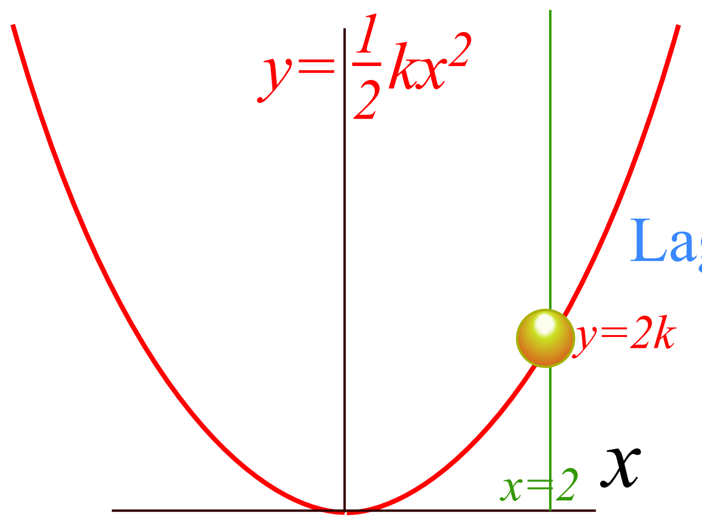
Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$\dot{p}_x=m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$

$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

$$f_x=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

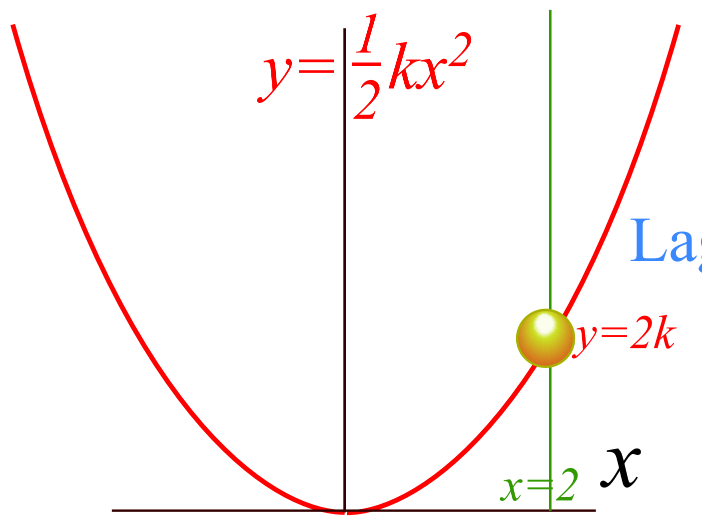
Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2$$

$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

$$f_x=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

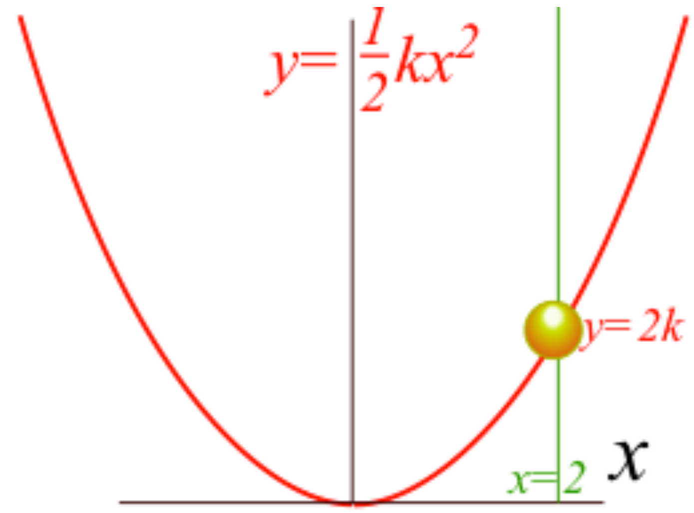
Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

$$\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx$$

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

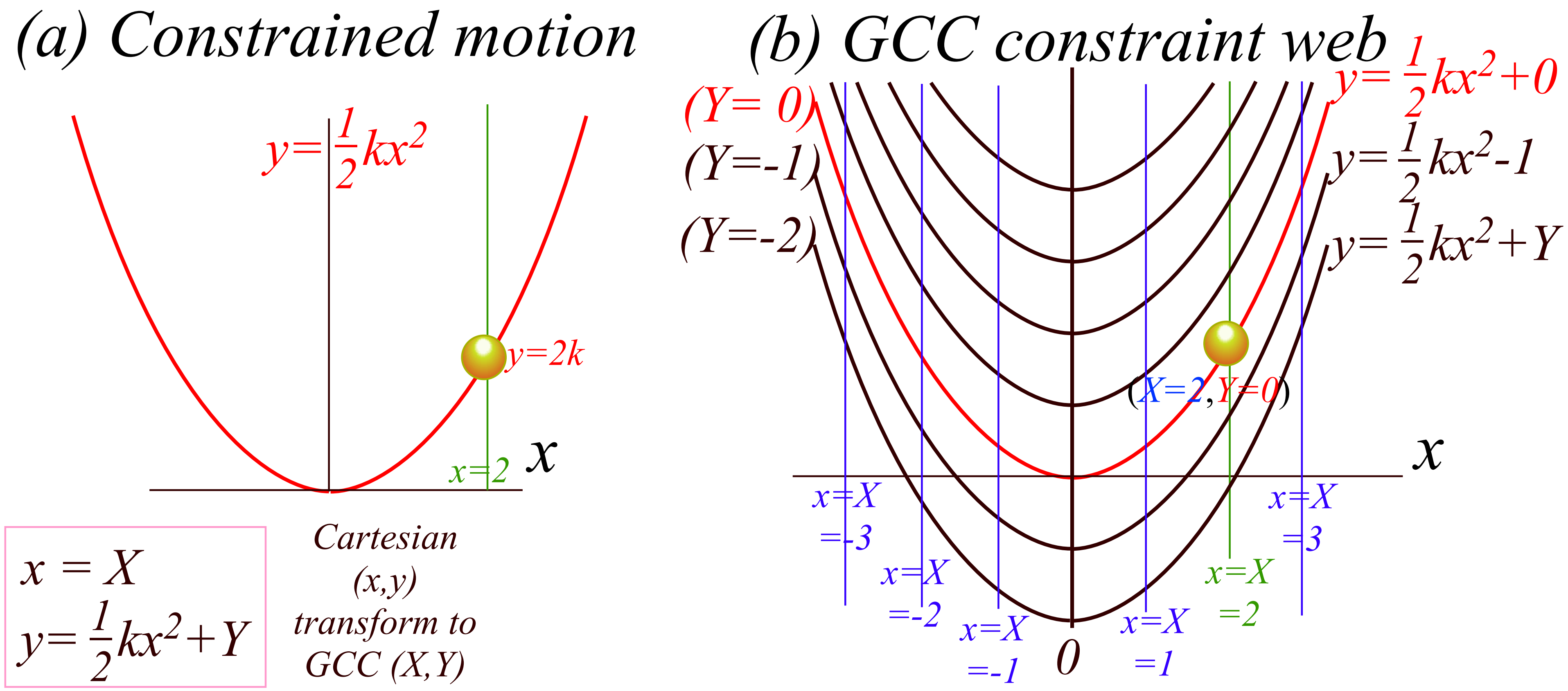
Way 1. Simple constraint insertion

→ *Way 2. GCC constraint webs*

Find covariant force equations

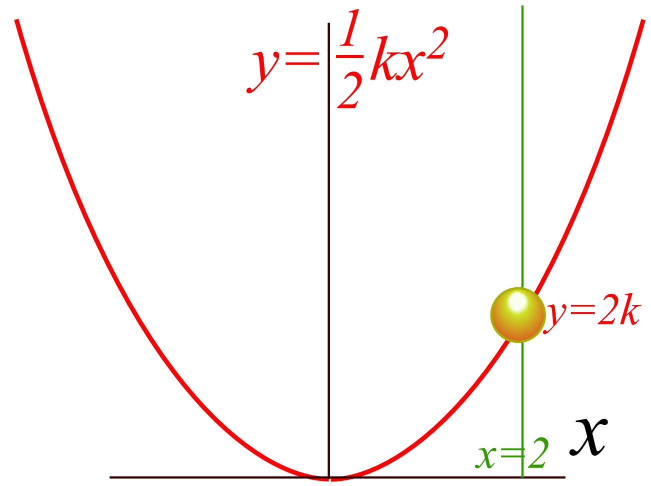
Compare covariant vs. contravariant forces

Way 2. *GCC constraint webs.*



Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

(a) Constrained motion

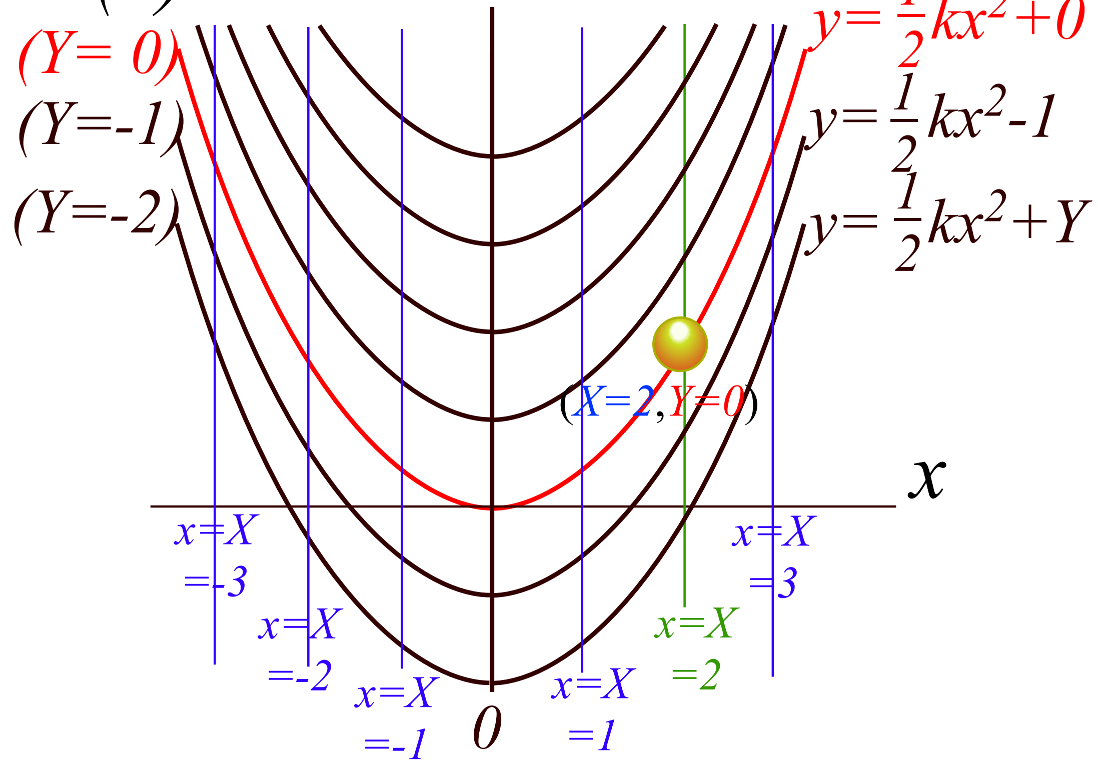


$x = X$

$y = \frac{1}{2}kx^2 + Y$

Cartesian (x,y) transform to GCC (X,Y)

(b) GCC constraint web



Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

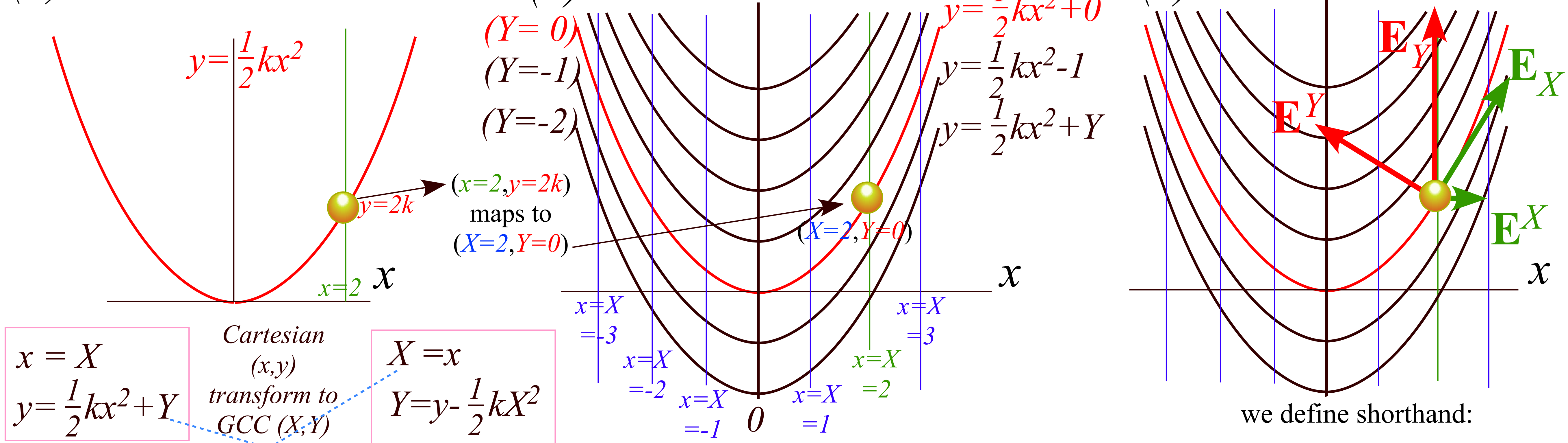
$x=q^1=X$ $\qquad$ $y=1/2kx^2+q^2= kX^2/2+Y$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Way 2. GCC constraint webs.

(a) Constrained motion (b) GCC constraint web (c) GCC **E**-vectors



we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

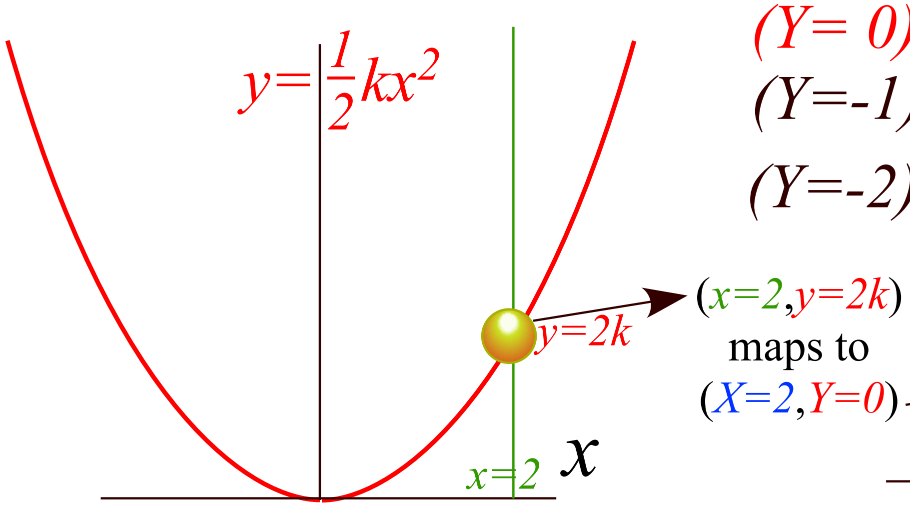
$$x=q^1=X \qquad y=1/2kx^2+q^2=kX^2/2+Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

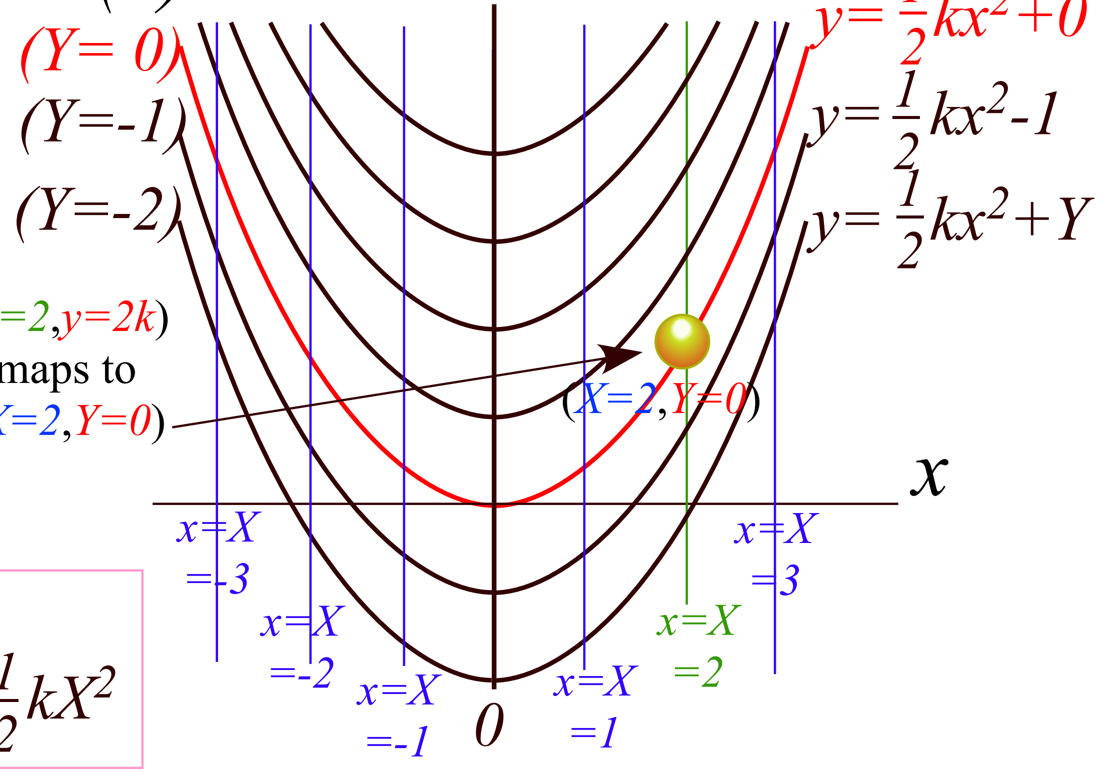
$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>Way 2.</u> GCC constraint webs.

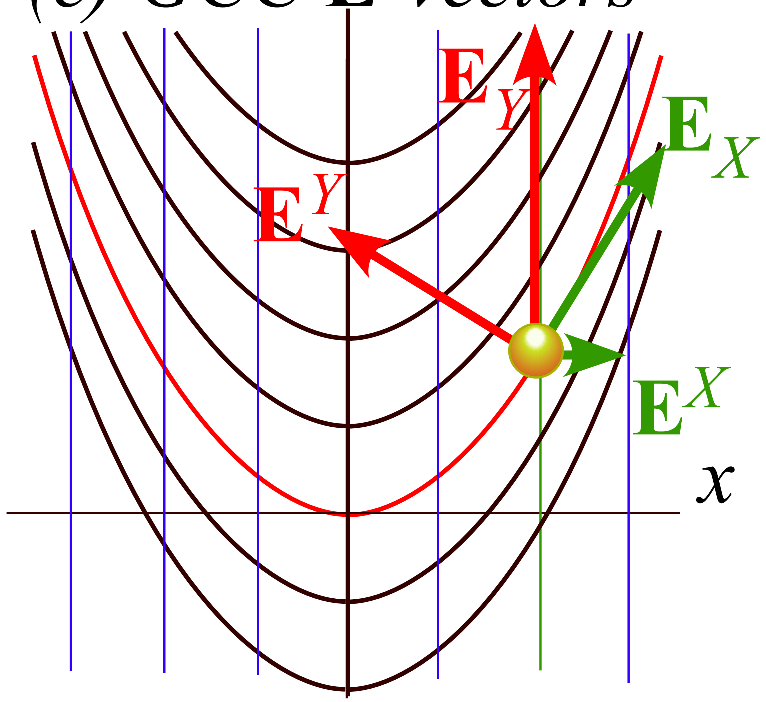
(a) Constrained motion



(b) GCC constraint web



(c) GCC **E**-vectors



$x = X$
$y = \frac{1}{2}kx^2 + Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$
$Y = y - \frac{1}{2}kX^2$

we define shorthand:

$X \equiv q^1$ and $Y \equiv q^2$ to avoid writing $q_{ueer}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x = q^1 = X$ $\quad$ $y = 1/2kx^2 + q^2 = kX^2/2 + Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

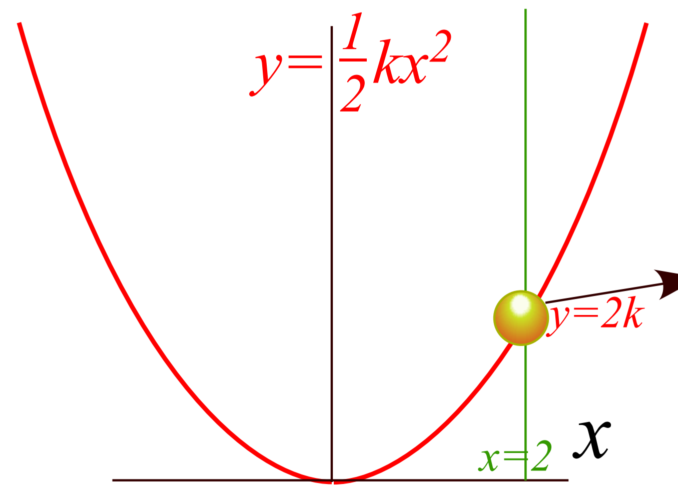
$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

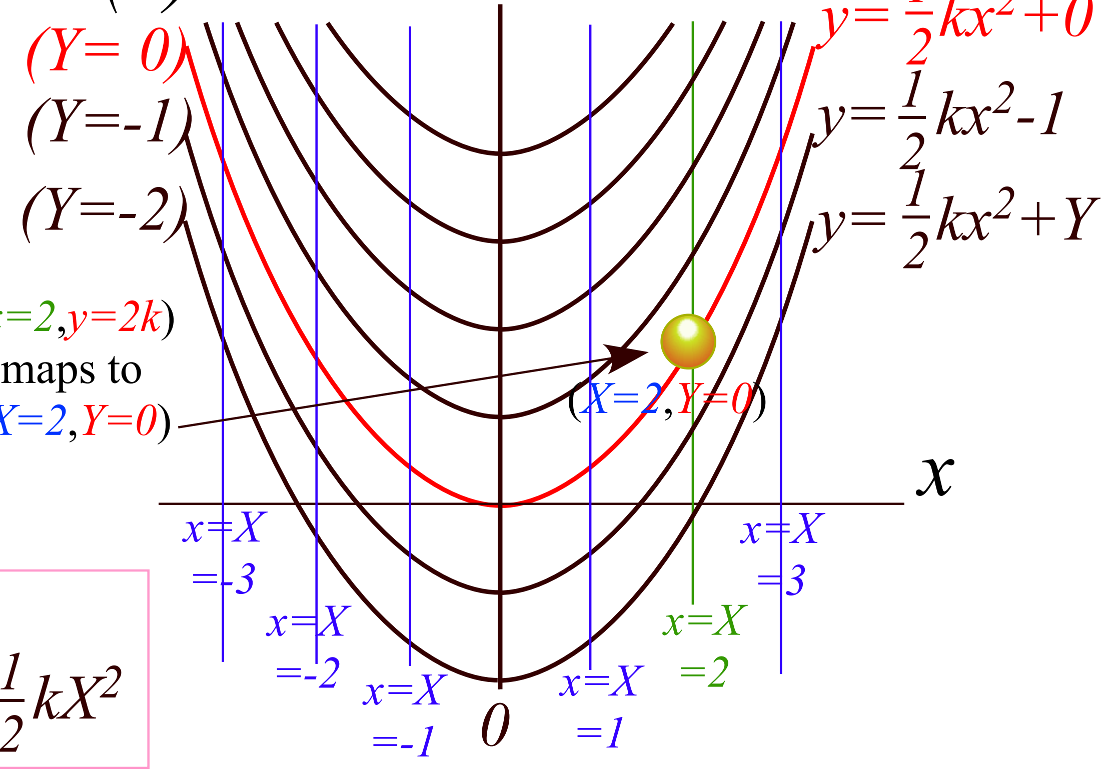
$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \begin{aligned} \mathbf{E}^X &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \mathbf{E}^Y &= \begin{pmatrix} -kx & 1 \end{pmatrix} \end{aligned}$$

Way 2. GCC constraint webs.

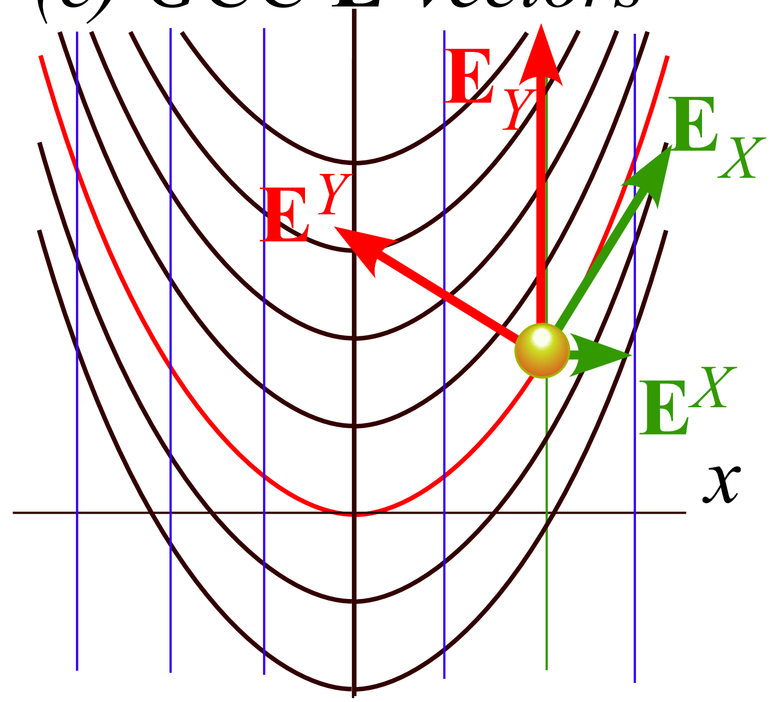
(a) Constrained motion



(b) GCC constraint web



(c) GCC **E**-vectors



$x = X$

$y = \frac{1}{2}kx^2 + Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$

$Y = y - \frac{1}{2}kX^2$

we define shorthand:

$X \equiv q^1$ and $Y \equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x = q^1 = X$ $\qquad$ $y = 1/2kx^2 + q^2 = kX^2/2 + Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

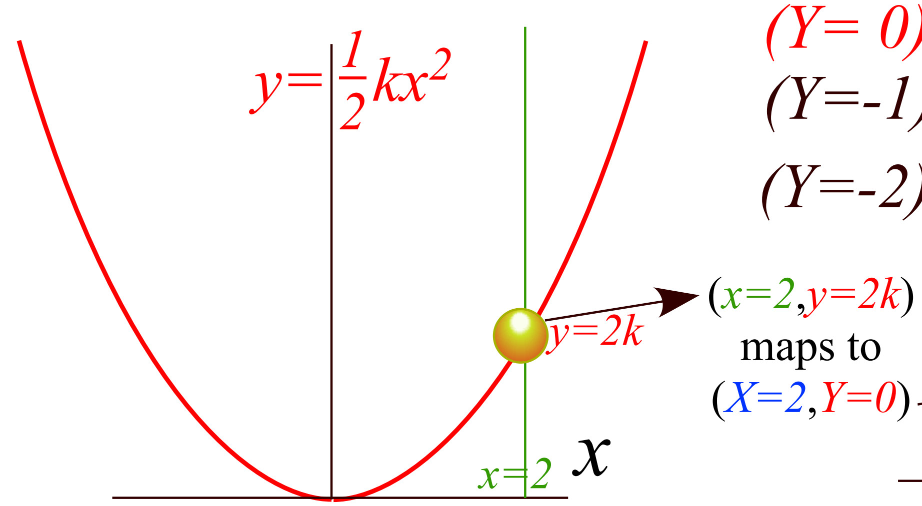
$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \begin{matrix} \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix} \end{matrix}$$

Find: 1st coordinate differentials and velocity relations:

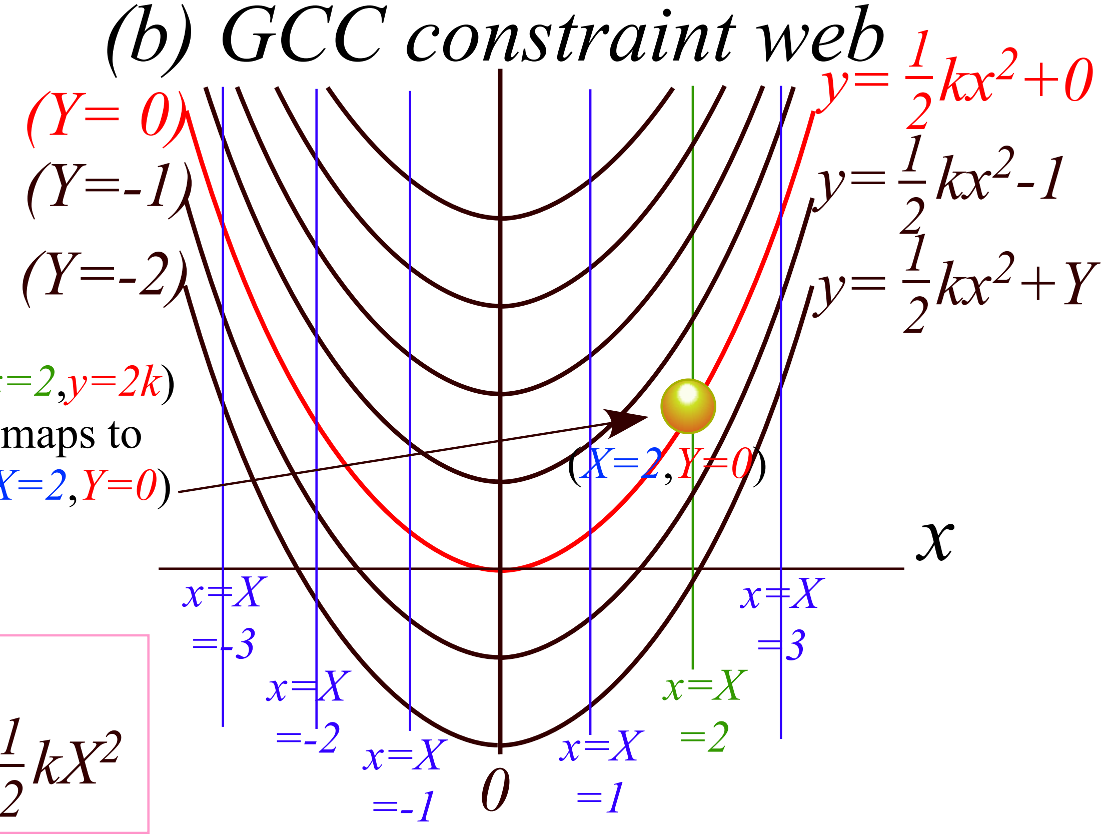
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Way 2. GCC constraint webs.

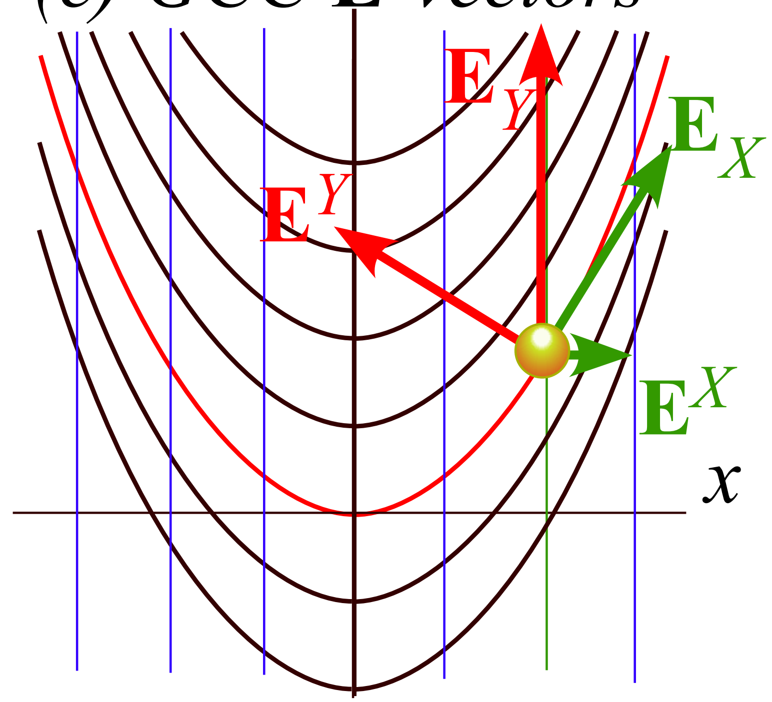
(a) Constrained motion



(b) GCC constraint web



(c) GCC E-vectors



$x = X$
$y = \frac{1}{2}kx^2 + Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$
$Y = y - \frac{1}{2}kX^2$

we define shorthand:

$X \equiv q^1$ and $Y \equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x = q^1 = X$ $\qquad y = 1/2kx^2 + q^2 = kX^2/2 + Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \; \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \begin{matrix} \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix} \end{matrix}$$

Find: 1st coordinate differentials and velocity relations:

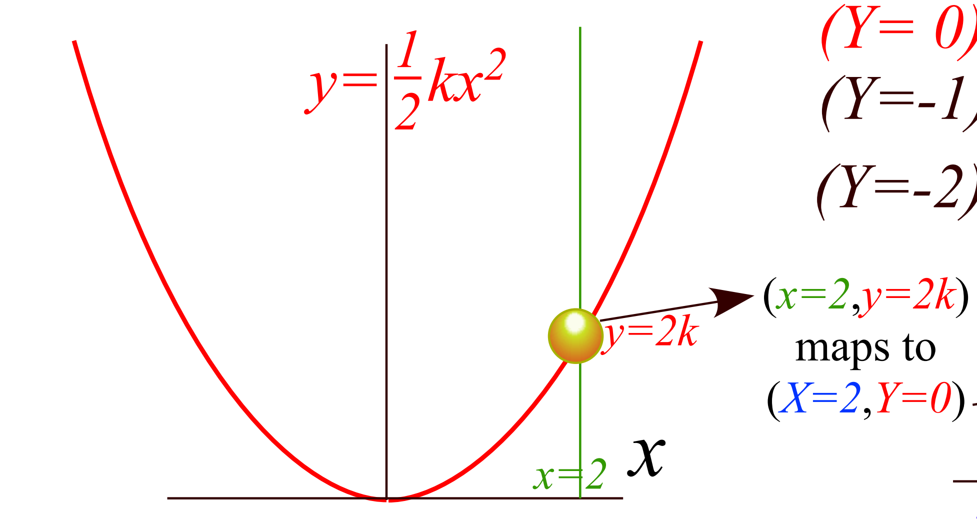
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB} = mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB} = J_{AC}J_{BC} = (JJ^\dagger)_{AB}$

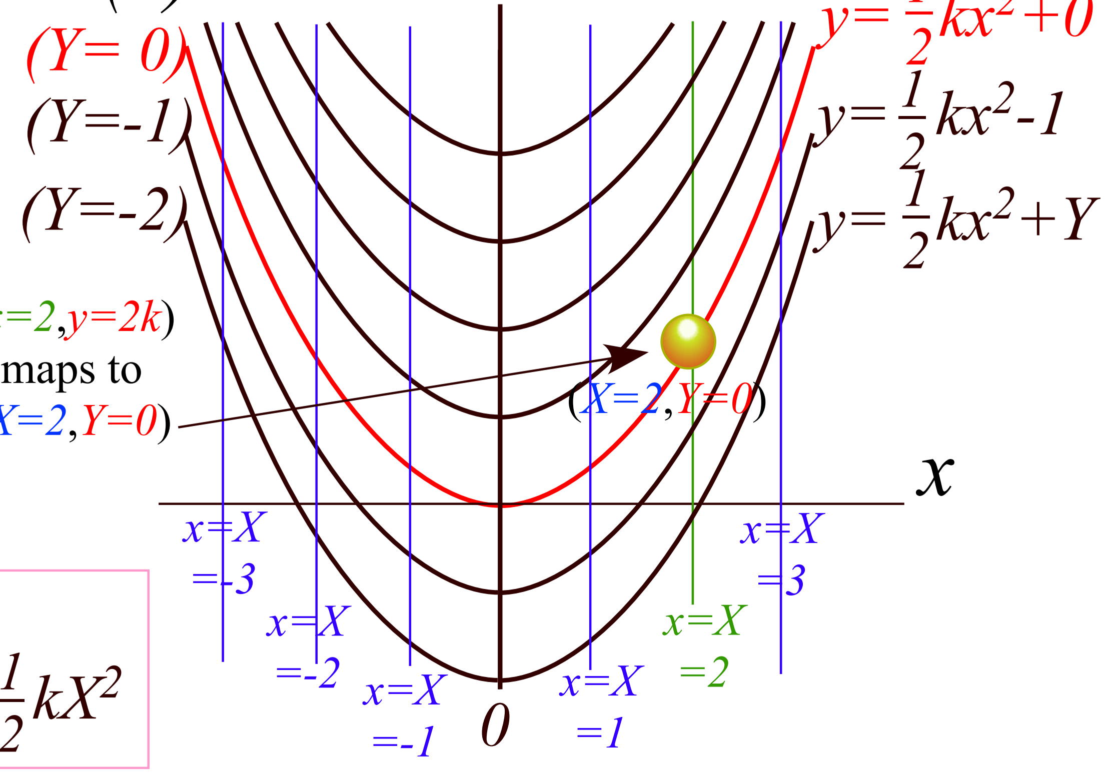
$$m \begin{pmatrix} \mathbf{E}_X \bullet \mathbf{E}_X & \mathbf{E}_X \bullet \mathbf{E}_Y \\ \mathbf{E}_Y \bullet \mathbf{E}_X & \mathbf{E}_Y \bullet \mathbf{E}_Y \end{pmatrix} = \begin{pmatrix} \gamma_{XX} & \gamma_{XY} \\ \gamma_{YX} & \gamma_{YY} \end{pmatrix} = m \begin{pmatrix} 1 + k^2x^2 & kx \\ kx & 1 \end{pmatrix}$$

Way 2. GCC constraint webs.

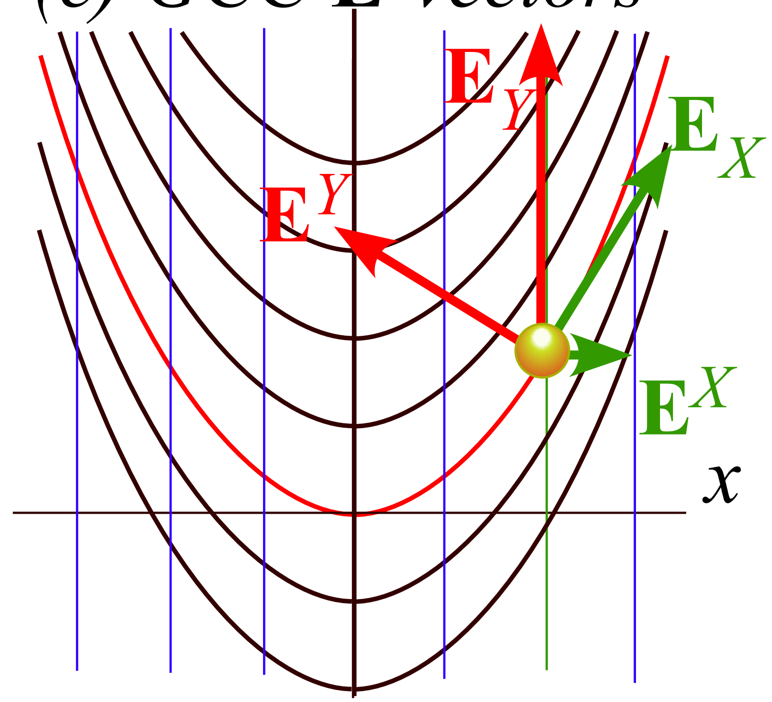
(a) Constrained motion



(b) GCC constraint web



(c) GCC **E**-vectors



$$x = X \qquad y = \frac{1}{2}kx^2 + Y$$

Cartesian (x,y) transform to GCC (X,Y)

$$X = x \qquad Y = y - \frac{1}{2}kX^2$$

we define shorthand:

$X \equiv q^1$ and $Y \equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$$x = q^1 = X \qquad y = 1/2kx^2 + q^2 = kX^2/2 + Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix}$$

Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB} = mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB} = J_{AC}J_{BC} = (JJ^{\dagger})_{AB}$

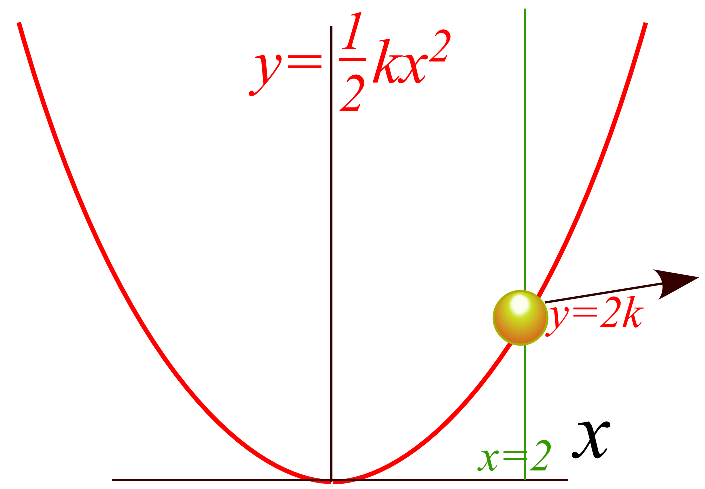
$$m\begin{pmatrix} \mathbf{E}_X \bullet \mathbf{E}_X & \mathbf{E}_X \bullet \mathbf{E}_Y \\ \mathbf{E}_Y \bullet \mathbf{E}_X & \mathbf{E}_Y \bullet \mathbf{E}_Y \end{pmatrix} = \begin{pmatrix} \gamma_{XX} & \gamma_{XY} \\ \gamma_{YX} & \gamma_{YY} \end{pmatrix} = m\begin{pmatrix} 1+k^2x^2 & kx \\ kx & 1 \end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix} \mathbf{E}^X \bullet \mathbf{E}^X & \mathbf{E}^X \bullet \mathbf{E}^Y \\ \mathbf{E}^Y \bullet \mathbf{E}^X & \mathbf{E}^Y \bullet \mathbf{E}^Y \end{pmatrix} = \begin{pmatrix} \gamma^{XX} & \gamma^{XY} \\ \gamma^{YX} & \gamma^{YY} \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 1 & -kx \\ -kx & 1+k^2x^2 \end{pmatrix}$$

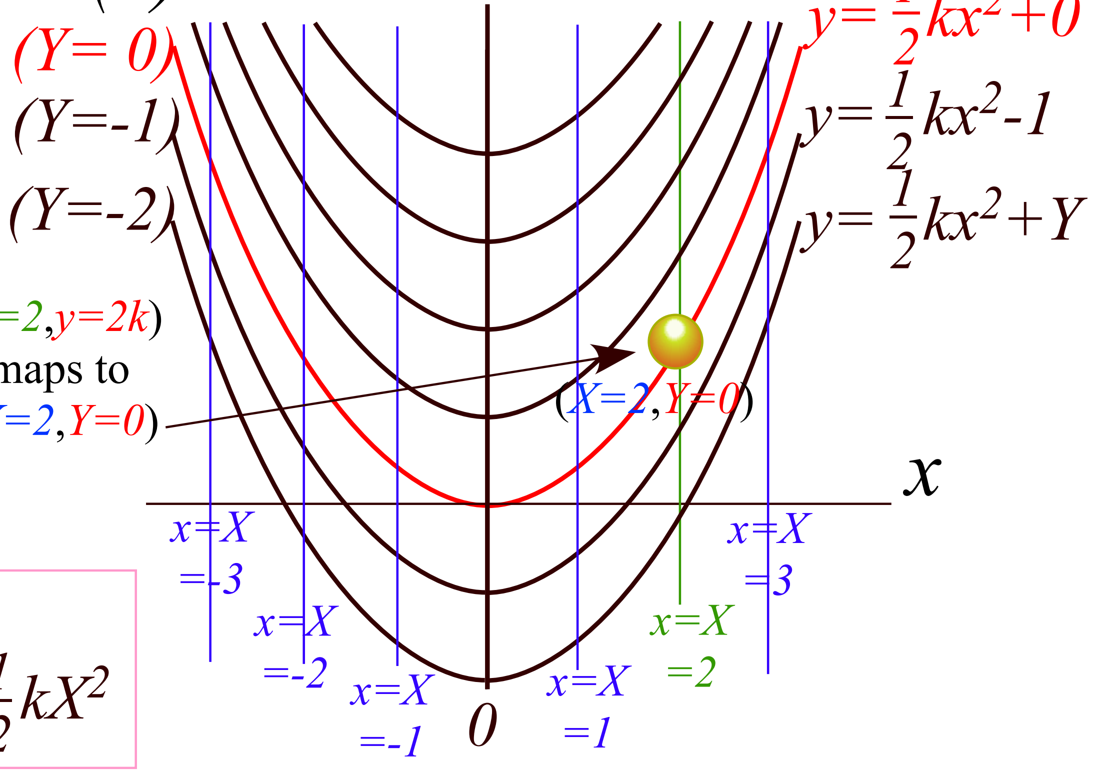
(Need contra-γ for Hamilton or Riemann equations)

Way 2. GCC constraint webs.

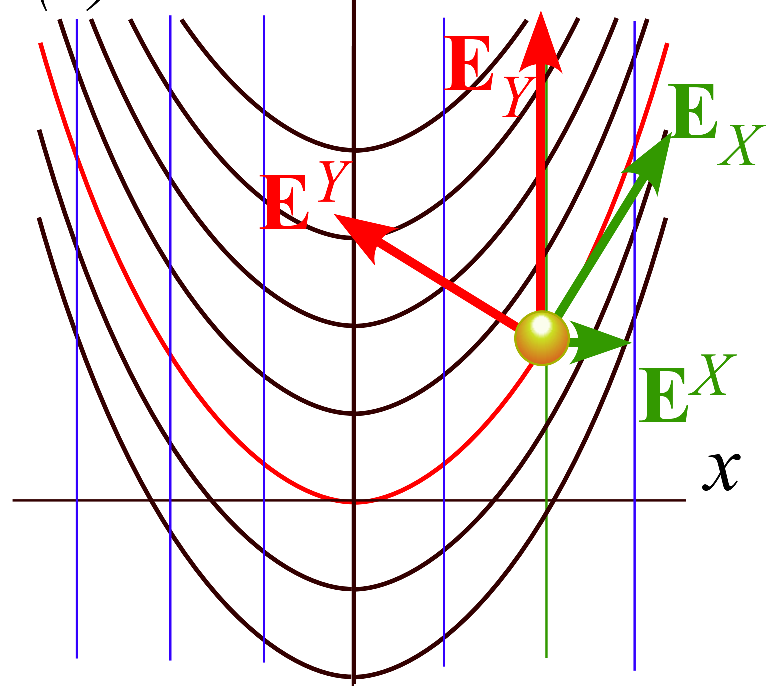
(a) Constrained motion



(b) GCC constraint web



(c) GCC E-vectors



$x = X$ $y=\frac{1}{2}kx^2+Y$	Cartesian (x,y) transform to GCC (X,Y)	$X=x$ $Y=y-\frac{1}{2}kX^2$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x=q^1=X$ $\quad$ $y=1/2kx^2+q^2=kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix} 1 \\ kx \end{pmatrix}, \mathbf{E}_Y=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \mathbf{E}^X=\begin{pmatrix} 1 & 0 \end{pmatrix} \quad \mathbf{E}^Y=\begin{pmatrix} -kx & 1 \end{pmatrix}$$

Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB}=mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB}=J_{AC}J_{BC}=(JJ^\dagger)_{AB}$

$$m\begin{pmatrix} \mathbf{E}_X\bullet\mathbf{E}_X & \mathbf{E}_X\bullet\mathbf{E}_Y \\ \mathbf{E}_Y\bullet\mathbf{E}_X & \mathbf{E}_Y\bullet\mathbf{E}_Y \end{pmatrix}=\begin{pmatrix} \gamma_{XX} & \gamma_{XY} \\ \gamma_{YX} & \gamma_{YY} \end{pmatrix}=m\begin{pmatrix} 1+k^2x^2 & kx \\ kx & 1 \end{pmatrix}$$

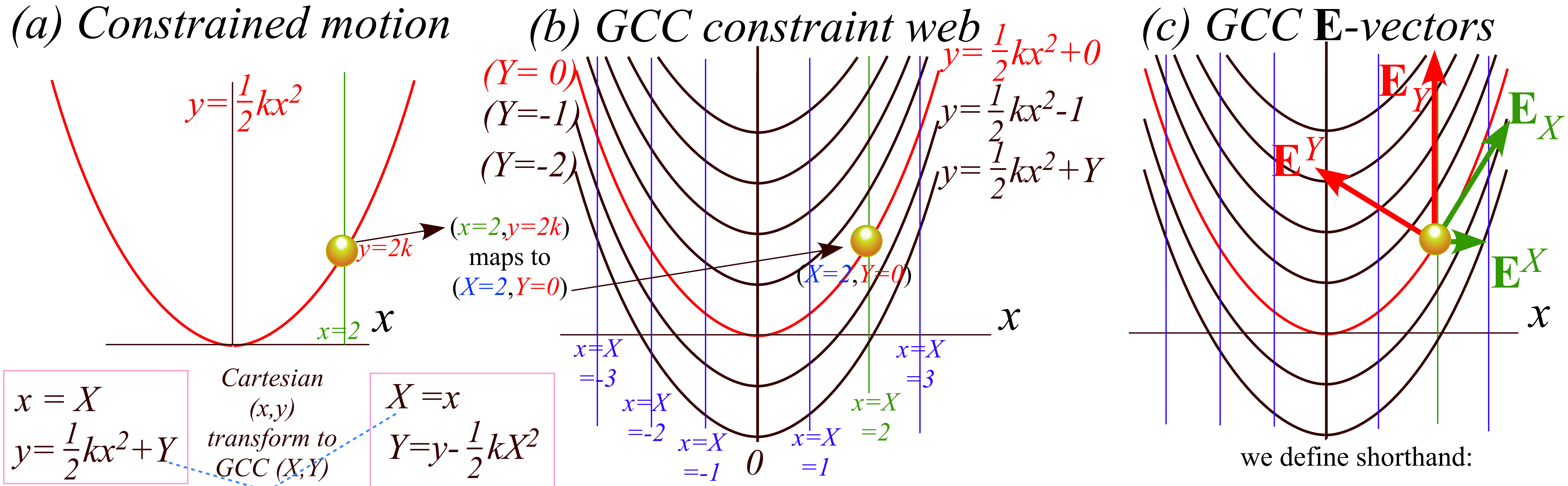
$$\frac{1}{m}\begin{pmatrix} \mathbf{E}^X\bullet\mathbf{E}^X & \mathbf{E}^X\bullet\mathbf{E}^Y \\ \mathbf{E}^Y\bullet\mathbf{E}^Y & \mathbf{E}^Y\bullet\mathbf{E}^Y \end{pmatrix}=\begin{pmatrix} \gamma^{XX} & \gamma^{XY} \\ \gamma^{YX} & \gamma^{YY} \end{pmatrix}=\frac{1}{m}\begin{pmatrix} 1 & -kx \\ -kx & 1+k^2x^2 \end{pmatrix}$$

(Need contra-γ for Hamilton or Riemann equations)

Find: Kinetic energy:

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{2}(\gamma_{XX}\dot{X}^2+2\gamma_{XY}\dot{X}\dot{Y}+\gamma_{YY}\dot{Y}^2)=m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2+kX\dot{X}\dot{Y}+\frac{1}{2}\dot{Y}^2\right]$$

Way 2. GCC constraint webs.



$$x = X \qquad y=\frac{1}{2}kx^2+Y$$

$$X=x \qquad Y=y-\frac{1}{2}kX^2$$

we define shorthand: $X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}{}^{Indices}$

Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$$x=q^1=X \qquad y=1/2kx^2+q^2=kX^2/2+Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix}1\\kx\end{pmatrix},\ \mathbf{E}_Y=\begin{pmatrix}0\\1\end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \mathbf{E}^X=\begin{pmatrix}1 & 0\end{pmatrix} \quad \mathbf{E}^Y=\begin{pmatrix}-kx & 1\end{pmatrix}$$

Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot X\\ \dot Y\end{pmatrix} \qquad \begin{pmatrix}\dot X\\ \dot Y\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB}=mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB}=J_{AC}J_{BC}=(JJ^\dagger)_{AB}$

$$m\begin{pmatrix}\mathbf{E}_X\bullet\mathbf{E}_X & \mathbf{E}_X\bullet\mathbf{E}_Y\\ \mathbf{E}_Y\bullet\mathbf{E}_X & \mathbf{E}_Y\bullet\mathbf{E}_Y\end{pmatrix}=\begin{pmatrix}\gamma_{XX} & \gamma_{XY}\\ \gamma_{YX} & \gamma_{YY}\end{pmatrix}=m\begin{pmatrix}1+k^2x^2 & kx\\ kx & 1\end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix}\mathbf{E}^X\bullet\mathbf{E}^X & \mathbf{E}^X\bullet\mathbf{E}^Y\\ \mathbf{E}^Y\bullet\mathbf{E}^Y & \mathbf{E}^Y\bullet\mathbf{E}^Y\end{pmatrix}=\begin{pmatrix}\gamma^{XX} & \gamma^{XY}\\ \gamma^{YX} & \gamma^{YY}\end{pmatrix}=\frac{1}{m}\begin{pmatrix}1 & -kx\\ -kx & 1+k^2x^2\end{pmatrix}$$

(Need contra-γ for Hamilton or Riemann equations)

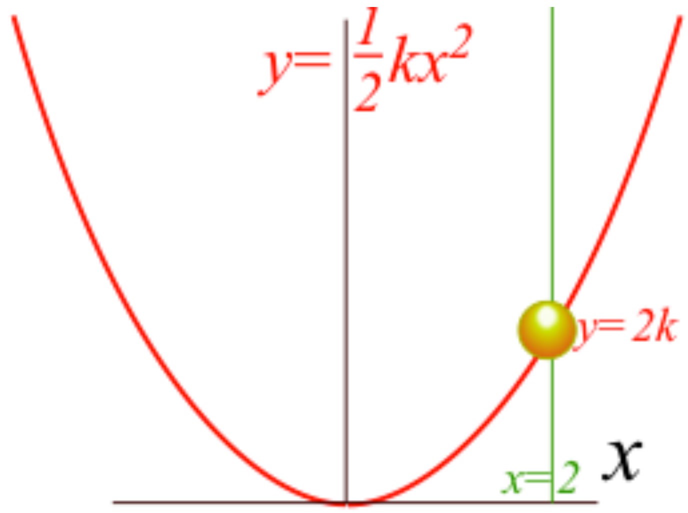
Find: Kinetic energy:

$$T=\frac{1}{2}m(\dot x^2+\dot y^2)=\frac{1}{2}(\gamma_{XX}\dot X^2+2\gamma_{XY}\dot X\dot Y+\gamma_{YY}\dot Y^2)=m\left[\frac{1}{2}(1+k^2X^2)\dot X^2+kX\dot X\dot Y+\frac{1}{2}\dot Y^2\right]$$

...and Lagrangian:

$$L=T-V=m\left[\frac{1}{2}(1+k^2X^2)\dot X^2+kX\dot X\dot Y+\frac{1}{2}\dot Y^2-gY-\frac{gk}{2}X^2\right] \qquad V=mgy=mg(Y+kX^2/2)$$

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

→ *Find covariant force equations*

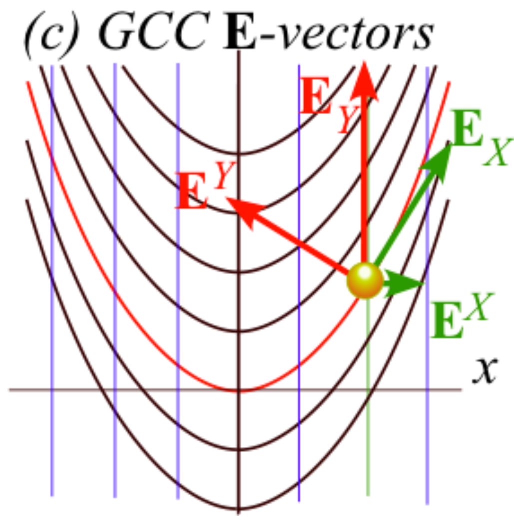
Compare covariant vs. contravariant forces

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$



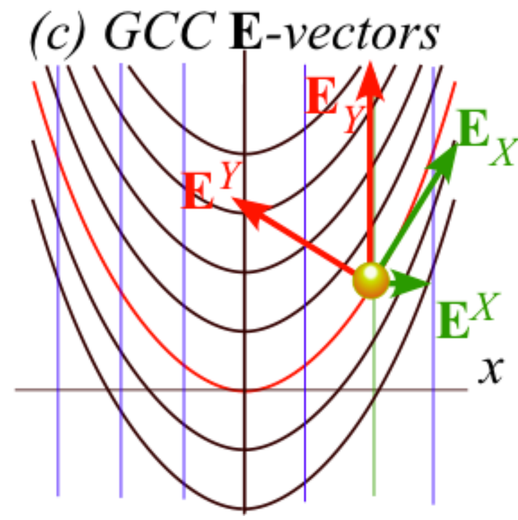
Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(metric γ_{AB})

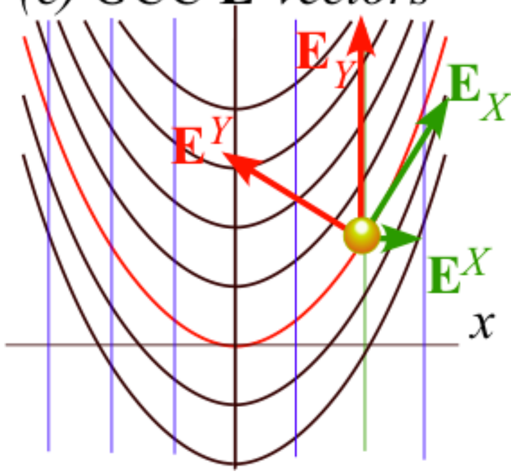
(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(metric γ_{AB})

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1[st] Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

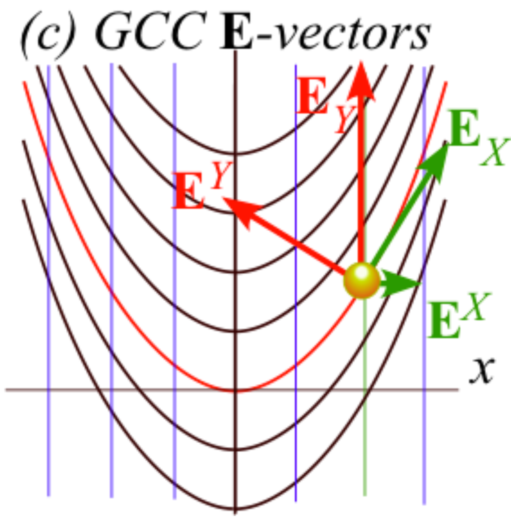
(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2[nd] Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. $(F_X^{cov} = 0 = F_Y^{cov})$



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

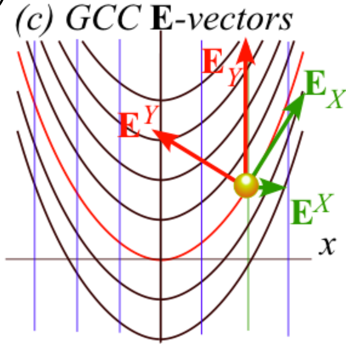
(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

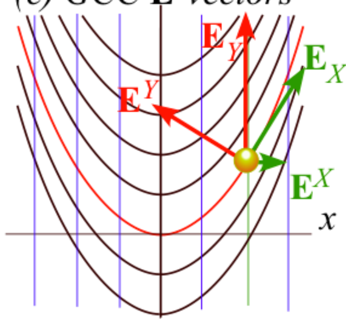
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov} = 0 = F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{(\text{metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{(\text{metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m \frac{d}{dt} \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

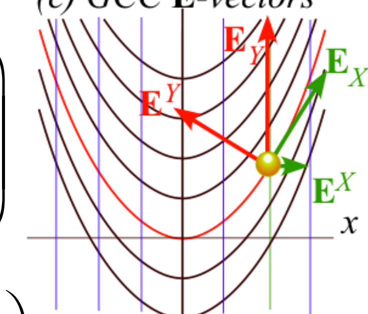
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. $(F_X^{cov} = 0 = F_Y^{cov})$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m \begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m \begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

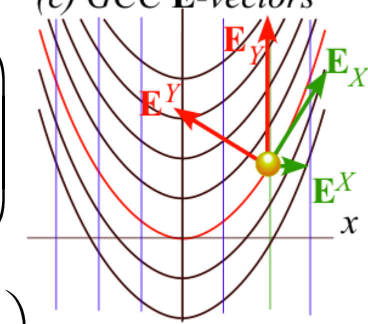
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

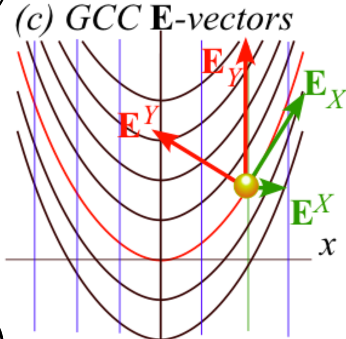
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m \frac{d}{dt} \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. $(F_X^{cov} = 0 = F_Y^{cov})$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m \begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m \begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y =_{const.}$)

$$\frac{1}{m} \underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{\text{(inverse of } \gamma_{AB})} \begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{\text{(inverse of } \gamma_{AB})} \begin{pmatrix} kX(k\dot{X}^2 + g) \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{(\text{metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{(\text{metric } \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

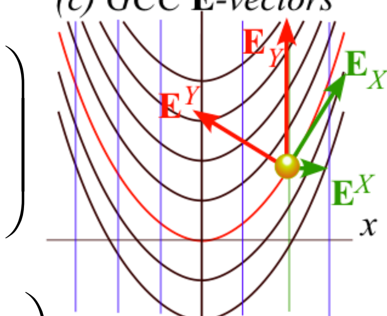
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

$$\frac{1}{m}\underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{(\text{inverse of } \gamma_{AB})}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{(\text{inverse of } \gamma_{AB})}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1^{st} Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2^{nd} Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

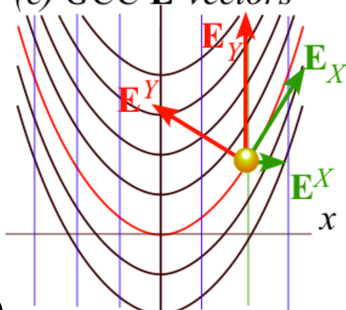
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB})

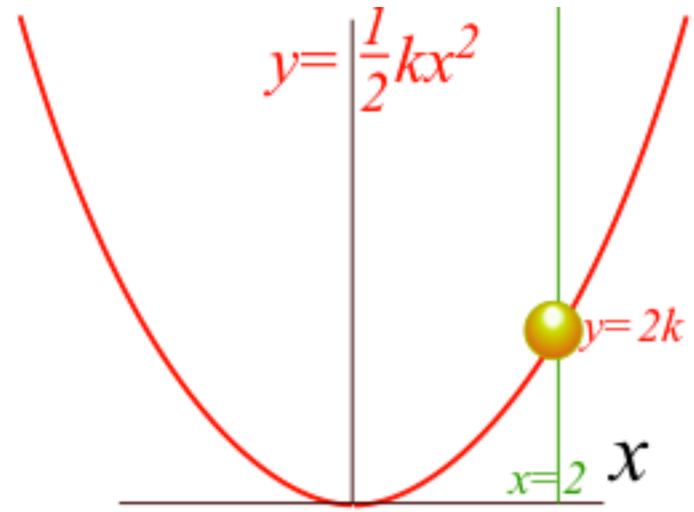
$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

(inverse of γ_{AB}) (cancelations)

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$\ddot{x} = 0 = \ddot{X}$

Simple constrained problem...



...and a variety of solutions

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

Find covariant force equations

→ *Compare covariant vs. contravariant forces*

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

(F_A are coefficients of ***normal*** *vectors* $\mathbf{E}^A$*)*

Frictional force components are contravariant

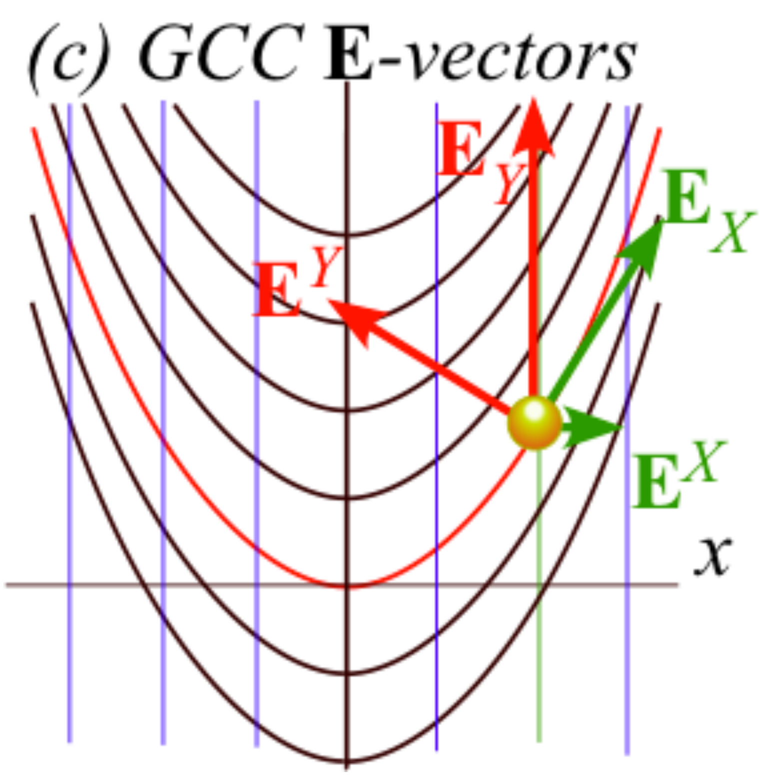
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

(F^A are coefficients of ***tangent*** *vectors* $\mathbf{E}_A$*)*

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

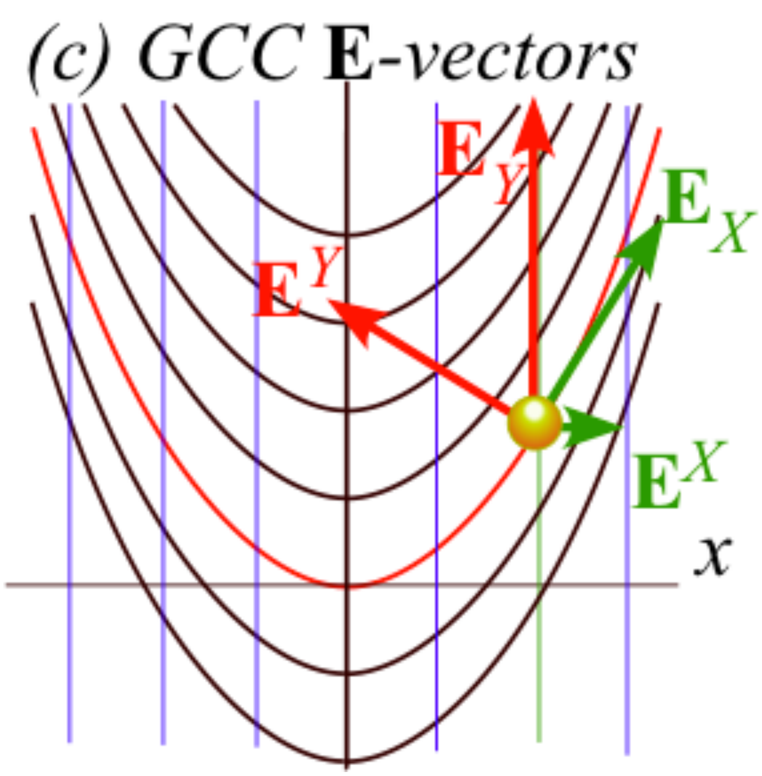
$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*



General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0 \cdot \nabla X + F_Y^{cov}\nabla Y$$
$$= 0 \cdot \mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

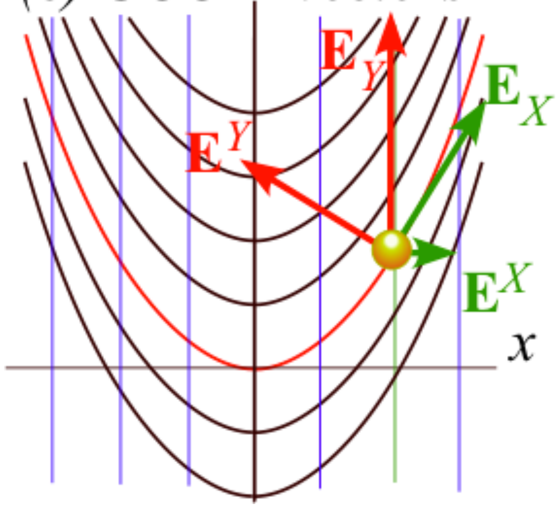
Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E***-vectors*



$\dot{Y} = 0 = \ddot{Y}$

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

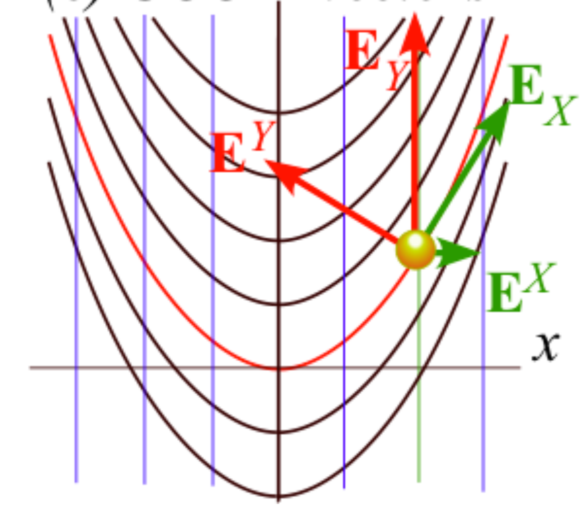
$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass m by parabola $Y=const.$ is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

(c) GCC **E**-vectors



$\dot{Y} = 0 = \ddot{Y}$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2+g}{1+k^2X^2}kX$$

FINALLY ! We get the Way 1. solution of p.12

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y}\end{pmatrix} = m\begin{pmatrix}(1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}F_X^{cov} \\ F_Y^{cov}\end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

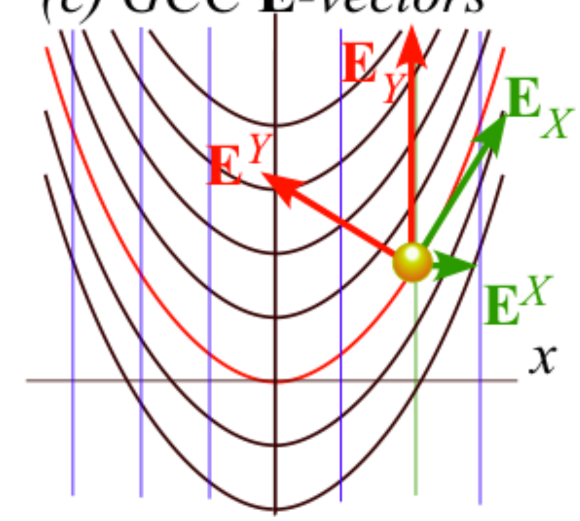
$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

(c) GCC **E**-vectors



$\dot{Y} = 0 = \ddot{Y}$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov}\,\mathbf{E}^Y = m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y}\end{pmatrix} = m\begin{pmatrix}(1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}F_X^{cov} \\ F_Y^{cov}\end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

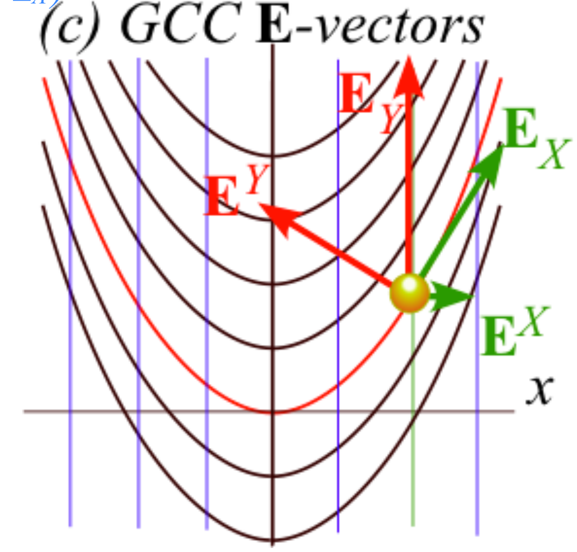
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$



$\dot{Y} = 0 = \ddot{Y}$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix} \rightarrow \ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = \quad F_Y^{cov} \quad \mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

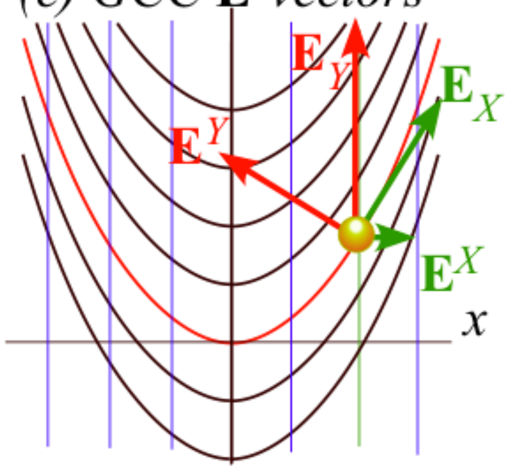
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$



$$\dot{Y} = 0 = \ddot{Y}$$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov}\ \mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

$$\begin{pmatrix}F_x \\ F_y\end{pmatrix} = \left(=\begin{pmatrix}0 \\ mk\dot{X}^2 + mg\end{pmatrix}\right)_{at:X=0}$$

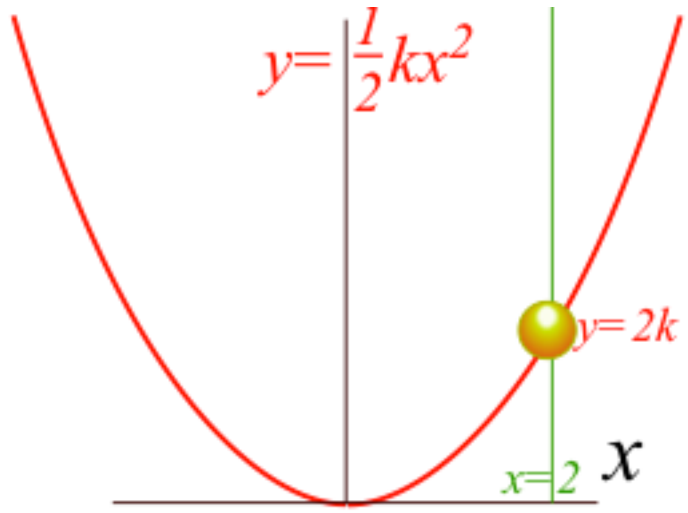
Centripetal force mkv^2+mg *(what roller-coaster rider feels at bottom)*

Recall: $x \equiv X$

$$\ddot{X} \equiv \ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx$$

$$-g = \ddot{y} = \frac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2 + Y\right)$$
$$= k\dot{X}^2 + kX\ddot{X} + \ddot{Y}\ (= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0)$$

Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

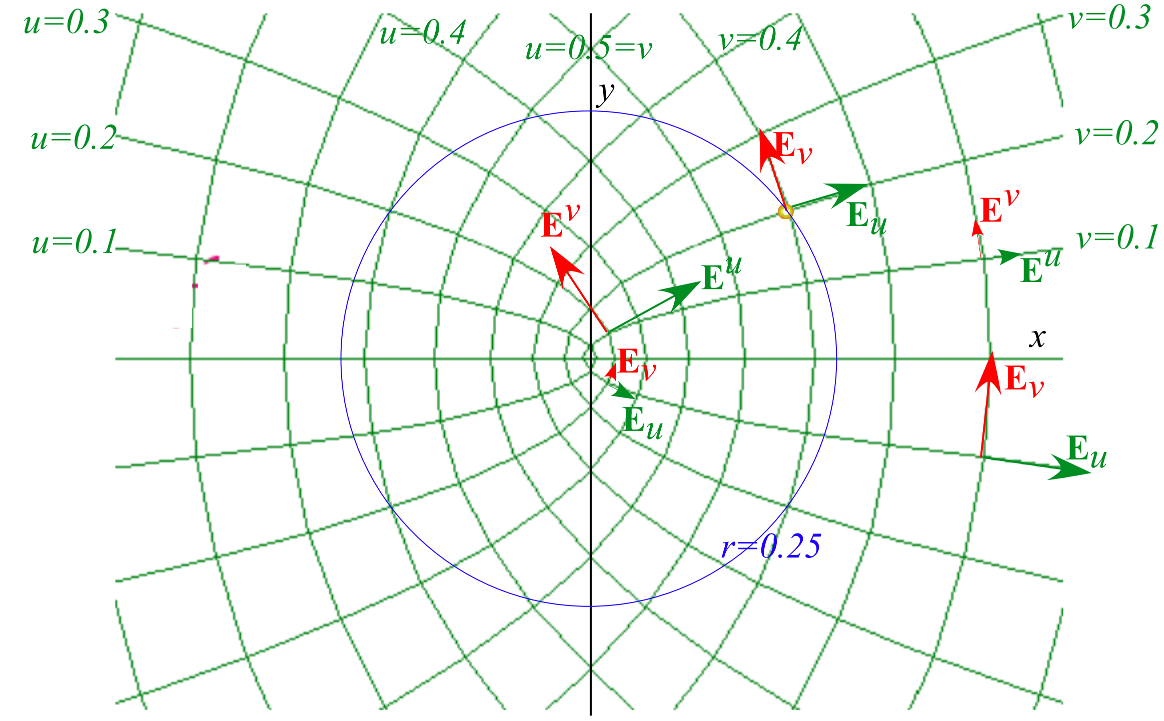
"Non-Holonomic" multipliers

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u + iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^* z = x^2 + y^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2u^2v^2$$



Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

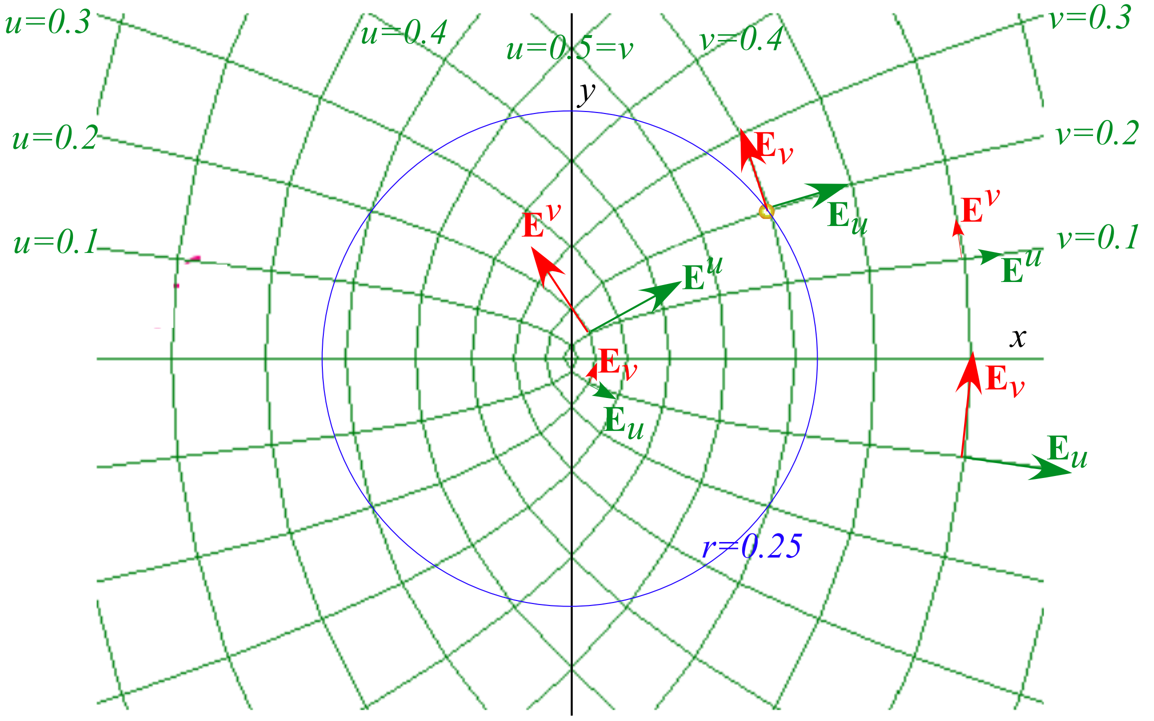
$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv$$

$$r^2 = z^* z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$



Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

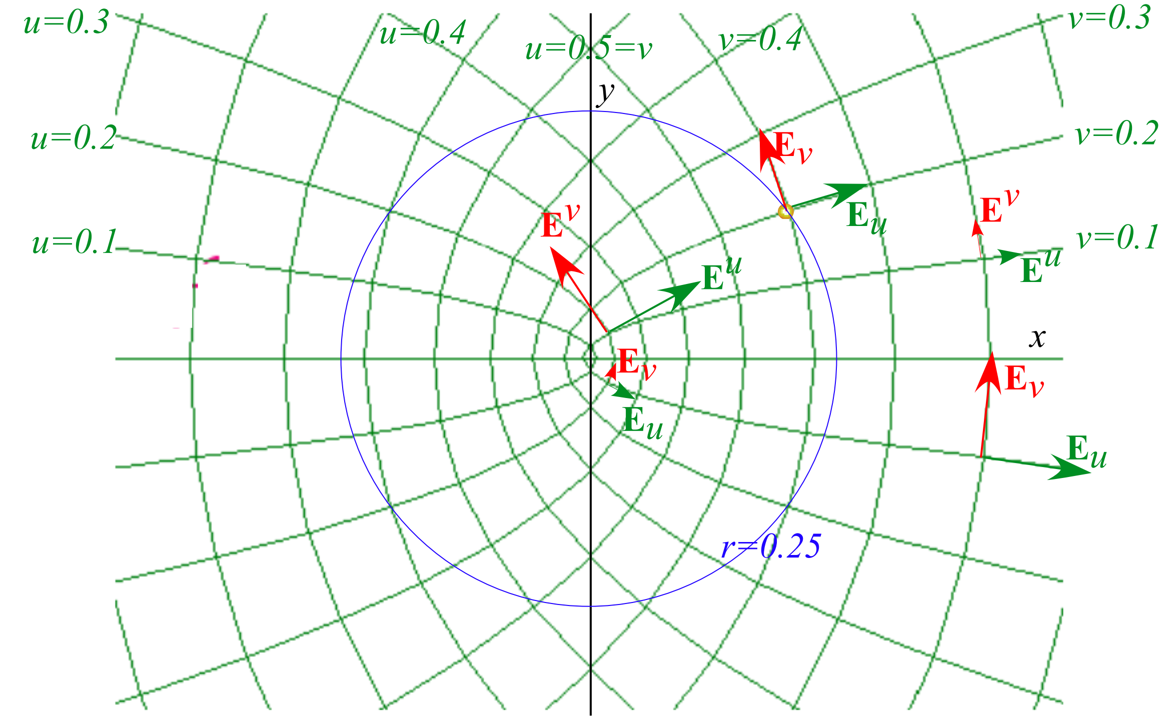
Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$

$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$



Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$

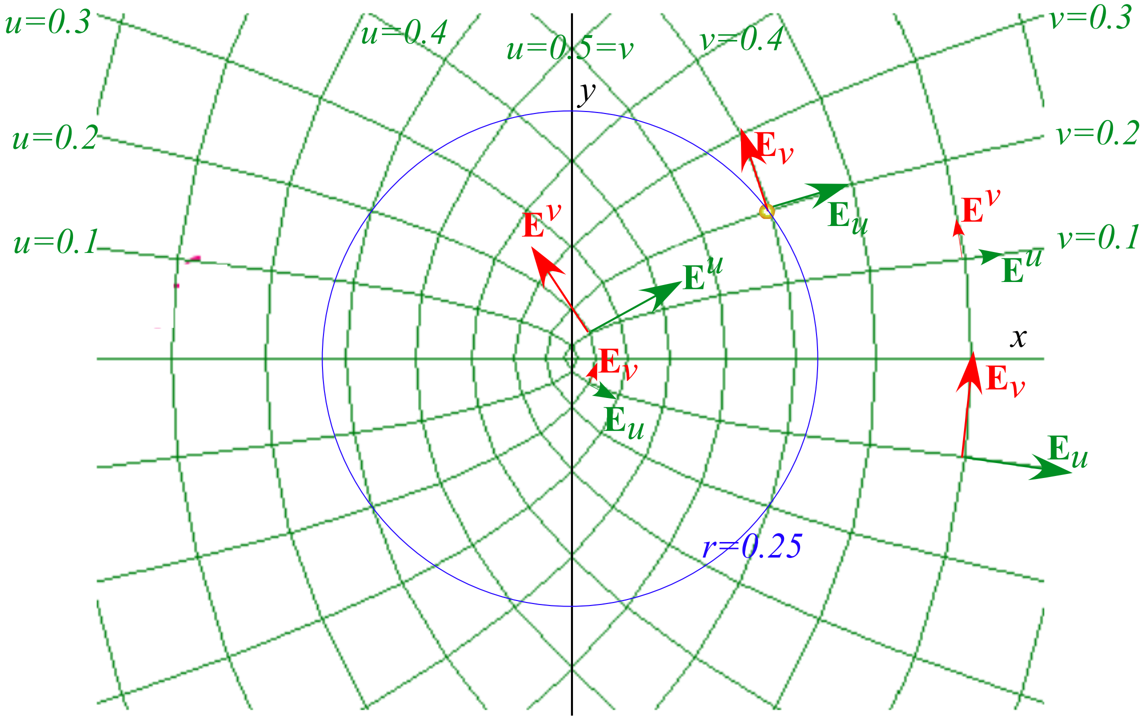
$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4v^2u^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2 \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

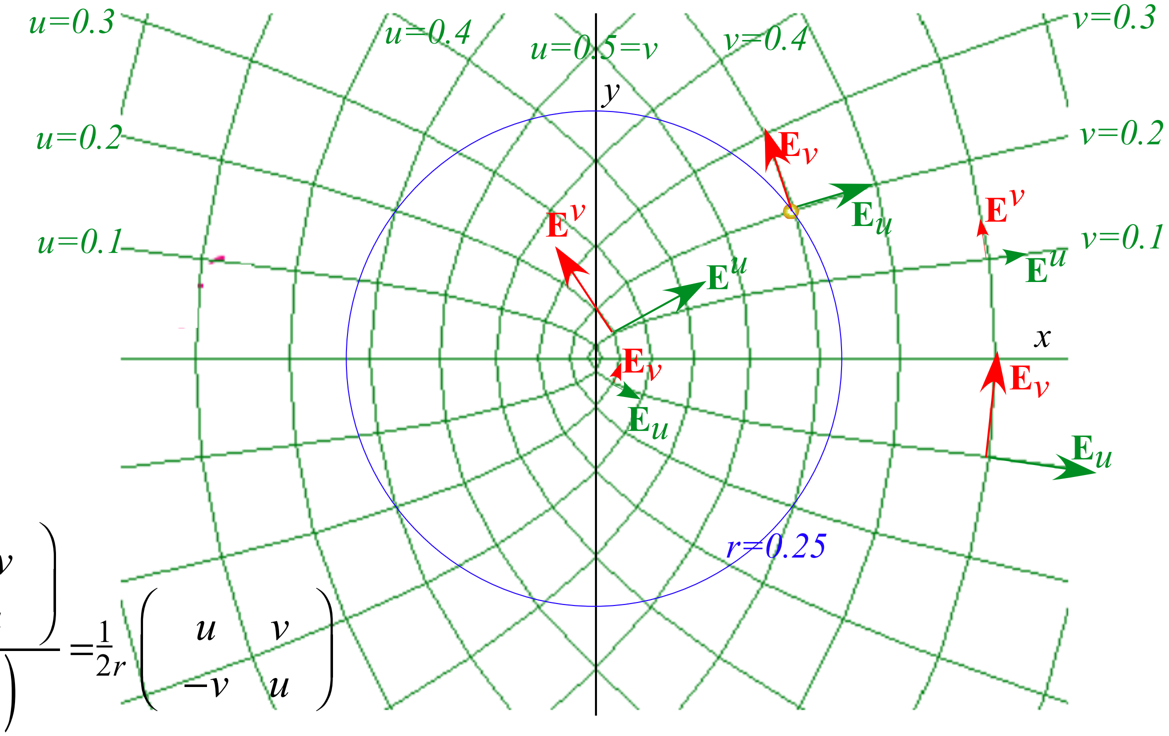
$$y = 2uv$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4u^2v^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Way 3. Parabolic OCC approach

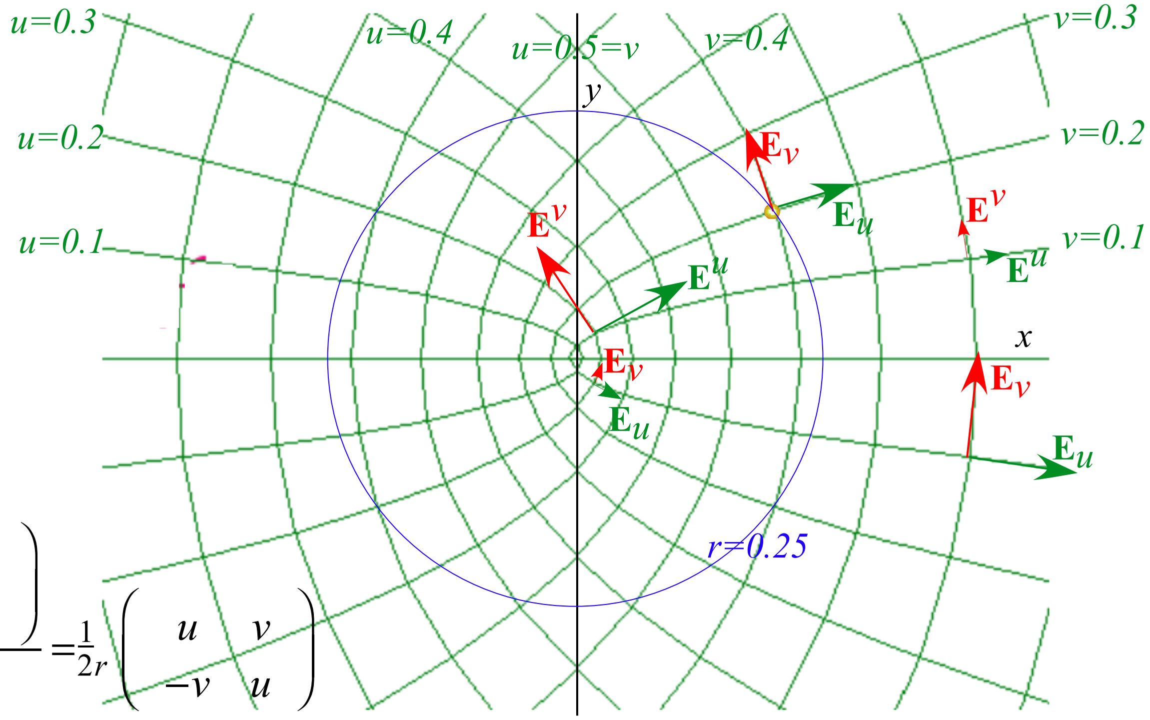
Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$
$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$
$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$
$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$
$$y^2 = 4u^2v^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$g_{uu} = \mathbf{E}_u\bullet\mathbf{E}_u = \mathbf{E}_v\bullet\mathbf{E}_v = g_{vv} = 4u^2 + 4v^2 = 4r \qquad g^{uu} = \mathbf{E}^u\bullet\mathbf{E}^u = \mathbf{E}^v\bullet\mathbf{E}^v = g^{vv} = \frac{1}{4u^2+4v^2} = \frac{1}{4r}$$

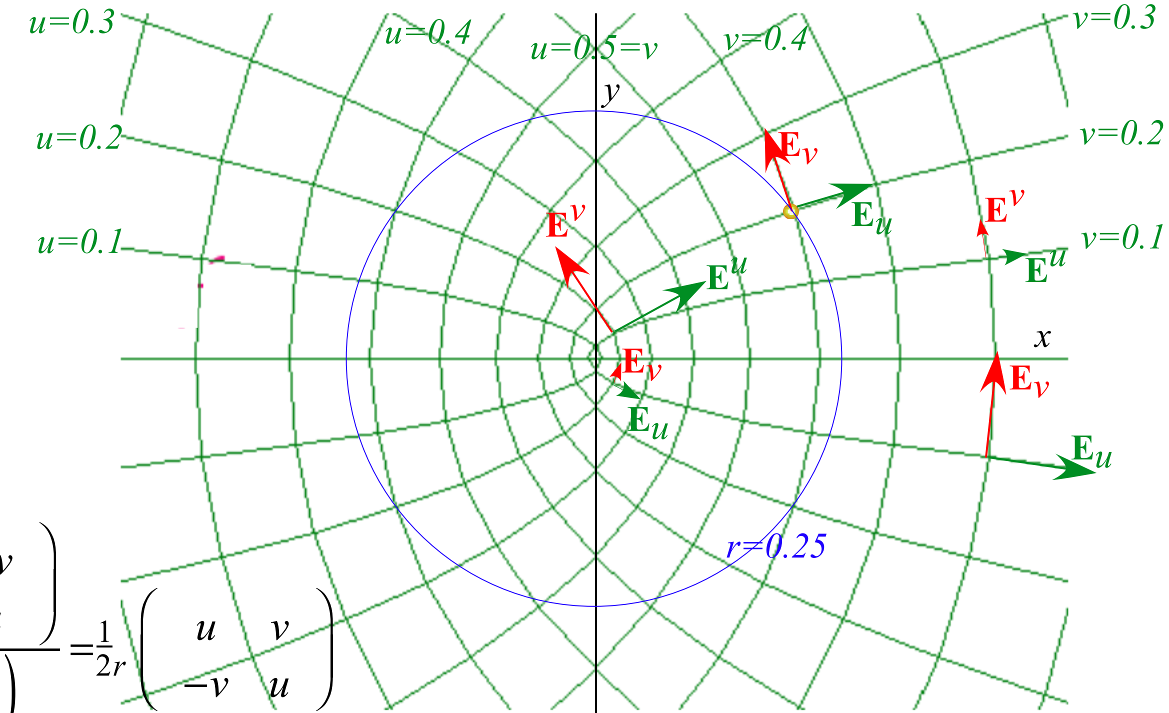
$$g_{uv} = \mathbf{E}_u\bullet\mathbf{E}_v = \mathbf{E}_v\bullet\mathbf{E}_u = g_{vu} = 0 \qquad g^{uv} = \mathbf{E}^u\bullet\mathbf{E}^v = \mathbf{E}^v\bullet\mathbf{E}^u = g^{vu} = 0$$

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x+iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^* z = x^2+y^2 = (u^2+v^2)^2 = u^4+v^4+2u^2v^2$$



$$x = u^2 - v^2$$

$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4u^2v^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. .

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$g_{uu} = \mathbf{E}_u\bullet\mathbf{E}_u = \mathbf{E}_v\bullet\mathbf{E}_v = g_{vv} = 4u^2 + 4v^2 = 4r$$

$$g_{uv} = \mathbf{E}_u\bullet\mathbf{E}_v = \mathbf{E}_v\bullet\mathbf{E}_u = g_{vu} = 0$$

Way 3. Parabolic OCC approach

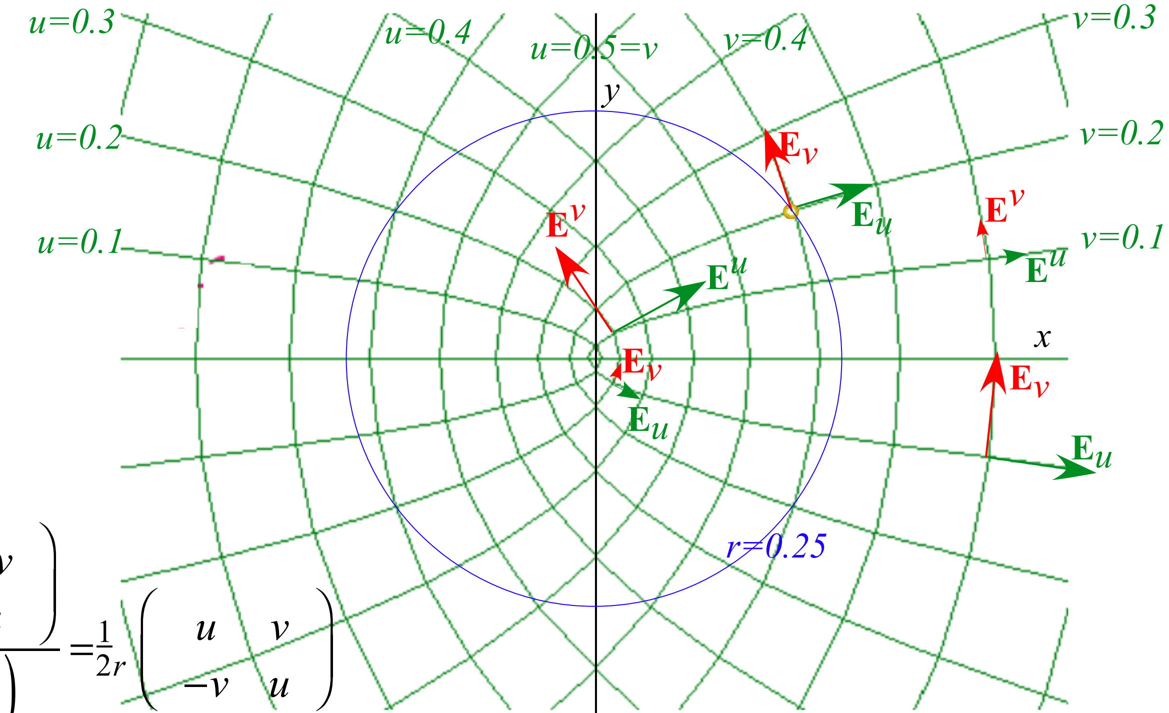
Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^* z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$
$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$
$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$
$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$
$$y^2 = 4u^2v^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

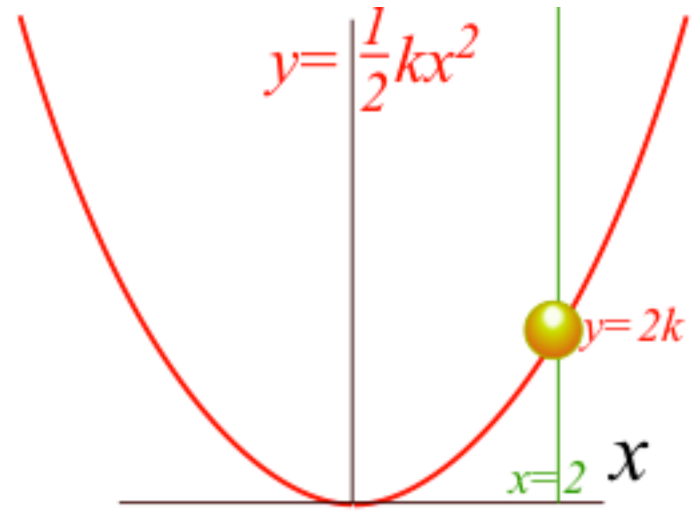
$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2 + p_v^2}{8m(u^2+v^2)} + V$$

$$g^{uu} = \mathbf{E}^u \bullet \mathbf{E}^u = \mathbf{E}^v \bullet \mathbf{E}^v = g^{vv} = \frac{1}{4u^2 + 4v^2} = \frac{1}{4r}$$

$$g^{uv} = \mathbf{E}^u \bullet \mathbf{E}^v = \mathbf{E}^v \bullet \mathbf{E}^u = g^{vu} = 0$$

Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

Way 3. OCC constraint webs

→ *Preview of atomic-Stark orbits*

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

"Non-Holonomic" multipliers

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x+iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2+y^2 = (u^2+v^2)^2 = u^4+v^4+2u^2v^2$$

$$x = u^2 - v^2 \qquad 2u^2 = r+x = \sqrt{x^2+y^2}+x$$

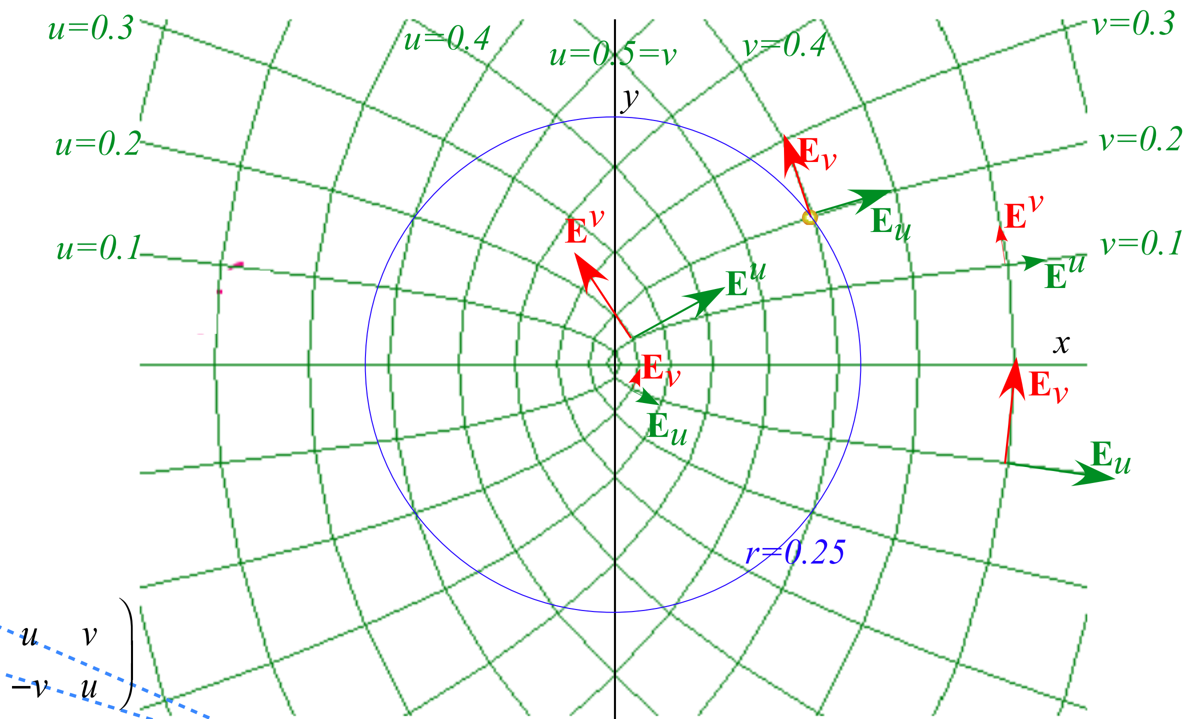
$$y = 2uv$$

$$r = u^2+v^2 \qquad 2v^2 = r-x = \sqrt{x^2+y^2}-x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2-x)$$

$$y^2 = 4u^2v^2 = 4v^2(v^2+x)$$

Gives confocal parabolics



$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_ap_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

Stark-Coulomb potential

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$

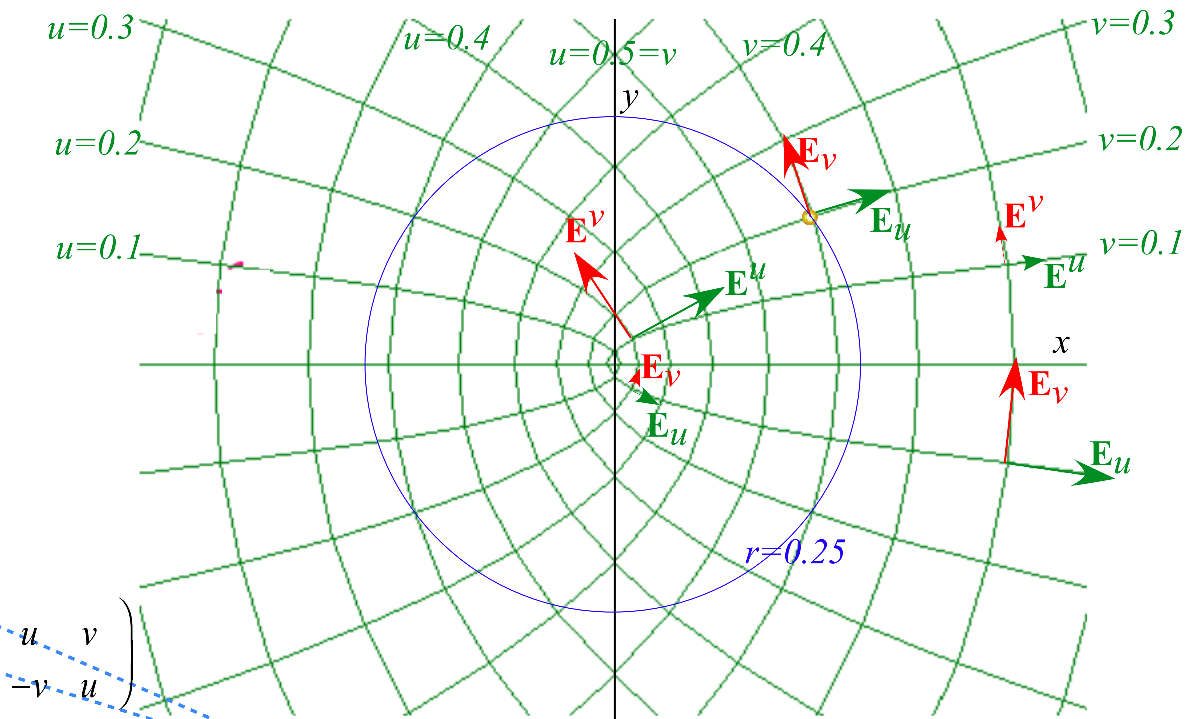
$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4u^2v^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_ap_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2 + p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2+p_v^2) + 4(u^4 - v^4)\varepsilon + 4k \quad \text{for: } H = E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2 - v^2) + \frac{k}{u^2+v^2}$$

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian *(x,y)* and OCC*(u,v)*

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$

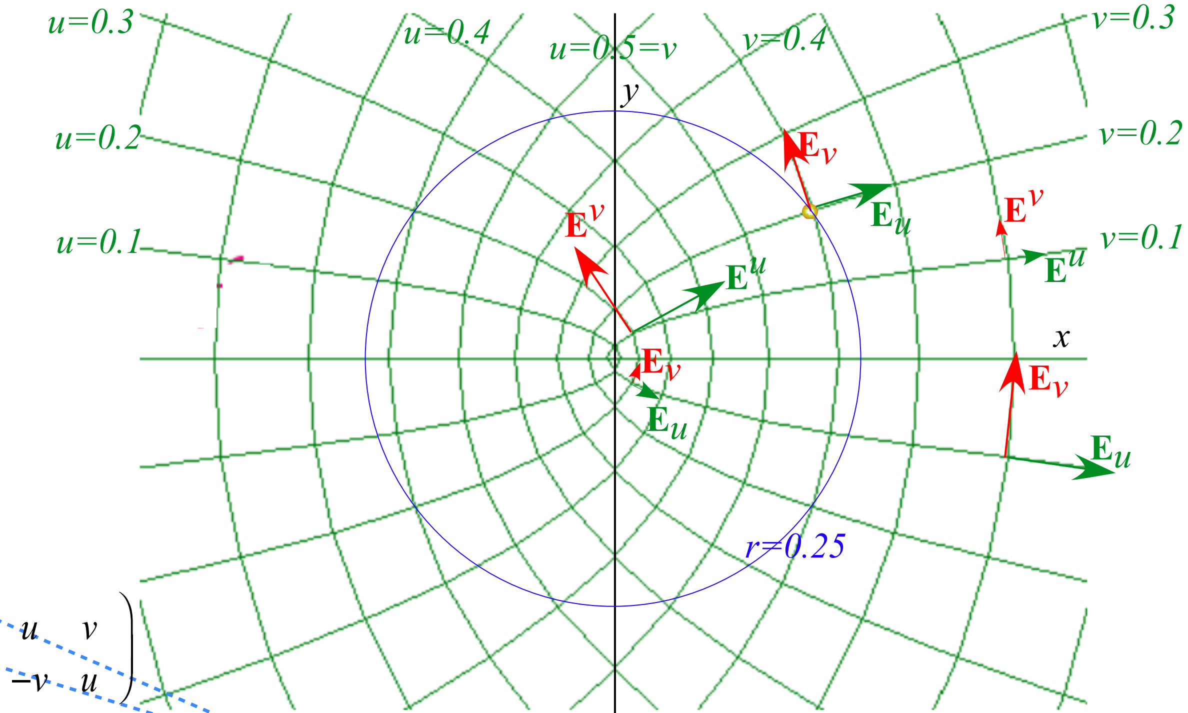
$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4u^2v^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2 + p_v^2}{8m(u^2+v^2)} + V \qquad V = \varepsilon x + k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2 + p_v^2) + 4(u^4 - v^4)\varepsilon + 4k \quad \text{for: } H = E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2 - v^2) + \frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0 = \tfrac{1}{2m}p_u^2 - 4Eu^2 + 4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2 - 4Ev^2 - 4\varepsilon v^4 + 4k \; = \; h_u \; + \; h_v$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

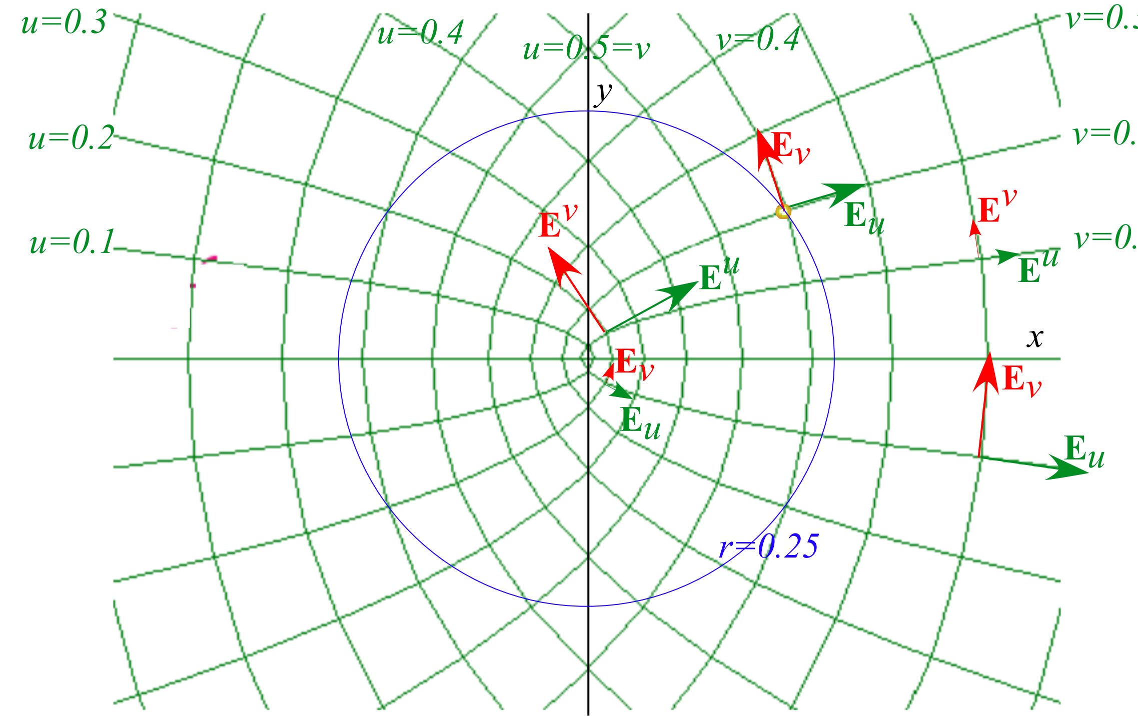
→ *Classical Hamiltonian separability*

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

"Non-Holonomic" multipliers



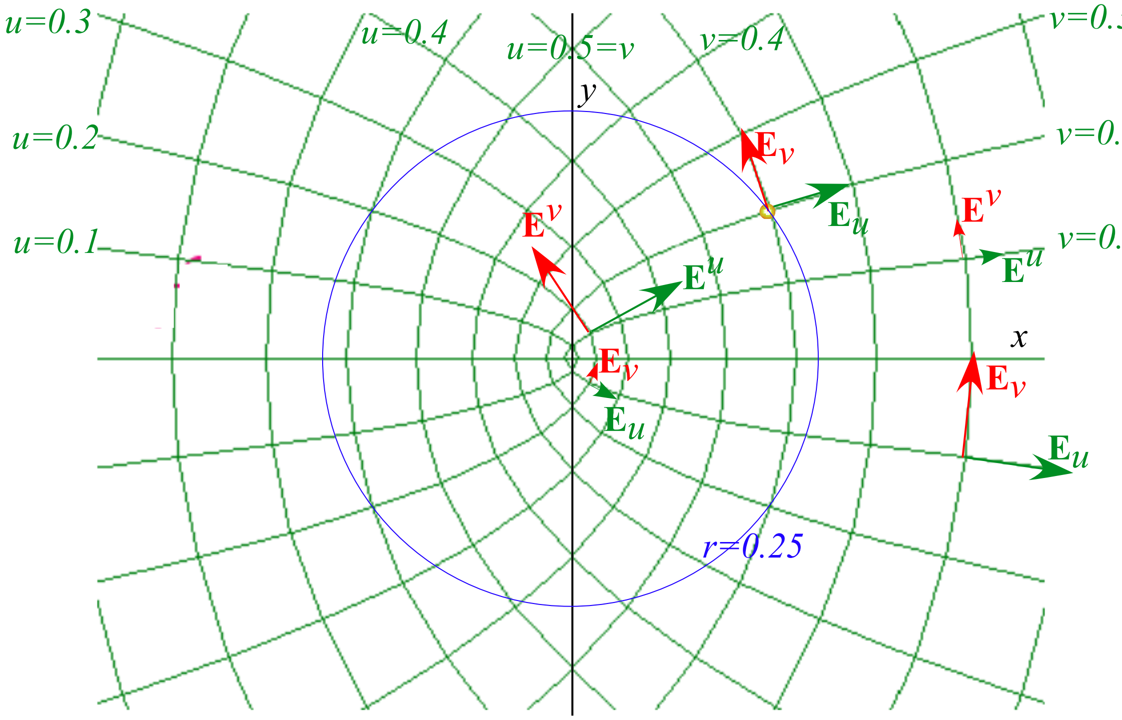
Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

Stark-Coulomb potential



Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

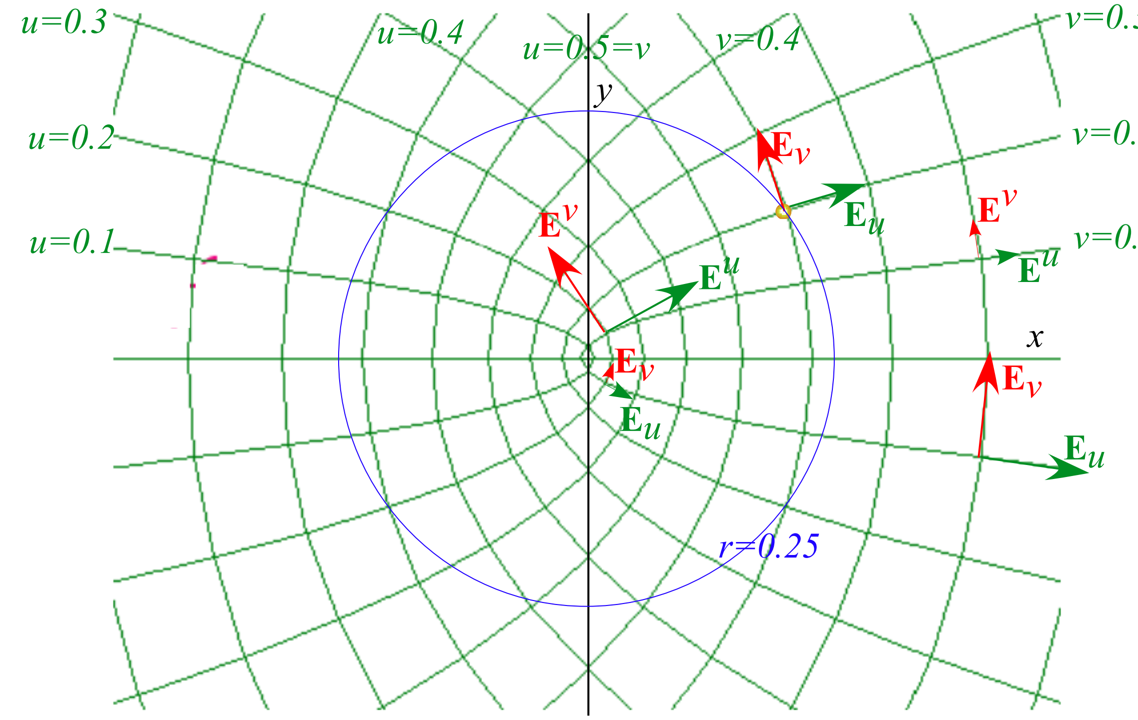
$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2+p_v^2) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H=E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$



Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

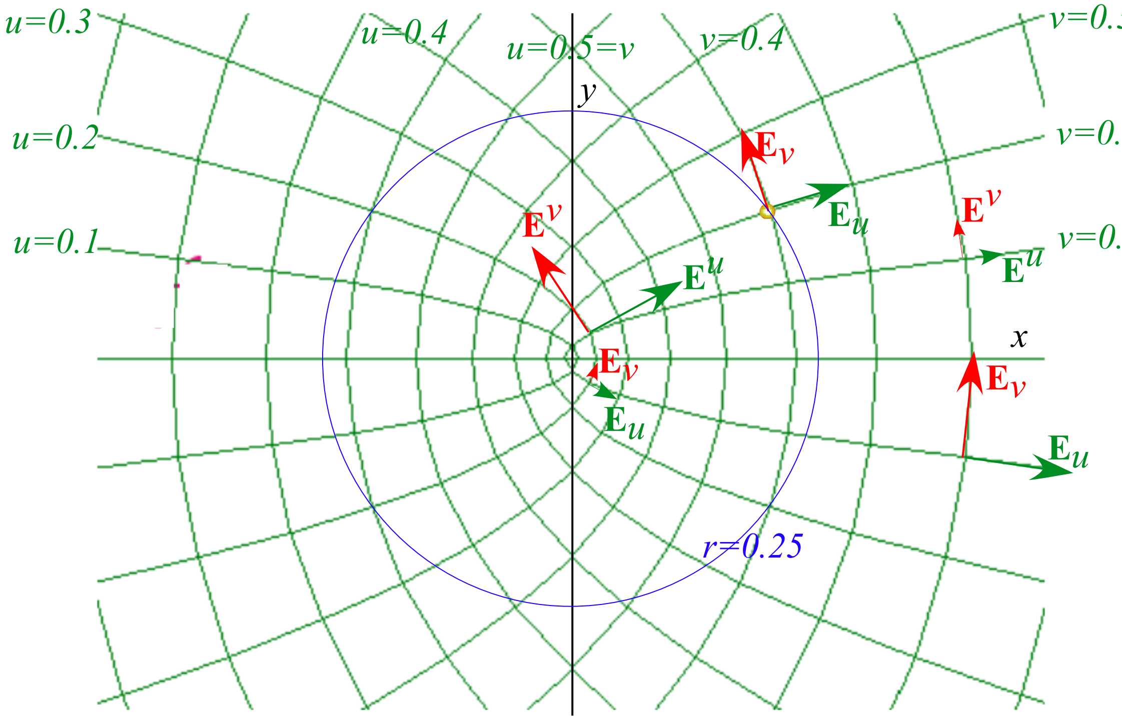
Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2+p_v^2) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H=E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0 = \tfrac{1}{2m}p_u^2 - 4Eu^2 + 4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2 - 4Ev^2 - 4\varepsilon v^4 + 4k \quad = \quad h_u \quad + \quad h_v$$



Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2+p_v^2) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H=E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0 = \tfrac{1}{2m}p_u^2 - 4Eu^2 + 4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2 - 4Ev^2 - 4\varepsilon v^4 + 4k \quad = \quad h_u \quad + \quad h_v$$

Zero Stark-field $(\varepsilon=0)$ gives h_u or h_v harmonic oscillation if $E<0$. It's unstable or anharmonic otherwise.

$$\dot{p}_u = -\frac{\partial h_u}{\partial u} = -8Eu + 16\varepsilon u^3 \qquad \dot{u} = \frac{\partial h_u}{\partial p_u} = p_u/m \qquad \dot{p}_v = -\frac{\partial h_v}{\partial v} = -8Ev - 16\varepsilon v^3 \qquad \dot{v} = \frac{\partial h_v}{\partial p_v} = p_v/m$$

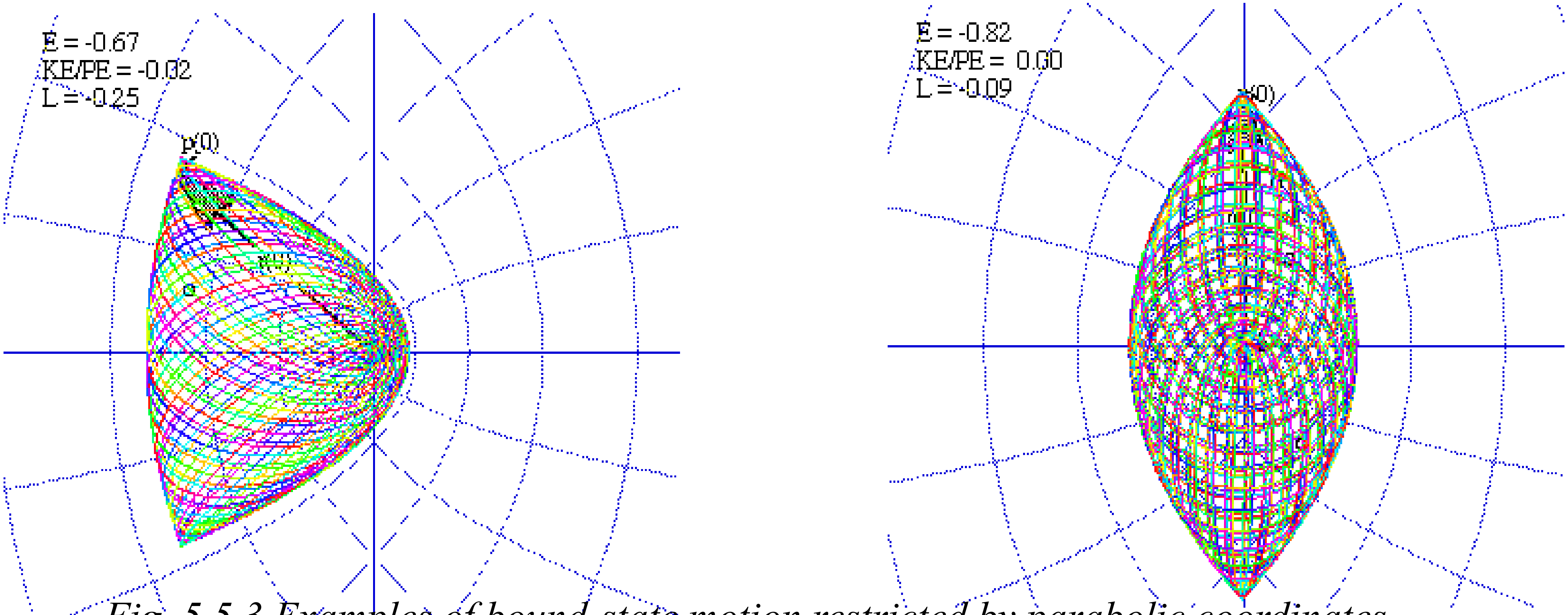


Fig. 5.5.3 Examples of bound-state motion restricted by parabolic coordinates

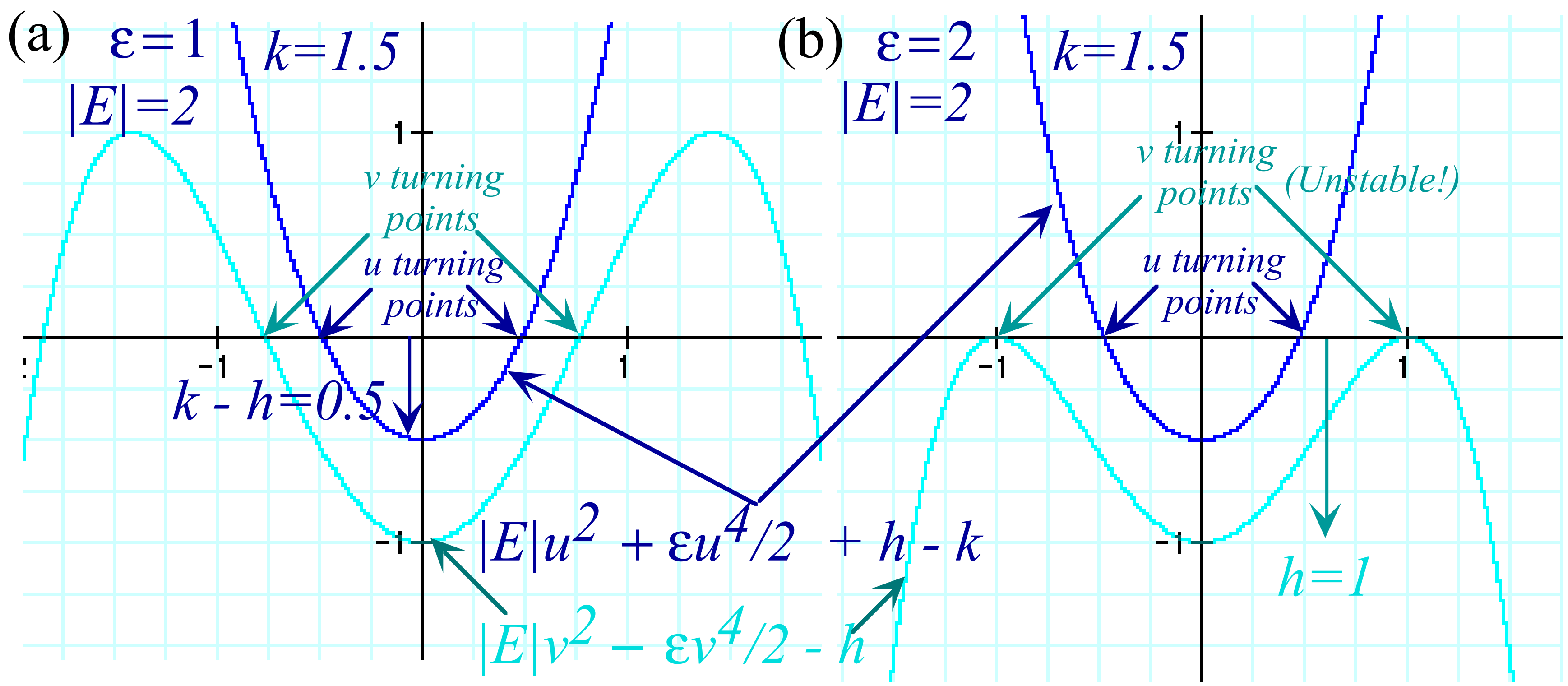
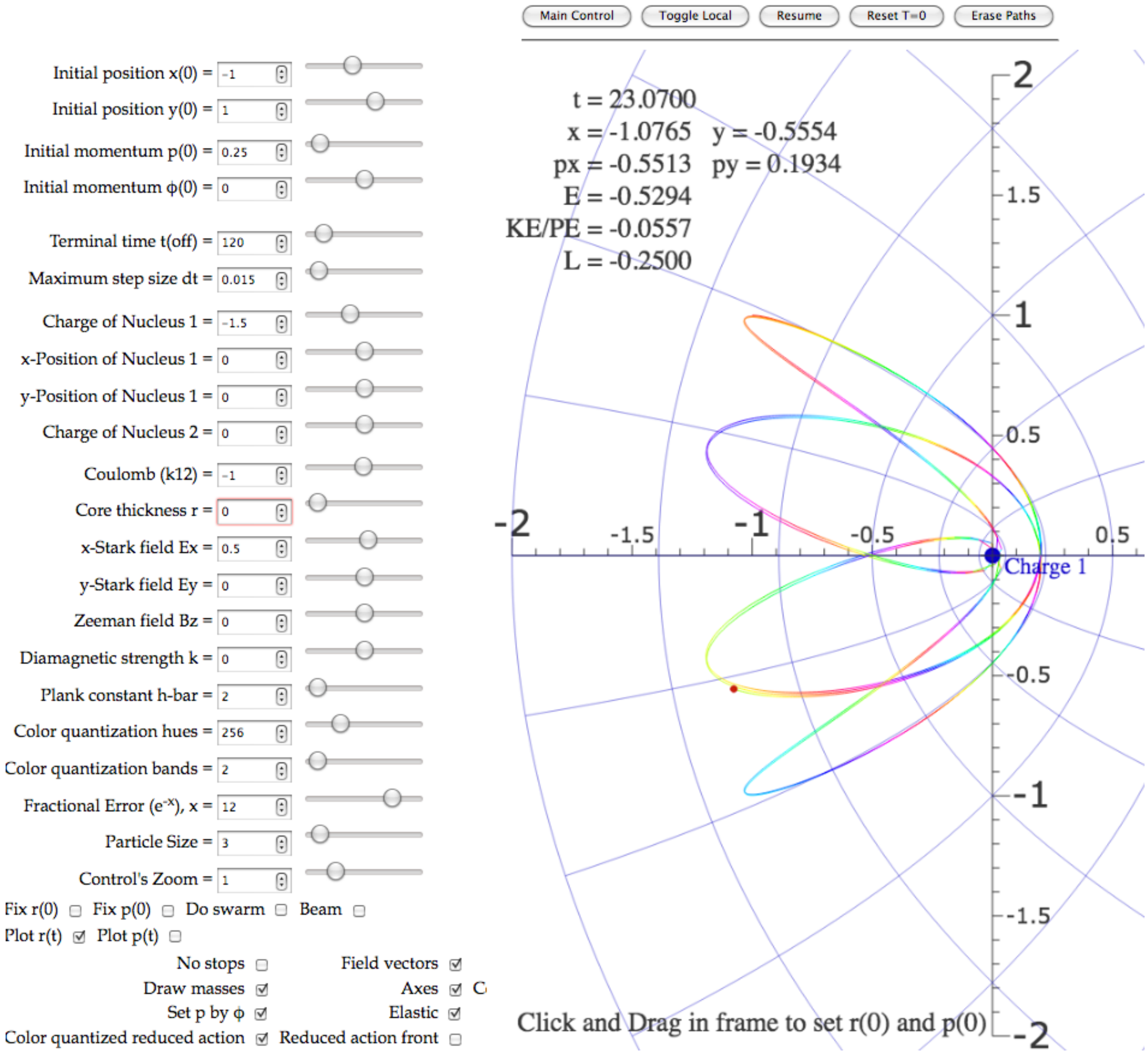


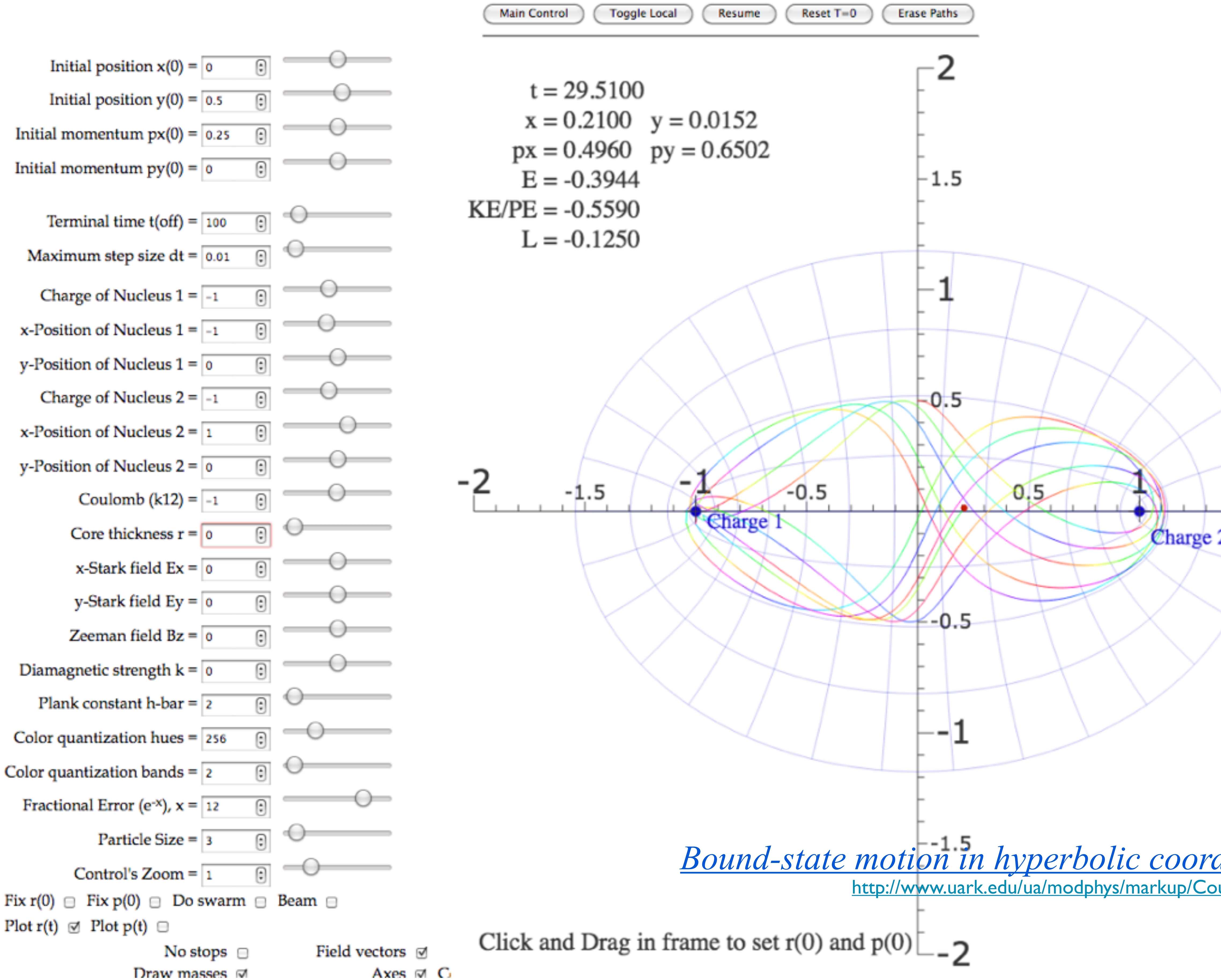
Fig. 5.5.2 Effective potentials for parabolic coordinates

Examples of bound-state motion restricted by parabolic coordinates
(H classical electronic Stark-field orbits with color-quantization)



[Bound-state motion in parabolic coordinates](http://www.uark.edu/ua/modphys/markup/CoulItWeb.html) http://www.uark.edu/ua/modphys/markup/CoulItWeb.html

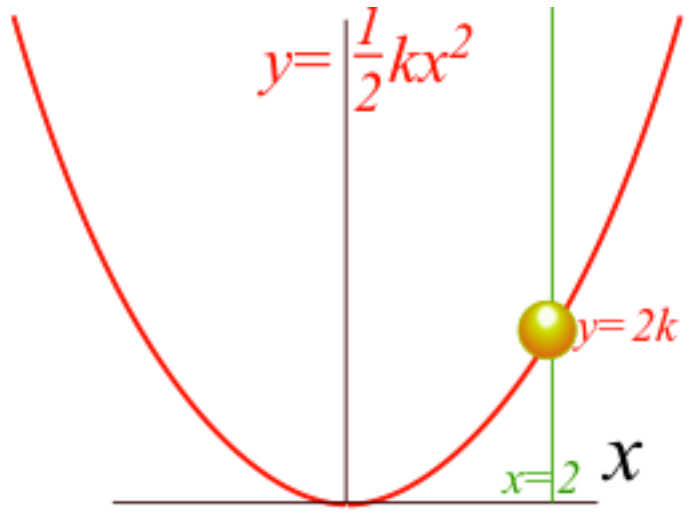
Examples of bound-state motion restricted by hyperbolic-elliptic coordinates (H2+-ion classical electronic orbits with color-quantization)



[Bound-state motion in hyperbolic coordinates](http://www.uark.edu/ua/modphys/markup/CoulItWeb.html)

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html

Simple constrained problem...



...and a variety of solutions

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

→ *Way 4. Lagrange multipliers*

Lagrange multiplier as eigenvalues

Multiple multipliers

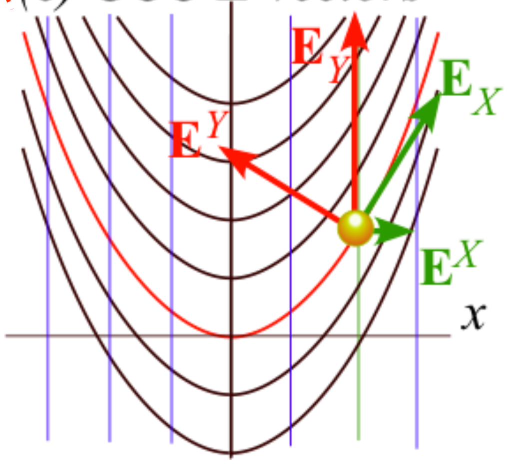
"Non-Holonomic" multipliers

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)



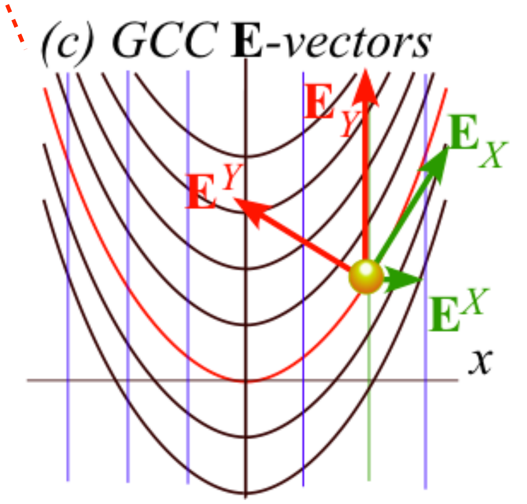
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

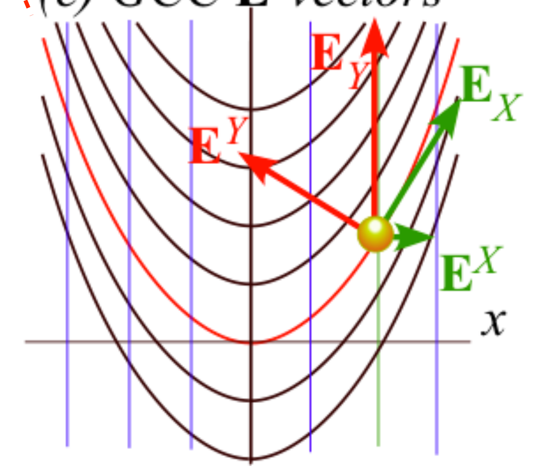
$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

(c) GCC **E***-vectors*



Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

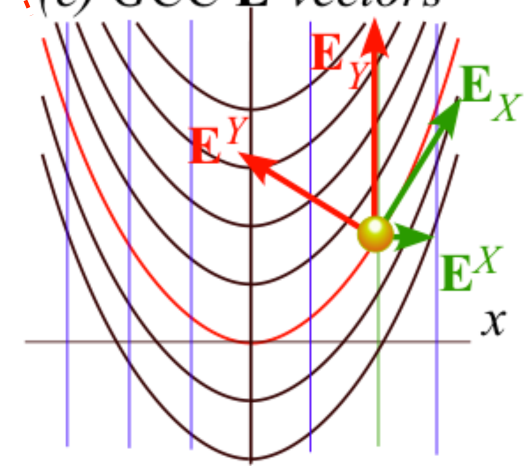
$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2} kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.



Lagrange multiplier approaches

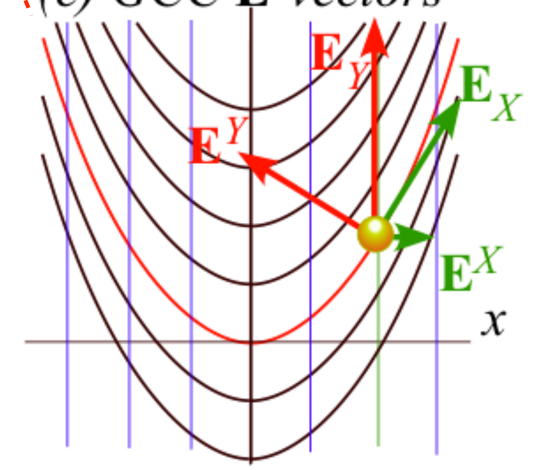
Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . ($\mathbf{F} \propto \nabla c^1$)

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2} kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$



Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

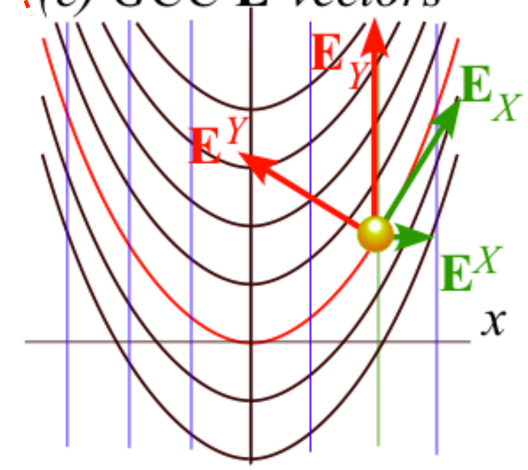
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2} kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

Lagrange multiplier approaches

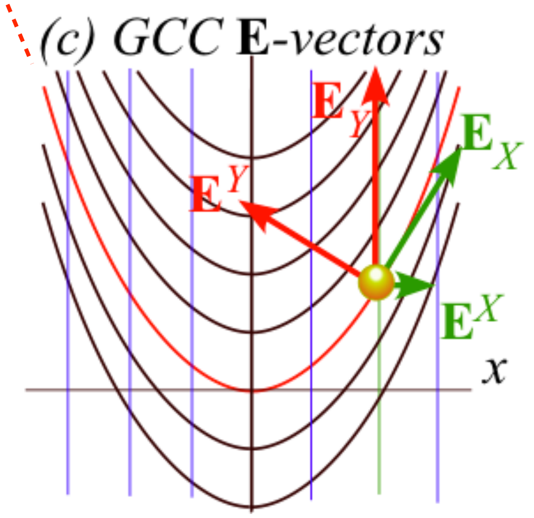
Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$



Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

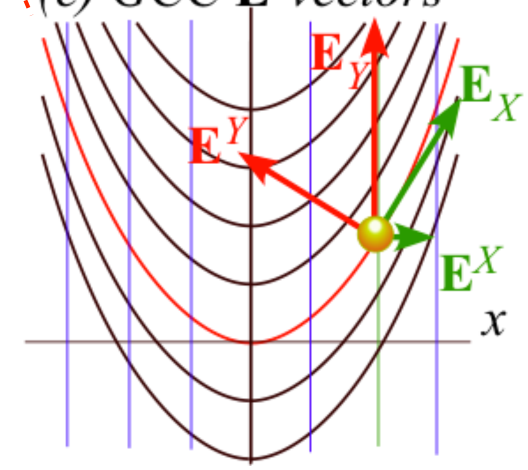
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$.

Lagrange multiplier approaches

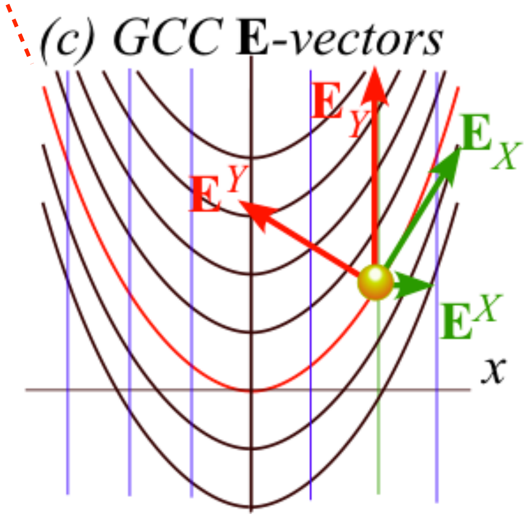
Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . ($\mathbf{F} \propto \nabla c^1$)

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$



Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

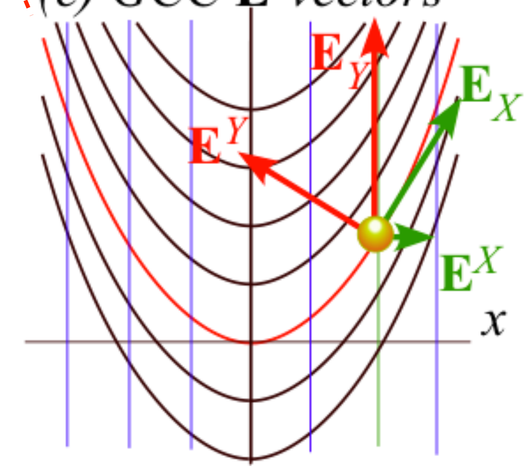
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

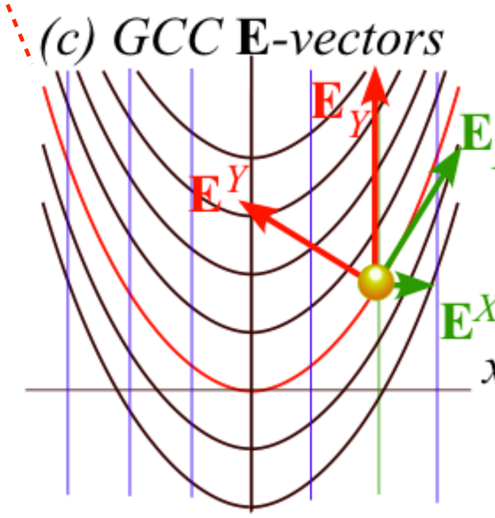
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola *y=1/2kx²* is defined as follows.

$$c^1 = \tfrac{1}{2} kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . ($\mathbf{F} \propto \nabla c^1$)



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function *y=1/2kx²* has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = m(-k\dot{x}^2 - kx\ddot{x} - g)$$

Lagrange multiplier approaches

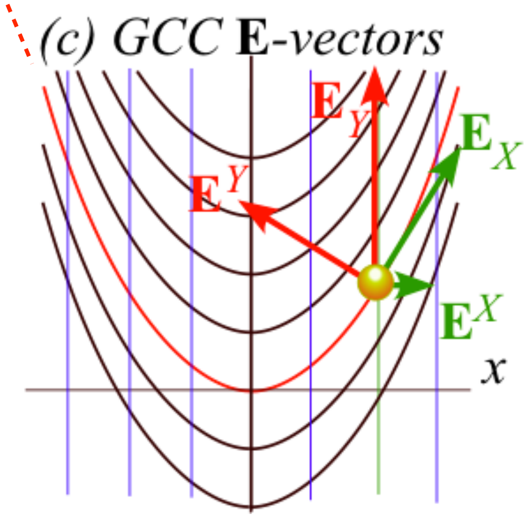
Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda\nabla c^1 = \lambda\nabla(\tfrac{1}{2}kx^2 - y) = \lambda\begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda\begin{pmatrix} kx \\ -1 \end{pmatrix}$$



Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda\begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained *x*-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx$$

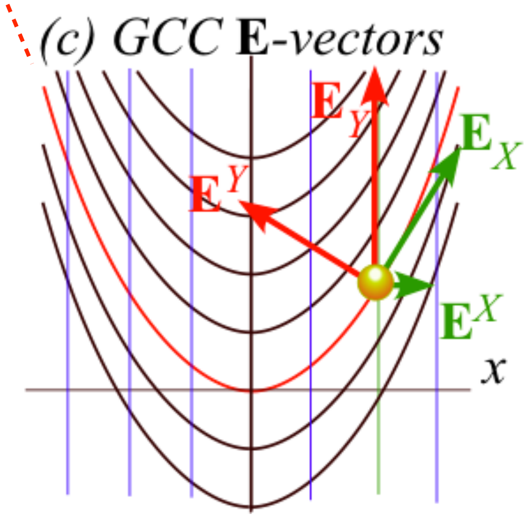
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . ($\mathbf{F} \propto \nabla c^1$)



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained x-equation of motion.

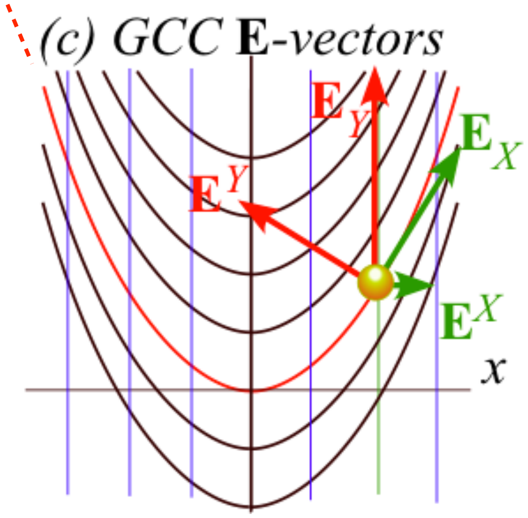
$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$ *(Back to "Stupid-Parabolic" GCC)*

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained x-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

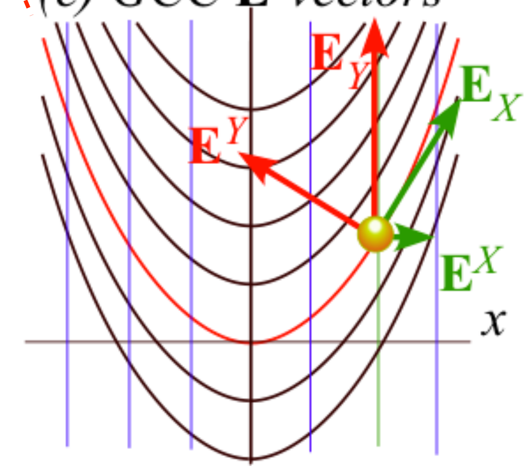
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient. $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained x-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

(Same equation as on p.12)

$$\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2x^2}kx$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

→ *Lagrange multiplier as eigenvalues*

Multiple multipliers

"Non-Holonomic" multipliers

Lagrange multiplier basics

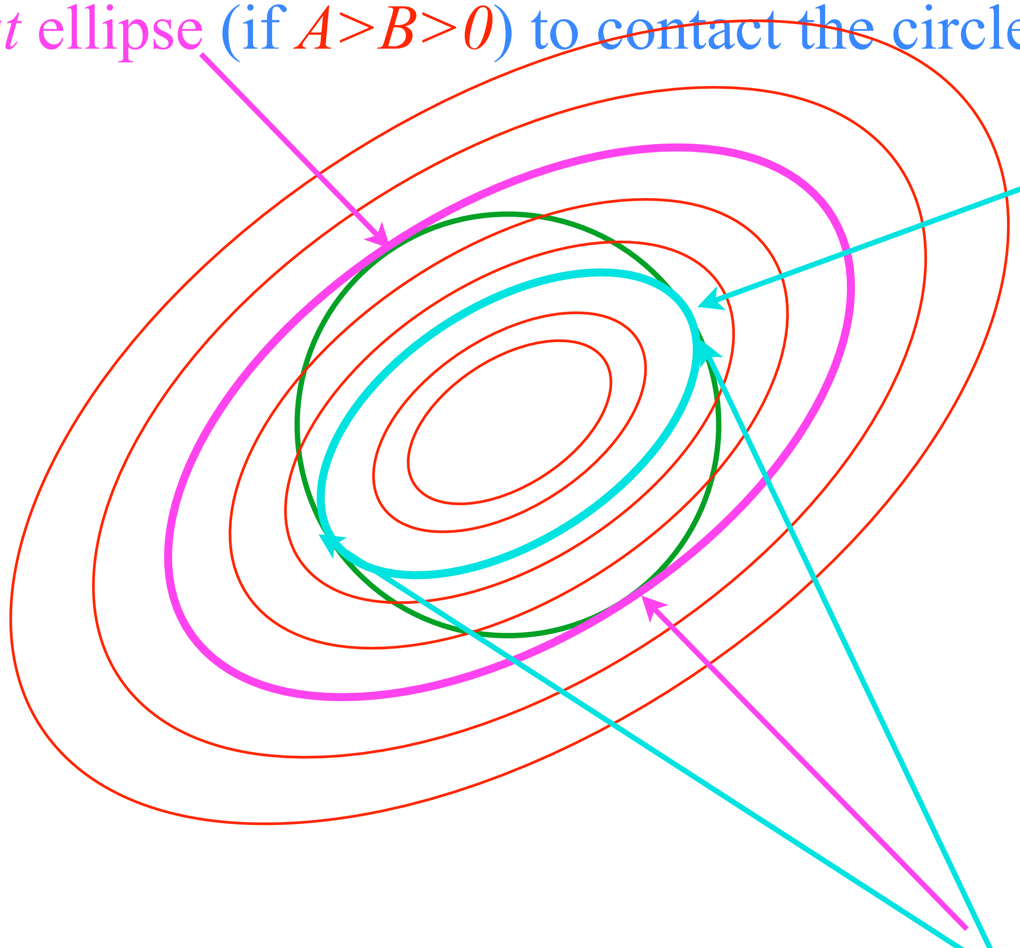
Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.

The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

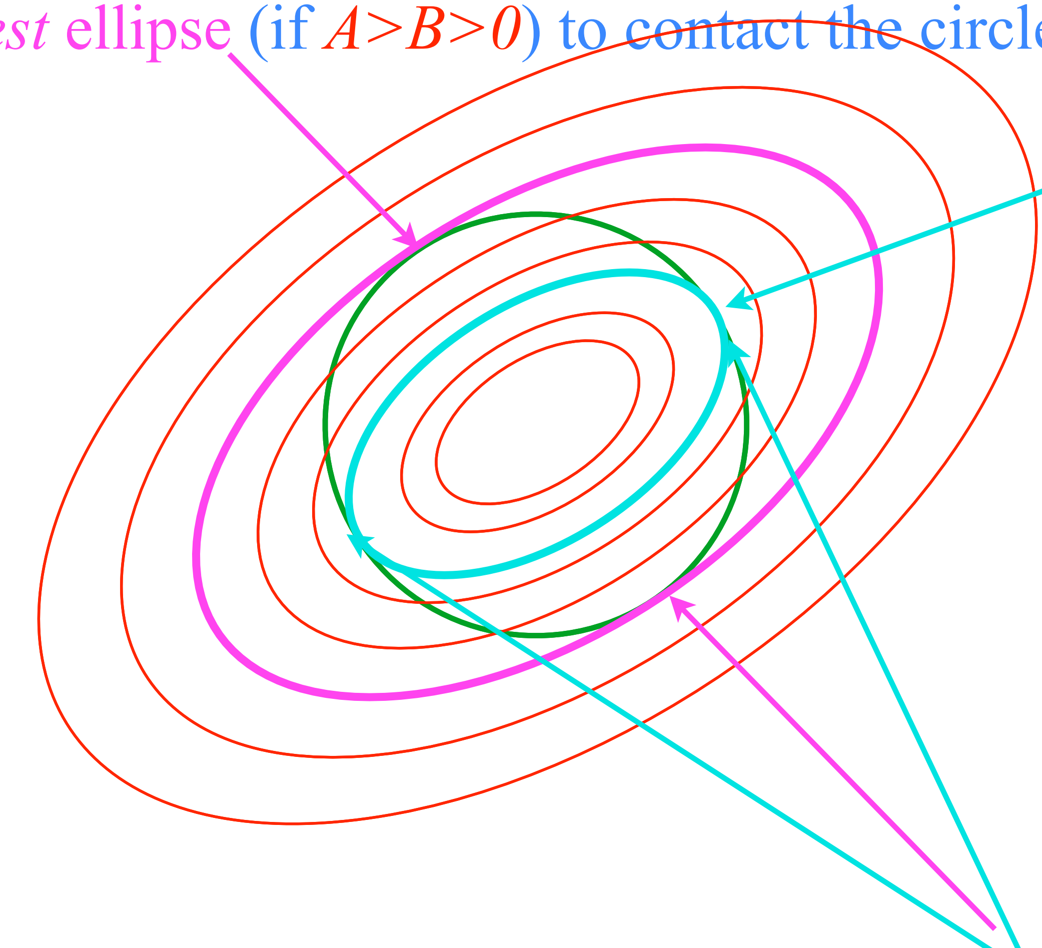
$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



Extreme cases occur only at *contact points*

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



The contact points satisfy gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Extreme cases occur only at *contact points*

This amounts to a *λ-eigenvalue-eigenvector* equation

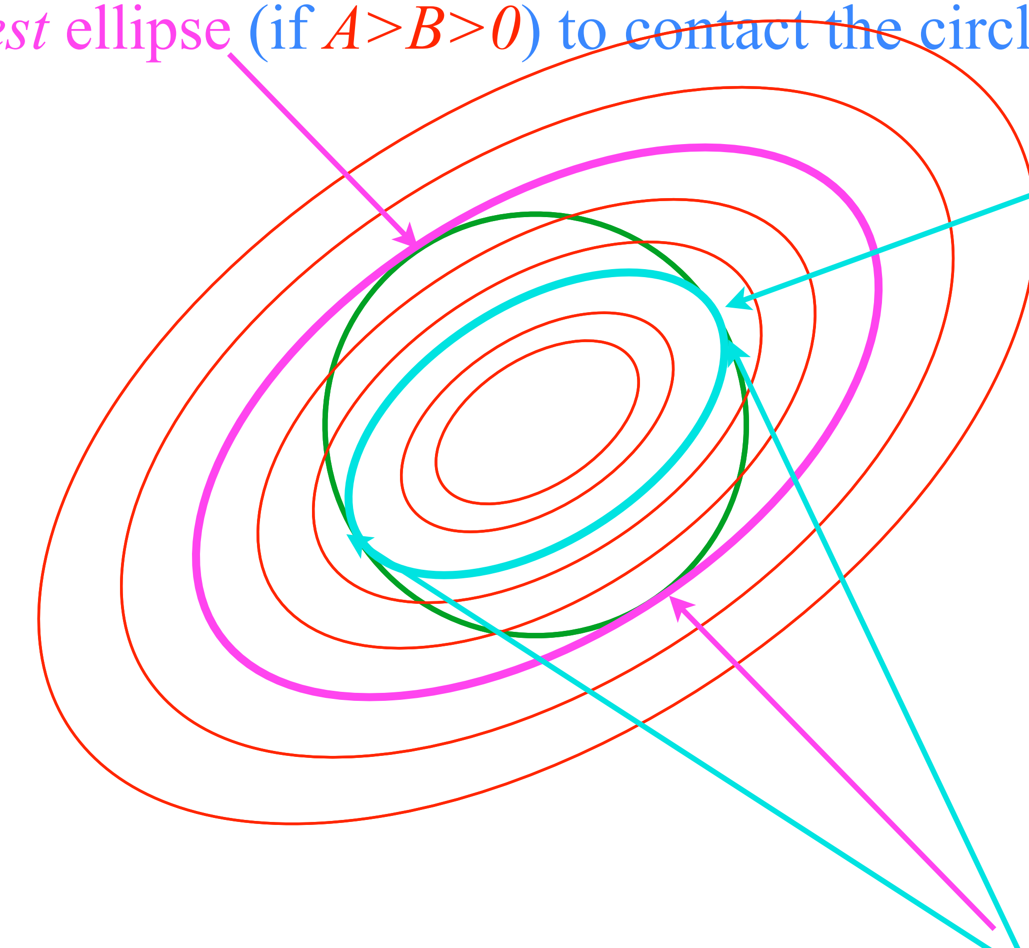
$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek "L")

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Extreme cases occur only at *contact points*

This amounts to a *λ-eigenvalue-eigenvector* equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek "L")

Eigenvalues λ are *extreme* matrix "own"-values $\langle\psi|M|\psi\rangle$ subject *Norm-constraint* $\langle\psi|\psi\rangle=1$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

→ *Multiple multipliers*

"Non-Holonomic" multipliers

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{\quad}{\partial q^k}\frac{\partial c}{\quad} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{\quad}{\partial q^k}\frac{\partial c}{\quad} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

Two or more constraints $c^1(q^k) = const., c^2(q^k) = const., \cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

→ *"Non-Holonomic" multipliers*

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C^1_1 dq^1 + C^1_2 dq^2 + \ldots$$

$$0 = C^2_1 dq^1 + C^2_2 dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 \frac{\partial c^1}{\partial q^2} + \lambda_2 \frac{\partial c^2}{\partial q^2} + \ldots$$

$$\vdots \qquad \vdots$$

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 C^1_1 + \lambda_2 C^2_1 + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 C^1_2 + \lambda_2 C^2_2 + \ldots$$

$$\vdots \qquad \vdots$$

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \dots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \dots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C^1_1 dq^1 + C^1_2 dq^2 + \dots$$

$$0 = C^2_1 dq^1 + C^2_2 dq^2 + \dots$$

$$\vdots \qquad \vdots$$

Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \dots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 \frac{\partial c^1}{\partial q^2} + \lambda_2 \frac{\partial c^2}{\partial q^2} + \dots$$

$$\vdots \qquad \vdots$$

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 C^1_1 + \lambda_2 C^2_1 + \dots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 C^1_2 + \lambda_2 C^2_2 + \dots$$

$$\vdots \qquad \vdots$$

If a differential can't be integrated to give a constraint function it's called a *non-holonomic constraint*.

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C_1^1 dq^1 + C_2^1 dq^2 + \ldots$$

$$0 = C_1^2 dq^1 + C_2^2 dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 \frac{\partial c^1}{\partial q^2} + \lambda_2 \frac{\partial c^2}{\partial q^2} + \ldots$$

$$\vdots \qquad \vdots$$

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 C_1^1 + \lambda_2 C_1^2 + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 C_2^1 + \lambda_2 C_2^2 + \ldots$$

$$\vdots \qquad \vdots$$

If a differential can't be integrated to give a constraint function it's called a *non-holonomic constraint*.

I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)
A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j}$$

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C_1^1 dq^1 + C_2^1 dq^2 + \ldots$$

$$0 = C_1^2 dq^1 + C_2^2 dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 \frac{\partial c^1}{\partial q^2} + \lambda_2 \frac{\partial c^2}{\partial q^2} + \ldots$$

$$\vdots \qquad \vdots$$

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 C_1^1 + \lambda_2 C_1^2 + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 C_2^1 + \lambda_2 C_2^2 + \ldots$$

$$\vdots \qquad \vdots$$

If a differential can't be integrated to give a constraint function it's called a *non-holonomic constraint*.

I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)

A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j}$$

Force components $F_k^\gamma = \frac{\partial c^\gamma}{\partial q^k} = C_k^\gamma$ must satisfy *reciprocity relations* to be gradients of a c^γ function.

Integral constraint differentials

$$\frac{\partial F_k^\gamma}{\partial q^j} = \frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial F_j^\gamma}{\partial q^k}$$

General differential constraint relations

$$\frac{\partial C_k^\gamma}{\partial q^j} \;\; \textit{may or may not be} \;\; \frac{\partial C_j^\gamma}{\partial q^k}$$