Lecture 16
Thur. 10.19.2017

Hamilton Equations for Trebuchet and Other Things (Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)
- *Hamiltonian definition from Lagrangian and γ_{mn} tensor*
- *Hamilton's equations and Poincare invariant relations*
- *Hamiltonian expression and contravariant γ^{mn} tensor*

Hamiltonian energy and momentum conservation and symmetry coordinates
- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *The multiple approaches to Mechanics (and physics in general)*

Chapter 1. The Trebuchet: A dream problem for Galileo?

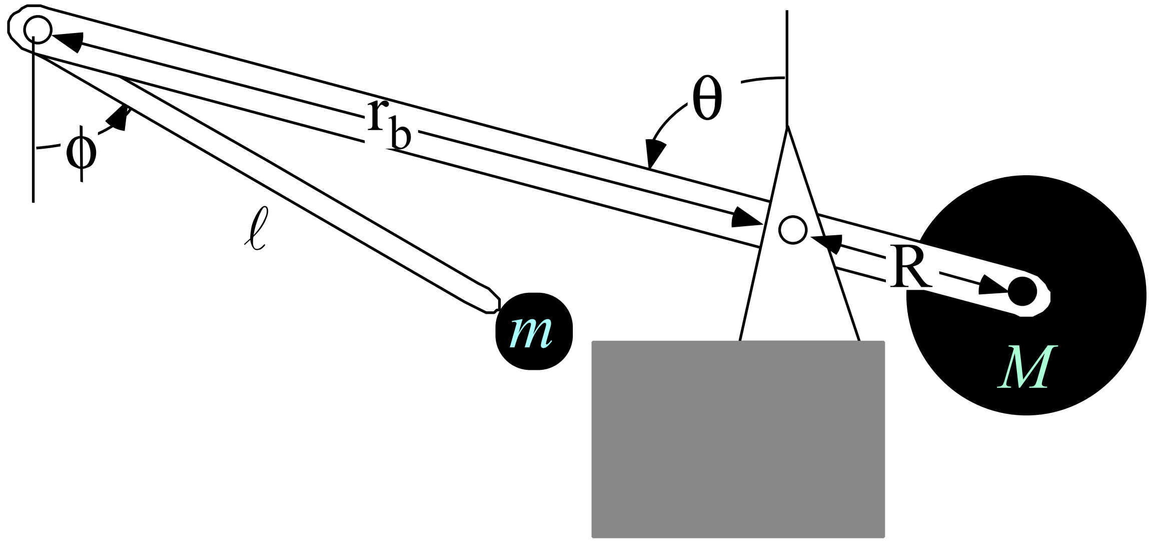
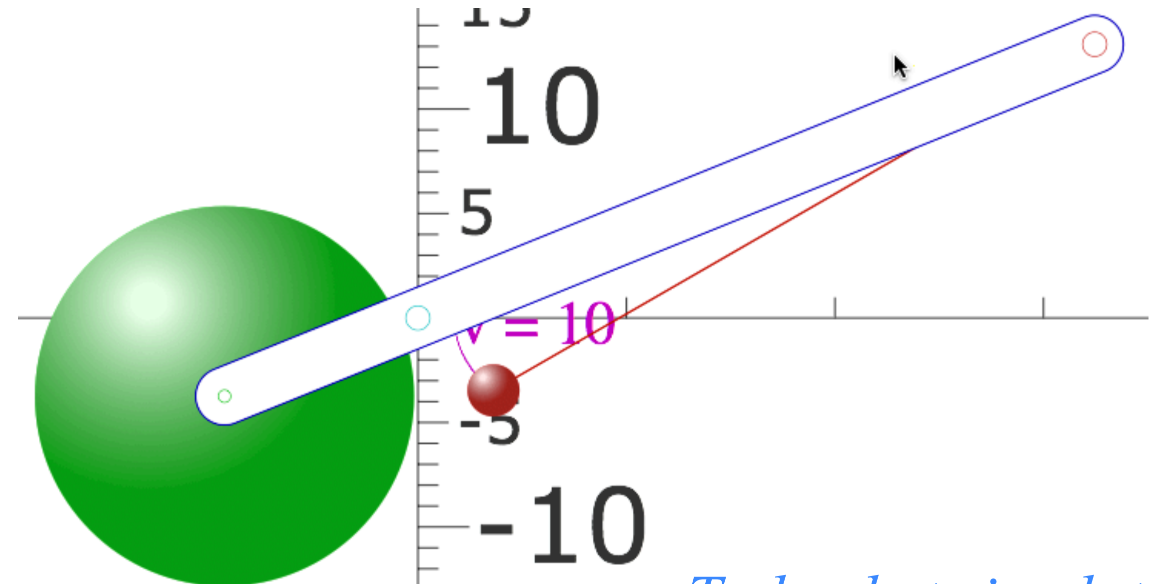
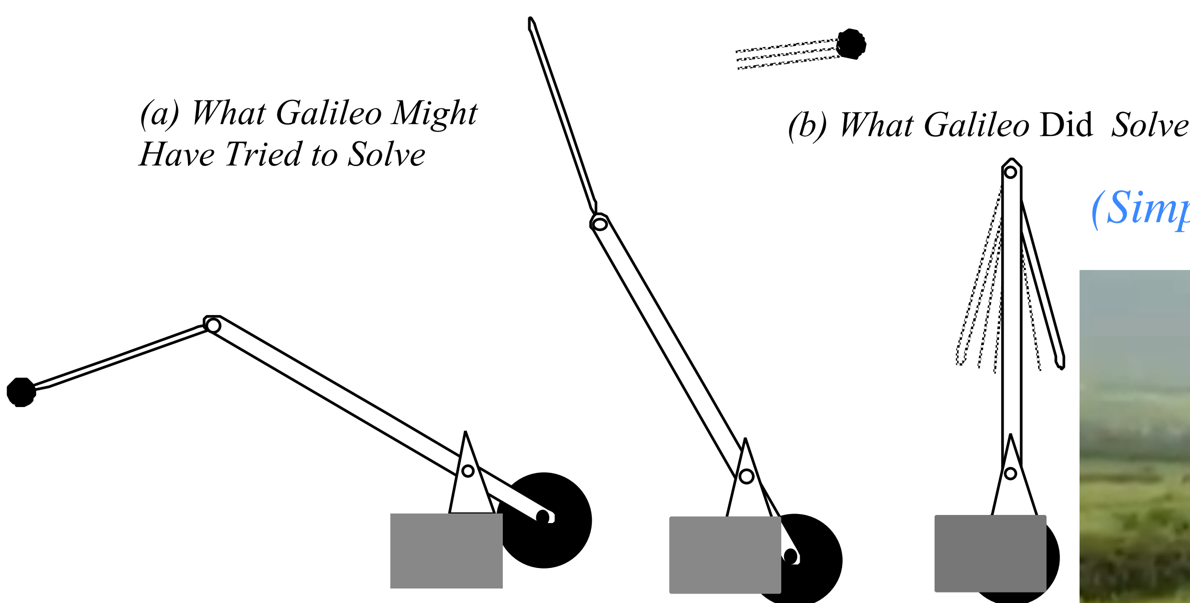


Fig. 2.1.1 An elementary ground-fixed trebuchet

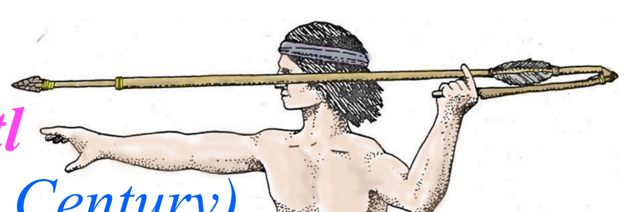


Trebuchet simulator
http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem



The Atlatl
(Cahokia, IL 12th Century)

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Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

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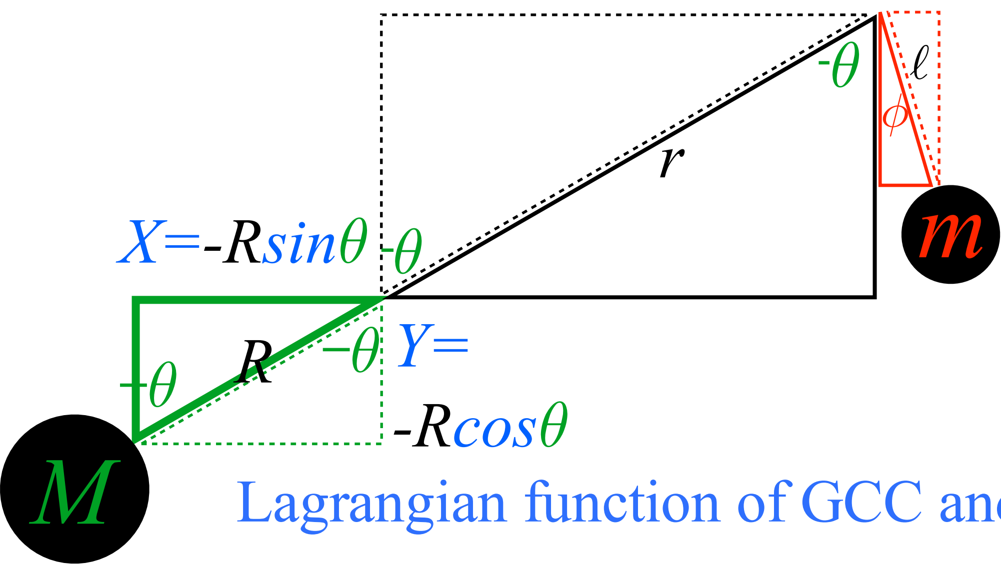

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Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

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1st differential chain

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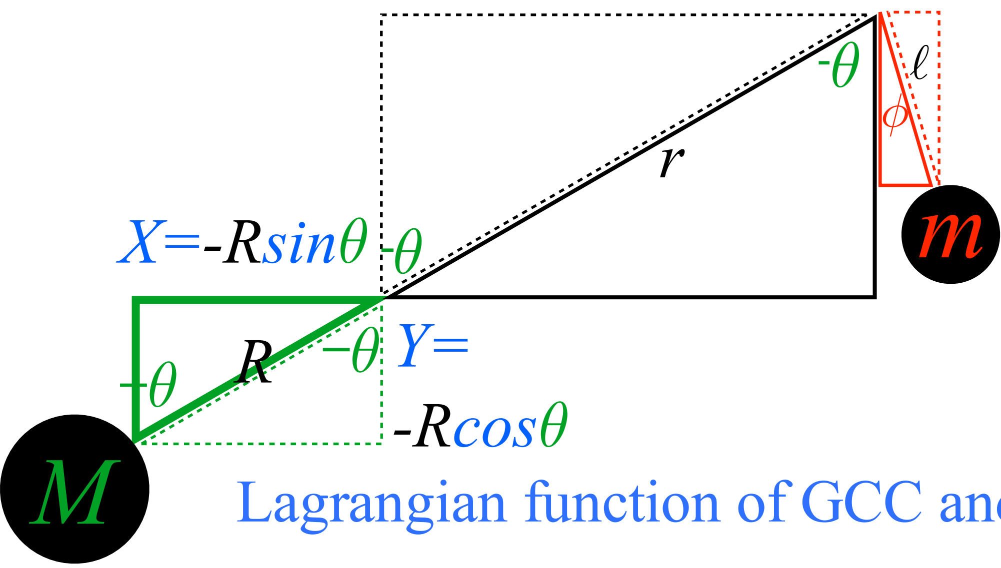

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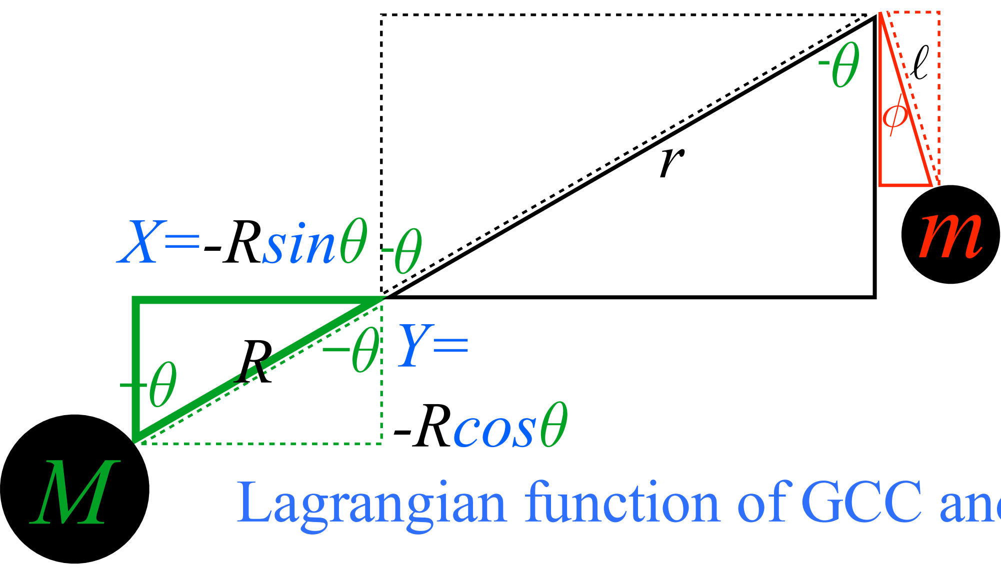

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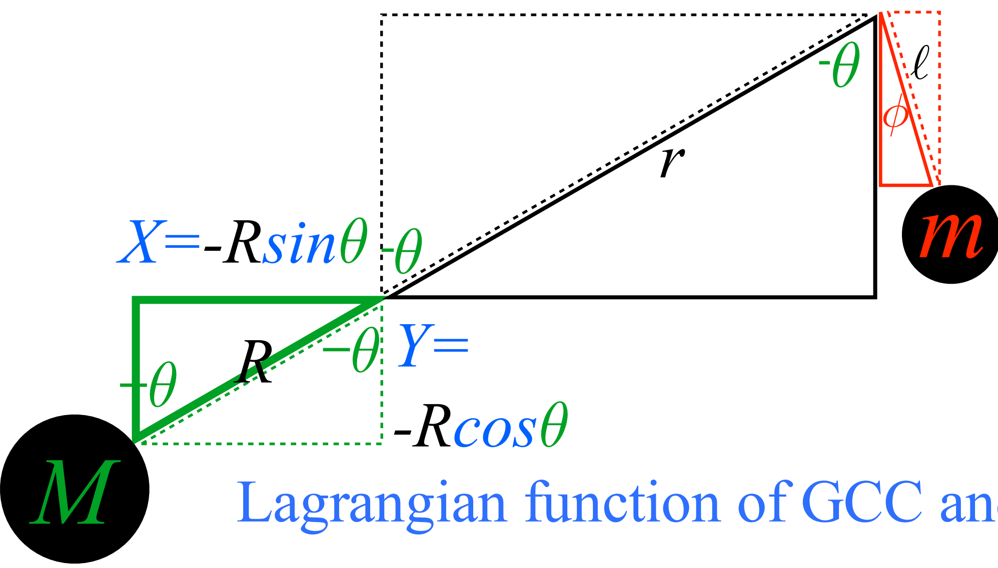
Lagrange equations



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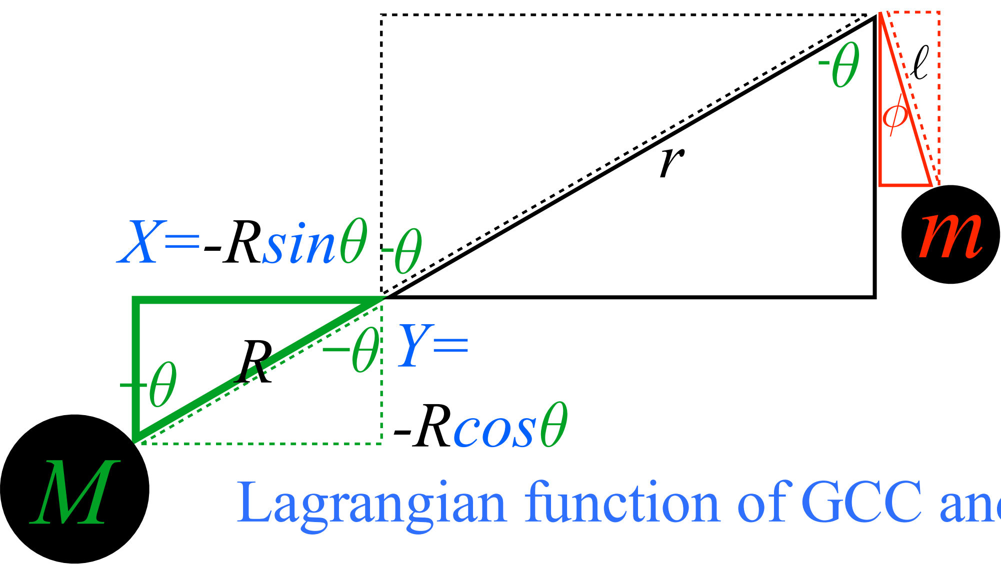

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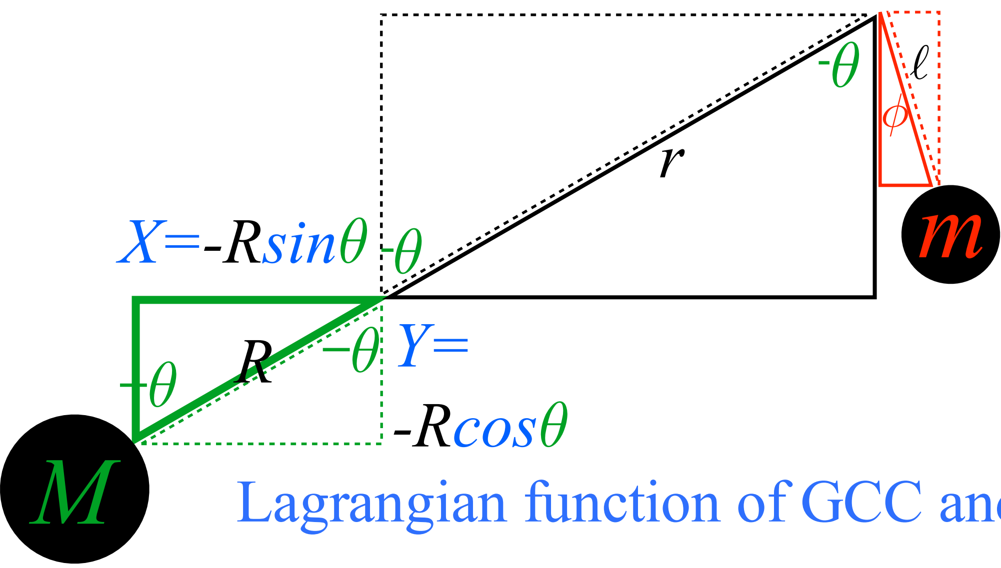

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Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

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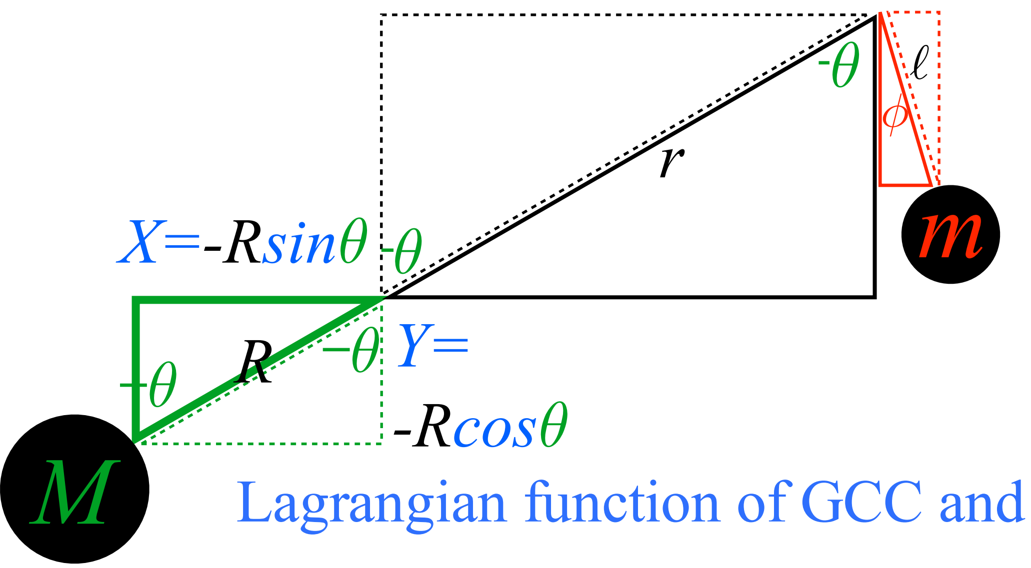

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Dynamic metric tensor γ_{mn} *in GCC* θ *and* ϕ

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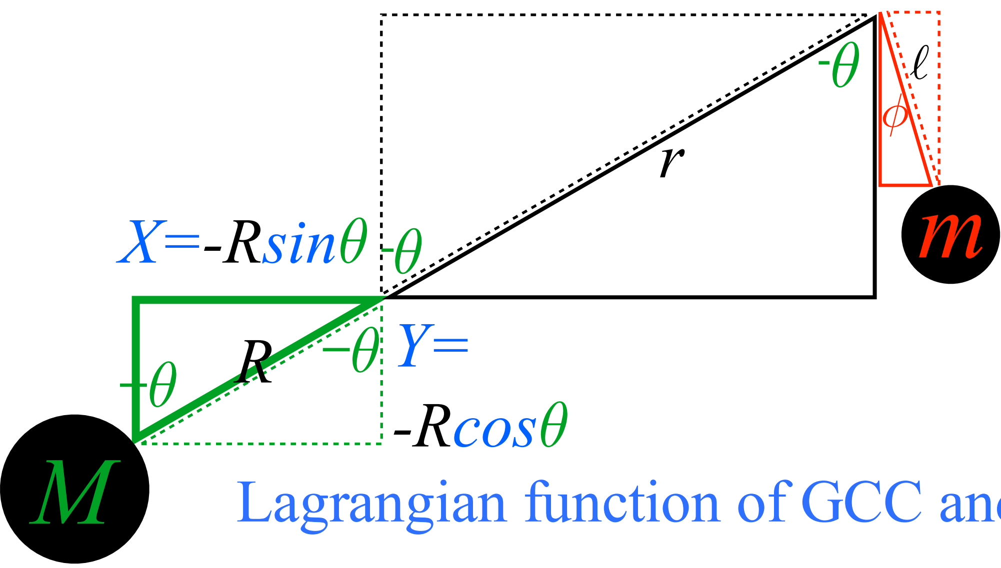

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by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta$$

by Lagrange equations

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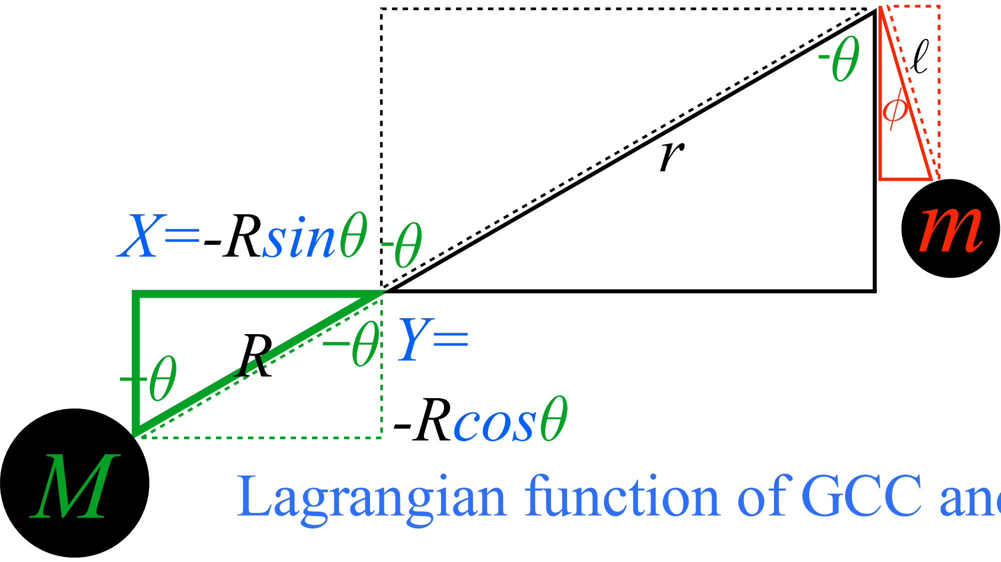


$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial \dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *velocity chain*

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *Lagrange equations*

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$ *(Consolidating)*

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)}_{\frac{dH}{dt}} = -\frac{\partial L}{\partial t}$$ *(Rearranging)*

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$ *Defining the Hamiltonian function*

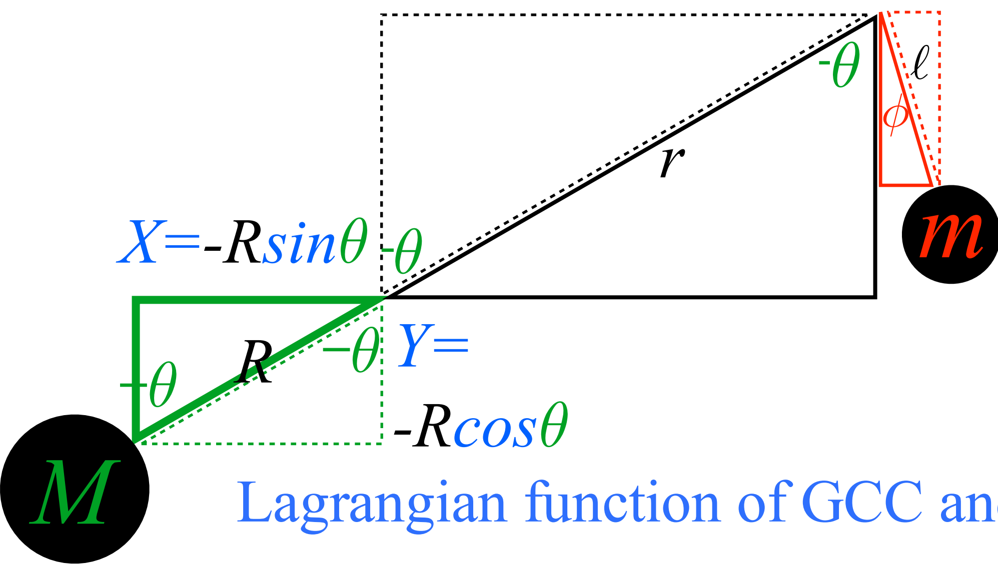
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} = \dot{\theta}$$

by Lagrange equations

$$Total\ KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

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$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *Lagrange equations*

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$ *(Consolidating)*

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$ *(Rearranging)*

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$ *Defining the Hamiltonian function*

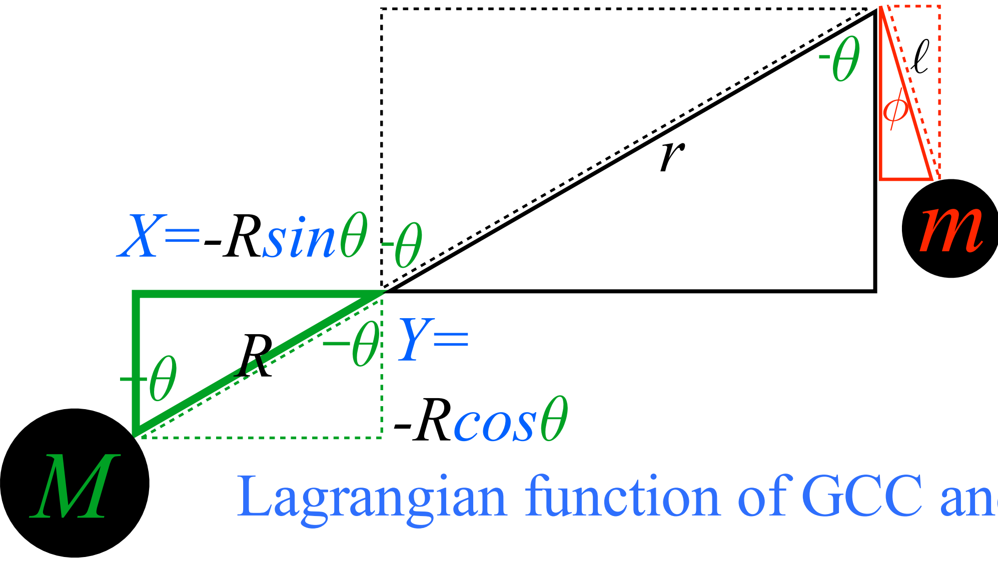
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} = -\frac{\partial L}{\partial\theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} = \dot{\theta} \qquad \frac{\partial H}{\partial\dot{\theta}} = p_\theta - \frac{\partial L}{\partial\dot{\theta}} = 0$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

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velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

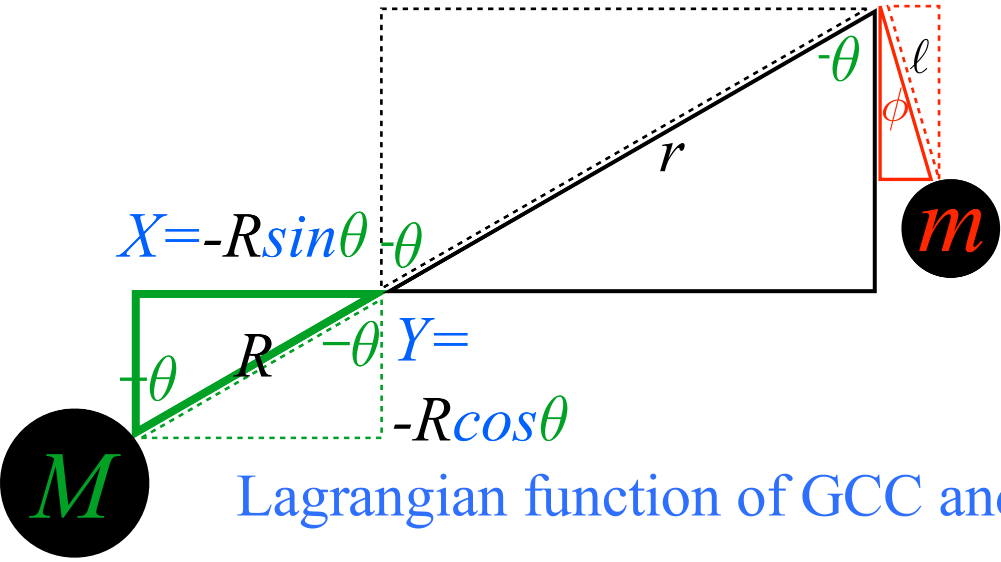
by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \cancel{\frac{\partial H}{\partial \dot{\theta}} \equiv 0} \quad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \cancel{\frac{\partial H}{\partial \dot{\phi}} \equiv 0}$$

Hamilton's equations

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Dynamic metric tensor γ_{mn} *in GCC* θ *and* ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

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$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *Lagrange equations*

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$ *(Consolidating)*

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$ *(Rearranging)*

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$ *Defining the Hamiltonian function*

Hamiltonian function of GCC and momenta: $\boxed{H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L}$ *Poincare-Legendre relation*

$$\boxed{\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \cancel{\frac{\partial H}{\partial \dot{\theta}}} \equiv 0 \qquad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \cancel{\frac{\partial H}{\partial \dot{\phi}}} \equiv 0}$$ *Hamilton's equations*

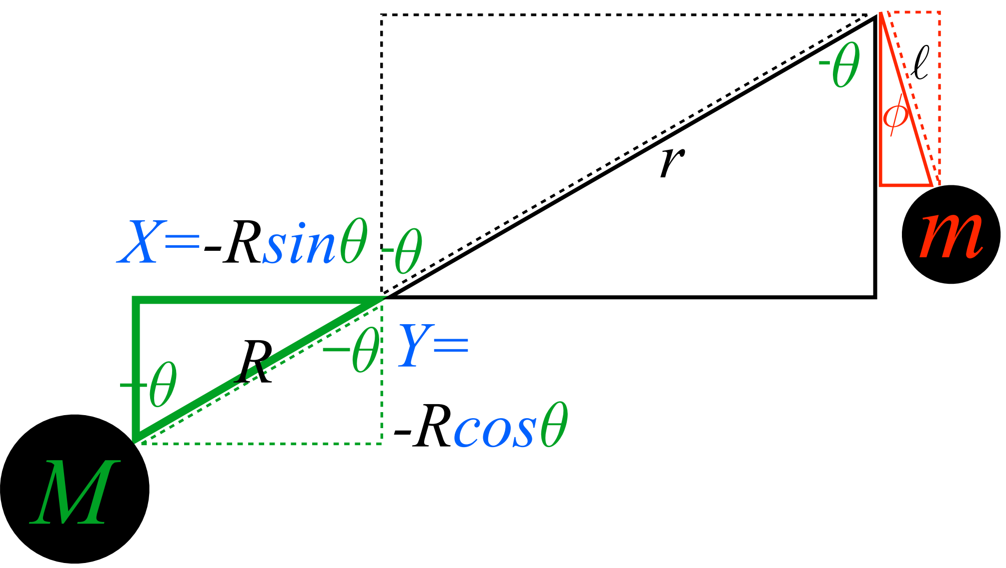
Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

$$Total\ KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$ *Covariant metric tensor* γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$ *Contravariant metric tensor* γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

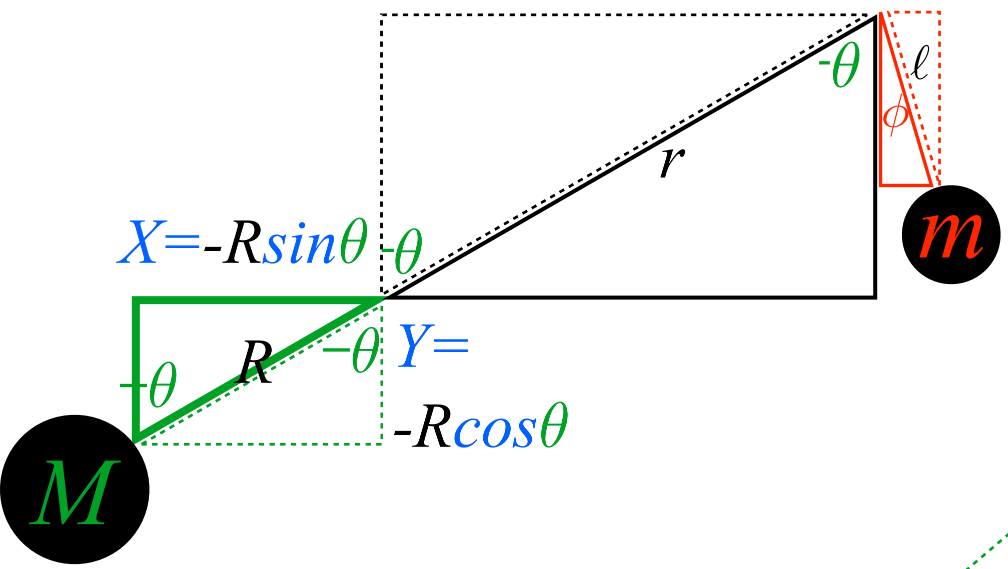
Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \quad + \quad p_\phi\dot{\phi} \quad - T \quad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

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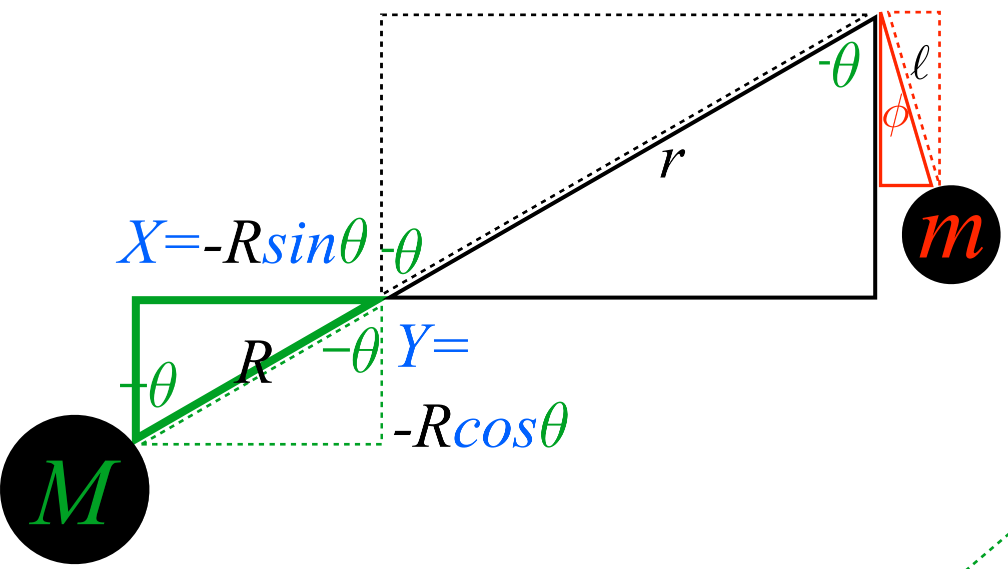
$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V$$

(Only correct numerically!)

$$Total\ KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

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$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

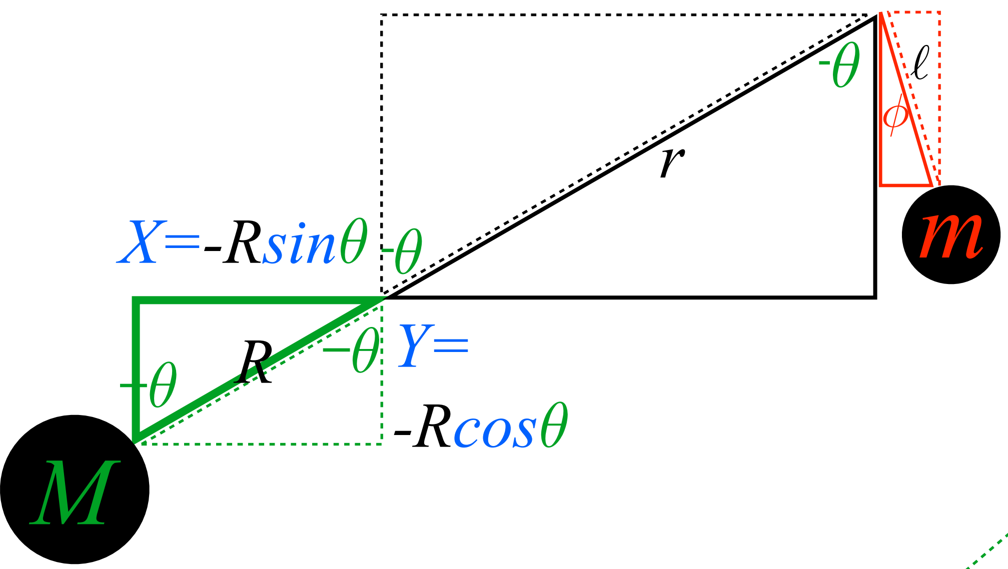
$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \quad = T + V \equiv E$$

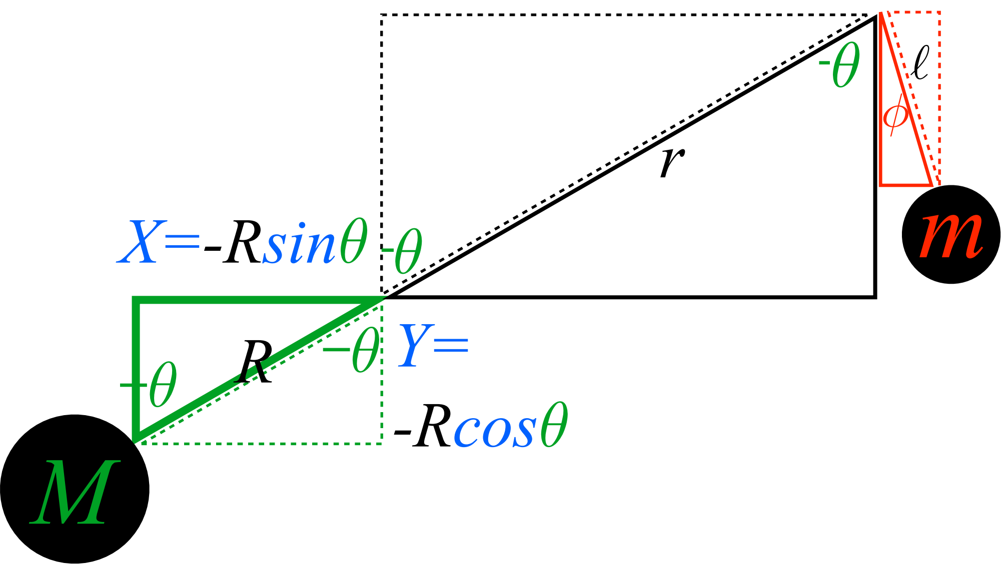
(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V = T + V$$

(Correct formally and numerically)

$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\,\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \quad = T + V \equiv E$$

(Only correct numerically!)

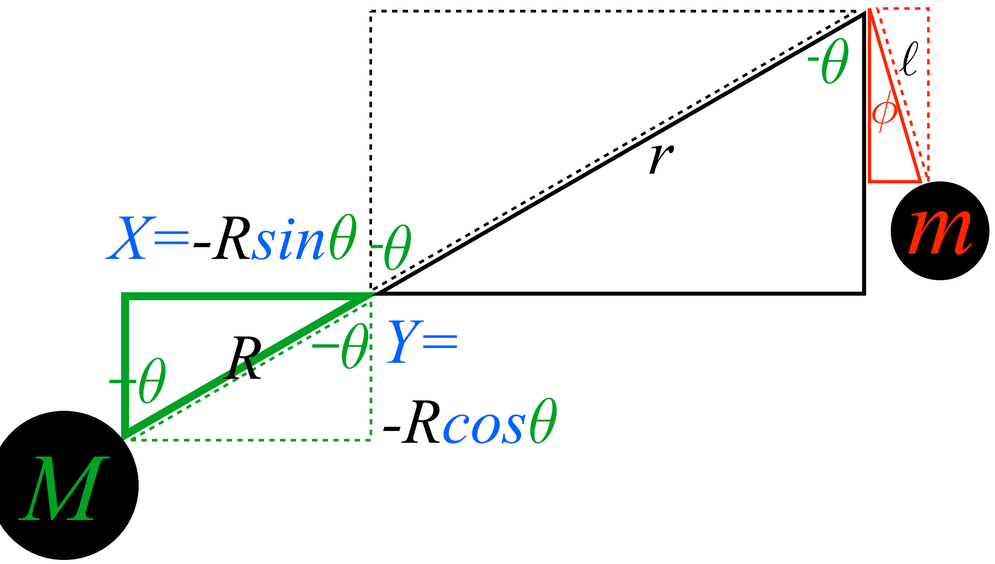
Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V \quad = T + V$$

(Correct formally and numerically)

$$H = \frac{m\ell^2 p_\theta p_\theta + 2mr\ell\cos(\theta-\phi)p_\theta p_\phi + (MR^2 + mr^2)p_\phi p_\phi}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} + V$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned} \dot\theta &= \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi \\ \dot\phi &= \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi \end{aligned}}$$

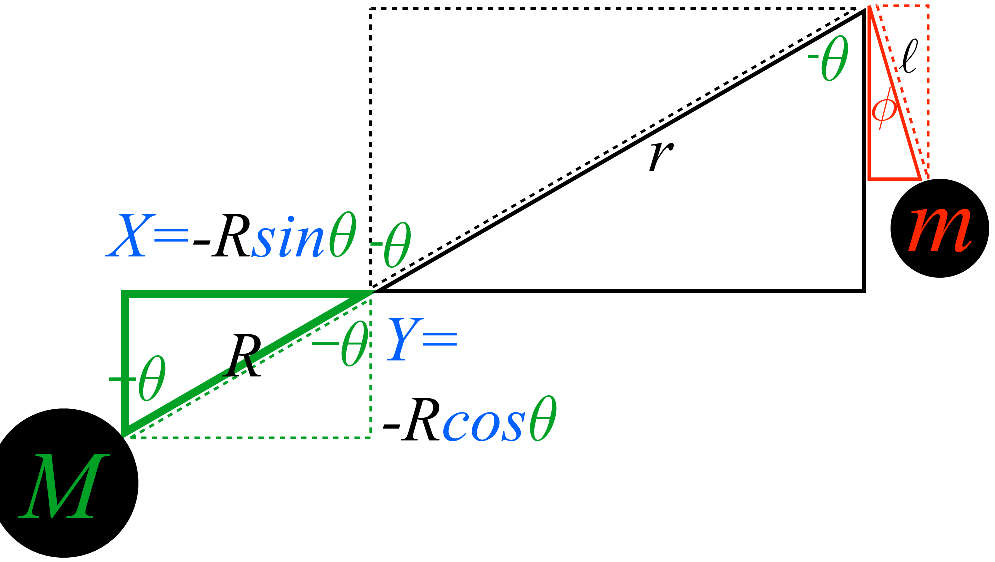
$$\boxed{\begin{aligned} p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi \\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi \end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2 \sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn} p_m p_n$$

$$\boxed{\begin{aligned} \dot{\theta} &= \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi \\ \dot{\phi} &= \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \end{aligned}}$$

$$\boxed{\begin{aligned} p_\theta &= \gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi} \\ p_\phi &= \gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi} \end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

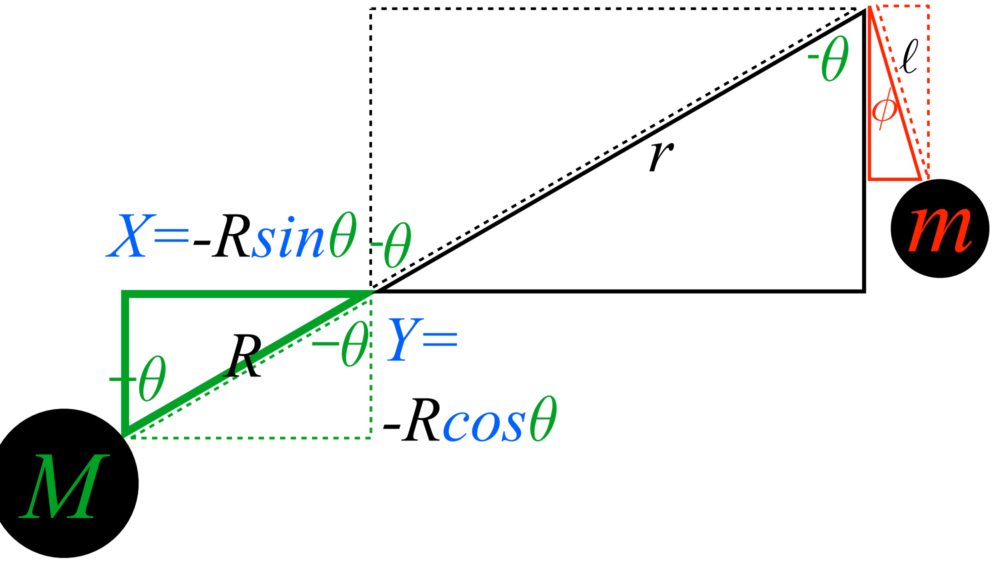
Momentum/force equations

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}$$
$$= mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}$$
$$= -mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\,\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot\theta &= \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot\phi &= \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta &= \gamma_{\theta\theta}\dot\theta+\gamma_{\theta\phi}\dot\phi\\ p_\phi &= \gamma_{\phi\theta}\dot\theta+\gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

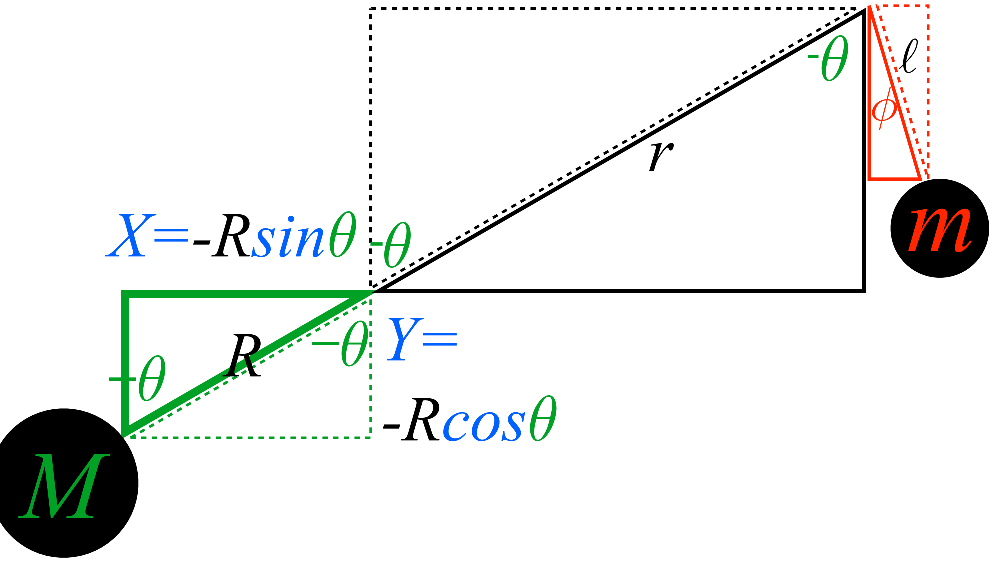
Momentum/force equations

$$\begin{aligned}\dot p_\theta &= -\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}\\ &= mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta\\ &= mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta\end{aligned}$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\begin{aligned}\dot p_\phi &= -\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}\\ &= -mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi\\ &= -mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi\end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot{\theta}&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot{\phi}&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\\ p_\phi&=\gamma_{\phi\theta}\dot{\theta}+\gamma_{\phi\phi}\dot{\phi}\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

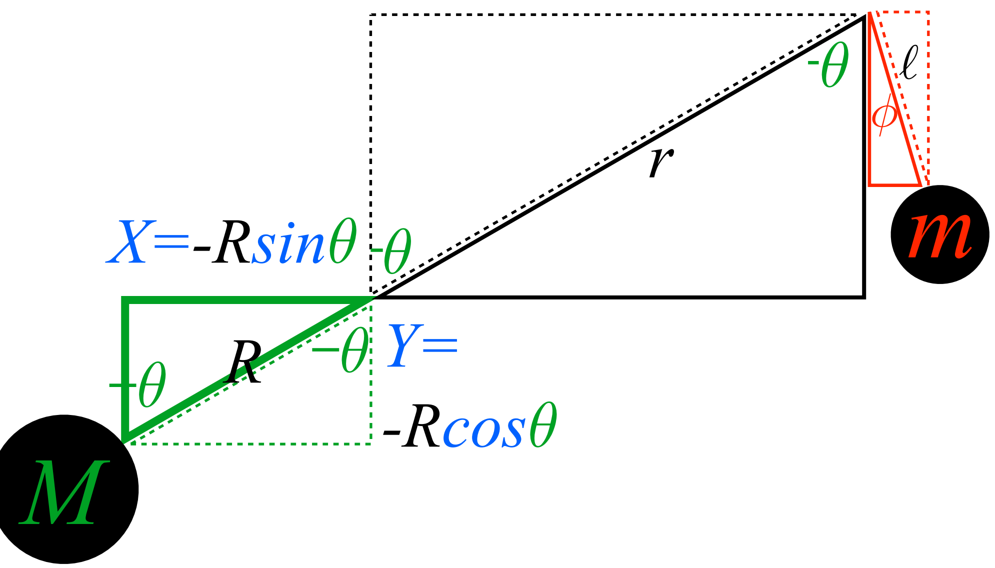
Momentum/force equations

$$\begin{aligned}\dot{p}_\theta&=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}\\ &=mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\theta\\ &=mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta\\ &=mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2+[\gamma^{\theta\theta}\gamma^{\phi\phi}+(\gamma^{\theta\phi})^2]p_\phi p_\theta+\gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi)+F_\theta\end{aligned}$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\begin{aligned}\dot{p}_\phi&=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}\\ &=\ \text{-}mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\phi\\ &=\text{-}mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi\\ &=\text{-[messy factor]}\sin(\theta-\phi)+F_\phi\end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn} p_m p_n$$

$$\boxed{\begin{aligned} \dot\theta &= \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi \\ \dot\phi &= \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \end{aligned}}$$

$$\boxed{\begin{aligned} p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi \\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi \end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

Momentum/force equations

$$\begin{aligned} \dot p_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} \\ &= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta \\ &= mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\theta \\ &= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta} p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2] p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi} p_\phi^2)\sin(\theta-\phi) + F_\theta \end{aligned}$$

$$\begin{aligned} \dot p_\phi &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} \\ &= -mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi \\ &= -mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\phi \\ &= -[\text{messy factor}]\sin(\theta-\phi) + F_\phi \end{aligned}$$

A lesson on Hamiltonian "elegance"...
...may be very elegant formally...but may not be so elegant algebraically!

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Lemma-1 from Lect.9 p.13

$$\begin{pmatrix} \frac{\partial \dot{\theta}_B}{\partial \dot{\theta}} & \frac{\partial \dot{\theta}_B}{\partial \dot{\phi}} \\ \frac{\partial \dot{\phi}_B}{\partial \dot{\theta}} & \frac{\partial \dot{\phi}_B}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix}$$

(Used in Lect.14 p.60)

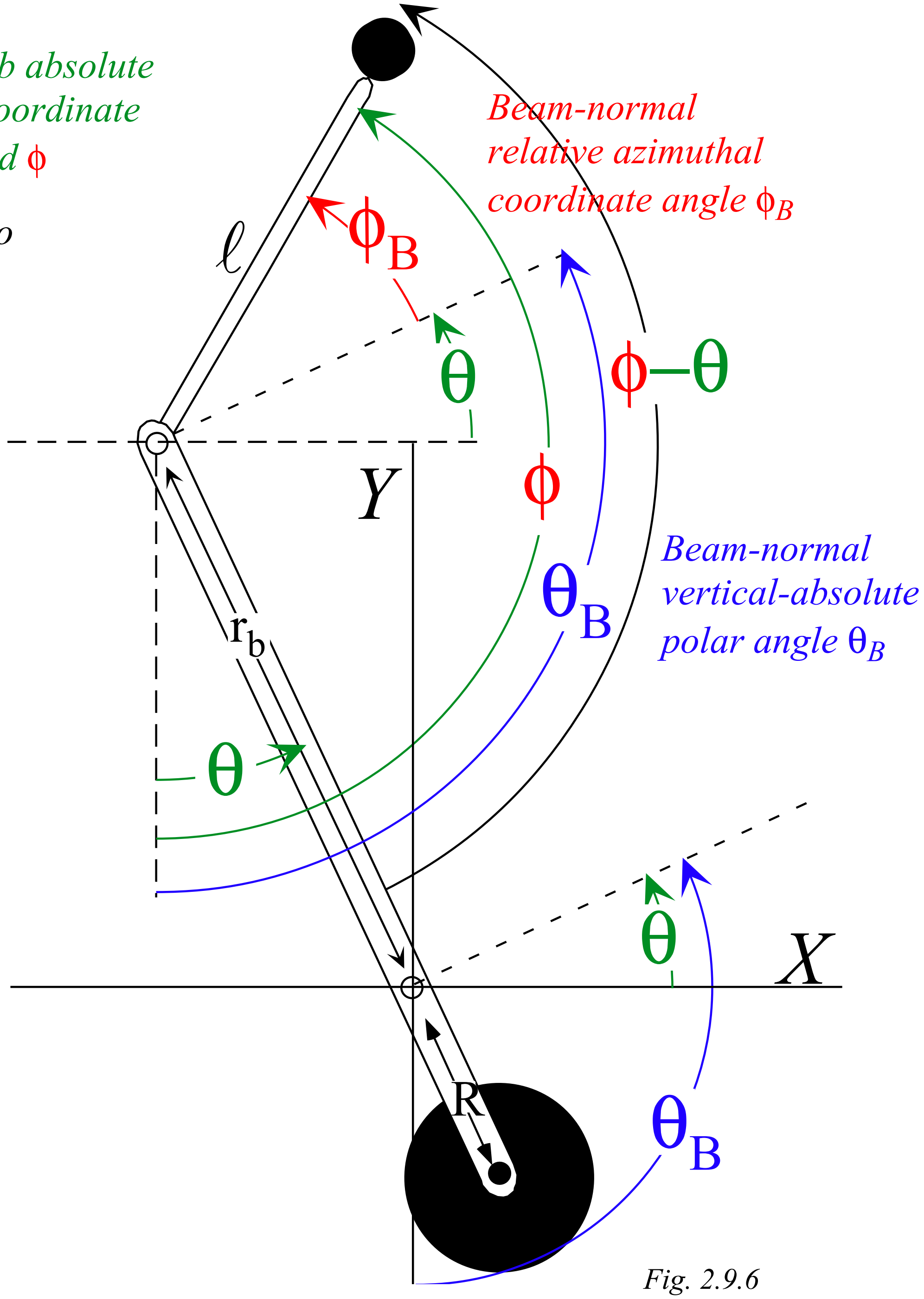


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \; -\pi/2$

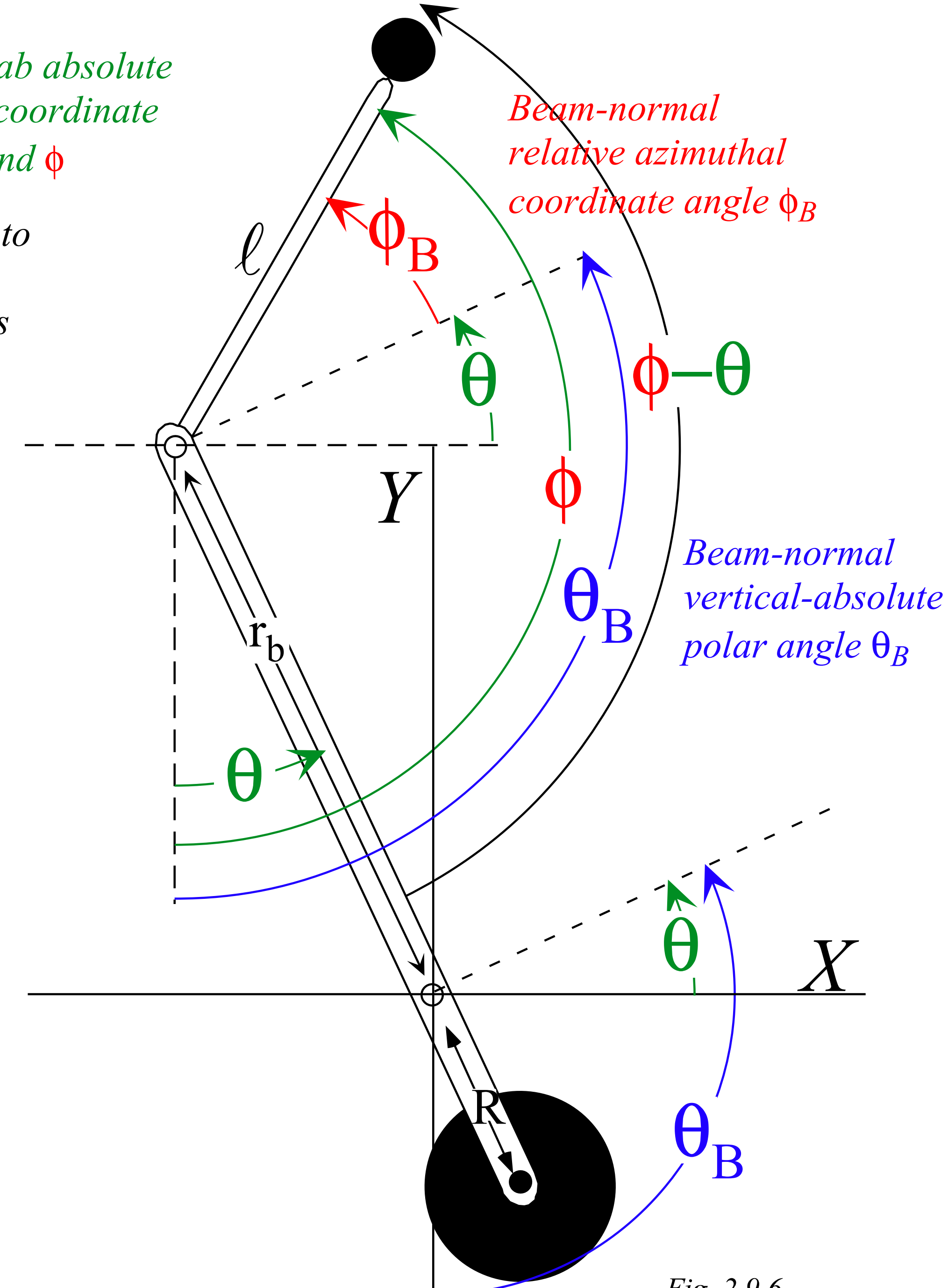
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



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Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \; -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Be careful with momentum.
Poincare invariance is ***crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

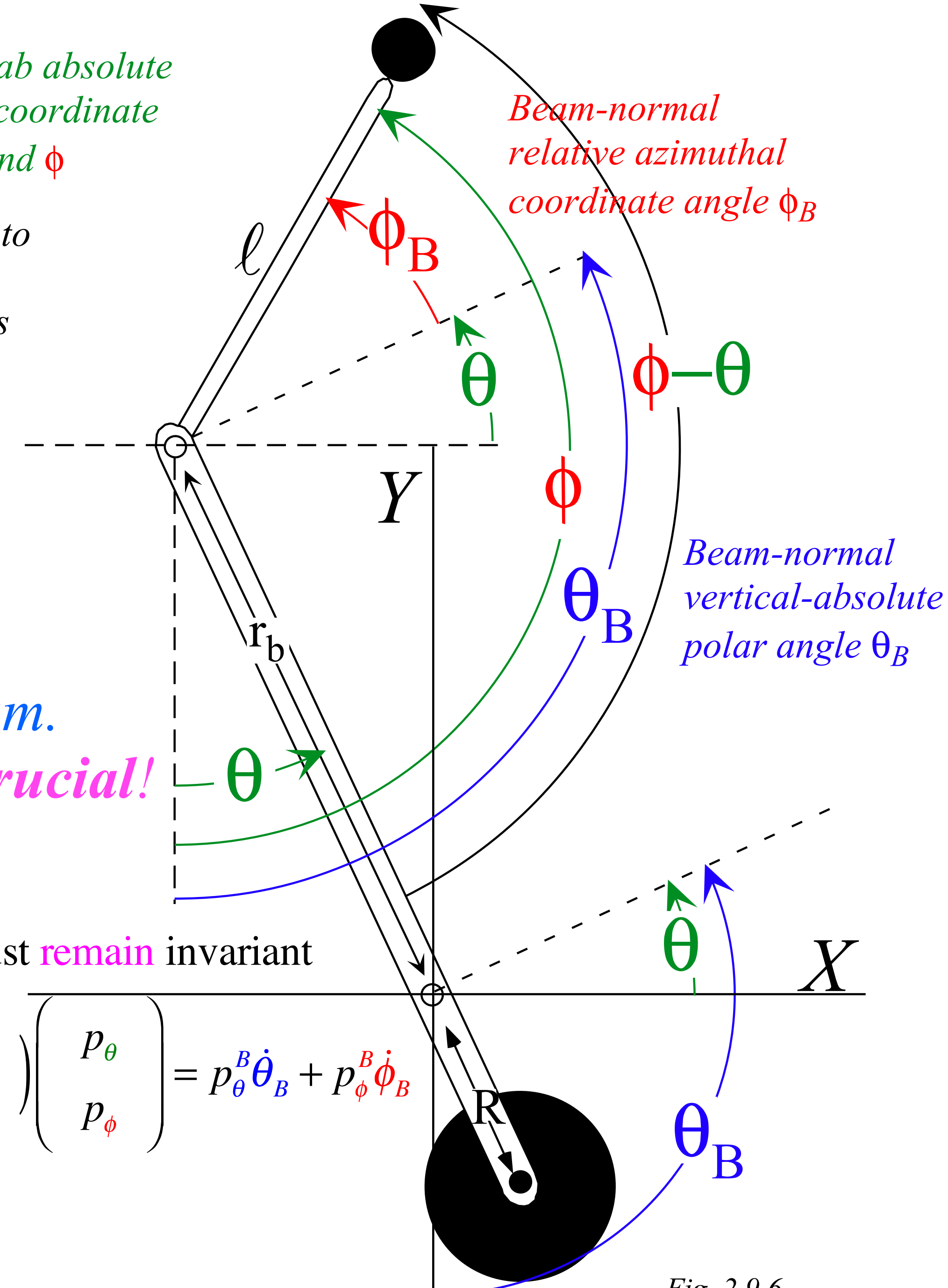

Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

$\phi = \theta_B + \phi_B$

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p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

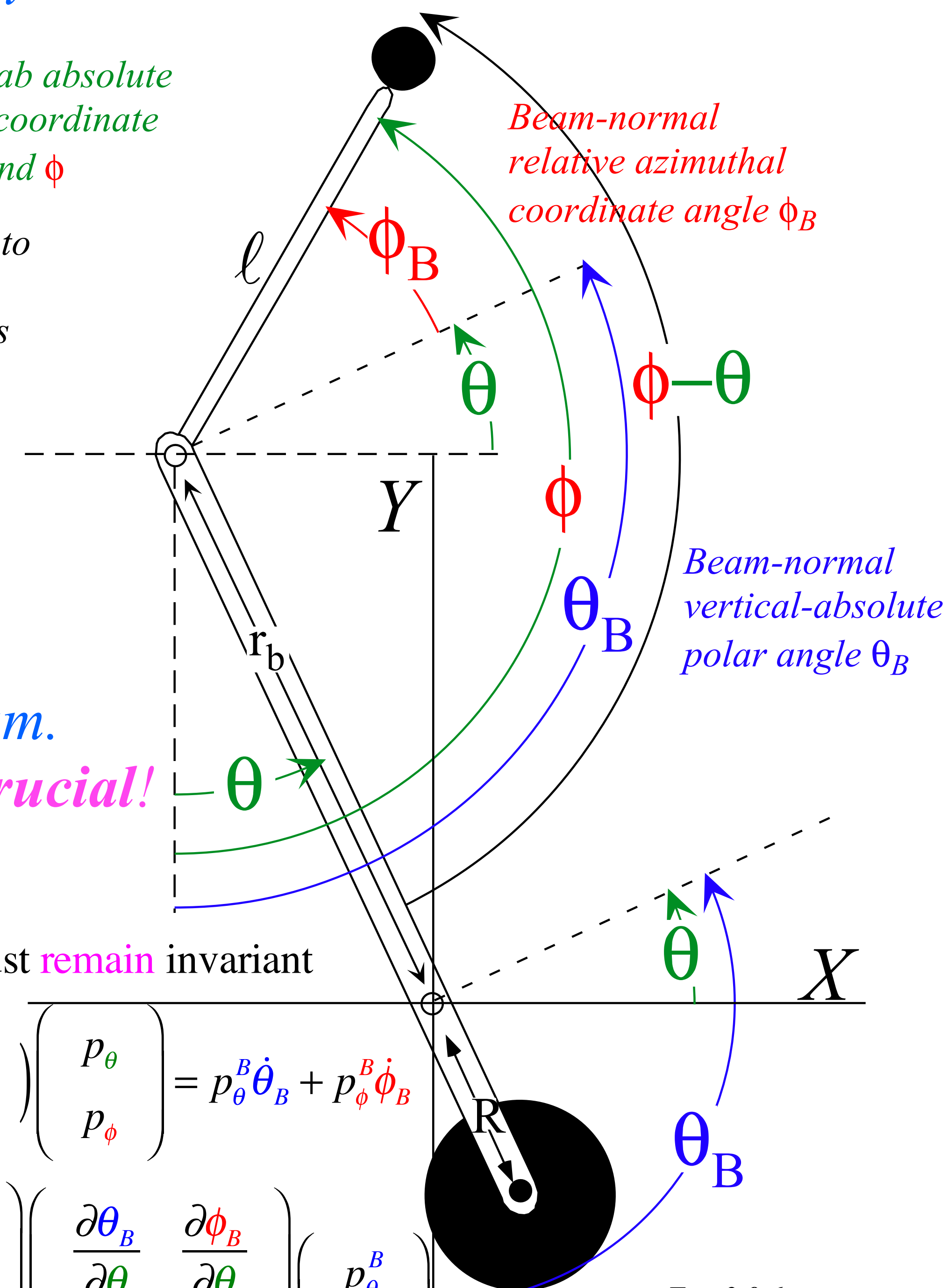
$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Be careful with momentum.
*Poincare invariance is **crucial**!*



Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix}\dot\theta_B & \dot\phi_B\end{pmatrix}\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

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$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

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Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

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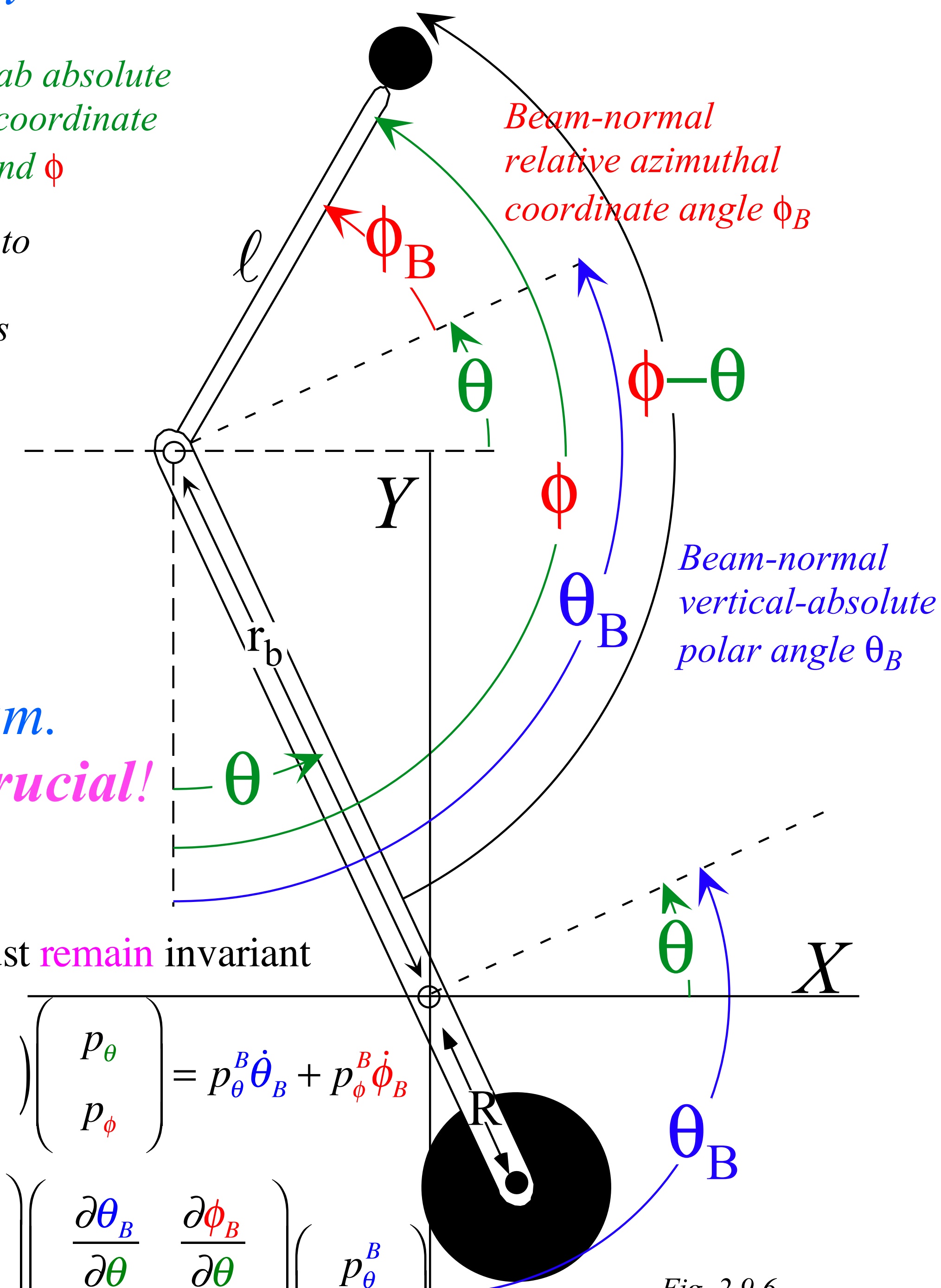

Poincare invariant must remain invariant

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$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

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Coordinate transformation helps reduce symmetric Hamiltonian

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Previous lab absolute trebuchet coordinate angles θ and φ

compared to

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p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix}p_\theta^B\\ p_\phi^B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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Be careful with momentum.
*Poincare invariance is **crucial**!*

Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p_\theta^B\dot\theta_B + p_\phi^B\dot\phi_B$$

$$\begin{pmatrix}\dot\theta_B & \dot\phi_B\end{pmatrix}\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p_\theta^B\\ p_\phi^B\end{pmatrix}$$

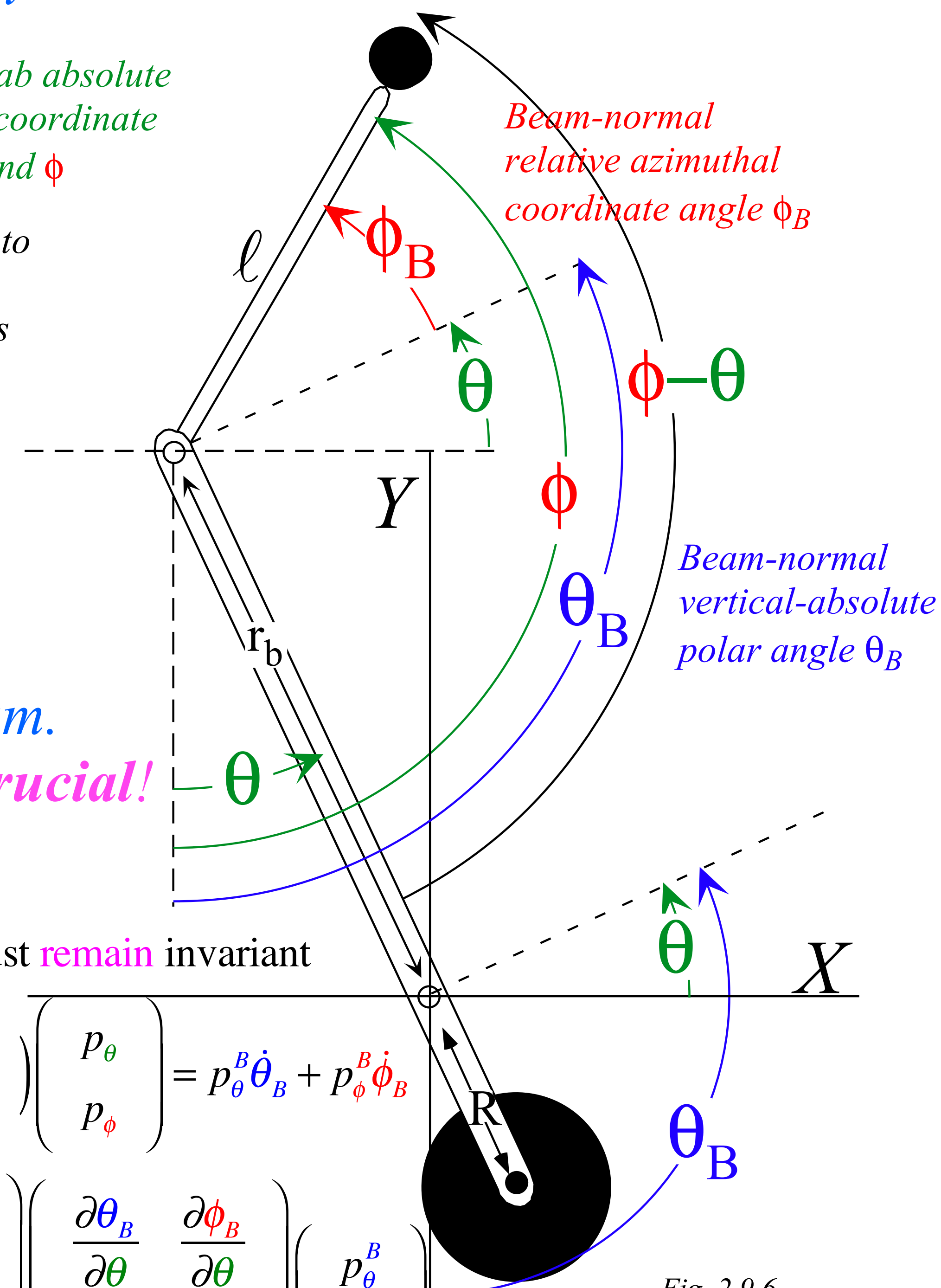

Fig. 2.9.6
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Original (ϕ,θ) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

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$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

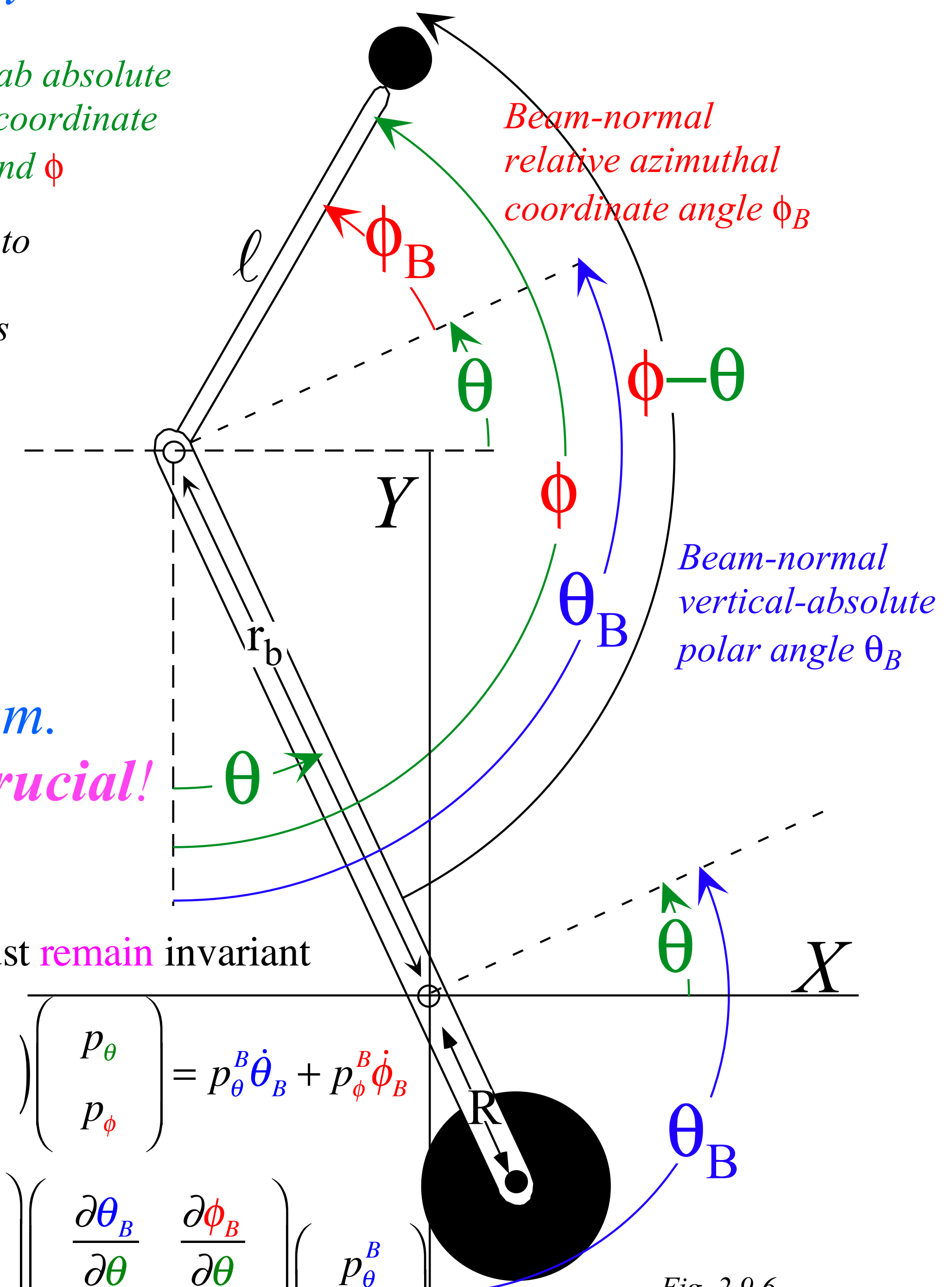
$p_\phi = \quad p_\phi^B$

Be careful with momentum.
*Poincare invariance is **crucial!***

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Poincare invariant must remain invariant

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$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

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(Each value is positive.)

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Original (ϕ,θ) Hamiltonian

(Use $\phi_B = \pi/2 - (\theta - \phi)$)

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} + V$$

Transformed (ϕ_B,θ_B) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian Lemma-1 definition: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \; -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

p_m transform is **TRANSPOSE INVERSE** *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

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Be careful with momentum.
Poincare invariance is ***crucial!***

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

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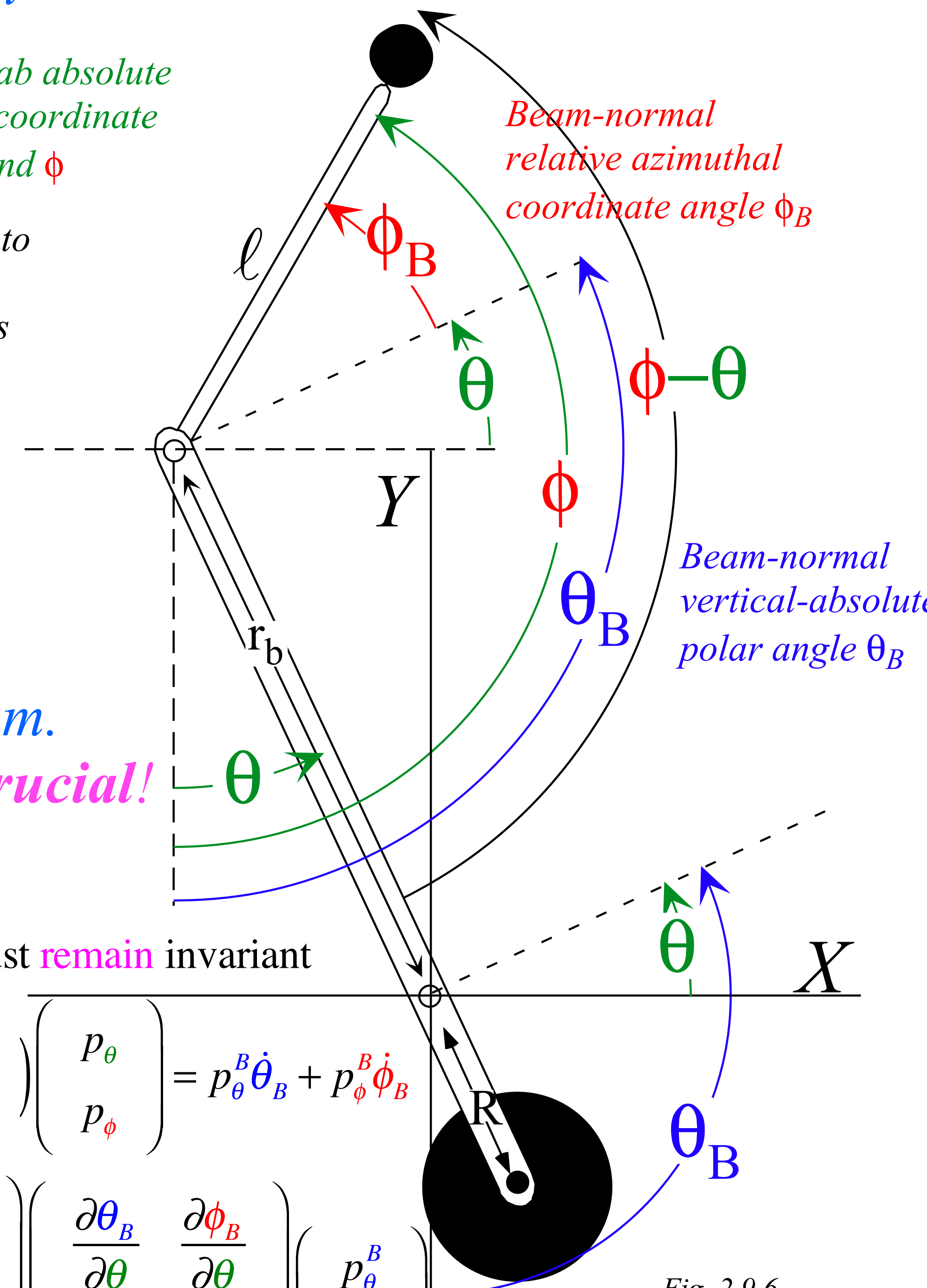

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

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(Each value is positive.)

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$\theta - \phi = -\pi/2 - \phi_B$

$F_\theta = -MgR\sin\theta + mgr\sin\theta$

$F_\phi = -mg\ell\sin\phi$

$$H = \frac{m\ell^2\left(p^B_\theta - p^B_\phi\right)^2 + \left(MR^2 + mr^2\right)\left(p^B_\phi\right)^2 - 2mr\ell p^B_\phi\left(p^B_\theta - p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} - \left(MR - mr\right)g\sin\theta_B \; - mg\ell\cos\left(\phi_B + \theta_B\right)$$

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

Algebraic approach

Direct approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

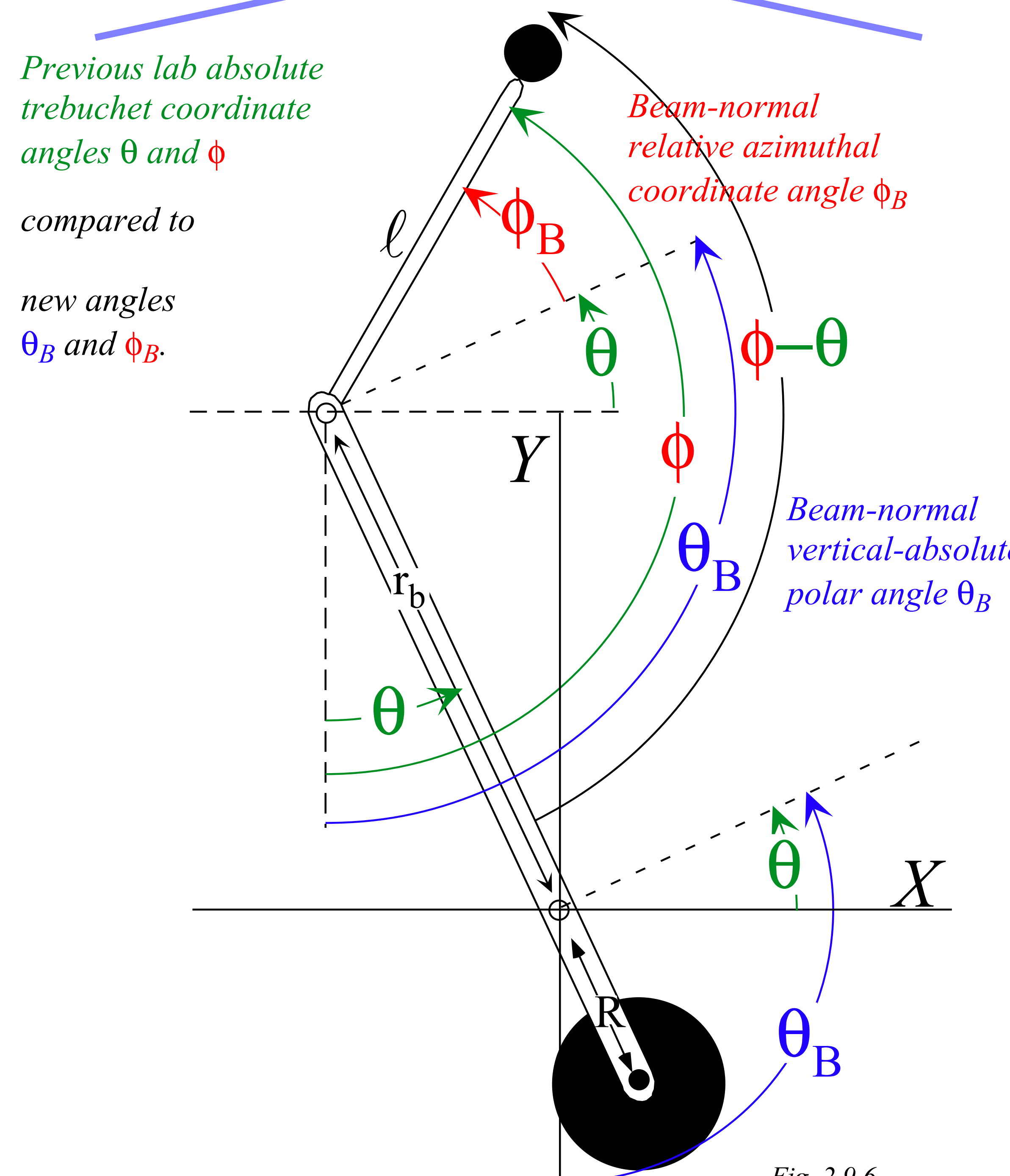
(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$



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Free-space trebuchet kinematics by symmetry: Algebraic approach

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$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Previous lab absolute trebuchet coordinate angles θ and φ

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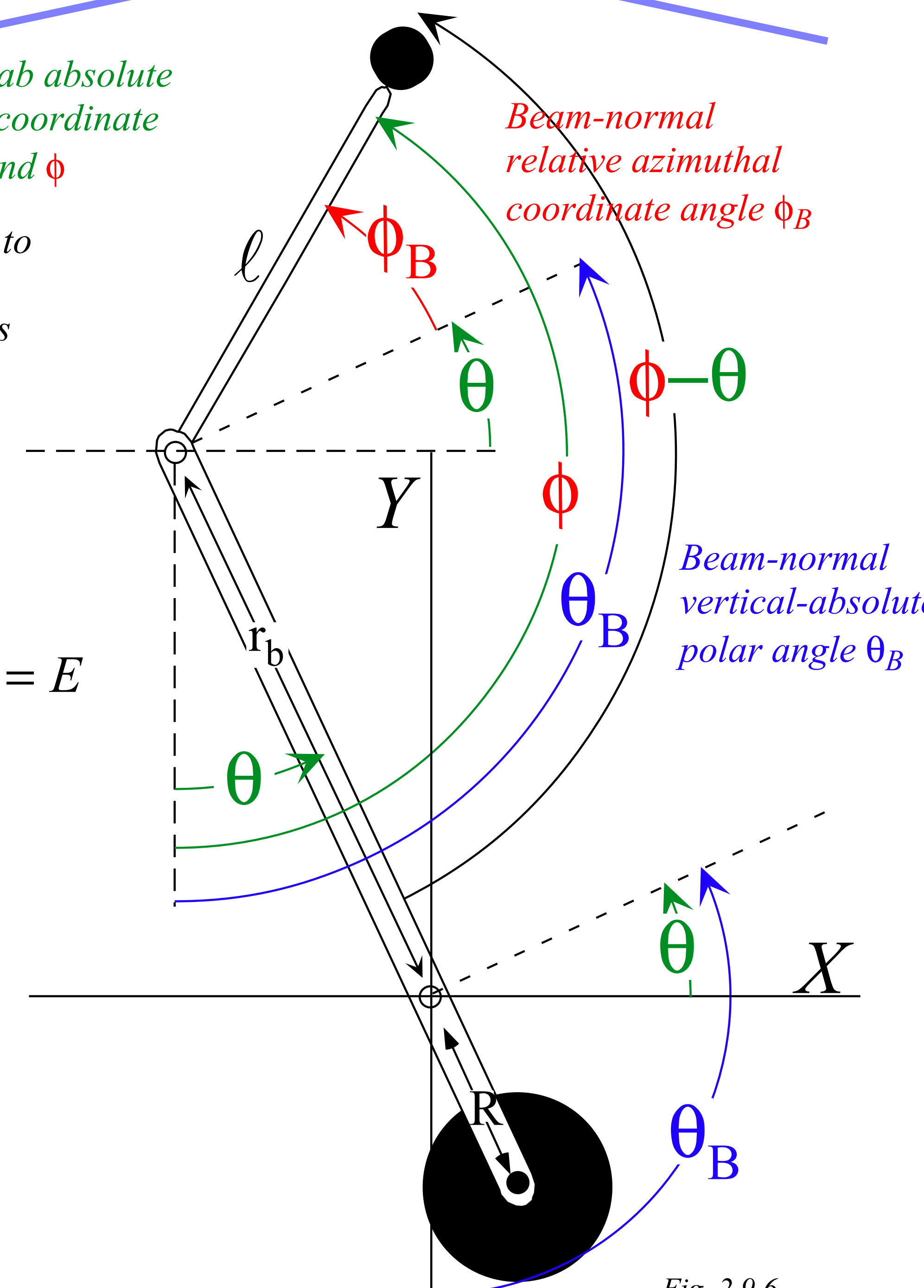

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

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$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

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$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

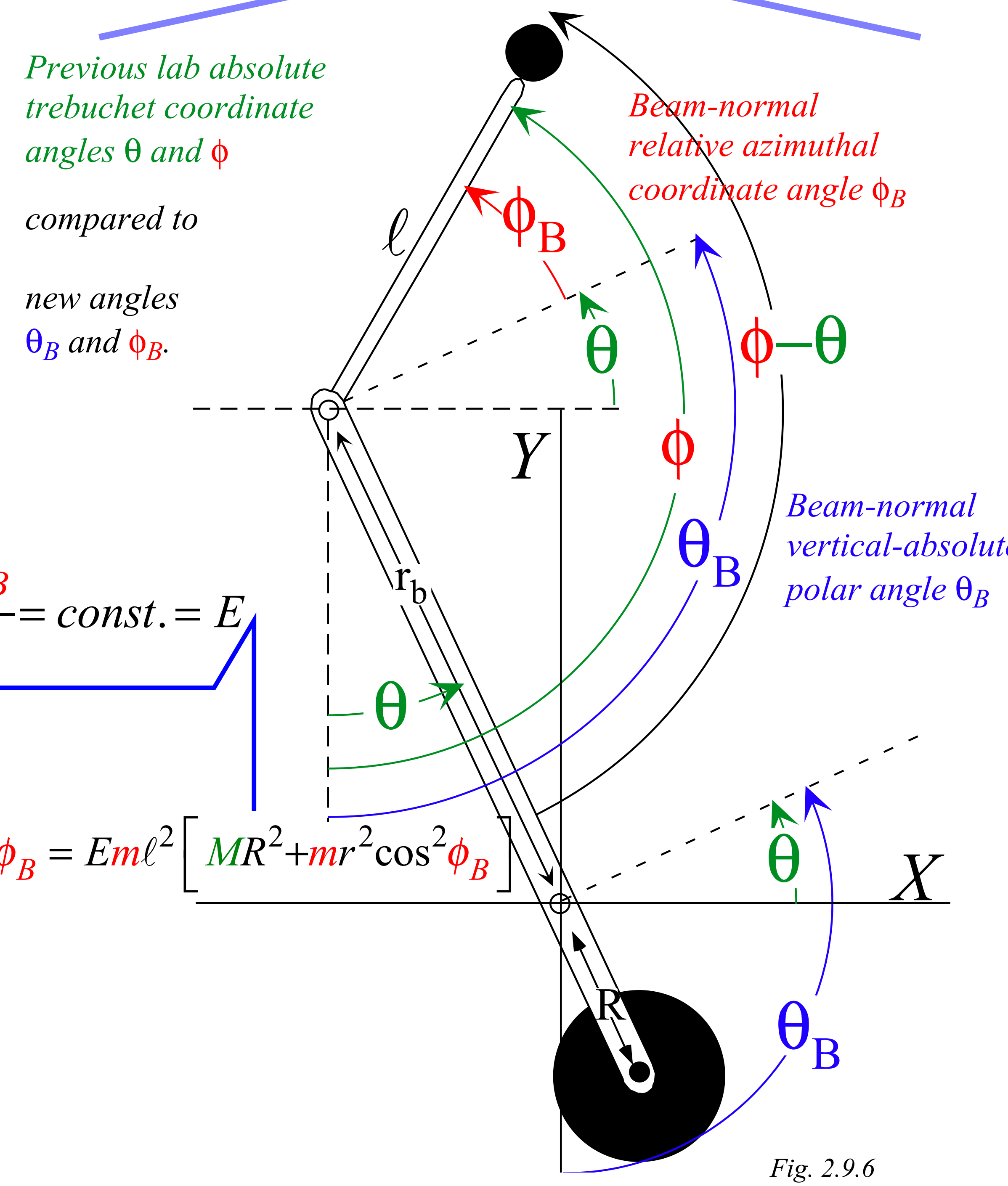


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ$_B$,ϕ$_B$) relative coordinates for trebuchet.

(Each value is positive.)

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

(Assume zero-gravity)

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

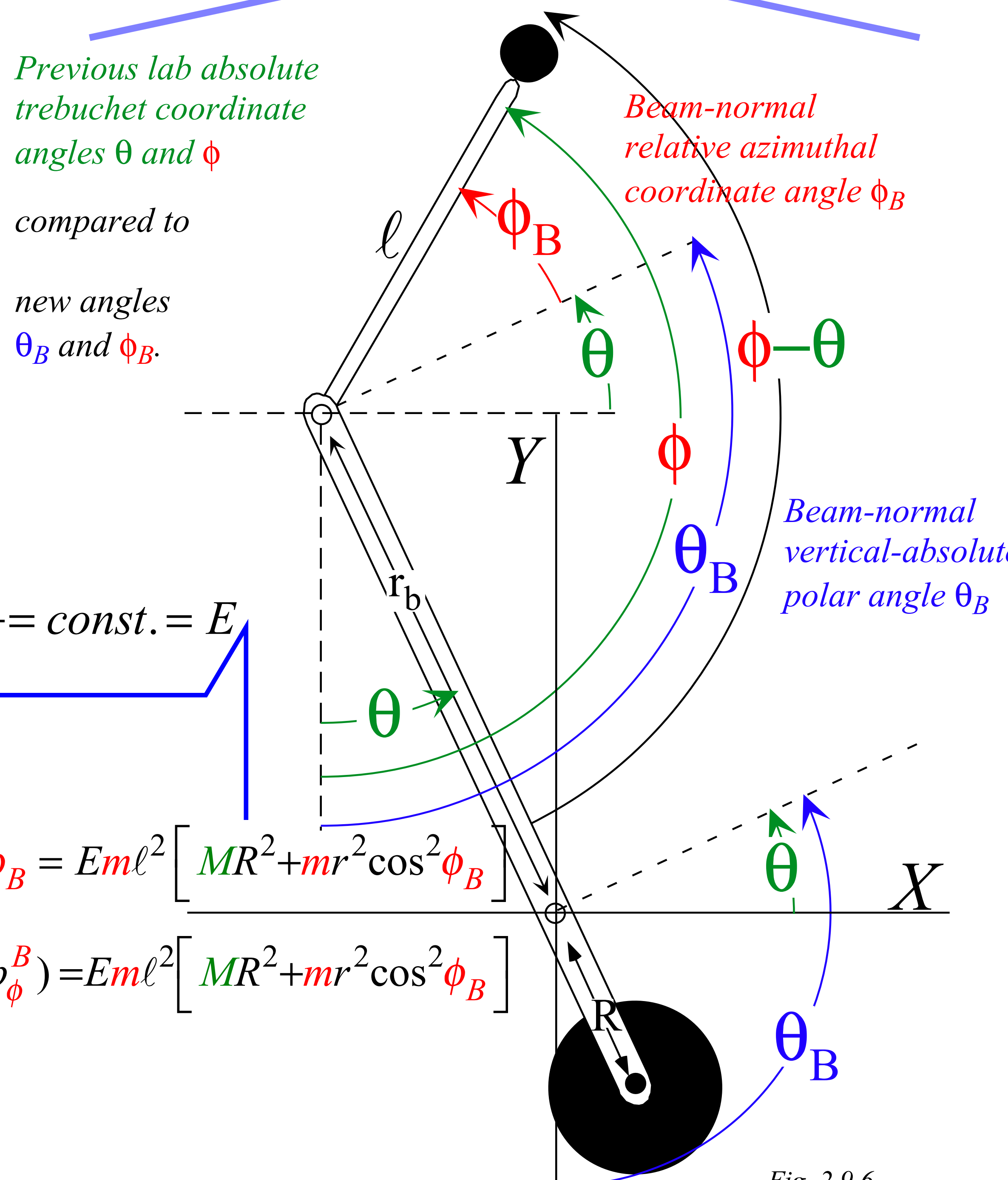


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

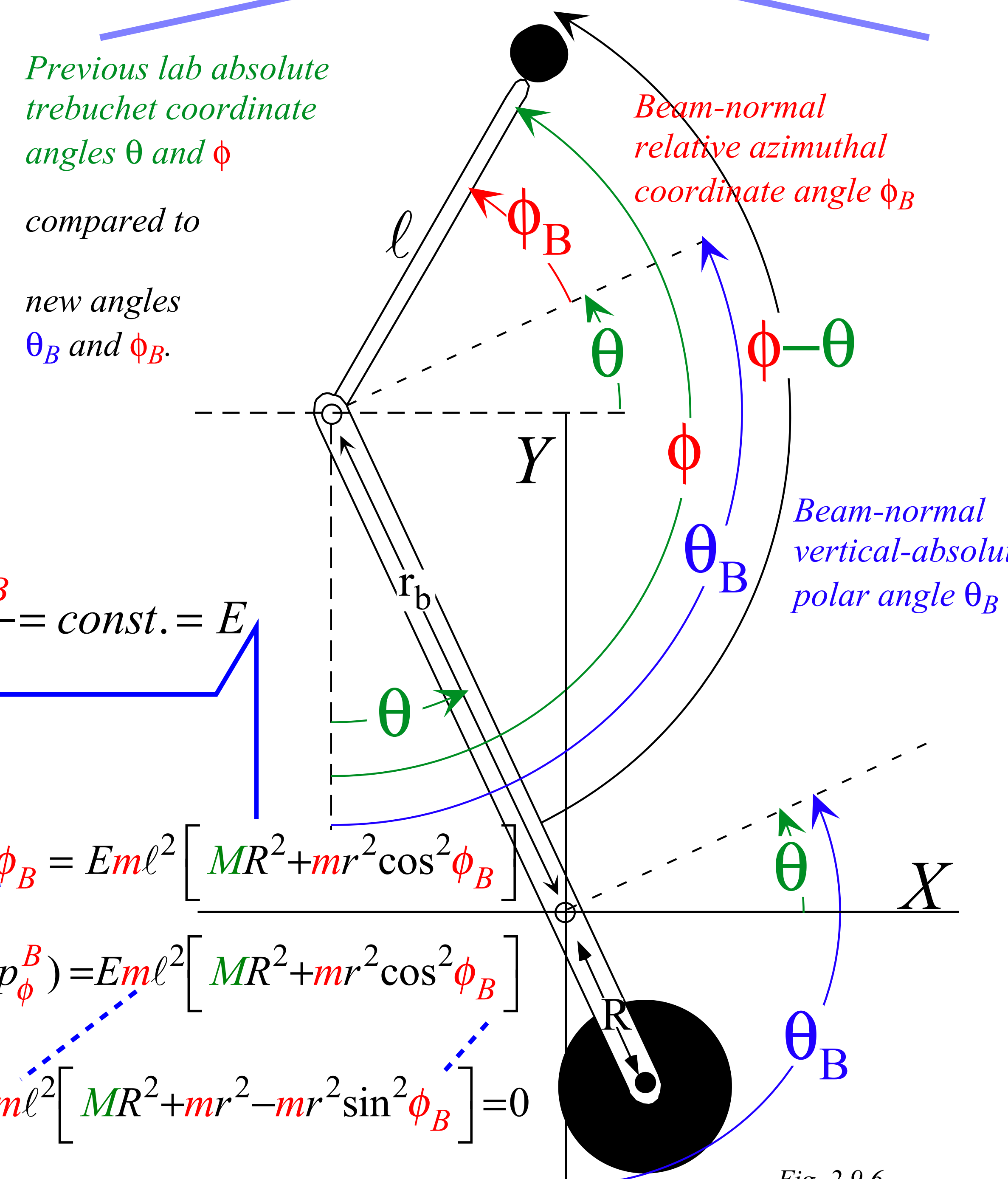


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

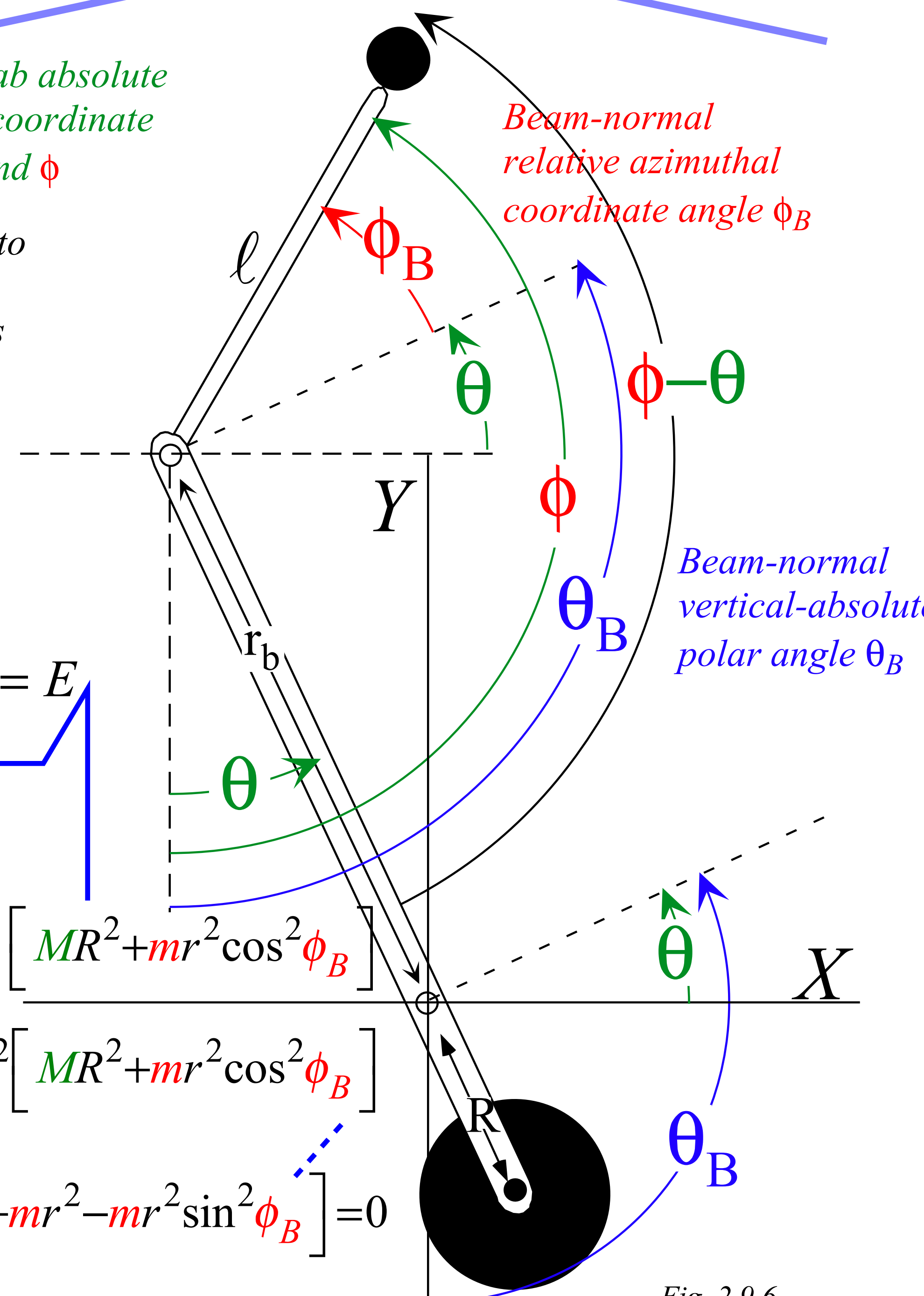


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

with: $J=\frac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p^B_\theta-p^B_\phi\right)^2+\left(MR^2+mr^2\right)\left(p^B_\phi\right)^2-2mr\ell p^B_\phi\left(p^B_\theta-p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}-\cancel{\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}^B_\theta=\frac{\partial H}{\partial\theta_B}=0$ and : $p^B_\theta=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p^B_\phi\right)^2+\left(MR^2+mr^2\right)\left(p^B_\phi\right)^2-2mr\ell p^B_\phi\left(\Lambda-p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p^B_\phi)+(p^B_\phi)^2\right)+\left(MR^2+mr^2\right)(p^B_\phi)^2-2mr\ell(p^B_\phi)\left(\Lambda-p^B_\phi\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p^B_\phi)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p^B_\phi)^2-2mr\ell\Lambda\sin\phi_B(p^B_\phi)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p^B_\phi)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p^B_\phi)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p^B_\phi)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p^B_\phi)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

(using quadratic solution: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$)

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

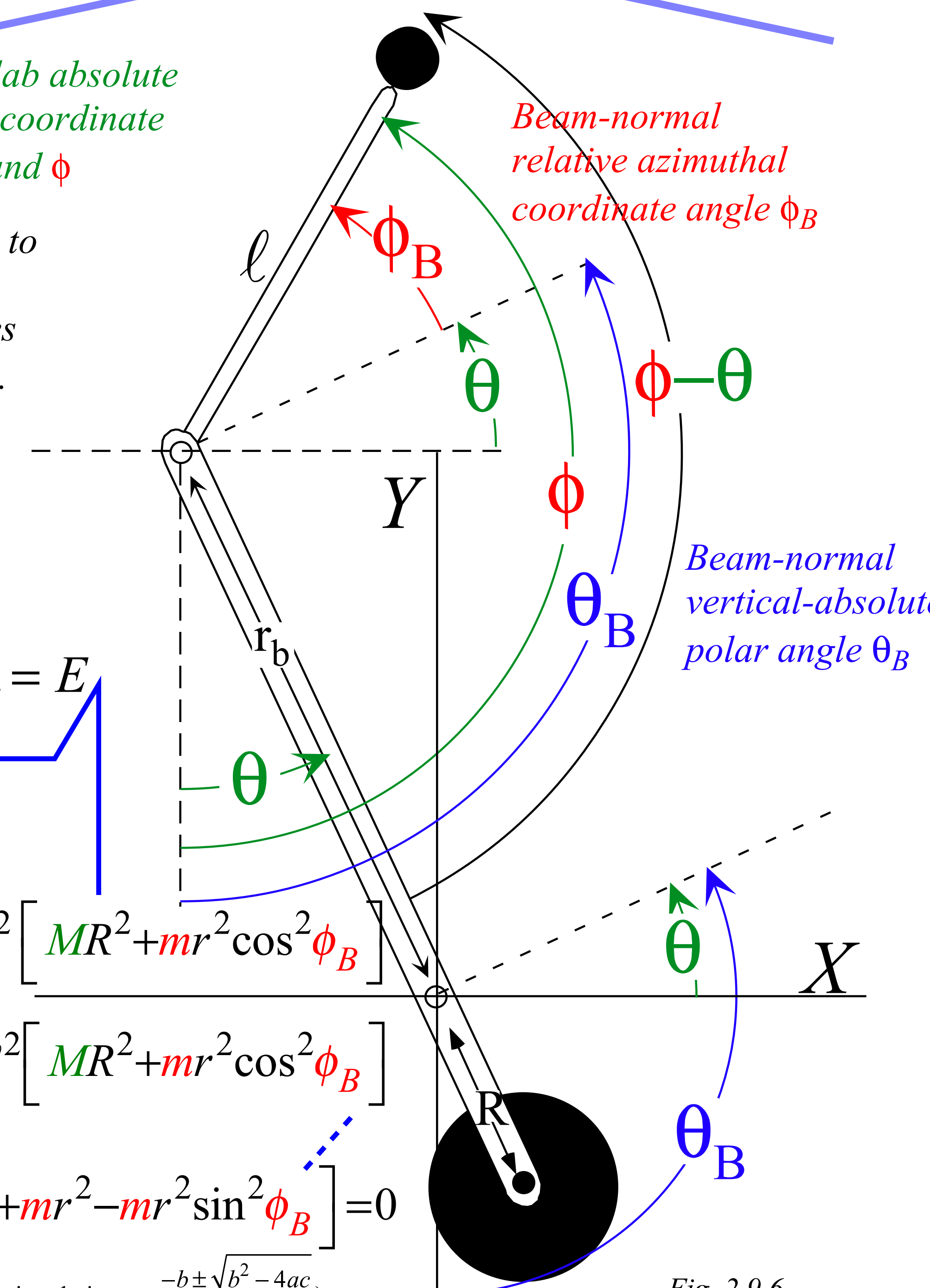


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

$$p^B_\phi=\frac{2\Lambda\left((r/\ell)\sin\phi_B+1\right)\pm\sqrt{4\Lambda^2\left((r/\ell)\sin\phi_B+1\right)^2-4\left(1+2(r/\ell)\sin\phi_B+J\right)\left(\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]\right)}}{2\left(1+2(r/\ell)\sin\phi_B+J\right)}$$

with: $J=\frac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

Algebraic approach

Direct approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

(Assume zero-gravity)

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

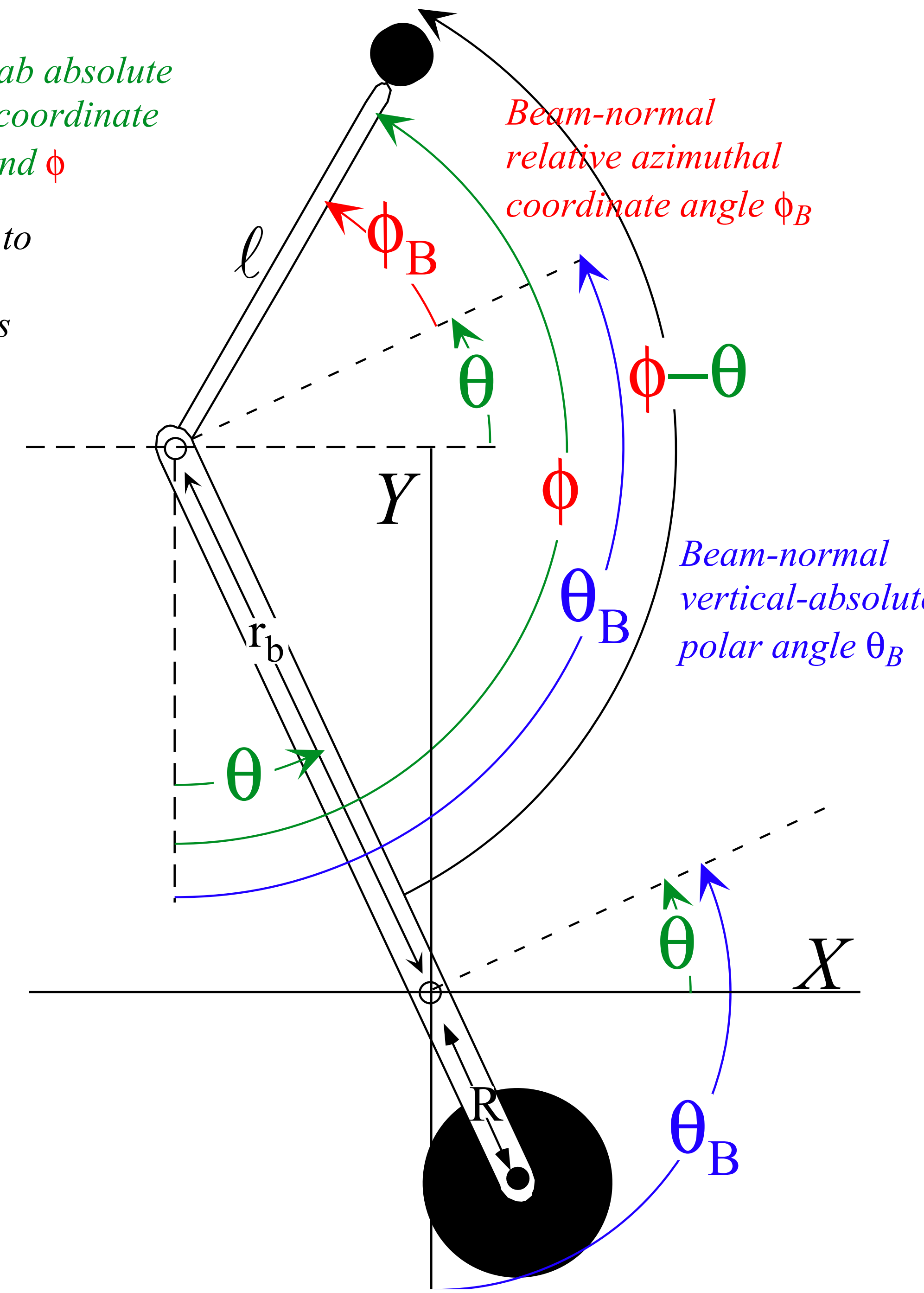
$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

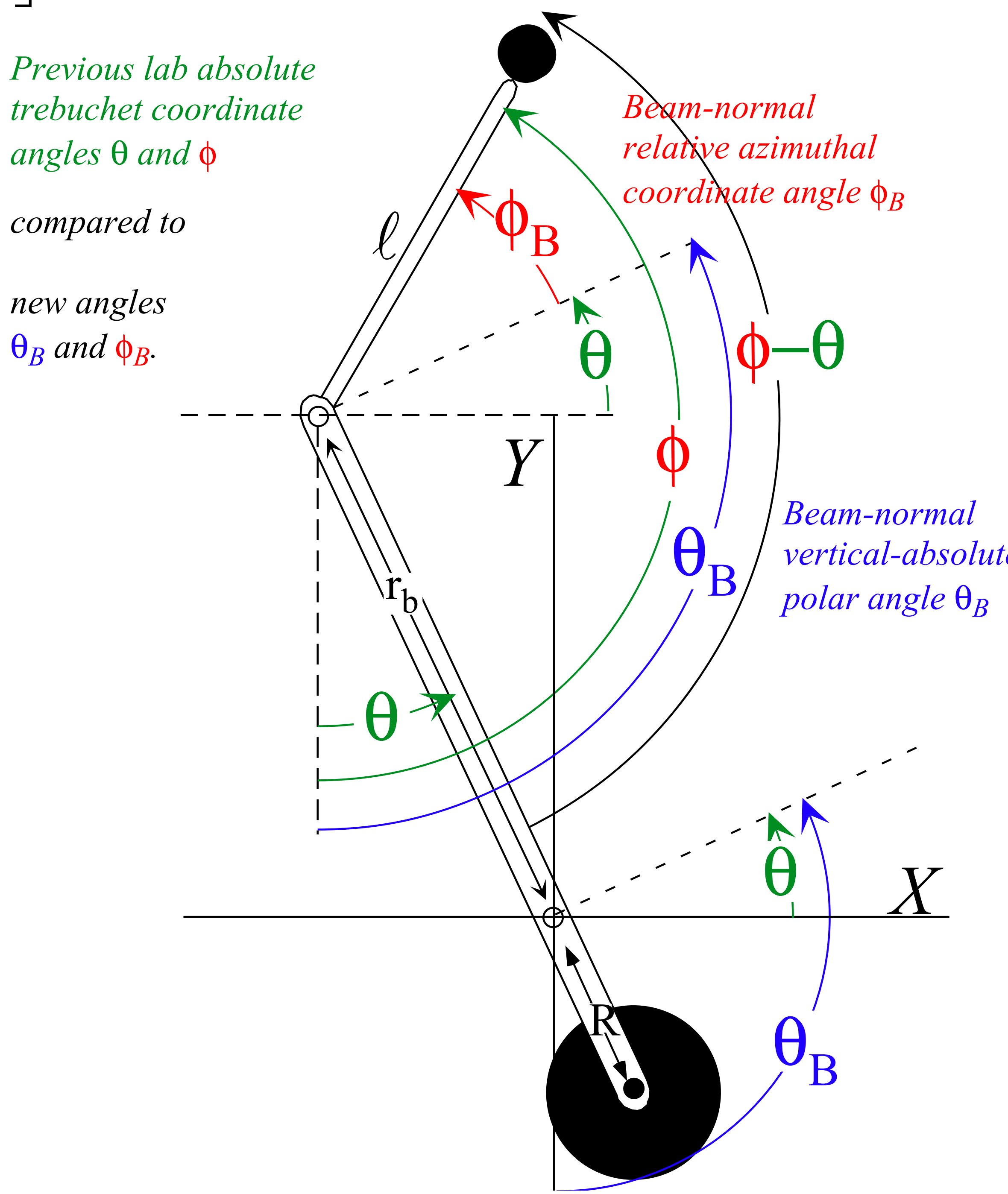
Transform to beam-relative coordinates and momenta

$$\boxed{\begin{aligned}\theta &= \theta_B - \pi/2\\ \phi &= \theta_B + \phi_B\end{aligned}}$$

$\theta - \phi = -\phi_B - \pi/2$

$$\boxed{\begin{aligned}\theta_B &= \theta + \pi/2\\ \phi_B &= -\theta + \phi - \pi/2\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta &= p_\theta^B - p_\phi^B\\ p_\phi &= p_\phi^B\end{aligned}}$$



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$\theta_B = \theta \quad +\pi/2$

$\phi_B = -\theta + \phi \; -\pi/2$

$p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

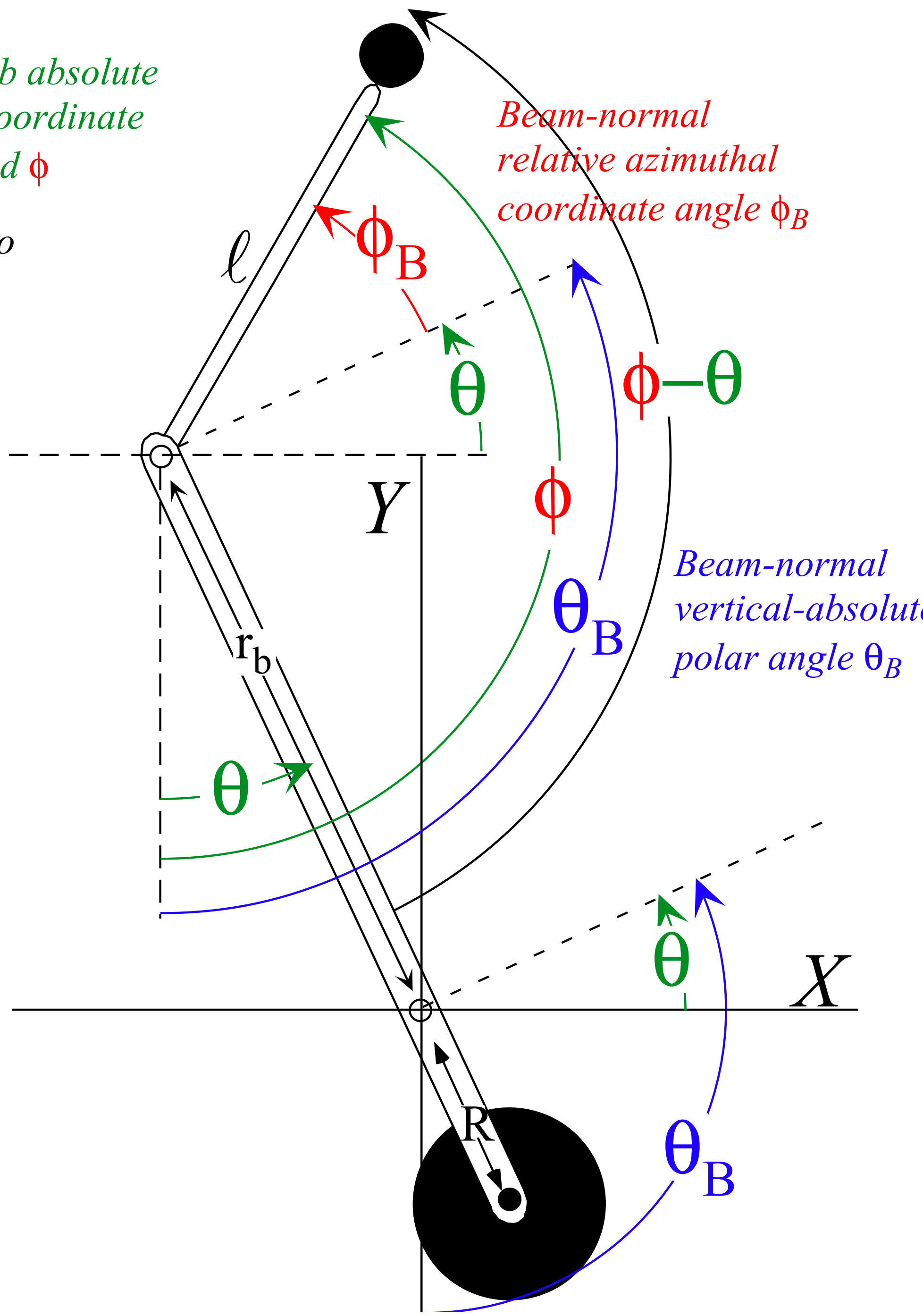
$\theta - \phi = -\phi_B - \pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$\theta_B = \theta \quad +\pi/2$

$\phi_B = -\theta + \phi \ -\pi/2$

$p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

$\theta - \phi = -\phi_B - \pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

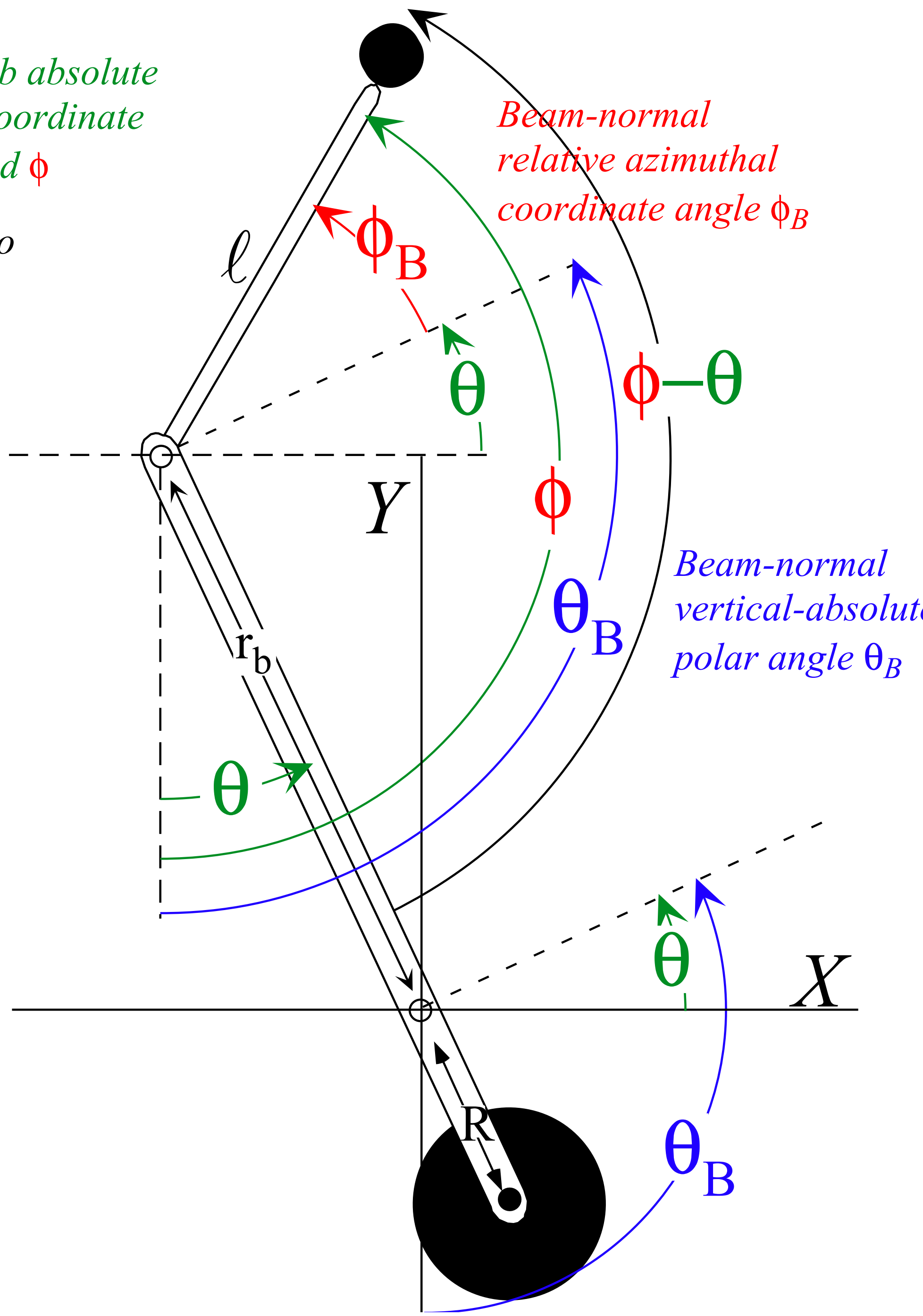
$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B\right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B\right)$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$$\theta = \theta_B - \pi/2 \qquad \phi = \theta_B + \phi_B$$

$$\theta_B = \theta + \pi/2 \qquad \phi_B = -\theta + \phi - \pi/2$$

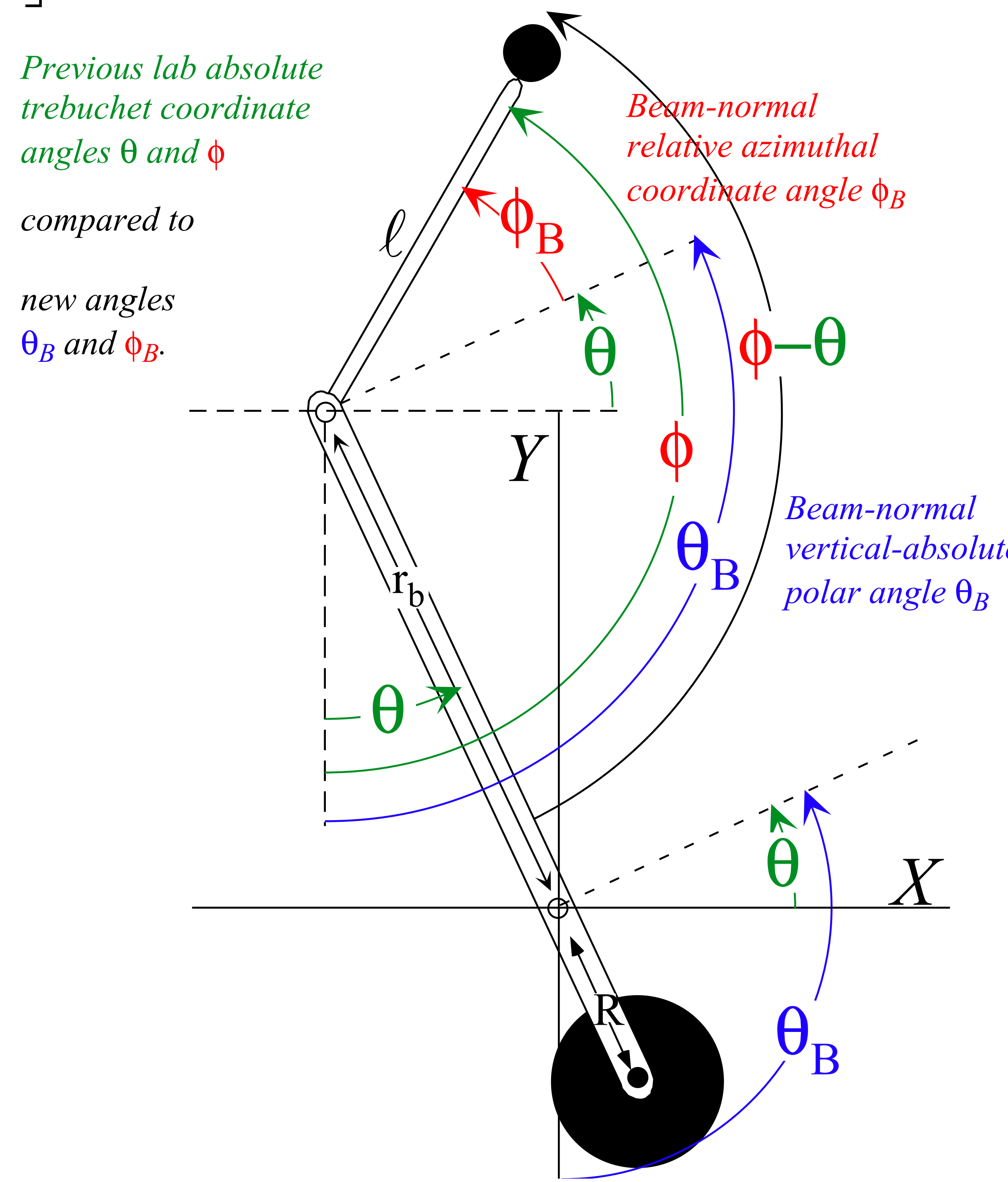
$$p_\theta = p_\theta^B - p_\phi^B \qquad p_\phi = p_\phi^B$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B\right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B\right)$$



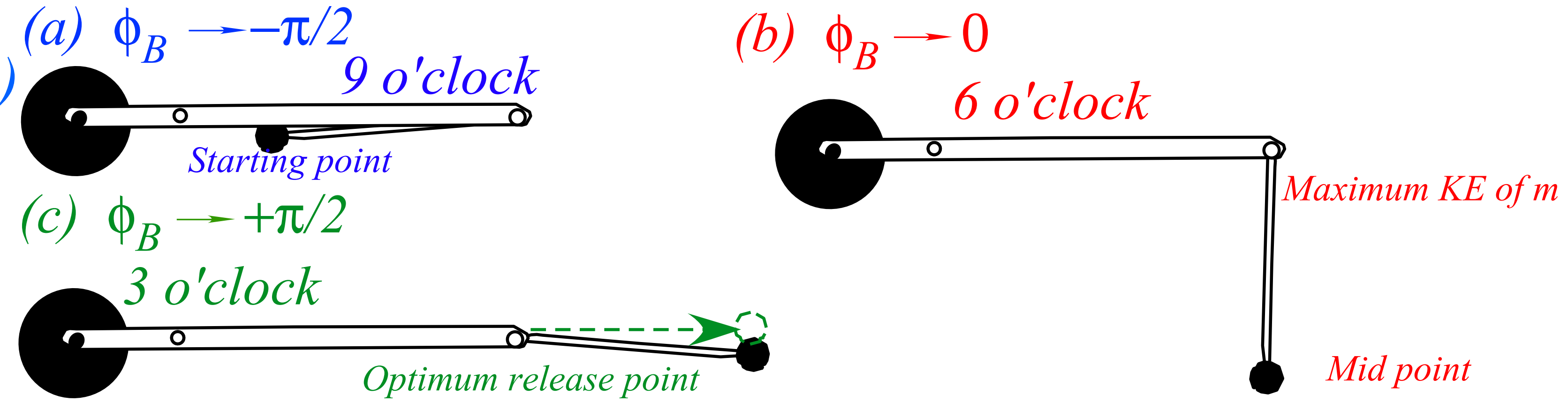
Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell\text{)}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

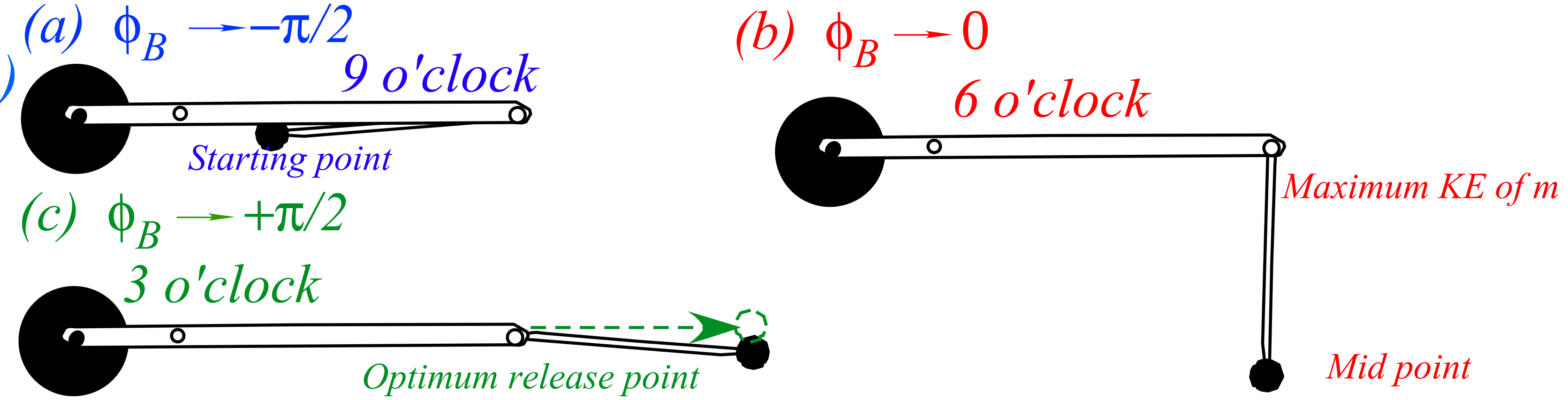
$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{ For: } r = \ell$$



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \begin{cases} 2E = MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{cases} \quad or : \begin{cases} 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$\sin\phi_B = -1$

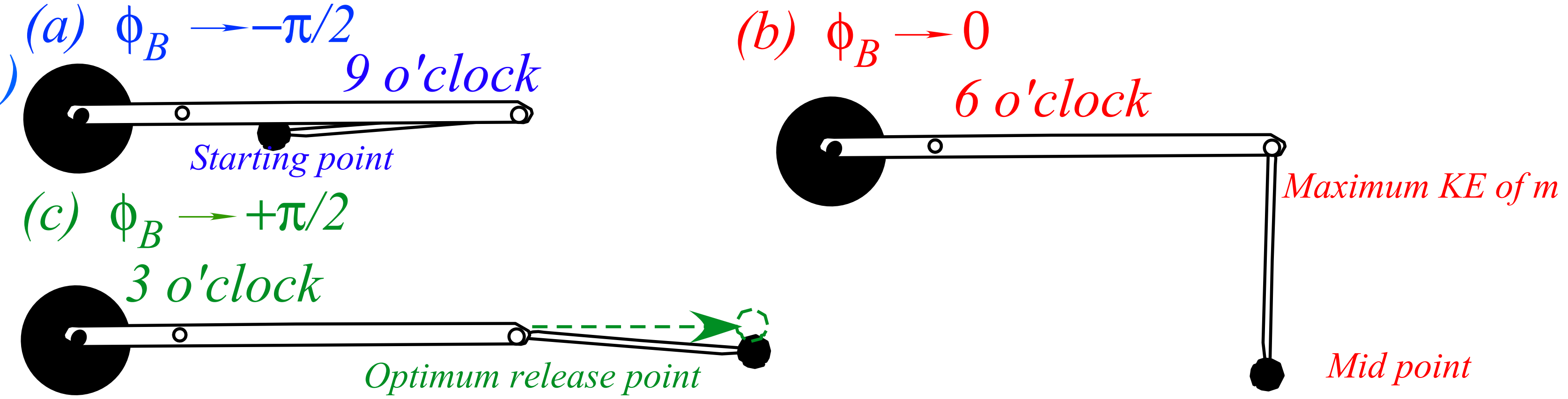
Conserved initial 2E

initial Λ

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned}2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right)\\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right)\end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned}2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2\\ \Lambda &= MR^2\dot{\theta}_{-\pi/2}\end{aligned}\right. \quad or : \left\{\begin{aligned}2E &= MR^2\omega^2\\ \Lambda &= MR^2\omega\end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$\sin\phi_B = -1$

Conserved initial $2E$

initial Λ

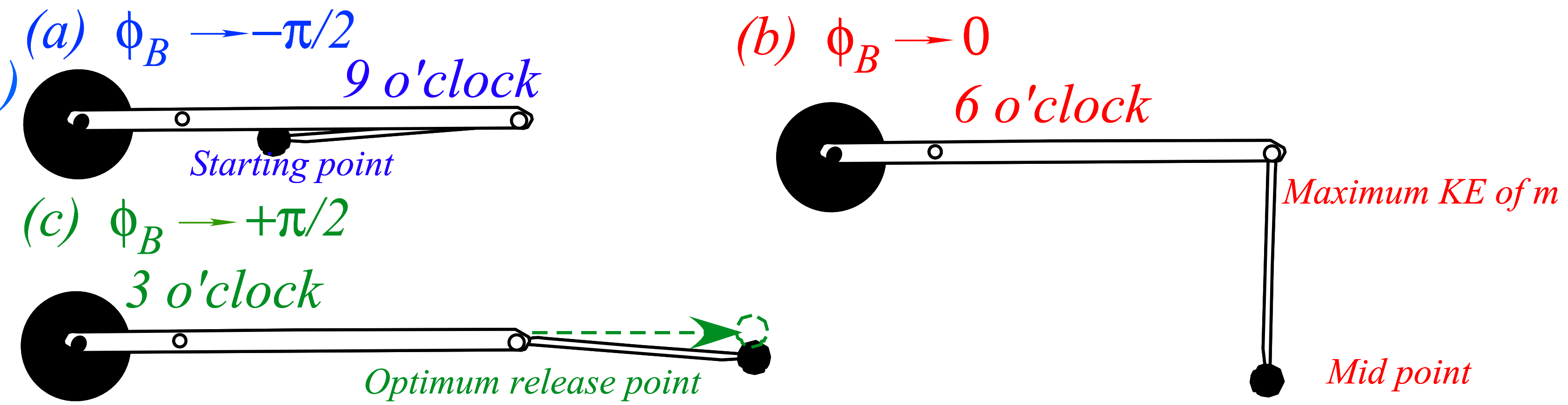
Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned}2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right)\\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right)\end{aligned}\right.$$

$\sin\phi_B = 0$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)



$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B = -1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} \overset{\text{Conserved initial } 2E}{2E = MR^2\omega^2} \\ \underset{\text{initial } \Lambda}{\Lambda = MR^2\omega} \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B = 0\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

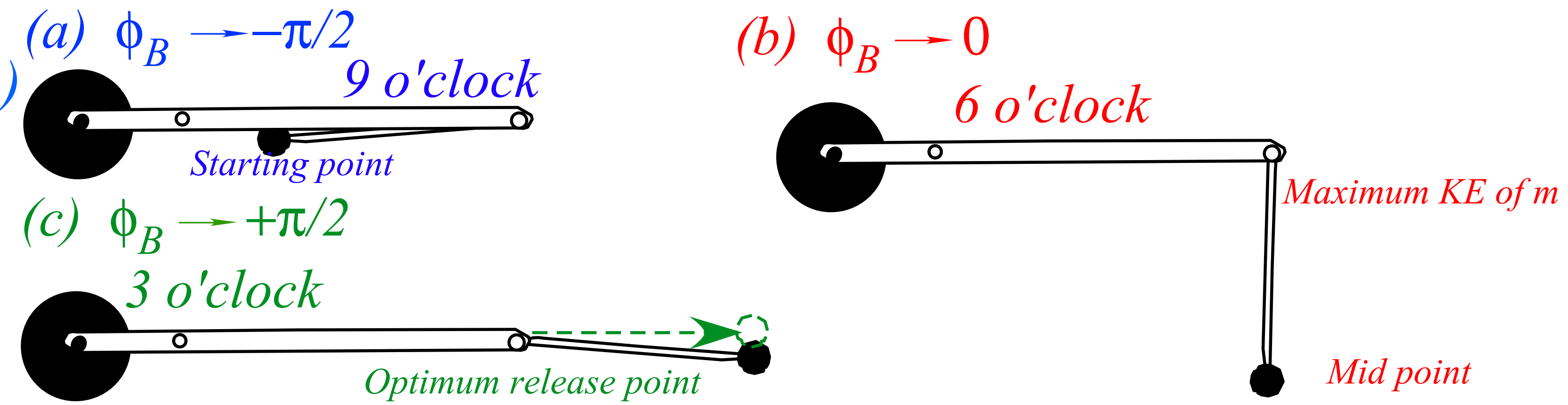
Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2: \\ \sin\phi_B = +1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \overset{\text{Conserved initial } 2E}{= MR^2\omega^2} \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) \underset{\text{initial } \Lambda}{= MR^2\omega} \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right)\\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

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Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B = 0\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = -\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) \quad \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = \frac{\pi}{2}\right)\end{aligned}\right.$$

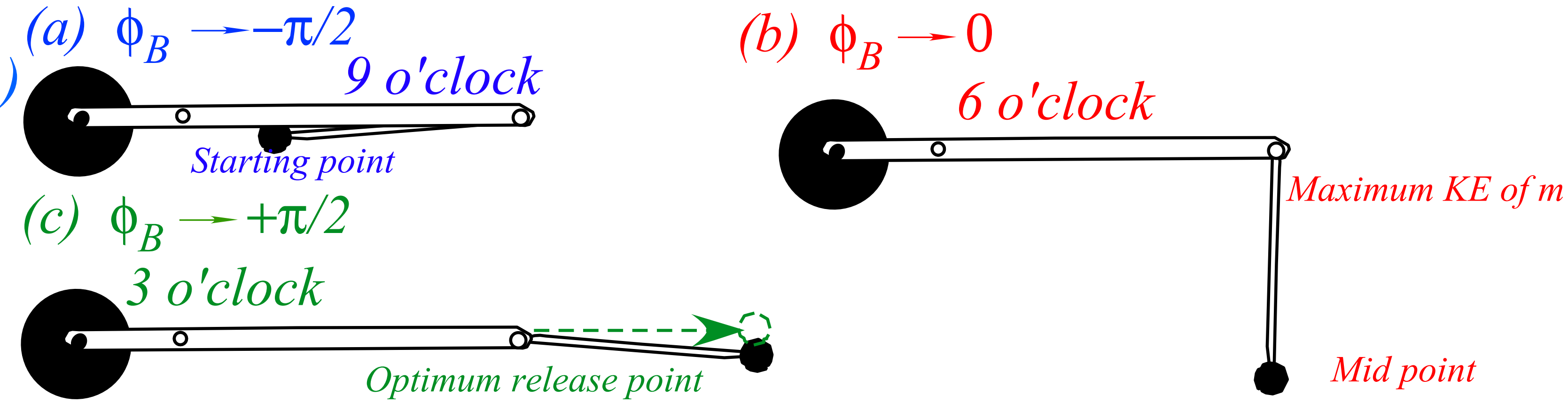
Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2: \\ \sin\phi_B = +1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 \underset{\text{Conserved initial } 2E}{=} MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) \underset{\text{initial } \Lambda}{=} MR^2\omega\end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

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$\sin\phi_B$=-1

Conserved initial 2E; initial Λ

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$\sin\phi_B$=0

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{ \begin{array}{ll} \frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{array}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved initial 2E

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

$\sin\phi_B$=+1

initial Λ

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

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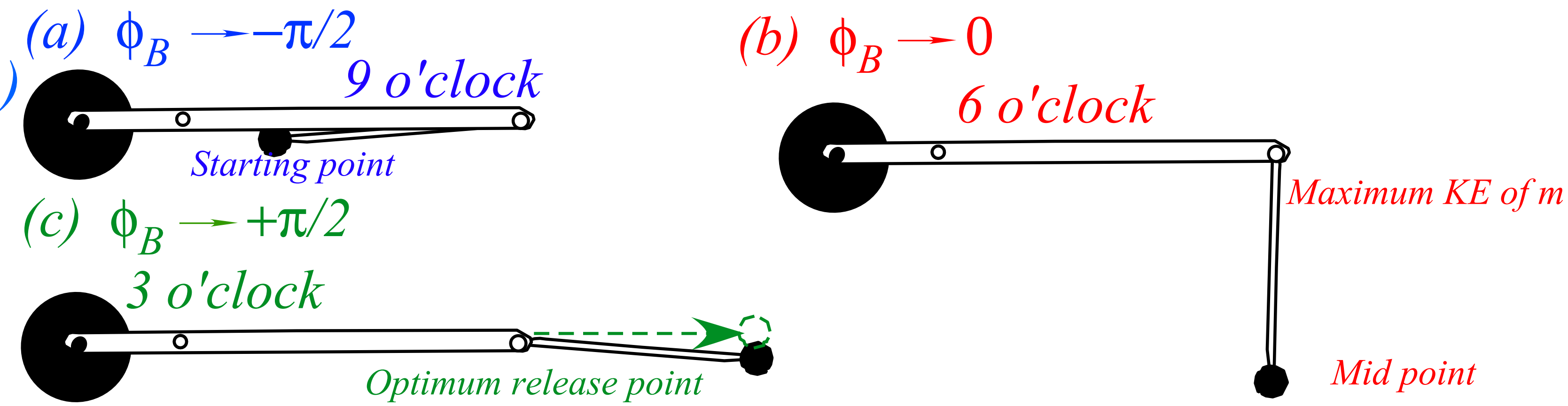


Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.$$

Conserved initial 2E ... *initial Λ*

For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right) = \begin{cases} \frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2: \quad \sin\phi_B = +1 \quad \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

Conserved initial 2E ... *initial Λ*

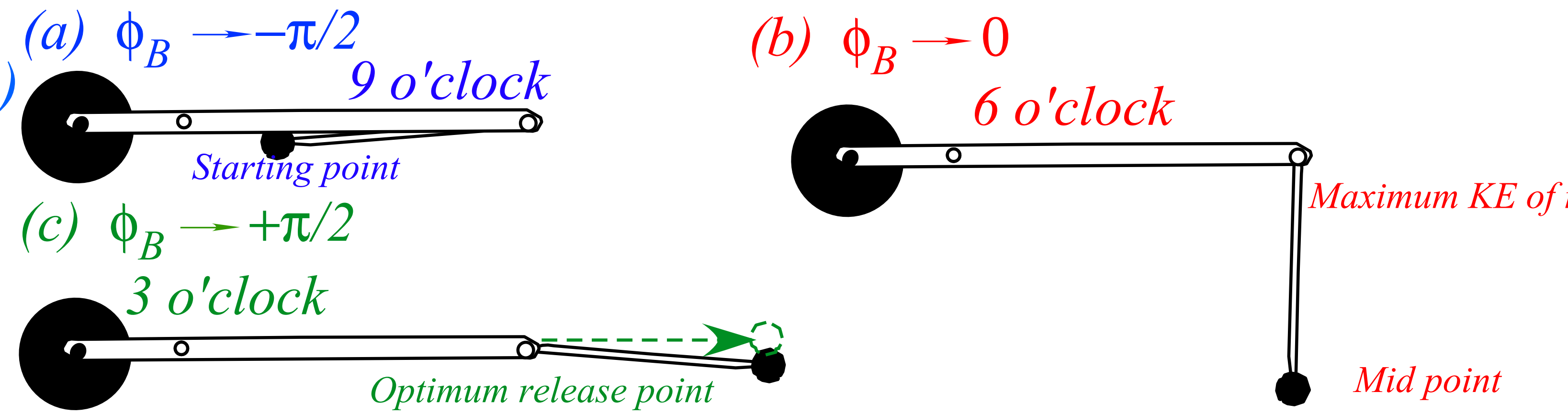
$$(\omega^2 - \dot{\theta}^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2$$

$$(\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

divide 2E by Λ

$$(\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\}\ \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}:\ \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \ \ \text{(Conserved initial 2E)} \\ \Lambda &= MR^2\omega \ \ \text{(initial } \Lambda\text{)} \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0:\ \sin\phi_B = 0 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \begin{cases} \dfrac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2:\ \sin\phi_B = +1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \ \ \text{(Conserved initial 2E)} \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \ \ \text{(initial } \Lambda\text{)} \end{aligned}\right.$$

$$(\omega^2 - \dot\theta^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$(\omega - \dot\theta_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

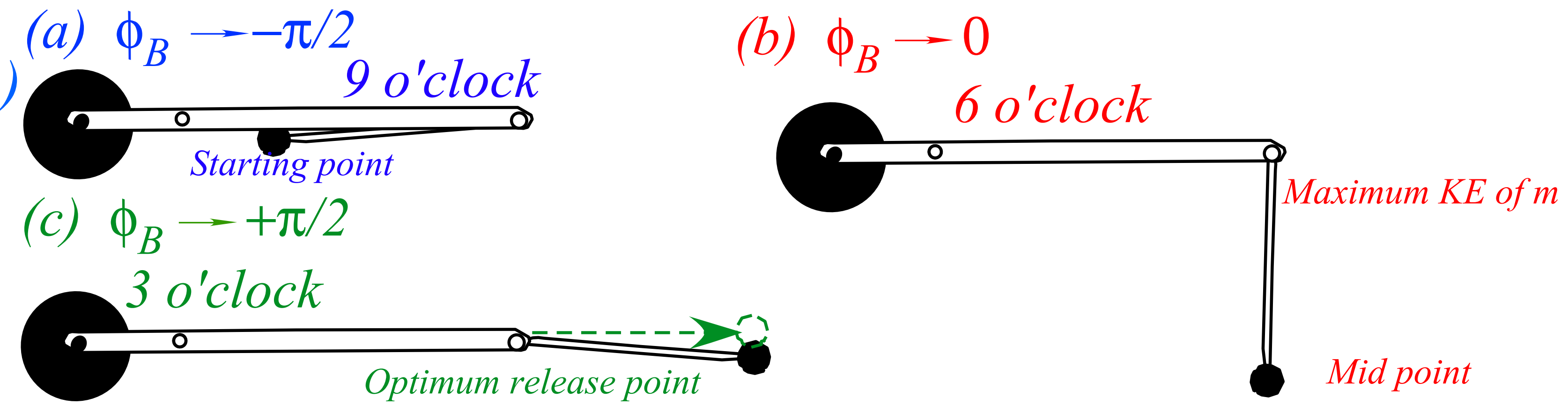
$$(\omega + \dot\theta_{\pi/2}) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda &= MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

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$\sin\phi_B$=-1

Conserved initial 2E ; initial Λ

$KE(m)=$

$$\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)$$

$$=\left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 && \left(\text{For: } \phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) && \left(\text{For: } \phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 && \left(\text{For: } \phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda &= MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$\sin\phi_B$=0

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B=\pi/2:\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 = MR^2\omega^2\\ \Lambda &= MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) = MR^2\omega\end{aligned}\right.$$

$\sin\phi_B$=+1

Conserved initial 2E ; initial Λ

$$(\omega^2-\dot\theta^2_{\pi/2})=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$(\omega-\dot\theta_{\pi/2})=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$(\omega+\dot\theta_{\pi/2})=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

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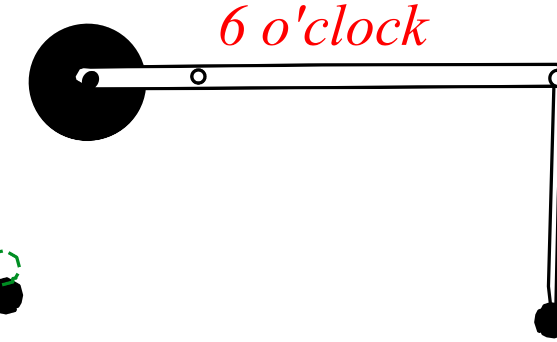
(a) $\phi_B \to -\pi/2$

9 o'clock

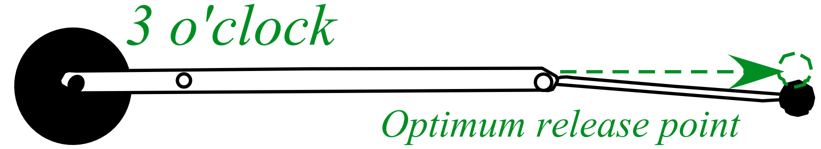
(b) $\phi_B \to 0$

6 o'clock



(c) $\phi_B \to +\pi/2$

3 o'clock



Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$\sin\phi_B = -1$

Conserved initial 2E

initial Λ

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$\sin\phi_B = 0$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

$\sin\phi_B = +1$

Conserved initial 2E

initial Λ

$$(\omega^2 - \dot\theta^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

divide 2E by Λ

$$(\omega + \dot\theta_{\pi/2}) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$(\omega - \dot\theta_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

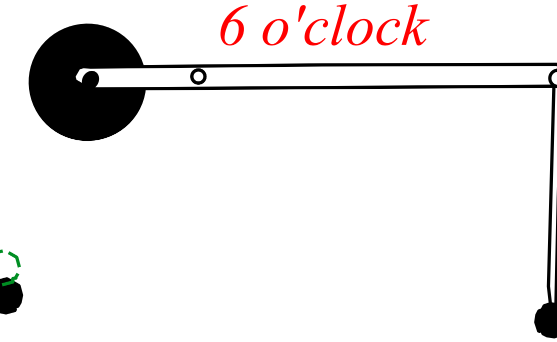
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(a) $\phi_B \to -\pi/2$

9 o'clock

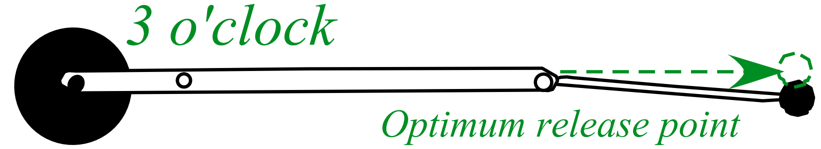
(b) $\phi_B \to 0$

6 o'clock



(c) $\phi_B \to +\pi/2$

3 o'clock



Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E&=MR^2\dot{\theta}^2+mr^2\left(\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B+\dot{\phi}^2\right)\\ \Lambda&=MR^2\dot{\theta}+mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right)\end{aligned}\right\}\text{ For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}:\ \sin\phi_B=-1\quad\left\{\begin{aligned} 2E&=MR^2\dot{\theta}^2_{-\pi/2}+mr^2\left(\dot{\phi}_{-\pi/2}-\dot{\phi}_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot{\theta}_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned} &\text{Conserved initial } 2E\quad 2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\quad\text{initial }\Lambda\end{aligned}\right.\quad\text{For: }\dot{\theta}_{-\pi/2}=\omega=\dot{\phi}_{-\pi/2}$$

$$KE(m)=\frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B\right)=\begin{cases}\frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: }\phi_B=-\frac{\pi}{2}\right)\\ \frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: }\phi_B=0\right)\\ \frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: }\phi_B=\frac{\pi}{2}\right)\end{cases}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\ \sin\phi_B=0\quad\left\{\begin{aligned} 2E&=MR^2\dot{\theta}_0^2+mr^2\left(\dot{\phi}_0^2+\dot{\theta}_0^2\right)\\ \Lambda&=MR^2\dot{\theta}_0+mr^2\left(\dot{\phi}_0+\dot{\theta}_0\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B=\pi/2:\ \sin\phi_B=+1\quad\left\{\begin{aligned} 2E&=MR^2\dot{\theta}^2_{\pi/2}+mr^2\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)^2 &&=MR^2\omega^2\ \text{(Conserved initial 2E)}\\ \Lambda&=MR^2\dot{\theta}_{\pi/2}+2mr^2\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right) &&=MR^2\omega\ \text{(initial }\Lambda)\end{aligned}\right.$$

$$\left(\omega^2-\dot{\theta}^2_{\pi/2}\right)=\frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)^2$$

$$\left(\omega-\dot{\theta}_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega+\dot{\theta}_{\pi/2}\right)=\frac{1}{2}\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)$$

$$\boxed{\dot{\phi}_{\pi/2}=\dot{\theta}_{\pi/2}+2\omega}$$

substitute

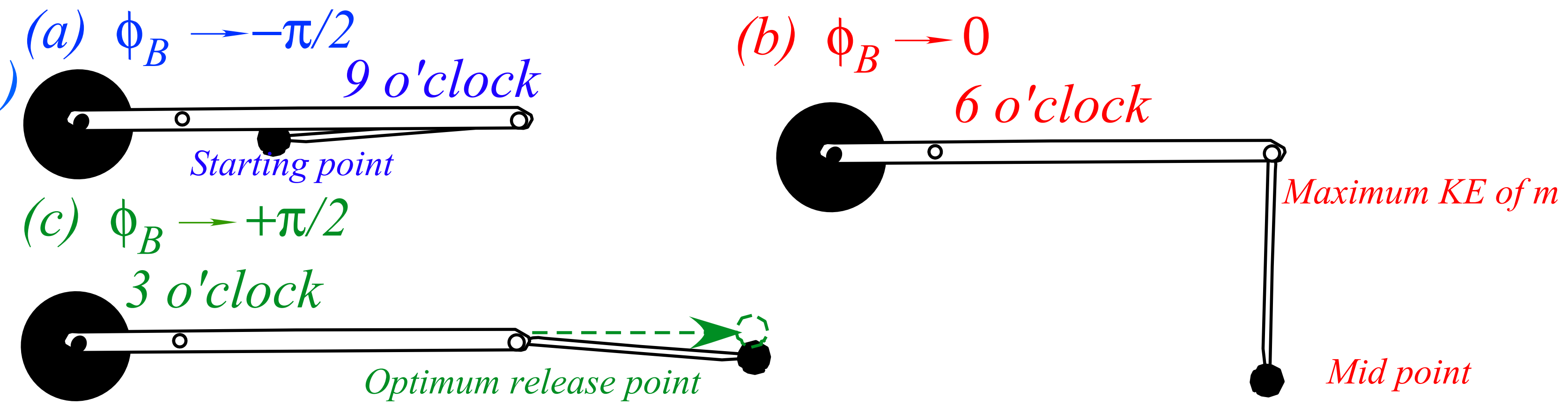
$$\omega-\dot{\theta}_{\pi/2}=\frac{2mr^2}{MR^2}\left(2\omega+2\dot{\theta}_{\pi/2}\right)$$

$$\omega-\frac{4mr^2}{MR^2}\omega=\dot{\theta}_{\pi/2}+\frac{4mr^2}{MR^2}\dot{\theta}_{\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \quad \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \begin{cases} 2E = MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot\theta_{-\pi/2} \end{cases} \quad or: \begin{cases} 2E = MR^2\omega^2 & \text{(Conserved initial 2E)} \\ \Lambda = MR^2\omega & \text{(initial } \Lambda\text{)} \end{cases}$$

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \begin{cases} 2E = MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{cases}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \begin{cases} \frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2: \quad \sin\phi_B = +1 \quad \begin{cases} 2E = MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 & \text{(Conserved initial 2E)} \\ \Lambda = MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega & \text{(initial } \Lambda\text{)} \end{cases}$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

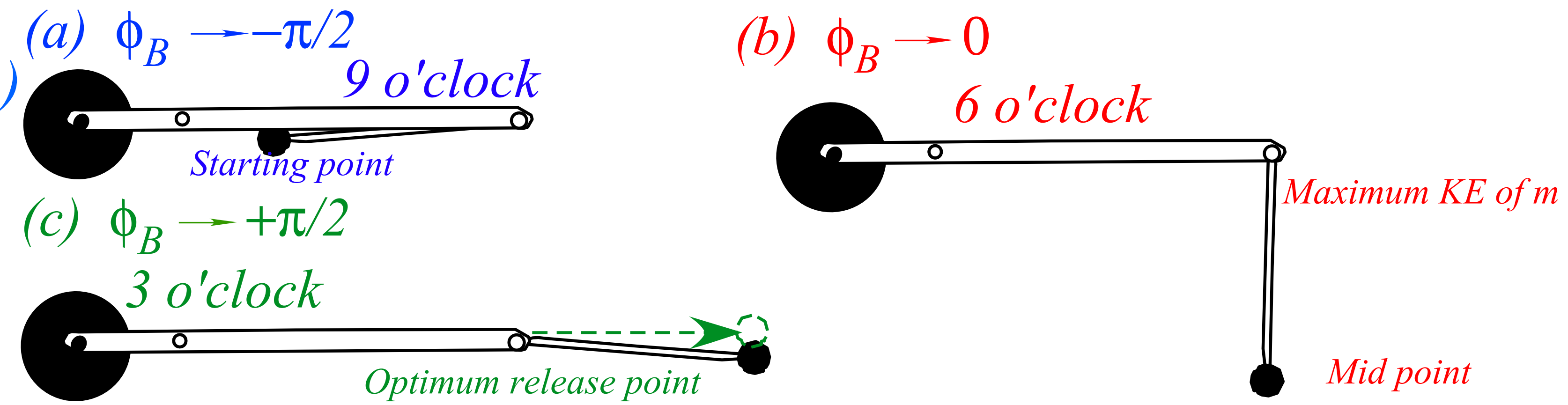
$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

solve

$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\,\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta + \dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or : \left\{\begin{aligned} 2E &= MR^2\omega^2 \quad \text{(Conserved initial 2E)} \\ \Lambda &= MR^2\omega \quad \text{(initial } \Lambda\text{)} \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$\sin\phi_B$=-1

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$\sin\phi_B$=0

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{(Conserved initial 2E)} \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{(initial } \Lambda\text{)} \end{aligned}\right.$$

$\sin\phi_B$=+1

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\xrightarrow{\text{divide 2E by }\Lambda} \left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

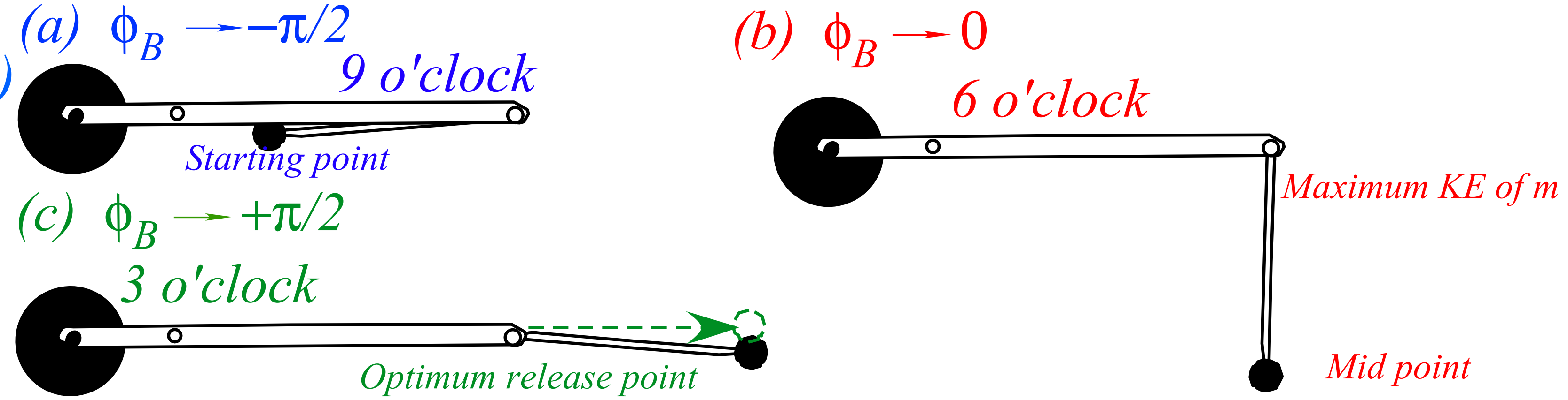
$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2} \xrightarrow{\text{solve}} \boxed{\dot\theta_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\omega}$$

Large $M \gg m$ case (Beam nearly constant ω)

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \quad \sin\phi_B = -1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right.$$

Conserved

$$or: \left\{\begin{aligned} &\text{initial } 2E: \quad 2E = MR^2\omega^2 \\ &\Lambda = MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right. \qquad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) \quad \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \quad \sin\phi_B = 0 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\phi_B = \pi/2: \quad \sin\phi_B = +1 \quad \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \quad \text{initial } 2E \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \quad \text{initial } \Lambda \end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

solve

$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\omega}$$

Large $M \gg m$ case

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Optimum $MR^2 = 4mr^2$ case

$$\boxed{\dot\phi_{\pi/2} = 0 + 2\omega = 2\omega}$$

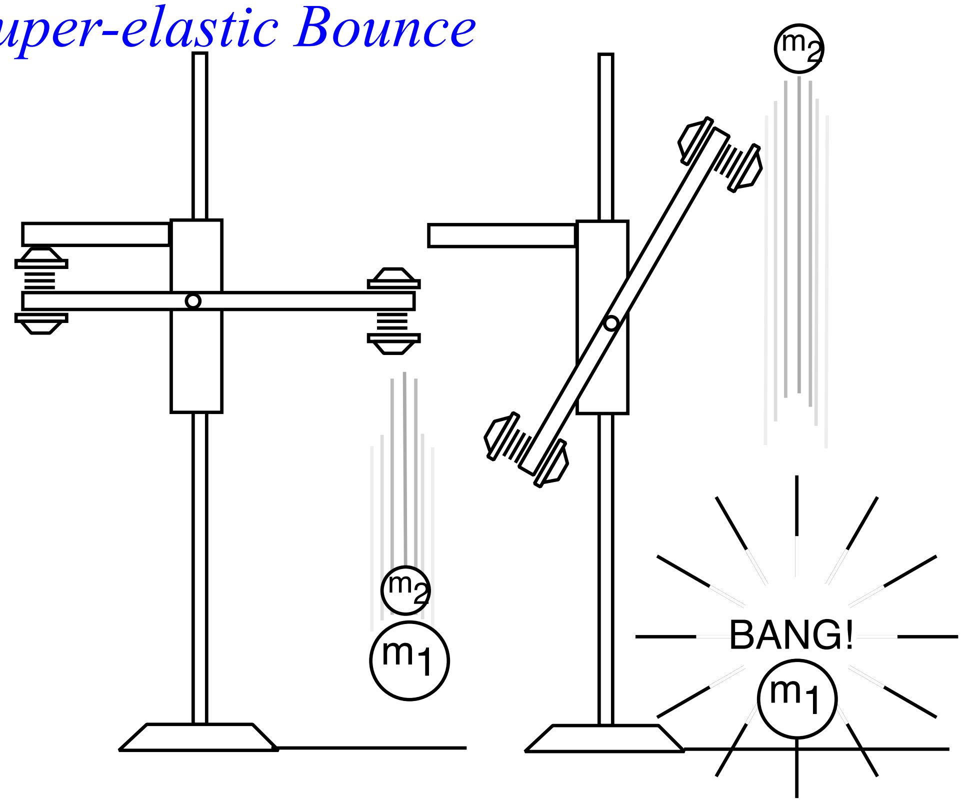
$$\boxed{\dot\theta_{\pi/2} = \frac{1-1}{1+1}\omega = 0}$$

Analogous Superball Models

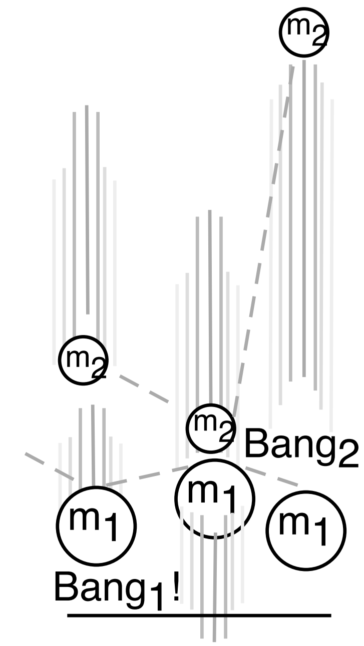
Similar in some ways to trebuchet models

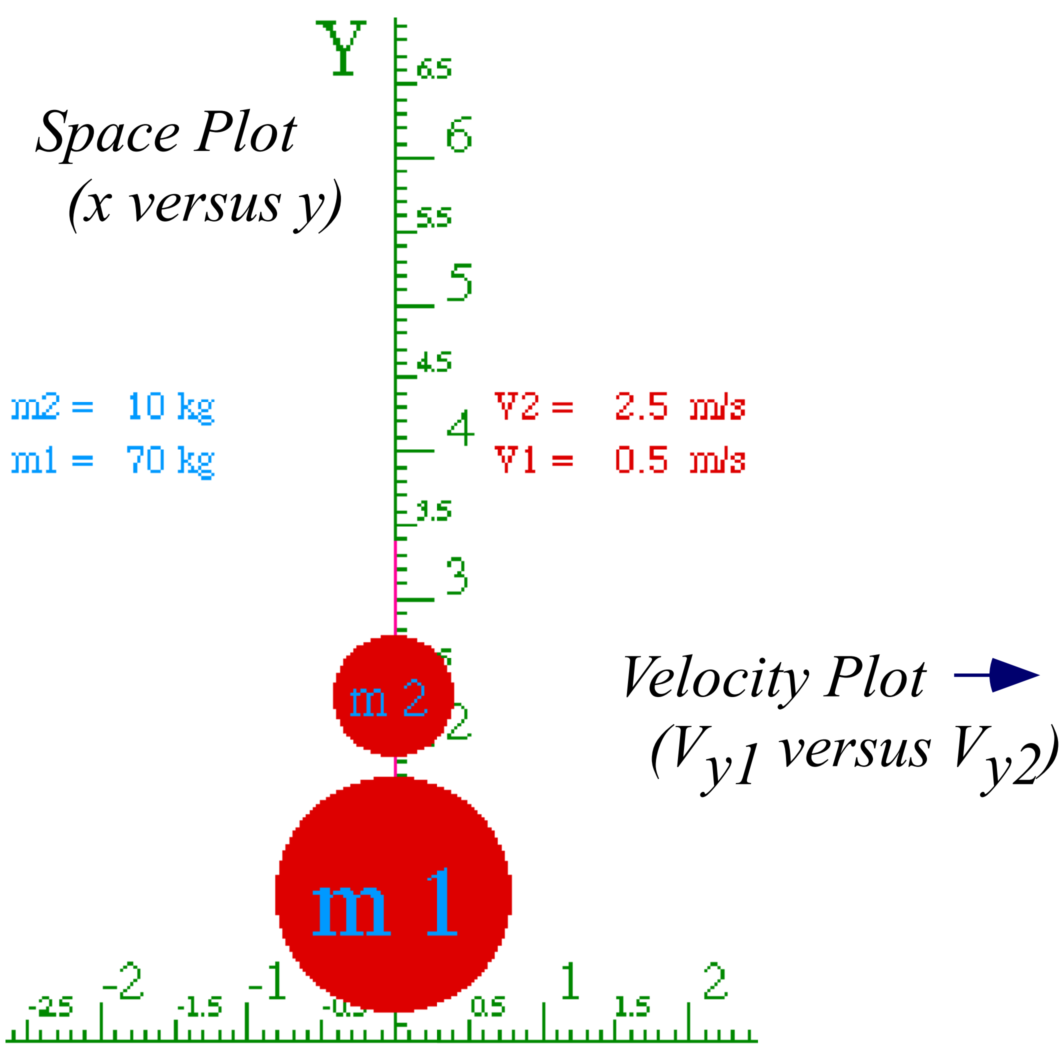
Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," Am. J. Phys. **39**, 656 (1971)
(A class project)

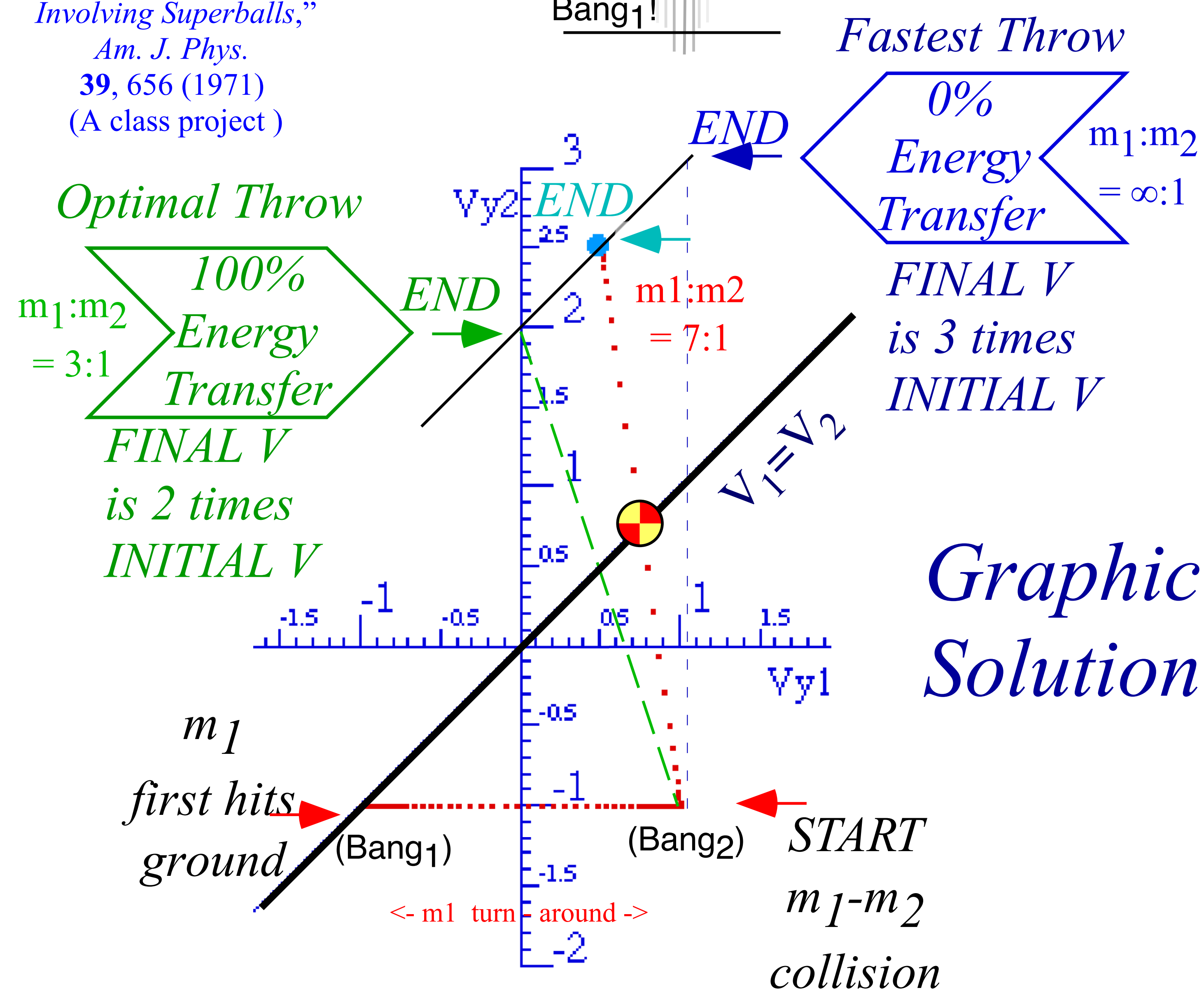
Super-elastic Bounce



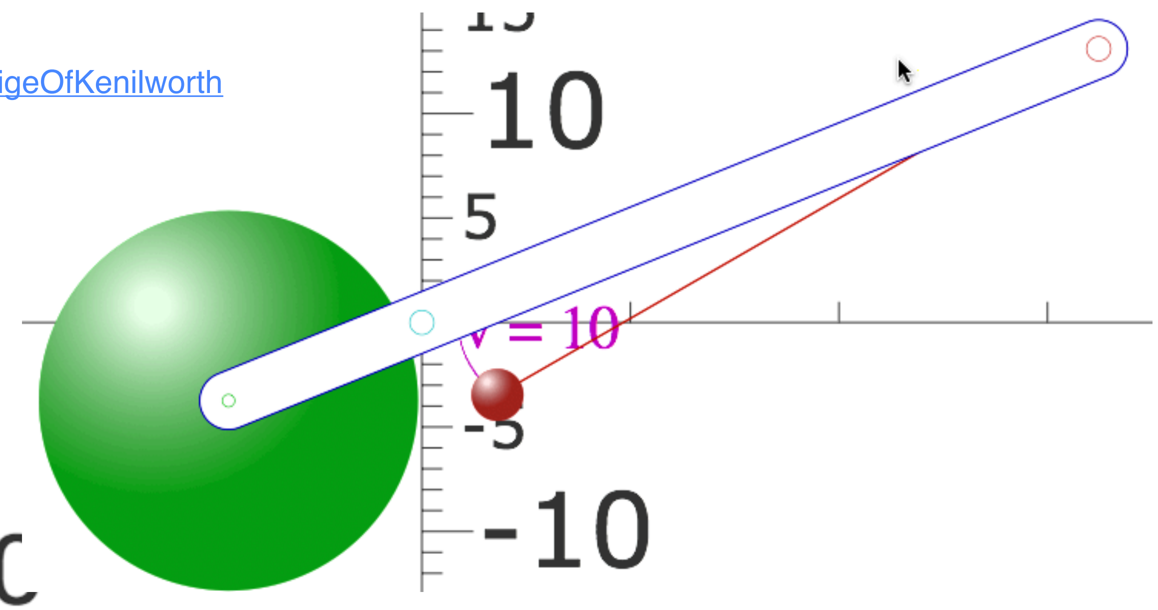
2-Bang Model

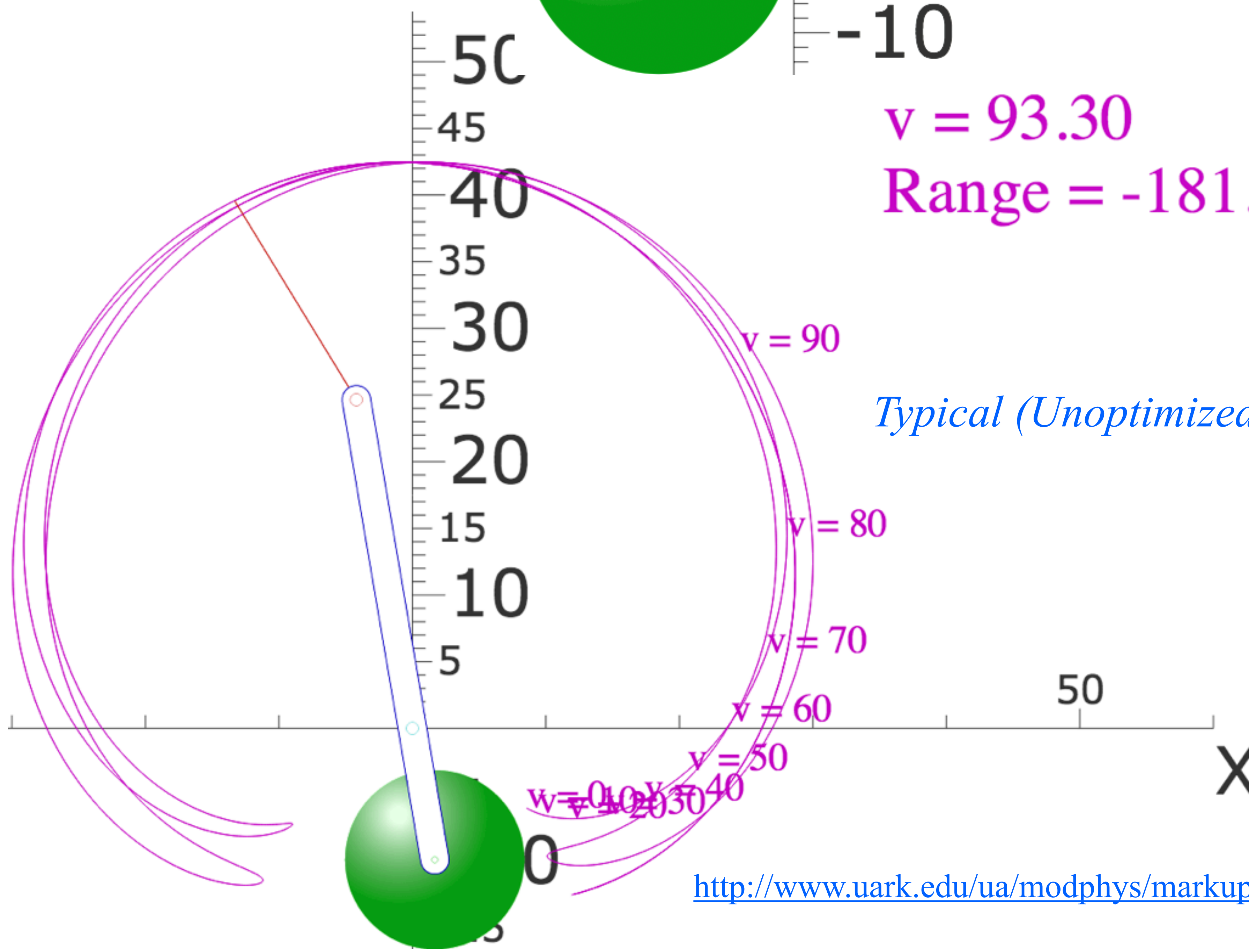






http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth

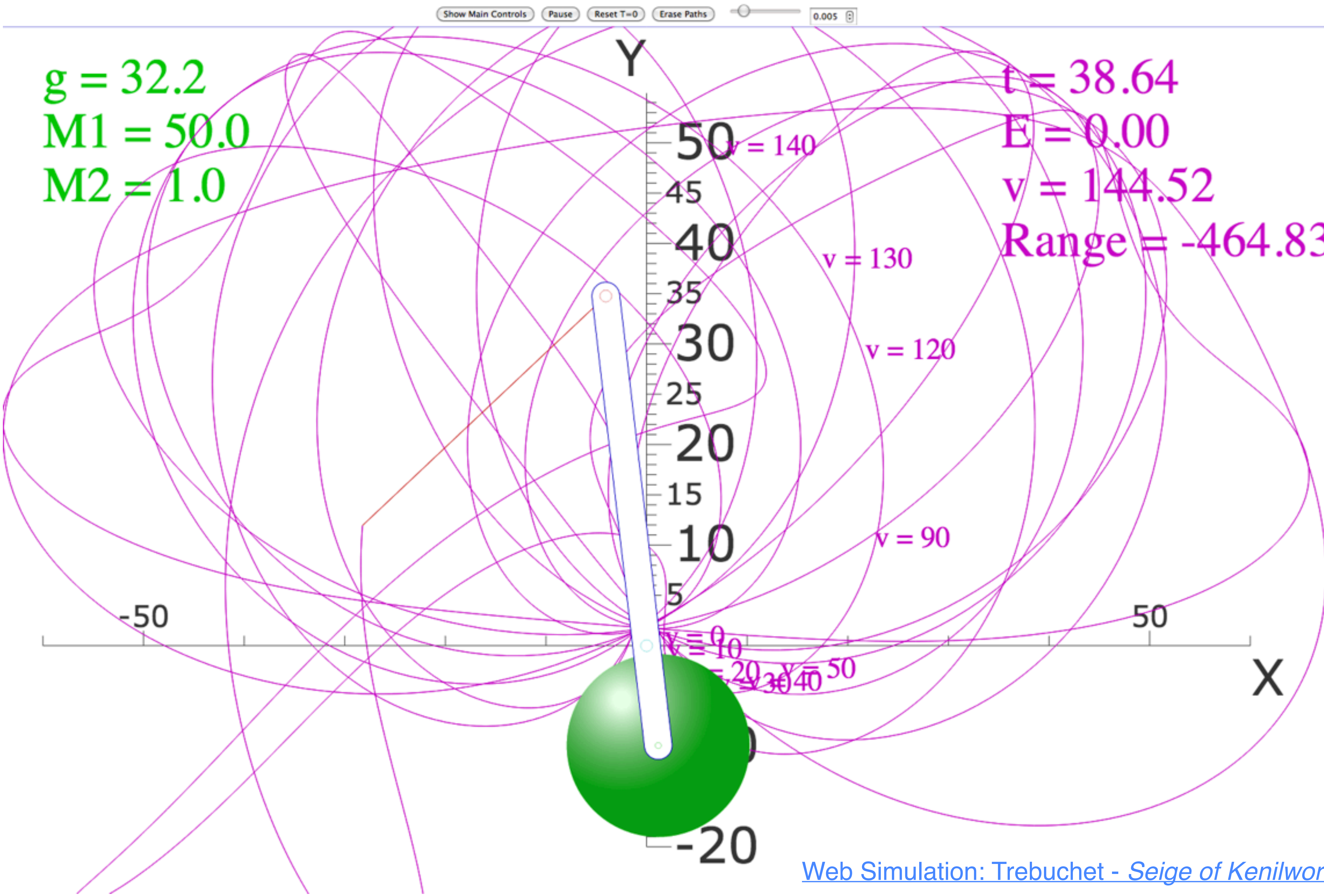




Typical (Unoptimized) Trebuchet

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

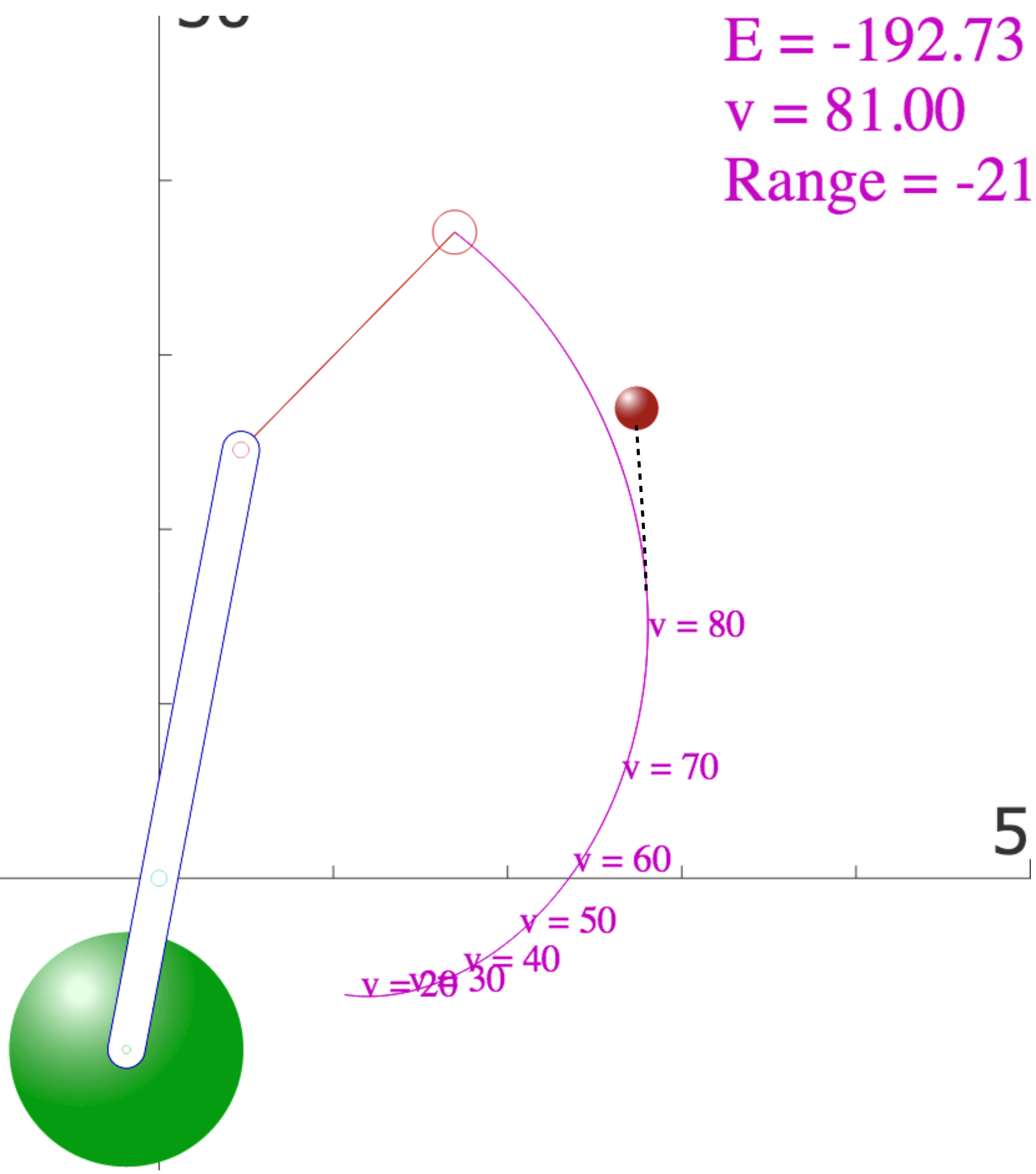
Trebuchet in Siege of Kenilworth 1215 ACE
(Re-enactment shown on NOVA-TV 2005)



Web Simulation: Trebuchet - *Seige of Kenilworth*

There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...

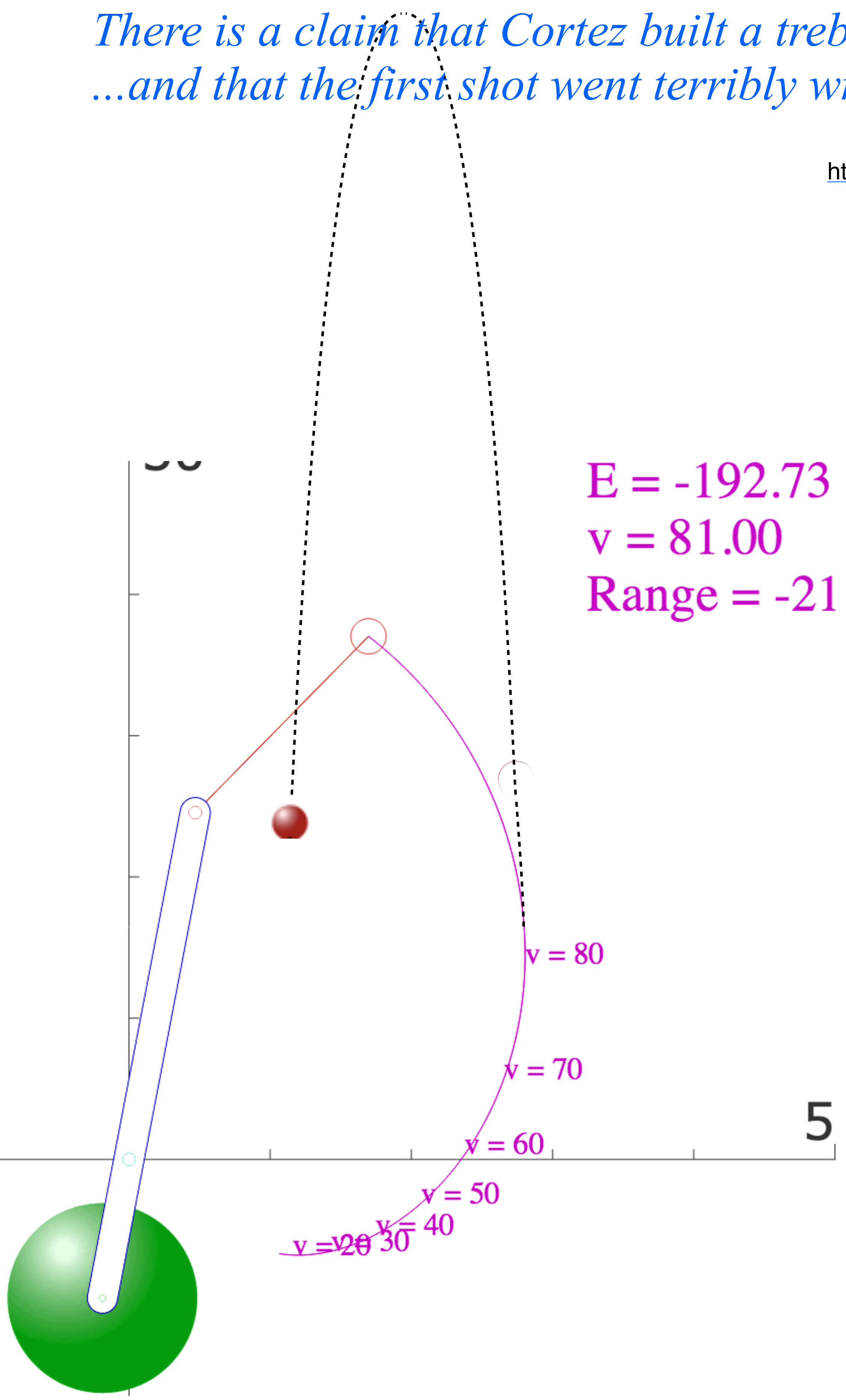
http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge



Web Simulation: Trebuchet - *Montezuma's Revenge*

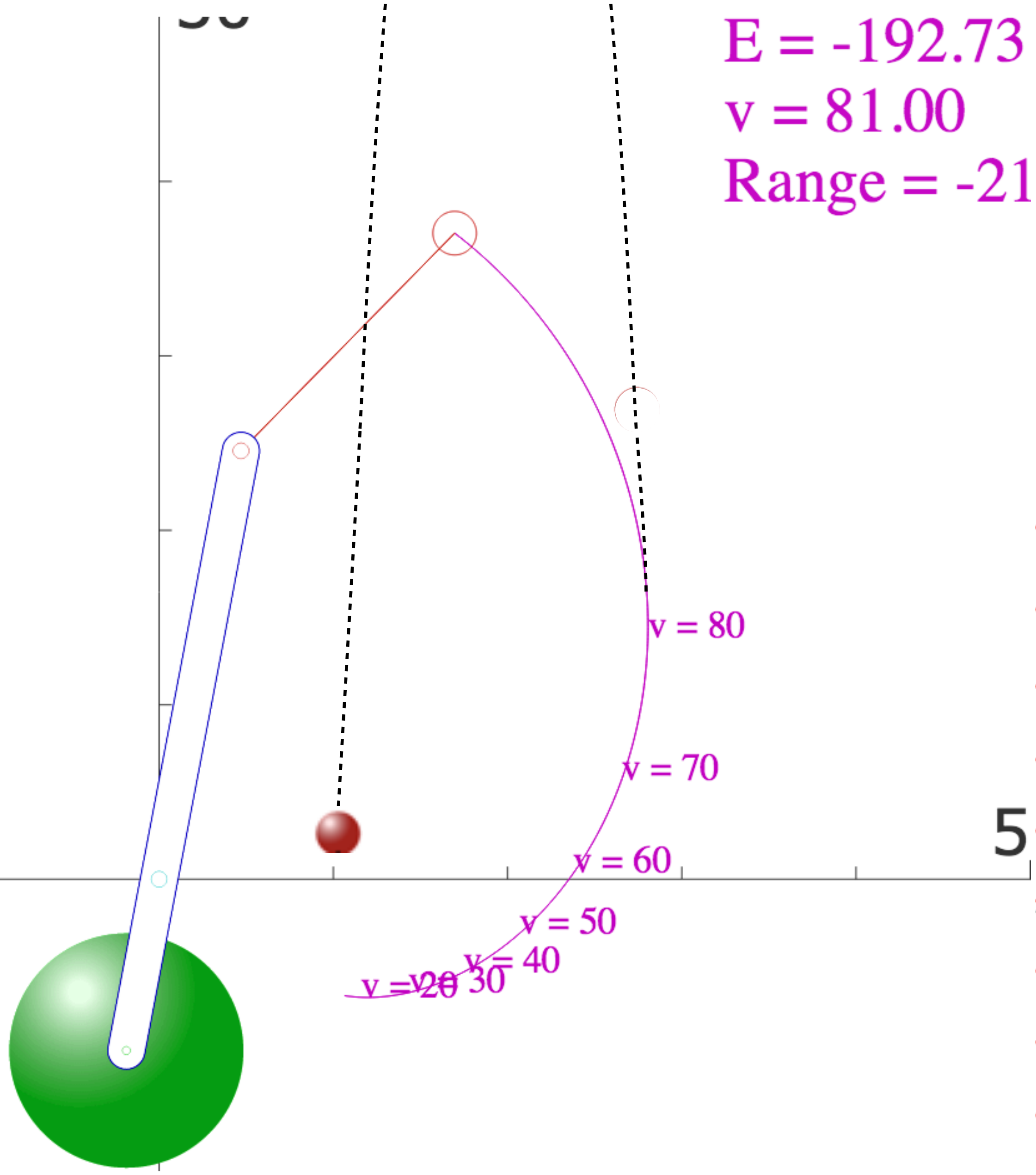
There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

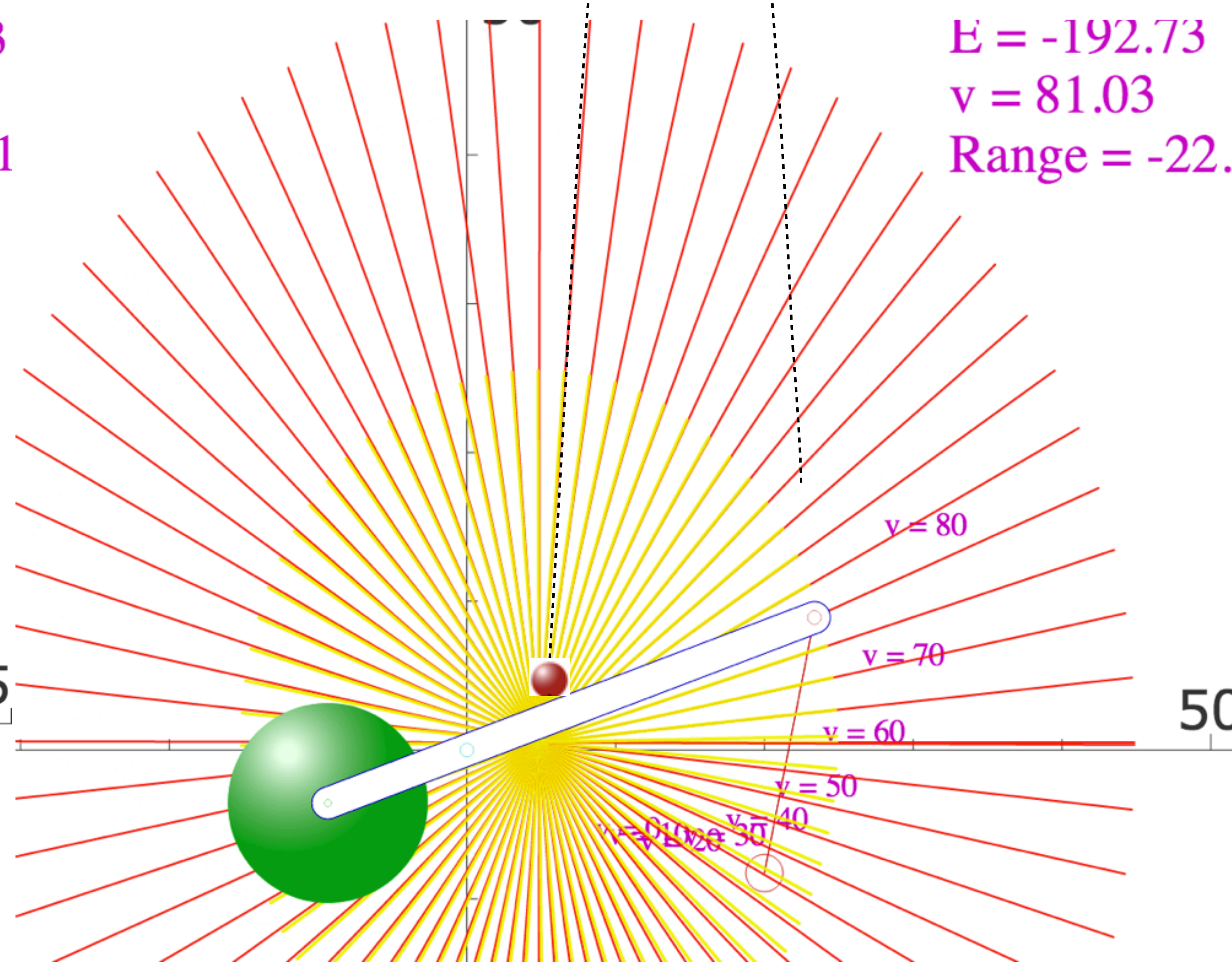
http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge



There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

...if this story is true, then it gives new meaning to the expression "Montezuma's Revenge"...

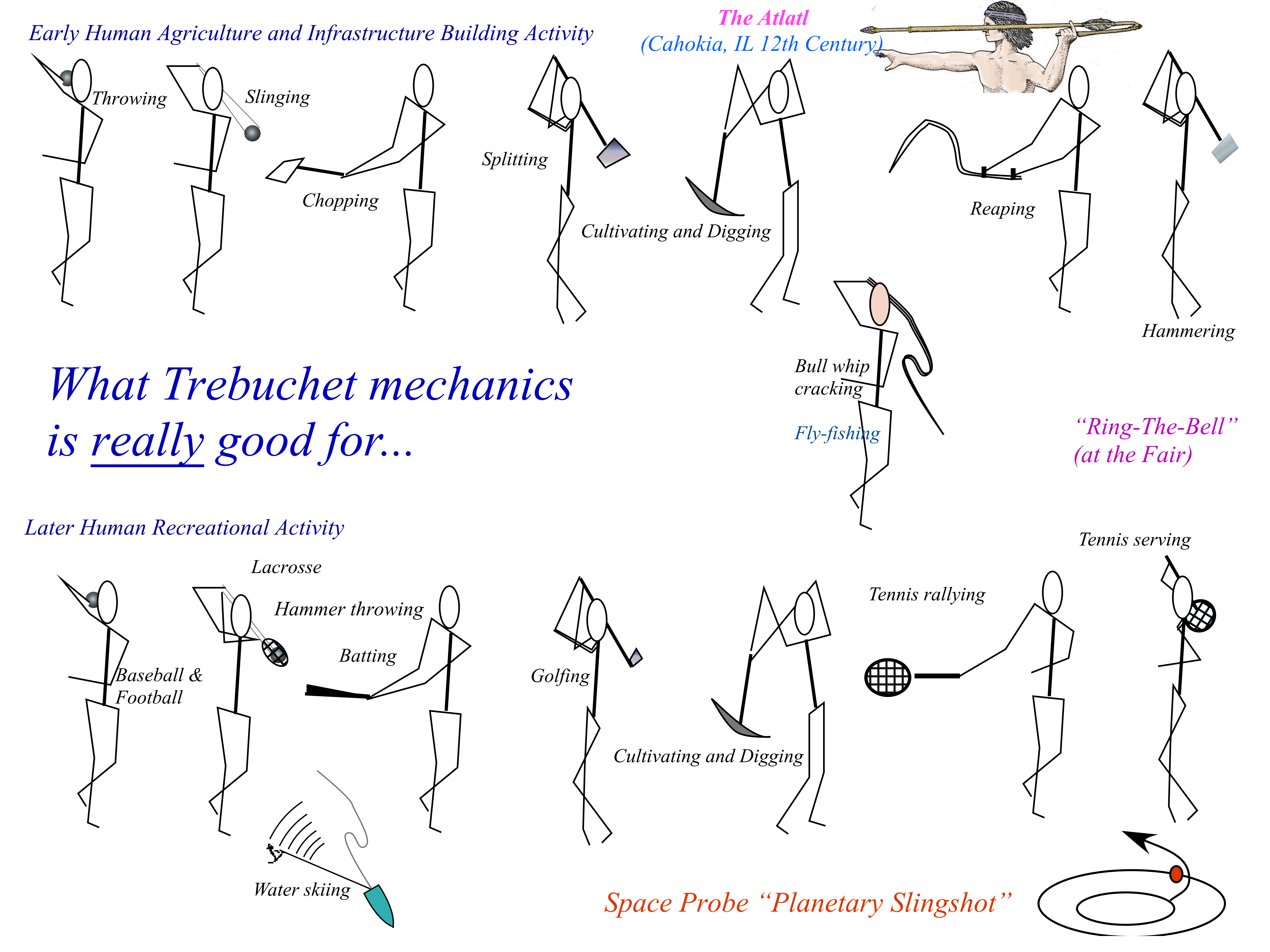




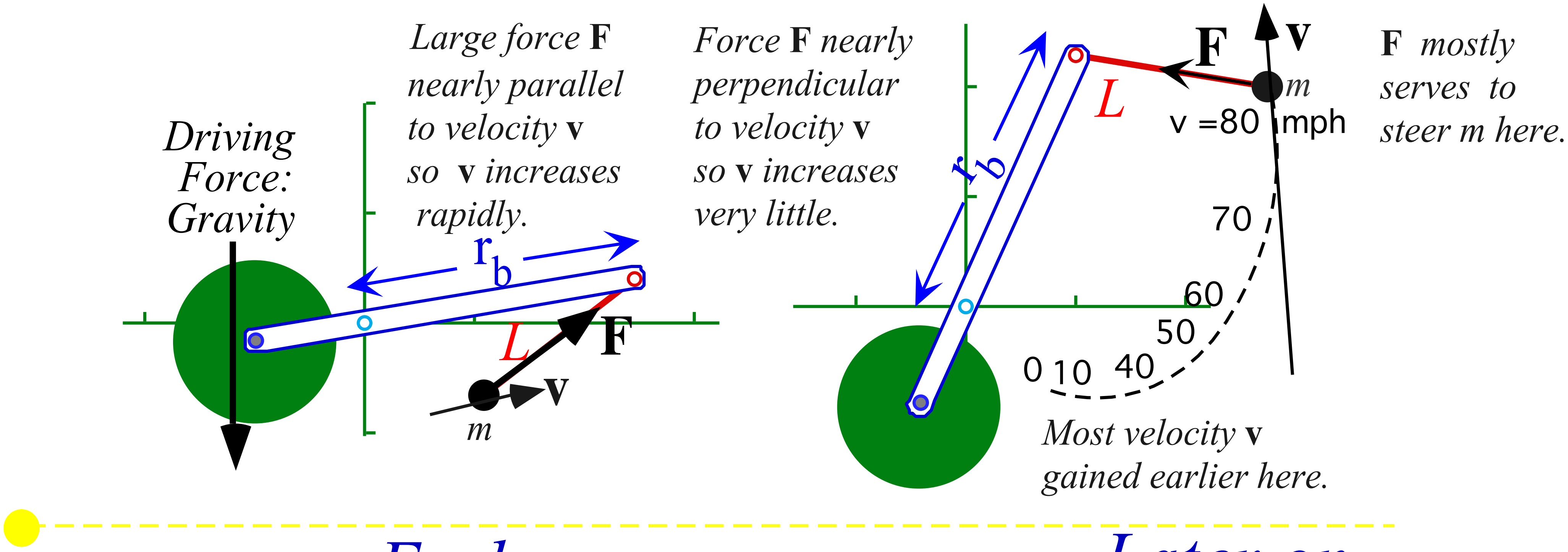
[Web Simulation: Trebuchet - *Montezuma's Revenge*]

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*


Early Human Agriculture and Infrastructure Building Activity
Throwing
Slinging
Chopping
Splitting
Cultivating and Digging
The Atlatl
(Cahokia, IL 12th Century)
Reaping
Hammering
Bull whip cracking
Fly-fishing
What Trebuchet mechanics is really good for...
"Ring-The-Bell" (at the Fair)
Later Human Recreational Activity
Lacrosse
Hammer throwing
Batting
Baseball & Football
Golfing
Cultivating and Digging
Tennis rallying
Tennis serving
Water skiing
Space Probe "Planetary Slingshot"

Trebuchet analogy with racquet swing - What we learn



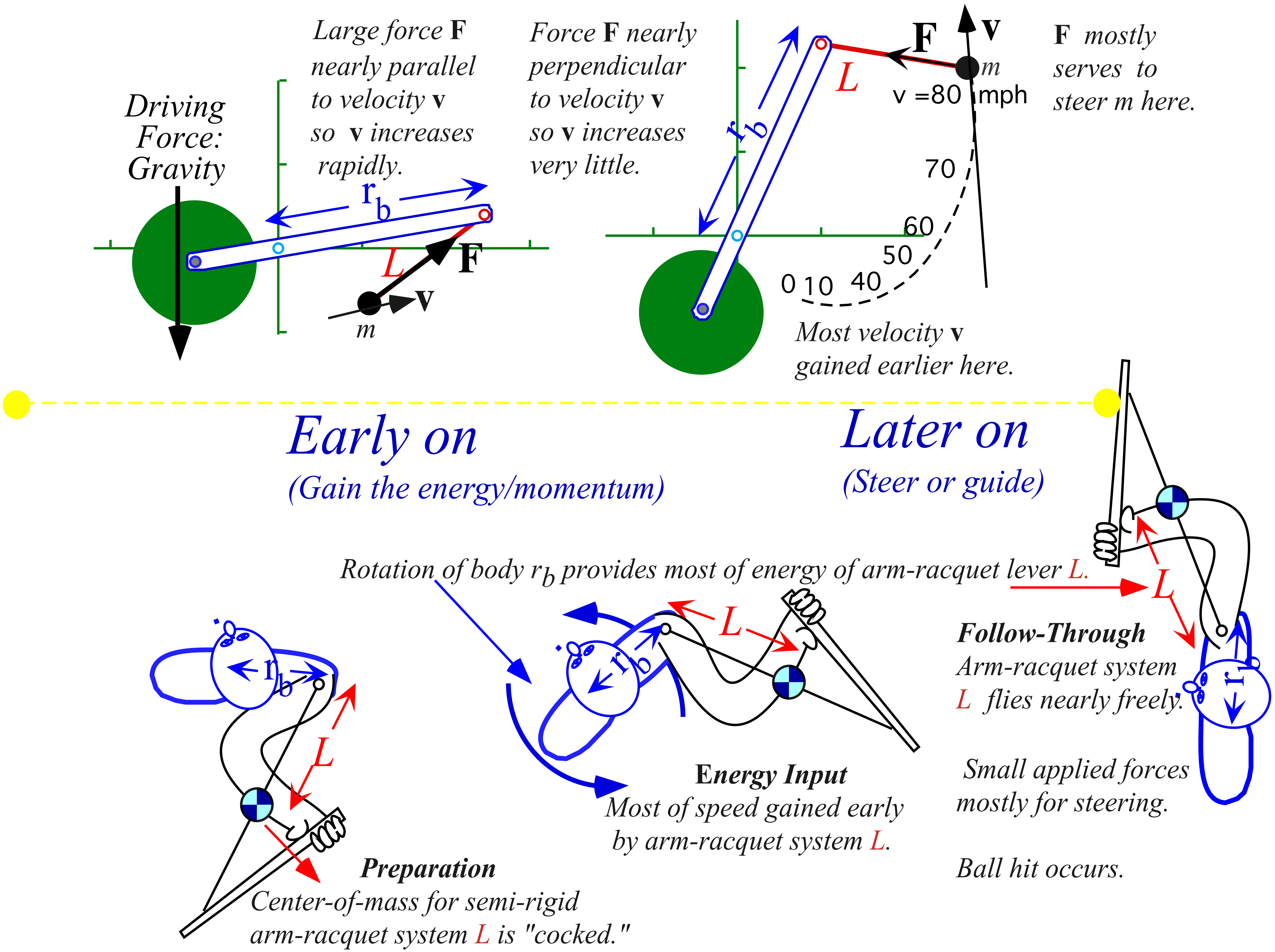
Early on

(Gain the energy/momentum)

Later on

(Steer or guide)

Trebuchet analogy with racquet swing - What we learn



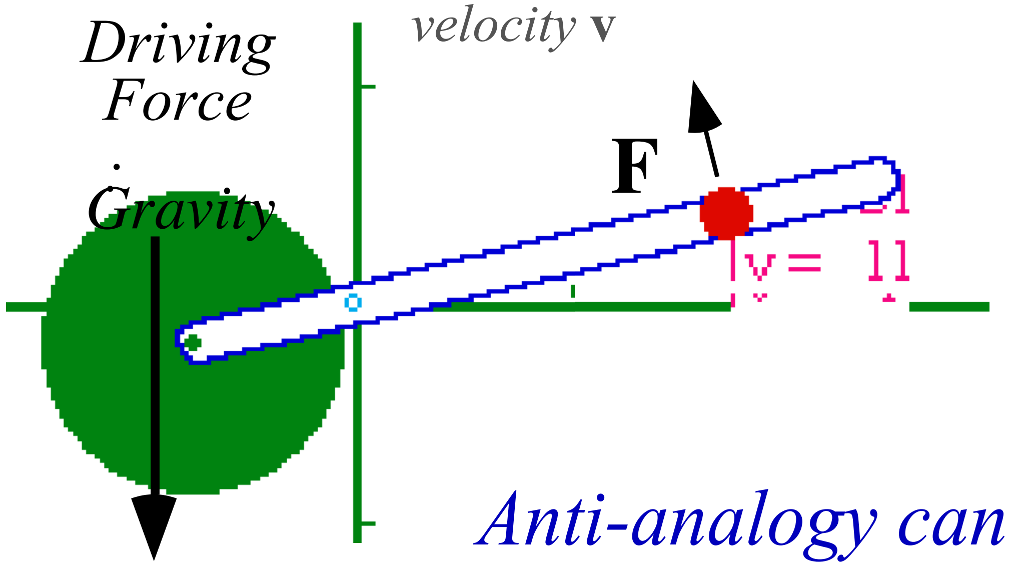
An Opposite to Trebuchet Mechanics- The "Flinger"

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=AnimateFlinger

Early on

(Not much happening)

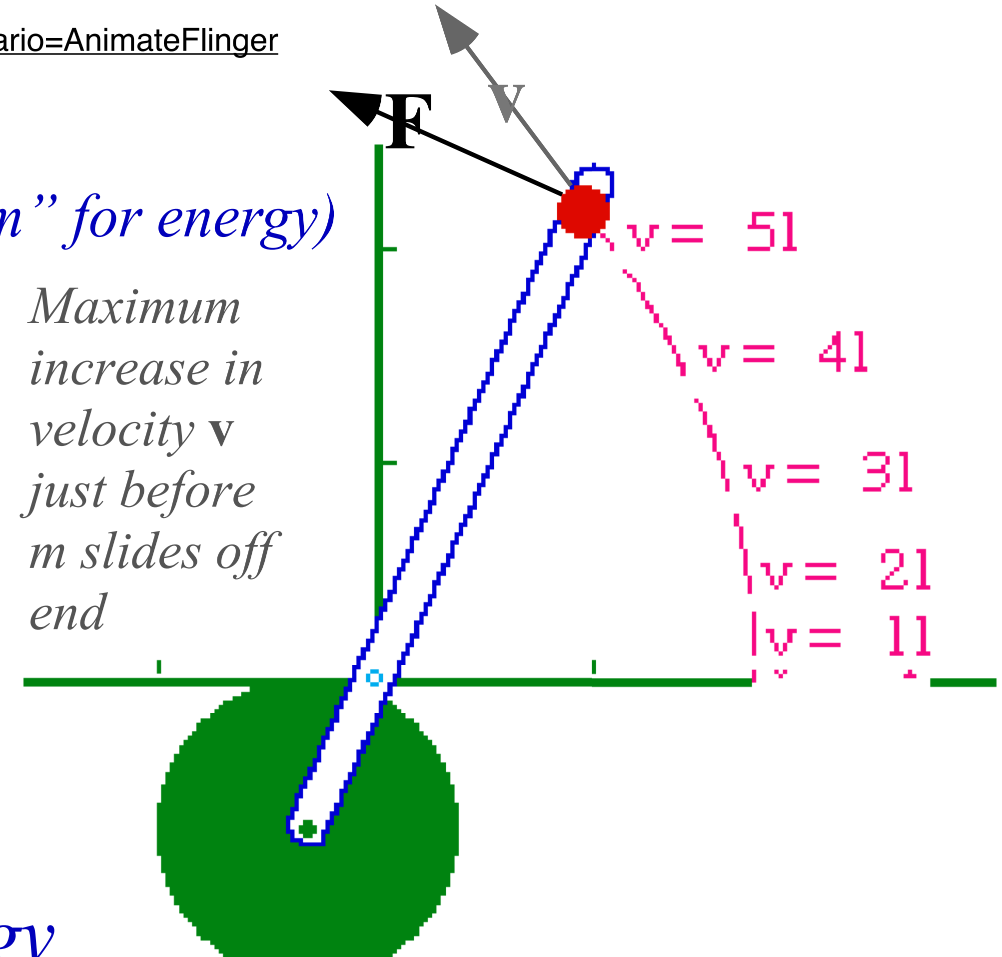
Not much increase in velocity **v**



Later on

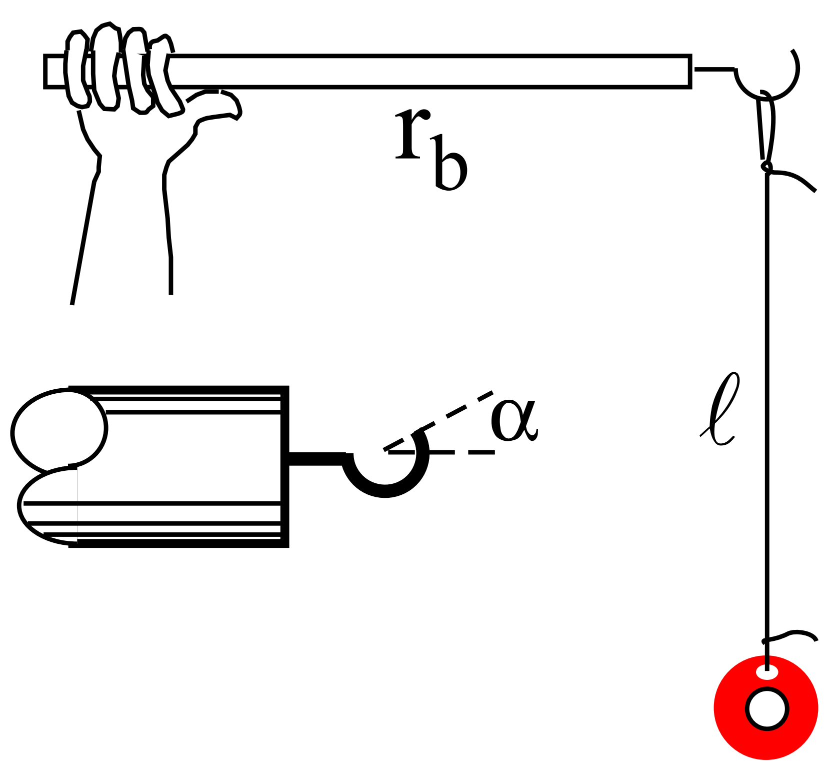
(Last-minute "cram" for energy)

Maximum increase in velocity **v** *just before m slides off end*



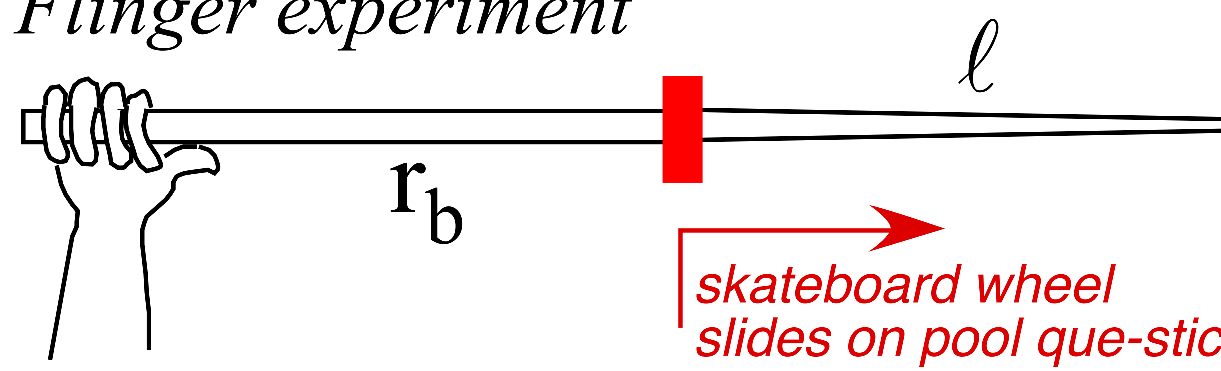
Anti-analogy can be useful pedagogy

Trebuchet-like experiment



skateboard wheel swings

Flinger experiment

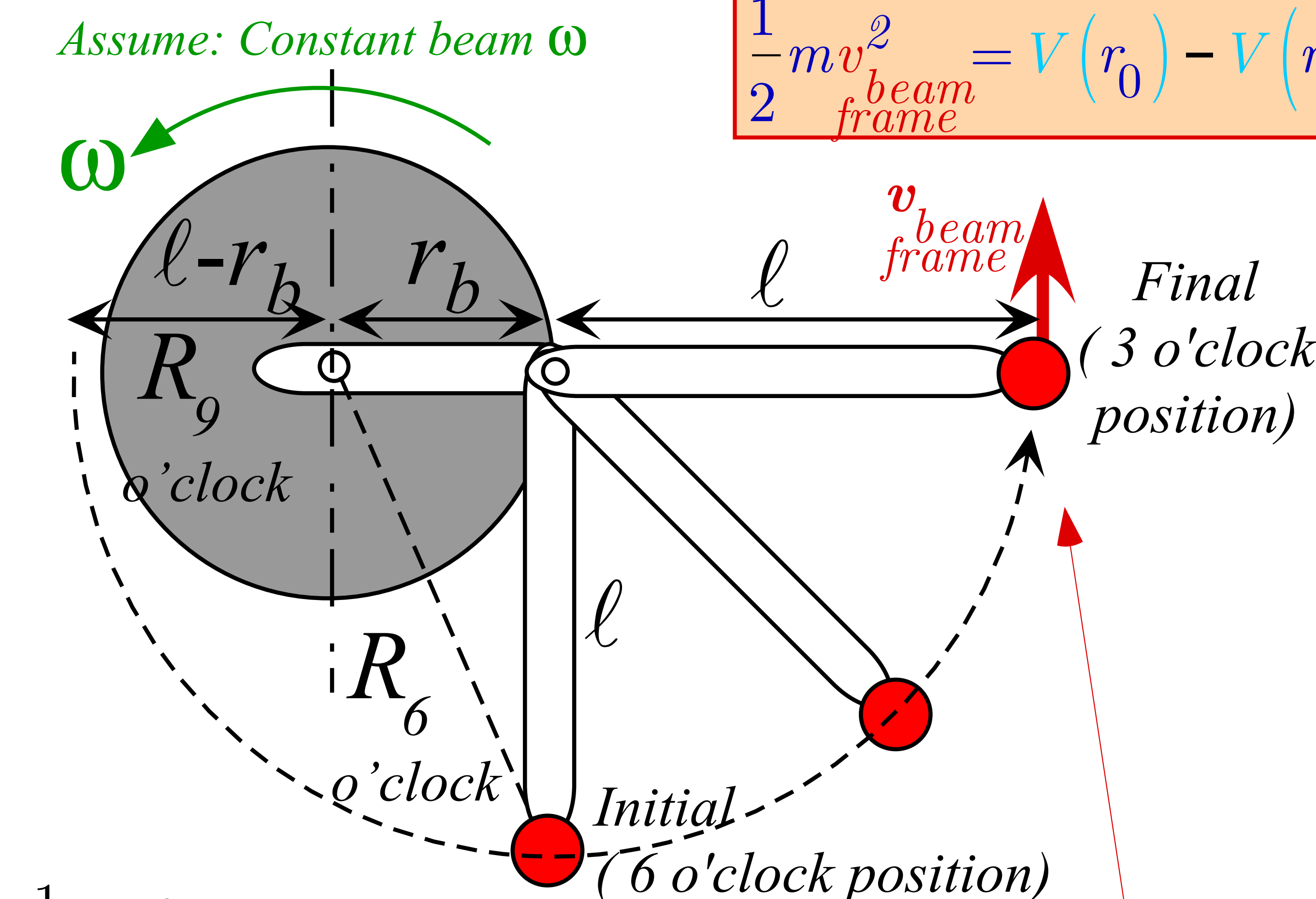


Generic URL - http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

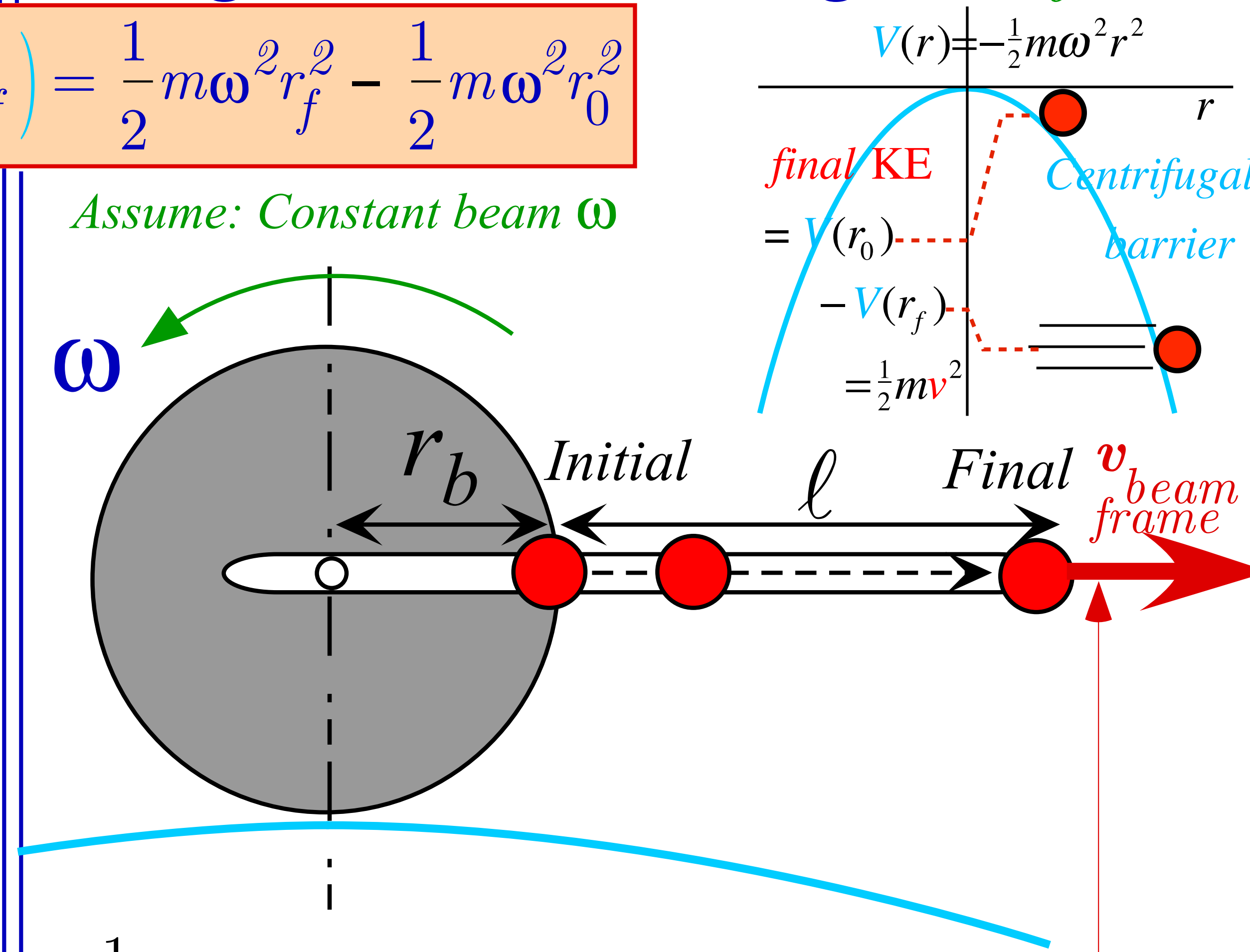
Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(trebuchet) = \textit{Final Trebuchet } KE_{\substack{beam\\frame}}$$

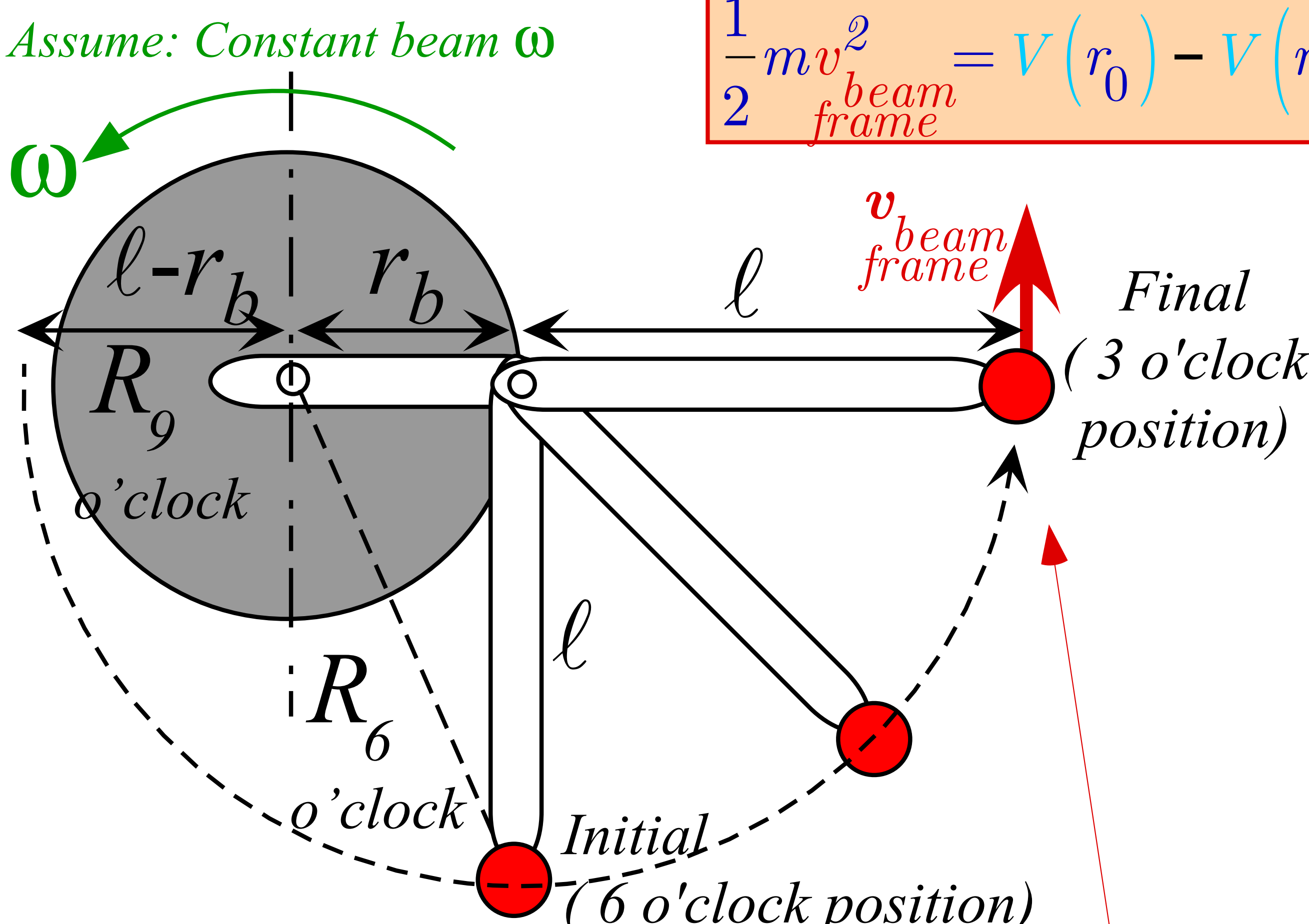


$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(flinger) = \textit{Final Flinger } KE_{\substack{beam\\frame}}$$

Trebuchet model in rotating beam frame

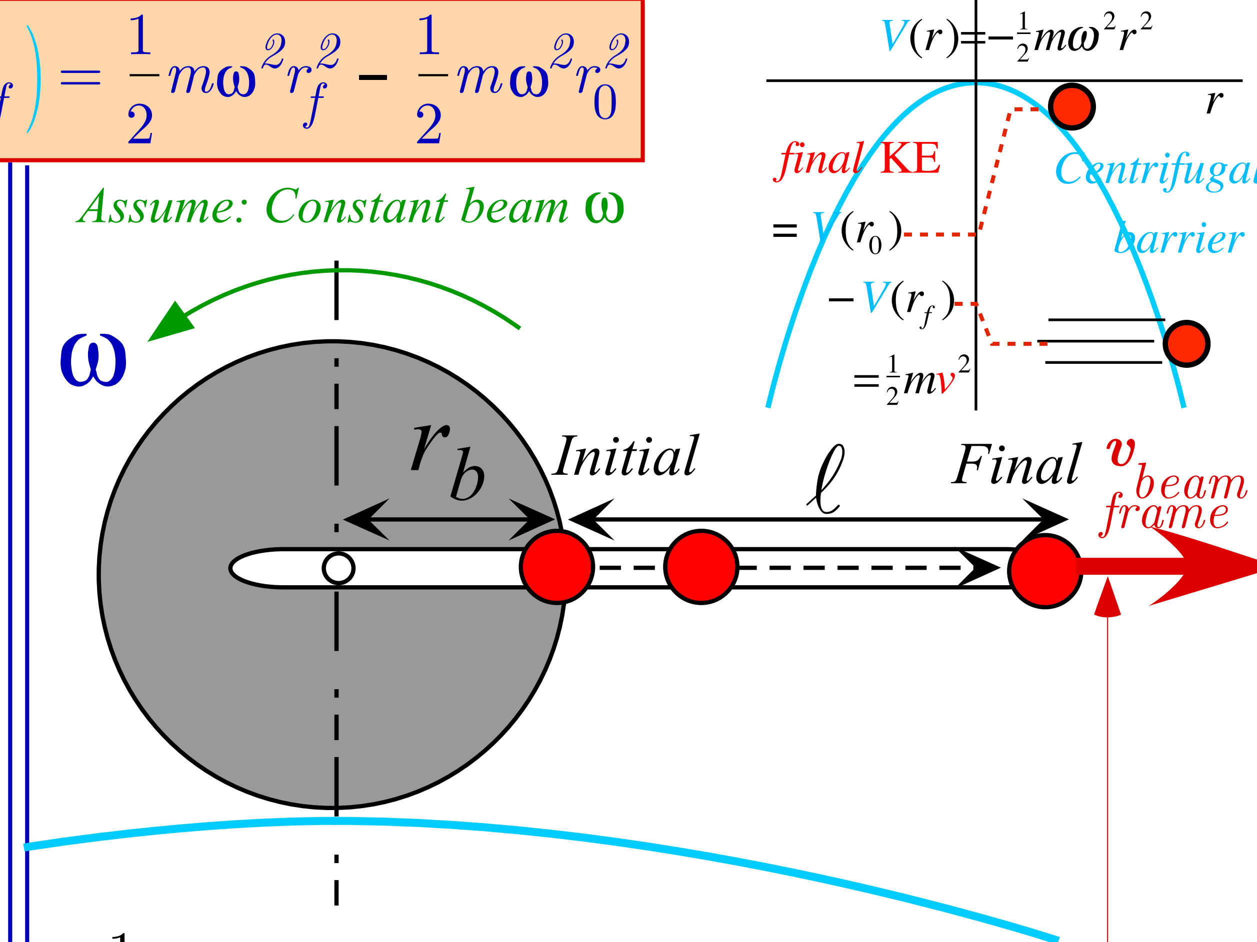
Flinger model in rotating beam frame

$$\boxed{\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2}$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\frac{1}{2}m\omega^2\underbrace{\left(r_b+\ell\right)^2}_{Final} - \frac{1}{2}m\omega^2\underbrace{\left(r_b^2+\ell^2\right)}_{Initial} = \frac{1}{2}m\omega^2\left(2r_b\,\ell\right)$$

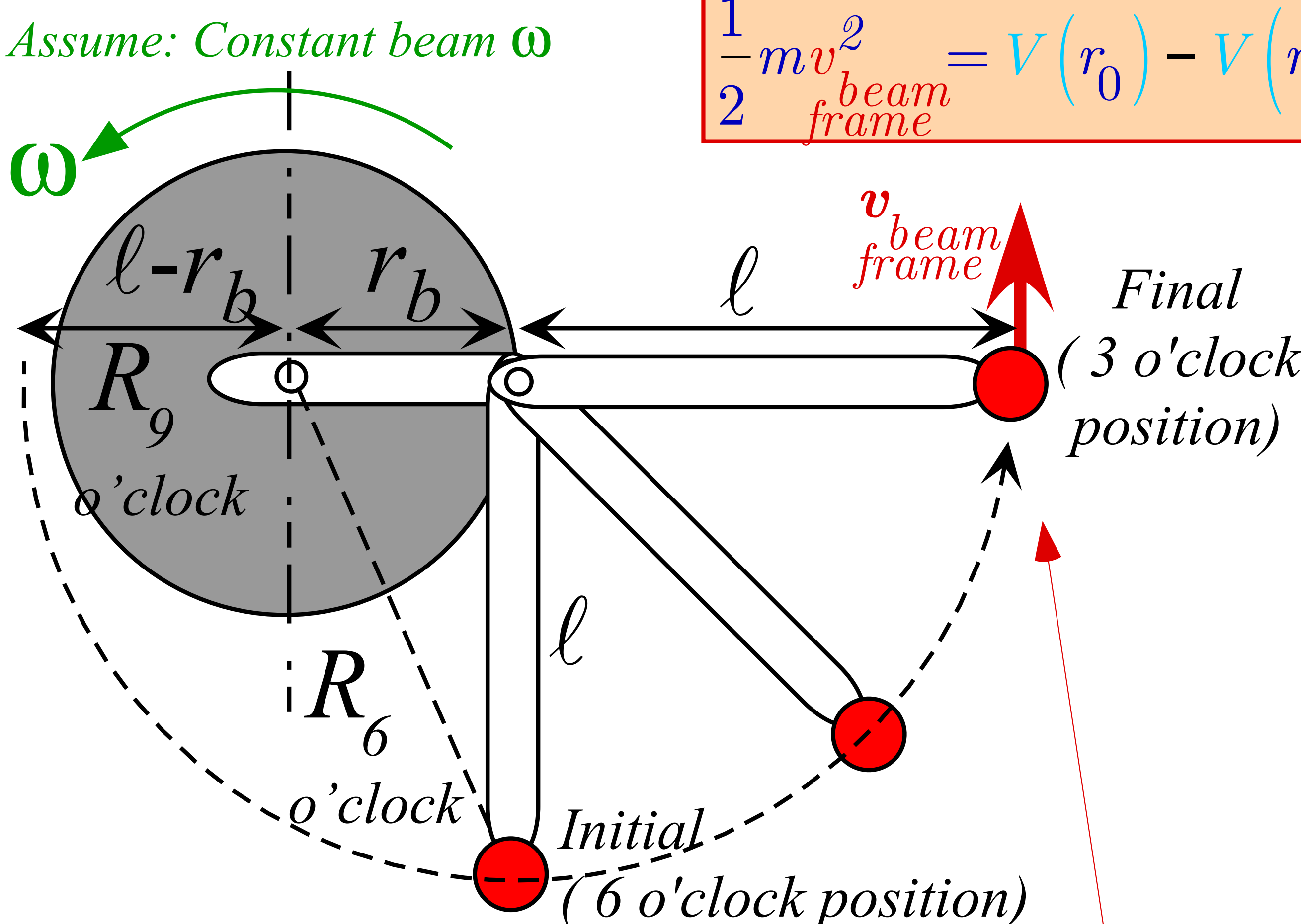


$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\frac{1}{2}m\omega^2\underbrace{\left(r_b+\ell\right)^2}_{Final} - \frac{1}{2}m\omega^2\underbrace{r_b^2}_{Initial} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

Trebuchet model in rotating beam frame | *Flinger model in rotating beam frame*

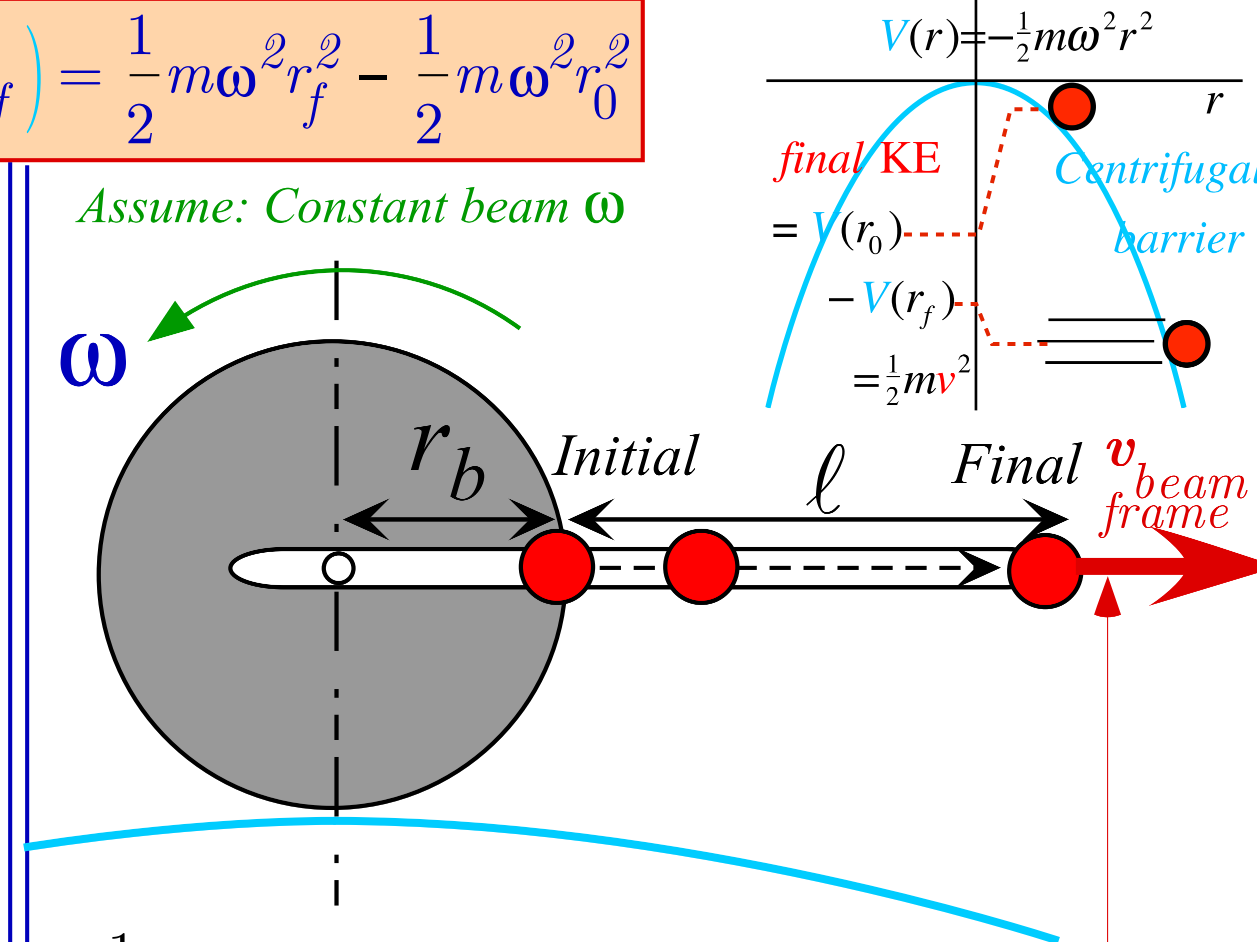
$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{\substack{Final\\3\ o'clock}} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)}_{\substack{Initial\\6\ o'clock}} = \frac{1}{2}m\omega^2\left(2r_b\,\ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2 + \ell^2$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

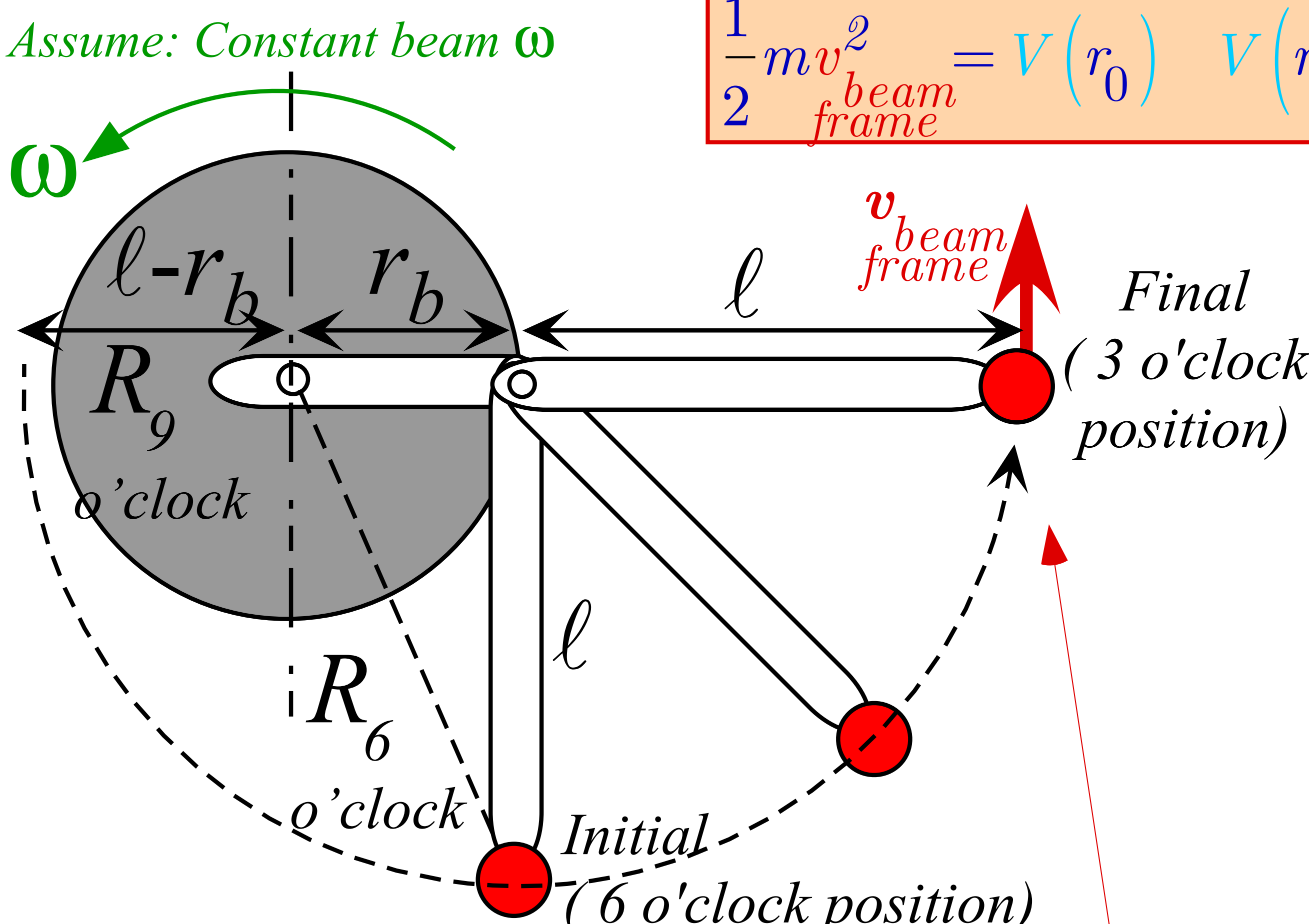
$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{\substack{Final\\3\ o'clock}} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{\substack{Initial\\3\ o'clock}} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

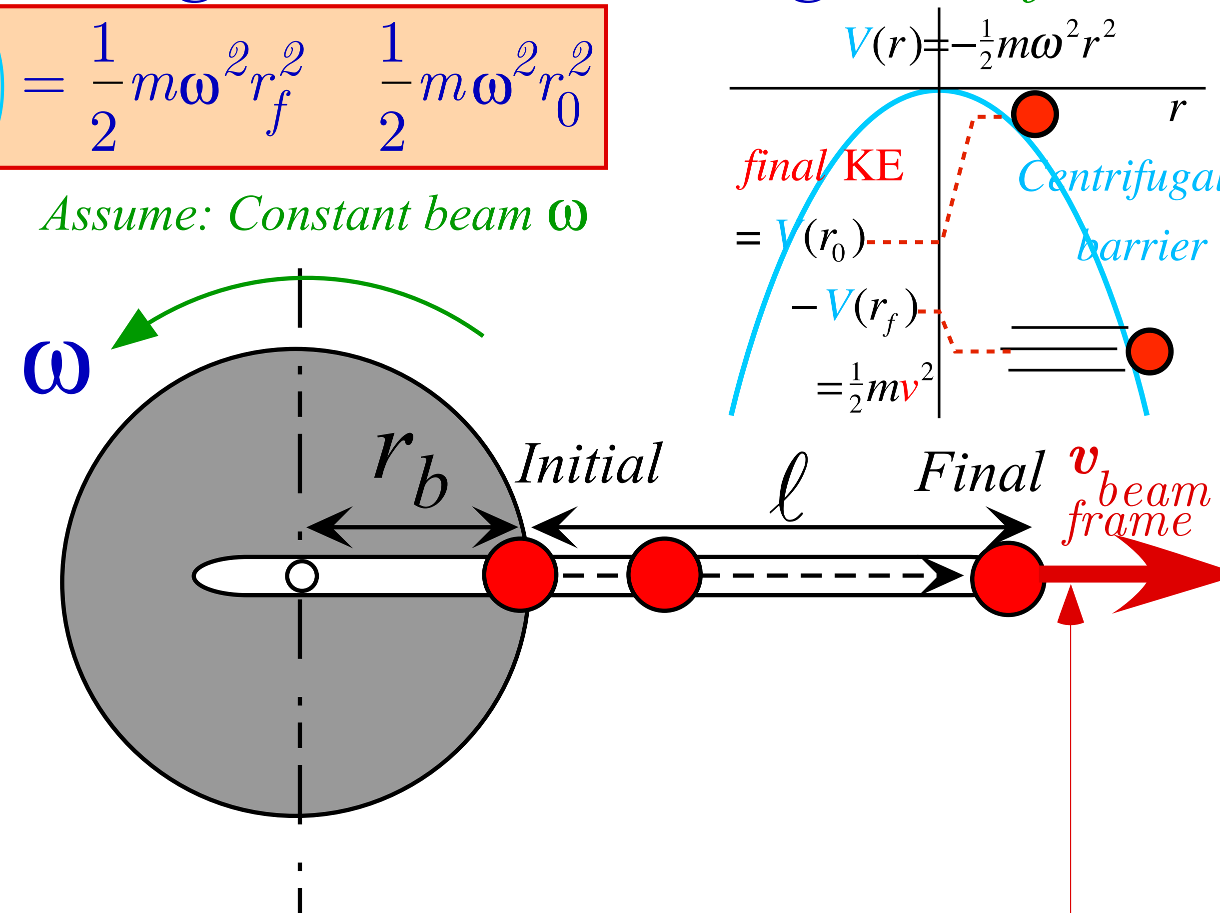
Flinger KE is $\frac{m\omega^2}{2}\ell^2$ *more than 6 o'clock trebuchet*
but misdirected

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) \quad V(r_f) = \frac{1}{2}m\omega^2 r_f^2 \quad \frac{1}{2}m\omega^2 r_0^2$$





$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2(r_b+\ell)^2}_{\substack{\text{Final}\\ \text{3 o'clock}}} - \underbrace{\frac{1}{2}m\omega^2(r_b^2+\ell^2)}_{\substack{\text{Initial}\\ \text{6 o'clock}}} = \frac{1}{2}m\omega^2(2r_b\ell)$$

$$R^2_{\substack{6\\ o'clock}} = r_b^2+\ell^2$$

$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2(r_b+\ell)^2}_{\substack{\text{Final}\\ \text{3 o'clock}}} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{\substack{\text{Initial}\\ \text{3 o'clock}}} = \frac{1}{2}m\omega^2\ell(2r_b+\ell)$$

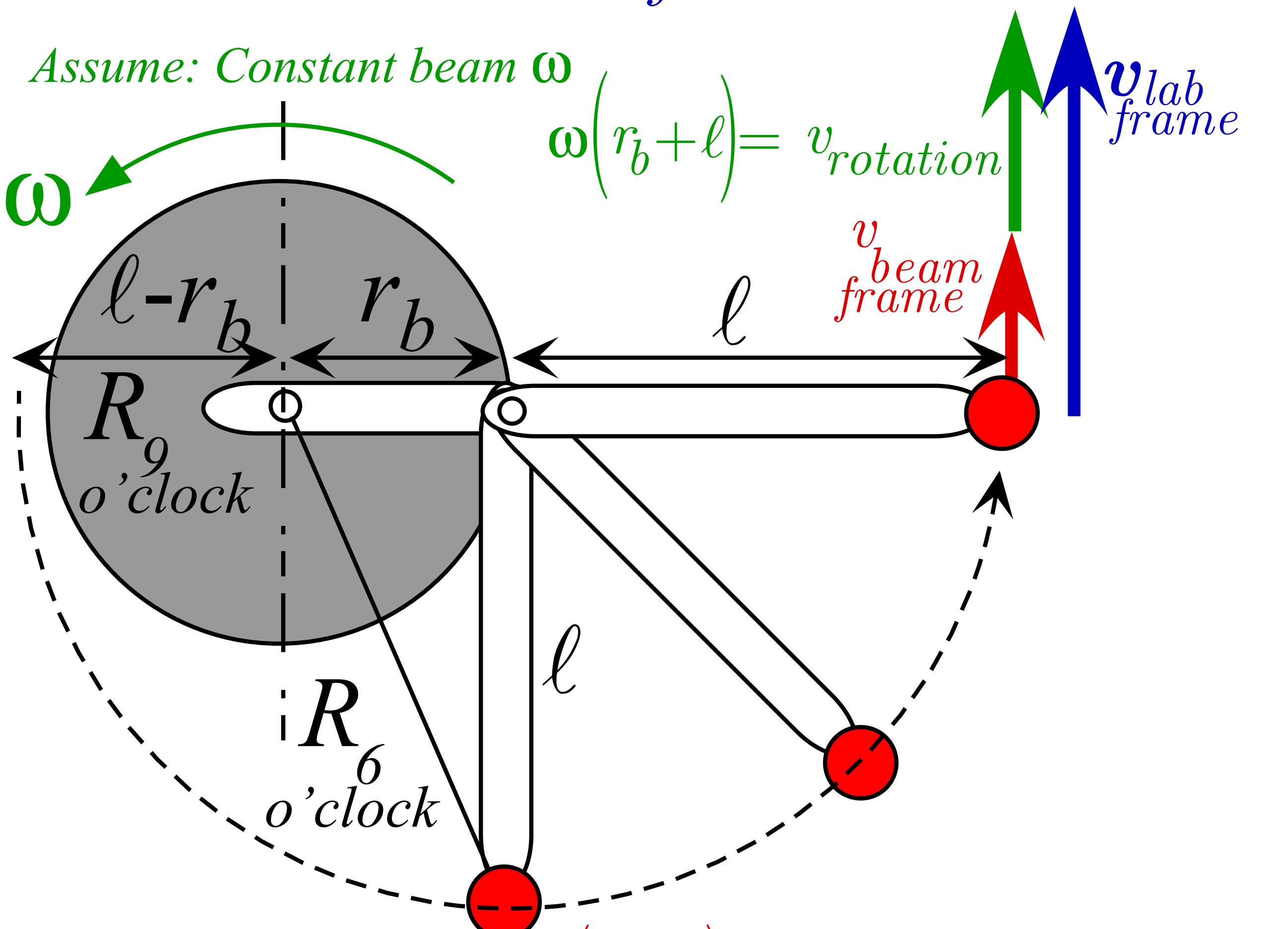
Flinger KE is $\frac{m\omega^2}{2}\ell^2$ *more than 6 o'clock trebuchet but <u>misdirected</u>*

$$\underbrace{}_{\substack{\text{Initial}\\ \text{9 o'clock}}} = \frac{1}{2}m\omega^2(4r_b\ell)$$

$$R^2_{\substack{9\\ o'clock}} = r_b^2+\ell^2-2r_b\ell$$

Flinger KE is $\frac{m\omega^2}{2}(2r_b\ell-\ell^2)$ *<u>less</u> than 9 o'clock trebuchet <u>and</u> <u>misdirected</u>*

Trebuchet model in lab frame



$$v^2_{\substack{beam\\frame}}(\text{trebuchet}) = \begin{cases} \omega^2\left(2r_b\,\ell\right) & \textit{half-cocked 6 o'clock} \\ \omega^2\left(4r_b\,\ell\right) & \textit{fully-cocked 9 o'clock} \end{cases}$$

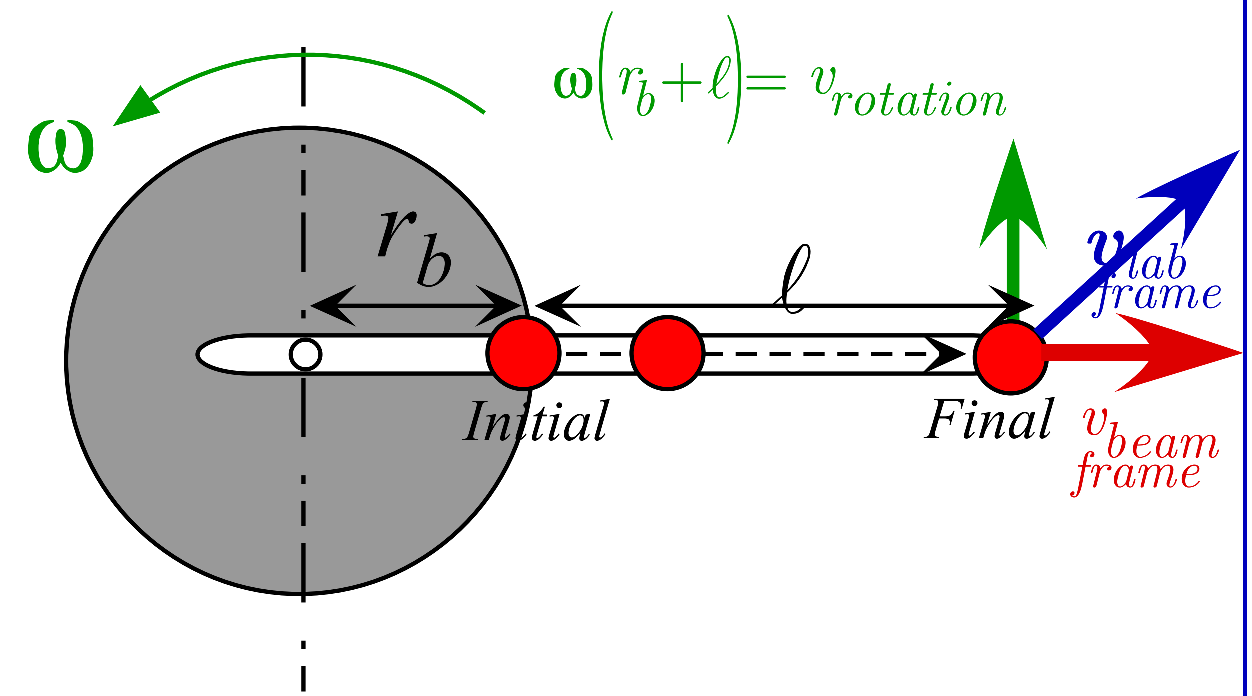
$$v_{lab\ \ frame}\left(\text{trebuchet}\right) =$$

$$\begin{cases} \omega\left(r_b+\ell+\sqrt{2\ell r_b}\right) & \textit{half-cocked 6 o'clock} \\ \omega\left(r_b+\ell+2\sqrt{\ell r_b}\right) & \textit{fully-cocked 9 o'clock} \end{cases}$$

$$=\begin{cases} 5.00\omega \\ 5.82\omega \end{cases} \qquad =\begin{cases} 5.16\omega \\ 6.00\omega \end{cases} \qquad =\begin{cases} 5.00\omega \\ 5.82\omega \end{cases}$$

$$\left(r_b=2\,,\ell=1\right),\ \left(r_b=1.5\,,\ell=1.5\right),\left(r_b=1\,,\ell=2\right)$$

Flinger model in lab frame



$$v^2_{\substack{beam\\frame}}(\text{flinger}) = \omega^2\ell\left(2r_b+\ell\right)$$

$$v_{lab\ \ frame}\left(\text{flinger}\right) =$$

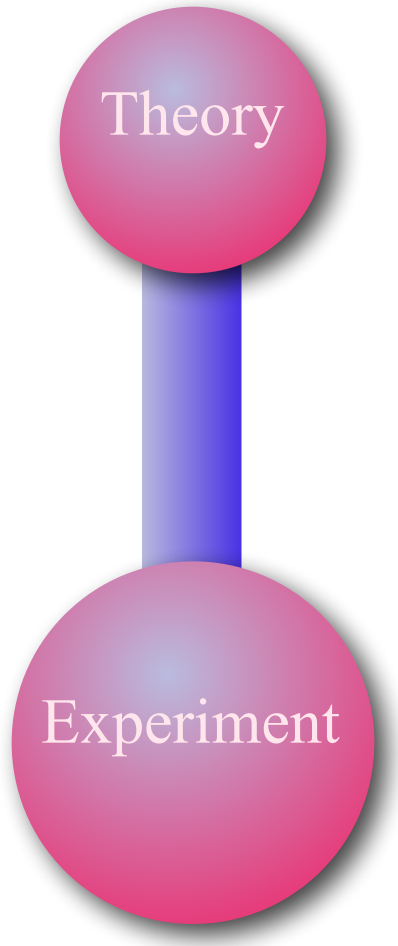
$$=\omega\sqrt{\left(r_b+\ell\right)^2+\ell\left(2r_b+\ell\right)}=\omega\sqrt{2\left(r_b+\ell\right)^2-r_b{}^2}$$

(compare)

$$=3.74\omega \qquad =3.96\omega \qquad =4.12\omega$$

$$\left(r_b=2\,,\ell=1\right),\ \left(r_b=1.5\,,\ell=1.5\right),\left(r_b=1\,,\ell=2\right)$$

Physics used to be pretty much bi-polar...



Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

Quick'n dirty
Newton F=Ma Equations
Cartesian coordinates

- French Approach

Tres elegant
Lagrange Equations
in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

- German Approach

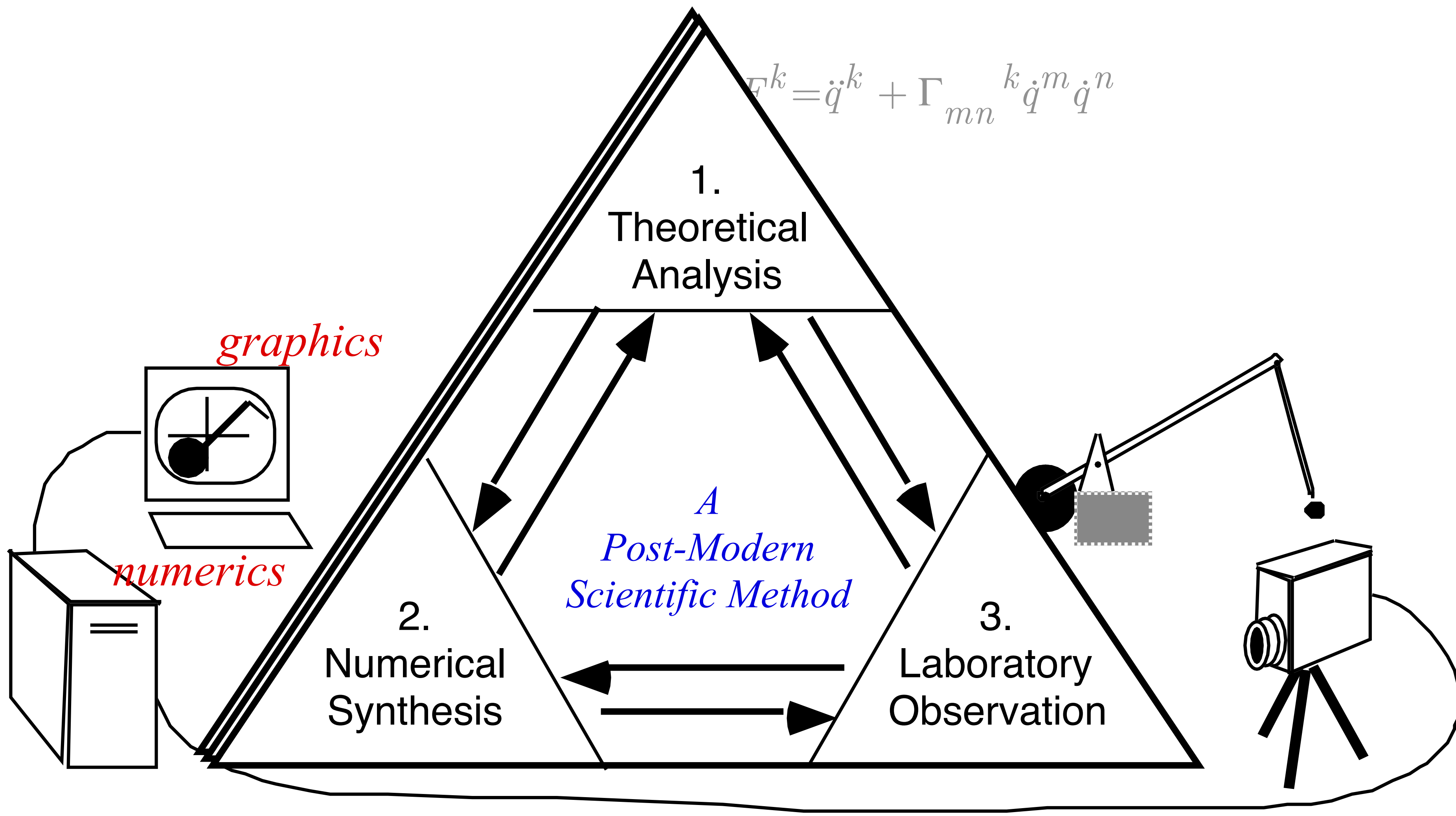
Pride and Precision
Riemann Christoffel Equations
in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

- Anglo-Irish Appproach

Powerfully Creative
Hamilton's Equations
Phase Space $\dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$

- Unified Approach



All approaches have one thing in common:
The *Art of Approximation*
Physics lives and dies by the art of
approximate models and analogs.

$dS = Ldt = p_\mu dq^\mu - Hdt$

Hamilton-Jacobi-Poincare: $p_\mu = \frac{\partial S}{\partial q^\mu}, \ -H = \frac{\partial S}{\partial t}$

Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

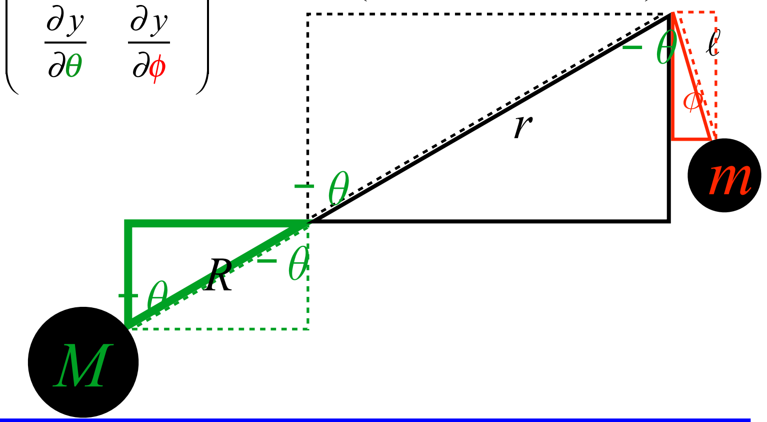
Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\ dX \quad = \quad M\ddot{X}\,dX \quad = \quad F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi \quad = \quad M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\ dY \quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\ dx \quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\ dy \quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$



Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

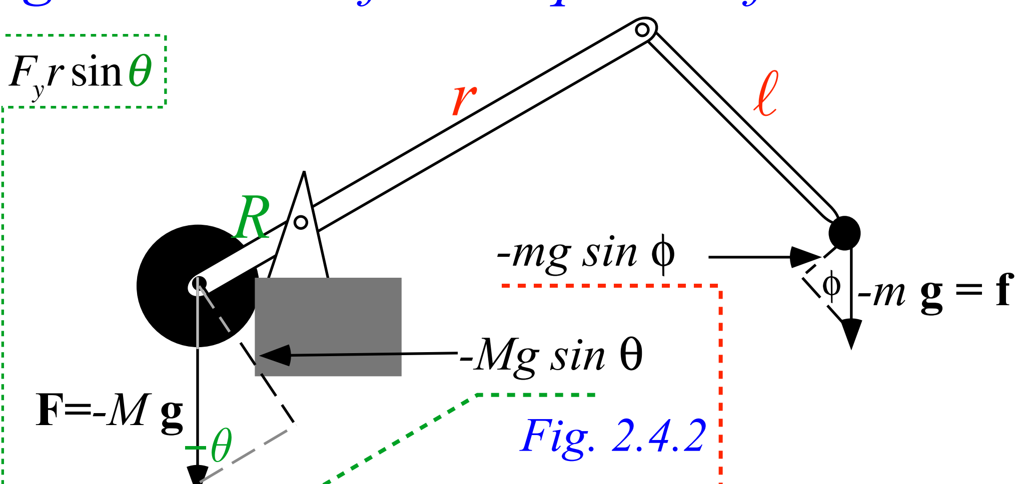


Fig. 2.4.2

Add F_θ gravity given ($F_X = 0$, $F_Y = -Mg$) ($F_x = 0$, $F_y = -mg$)

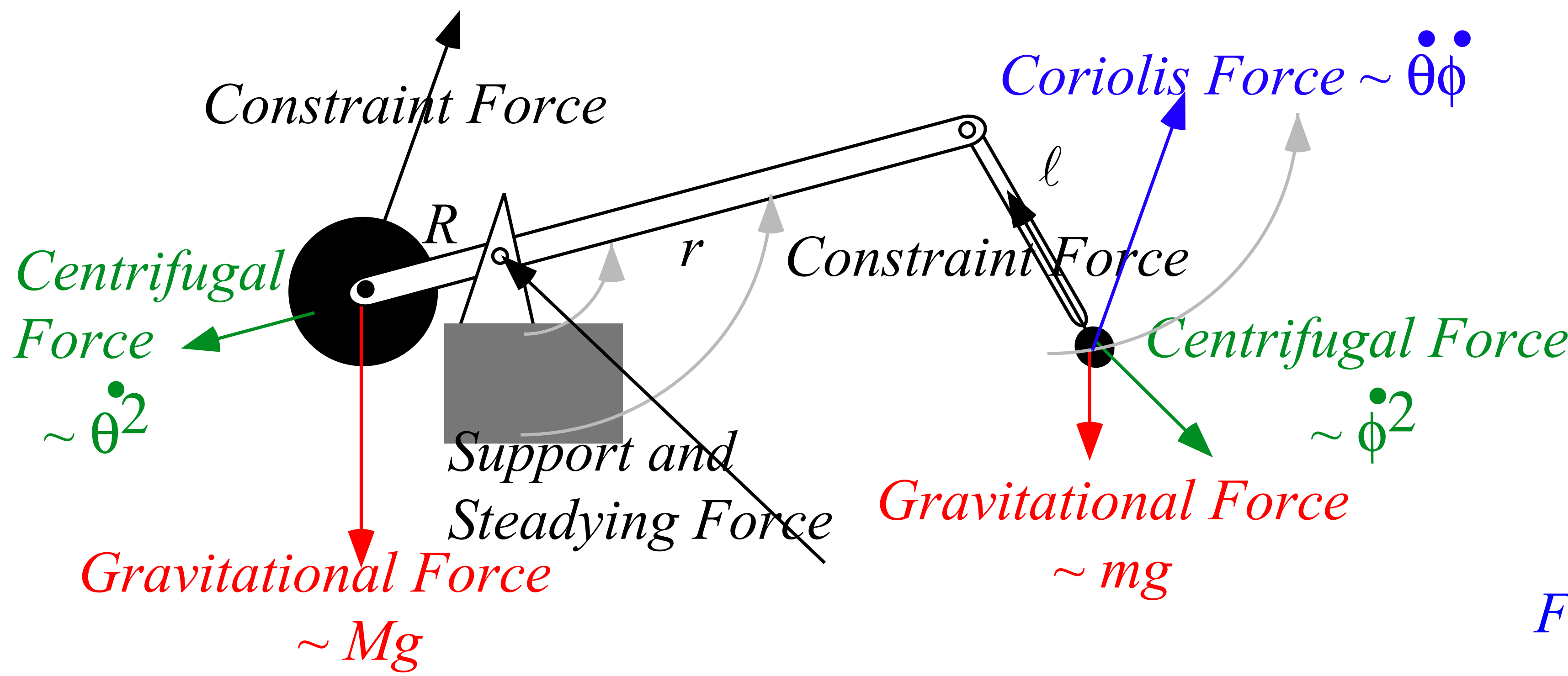
$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

Add F_ϕ gravity given ($F_X = 0$, $F_Y = -Mg$) ($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... ... and a torque on throwing lever ℓ

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

For conservative forces

where: $F_\theta = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations

$L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.