

Lecture 15
Thur.10.12.2017

**treb-yew-shay*

GCC Lagrange and Riemann Equations for Trebuchet

(Ch. 1-5 of Unit 2 and Unit 3)

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued*

Toroidal “rolled-up” ($q^1=\theta$, $q^2=\phi$)-manifold and “Flat” ($x=\theta$, $y=\phi$)-graph

Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors: Jacobian J vs. Kjobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

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Riemann equation force analysis

2nd-guessing Riemann equation?

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Chapter 1. The Trebuchet: A dream problem for Galileo?

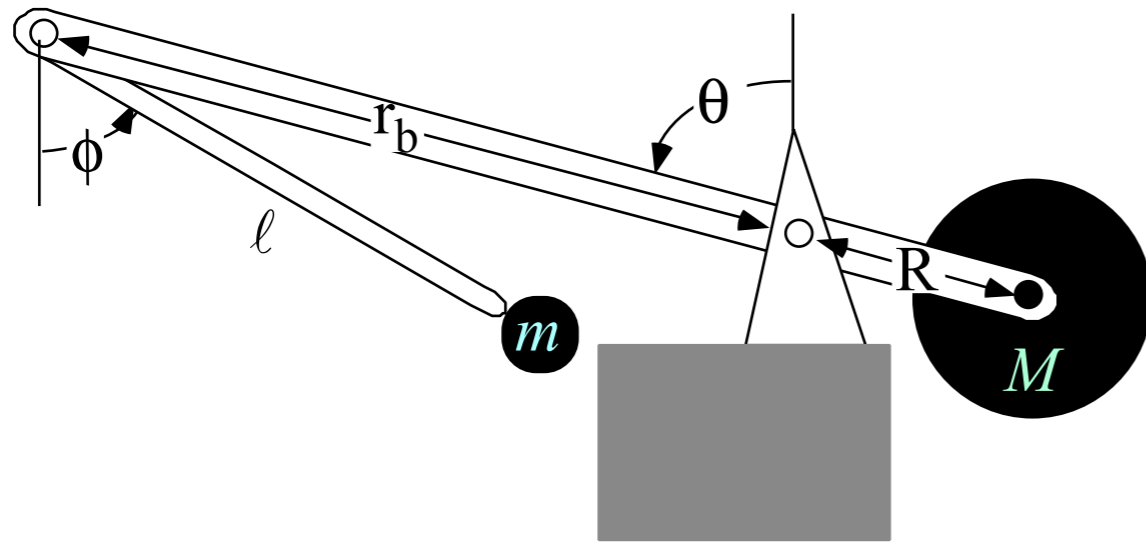
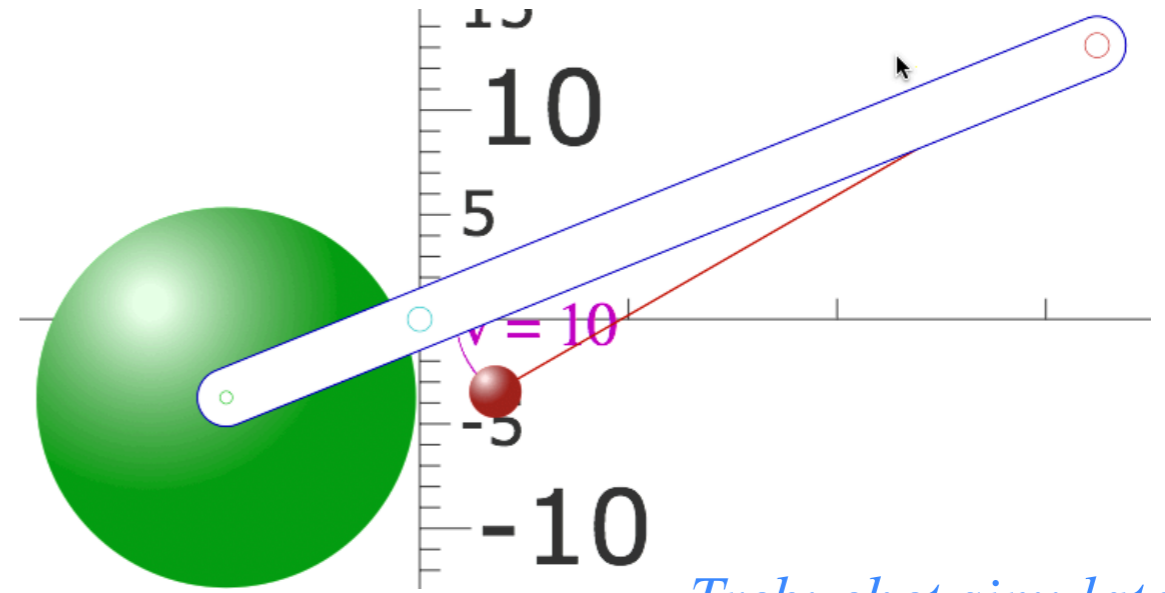


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html>

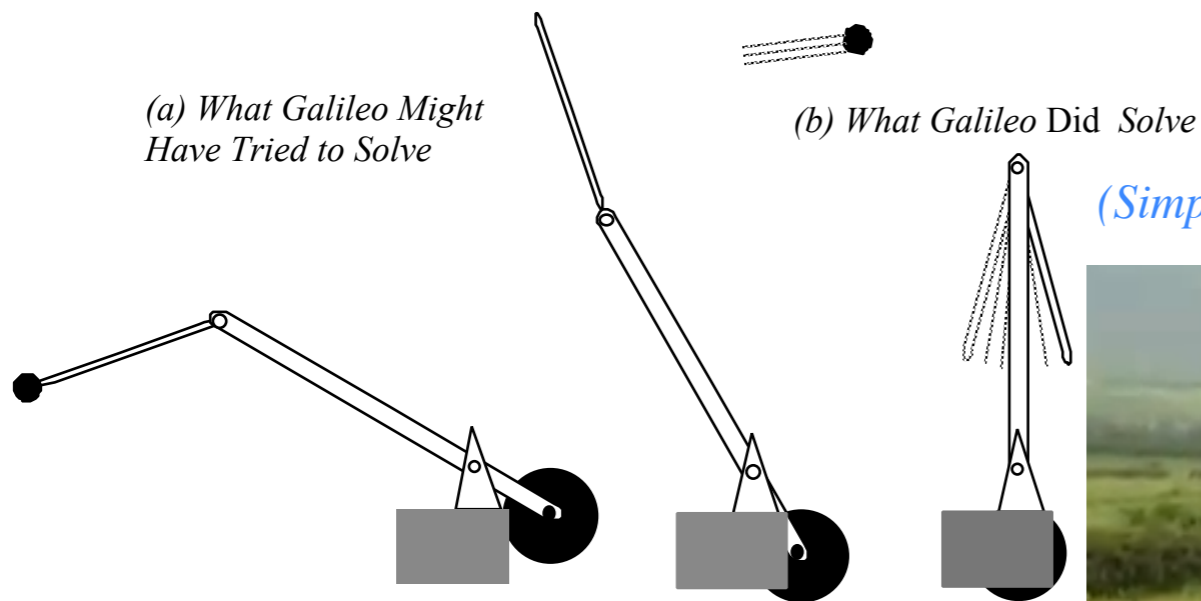



Fig. 2.1.2 Galileo's (supposed) problem



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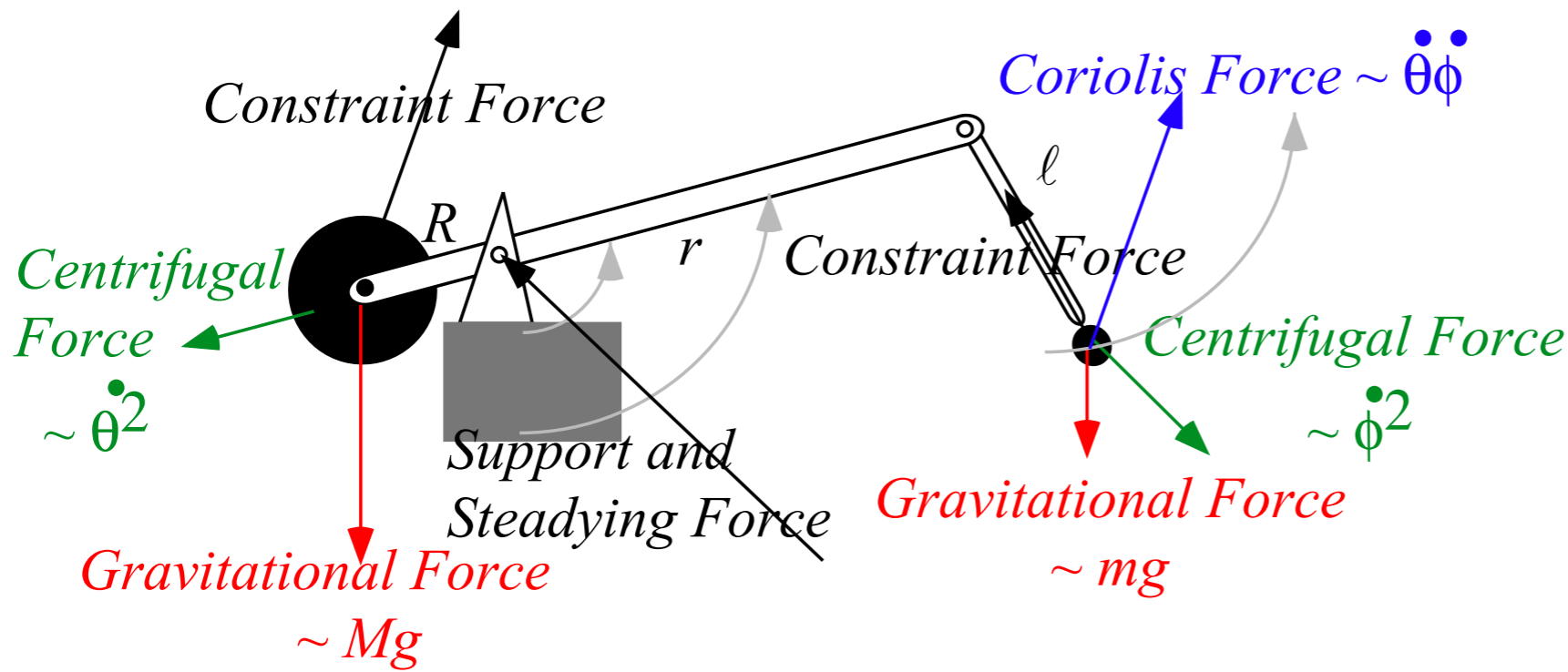
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 15 p. 69)

Riemann equation derivation for trebuchet model

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2nd-guessing Riemann equation?

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:

Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

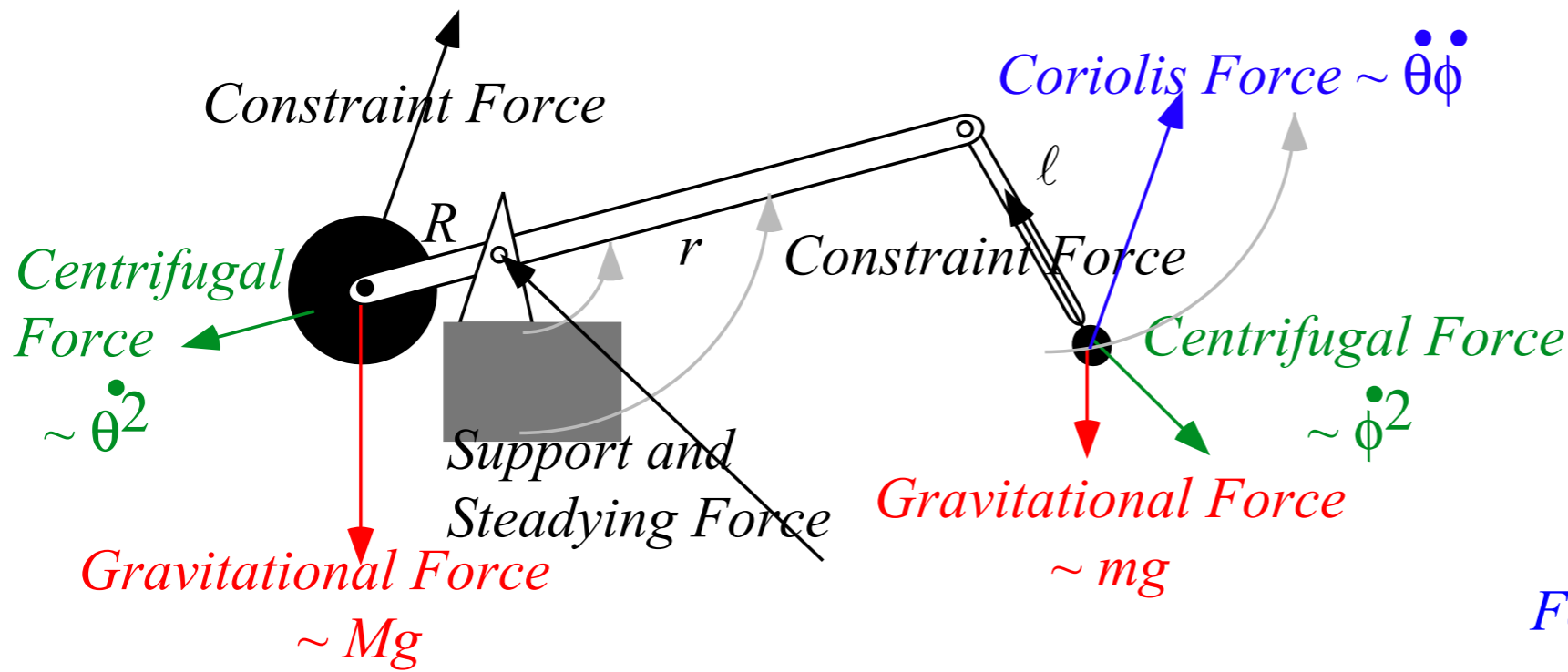
$$\dot{p}_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23, Unit 2)

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

For conservative forces

where: $F_{\theta} = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$
 $F_{\phi} = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$\dot{p}_{\theta} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_{\theta} + 0$$

$$\dot{p}_{\phi} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_{\phi} + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23, Unit 2)

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} \quad \dot{p}_{\theta} = \frac{\partial L}{\partial \theta}$$

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} \quad \dot{p}_{\phi} = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
 $L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

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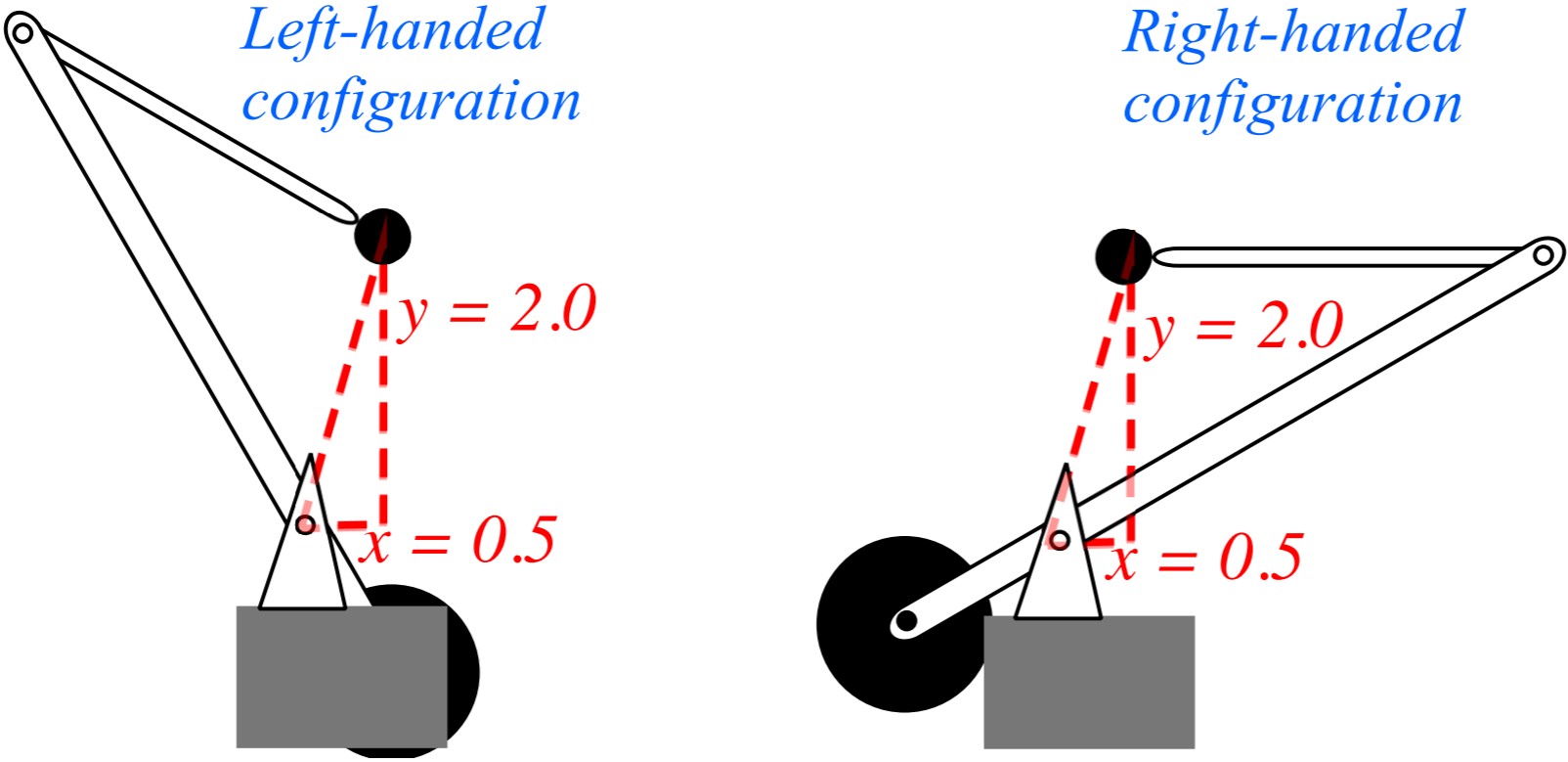


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m .

Trebuchet Cartesian projectile coordinates are double-valued... (Belong to 2 distinct manifolds)

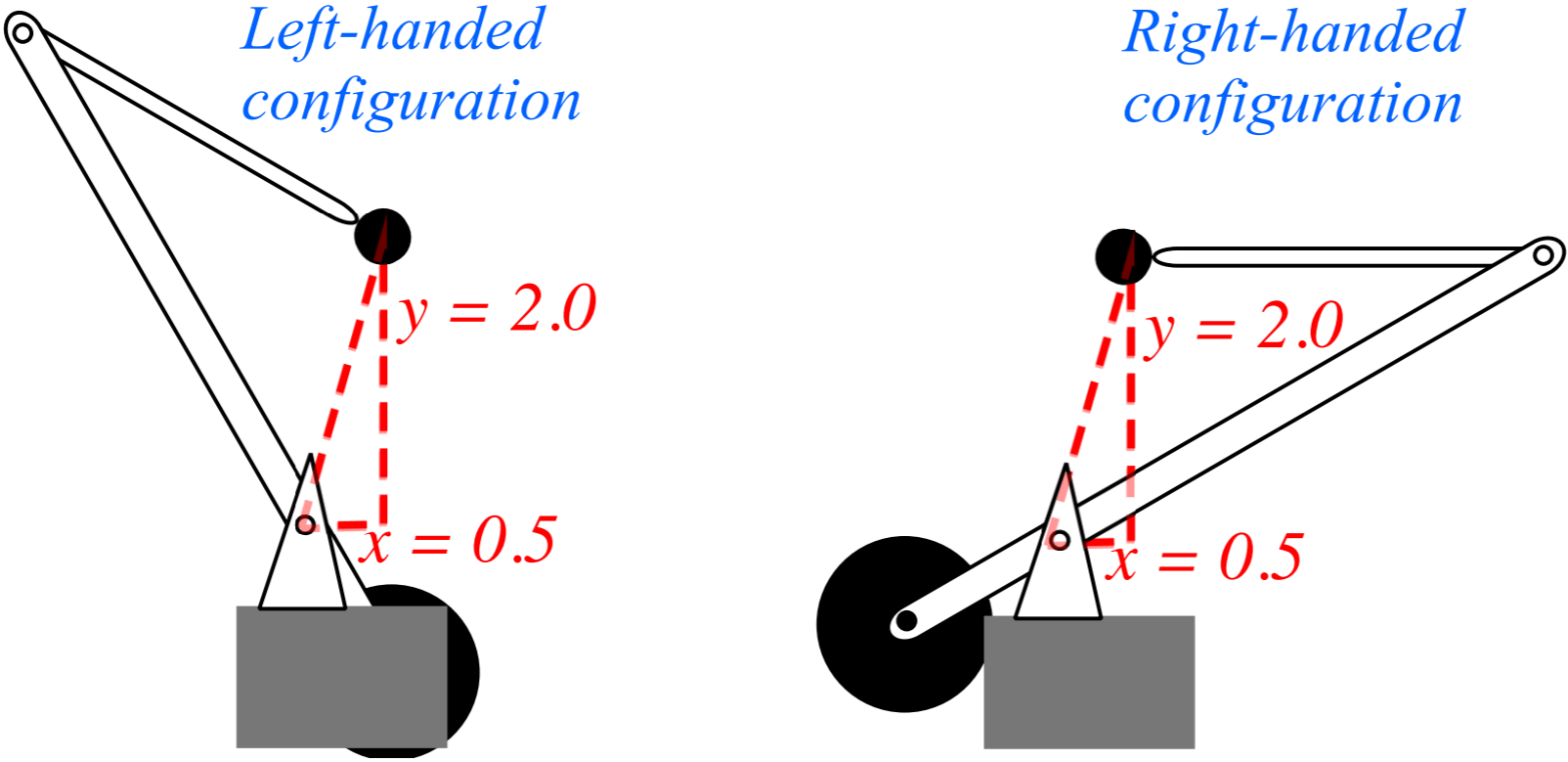


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m .

So, for example, are polar coordinates ... (for each angle there are two r -values)

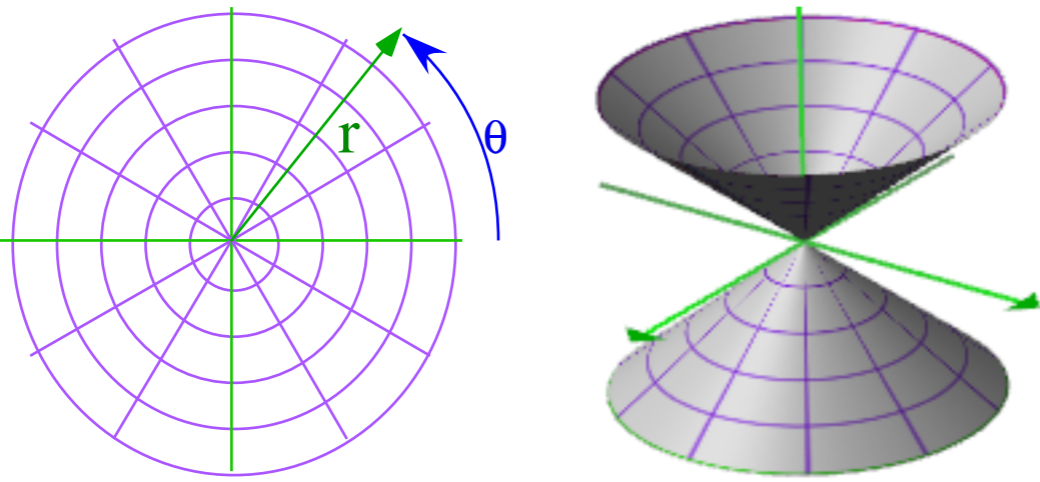



Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

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“Flat” ($q^1=\theta, q^2=\phi$)-graph of trebuchet loci compared to “rolled-up” toroidal manifold

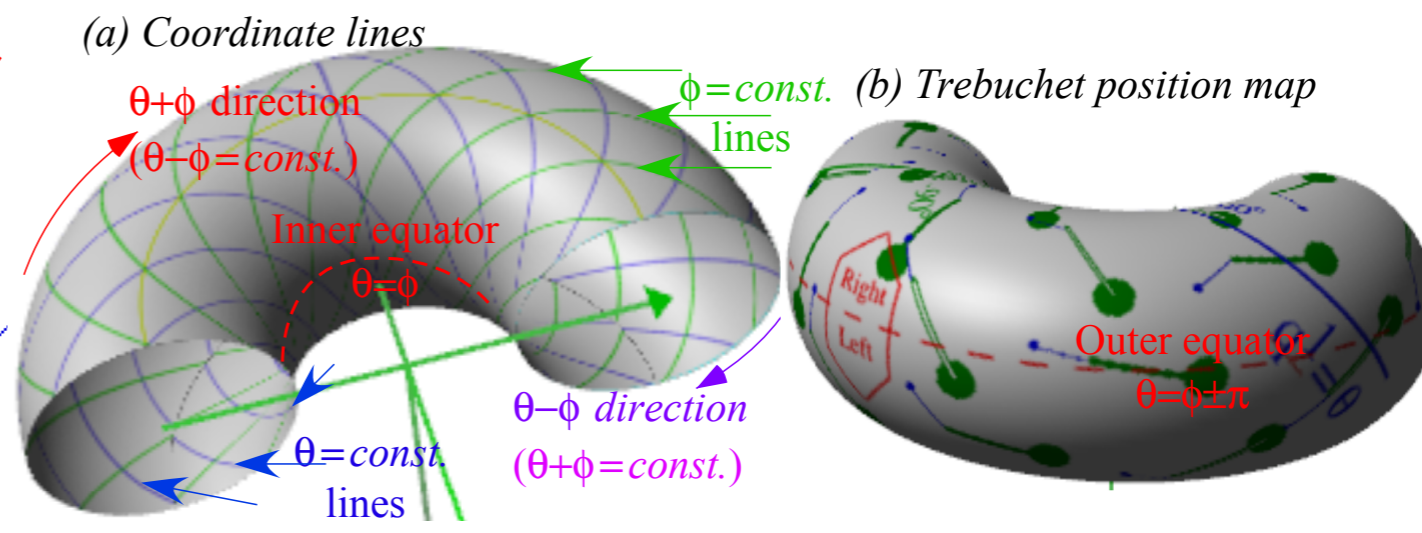
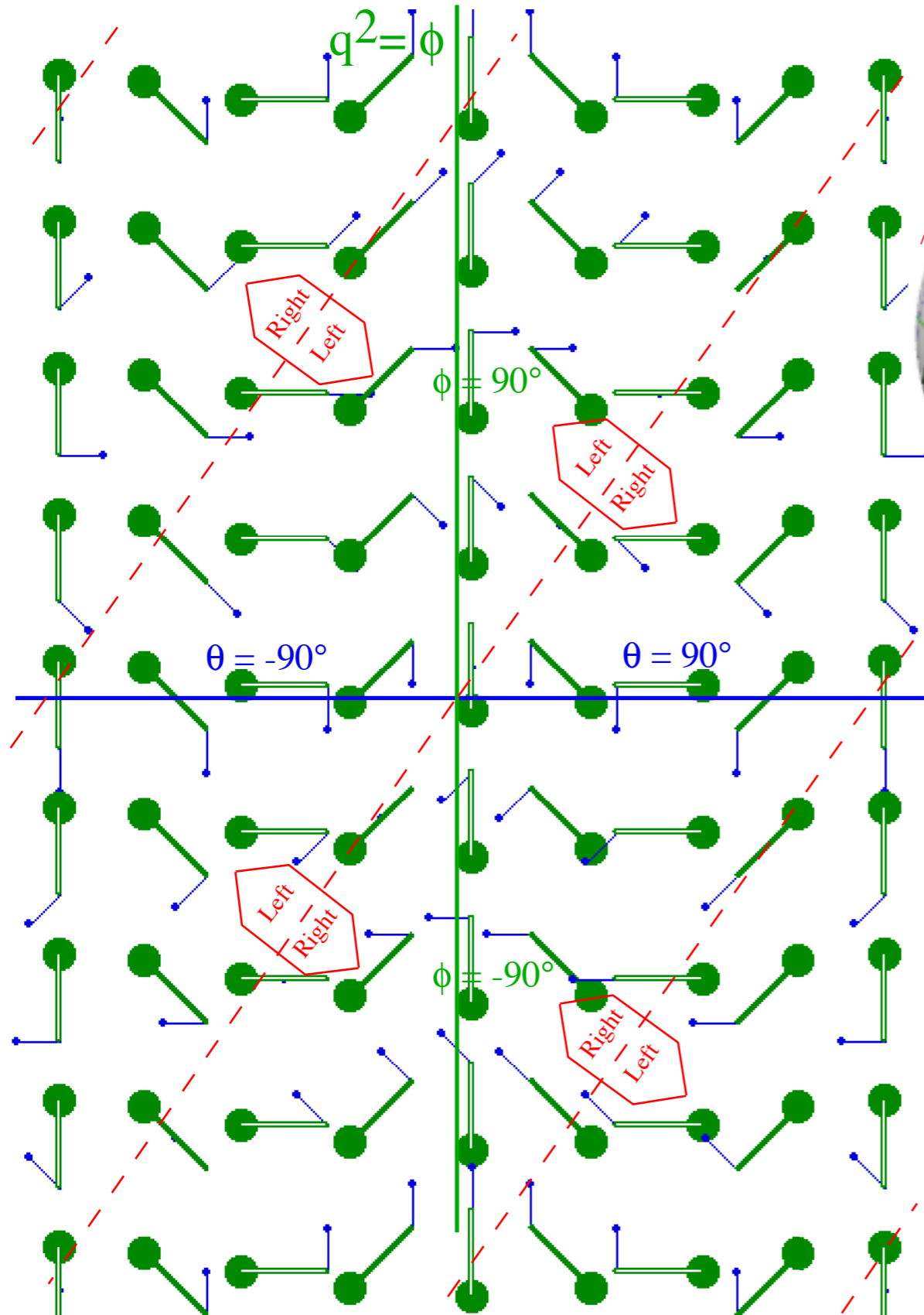


Fig. 3.1.2 Trebuchet torus.
 (a) ($q^1=\theta, q^2=\phi$) coordinate lines. (b) Trebuchet position map and equators.

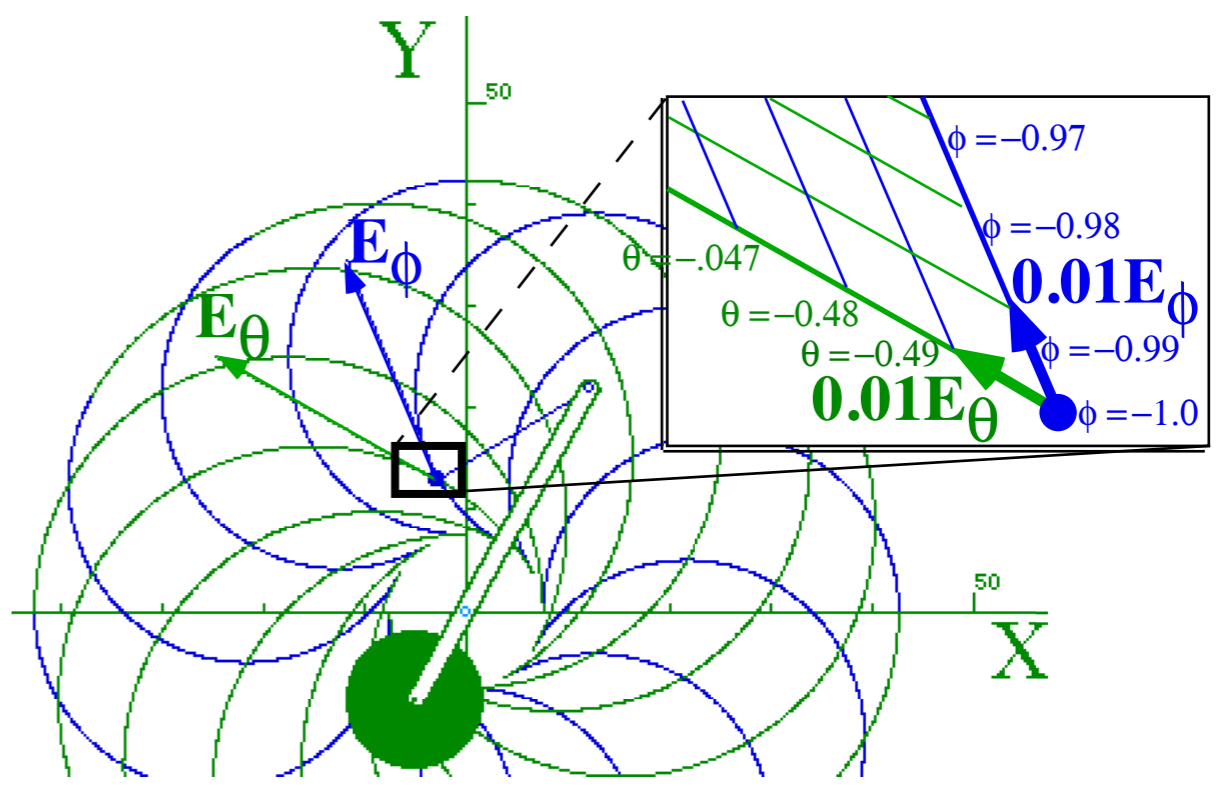
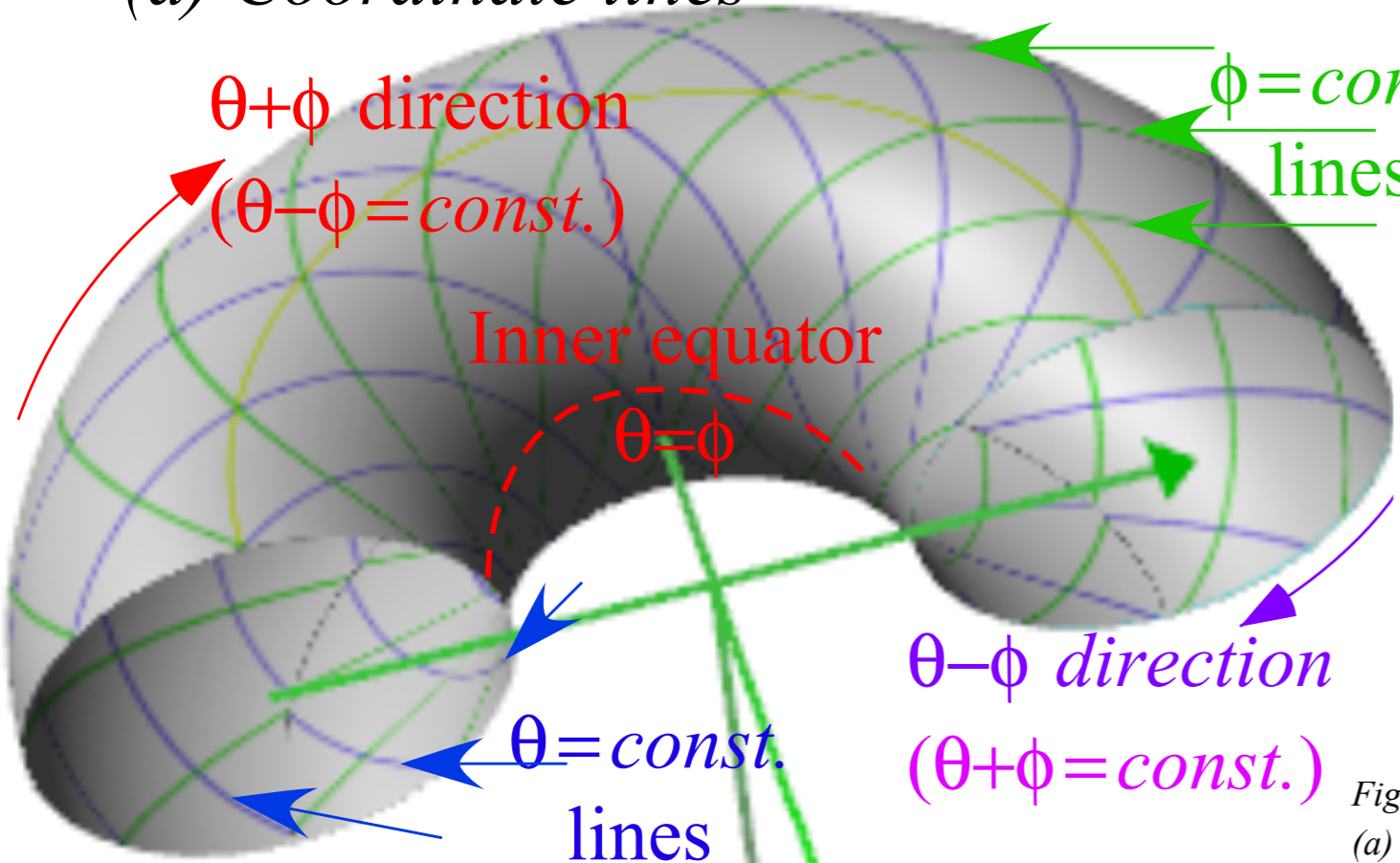


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Fig. 3.1.3 "Flattened" ($q^1=\theta, q^2=\phi$) coordinate manifold for trebuchet

Toroidal “rolled-up” ($q^1=\theta, q^2=\phi$)-manifold of trebuchet positions

(a) Coordinate lines



(b) Trebuchet position map

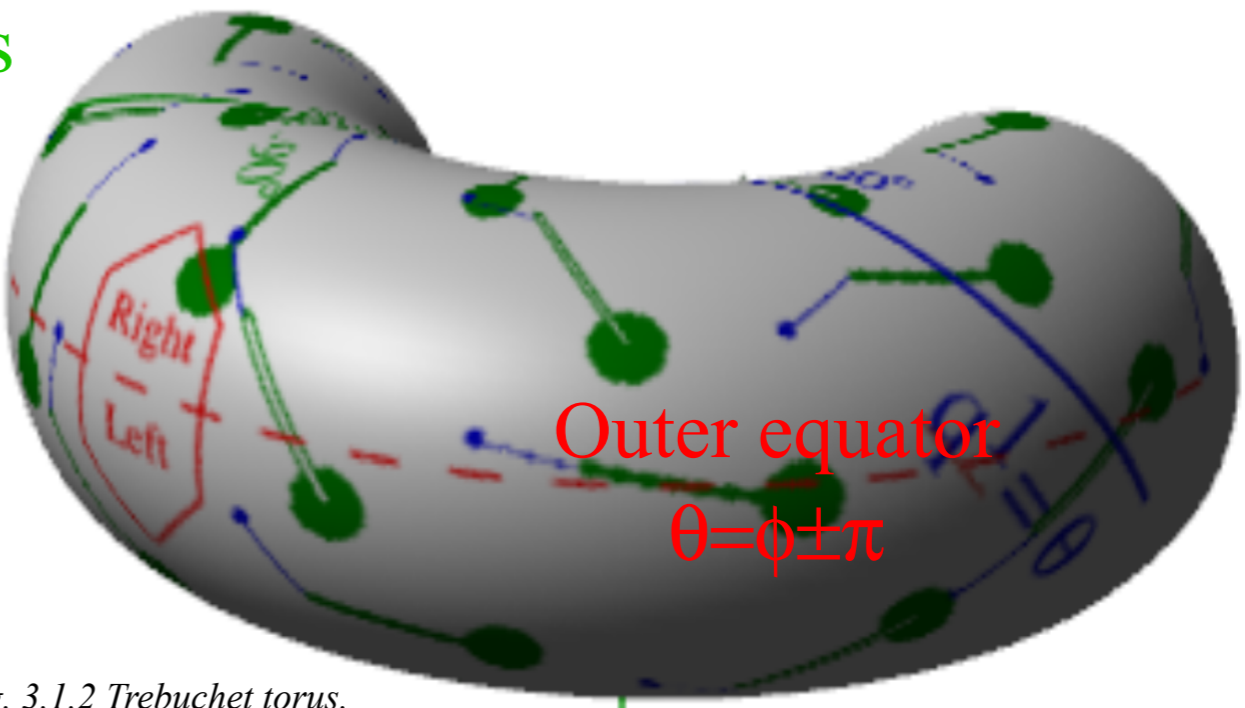


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Covariant tangent-space
GCC vectors
 $\mathbf{E}_1=\mathbf{E}_\theta$ and $\mathbf{E}_2=\mathbf{E}_\phi$

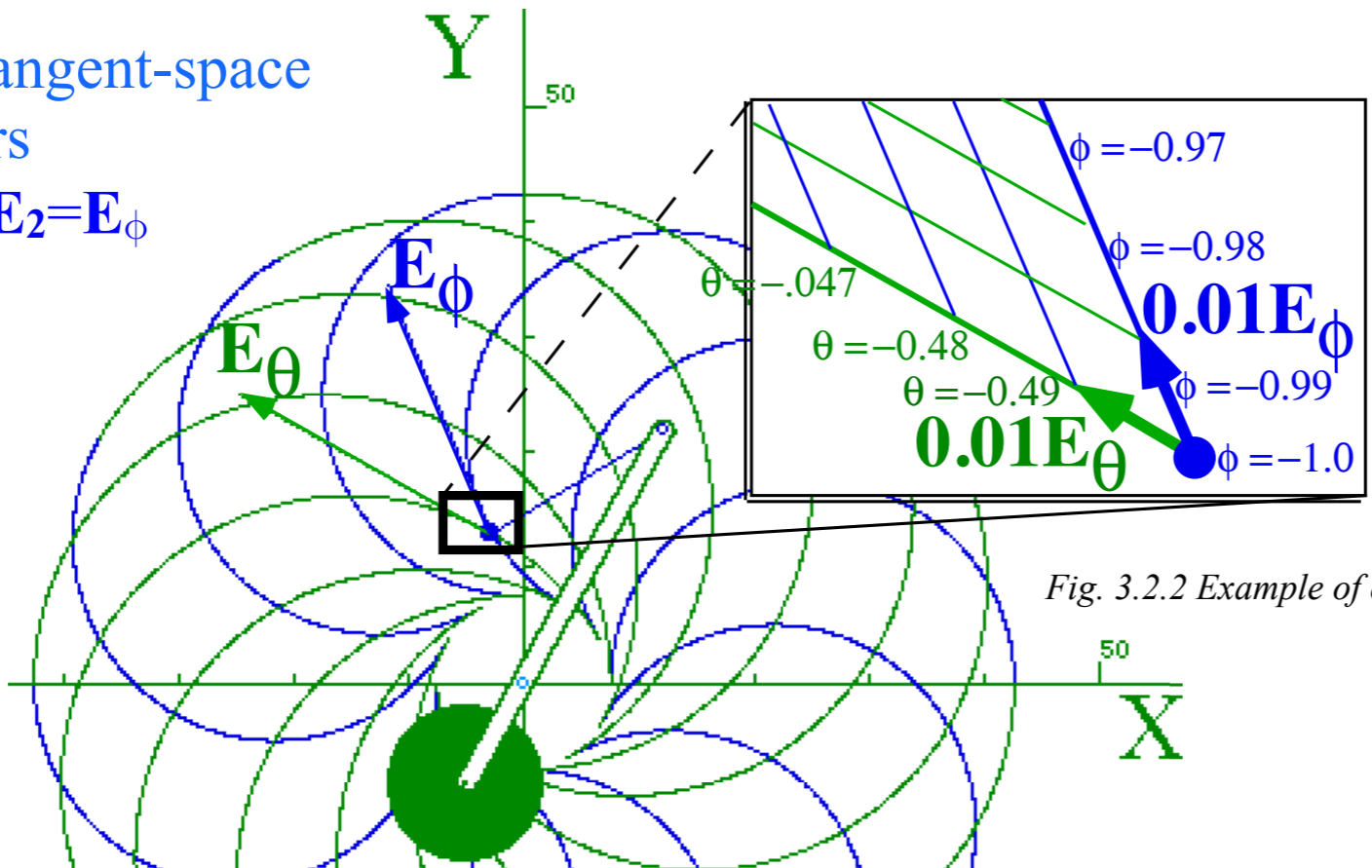


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Toroidal “rolled-up” ($q^1=\theta, q^2=\phi$)-manifold of trebuchet positions and “Flat” ($q^1=\theta, q^2=\phi$)-graph

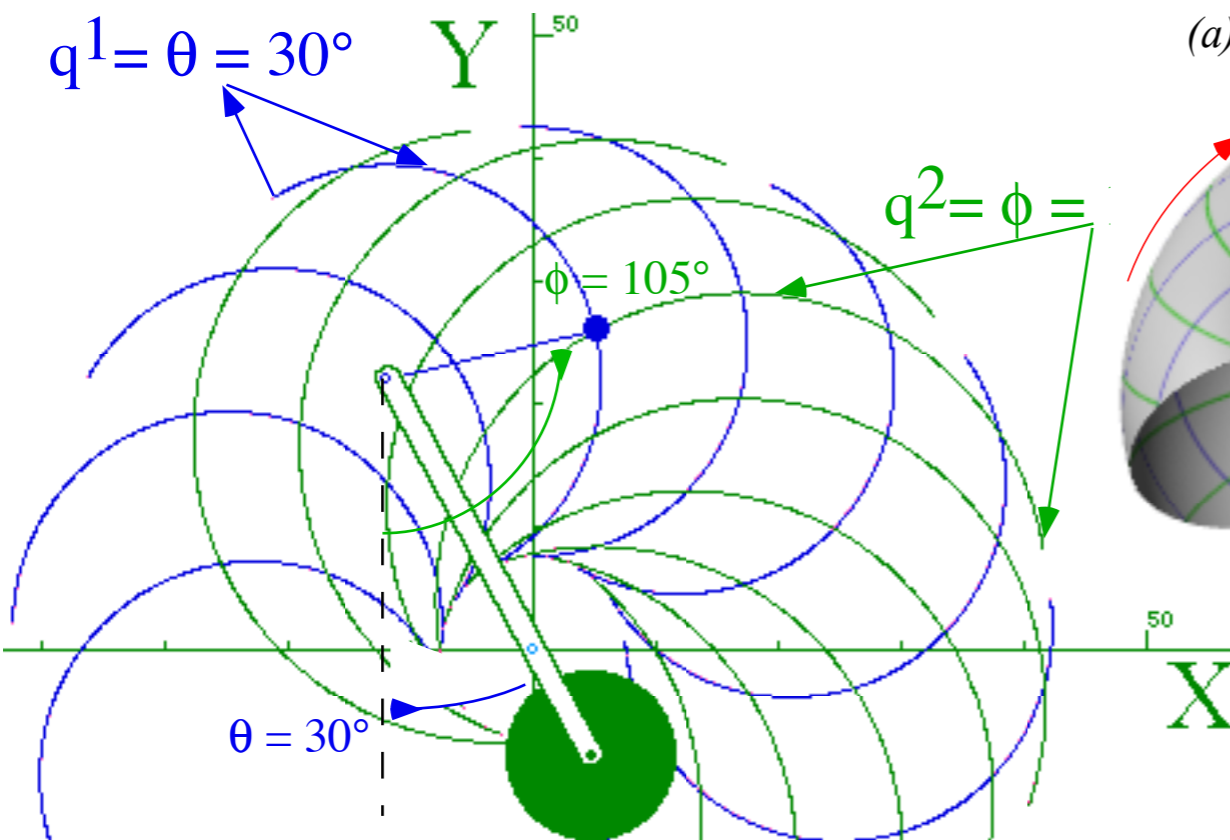


Fig. 3.1.1a ($q^1=\theta, q^2=\phi$) Coordinate manifold for trebuchet (Left handed sheet.)

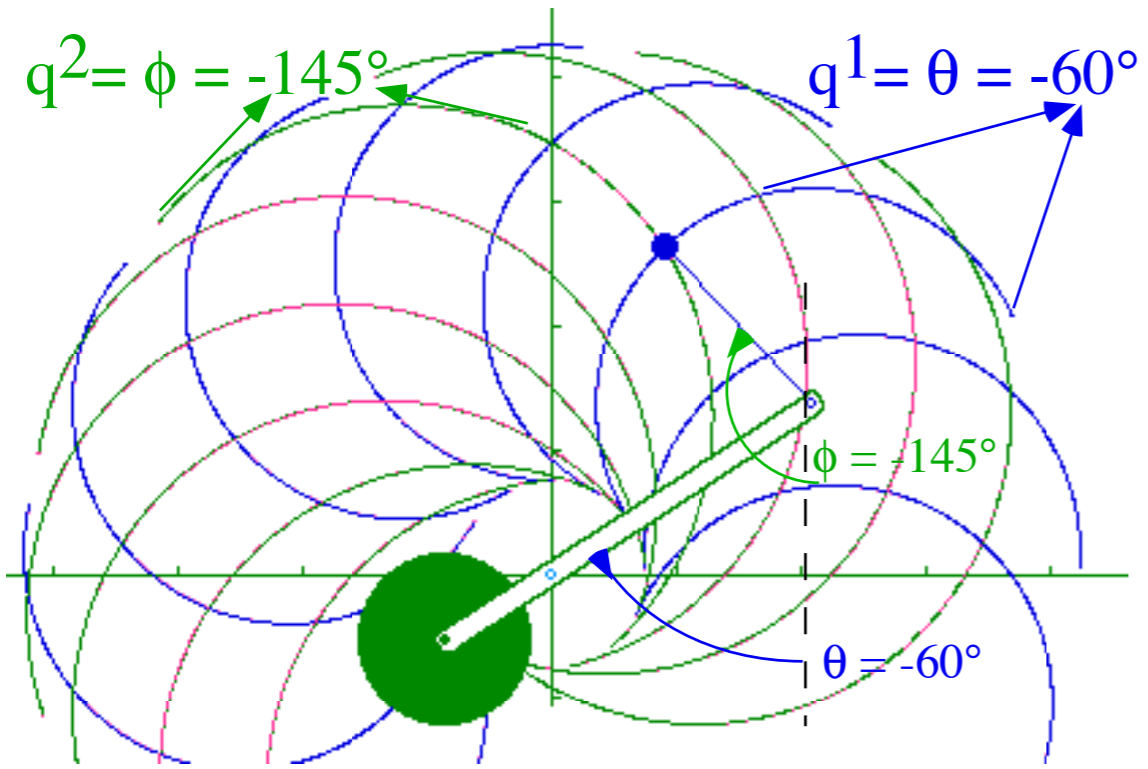


Fig. 3.1.1b ($q^1=\theta, q^2=\phi$) Coordinate manifold for trebuchet (Right handed sheet.)

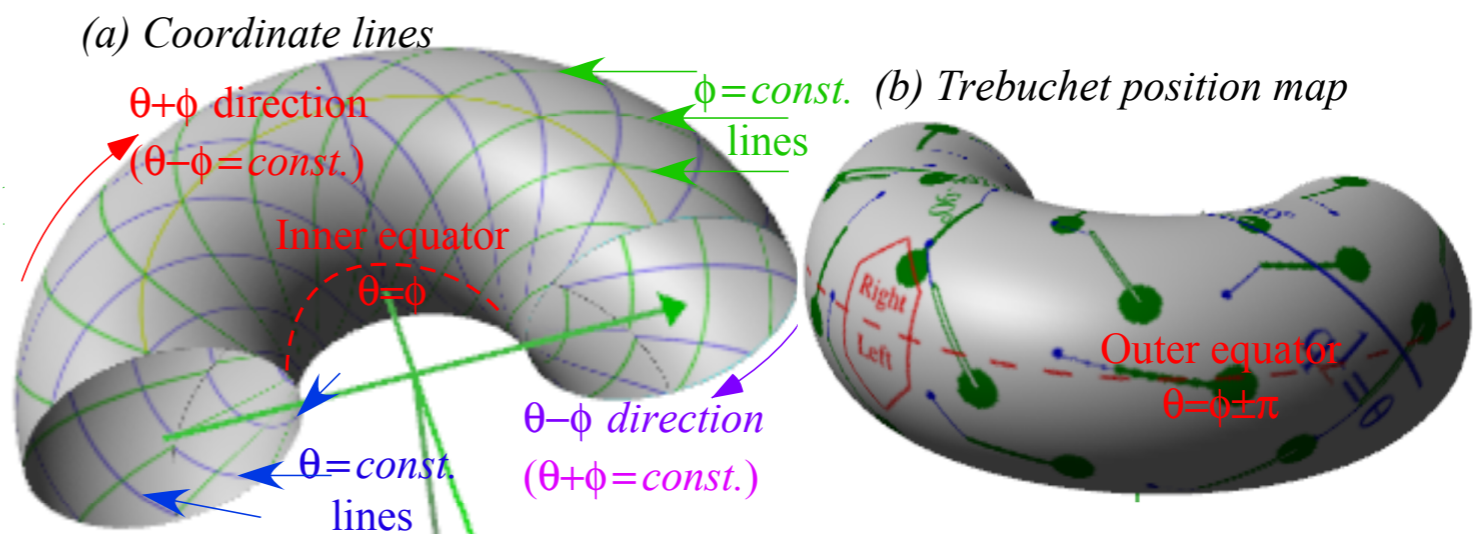
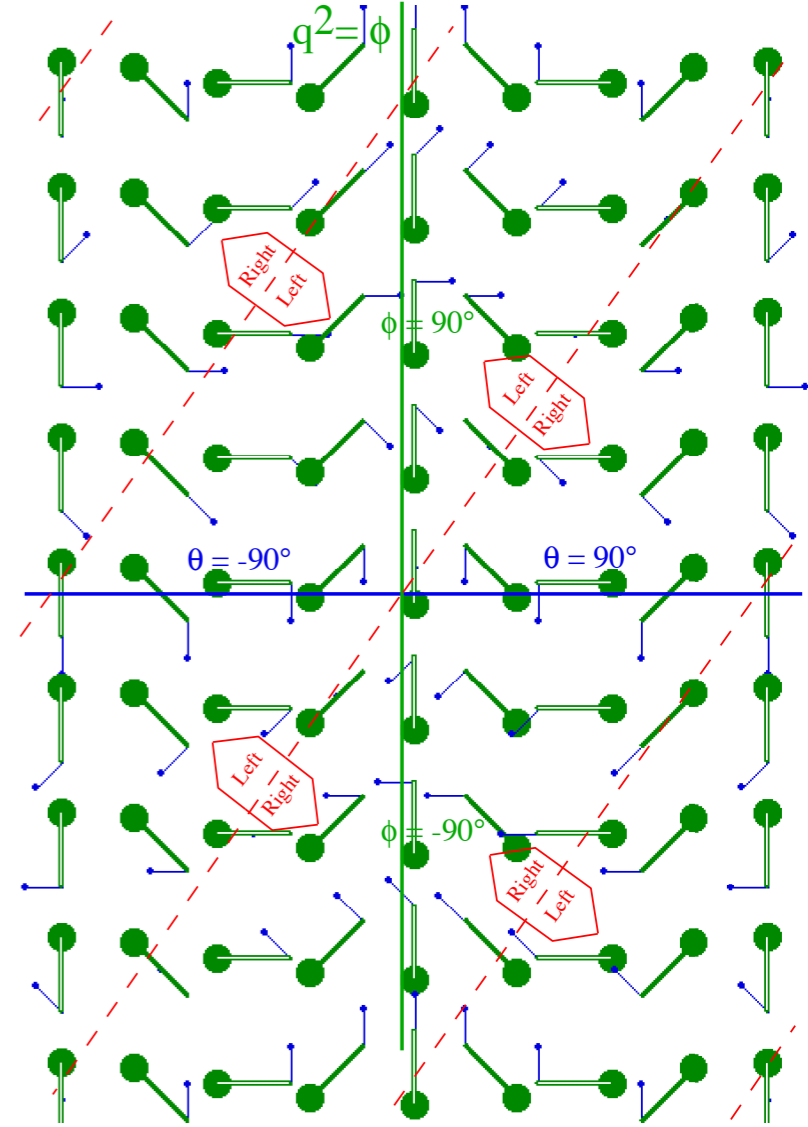


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2nd-guessing Riemann equation?

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

J-Columns are *covariant vectors* $\{\mathbf{E}_1=\mathbf{E}_r \ \mathbf{E}_2=\mathbf{E}_\phi\}$

K-Rows are *contravariant vectors* $\{\mathbf{E}^1=\mathbf{E}^r \ \mathbf{E}^2=\mathbf{E}^\phi\}$

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\ \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix}$$

$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad \uparrow \mathbf{E}_r \qquad \uparrow \mathbf{E}_\phi$

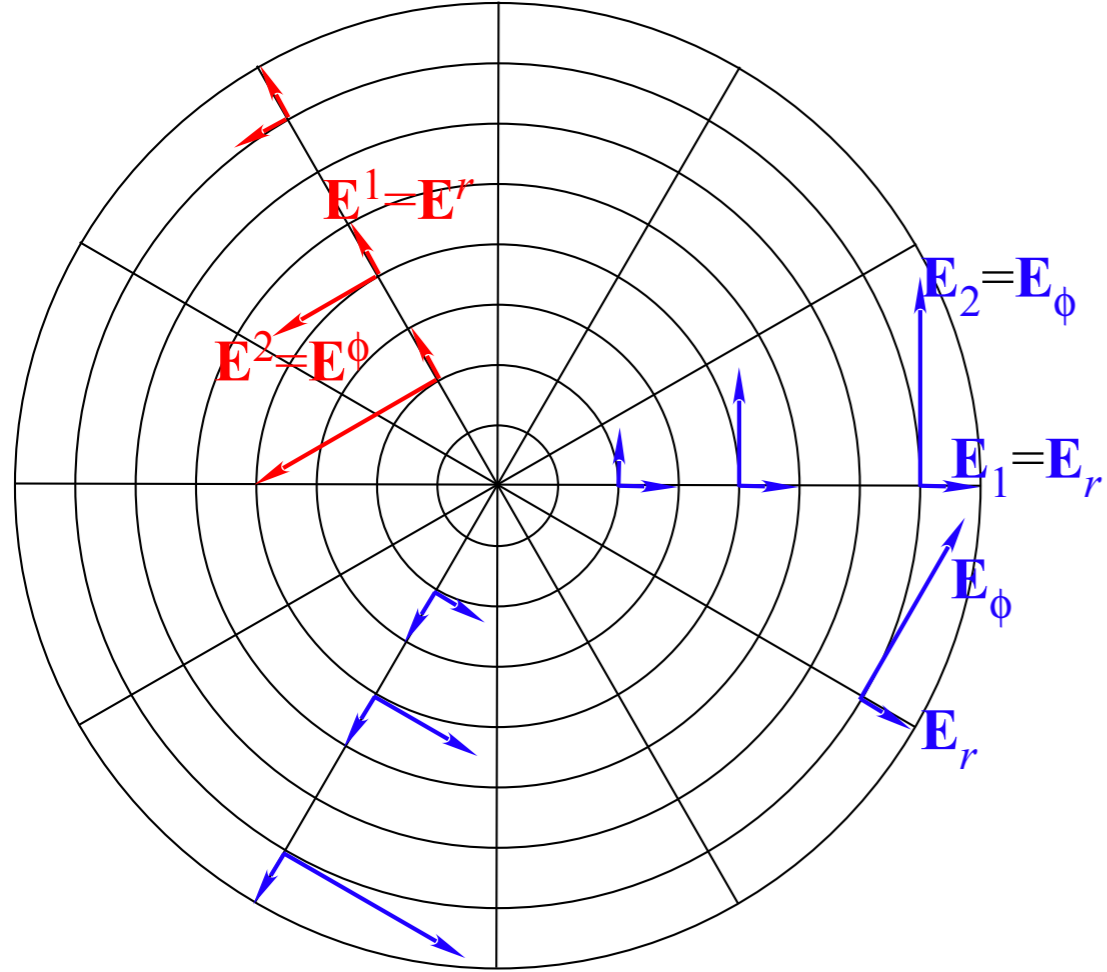
$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix}$$

$\leftarrow \mathbf{E}^r = \mathbf{E}^1$
 $\leftarrow \mathbf{E}^\phi = \mathbf{E}^2$

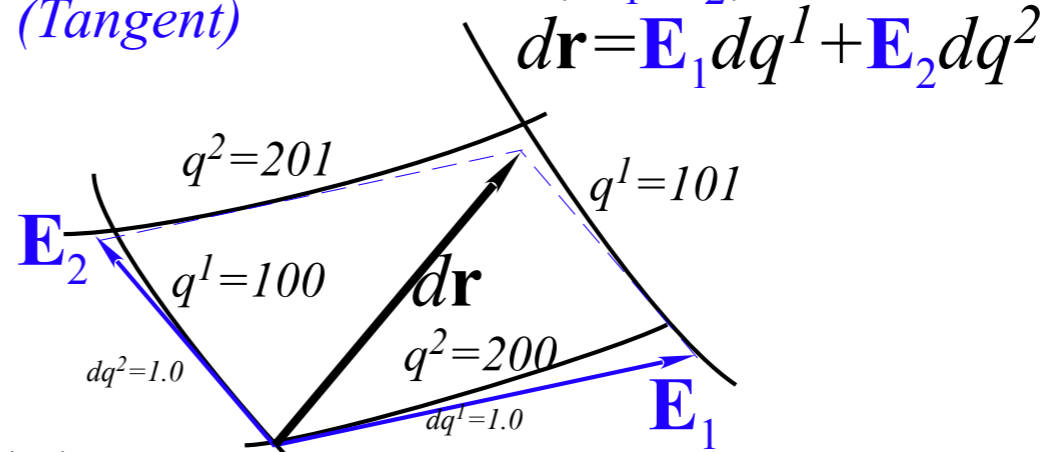
Inverse polar definition:
 $r^2=x^2+y^2$ and $\phi = \text{atan2}(y,x)$

Derived from polar definition: $x=r \cos \phi$ and $y=r \sin \phi$

(a) Polar coordinate bases

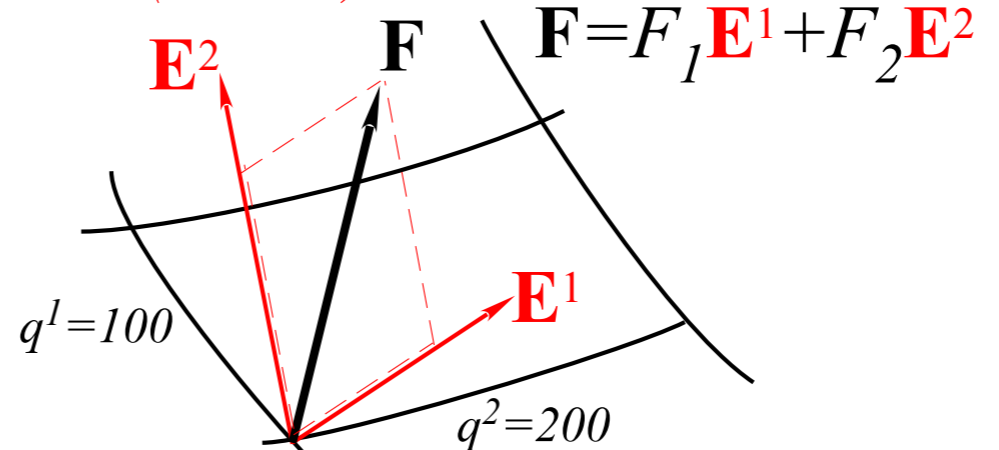


(b) Covariant bases $\{\mathbf{E}_1 \ \mathbf{E}_2\}$
 (Tangent)



NOTE: These are 2D drawings!
 No 3D perspective

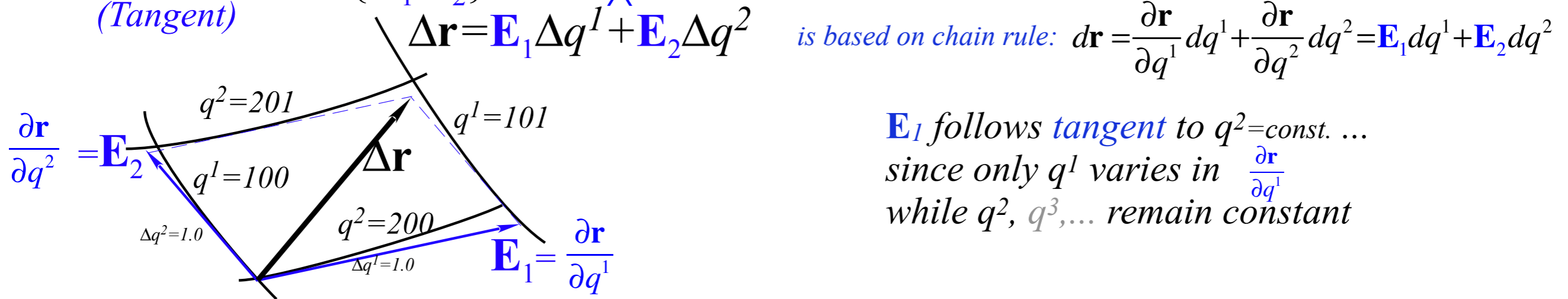
(c) Contravariant bases $\{\mathbf{E}^1 \ \mathbf{E}^2\}$
 (Normal)



Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

from p. 49 of Lect. 9

Covariant bases $\{\mathbf{E}_1, \mathbf{E}_2\}$ match ^{geometric unit} cell walls
(Tangent)



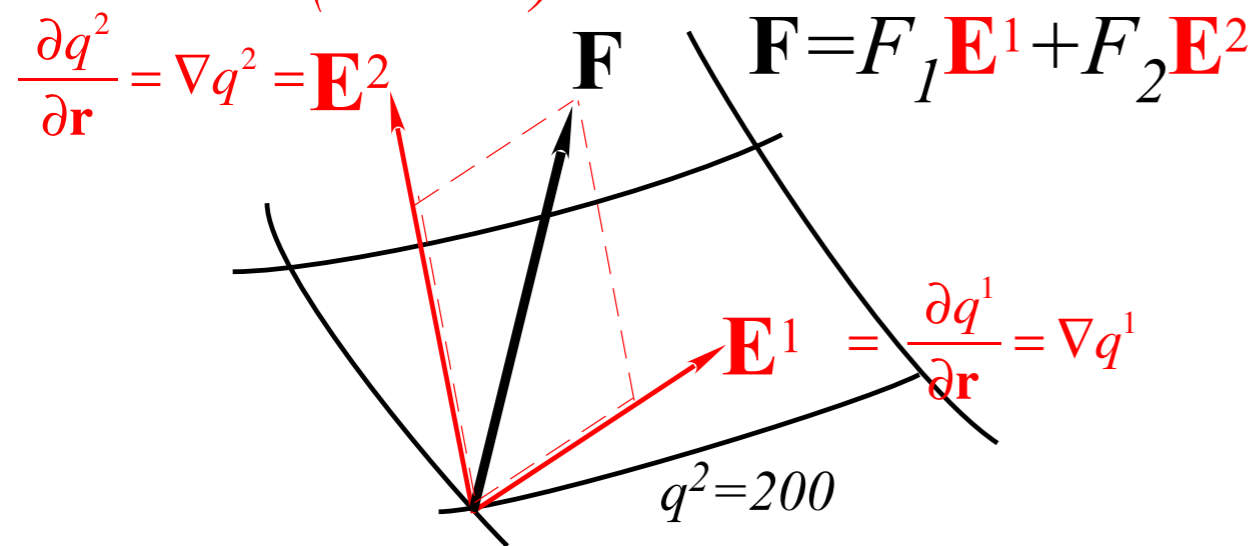
\mathbf{E}_1 follows *tangent* to $q^2 = \text{const.}$...
since only q^1 varies in $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3, \dots remain constant

\mathbf{E}_m are convenient bases for *extensive* quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

Contravariant $\{\mathbf{E}^1, \mathbf{E}^2\}$ match reciprocal cells

(Normal)



NOTE: These are 2D drawings!
No 3D perspective

\mathbf{E}^1 is *normal* to $q^2 = \text{const.}$ since **gradient** of q^1 is vector sum $\nabla q^1 = \left(\begin{matrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{matrix} \right)$ of all its partial derivatives

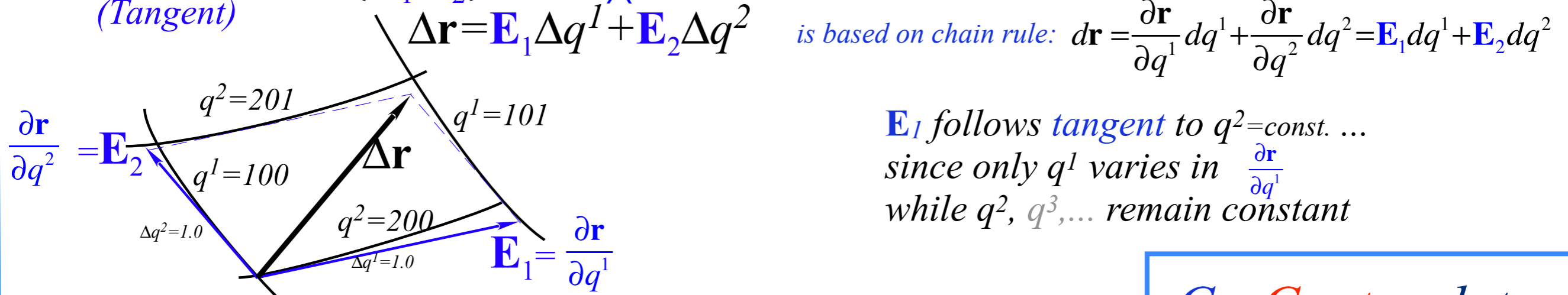
\mathbf{E}^m are convenient bases for *intensive* quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

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\mathbf{E}_m are convenient bases for extensive quantities like distance and velocity.

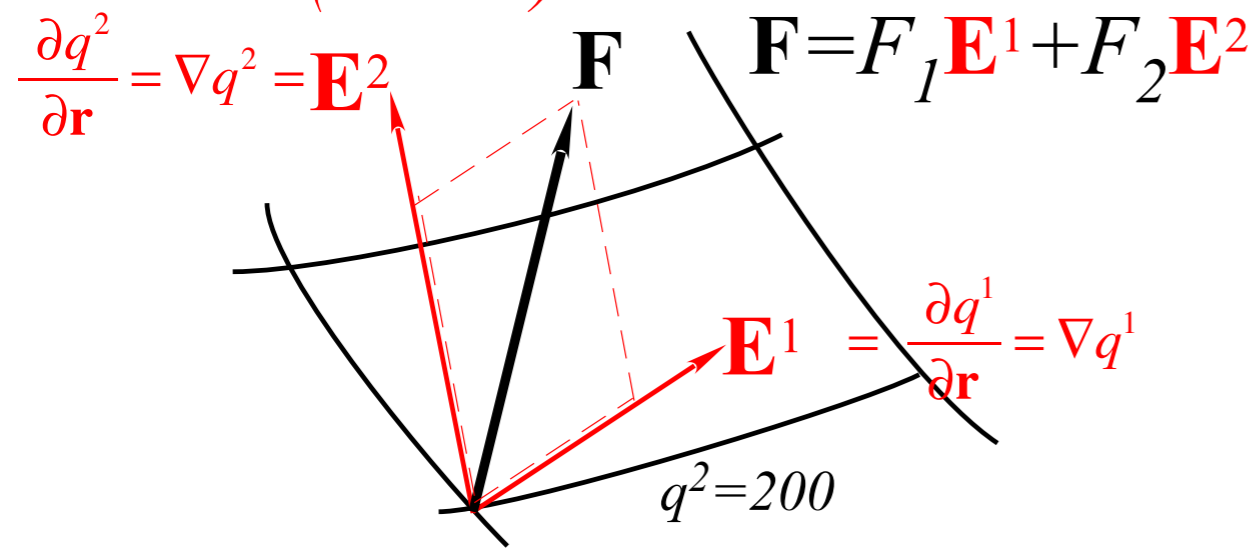
$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

Co-Contr dot products $\mathbf{E}_m \cdot \mathbf{E}^n$ are orthonormal:

$$\mathbf{E}_m \cdot \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

Contravariant $\{\mathbf{E}^1, \mathbf{E}^2\}$ match reciprocal cells

(Normal)



\mathbf{E}^1 is normal to $q^1 = \text{const.}$ since gradient of q^1 is vector sum $\nabla q^1 =$

$$\left(\begin{array}{c} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{array} \right)$$

\mathbf{E}^m are convenient bases for intensive quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

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→ *Review of **covariant \mathbf{E}_n** and contravariant \mathbf{E}^m vectors. **Jacobian J** vs. **Kajobian K***

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

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2nd-guessing Riemann equation?

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):

$$X = R \sin \theta$$

$$Y = -R \cos \theta$$

$$y_r = r \cos \theta$$

$$X = -R \sin \theta$$

$$Y = -R \cos \theta$$

$$x = -r \sin \theta + l \sin \phi$$

$$x_r = -r \sin \theta \quad x_\ell = l \sin \phi$$

$$y_\ell = -l \cos \phi$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 18 of Lect. 14

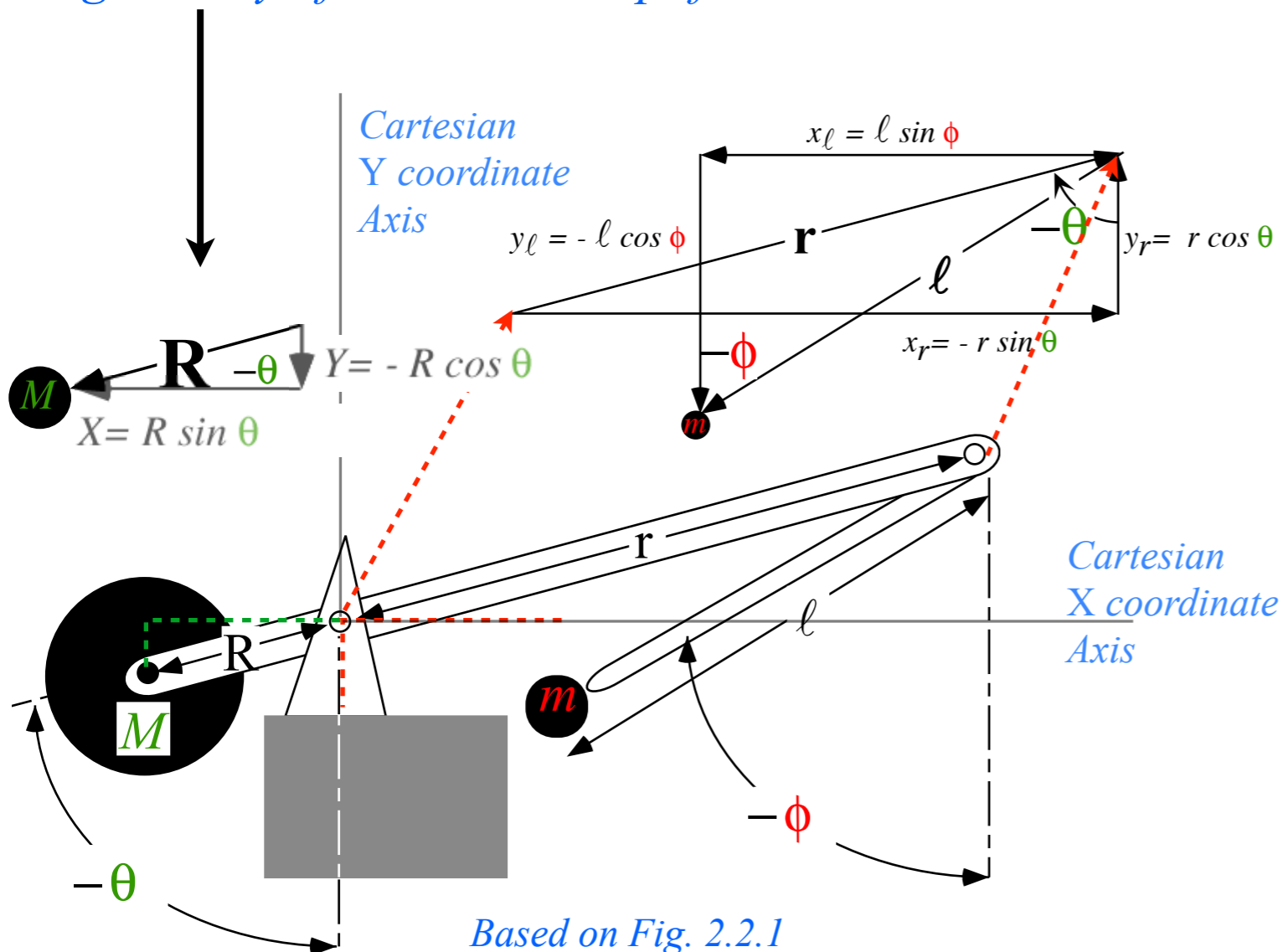
Coordinates of mass m
(Payload or projectile):

$$x = x_r + x_\ell = -r \sin \theta + l \sin \phi$$

$$y = y_r + y_\ell = r \cos \theta - l \cos \phi$$



geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\left(\begin{array}{l} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{array} \right)$$

from p. 89 of Lect. 14

$$\begin{array}{c|ccc} \mathbf{E}_1 & \mathbf{E}_2 & \dots & \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots & \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots & \\ \vdots & \vdots & \ddots & \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}$$

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{array}{c|ccc} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{c|cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{array}$$

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{array}{c|ccc} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{c|cc} & \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline & -r \cos \theta & l \cos \phi \\ & -r \sin \theta & l \sin \phi \end{array}$$

Covariant vectors \mathbf{E}_n

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{array}{c|ccc} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{c|cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{array}$$

Covariant vectors \mathbf{E}_n

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

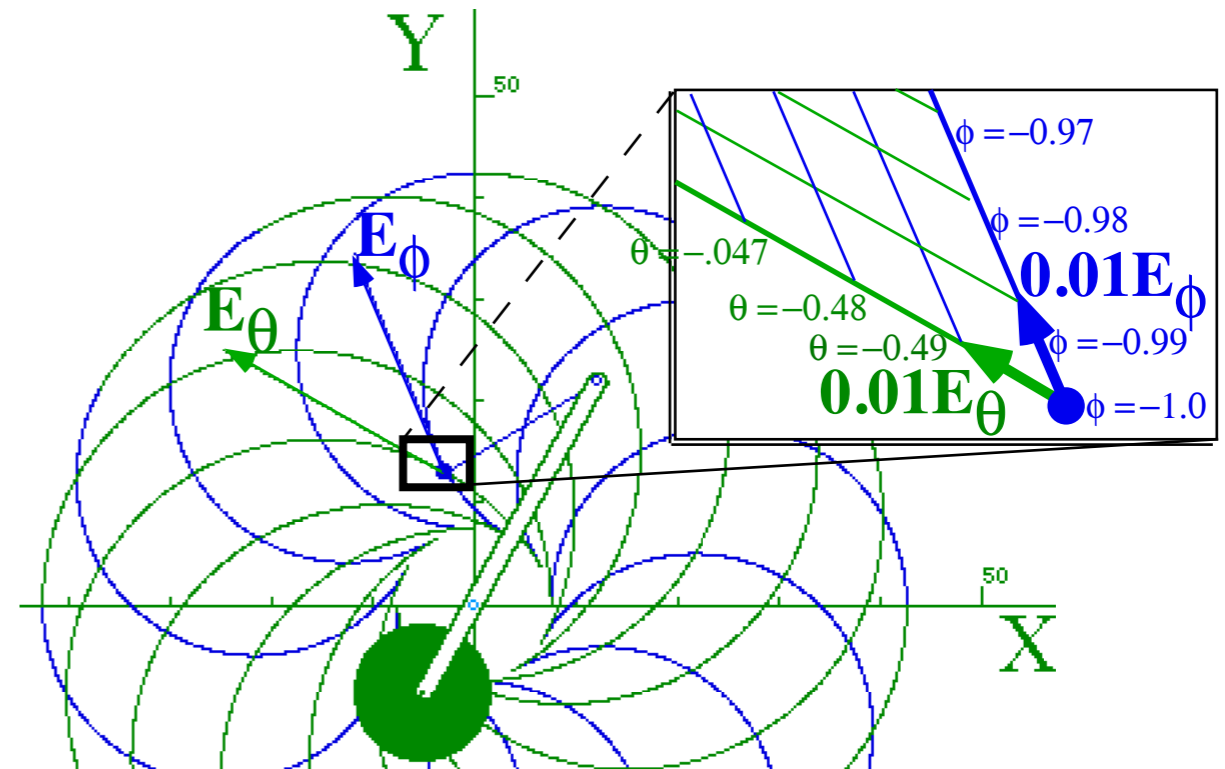


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued*

Toroidal “rolled-up” ($q^1=\theta$, $q^2=\phi$)-manifold and “Flat” ($x=\theta$, $y=\phi$)-graph

→ *Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors: Jacobian J vs. Kajobian K*

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{AD-BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{1}{lr \sin \theta \cos \phi - lr \sin \phi \cos \theta} \begin{pmatrix} l \sin \phi & -l \cos \phi & \mathbf{E}^\theta \\ r \sin \theta & -r \cos \theta & \mathbf{E}^\phi \end{pmatrix}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix}$$

Covariant vectors \mathbf{E}_n

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

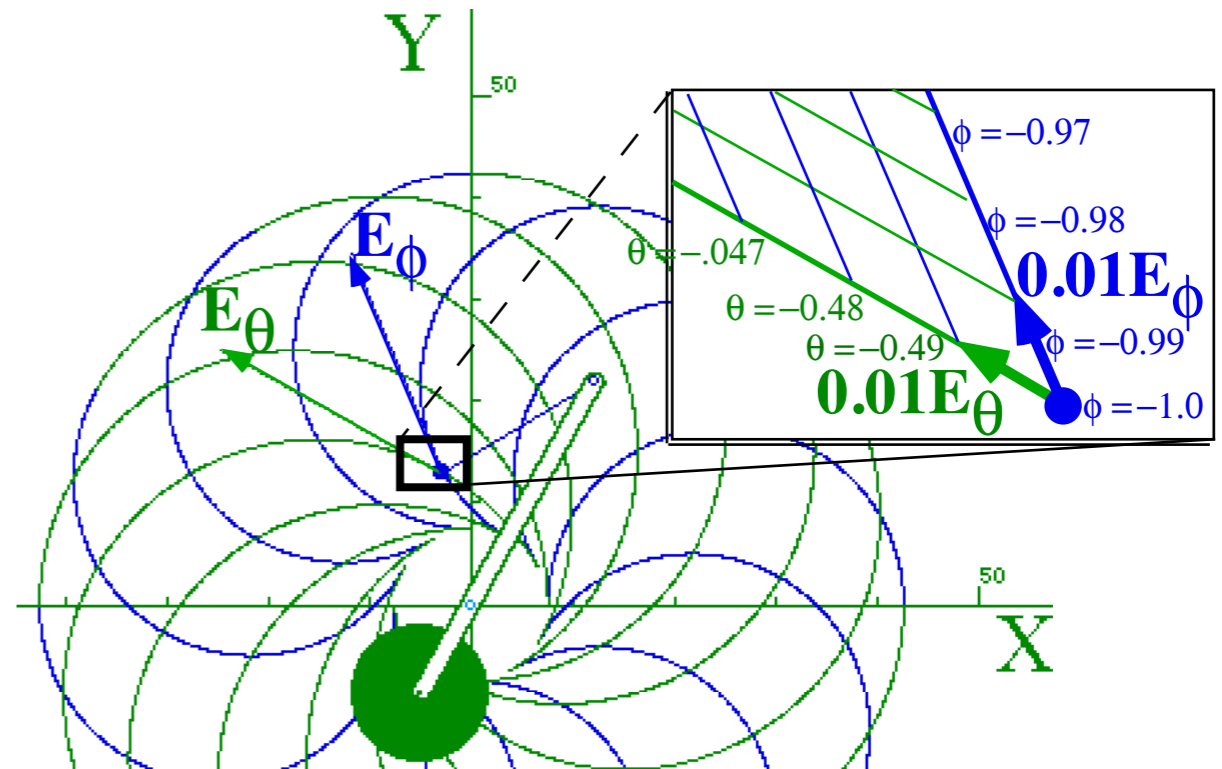


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{AD-BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{1}{rl \sin(\theta - \phi)} \begin{pmatrix} l \sin \phi & -l \cos \phi \\ r \sin \theta & -r \cos \theta \end{pmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}$$

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

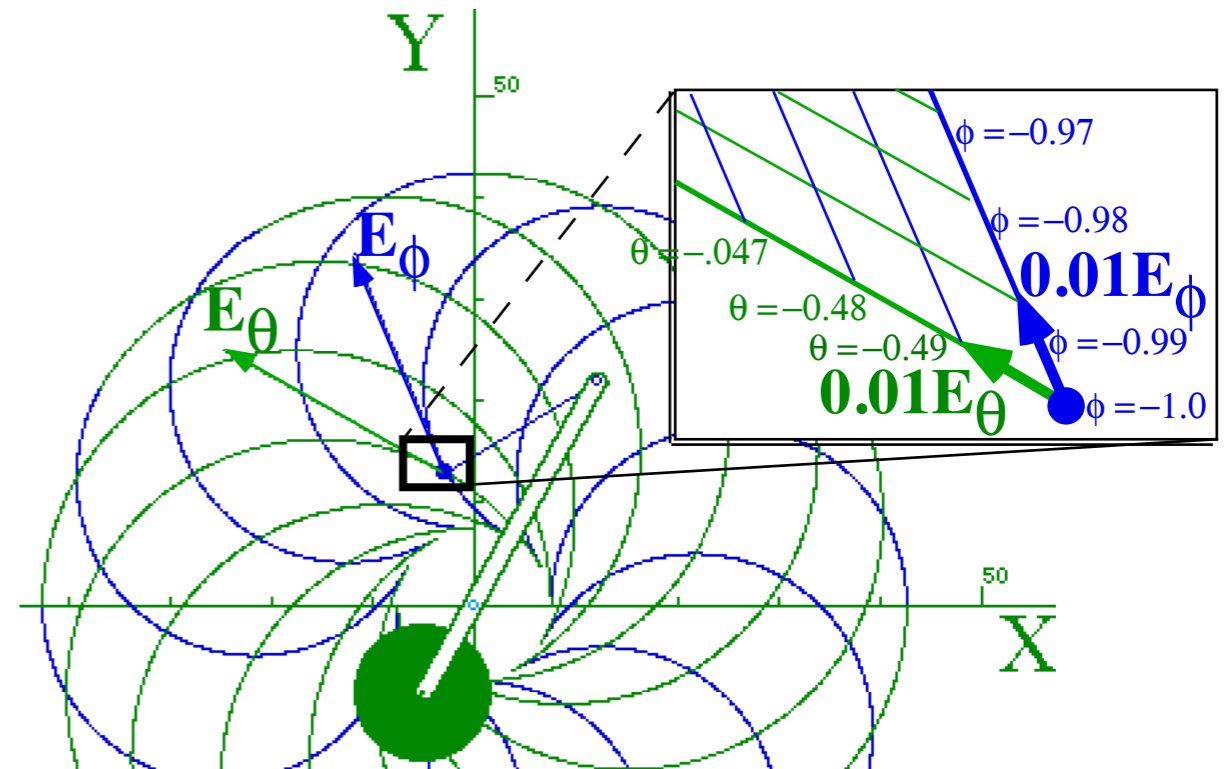


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \text{Using } 2 \times 2 \text{ inverse } \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}^{-1} = \frac{1}{AD-BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{1}{rl \sin(\theta - \phi)} \begin{pmatrix} l \sin \phi & -l \cos \phi \\ r \sin \theta & -r \cos \theta \end{pmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}$$

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

Covariant tangent-space

GCC vectors

$$\mathbf{E}_1 = \mathbf{E}_\theta \text{ and } \mathbf{E}_2 = \mathbf{E}_\phi$$

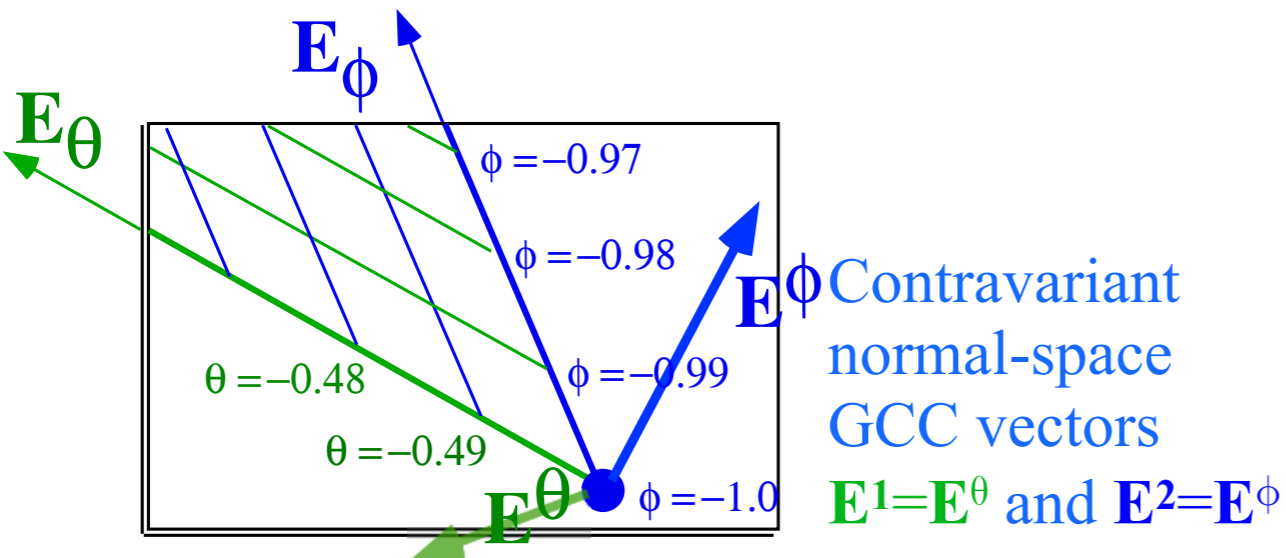


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

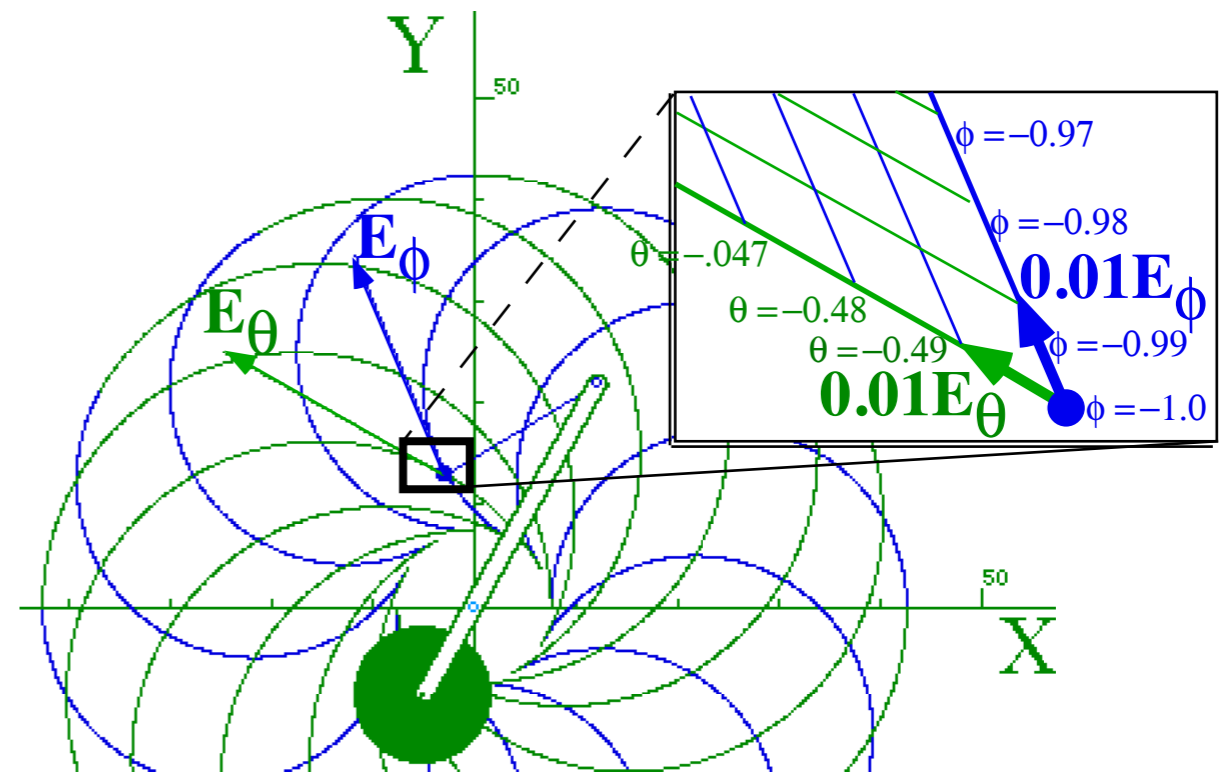


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{AD-BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{1}{rl \sin(\theta - \phi)} \begin{pmatrix} l \sin \phi & -l \cos \phi \\ r \sin \theta & -r \cos \theta \end{pmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}$$

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{pmatrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \cdot \mathbf{E}^\phi$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \cdot \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

Covariant tangent-space

GCC vectors

$$\mathbf{E}_1 = \mathbf{E}_\theta \text{ and } \mathbf{E}_2 = \mathbf{E}_\phi$$

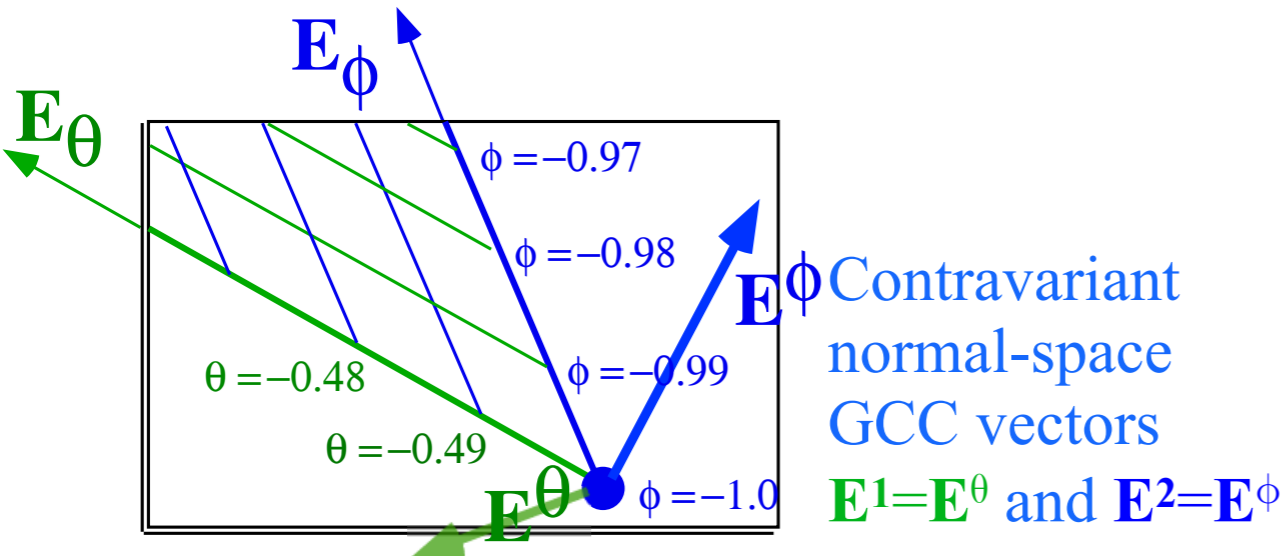


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

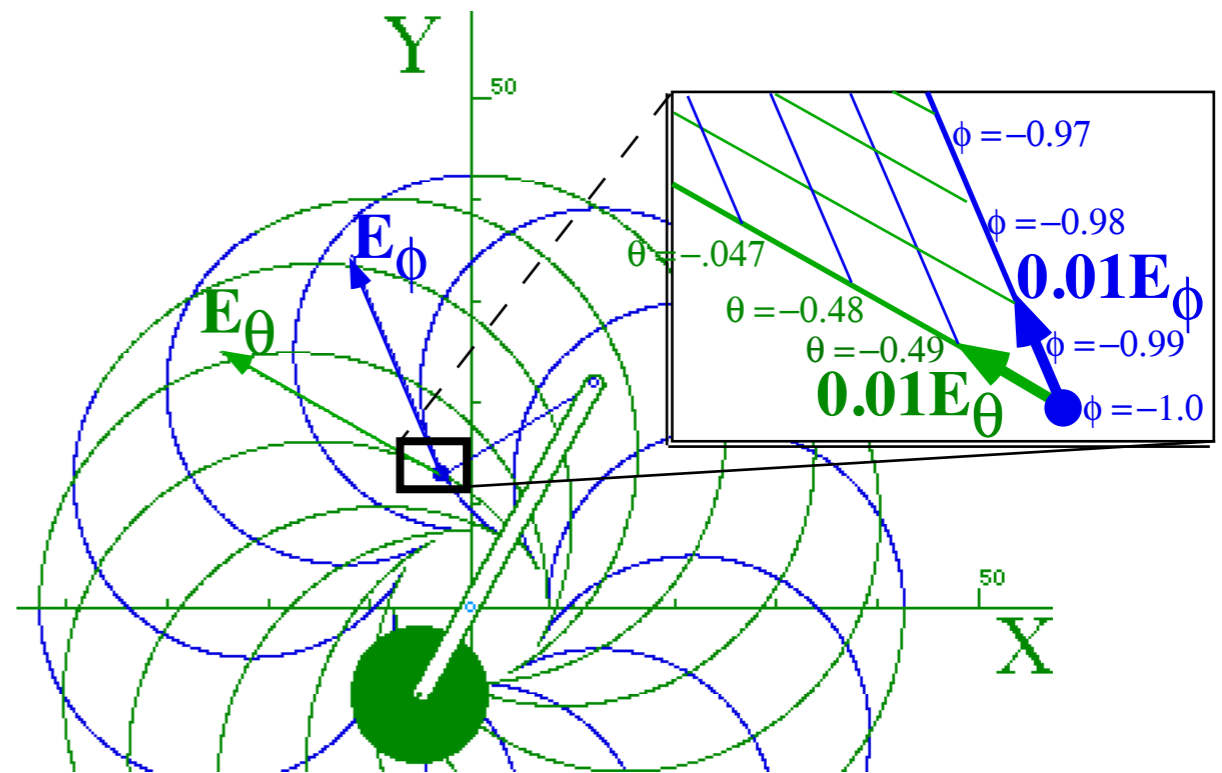


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued*

Toroidal “rolled-up” ($q^1 = \theta$, $q^2 = \phi$)-manifold and “Flat” ($x = \theta$, $y = \phi$)-graph

*Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors. **Jacobian J** vs. **Kajobian K***

 **Covariant metric g_{mn}** vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

$$\mathbf{E}_m \cdot \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \cdot \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \cdot \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \cdot \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor

g_{mn}

Invariant
Kronecker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor

g^{mn}

from p. 53 of Lect. 9

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\ \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix} \quad \langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix}$$

$\uparrow \mathbf{E}_1$ $\uparrow \mathbf{E}_2$ $\uparrow \mathbf{E}_r$ $\uparrow \mathbf{E}_\phi$

$\leftarrow \mathbf{E}^r = \mathbf{E}^1$
 $\leftarrow \mathbf{E}^\phi = \mathbf{E}^2$

Covariant g_{mn}

Invariant δ_m^n

Contravariant g^{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}^r & \mathbf{E}_r \cdot \mathbf{E}^\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}^r & \mathbf{E}_\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \cdot \mathbf{E}^r & \mathbf{E}^r \cdot \mathbf{E}^\phi \\ \mathbf{E}^\phi \cdot \mathbf{E}^r & \mathbf{E}^\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{vmatrix} D & -B \\ -C & A \end{vmatrix} / (AD - BC)$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} l \sin \phi & -l \cos \phi \\ r \sin \theta & -r \cos \theta \end{vmatrix}}{rl \sin(\theta - \phi)} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}$$

$$\begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{vmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{vmatrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \cdot \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\theta = 1 = \mathbf{E}_\theta \cdot \mathbf{E}^\theta$$

Contravariant metric $g^{mn} = \mathbf{E}^m \cdot \mathbf{E}^n = g^{nm}$

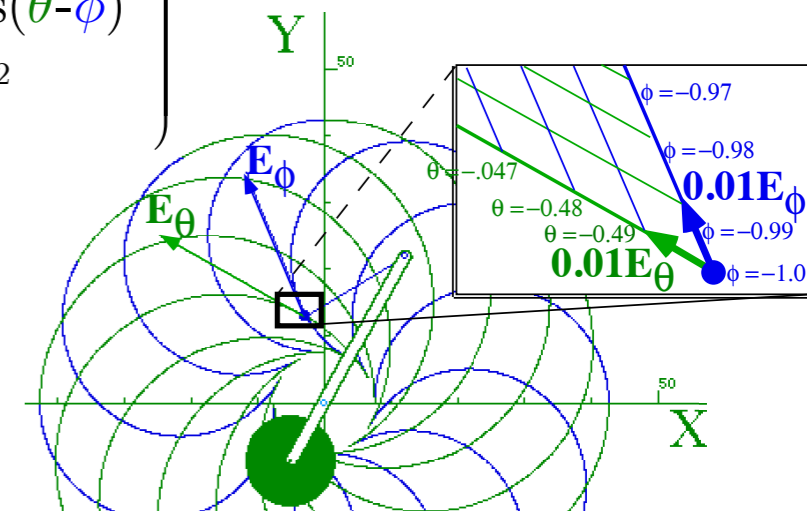
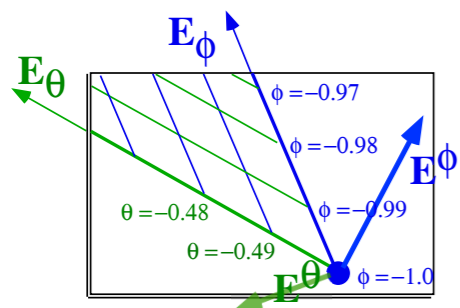
versus

Covariant metric $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \cdot \mathbf{E}_\theta & \mathbf{E}_\theta \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_\theta & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -rl(\cos \theta \cos \phi + \sin \theta \sin \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix}$$

$$= \begin{pmatrix} r^2 & -rl \cos(\theta - \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix}$$



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Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

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*Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors: Jacobian J vs. **Kajobian K***

 *Covariant metric g_{mn} vs. **contravariant metric g^{mn}** (Lect. 10 p.43-49)*

Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 15 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 15 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{AD-BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{matrix} l \sin \phi & -l \cos \phi & \mathbf{E}^\theta \\ r \sin \theta & -r \cos \theta & \mathbf{E}^\phi \end{matrix}}{rl \sin(\theta - \phi)}$$

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{matrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \cdot \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\theta = 1 = \mathbf{E}_\theta \cdot \mathbf{E}^\theta$$

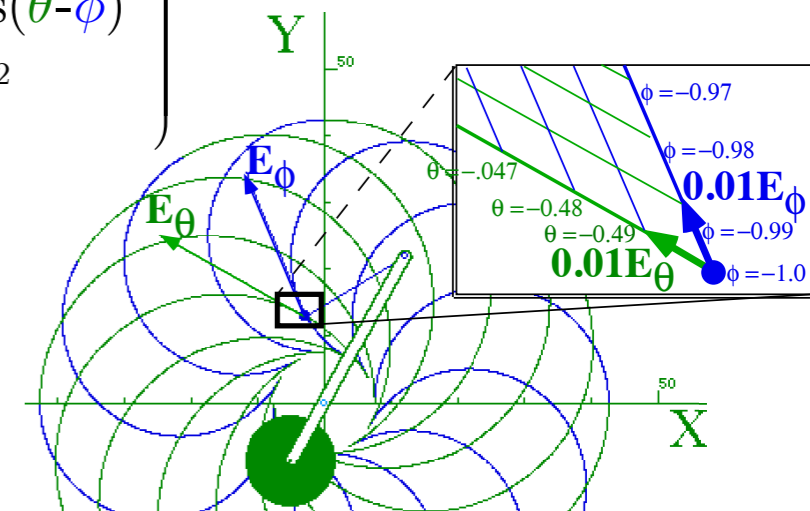
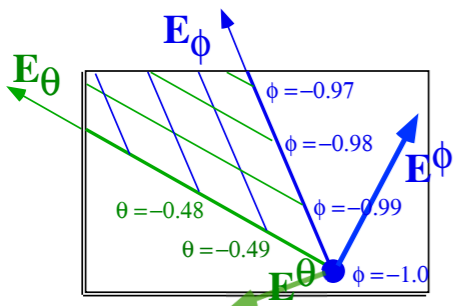
Contravariant metric $g^{mn} = \mathbf{E}^m \cdot \mathbf{E}^n = g^{nm}$

versus

Covariant metric $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \cdot \mathbf{E}^\theta & \mathbf{E}^\theta \cdot \mathbf{E}^\phi \\ \mathbf{E}^\phi \cdot \mathbf{E}^\theta & \mathbf{E}^\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} l^2 & rl(\sin \phi \sin \theta + \cos \phi \cos \theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2 l^2 \sin^2(\theta - \phi) = \begin{pmatrix} l^2 & rl \cos(\theta - \phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2 l^2 \sin^2(\theta - \phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \cdot \mathbf{E}_\theta & \mathbf{E}_\theta \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_\theta & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} r^2 & -rl(\cos \theta \cos \phi + \sin \theta \sin \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix} = \begin{pmatrix} r^2 & -rl \cos(\theta - \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix}$$



Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{AD-BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r \sin \theta + l \sin \phi \\ y = r \cos \theta - l \cos \phi \end{pmatrix}$$

from p. 89 of Lect. 14

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{matrix} l \sin \phi & -l \cos \phi & \mathbf{E}^\theta \\ r \sin \theta & -r \cos \theta & \mathbf{E}^\phi \end{matrix}}{rl \sin(\theta - \phi)}$$

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{matrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \cdot \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \cdot \mathbf{E}^\phi$$

Contravariant metric $g^{mn} = \mathbf{E}^m \cdot \mathbf{E}^n = g^{nm}$

versus

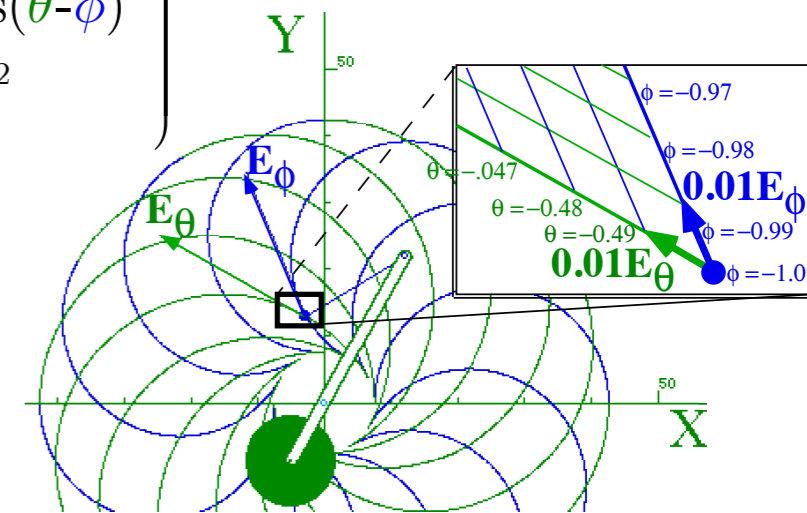
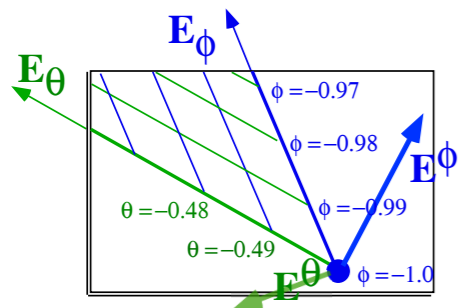
Covariant metric $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \cdot \mathbf{E}^\theta & \mathbf{E}^\theta \cdot \mathbf{E}^\phi \\ \mathbf{E}^\phi \cdot \mathbf{E}^\theta & \mathbf{E}^\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} l^2 & rl(\sin \phi \sin \theta + \cos \phi \cos \theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2 l^2 \sin^2(\theta - \phi) = \begin{pmatrix} l^2 & rl \cos(\theta - \phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2 l^2 \sin^2(\theta - \phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \cdot \mathbf{E}_\theta & \mathbf{E}_\theta \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_\theta & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} r^2 & -rl(\cos \theta \cos \phi + \sin \theta \sin \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix} = \begin{pmatrix} r^2 & -rl \cos(\theta - \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix}$$

Jacobian $J^T J$ -product gives g_{mn}

$$J^T J = \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \mathbf{E}_\phi & \mathbf{E}_\theta \end{matrix} \begin{matrix} -r \cos \theta & -r \sin \theta \\ l \cos \phi & l \sin \phi \end{matrix} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$



Kajobian transformation matrix

versus

Jacobian transformation matrix

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

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from p. 89 of Lect. 14

$$\begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \dots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{matrix} l \sin \phi & -l \cos \phi & \mathbf{E}^\theta \\ r \sin \theta & -r \cos \theta & \mathbf{E}^\phi \end{matrix}}{rl \sin(\theta - \phi)}$$

$$\begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \dots \\ \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \dots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{matrix}$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

$$\mathbf{E}^\theta = \begin{pmatrix} l \sin \phi & -l \cos \phi \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \cdot \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} l \cos \phi \\ l \sin \phi \end{pmatrix}$$

$$\mathbf{E}^\phi = \begin{pmatrix} r \sin \theta & -r \cos \theta \end{pmatrix} / rl \sin(\theta - \phi)$$

$$\mathbf{E}^\theta \cdot \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \cdot \mathbf{E}^\phi$$

Contravariant metric $g^{mn} = \mathbf{E}^m \cdot \mathbf{E}^n = g^{nm}$

versus

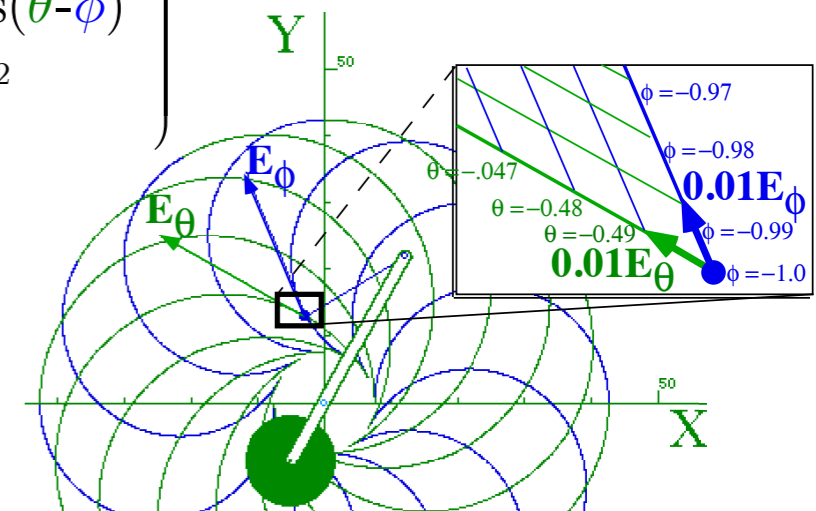
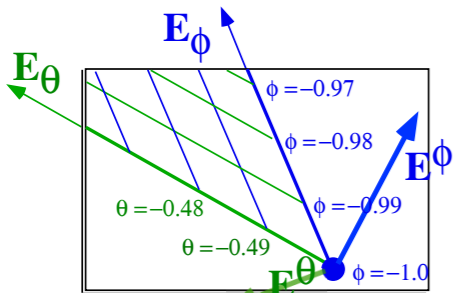
Covariant metric $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^\theta \cdot \mathbf{E}^\theta & \mathbf{E}^\theta \cdot \mathbf{E}^\phi \\ \mathbf{E}^\phi \cdot \mathbf{E}^\theta & \mathbf{E}^\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} l^2 & rl(\sin \phi \sin \theta + \cos \phi \cos \theta) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2 l^2 \sin^2(\theta - \phi) = \begin{pmatrix} l^2 & rl \cos(\theta - \phi) \\ g^{\phi\theta} & r^2 \end{pmatrix} / r^2 l^2 \sin^2(\theta - \phi)$$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \cdot \mathbf{E}_\theta & \mathbf{E}_\theta \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_\theta & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} r^2 & -rl(\cos \theta \cos \phi + \sin \theta \sin \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix} = \begin{pmatrix} r^2 & -rl \cos(\theta - \phi) \\ g_{\phi\theta} & l^2 \end{pmatrix}$$

Jacobian $J^T J$ -product gives g_{mn}

$$J^T J = \begin{matrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ -r \cos \theta & l \cos \phi \\ -r \sin \theta & l \sin \phi \end{matrix} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$



Kajobian KK^T -product would give g^{mn}

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued*

Toroidal “rolled-up” ($q^1=\theta$, $q^2=\phi$)-manifold and “Flat” ($x=\theta$, $y=\phi$)-graph

Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

 *Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space*

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

$$\mathbf{V} = V^m \mathbf{E}_m = V_n \mathbf{E}^n = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

where the U^m, V^m, \dots are *contravariant components*

and the U_n, V_n, \dots are *covariant components*

$$U^m = \mathbf{U} \cdot \mathbf{E}^m, \quad V^m = \mathbf{V} \cdot \mathbf{E}^m, \quad \text{and} \quad \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \quad \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}},$$

$$U_n = \mathbf{U} \cdot \mathbf{E}_n, \quad V_n = \mathbf{V} \cdot \mathbf{E}_n, \quad \text{and} \quad \bar{U}_{\bar{n}} = \mathbf{U} \cdot \bar{\mathbf{E}}_{\bar{n}}, \quad \text{etc.}$$

Normal space (Contravariant)

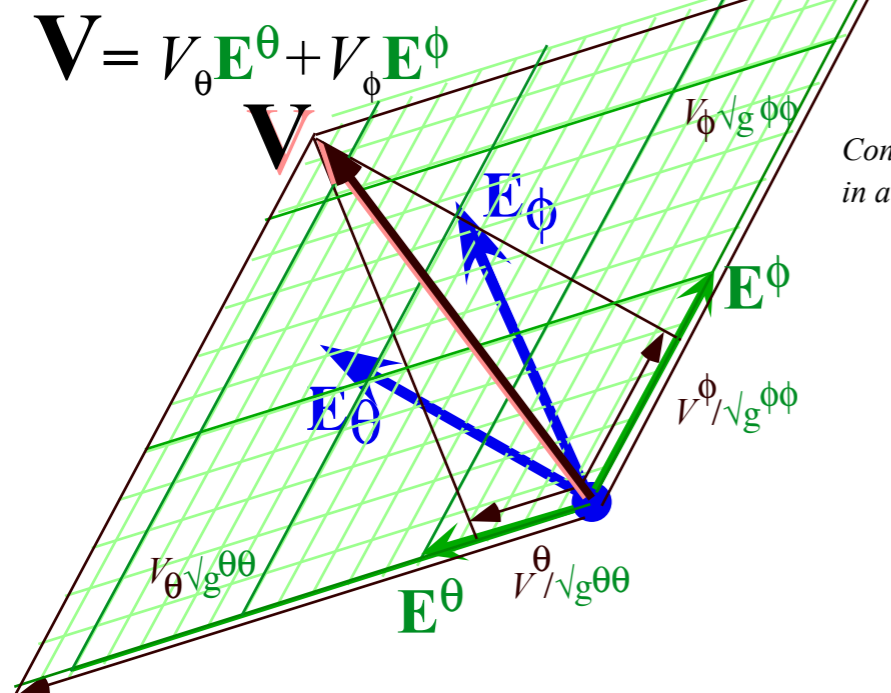


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

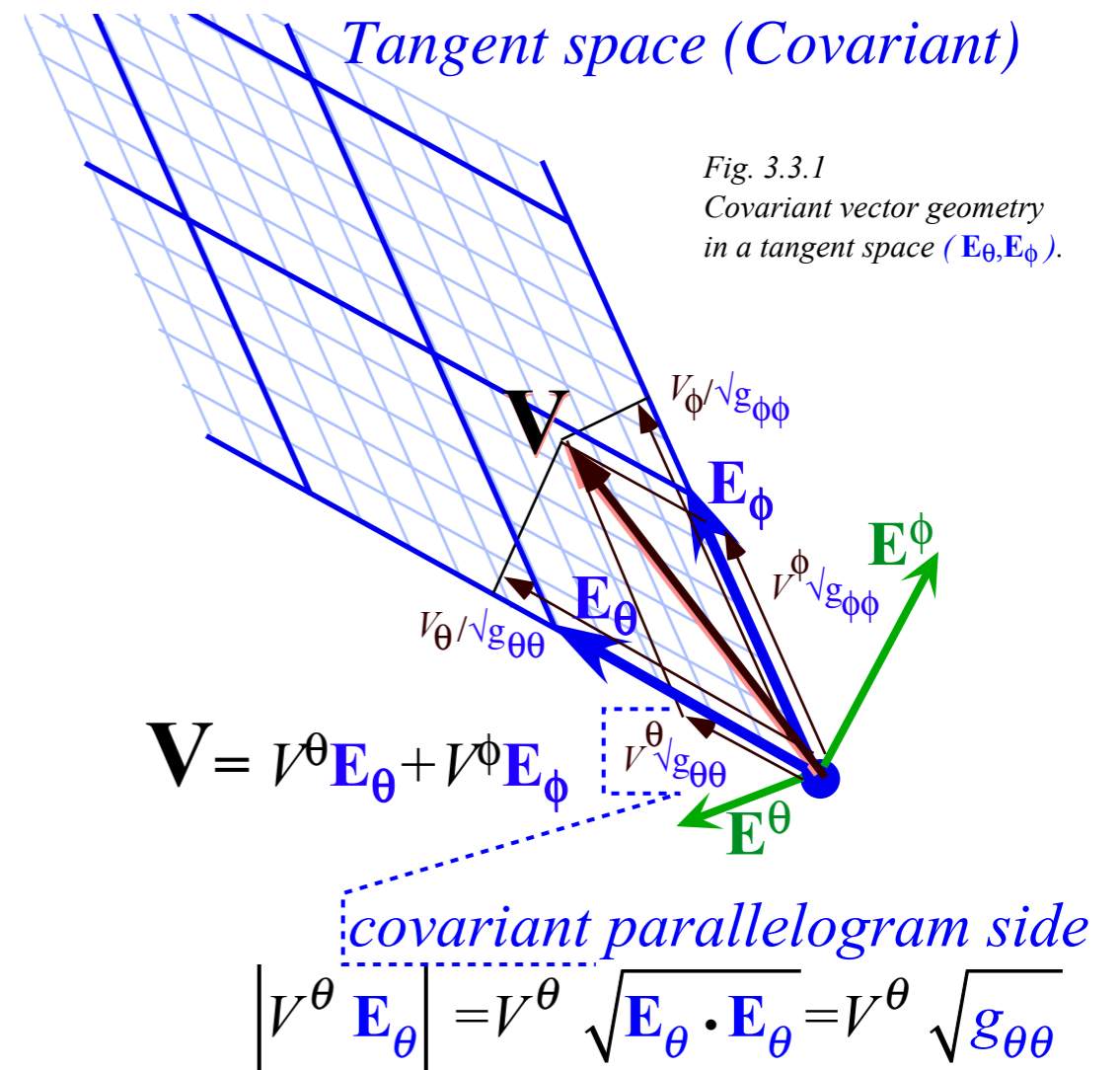


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

$$\mathbf{V} = V^m \mathbf{E}_m = V_n \mathbf{E}^n = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

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$$U^m = \mathbf{U} \cdot \mathbf{E}^m, \quad V^m = \mathbf{V} \cdot \mathbf{E}^m, \quad \text{and} \quad \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \quad \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}},$$

$$U_n = \mathbf{U} \cdot \mathbf{E}_n, \quad V_n = \mathbf{V} \cdot \mathbf{E}_n, \quad \text{and} \quad \bar{U}_{\bar{n}} = \mathbf{U} \cdot \bar{\mathbf{E}}_{\bar{n}}, \quad \text{etc.}$$

Normal space (Contravariant)

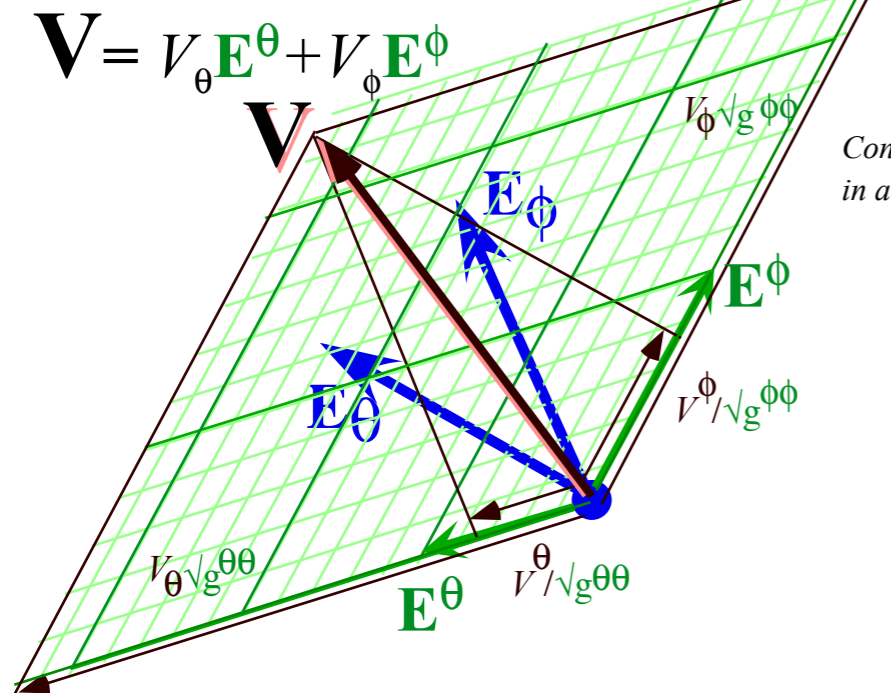


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

covariant projection

$$|\mathbf{V} \cdot \mathbf{E}_\theta| = \mathbf{V} \cdot \hat{\mathbf{E}}_\theta = \mathbf{V} \cdot \mathbf{E}_\theta / \sqrt{g_{\theta\theta}} = V_\theta / \sqrt{g_{\theta\theta}}$$

Tangent space (Covariant)

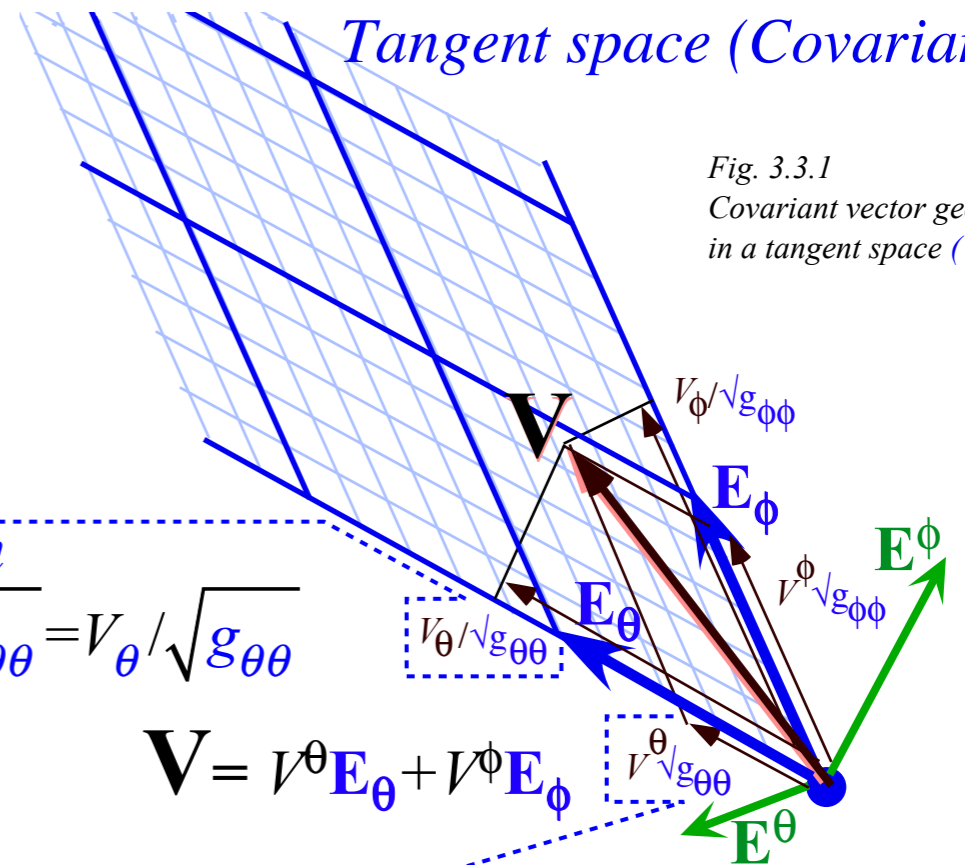


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

$$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$$

covariant parallelogram side

$$|V^\theta \mathbf{E}_\theta| = V^\theta \sqrt{\mathbf{E}_\theta \cdot \mathbf{E}_\theta} = V^\theta \sqrt{g_{\theta\theta}}$$

Contravariant vector \mathbf{E}^m is written in terms of covariant vectors \mathbf{E}_n
as would any vector $\mathbf{V} = V^n \mathbf{E}_n$ using dot product $V^n = \mathbf{V} \cdot \mathbf{E}^n$ and metric g_{mn} or $g^{mn} \dots$

Metric g_{mn} or g^{mn} tensor geometric
relations to length, area, and volume

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

$$\mathbf{V} = V^m \mathbf{E}_m = V_n \mathbf{E}^n = \bar{V}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{V}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

where the U^m, V^m, \dots are *contravariant components*

and the U_n, V_n, \dots are *covariant components*

$$U^m = \mathbf{U} \cdot \mathbf{E}^m, \quad V^m = \mathbf{V} \cdot \mathbf{E}^m, \quad \text{and} \quad \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \quad \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}},$$

$$U_n = \mathbf{U} \cdot \mathbf{E}_n, \quad V_n = \mathbf{V} \cdot \mathbf{E}_n, \quad \text{and} \quad \bar{U}_{\bar{n}} = \mathbf{U} \cdot \bar{\mathbf{E}}_{\bar{n}}, \quad \text{etc.}$$

Normal space (Contravariant)

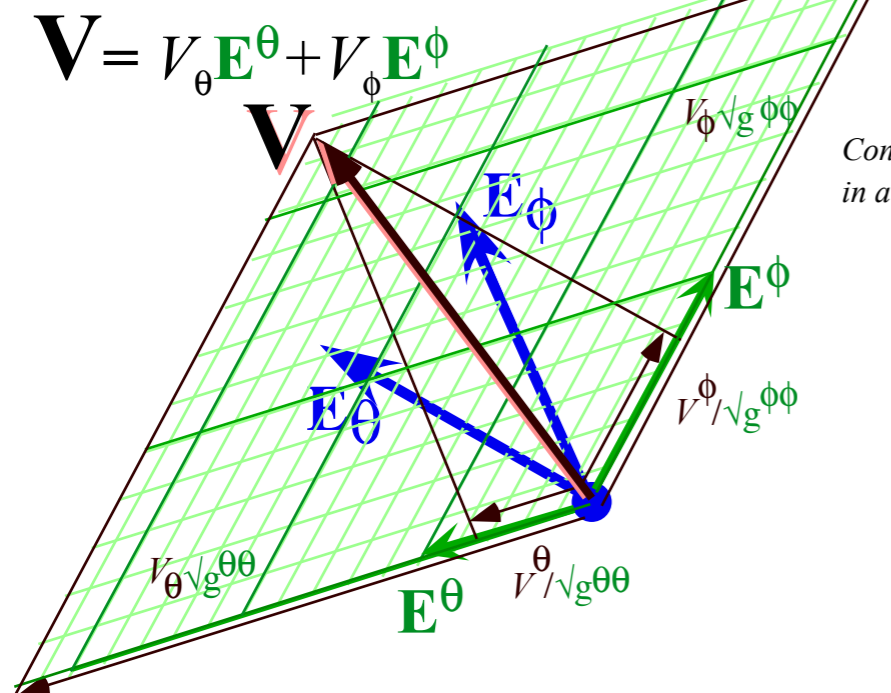


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

covariant projection

$$|\mathbf{V} \cdot \mathbf{E}_\theta| = \mathbf{V} \cdot \hat{\mathbf{E}}_\theta = \mathbf{V} \cdot \mathbf{E}_\theta / \sqrt{g_{\theta\theta}} = V_\theta / \sqrt{g_{\theta\theta}}$$

Tangent space (Covariant)

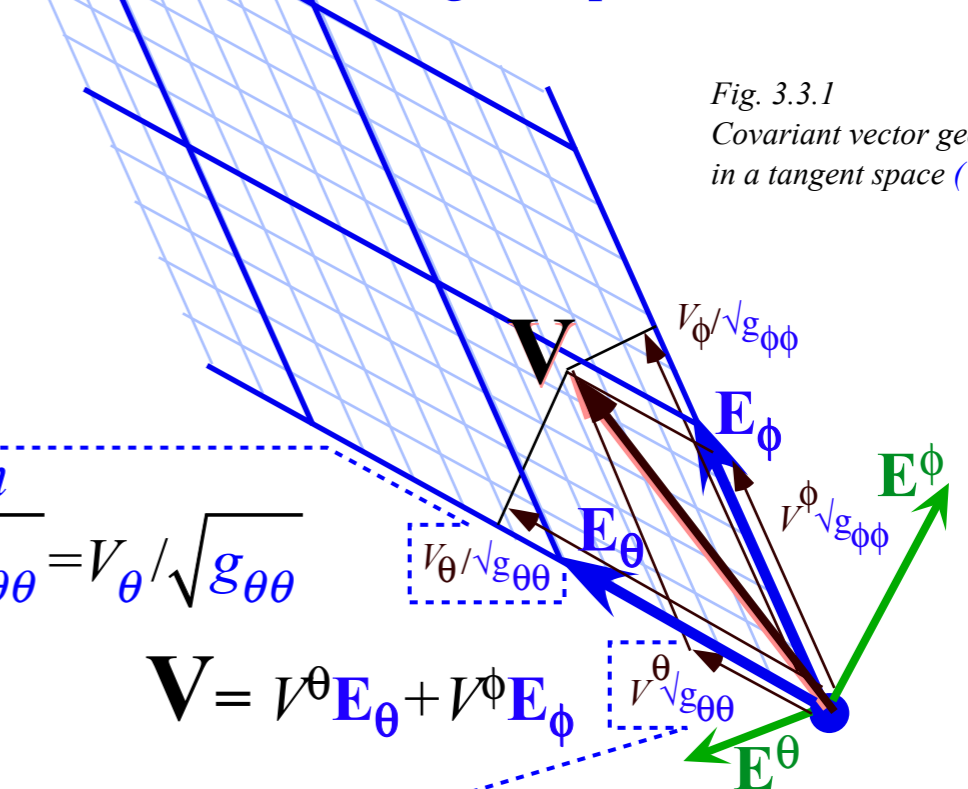


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

$$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$$

covariant parallelogram side

$$|V^\theta \mathbf{E}_\theta| = V^\theta \sqrt{\mathbf{E}_\theta \cdot \mathbf{E}_\theta} = V^\theta \sqrt{g_{\theta\theta}}$$

Contravariant vector \mathbf{E}^m is written in terms of covariant vectors \mathbf{E}_n

as would any vector $\mathbf{V} = V^n \mathbf{E}_n$ using dot product $V^n = \mathbf{V} \cdot \mathbf{E}^n$ and metric g_{mn} or $g^{mn} \dots$

$$\mathbf{E}^m = (\mathbf{E}^m)^n \mathbf{E}_n \quad \text{implies:} \quad (\mathbf{E}^m)^n = \mathbf{E}^m \cdot \mathbf{E}^n = g^{mn}$$

$$\text{so: } \mathbf{E}^m = g^{mn} \mathbf{E}_n$$

Contravariant vectors \mathbf{E}^m

versus

Covariant vectors \mathbf{E}_n

Any vector $\mathbf{U}, \mathbf{V}, \dots$ is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m \mathbf{E}_m = U_n \mathbf{E}^n = \bar{U}^{\bar{m}} \bar{\mathbf{E}}_{\bar{m}} = \bar{U}_{\bar{n}} \bar{\mathbf{E}}^{\bar{n}}$$

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where the U^m, V^m, \dots are **contravariant components**

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$$U^m = \mathbf{U} \cdot \mathbf{E}^m, V^m = \mathbf{V} \cdot \mathbf{E}^m, \text{ and } \bar{U}^{\bar{m}} = \mathbf{U} \cdot \bar{\mathbf{E}}^{\bar{m}}, \bar{V}^{\bar{m}} = \mathbf{V} \cdot \bar{\mathbf{E}}^{\bar{m}}, ,$$

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Normal space (Contravariant)

$$\mathbf{V} = V_\theta \mathbf{E}^\theta + V_\phi \mathbf{E}^\phi$$

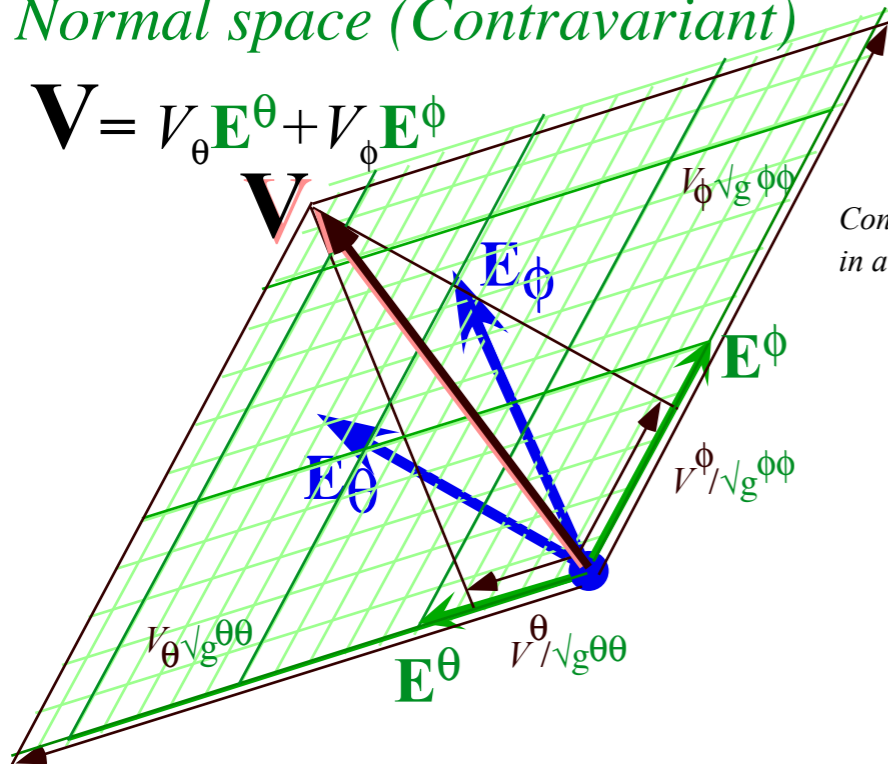


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

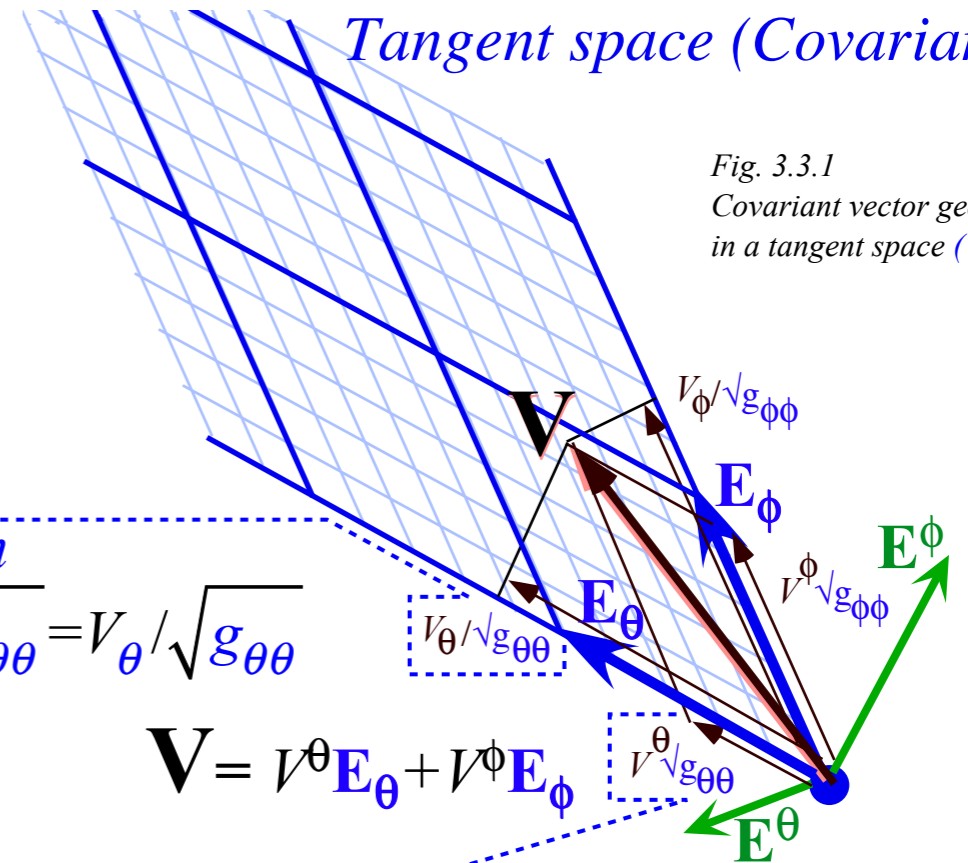


Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

covariant projection

$$|\mathbf{V} \cdot \mathbf{E}_\theta| = \mathbf{V} \cdot \hat{\mathbf{E}}_\theta = \mathbf{V} \cdot \mathbf{E}_\theta / \sqrt{g_{\theta\theta}} = V_\theta / \sqrt{g_{\theta\theta}}$$

$$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$$

covariant parallelogram side

$$|V^\theta \mathbf{E}_\theta| = V^\theta \sqrt{\mathbf{E}_\theta \cdot \mathbf{E}_\theta} = V^\theta \sqrt{g_{\theta\theta}}$$

Contravariant vector \mathbf{E}^m is written in terms of **covariant vectors \mathbf{E}_n**
as would any vector $\mathbf{V} = V^n \mathbf{E}_n$ using dot product $V^n = \mathbf{V} \cdot \mathbf{E}^n$ and metric g_{mn} or $g^{mn} \dots$

$$\mathbf{E}^m = (\mathbf{E}^m)^n \mathbf{E}_n \text{ implies: } (\mathbf{E}^m)^n = \mathbf{E}^m \cdot \mathbf{E}^n = g^{mn}$$

so: $\mathbf{E}^m = g^{mn} \mathbf{E}_n$

and: $\mathbf{E}_n = g_{mn} \mathbf{E}^m$...the same for **covariant vectors**

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

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Normal space (Contravariant)

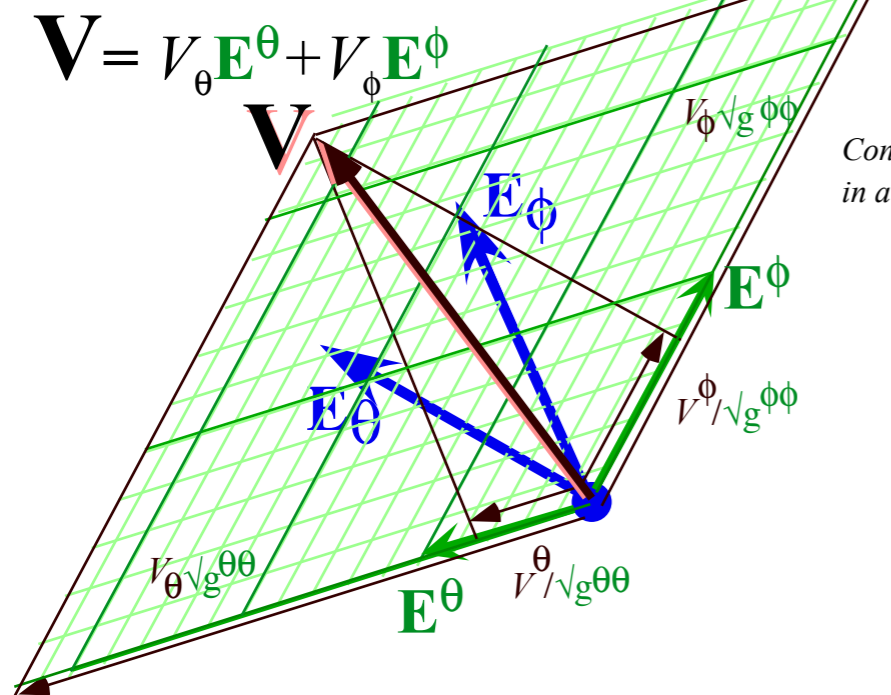


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Tangent space (Covariant)

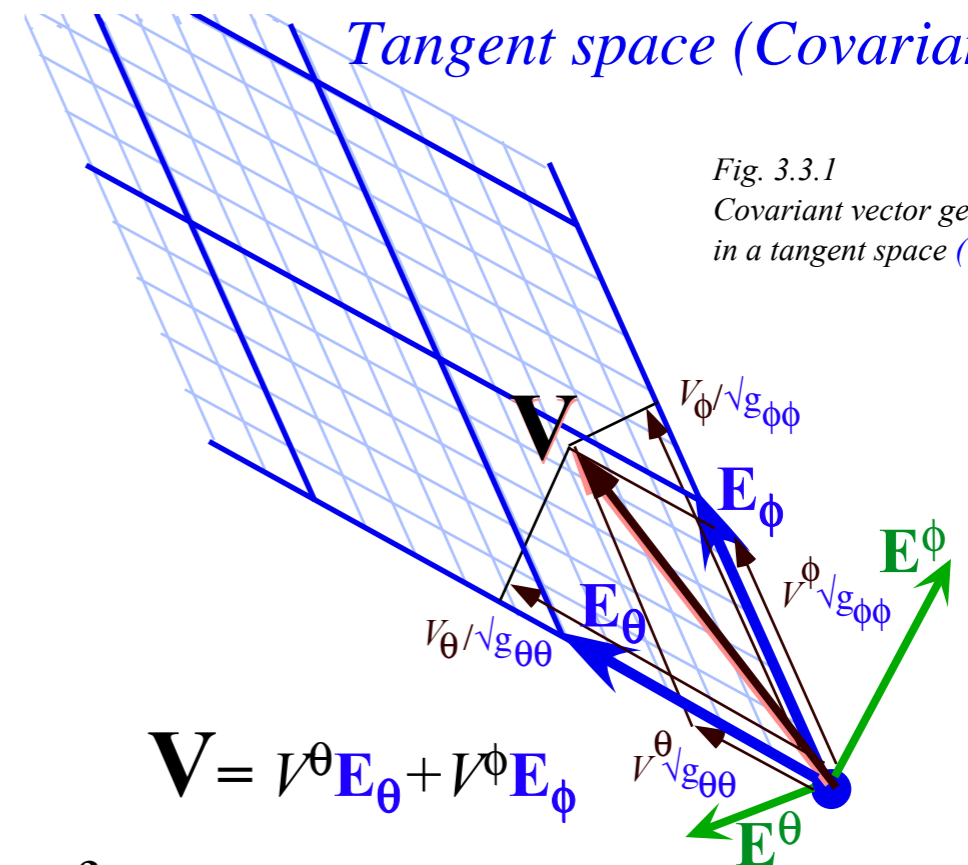


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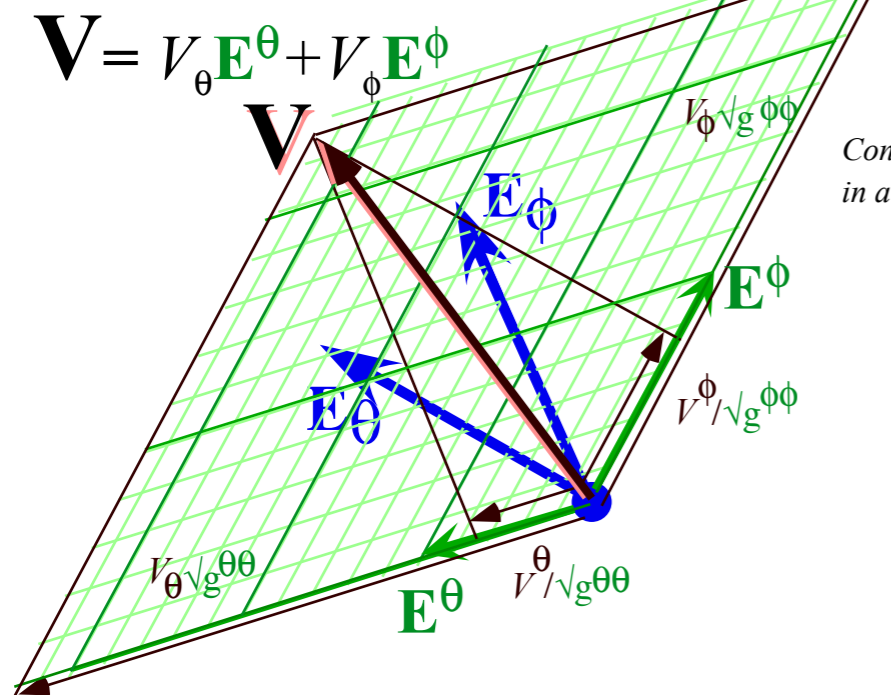


Fig. 3.3.2
Contravariant vector geometry
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Tangent space (Covariant)

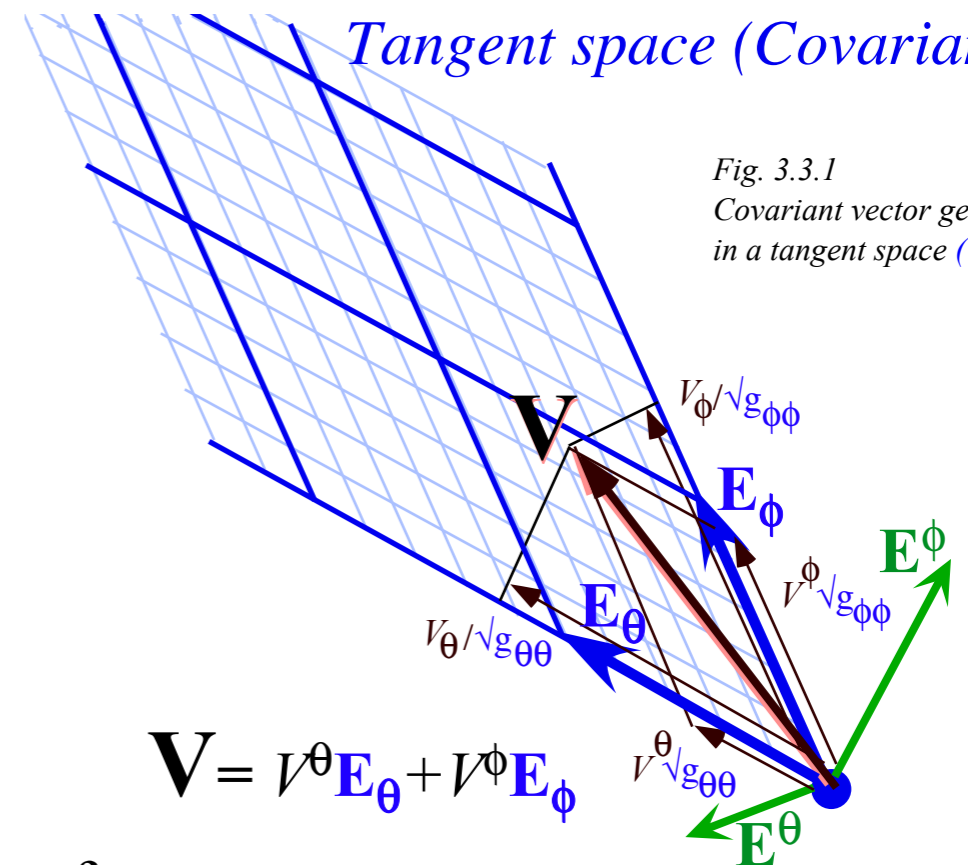


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$$\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}, \quad \text{or:} \quad \boxed{\mathbf{E}_m = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \bar{\mathbf{E}}_{\bar{m}}}$$

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Normal space (Contravariant)

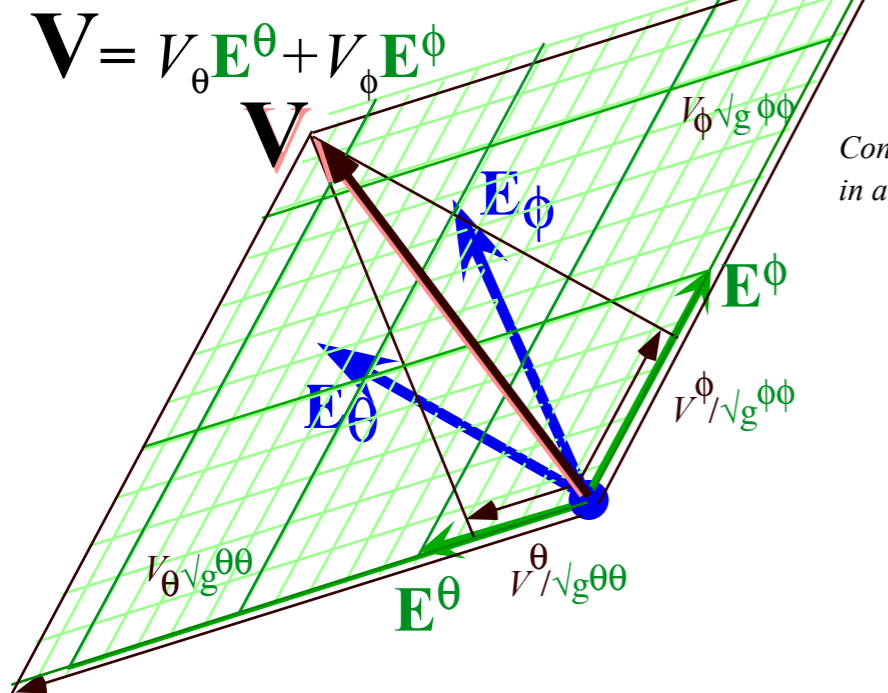


Fig. 3.3.2
Contravariant vector geometry
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Tangent space (Covariant)

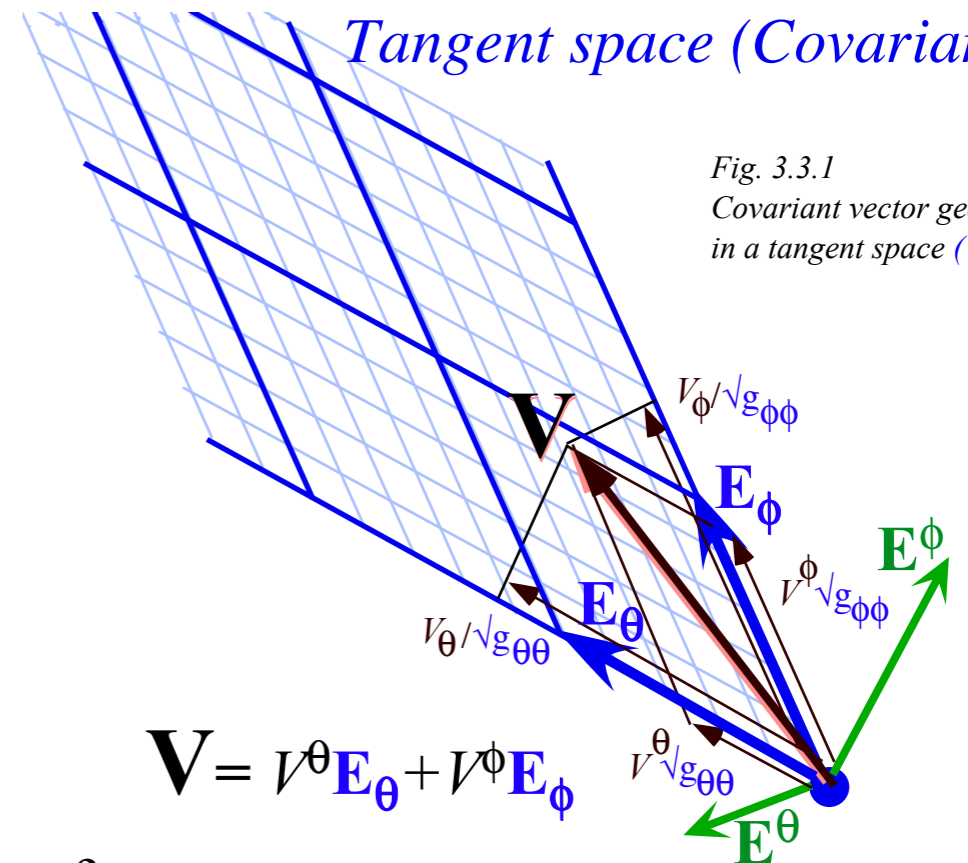


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Normal space (Contravariant)

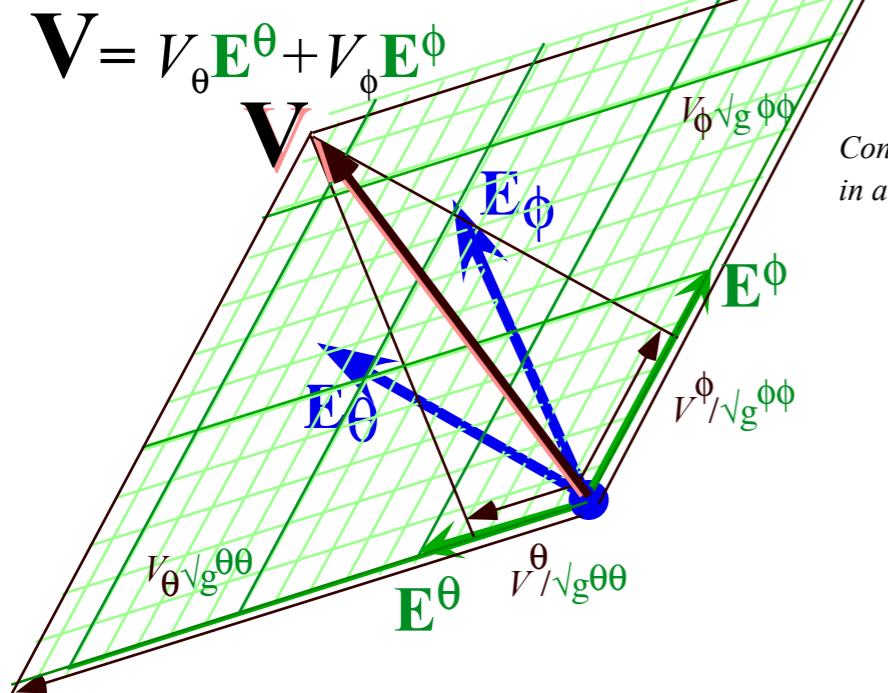


Fig. 3.3.2
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Tangent space (Covariant)

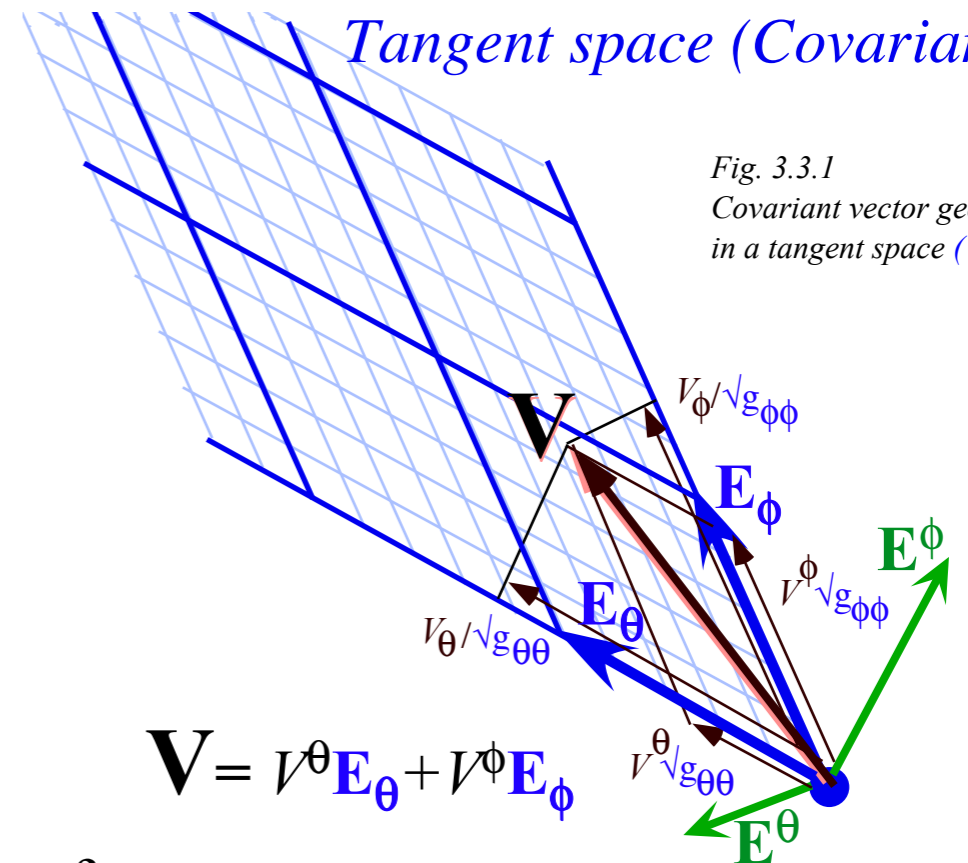


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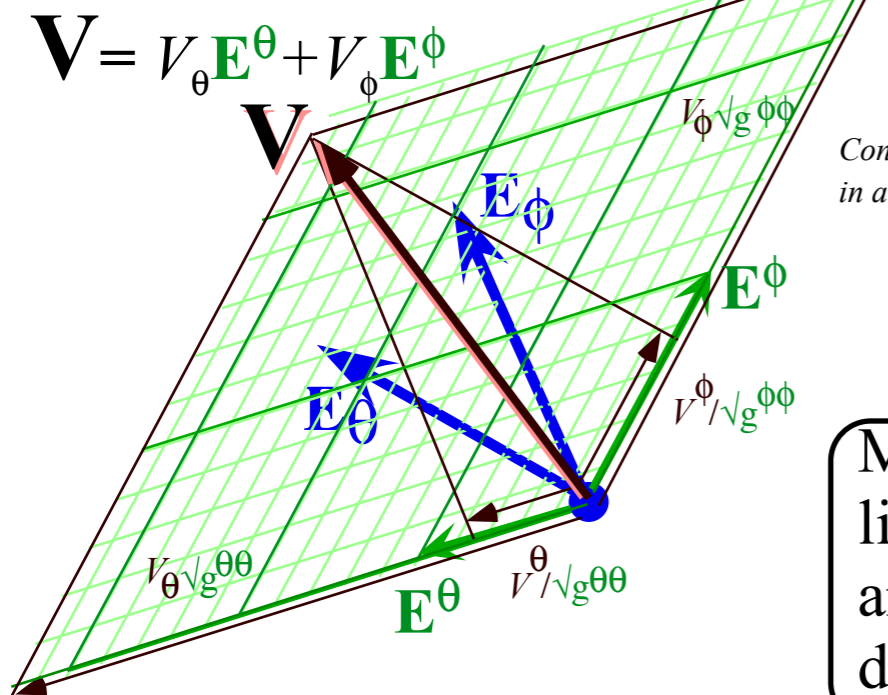


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Metric relations
like: $\mathbf{E}^m = g^{mn} \mathbf{E}_n$
and: $\mathbf{E}_n = g_{mn} \mathbf{E}^m$
don't exist for “ $\langle \text{bra-kets} \rangle$ ”

Tangent space (Covariant)

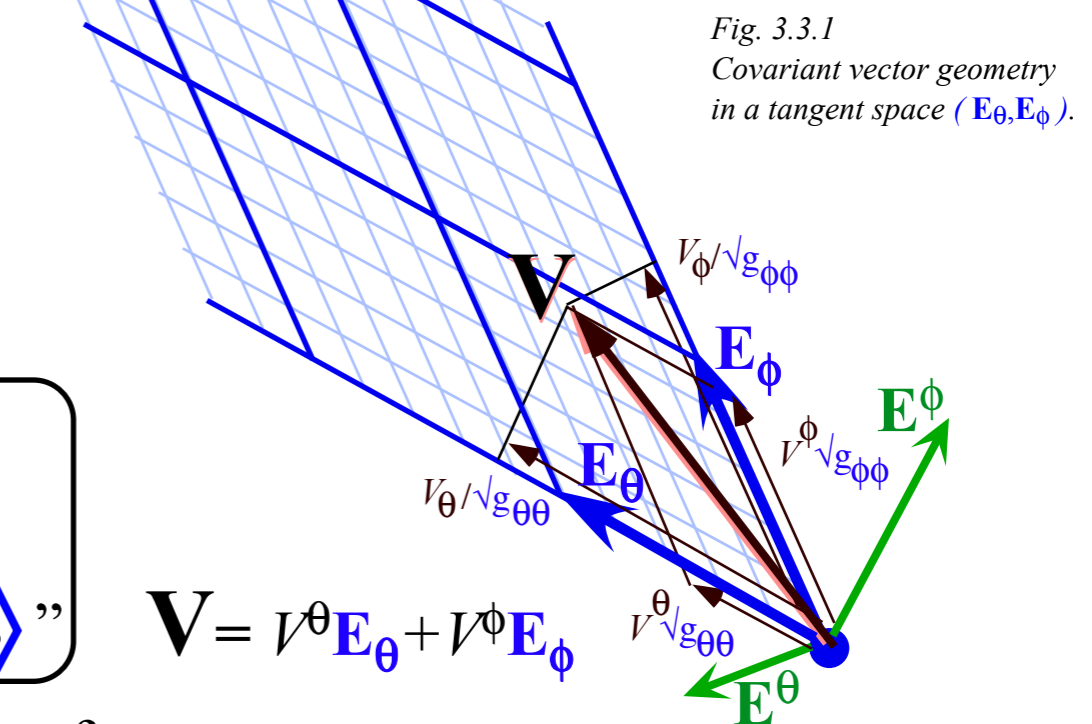


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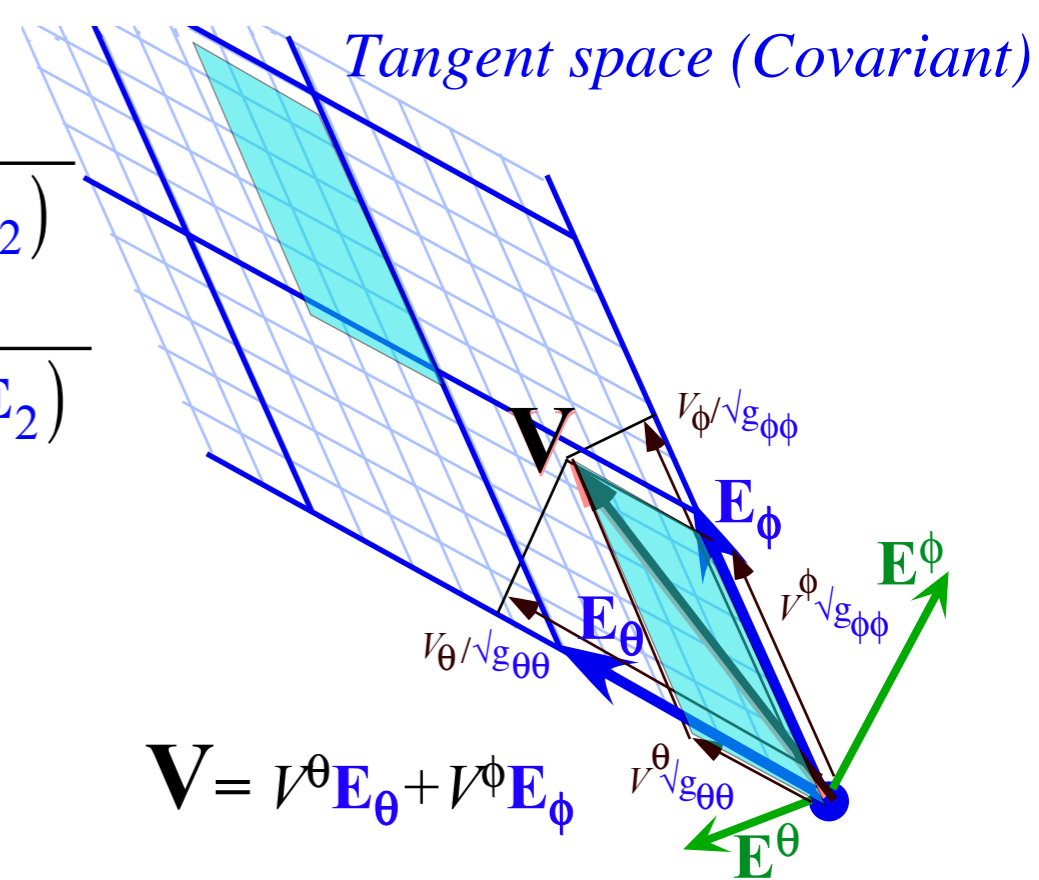
Riemann equation force analysis

2nd-guessing Riemann equation?

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2|\mathbf{E}_1 \times \mathbf{E}_2| = V^1V^2\sqrt{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}$$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2\sqrt{(\mathbf{E}_1 \cdot \mathbf{E}_1)(\mathbf{E}_2 \cdot \mathbf{E}_2) - (\mathbf{E}_1 \cdot \mathbf{E}_2)^2}$$



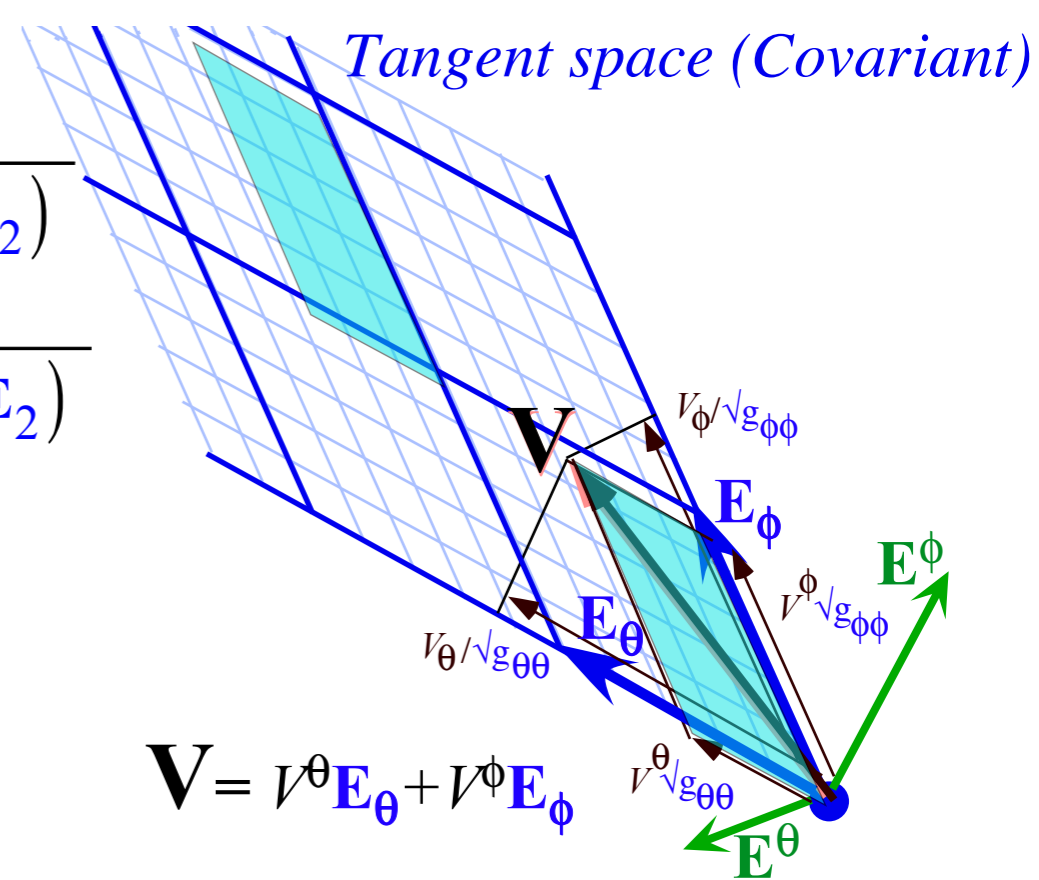
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Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2|\mathbf{E}_1 \times \mathbf{E}_2| = V^1V^2\sqrt{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}$$

$$\begin{aligned} \text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) &= V^1V^2\sqrt{(\mathbf{E}_1 \cdot \mathbf{E}_1)(\mathbf{E}_2 \cdot \mathbf{E}_2) - (\mathbf{E}_1 \cdot \mathbf{E}_2)(\mathbf{E}_1 \cdot \mathbf{E}_2)} \\ &= V^1V^2\sqrt{g_{11}g_{22} - g_{12}g_{21}} \end{aligned}$$

where: $g_{12} = \mathbf{E}_1 \cdot \mathbf{E}_2 = g_{21}$



Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

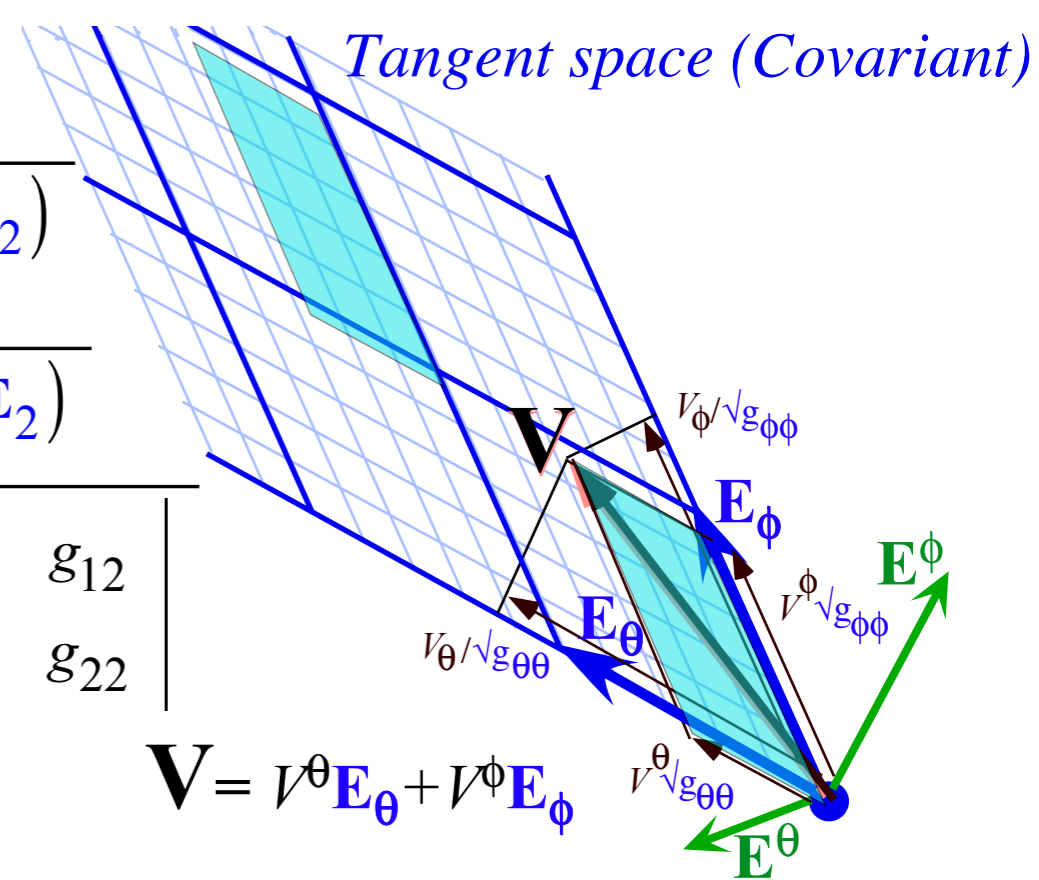
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2|\mathbf{E}_1 \times \mathbf{E}_2| = V^1V^2\sqrt{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}$$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2\sqrt{(\mathbf{E}_1 \cdot \mathbf{E}_1)(\mathbf{E}_2 \cdot \mathbf{E}_2) - (\mathbf{E}_1 \cdot \mathbf{E}_2)(\mathbf{E}_1 \cdot \mathbf{E}_2)}$$

$$= V^1V^2\sqrt{g_{11}g_{22} - g_{12}g_{21}} = V^1V^2\sqrt{\det \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1 \cdot \mathbf{E}_2 = g_{21}$



$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

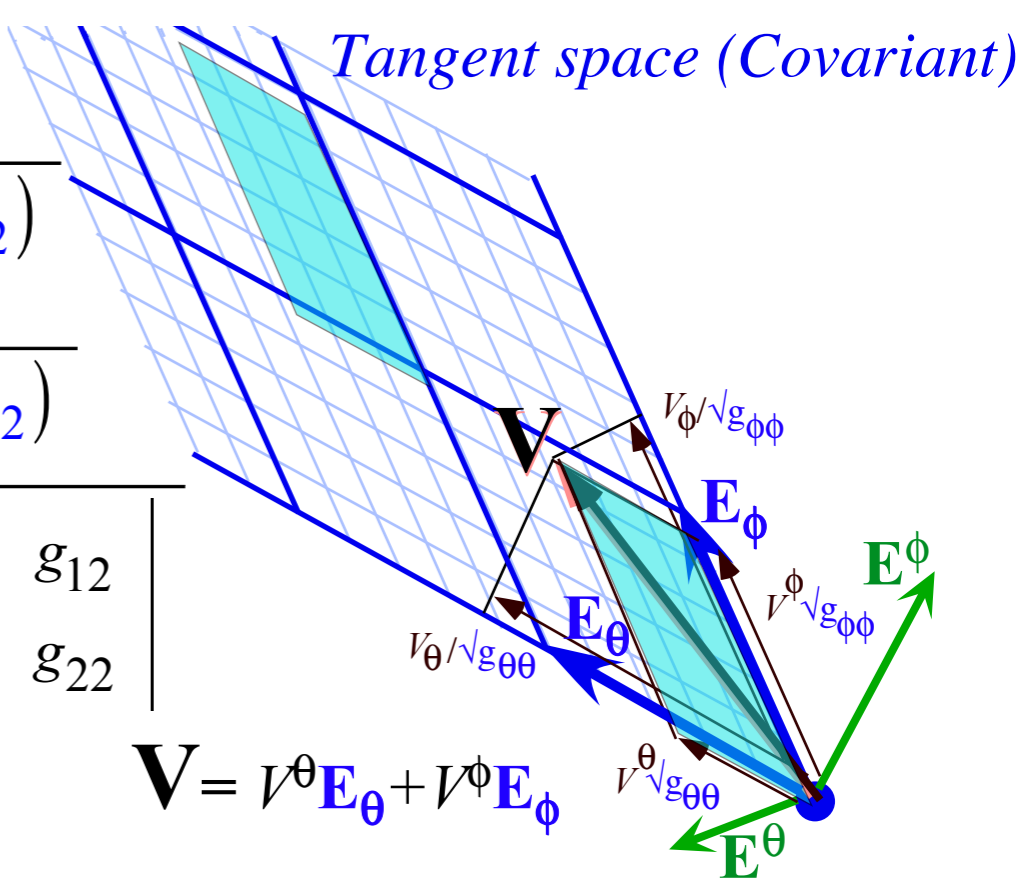
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2|\mathbf{E}_1 \times \mathbf{E}_2| = V^1V^2\sqrt{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}$$

$$\text{Area}(V^1\mathbf{E}_1, V^2\mathbf{E}_2) = V^1V^2\sqrt{(\mathbf{E}_1 \cdot \mathbf{E}_1)(\mathbf{E}_2 \cdot \mathbf{E}_2) - (\mathbf{E}_1 \cdot \mathbf{E}_2)(\mathbf{E}_1 \cdot \mathbf{E}_2)}$$

$$= V^1V^2\sqrt{g_{11}g_{22} - g_{12}g_{21}} = V^1V^2\sqrt{\det \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1 \cdot \mathbf{E}_2 = g_{21}$

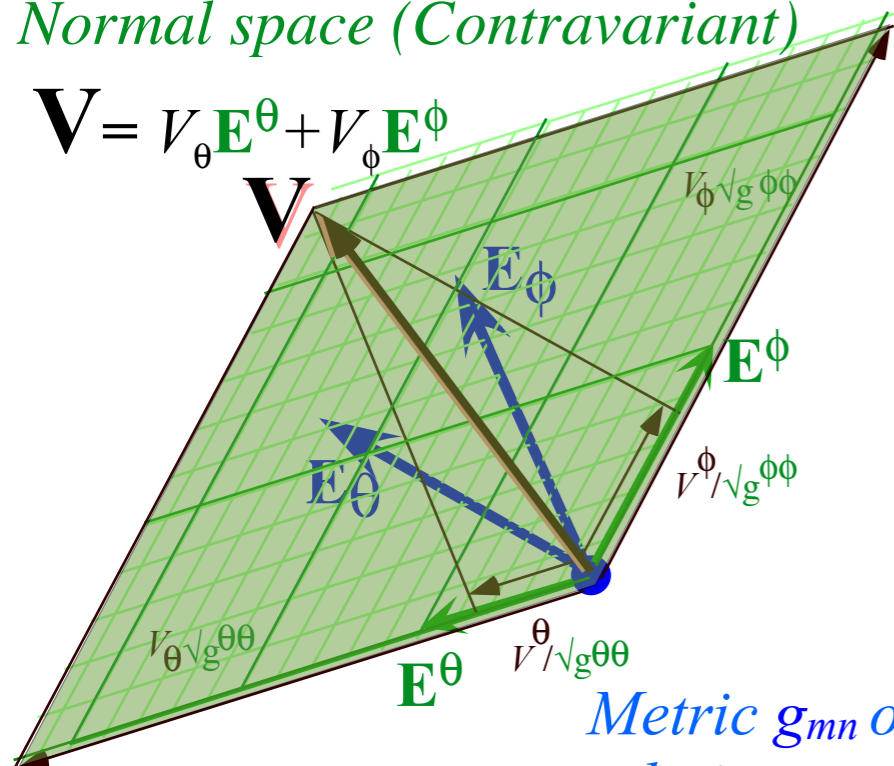


$$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$$

Normal space (Contravariant) area spanned by $V_1\mathbf{E}^1$ and $V_2\mathbf{E}^2$

Normal space (Contravariant)

$$\mathbf{V} = V_\theta \mathbf{E}^\theta + V_\phi \mathbf{E}^\phi$$



$$\text{Area}(V_1\mathbf{E}^1, V_2\mathbf{E}^2) = V_1V_2|\mathbf{E}^1 \times \mathbf{E}^2| = V_1V_2\sqrt{(\mathbf{E}^1 \times \mathbf{E}^2) \cdot (\mathbf{E}^1 \times \mathbf{E}^2)}$$

$$\text{Area}(V_1\mathbf{E}^1, V_2\mathbf{E}^2) = V_1V_2\sqrt{(\mathbf{E}^1 \cdot \mathbf{E}^1)(\mathbf{E}^2 \cdot \mathbf{E}^2) - (\mathbf{E}^1 \cdot \mathbf{E}^2)(\mathbf{E}^1 \cdot \mathbf{E}^2)}$$

$$= V_1V_2\sqrt{g^{11}g^{22} - g^{12}g^{21}} = V_1V_2\sqrt{\det \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix}}$$

where: $g^{12} = \mathbf{E}^1 \cdot \mathbf{E}^2 = g^{21}$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

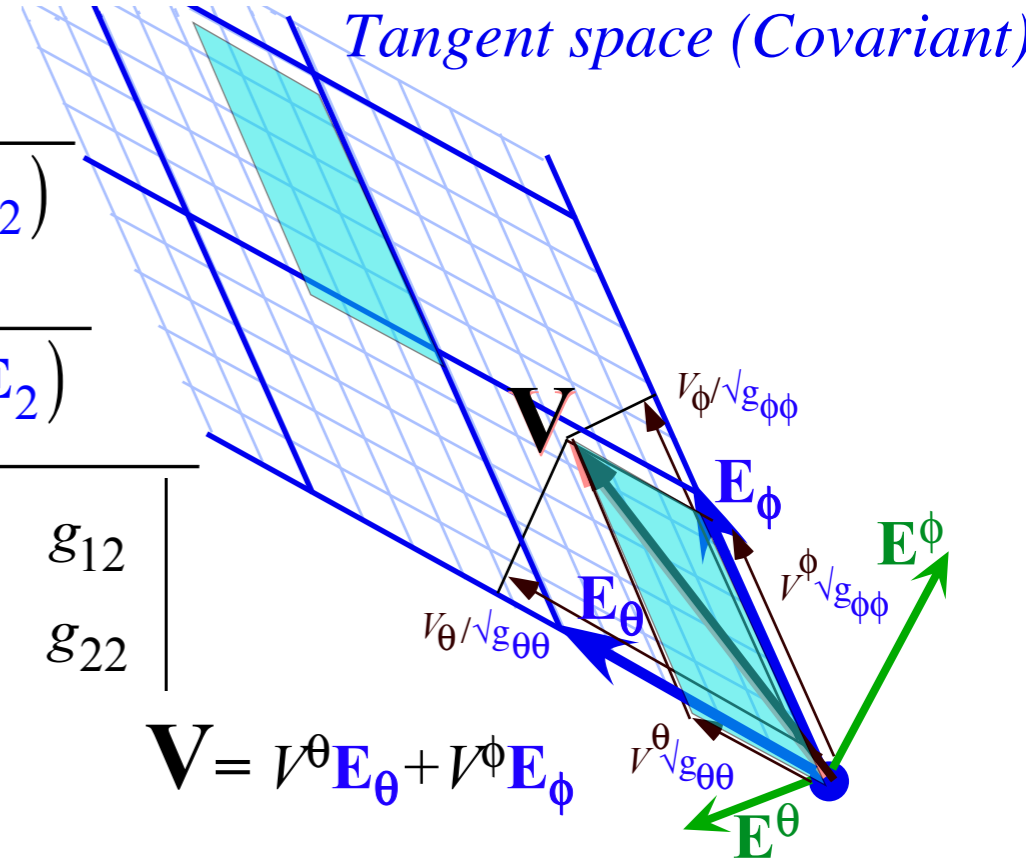
$$\text{Area}(V^1 \mathbf{E}_1, V^2 \mathbf{E}_2) = V^1 V^2 |\mathbf{E}_1 \times \mathbf{E}_2| = V^1 V^2 \sqrt{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}$$

$$\text{Area}(V^1 \mathbf{E}_1, V^2 \mathbf{E}_2) = V^1 V^2 \sqrt{(\mathbf{E}_1 \cdot \mathbf{E}_1)(\mathbf{E}_2 \cdot \mathbf{E}_2) - (\mathbf{E}_1 \cdot \mathbf{E}_2)(\mathbf{E}_1 \cdot \mathbf{E}_2)}$$

$$= V^1 V^2 \sqrt{g_{11}g_{22} - g_{12}g_{21}} = V^1 V^2 \sqrt{\det \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1 \cdot \mathbf{E}_2 = g_{21}$

$$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$$



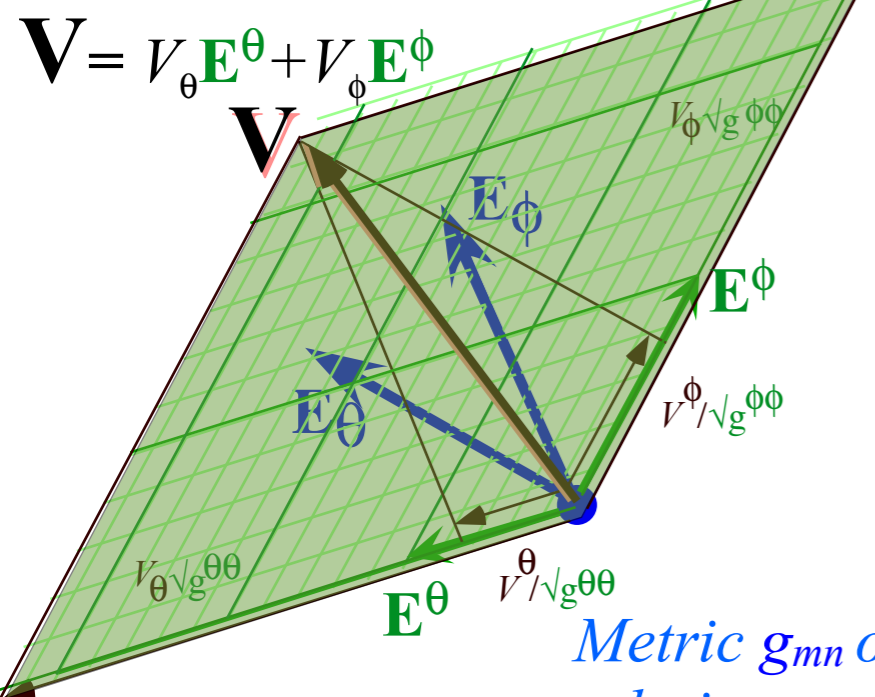
Determinant product rule: $\det|\mathbf{g}_{cov}| \cdot \det|\mathbf{g}^{cont}| = 1$ since $(\mathbf{g}_{cov})^{-1} = \mathbf{g}^{cont}$ or :

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(Recall: $\mathbf{E}_n \cdot \mathbf{E}^n = 1$)

$(\mathbf{g}_{cov}) \cdot (\mathbf{g}^{cont}) = (\mathbf{1})$

Normal space (Contravariant)



$$\text{Area}(V_1 \mathbf{E}^1, V_2 \mathbf{E}^2) = V_1 V_2 |\mathbf{E}^1 \times \mathbf{E}^2| = V_1 V_2 \sqrt{(\mathbf{E}^1 \times \mathbf{E}^2) \cdot (\mathbf{E}^1 \times \mathbf{E}^2)}$$

$$\text{Area}(V_1 \mathbf{E}^1, V_2 \mathbf{E}^2) = V_1 V_2 \sqrt{(\mathbf{E}^1 \cdot \mathbf{E}^1)(\mathbf{E}^2 \cdot \mathbf{E}^2) - (\mathbf{E}^1 \cdot \mathbf{E}^2)(\mathbf{E}^1 \cdot \mathbf{E}^2)}$$

$$= V_1 V_2 \sqrt{g^{11}g^{22} - g^{12}g^{21}} = V_1 V_2 \sqrt{\det \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12} = \mathbf{E}^1 \cdot \mathbf{E}^2 = g^{21}$

3D Covariant Jacobian determinant J -columns are \mathbf{E}_1 , \mathbf{E}_2 and \mathbf{E}_3 .

$$Volume(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3) = V^1V^2V^3 |\mathbf{E}_1 \times \mathbf{E}_2 \cdot \mathbf{E}_3| = V^1V^2V^3 \det \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J -columns are \mathbf{E}_1 , \mathbf{E}_2 and \mathbf{E}_3 .

$$\text{Volume}(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3) = V^1V^2V^3 |\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3| = V^1V^2V^3 \det \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J -matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J -columns are \mathbf{E}_1 , \mathbf{E}_2 and \mathbf{E}_3 .

$$Volume(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3) = V^1V^2V^3 |\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3| = V^1V^2V^3 \det \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J -matrix and its transpose J^T

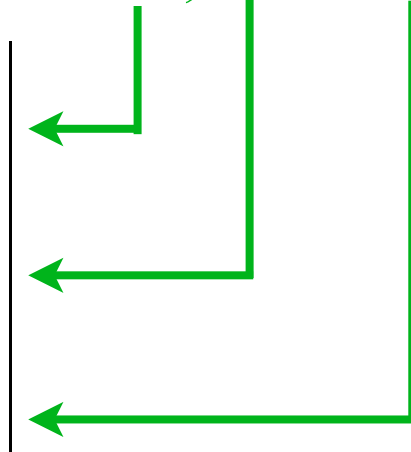
$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Then determinant product ($\det|A| \det|B| = \det|A \bullet B|$) and symmetry ($\det|A^T| = \det|A|$) gives:

$$Volume(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3) = V^1V^2V^3 \det|J| = V^1V^2V^3 \sqrt{\det|\mathbf{g}_{cov}|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Contravariant Kjobian determinant K -rows are \mathbf{E}^1 , \mathbf{E}^2 and \mathbf{E}^3 .

$$Volume(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3) = V_1V_2V_3 |\mathbf{E}^1 \times \mathbf{E}^2 \bullet \mathbf{E}^3| = V_1V_2V_3 \det \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{pmatrix}$$


Contravariant metric matrix is product of K -matrix and its transpose K^T

$$\mathbf{g}^{cont} \equiv \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^2}{\partial x^1} & \frac{\partial q^3}{\partial x^1} \\ \frac{\partial q^1}{\partial x^2} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^3}{\partial x^2} \\ \frac{\partial q^1}{\partial x^3} & \frac{\partial q^2}{\partial x^3} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} = K \bullet K^T$$

Then determinant product ($\det|A| \det|B| = \det|A \bullet B|$) and symmetry ($\det|A^T| = \det|A|$) gives:

$$Volume(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3) = V_1V_2V_3 \det|K| = V_1V_2V_3 \sqrt{\det|\mathbf{g}^{cont}|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued*

Toroidal “rolled-up” ($q^1=\theta$, $q^2=\phi$)-manifold and “Flat” ($x=\theta$, $y=\phi$)-graph

Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors: Jacobian J vs. Kadjobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

 *Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)*

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Canonical momentum and γ_{mn} tensor Review of p_θ, p_ϕ vs γ_{mn} from p. 77 of Lect. 14

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2} \left[(MR^2 + mr^2) \dot{\theta}^2 - 2mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + ml^2 \dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2) \dot{\theta} - mrl \dot{\phi} \cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mrl \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right)$$

$$= ml^2 \dot{\phi} - mrl \dot{\theta} \cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

where: γ_{mn} tensor is $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix}$

Momentum γ_{mn} -matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mrl \cos(\theta - \phi) \\ -mrl \cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn} -tensor theorem: (proof here)

$$p_m = \gamma_{mn} \dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m} \frac{1}{2} \gamma_{jk} \dot{q}^j \dot{q}^k = \frac{1}{2} \gamma_{jk} \frac{\partial \dot{q}^j}{\partial \dot{q}^m} \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2} \gamma_{jk} \delta_m^j \dot{q}^k + \frac{1}{2} \gamma_{jk} \dot{q}^j \delta_m^k = \frac{1}{2} \gamma_{mk} \dot{q}^k + \frac{1}{2} \gamma_{jm} \dot{q}^j$$

$$= \gamma_{mn} \dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm} \quad \text{QED}$$

Lagrange equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt} \left((MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi) \right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

*p-dot part of
Lagrange
2nd equations*

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt} \left(m\ell^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi) \right)$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} m\ell^2 \dot{\phi}^2 \right)$$
$$= (MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} m\ell^2 \dot{\phi}^2 \right)$$
$$= m\ell^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi)$$

*From preceding
Lagrange
1st equations*

Lagrange equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\begin{aligned} \dot{p}_\theta &= \frac{d}{dt} p_\theta = \frac{d}{dt} \left((MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi) \right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}] \\ &= (MR^2 + mr^2) \ddot{\theta} - mr\ell \ddot{\phi} \cos(\theta - \phi) + mr\ell \dot{\phi} (\dot{\theta} - \dot{\phi}) \sin(\theta - \phi) \end{aligned}$$

*p-dot part of
Lagrange
2nd equations*

$$\begin{aligned} \dot{p}_\phi &= \frac{d}{dt} p_\phi = \frac{d}{dt} \left(ml^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi) \right) \\ &= ml^2 \ddot{\phi} - mr\ell \ddot{\theta} \cos(\theta - \phi) + mr\ell \dot{\theta} (\dot{\theta} - \dot{\phi}) \sin(\theta - \phi) \end{aligned}$$

$$\begin{aligned} p_\theta &= \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right) \\ &= (MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi) \end{aligned}$$

$$\begin{aligned} p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 - mr\ell \cos(\theta - \phi) \dot{\theta} \dot{\phi} + \frac{1}{2} ml^2 \dot{\phi}^2 \right) \\ &= ml^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi) \end{aligned}$$

*From preceding
Lagrange
1st equations*

Lagrange equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

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*p-dot part of
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$$= (MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi)$$

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$$= ml^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi)$$

*From preceding
Lagrange
1st equations*

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
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Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

 *Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)*

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Force, Work, and Acceleration

$$dW = F_X dX + F_Y dY + F_x dx + F_y dy$$

Review of F_θ, F_ϕ vs F_x, F_y, F_X, F_Y from p. 69 of Lect. 14

$$= M\ddot{X}dX + M\ddot{Y}dY + m\ddot{x}dx + m\ddot{y}dy$$

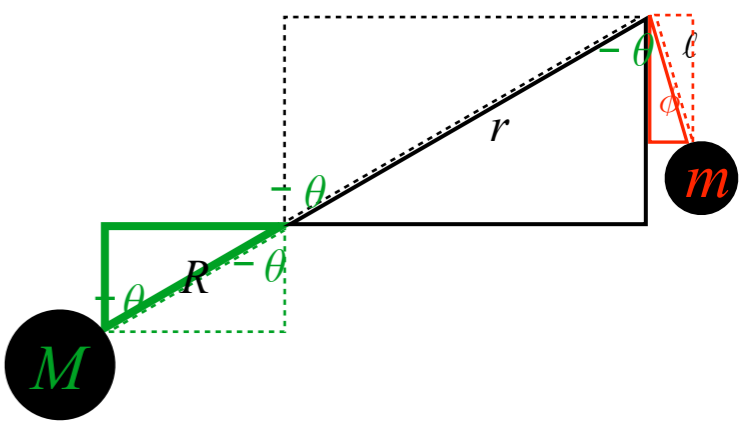
Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X dX = M\ddot{X}dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y dY + M\ddot{Y}dY + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x dx + m\ddot{x}dx + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y dy + m\ddot{y}dy + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$



Lagrange trickery:

STEP D

Add up first and last columns for each variable θ and ϕ for:

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

Let: $F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$ (Defines F_θ)

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Let: $F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$ (Defines F_ϕ)

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R \cos \theta + F_Y R \sin \theta - F_x r \cos \theta - F_y r \sin \theta$$

$$\equiv F_\theta = \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Add F_θ gravity given
 $(F_X=0, F_Y=-Mg)$
 $(F_x=0, F_y=-mg)$

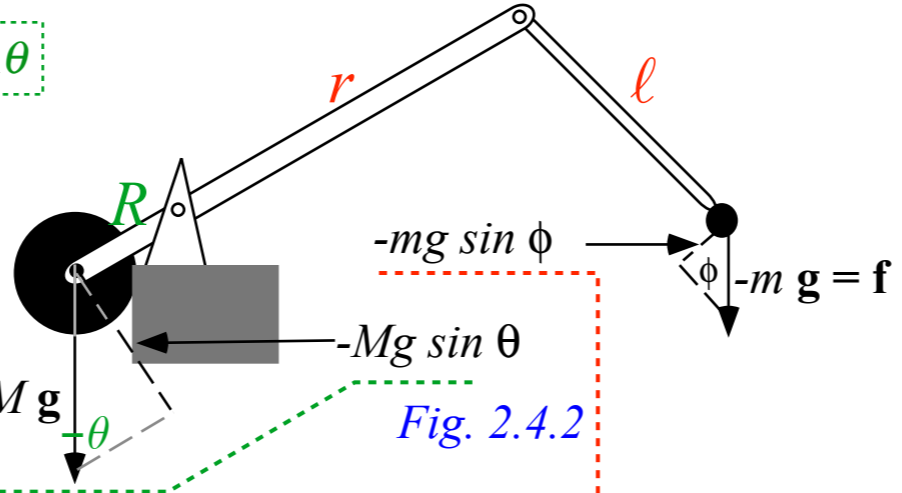


Fig. 2.4.2

Q: Are there \pm sign errors here?
 A: No. Beam in $-\theta$ position.

$$F_X \cdot 0 + F_Y \cdot 0 + F_x l \cos \phi + F_y l \sin \phi$$

$$\equiv F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Add F_ϕ gravity given
 $(F_X=0, F_Y=-Mg)$
 $(F_x=0, F_y=-mg)$

$$F_\phi = \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = -mgl \sin \phi$$

These are competing torques on main beam R...

... and a torque on throwing lever l

Lagrange equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

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$$F_\theta = -MgR \sin \theta + mgr \sin \theta$$

$$F_\phi = -mgl \sin \phi$$

Lagrange equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt} \left((MR^2 + mr^2) \dot{\theta} - mr\ell \dot{\phi} \cos(\theta - \phi) \right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

$$= (MR^2 + mr^2) \ddot{\theta} - mr\ell \ddot{\phi} \cos(\theta - \phi) + mr\ell \dot{\phi} (\dot{\theta} - \dot{\phi}) \sin(\theta - \phi)$$

$$= (MR^2 + mr^2) \ddot{\theta} - mr\ell \ddot{\phi} \cos(\theta - \phi) - mr\ell \dot{\phi}^2 \sin(\theta - \phi) = F_\theta = -MgR \sin \theta + mgr \sin \theta$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt} \left(ml^2 \dot{\phi} - mr\ell \dot{\theta} \cos(\theta - \phi) \right)$$

$$= ml^2 \ddot{\phi} - mr\ell \ddot{\theta} \cos(\theta - \phi) + mr\ell \dot{\theta} (\dot{\theta} - \dot{\phi}) \sin(\theta - \phi)$$

$$= ml^2 \ddot{\phi} - mr\ell \ddot{\theta} \cos(\theta - \phi) + mr\ell \dot{\theta}^2 \sin(\theta - \phi) = F_\phi = -mgl \sin \phi$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T / \partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 + \frac{1}{2} ml^2 \dot{\phi}^2 - mr\ell \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right)$$

$$= F_\theta + mr\ell \dot{\theta} \dot{\phi} \sin(\theta - \phi)$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi} \left(\frac{1}{2} (MR^2 + mr^2) \dot{\theta}^2 + \frac{1}{2} ml^2 \dot{\phi}^2 - mr\ell \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right)$$

$$= F_\phi - mr\ell \dot{\theta} \dot{\phi} \sin(\theta - \phi)$$

gravity forces F_μ from p.69 of Lect. 14 (or p.63 above)

$$F_\theta = -MgR \sin \theta + mgr \sin \theta$$

$$F_\phi = -mgl \sin \phi$$

Lagrange equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

$$\dot{p}_\theta = \left(MR^2 + mr^2 \right) \ddot{\theta} - mr\ell \ddot{\phi} \cos(\theta - \phi) - mr\ell \dot{\phi}^2 \sin(\theta - \phi) = F_\theta = -MgR \sin \theta + mgr \sin \theta$$

$$\dot{p}_\phi = m\ell^2 \ddot{\phi} - mr\ell \ddot{\theta} \cos(\theta - \phi) + mr\ell \dot{\theta}^2 \sin(\theta - \phi) = F_\phi = -mg\ell \sin \phi$$

*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued*

Toroidal “rolled-up” ($q^1=\theta$, $q^2=\phi$)-manifold and “Flat” ($x=\theta$, $y=\phi$)-graph

Review of covariant \mathbf{E}_n and contravariant \mathbf{E}^m vectors: Jacobian J vs. Kadjobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$ space vs. Normal $\{\mathbf{E}^m\}$ space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

 *Riemann equation derivation for trebuchet model*

Riemann equation force analysis

2nd-guessing Riemann equation?

Riemann equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ and $\ddot{\phi}$

$$\dot{p}_\theta = \left(MR^2 + mr^2 \right) \ddot{\theta} - mr\ell \ddot{\phi} \cos(\theta - \phi) - mr\ell \dot{\phi}^2 \sin(\theta - \phi) = F_\theta = -MgR \sin \theta + mgr \sin \theta$$

$$\dot{p}_\phi = m\ell^2 \ddot{\phi} - mr\ell \ddot{\theta} \cos(\theta - \phi) + mr\ell \dot{\theta}^2 \sin(\theta - \phi) = F_\phi = -mg\ell \sin \phi$$

Riemann equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ and $\ddot{\phi}$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta - \phi) \\ -mr\ell\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ and $\ddot{\phi}$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta - \phi) \\ -mr\ell\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ and $\ddot{\phi}$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta - \phi) \\ -mr\ell\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor:

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix}$$

Need to invert the γ_{mn} -matrix... Let's consolidate ...

Riemann equation force analysis

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mrl\ddot{\phi}\cos(\theta - \phi) - mrl\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = ml^2\ddot{\phi} - mrl\ddot{\theta}\cos(\theta - \phi) + mrl\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mrl\dot{\phi}^2\sin(\theta - \phi) \\ -mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor:

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix}$$

Need to invert the γ_{mn} -matrix...

Riemann equation force analysis $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mrl\ddot{\phi}\cos(\theta - \phi) - mrl\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = ml^2\ddot{\phi} - mrl\ddot{\theta}\cos(\theta - \phi) + mrl\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mrl\dot{\phi}^2\sin(\theta - \phi) \\ -mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor:

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix}$$

Need to invert the γ_{mn} -matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} ml^2 & mrl\cos(\theta - \phi) \\ mrl\cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix}}{ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} \leftarrow \begin{matrix} \text{"Super-Inertia"} \\ I_S \end{matrix}$$

Riemann equation force analysis $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu} \dot{p}_\mu = \ddot{q}^\nu \dots$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mrl\ddot{\phi}\cos(\theta - \phi) - mrl\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = ml^2\ddot{\phi} - mrl\ddot{\theta}\cos(\theta - \phi) + mrl\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mrl\dot{\phi}^2\sin(\theta - \phi) \\ -mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor:

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix}$$

Need to invert the γ_{mn} -matrix...

... and apply it...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} ml^2 & mrl\cos(\theta - \phi) \\ mrl\cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix}}{ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} \leftarrow \text{“Super-Inertia” } I_S$$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} \quad \text{Riemann equation form}$$

Riemann equation force analysis $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu} \dot{p}_\mu = \ddot{q}^\nu \dots$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mrl\ddot{\phi}\cos(\theta - \phi) - mrl\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = ml^2\ddot{\phi} - mrl\ddot{\theta}\cos(\theta - \phi) + mrl\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mrl\dot{\phi}^2\sin(\theta - \phi) \\ -mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor:

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix}$$

Need to invert the γ_{mn} -matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} ml^2 & mrl\cos(\theta - \phi) \\ mrl\cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix}}{ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} \leftarrow \text{“Super-Inertia” } I_S$$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} \quad \text{Riemann equation form}$$

Gravity-free case:

$$F_\theta = 0 = F_\phi \quad I_S \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mrl\sin(\theta - \phi)$$

Riemann equation force analysis $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu} \dot{p}_\mu = \ddot{q}^\nu \dots$

$$\dot{p}_\theta = (MR^2 + mr^2)\ddot{\theta} - mrl\ddot{\phi}\cos(\theta - \phi) - mrl\dot{\phi}^2\sin(\theta - \phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = ml^2\ddot{\phi} - mrl\ddot{\theta}\cos(\theta - \phi) + mrl\dot{\theta}^2\sin(\theta - \phi) = F_\phi = -mgl\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} (MR^2 + mr^2) & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mrl\dot{\phi}^2\sin(\theta - \phi) \\ -mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor:

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mrl\cos(\theta - \phi) \\ -mrl\cos(\theta - \phi) & ml^2 \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mgl\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix}$$

Need to invert the γ_{mn} -matrix...

$$I_s = ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)] \quad \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} ml^2 & mrl\cos(\theta - \phi) \\ mrl\cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix}}{ml^2 [MR^2 + mr^2 \sin^2(\theta - \phi)]} \leftarrow \text{“Super-Inertia” } I_s$$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mrl\dot{\phi}^2\sin(\theta - \phi) \\ F_\phi - mrl\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} \quad \text{Riemann equation form}$$

Gravity-free case:

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*Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
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2nd-guessing Riemann equation?

Riemann equation force analysis $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu} \dot{p}_\mu = \ddot{q}^\nu \dots$

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Trying to 2nd-guess Riemann results

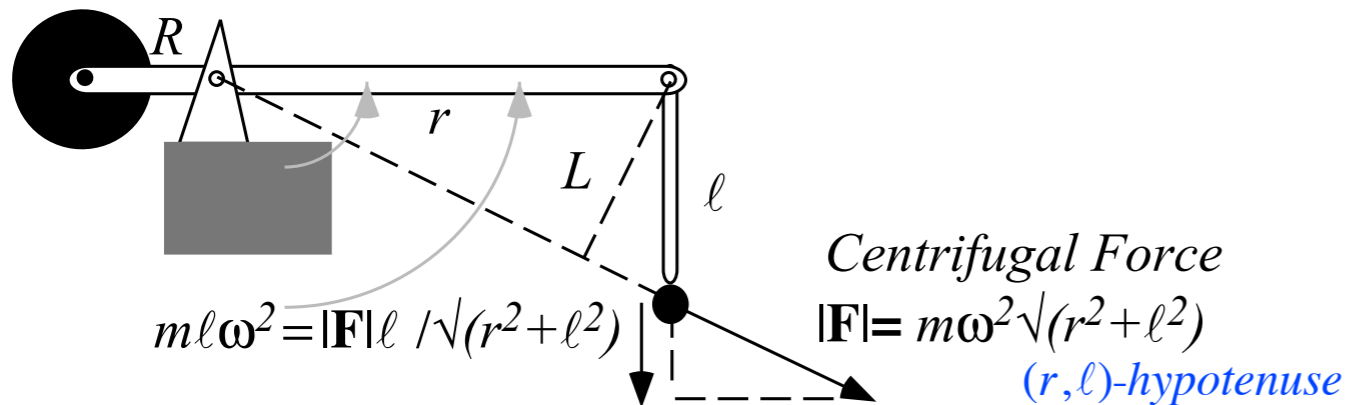


Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = -\frac{\pi}{2}$, $\phi = 0$)

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Trying to 2nd-guess Riemann results

The ϕ -torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L = r \cdot \ell / \sqrt{r^2 + \ell^2}$.

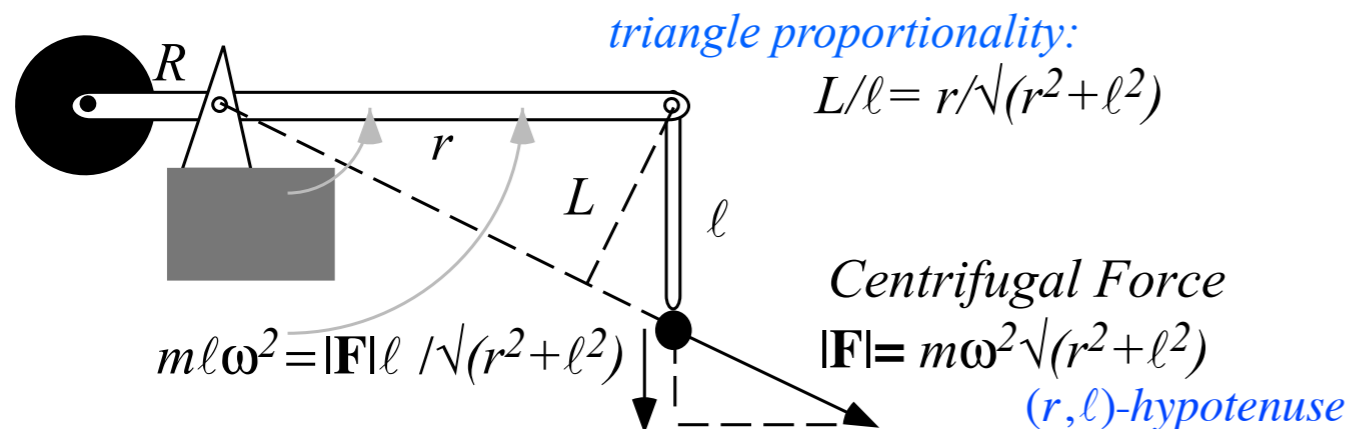


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Trying to 2nd-guess Riemann results (Gravity-free case)

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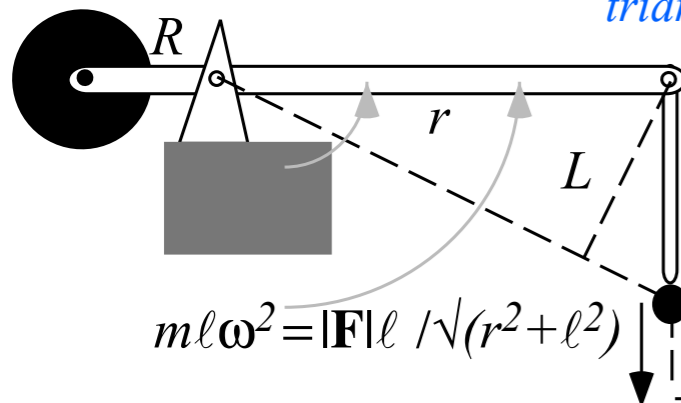
This is the rate of change of ϕ -angular momentum around the pivot at the top of ℓ .

triangle proportionality:

$$L/\ell = r/\sqrt{r^2 + \ell^2}$$

$$m\ell^2 \ddot{\phi} = FL = m\omega^2 \sqrt{r^2 + \ell^2} \frac{r\ell}{\sqrt{r^2 + \ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL / m\ell^2 = \omega^2 r / \ell$



$$m\ell\omega^2 = |\mathbf{F}| \ell / \sqrt{r^2 + \ell^2}$$

Centrifugal Force

$$|\mathbf{F}| = m\omega^2 \sqrt{r^2 + \ell^2}$$

(r, ℓ)-hypotenuse

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = -\frac{\pi}{2}$, $\phi = 0$)

[Move to top of page...](#)

Riemann equation force analysis $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu} \dot{p}_\mu = \ddot{q}^\nu \dots$

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Trying to 2nd-guess Riemann results (Gravity-free case)

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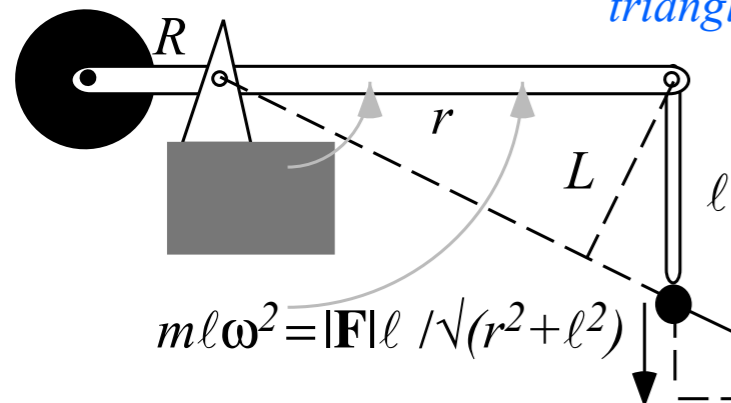
This is the rate of change of ϕ -angular momentum around the pivot at the top of ℓ .

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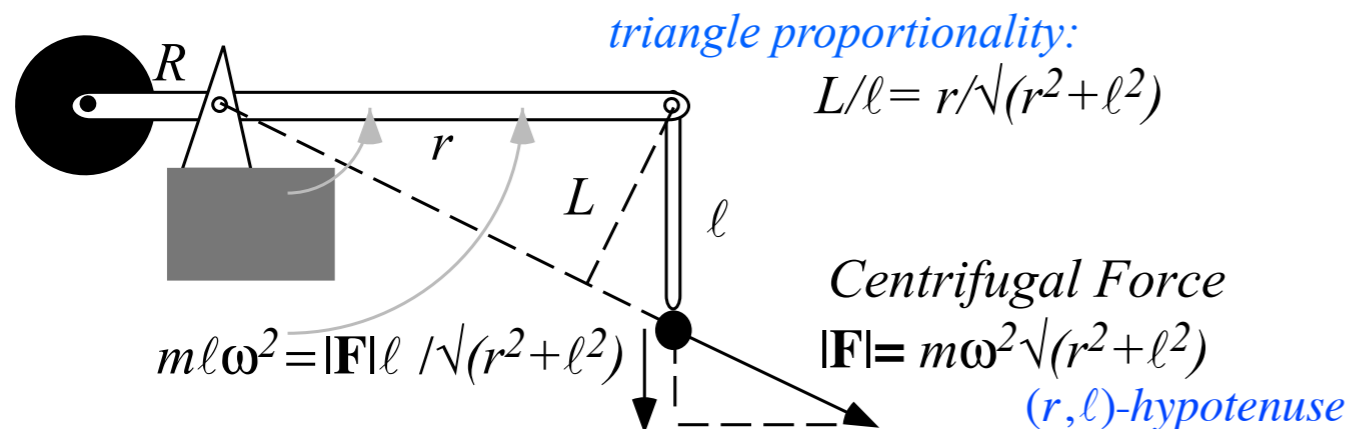
(r, l)-hypotenuse

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Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = -\frac{\pi}{2}$, $\phi = 0$)

Trying to 2nd-guess Riemann results (Gravity-free case) [Move to top of page...](#)

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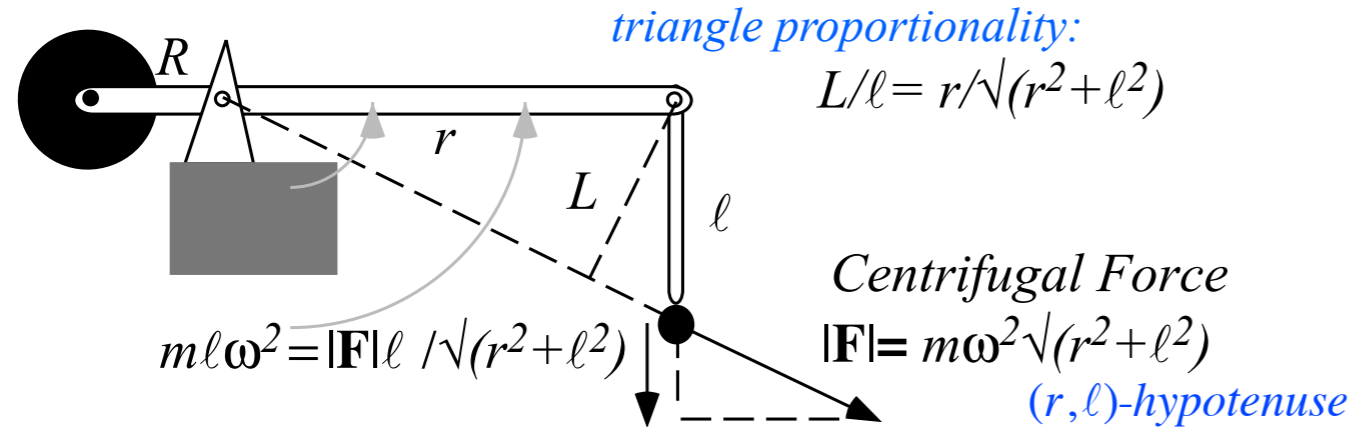
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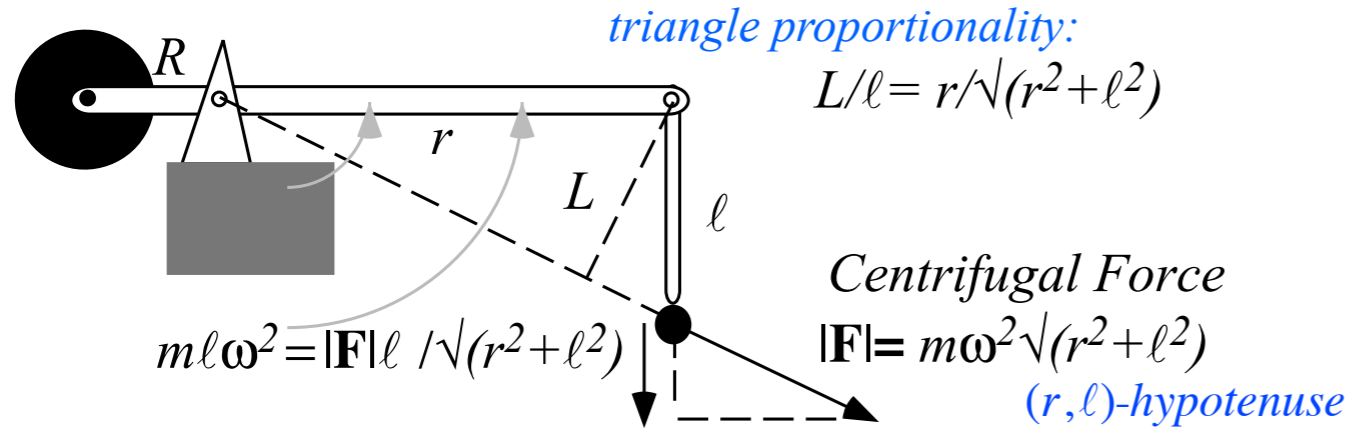
Riemann equation force analysis

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 Riemann
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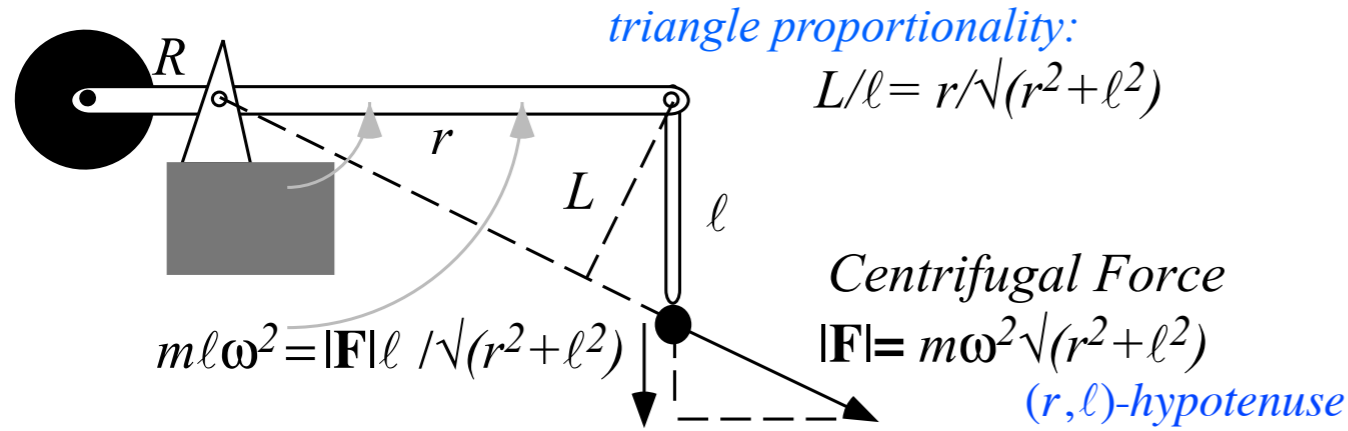
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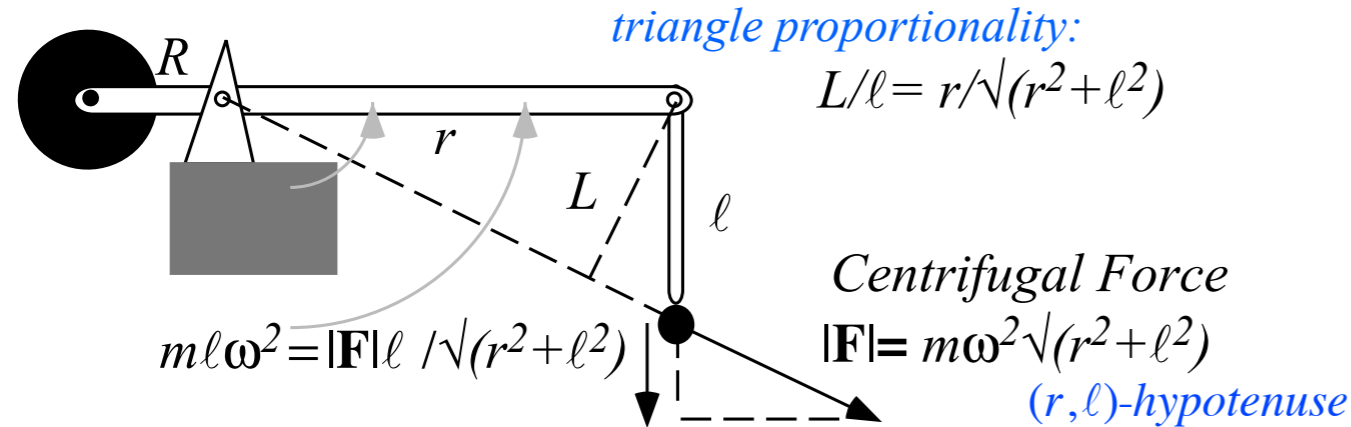
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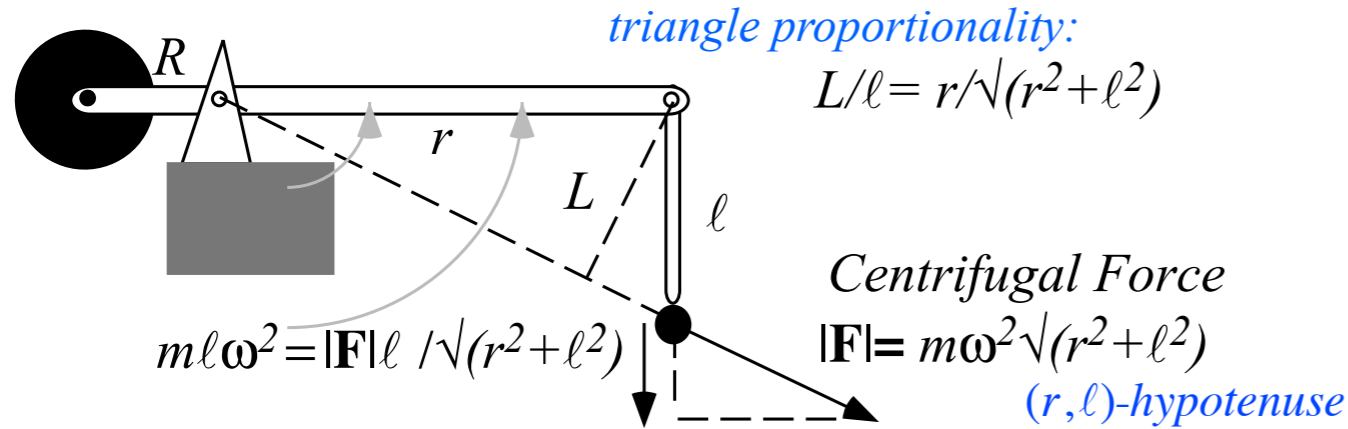
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It may seem paradoxical that the θ -coordinate for main r -arm feels any torque or acceleration at all. Indeed, if the device is rigid there can be none since the centrifugal force has no moment; (Its line of action hits the θ -axis of the R -arm.)

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or: $\ddot{\phi} = FL / m\ell^2 = \omega^2 r / \ell$

Now...
 2nd-guess
 Riemann
 results:

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2 + mr^2} \\ \omega^2 r / \ell \end{pmatrix}$$

Fig. 2.5.1 Centrifugal force for state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = -\frac{\pi}{2}$, $\phi = 0$)

It may seem paradoxical that the θ -coordinate for main r -arm feels any torque or acceleration at all. Indeed, if the device is rigid there can be none since the centrifugal force has no moment;

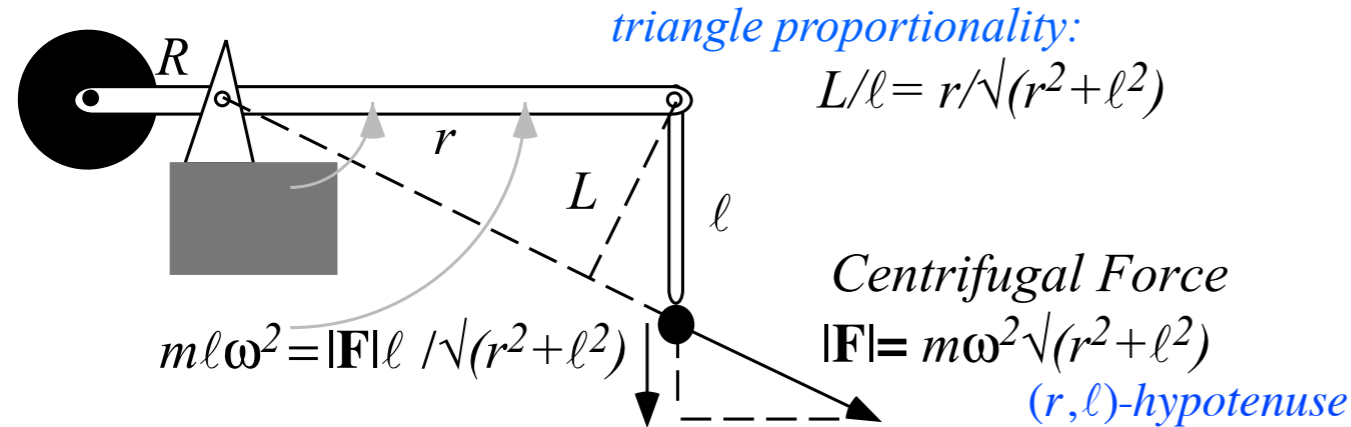
(Its line of action hits the θ -axis of the R -arm.)

However, this device isn't rigid. The ℓ -leg pivot is frictionless and can only transmit a component $m\cdot\ell\omega^2$ of force along ℓ .

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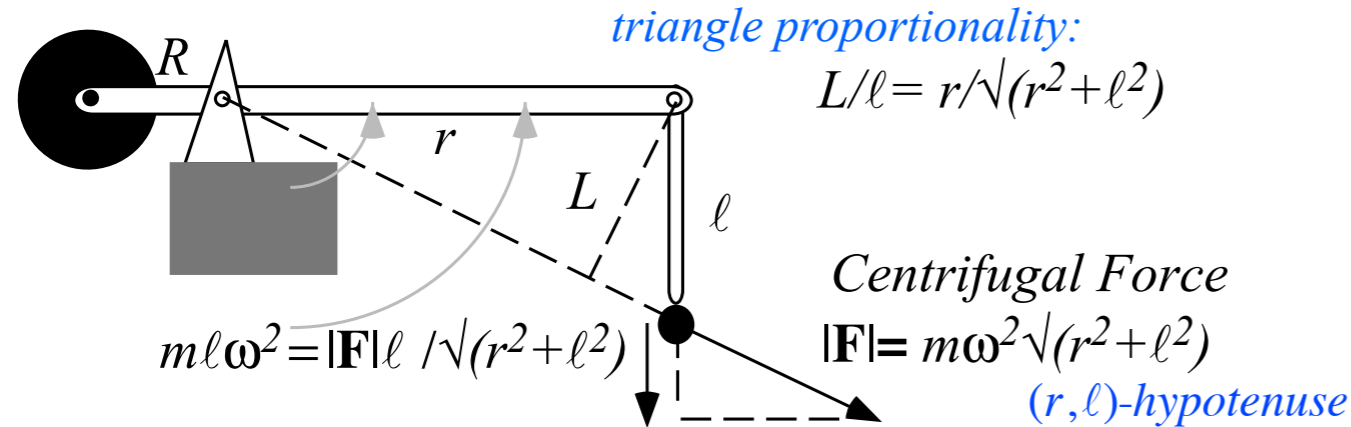
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It reduces θ -angular momentum to exactly cancel the rate of increase in ϕ -momentum.

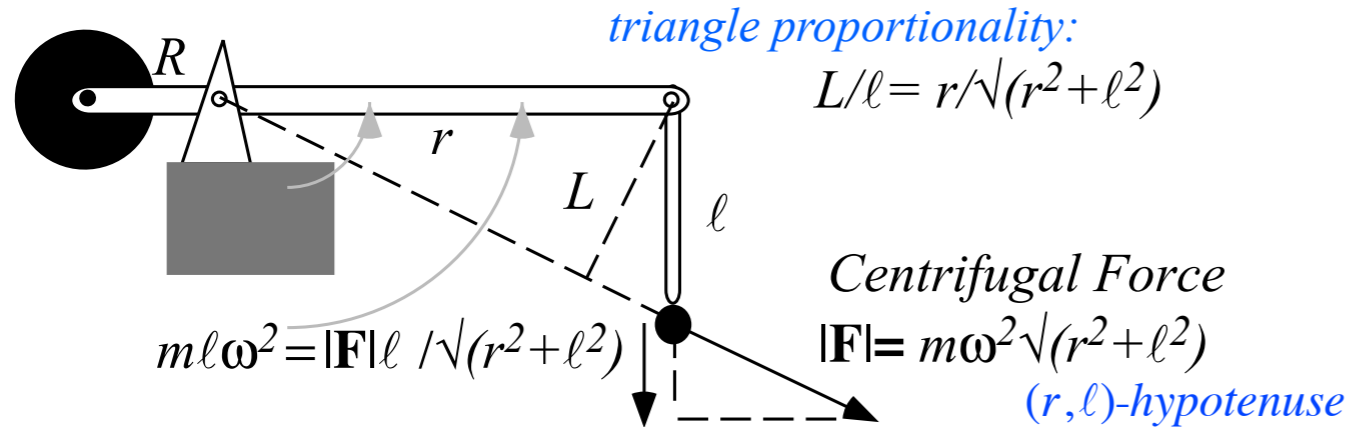
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Checks with $\ddot{\theta}$ Riemann equation

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Checks with $\ddot{\theta}$ Riemann equation

Note the time derivative of total momentum is zero if outside torques are zero. (twirling skater analogy)

$$\dot{p}_{\theta} + \dot{p}_{\phi} = 0, \text{ if } F_{\theta} = 0 = F_{\phi}$$