Lecture 13
Thur. 10.05.2017

Complex Variables, Series, and Field Coordinates II.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
   How good are those power series?
   Taylor-Maclaurin series, imaginary interest, and complex exponentials
2. What good are complex exponentials?
   Easy trig
   Easy 2D vector analysis
   Easy oscillator phase analysis
   Easy rotation and "dot" or "cross" products
3. Easy 2D vector calculus
   Easy 2D vector derivatives
   Easy 2D source-free field theory
   Easy 2D vector field-potential theory
4. Riemann-Cauchy relations (What's analytic? What's not?)
   Easy 2D curvilinear coordinate discovery
   Easy 2D circulation and flux integrals
   Easy 2D monopole, dipole, and 2^n-pole analysis
   Easy 2^n-multipole field and potential expansion
   Easy stereo-projection visualization
   Cauchy integrals, Laurent-Maclaurin series
5. Mapping and Non-analytic 2D source field analysis

Lecture 13 Thur. 10.05.17
Starts review here

Lect. 12
ended here

1. Complex numbers provide "automatic trigonometry"
2. Complex numbers add like vectors.
3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.
4. Complex products provide 2D rotation operations.
5. Complex products provide 2D "dot"(•) and "cross"(x) products.
6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field
7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]
8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
   The half-n'-half results: (Riemann-Cauchy Derivative Relations)
9. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field
10. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)
11. Complex integrals define 2D monopole fields and potentials
12. Complex derivatives give 2D dipole fields
13. More derivatives give 2D 2^N-pole fields…
14. ...and 2^N-pole multipole expansions of fields and potentials...
15. ...and Laurent Series...
16. ...and non-analytic source analysis.
17. …and mapping…

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

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Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla \cdot \mathbf{f}$ and imaginary part has curl $\nabla \times \mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\underbrace{\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)} + \frac{i}{2}\underbrace{\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)} = \frac{1}{2}\nabla \bullet \mathbf{f} + \frac{i}{2}|\nabla \times \mathbf{f}|_{Z\perp(x,y)}$$

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7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function $f(z)$, conjugate it (change all i's to $-i$) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz}=0}$.

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For example: if *f(z)=a·z* then *f*(z*)=a·z*=a(x-iy)* is not function of *z* so it has *zero z-derivative*.

F=*(F$_x$,F$_y$)=(f*$_x$,f*$_y$)=(a·x,-a·y) has zero divergence:* $\nabla\bullet\mathbf{F}=0$ *and has zero curl:* $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

What Good Are Complex Exponentials? (contd.) *(Review of Lecture 12.)*

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For example: if $f(z)=a\cdot z$ then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ has *zero divergence*: $\nabla\bullet\mathbf{F}=0$ and has *zero curl*: $|\nabla\times\mathbf{F}|=0$.

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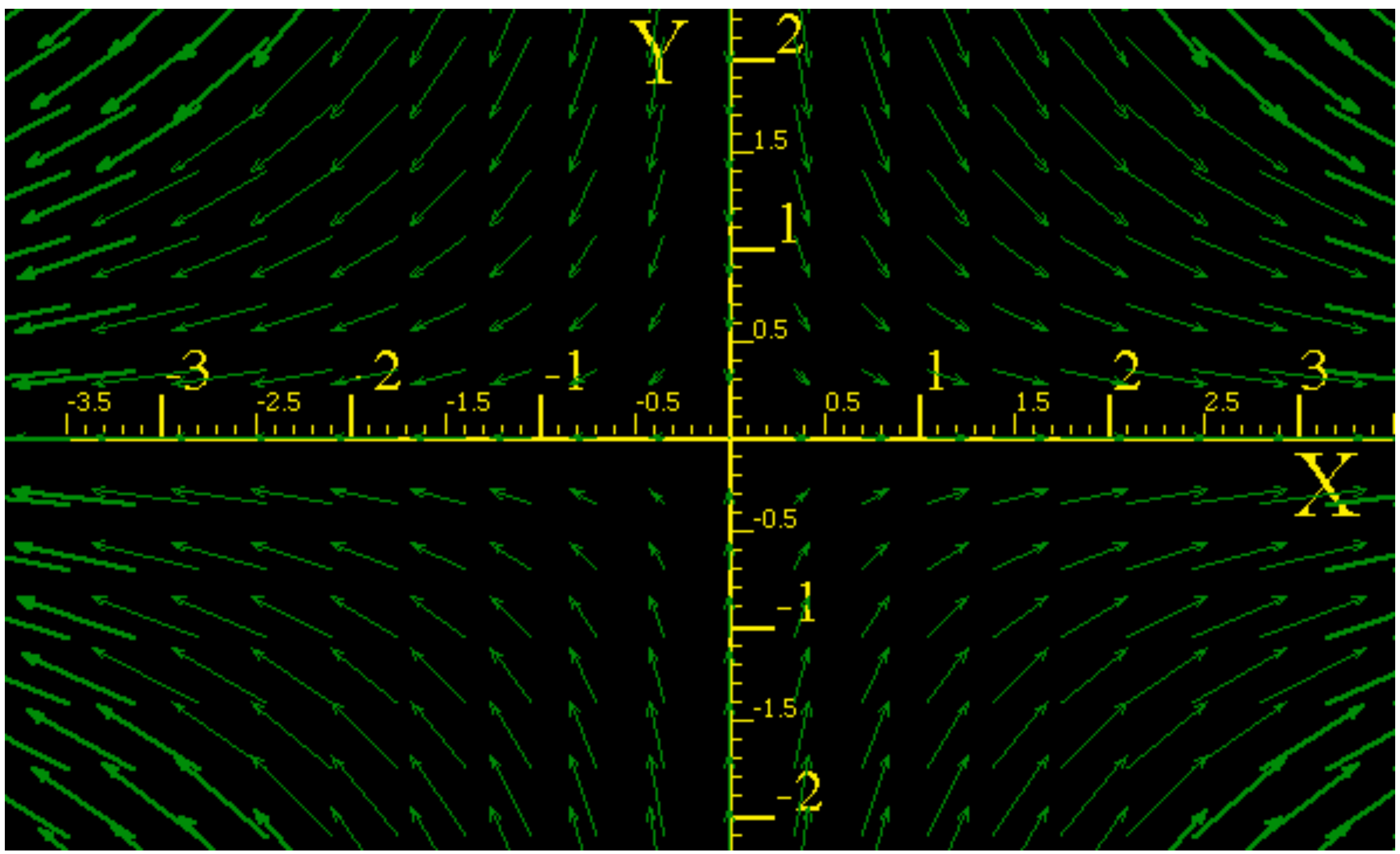

precursor to
Unit 1
Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

*8. Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

Any *DFL* field **F** is a gradient of a *scalar potential field* **Φ** or a curl of a *vector potential field* **A**.

$$\mathbf{F} = \nabla\Phi \qquad\qquad \mathbf{F} = \nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\Phi(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\Phi(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real $(\text{Re}\,\phi =\mathbf{\Phi})$ and imaginary $(\text{Im}\,\phi =\mathbf{A})$ parts.

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$$f(z)=\tfrac{d\phi}{dz} \quad\Rightarrow\quad \phi = \quad \Phi \quad + i \;\; \mathbf{A} = \int f\cdot dz = \int az\cdot dz = \tfrac{1}{2}\,az^2$$

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To find $\phi=\Phi+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real (Re $\phi=\Phi$) and imaginary (Im $\phi=\mathbf{A}$) parts.

$$f(z)=\frac{d\phi}{dz} \quad\Rightarrow\quad \phi = \overbrace{\tfrac{1}{2}a(x^2-y^2)}^{\Phi} + i\,\overbrace{axy}^{\mathbf{A}} = \int f\cdot dz = \int az\cdot dz = \tfrac{1}{2}az^2 = \tfrac{1}{2}a(x+iy)^2$$

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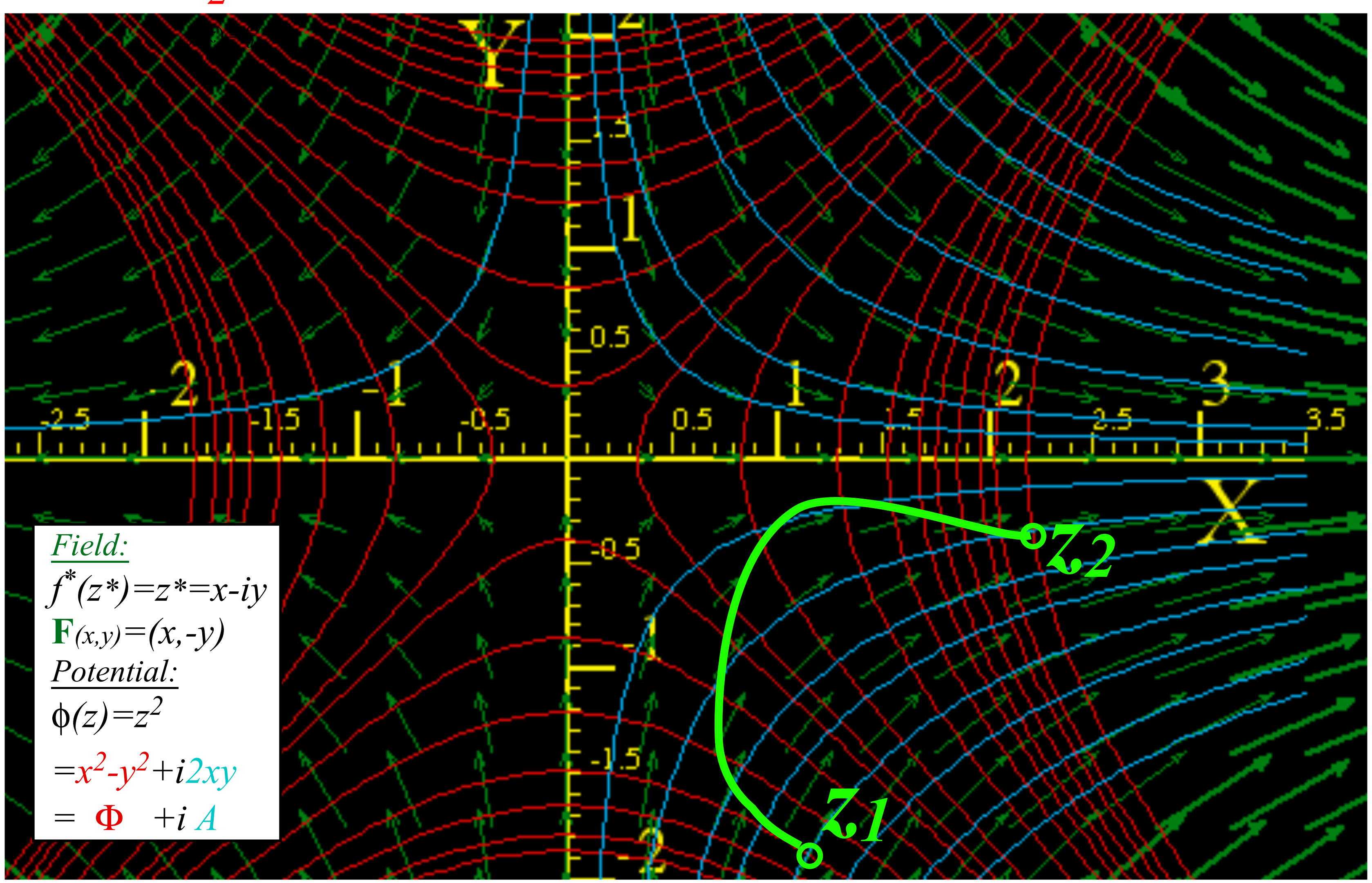

Unit 1
Fig. 10.7

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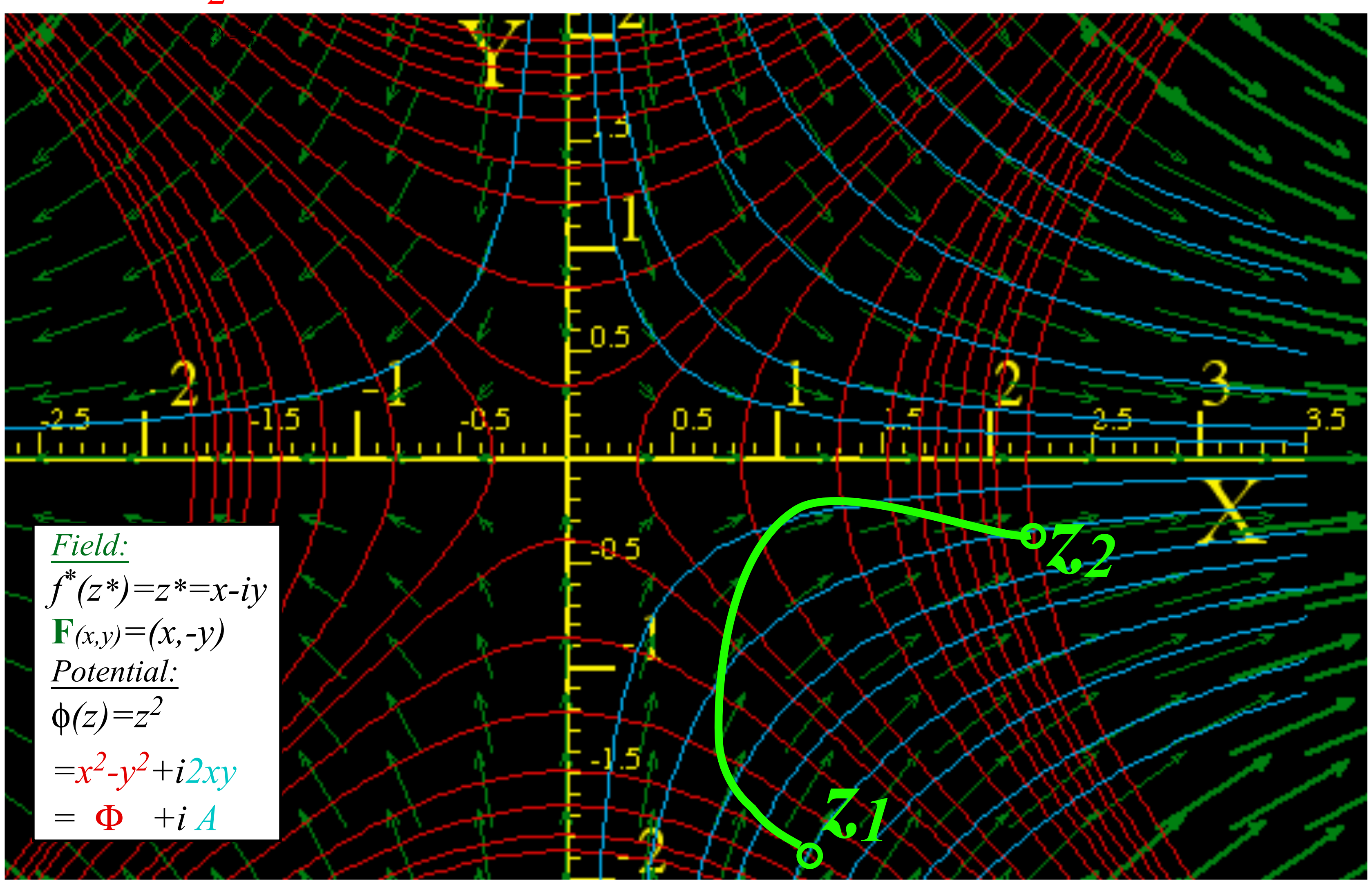
$$f(z)=\frac{d\phi}{dz} \quad \Rightarrow \quad \phi = \overbrace{\Phi}^{} + i\,\mathbf{A} = \int f\cdot dz = \int az\cdot dz = \tfrac{1}{2}\,az^2 = \tfrac{1}{2}\,a(x+iy)^2$$
$$= \underbrace{\tfrac{1}{2}\,a(x^2-y^2)}_{\Phi} + i\,\underbrace{axy}_{\mathbf{A}}$$

BONUS!
Get a free coordinate system!



Unit 1
Fig. 10.7

The (Φ,A) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2 = const.$

$q^2=\mathbf{A}=(xy)= const.$

*Actually it's OCC.

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Easy 2D vector derivatives

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The half-n'-half results: (Riemann-Cauchy Derivative Relations)

*8. (contd.) Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi*}{dz*}$ has 2D gradient $\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial\mathbf{A}}{\partial y} \\ -\frac{\partial\mathbf{A}}{\partial y} \end{pmatrix}$ of vector **A** (*and they're <u>equal!</u>*)

$f(z) = \frac{d\phi}{dz} \quad \Rightarrow$

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\underbrace{\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right)} + \frac{1}{2}\underbrace{\left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x}\right)} = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

What Good Are Complex Exponentials? (contd.)

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Note, *mathematician definition* of force field **F**=+$\nabla\mathbf{\Phi}$ replaces usual physicist's definition **F**=-$\nabla\mathbf{\Phi}$

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Given ϕ:

$$\boxed{\begin{aligned}\phi &= \Phi + i\,\mathbf{A}\\ &= \tfrac{1}{2}a(x^2 - y^2) + i\,axy\end{aligned}}$$

The half-n'-half result

find:

$$\nabla\Phi = \begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2)\end{pmatrix} = \begin{pmatrix}ax\\ -ay\end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix} = \begin{pmatrix}ax\\ -ay\end{pmatrix} = \mathbf{F}$$

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Given ϕ:

$$\phi = \quad \boldsymbol{\Phi} \quad + i \quad \mathbf{A}$$
$$= \tfrac{1}{2}a(x^2 - y^2) \quad + i \quad axy$$

The half-n'-half result

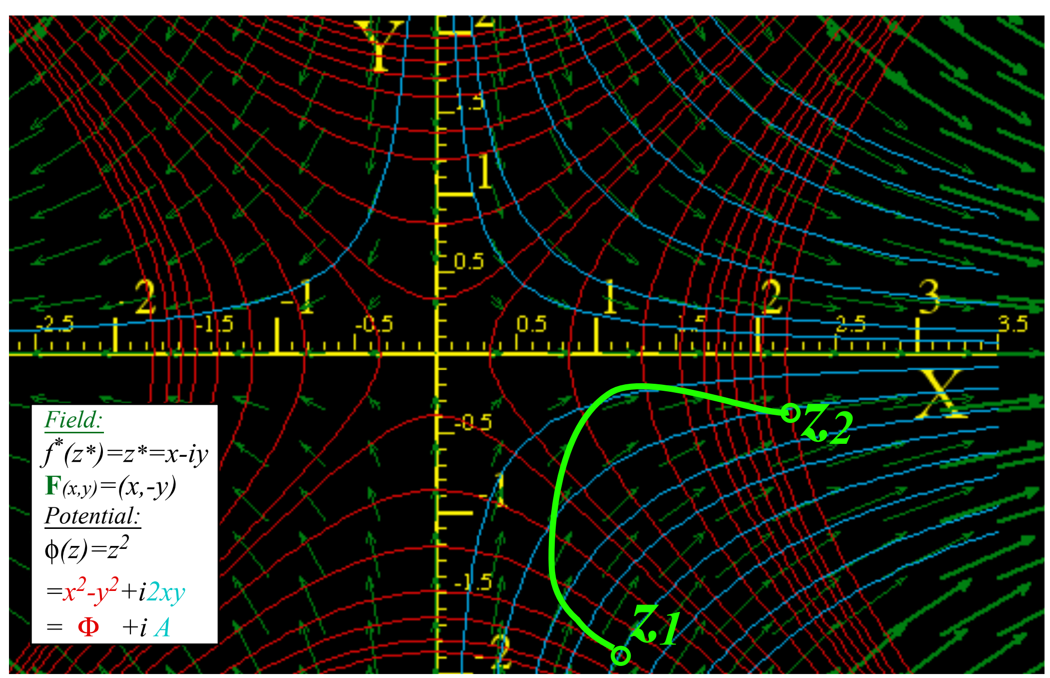
find:

$$\nabla\boldsymbol{\Phi} = \begin{pmatrix} \frac{\partial \boldsymbol{\Phi}}{\partial x} \\ \frac{\partial \boldsymbol{\Phi}}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

Scalar *static potential lines* $\boldsymbol{\Phi}$=*const.* and vector *flux potential lines* **A**=*const.* define *DFL field-net*.



What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains "scalar"(**F**=$\nabla\Phi$) and "vector"(**F**=∇x**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial y} \end{pmatrix}$ of vector **A** (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x}\right) = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+$\nabla\Phi$ replaces usual physicist's definition **F**=-$\nabla\Phi$

Given ϕ:

$$\boxed{\begin{aligned} \phi &= \Phi + i\,\mathbf{A} \\ &= \tfrac{1}{2}a(x^2 - y^2) + i\,axy \end{aligned}}$$

The half-n'-half result

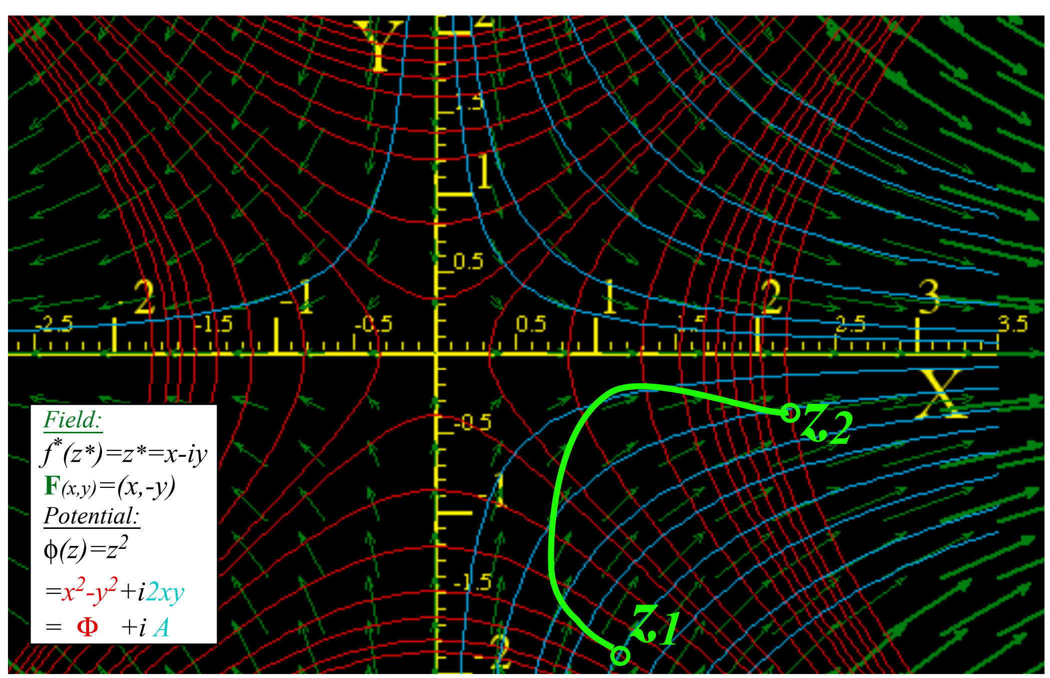
find:

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial\mathbf{A}}{\partial y} \\ -\frac{\partial\mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* **A**=*const.* define *DFL field-net*.



The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\mathbf{A}}{\partial y} \quad \text{is:} \quad \frac{\partial \mathrm{Re} f(z)}{\partial x} = \frac{\partial \mathrm{Im} f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial\mathbf{A}}{\partial x} \quad \text{is:} \quad \frac{\partial \mathrm{Re} f(z)}{\partial y} = -\frac{\partial \mathrm{Im} f(z)}{\partial x}$$

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must <u>*not*</u> *be a function of $z^*=x-iy$, that is:* $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies: } \boxed{\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

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$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

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Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z*)*** *of $z^*=x-iy$:*

First, f(z) must <u>not</u> be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies: } \boxed{\frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's <u>not</u>?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z=x+iy$?

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

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$$= z+z^* \quad + \quad (2z-2z^*)$$
$$= 3z-z^*$$

A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

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$$= 3z-z^*$$

A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: *NO!* *$r(xy)=z^*z$ is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

$$
\begin{aligned}
f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\
&= z+z^* + (2z-2z^*) \\
&= 3z-z^*
\end{aligned}
$$

A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: *NO!* $r(xy)=z^*z$ *is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=x+iy$?

A: *YES!* $s(xy)=(x+iy)^2 = z^2$ *is analytic function of z.* *(Yay!)*

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

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Easy stereo-projection visualization

What Good Are Complex Exponentials? (contd.)

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F**•d**r**) and "flux"(∫**F**x d**r**)*

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

9. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int \mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int \mathbf{F}\times d\mathbf{r}$)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\underbrace{[\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)]}_{\Delta\mathbf{A}}$$

$$\Delta\phi = \qquad \Delta\mathbf{\Phi} \qquad + i \qquad \Delta\mathbf{A}$$

In *DFL*-field $\mathbf{F}$, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

$$\int f(z)dz = \int\left(f^*(z^*)\right)^* dz = \int\left(f^*(z^*)\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* + i\,f_y^*\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* - i\,f_y^*\right)\left(dx + i\,dy\right)$$

$$= \int(f_x^*\,dx + f_y^*\,dy) + i\int(f_x^*\,dy - f_y^*\,dx)$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{S} \qquad \text{where:} \qquad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

What Good Are Complex Exponentials? (contd.)

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫F×dr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\Phi(x_2,y_2) - \Phi(x_1,y_1)}_{\Delta\Phi} + i\,[\underbrace{A(x_2,y_2) - A(x_1,y_1)}_{\Delta A}]$$

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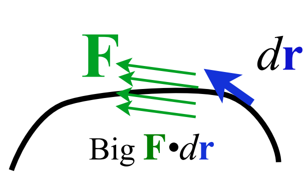
$$= \int(f_x^*\,dx + f_y^*\,dy) + i\int(f_x^*\,dy - f_y^*\,dx)$$

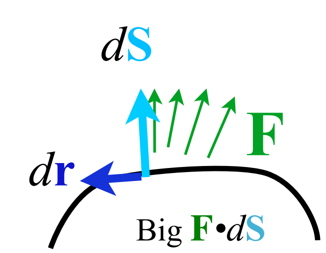
$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \boxed{\int \mathbf{F}\bullet d\mathbf{r}} \quad \boxed{+\,i\int \mathbf{F}\bullet d\mathbf{S}} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$



Real part $\int_1^2 \mathbf{F}\bullet d\mathbf{r} = \Delta\Phi$
sums **F** projections *along* path $d\mathbf{r}$ that is, *circulation* on path to get $\Delta\Phi$.

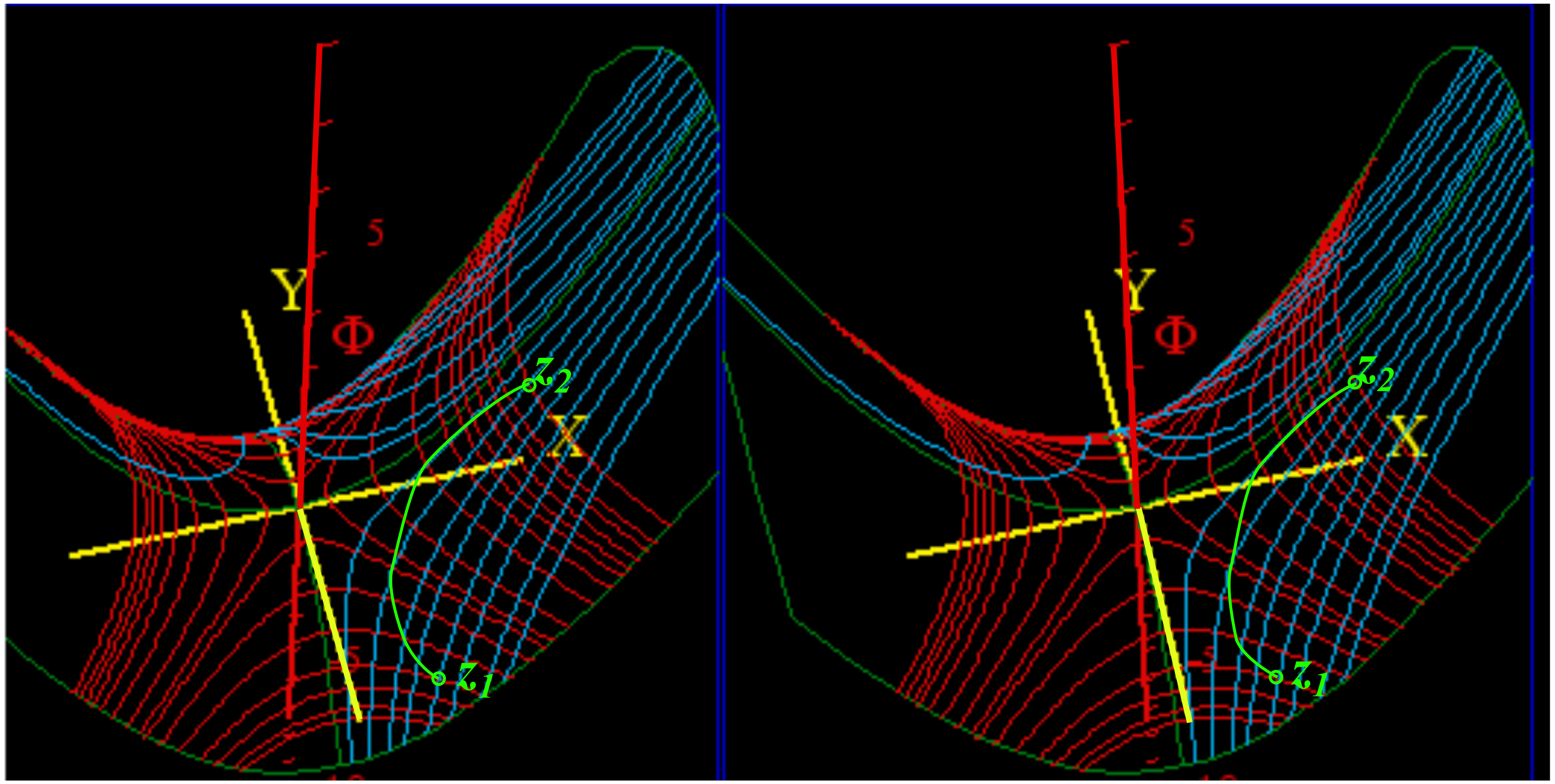


Imaginary part $\int_1^2 \mathbf{F}\bullet d\mathbf{S} = \Delta A$
sums **F** projection *across* path $d\mathbf{r}$ that is, *flux* thru surface elements $d\mathbf{S} = d\mathbf{r}\times\mathbf{e}_Z$ normal to $d\mathbf{r}$ to get ΔA.

Here the scalar potential $\Phi=(x^2-y^2)/2$ is stereo-plotted vs. (x,y)

The $\Phi=(x^2-y^2)/2=const.$ curves are topography lines

The $A=(xy)=const.$ curves are streamlines normal to topography lines



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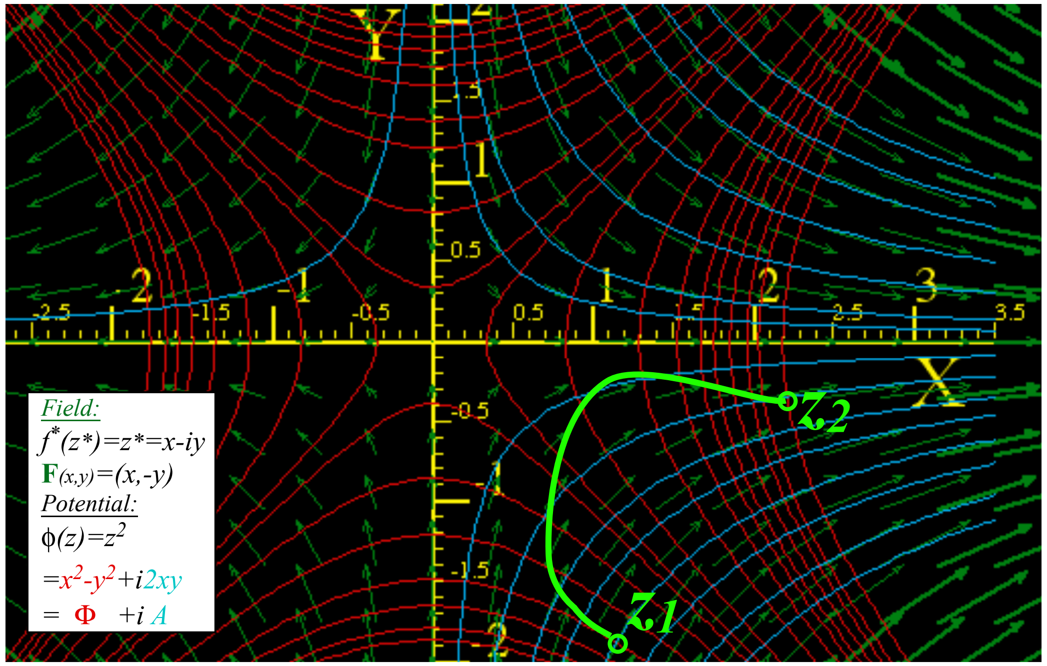
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \uparrow \quad \uparrow \\ \mathbf{E}_{\Phi} \quad \mathbf{E}_{A} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \uparrow \quad \uparrow \\ \mathbf{E}_{\Phi} \quad \mathbf{E}_{A} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

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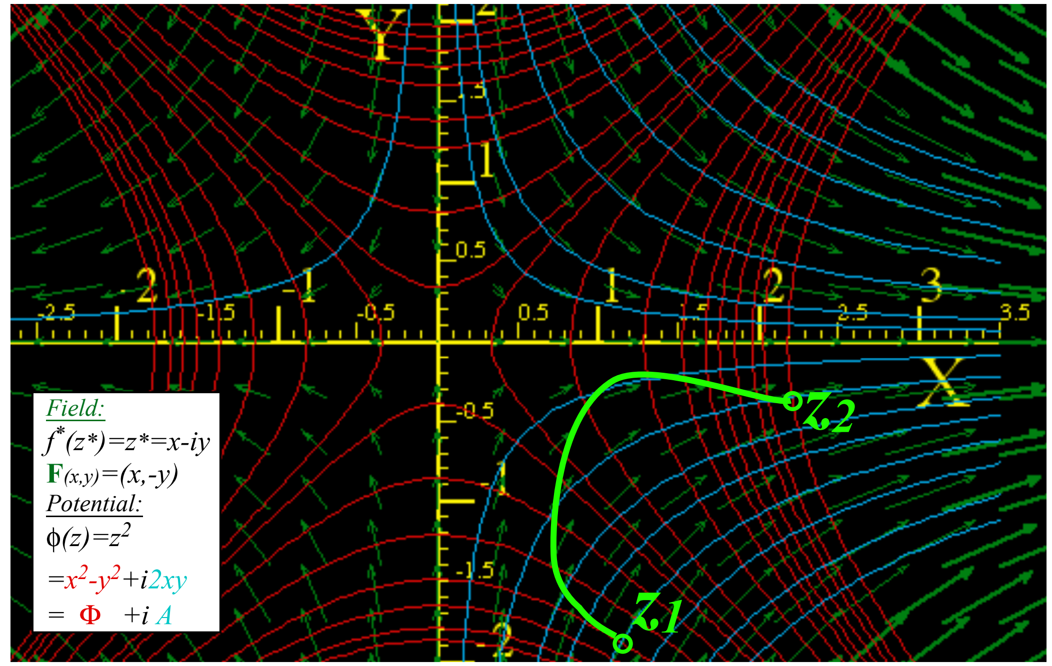
*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y}$$
$$= -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

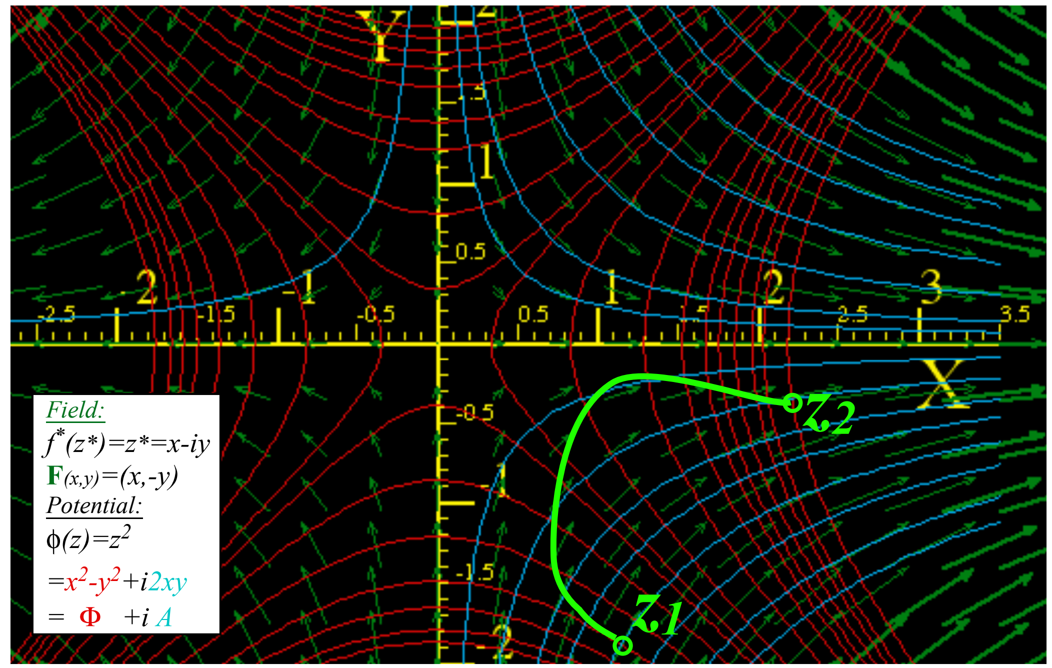
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



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or Riemann-Cauchy *Zero divergence requirement:*

$$0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

and so does A *potential Φ obeys Laplace equation*

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}=z^{-1}$ $\qquad$ $f(z)=\frac{a}{z}=az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

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$$\phi(z)=\quad \Phi \quad + \quad i\mathbf{A}=\int f(z)dz=\int \frac{a}{z}dz=a\ln(z)$$

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$$\phi(z)=\underbrace{\mathbf{\Phi}}_{=a\ln(r)}+i\underbrace{\mathbf{A}}_{a\theta}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

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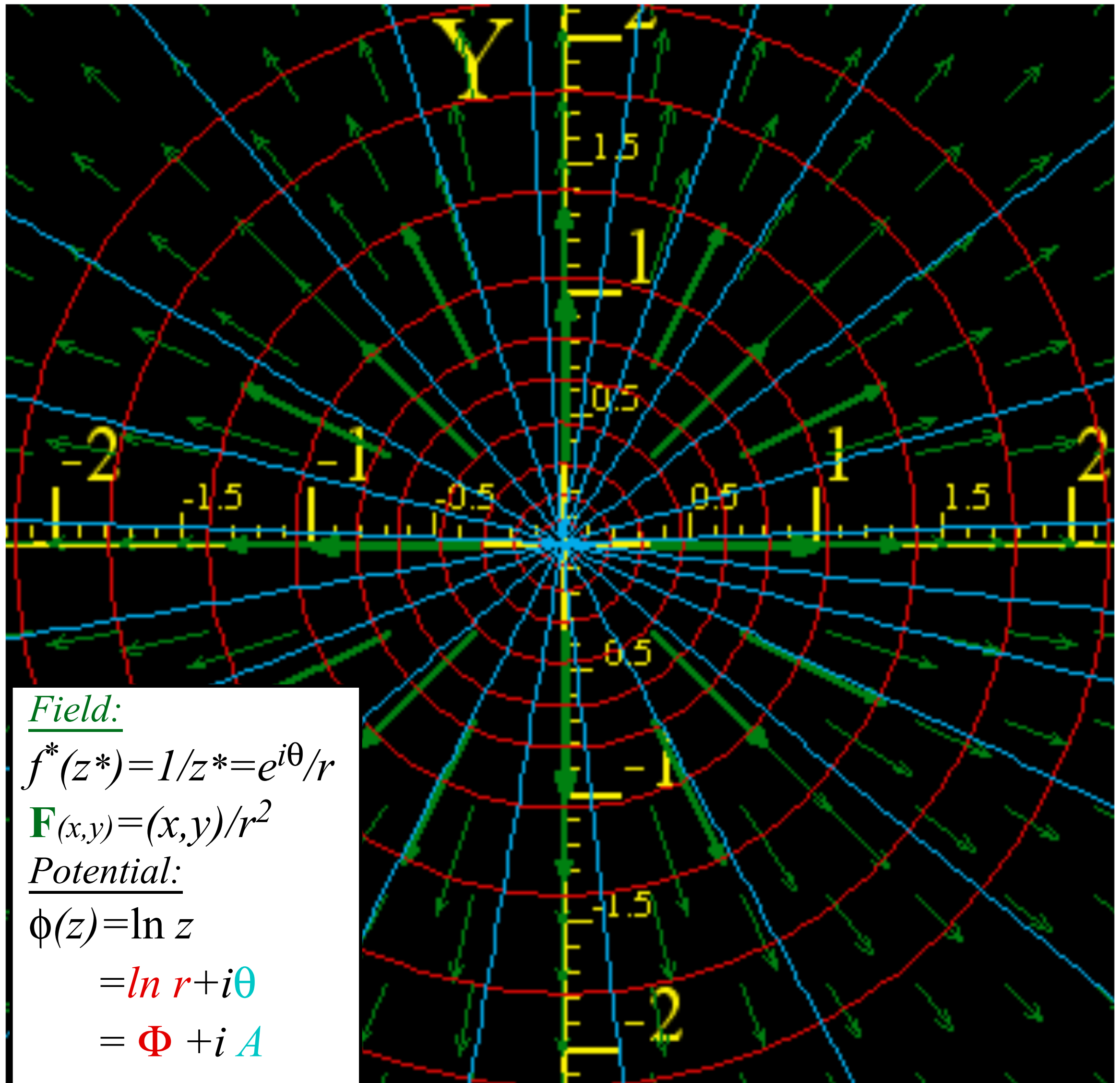
$$\phi(z)=\Phi+i\mathbf{A}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$
$$=\overbrace{a\ln(r)}^{\Phi}+i\overbrace{a\theta}^{\mathbf{A}}$$

(a) Unit Z-line-flux field f(z)=1/z



Field:
$f^*(z^*)=1/z^*=e^{i\theta}/r$
$\mathbf{F}(x,y)=(x,y)/r^2$
Potential:
$\phi(z)=\ln z$
$=\ln r+i\theta$
$=\Phi+iA$

Lecture 12 Tue. 10.03
May end here

11. Complex integrals define 2D monopole fields and potentials

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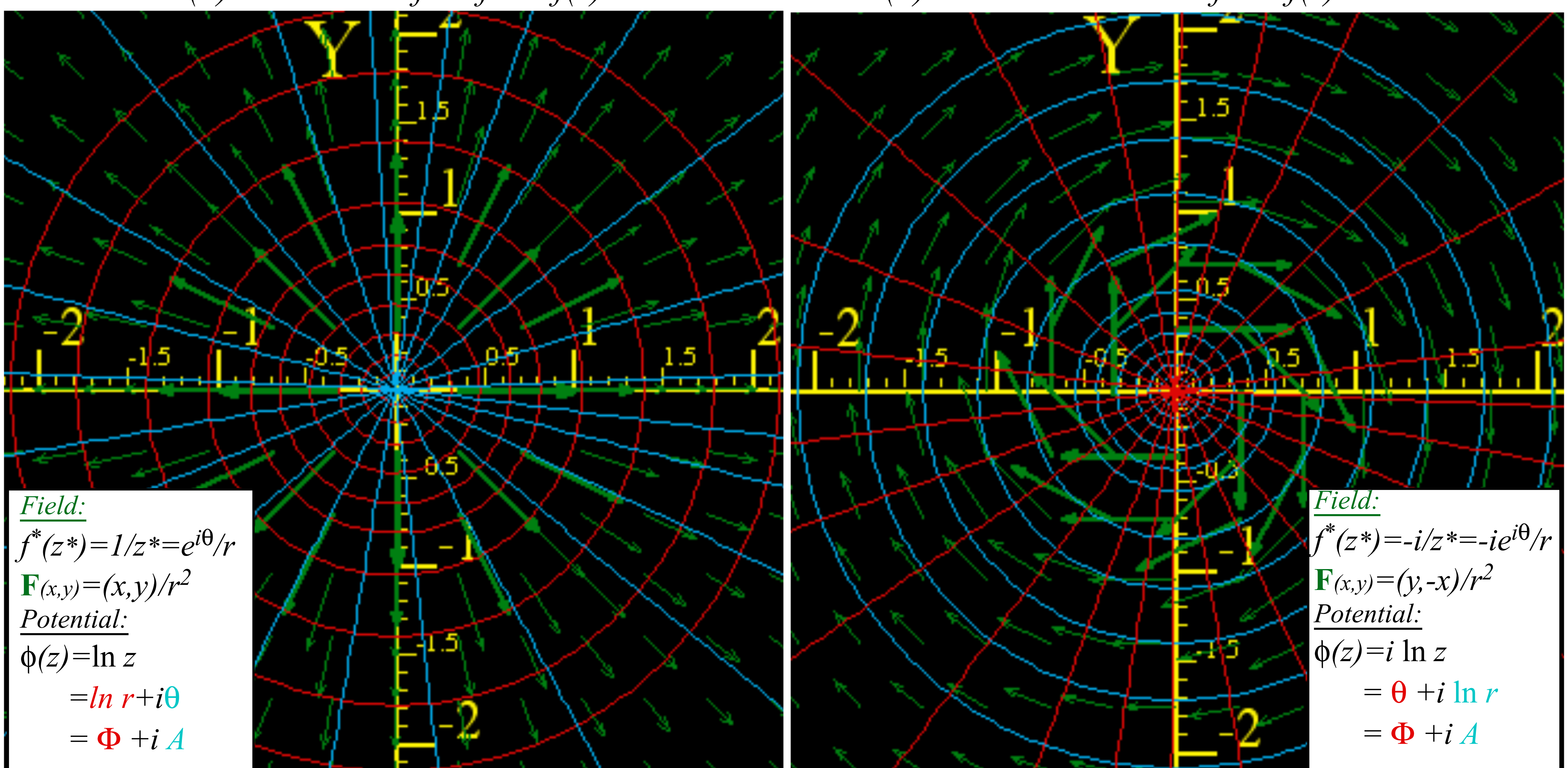
$$=a\ln(r)+i\,a\theta$$

(a) *Unit Z-line-flux field $f(z)=1/z$* $\qquad$ (b) *Unit Z-line-vortex field $f(z)=i/z$*



11. Complex integrals define 2D monopole fields and potentials

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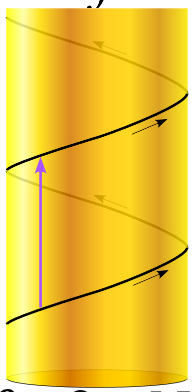
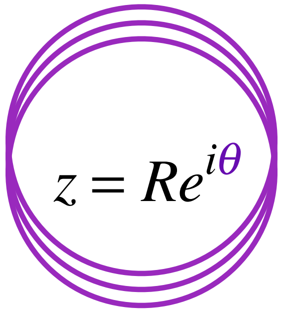
$$=a\ln(r)+i\,a\theta$$

A monopole field is the only power-law field whose integral (potential) depends on path of integration.

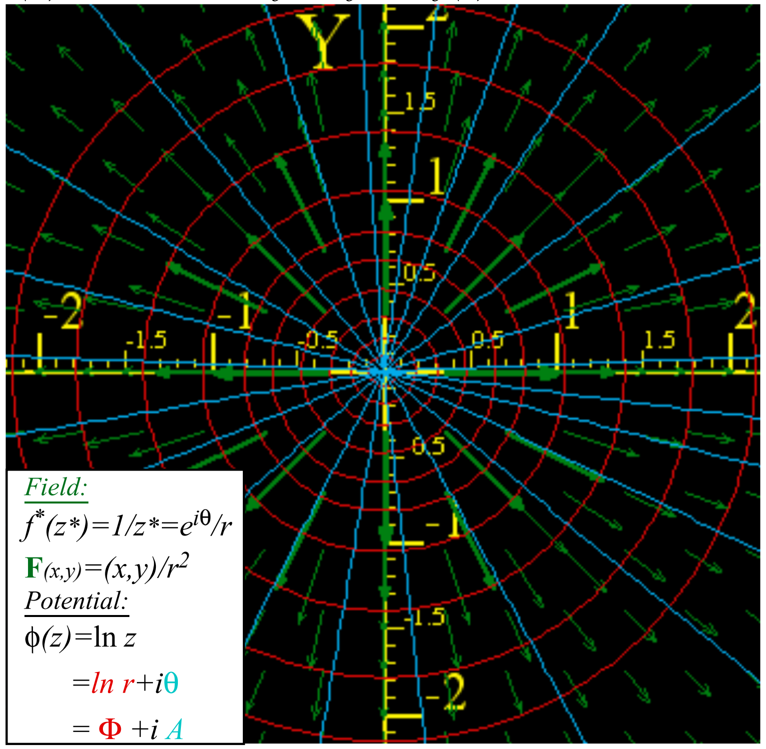




path that goes N times around origin (r=0) at constant r = R.

$$\Delta\phi=\oint f(z)dz=a\oint\frac{dz}{z}=a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}}=a\int_{\theta=0}^{\theta=2\pi N}id\theta=ai\,\theta\Big|_0^{2\pi N}=2a\pi iN$$

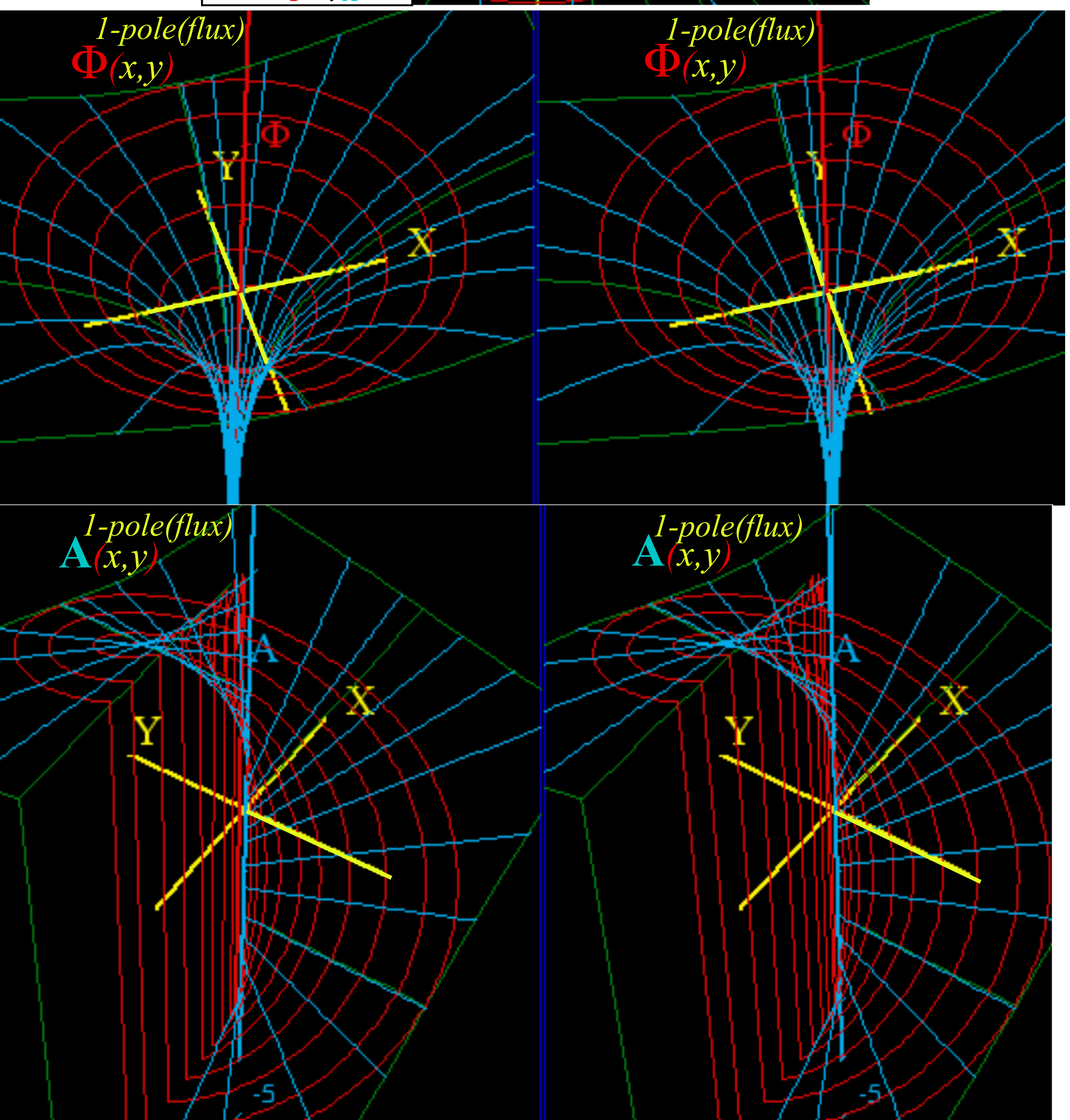
(a) Unit Z-line-flux field f(z)=1/z



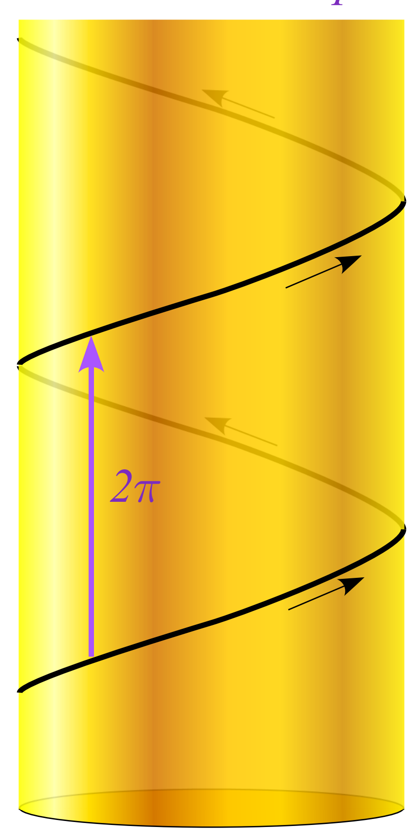
$$\phi(z) = \underbrace{\Phi}_{=\ln(r)} + i\underbrace{\mathbf{A}}_{\theta} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(re^{i\theta})$$

$$= \ln(r) + i\theta$$

(For a=1)

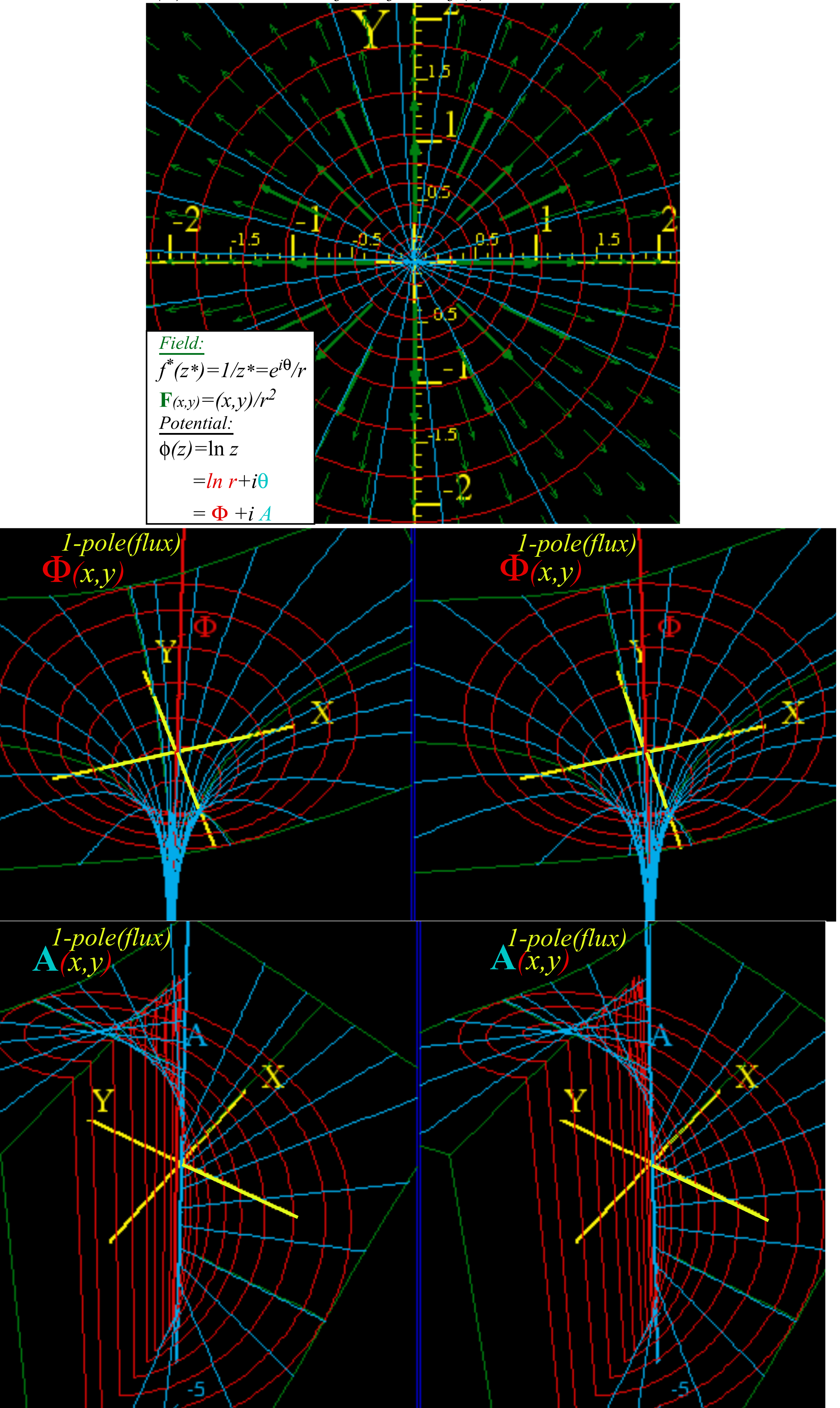


*Each turn around origin adds 2πi to vector potential i**A***

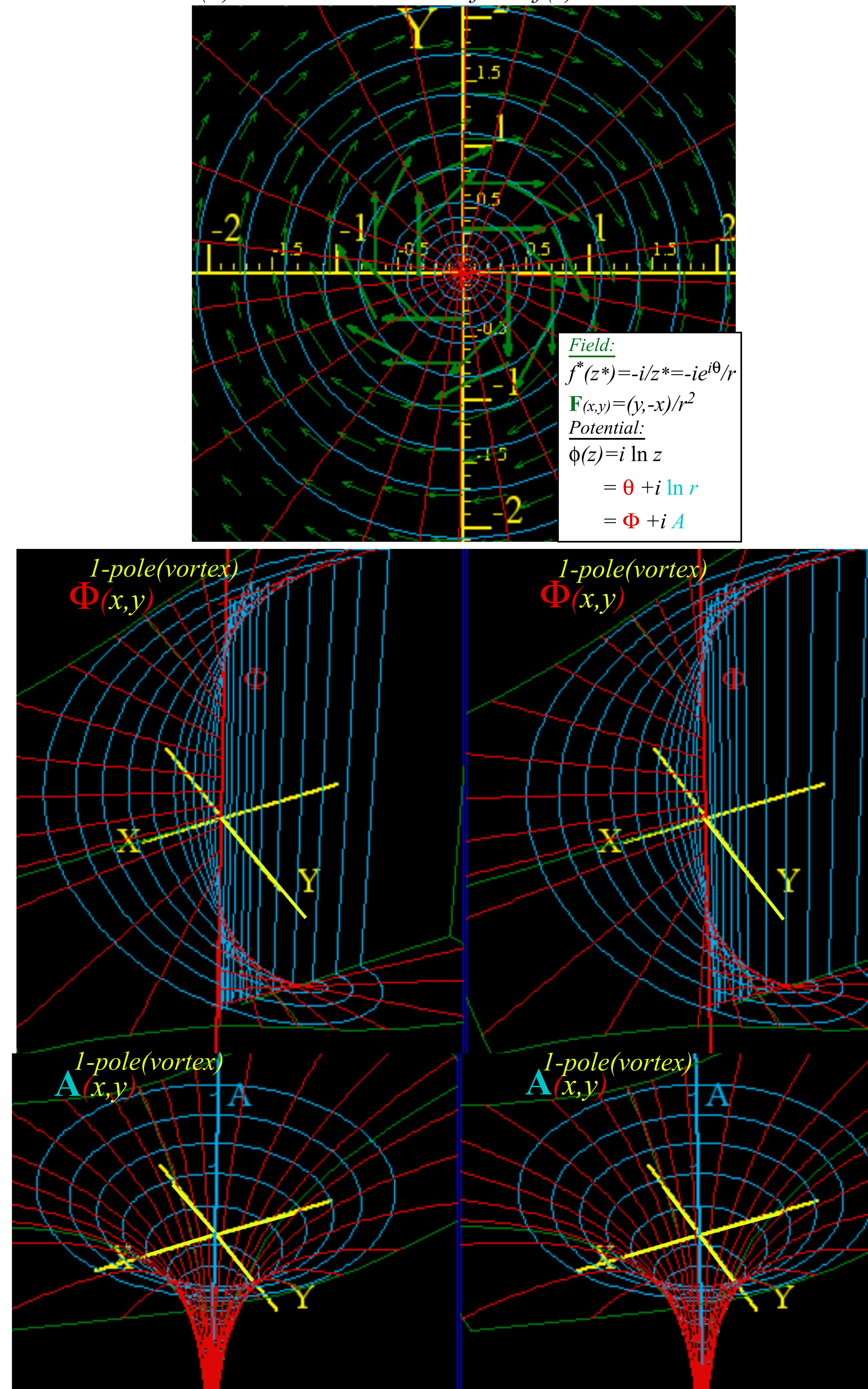


(For a=1)

(a) Unit Z-line-flux field $f(z)=1/z$



(b) Unit Z-line-vortex field $f(z)=i/z$

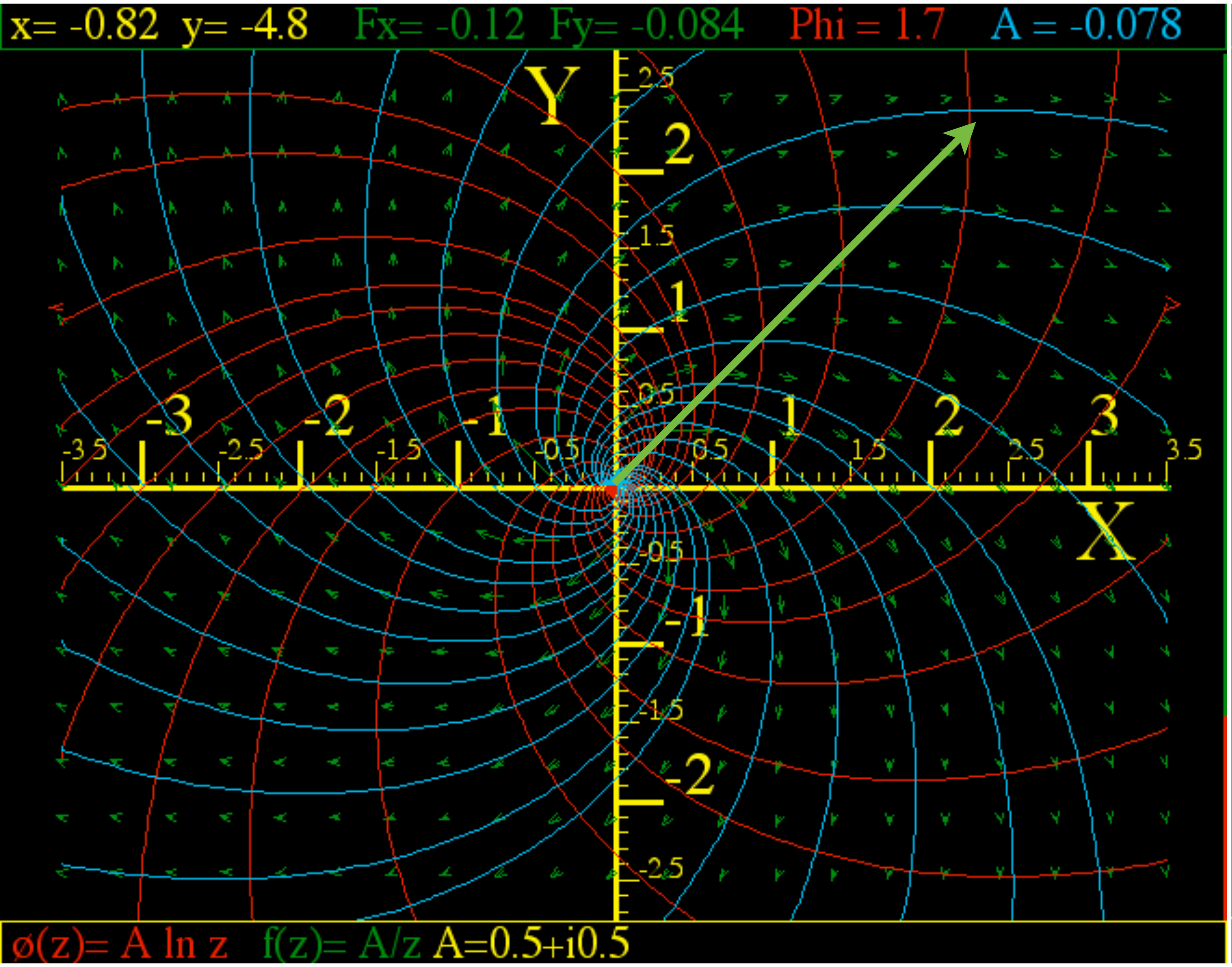


What Good Are Complex Exponentials? (contd.)

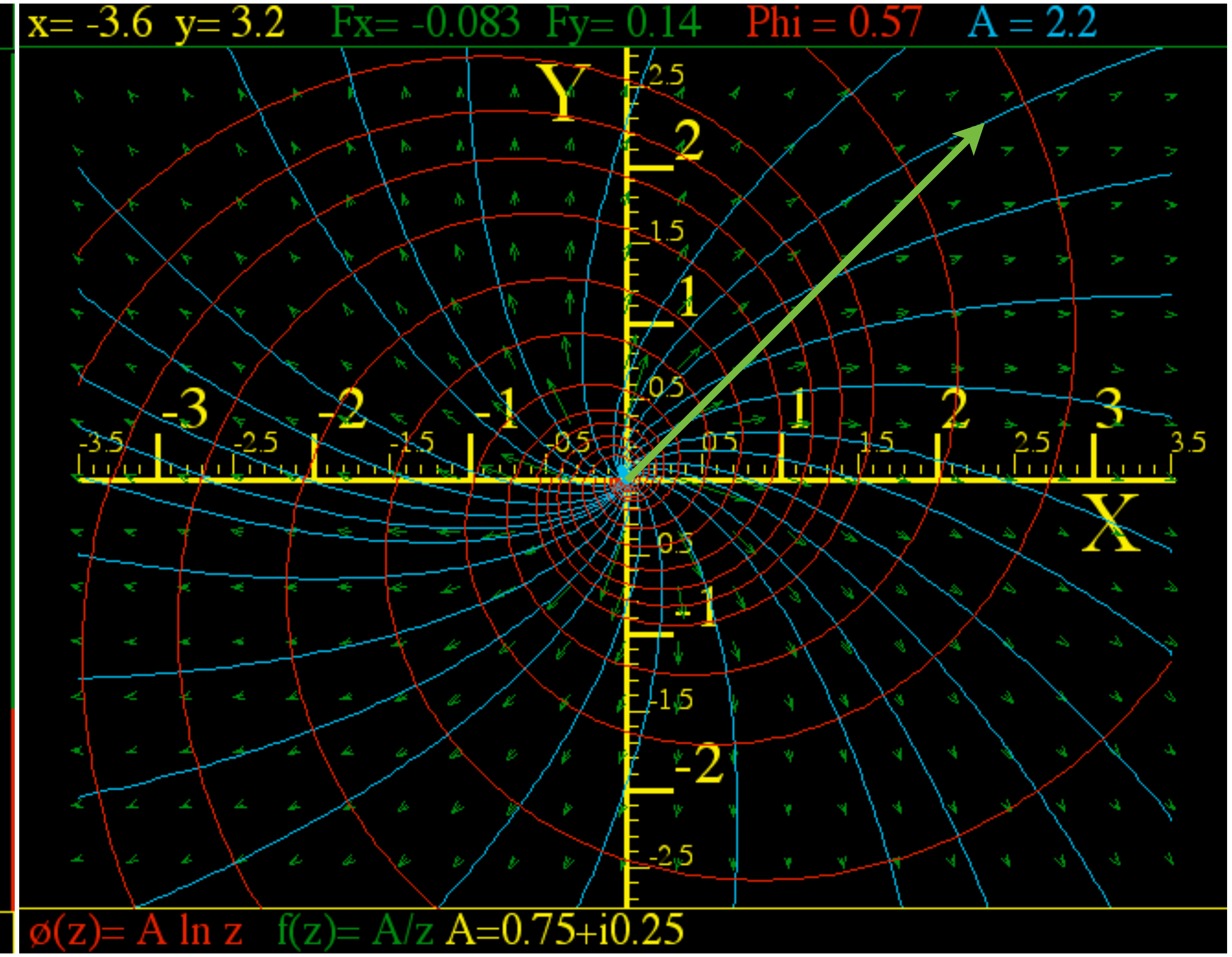
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

$f(z) = (0.75 + i0.25)/z = e^{i18^{\circ}}/z\sqrt{n}$

"Vortex"



"Hurricane"



4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

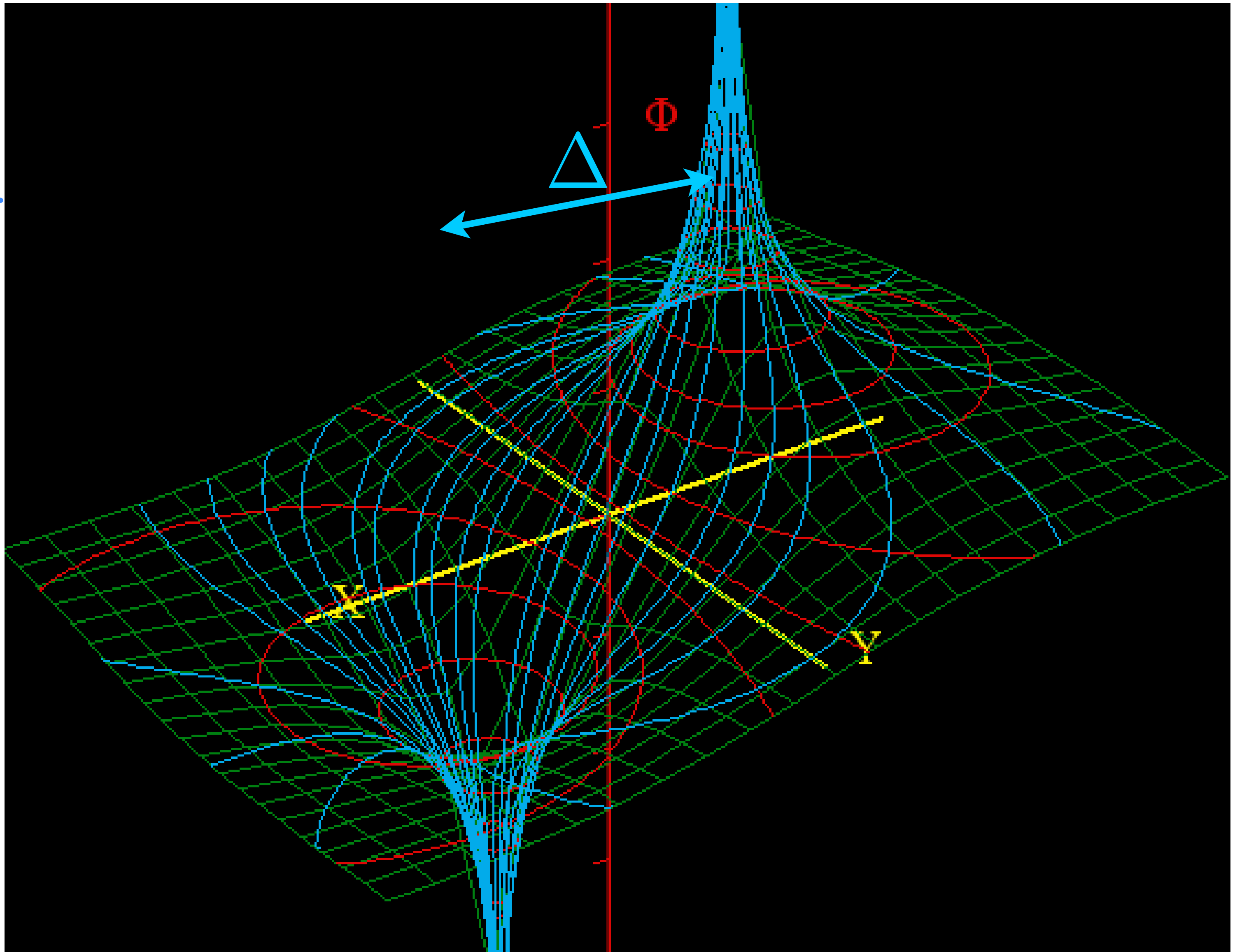
This is like the derivative definition:

$$\frac{df}{dz}=\frac{f(z+\Delta)-f(z)}{\Delta}$$

or:

$$\frac{df}{dz}=\frac{f(z+\frac{\Delta}{2})-f(z-\frac{\Delta}{2})}{\Delta}$$

if Δ is infinitesimal ($\Delta\to 0$)



So-called "physical dipole" has finite Δ

(+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

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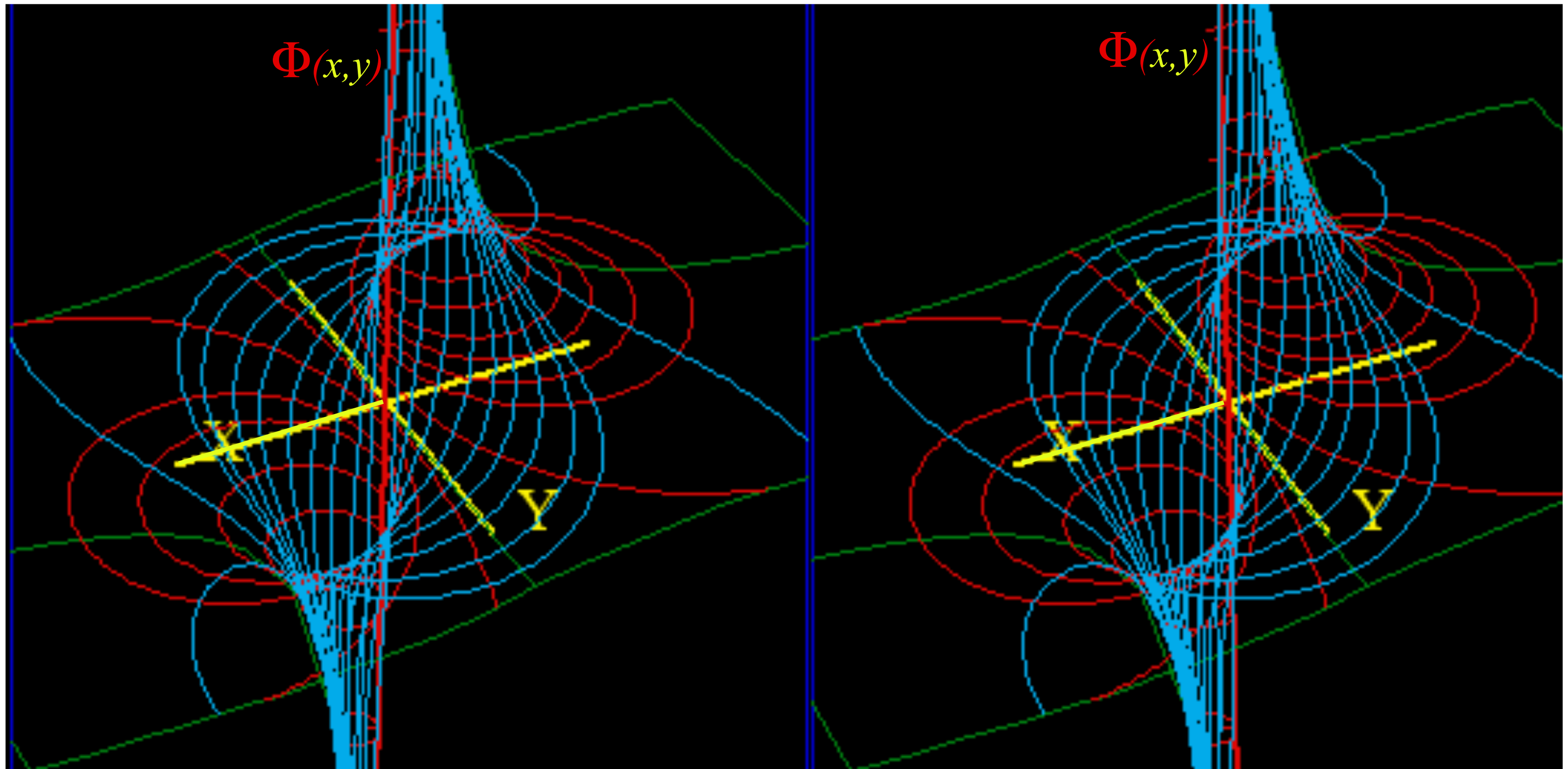

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A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

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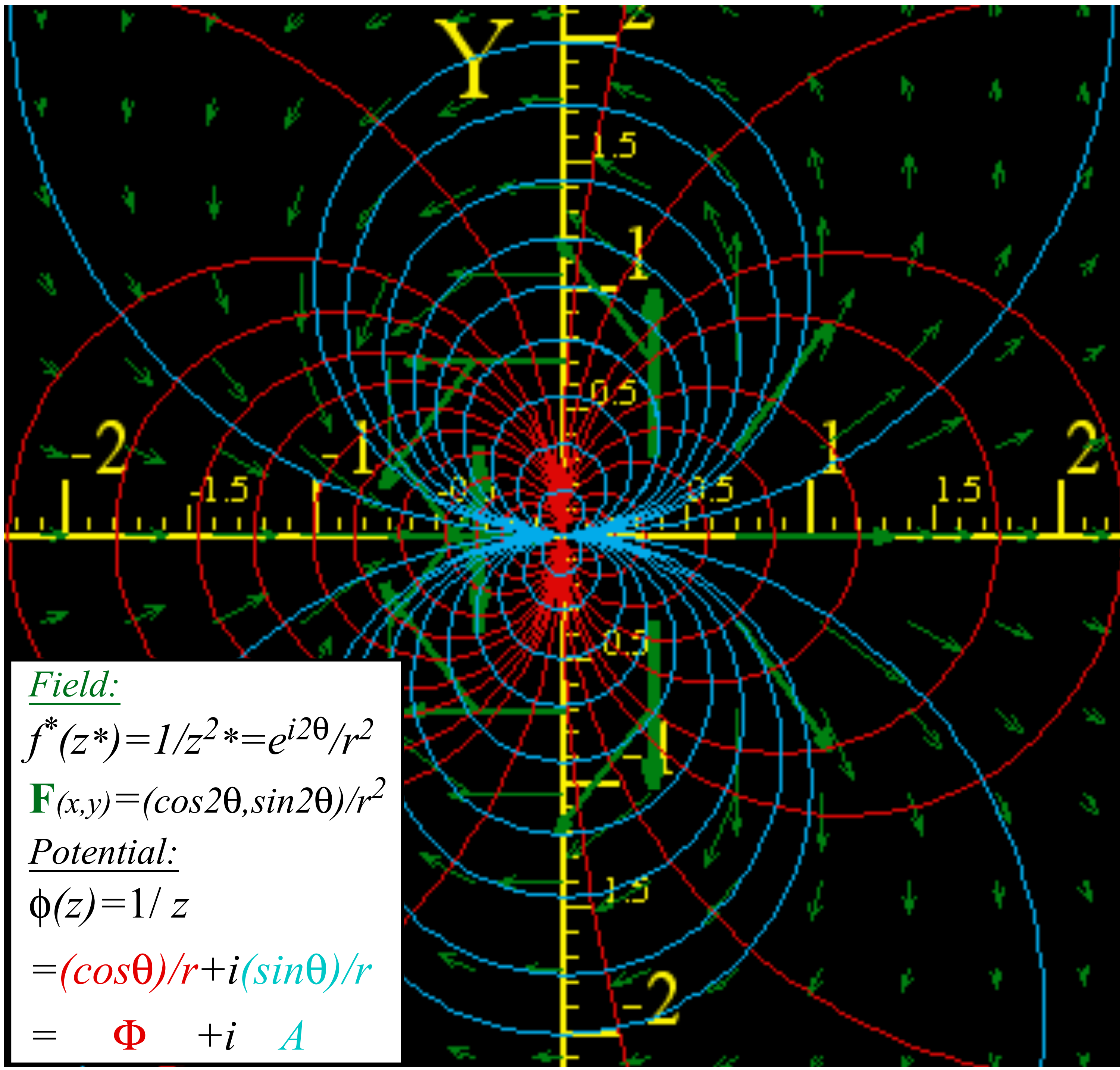
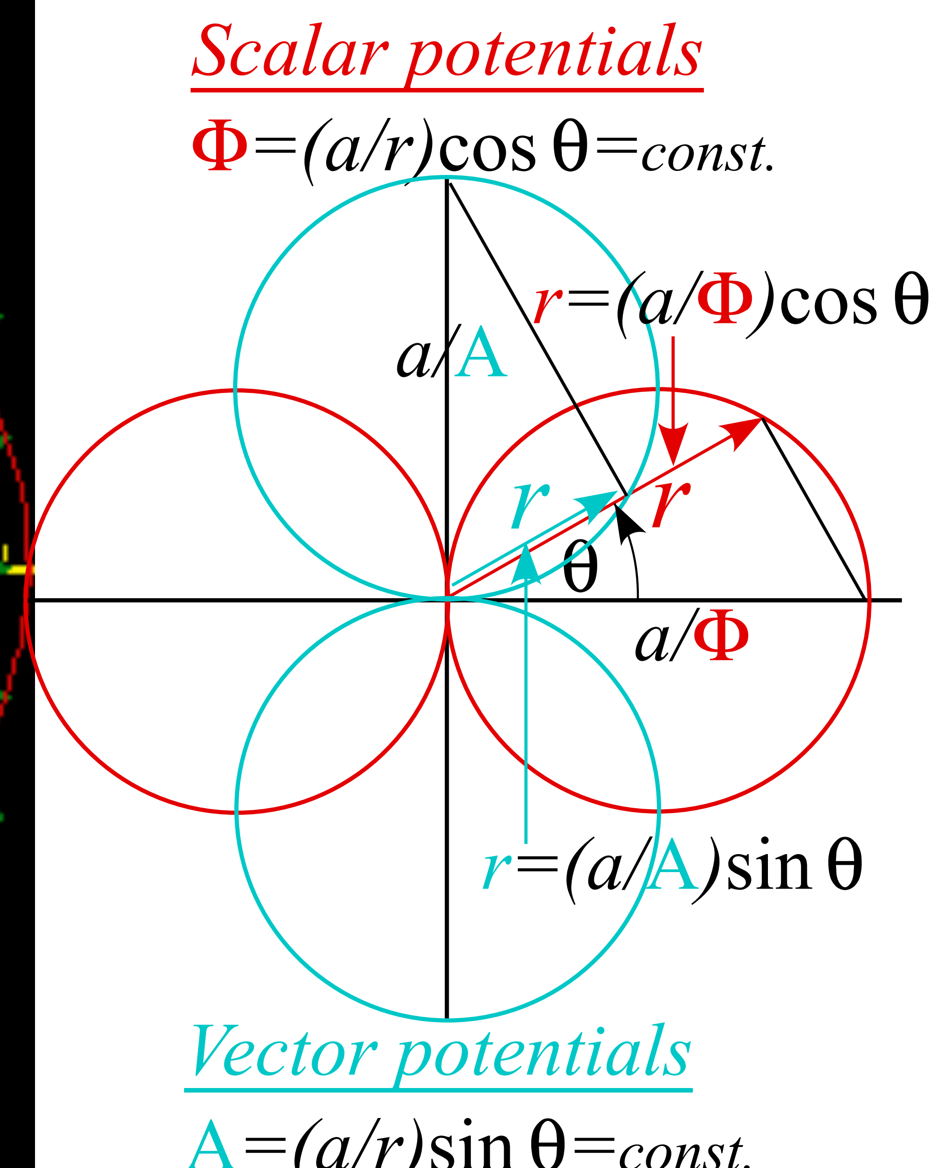




2^n*-pole analysis* *(quadrupole:2^2=4-pole, octapole:2^3=8-pole,…, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\,4\text{-}pole}$ and potential $\phi^{4\text{-}pole}$.

Each a z-derivative of $f^{\,2\text{-}pole}$ and $\phi^{2\text{-}pole}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad\qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$

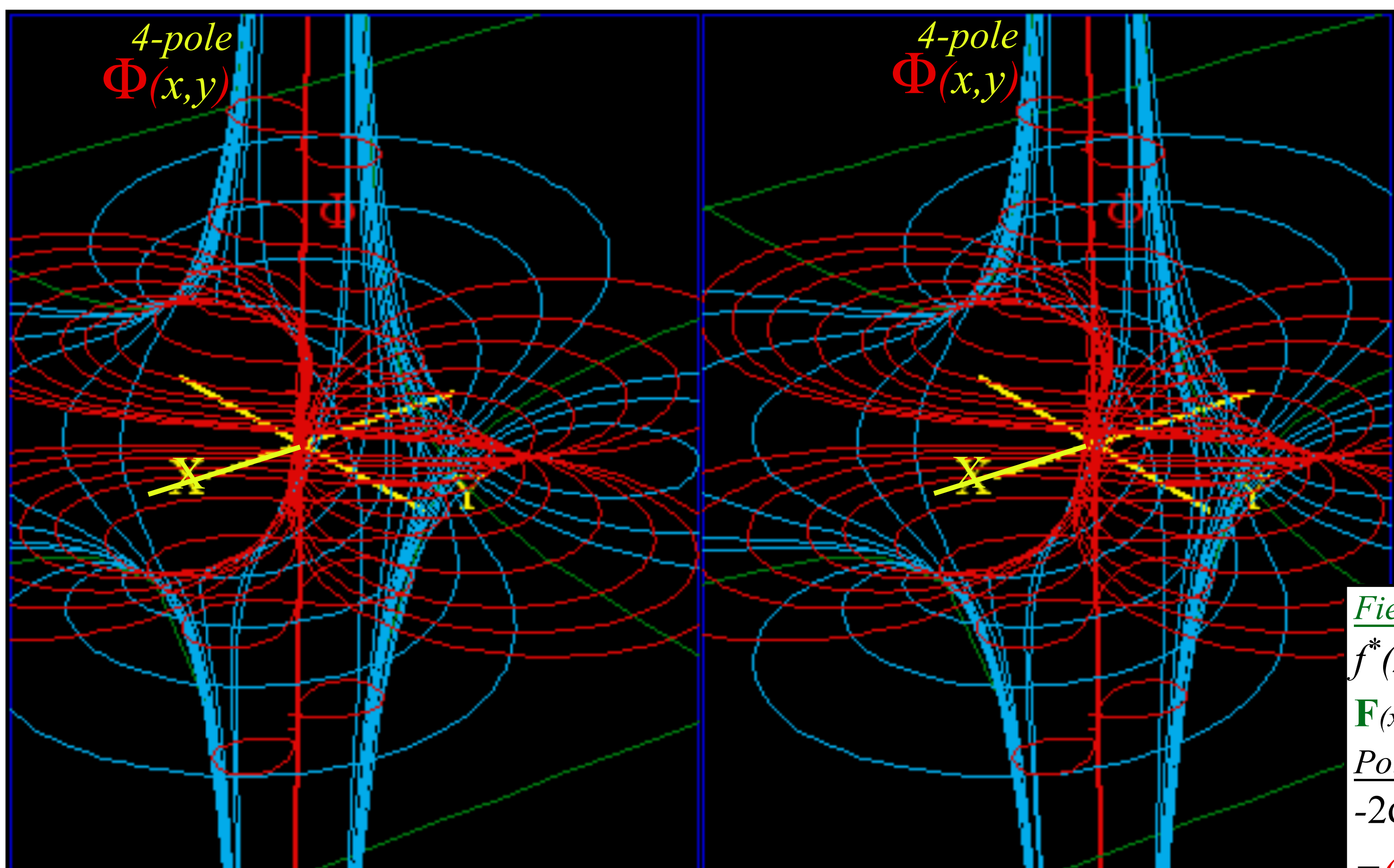
2^n-pole analysis *(quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, …, pole dancer,*

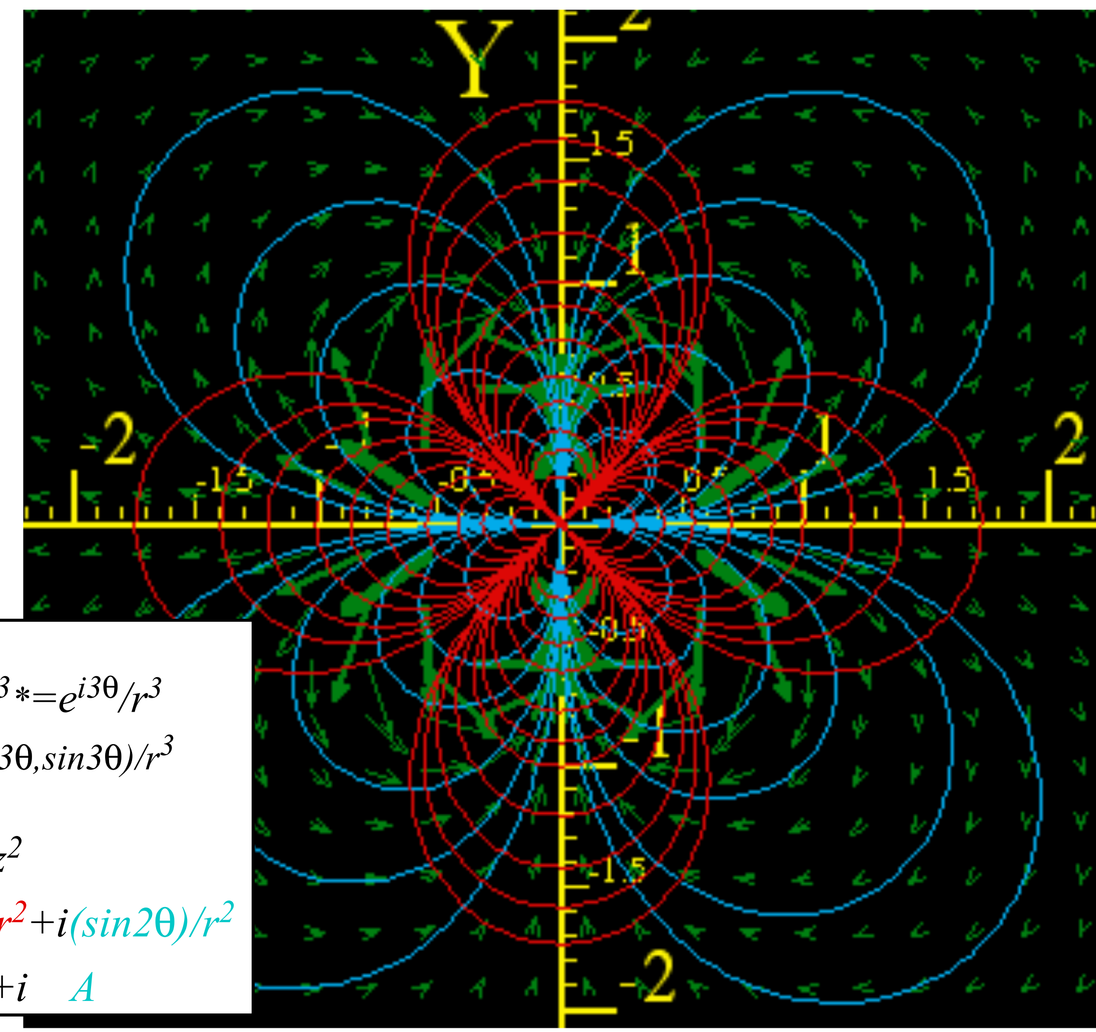
What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a *z*-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad\qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$





Field:
$f^*(z^*)=1/z^3{}^*=e^{i3\theta}/r^3$
$\mathbf{F}_{(x,y)}=(\cos 3\theta,\sin 3\theta)/r^3$
Potential:
$-2\phi(z)=1/z^2$
$=(\cos 2\theta)/r^2+i(\sin 2\theta)/r^2$
$= \quad \Phi \quad +i \quad A$

4. Riemann-Cauchy conditions *What's analytic? (...and what's* ***not*** *?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

→ *Easy 2^n-multipole field and potential expansion*

Easy stereo-projection visualization

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function $f(z)$ around $z=0$.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

$\cdots 2^2$-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz = \phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a 2^m*-pole*.

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

(octapole)$_0$	*(quadrupole)$_0$*	*(dipole)$_0$*	*(monopole)*	*(dipole)$_\infty$*	*(quadrupole)$_\infty$*	*(octapole)$_\infty$*

$$\phi(z)=\ldots\frac{a_{-4}}{-3}z^{-3} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \ldots$$

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at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz =$$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

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$$\phi(z)=\ldots\frac{a_{-4}}{-3}z^{-3} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \ldots$$

$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3} + \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-2}}{-1}w^{-1} + a_{-1}\ln w + a_0 w + \frac{a_1}{2}w^2 + \frac{a_2}{3}w^3 + \ldots$$

(*with* $z=w^{-1}$)

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at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

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(*with* $z\rightarrow w$)

$$=\ldots\frac{a_2}{3}z^{-3}+\frac{a_1}{2}z^{-2}+a_0z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^2+\frac{a_{-4}}{-3}z^3+\ldots$$

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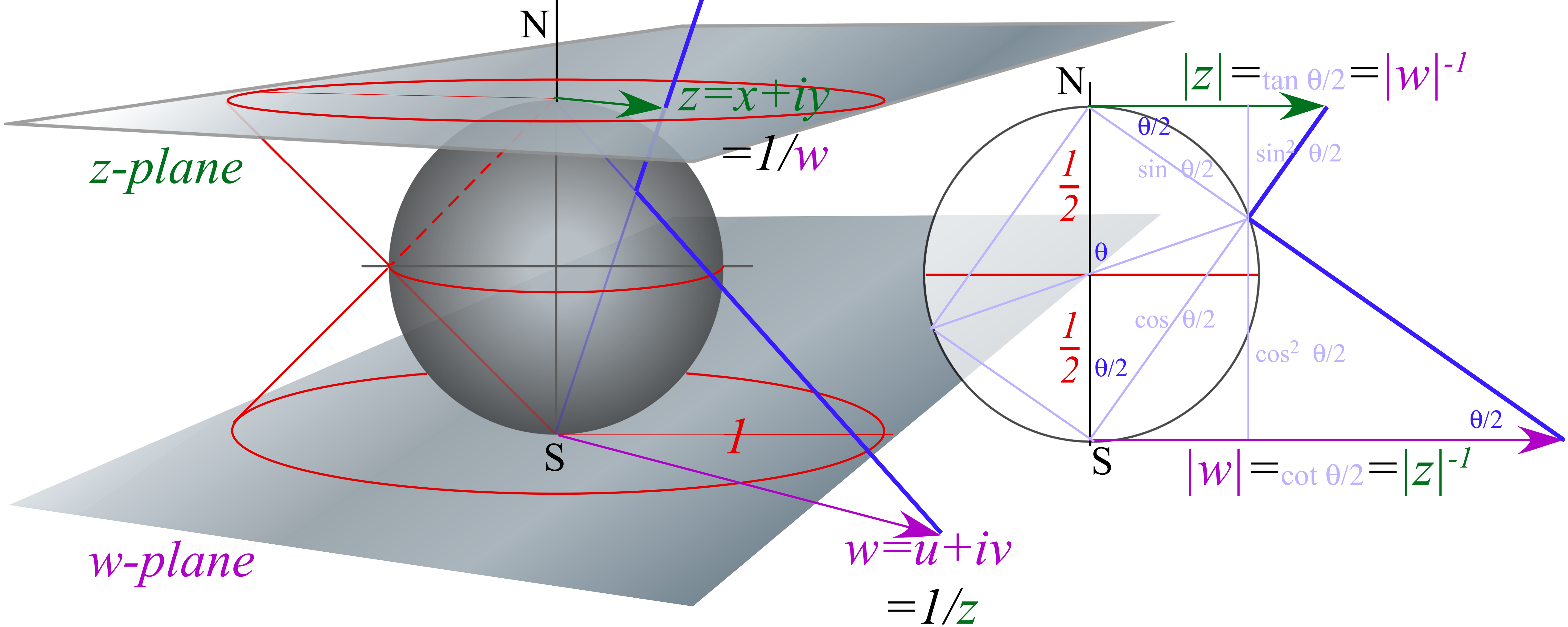

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(with $z \to w$)

$$= \ldots \frac{a_2}{3} z^{-2} + \frac{a_1}{2} z^{-2} + a_0 z^{-1} - a_{-1}\ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^2 + \frac{a_{-4}}{-3} z^3 + \ldots$$

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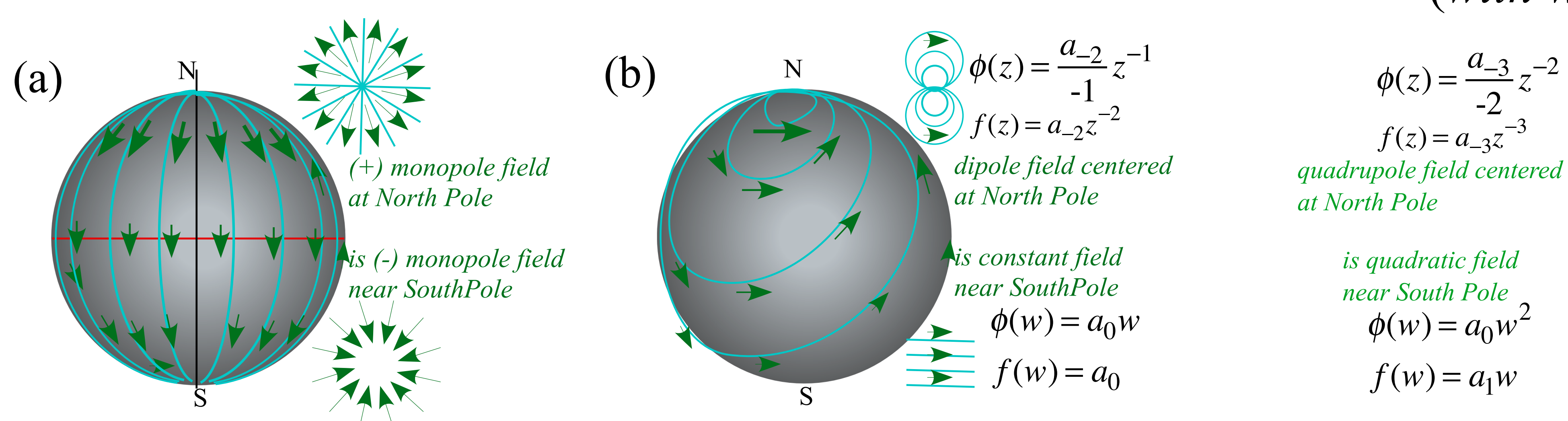

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Of all *2^m-pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

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(but any contour that doesn't "touch a gives same answer)

$$\boxed{f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz}$$

The $f(a)$ result is called a *Cauchy integral*. Then repeated a-derivatives gives a sequence of them.

$$\frac{df(a)}{da} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz\ ,\ \frac{d^2 f(a)}{da^2} = \frac{2}{2\pi i}\oint \frac{f(z)}{(z-a)^3}dz\ ,\ \frac{d^3 f(a)}{da^3} = \frac{3!}{2\pi i}\oint \frac{f(z)}{(z-a)^4}dz, \cdots, \frac{d^n f(a)}{da^n} = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz$$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

Of all 2^m*-pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$$

This $m=1$*-pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\cdots a_{-3} = \frac{1}{2\pi i}\oint z^2 f(z)dz \;,\; a_{-2} = \frac{1}{2\pi i}\oint z^1 f(z)dz \;,\; a_{-1} = \frac{1}{2\pi i}\oint f(z)dz \;,\; a_0 = \frac{1}{2\pi i}\oint \frac{f(z)}{z}dz \;,\; a_1 = \frac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \;,\cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

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(quadrupole)₀ *(dipole)₀* *(monopole)* *(dipole)∞* *(quadrupole)∞* *(octapole)∞* *(hexadecapole)∞* ...

$$f(z) = \ldots a_{-3}z^{-3} + \underbrace{a_{-2}z^{-2}}_{\text{dipole moment}} + \underbrace{a_{-1}z^{-1}}_{\text{monopole moment}} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

5. Mapping and Non-analytic 2D source field analysis

The *half*-n'-*half* results

are called

Riemann-Cauchy

Derivative Relations

$$\boxed{\begin{array}{l}\frac{\partial\Phi}{\partial x}=\frac{\partial A}{\partial y}\\ \frac{\partial\Phi}{\partial y}=-\frac{\partial A}{\partial x}\end{array}}\begin{array}{l}\text{is:}\\ \text{is:}\end{array}\boxed{\begin{array}{l}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{array}}\begin{array}{l}\text{or:}\\ \text{or:}\end{array}\boxed{\begin{array}{l}\frac{\partial \mathrm{Re}\,f(z)}{\partial x}=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y}=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{array}}\begin{array}{l}\text{is:}\\ \text{is:}\end{array}\boxed{\begin{array}{l}\frac{\partial f_x(z)}{\partial x}=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}=-\frac{\partial f_y(z)}{\partial x}\end{array}}$$

RC applies to analytic potential $\phi(z)=\Phi+iA$ *and analytic field* $f(z)=f_x+if_y$ *and any analytic function*

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Derivative Relations

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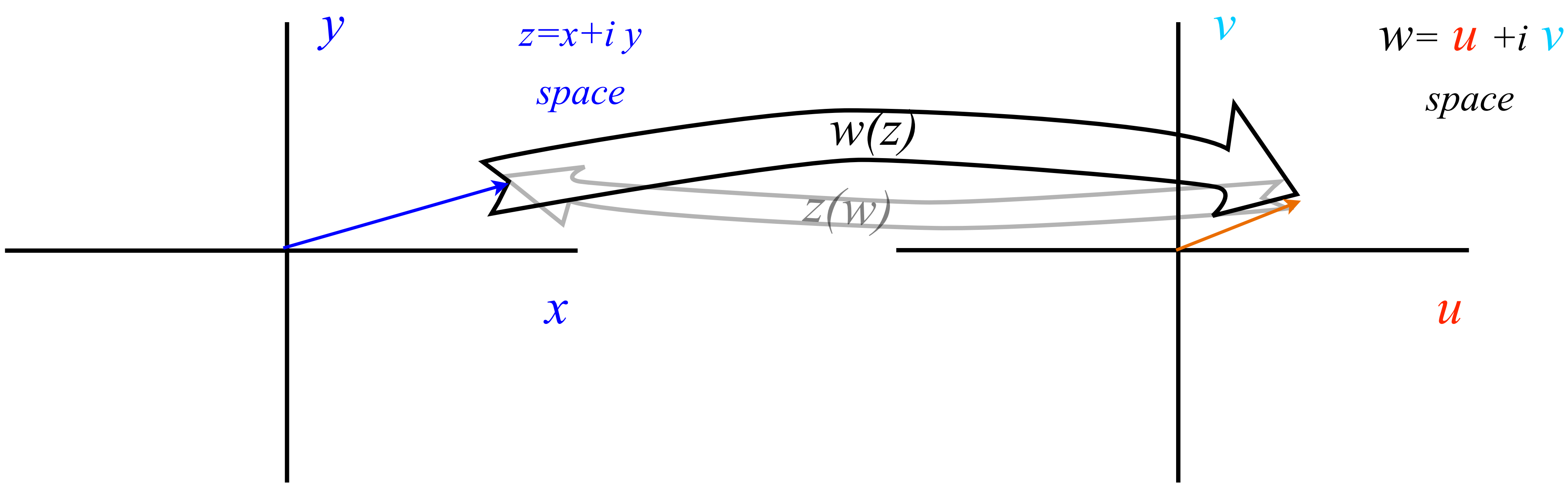
RC applies to analytic potential $\phi(z)=\Phi+i\mathbf{A}$ *and analytic field* $f(z)=f_x+if_y$ *and any analytic function*

Common notation for mapping: $w(z)=u+iv$



The *half*-n'-*half* results

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Derivative Relations

$$\boxed{\begin{aligned}\frac{\partial\Phi}{\partial x}&=\frac{\partial A}{\partial y}\\ \frac{\partial\Phi}{\partial y}&=-\frac{\partial A}{\partial x}\end{aligned}}\ \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial\,\mathrm{Re}\,\phi(z)}{\partial x}&=\frac{\partial\,\mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial\,\mathrm{Re}\,\phi(z)}{\partial y}&=-\frac{\partial\,\mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}}\ \begin{aligned}&\text{or:}\\&\text{or:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial\,\mathrm{Re}\,f(z)}{\partial x}&=\frac{\partial\,\mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial\,\mathrm{Re}\,f(z)}{\partial y}&=-\frac{\partial\,\mathrm{Im}\,f(z)}{\partial x}\end{aligned}}\ \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x}&=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}&=-\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

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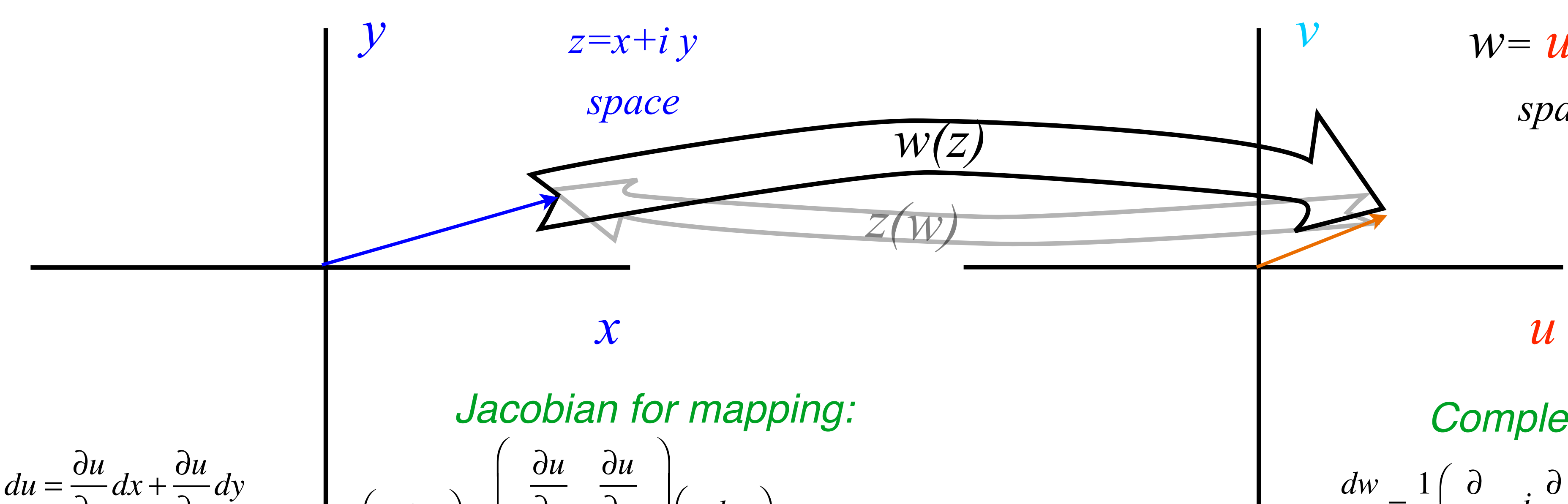

Jacobian for mapping:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$
$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix}du\\dv\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

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Riemann-Cauchy

Derivative Relations

$$\boxed{\begin{aligned}\frac{\partial \Phi}{\partial x}&=\frac{\partial A}{\partial y}\\ \frac{\partial \Phi}{\partial y}&=-\frac{\partial A}{\partial x}\end{aligned}}\ \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}}\ \begin{aligned}&\text{or:}\\&\text{or:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,f(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{aligned}}\ \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x}&=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}&=-\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

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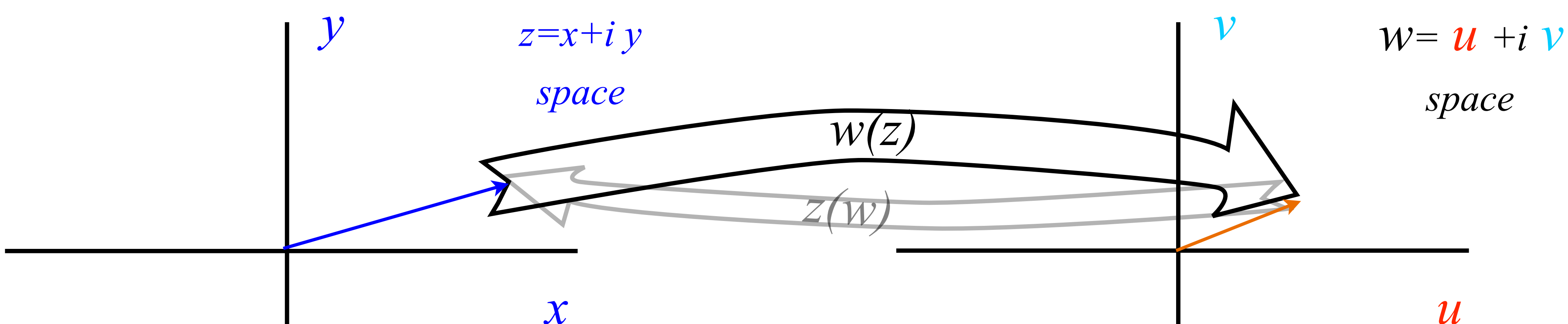

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

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$$\begin{pmatrix} du\\ dv\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix} dx\\ dy\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x}\end{pmatrix}\begin{pmatrix} dx\\ dy\end{pmatrix}=\begin{pmatrix}\frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix} dx\\ dy\end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

The half-n'-half results

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Riemann-Cauchy

Derivative Relations

$$\boxed{\begin{aligned}\frac{\partial\Phi}{\partial x}&=\frac{\partial A}{\partial y}\\ \frac{\partial\Phi}{\partial y}&=-\frac{\partial A}{\partial x}\end{aligned}}\;\begin{matrix}\text{is:}\\ \text{is:}\end{matrix}\;\boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}}\;\begin{matrix}\text{or:}\\ \text{or:}\end{matrix}\;\boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,f(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{aligned}}\;\begin{matrix}\text{is:}\\ \text{is:}\end{matrix}\;\boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x}&=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}&=-\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

RC applies to analytic potential $\phi(z)=\Phi+i\mathbf{A}$ and analytic field $f(z)=f_x+if_y$ and any analytic function

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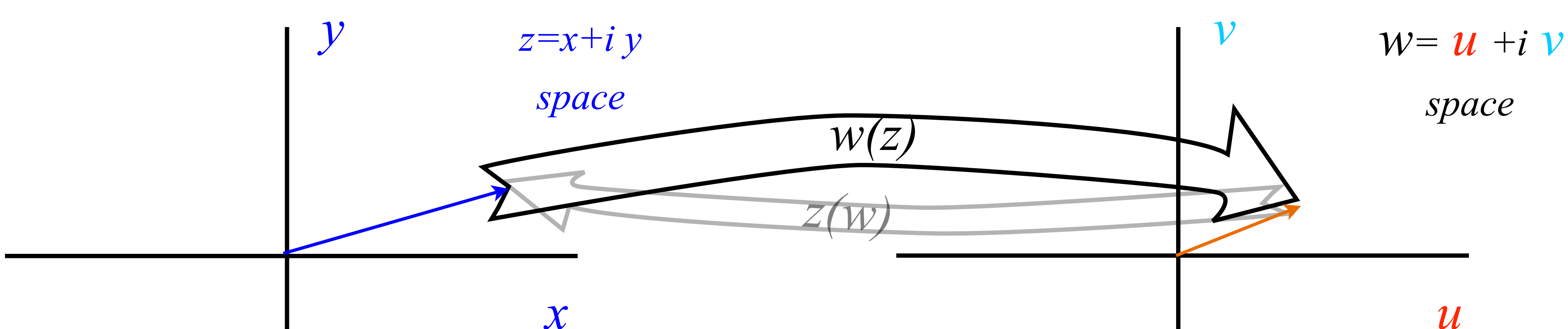

Jacobian for mapping:

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$$=\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ -\frac{\partial u}{\partial y}&\frac{\partial u}{\partial x}\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}=\begin{pmatrix}\frac{\partial v}{\partial y}&-\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

The *half*-n'-*half* results

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Riemann-Cauchy

Derivative Relations

$\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y}$	is:	$\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x} = \frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}$	or:	$\frac{\partial \mathrm{Re}\,f(z)}{\partial x} = \frac{\partial \mathrm{Im}\,f(z)}{\partial y}$	is:	$\frac{\partial f_x(z)}{\partial x} = \frac{\partial f_y(z)}{\partial y}$
$\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x}$	is:	$\frac{\partial \mathrm{Re}\,\phi(z)}{\partial y} = -\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}$	or:	$\frac{\partial \mathrm{Re}\,f(z)}{\partial y} = -\frac{\partial \mathrm{Im}\,f(z)}{\partial x}$	is:	$\frac{\partial f_x(z)}{\partial y} = -\frac{\partial f_y(z)}{\partial x}$

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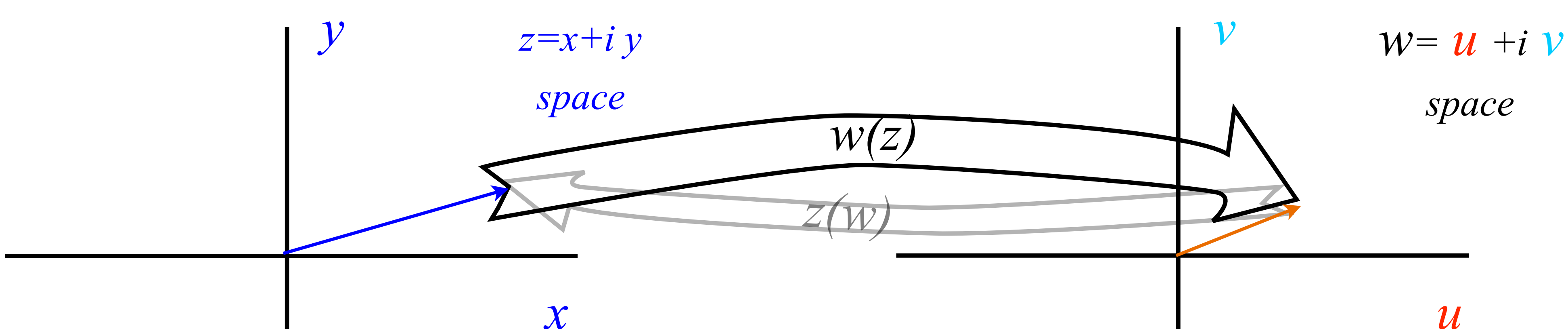


Important result:

$$dw = \sqrt{J} \cdot e^{i\theta} \cdot dz$$

is scaled rotation of dz.

Jacobian for mapping is scaled rotation:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \sqrt{\det J} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

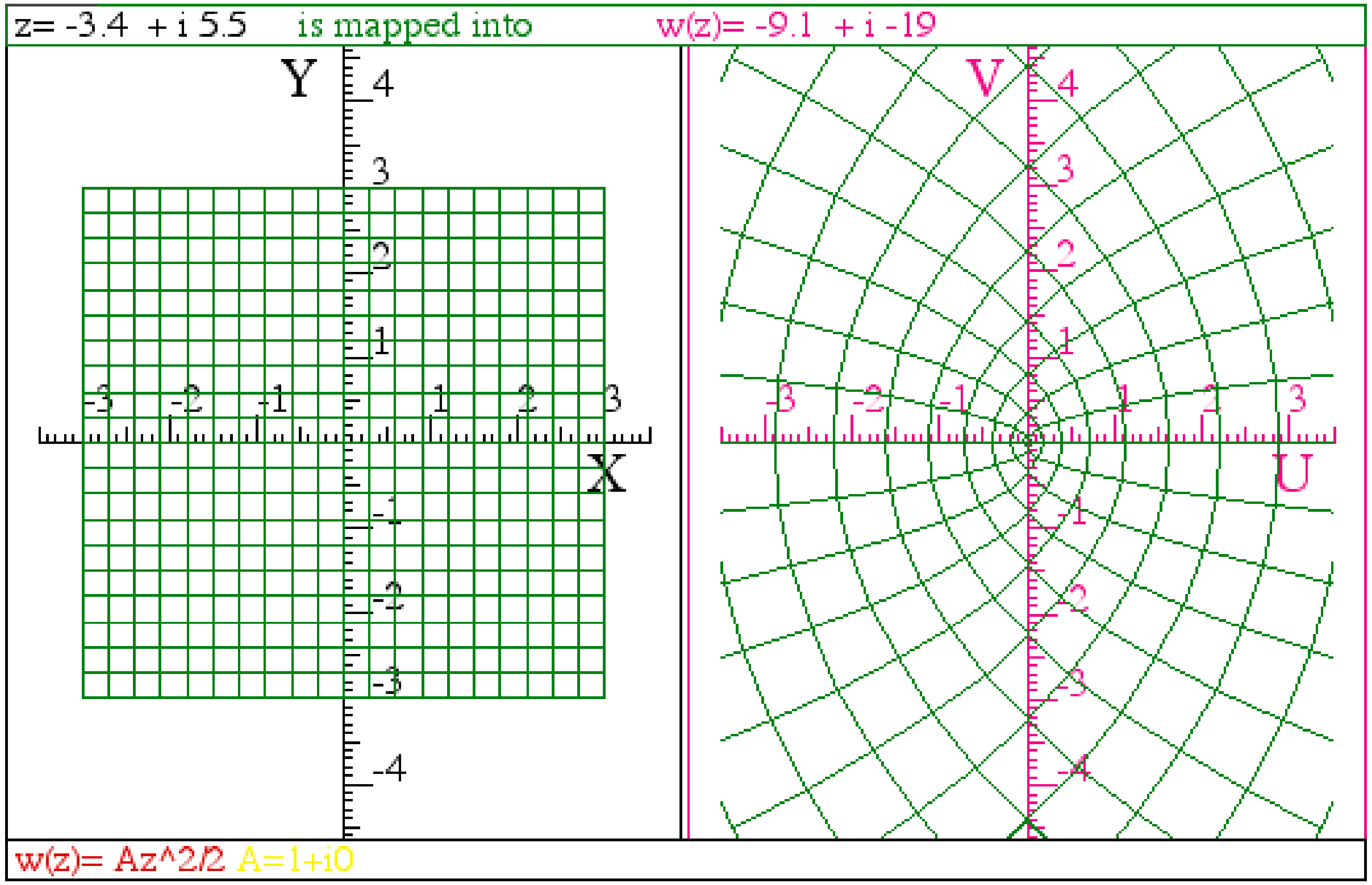
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

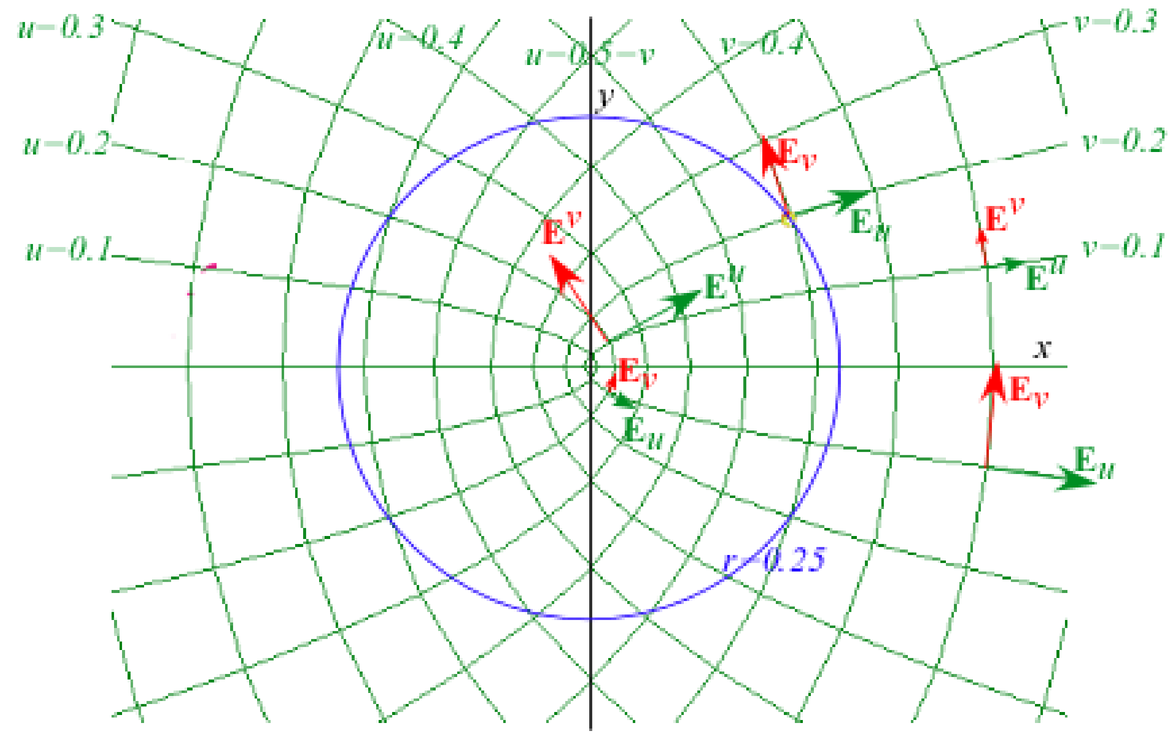
Complex derivative abs-square:

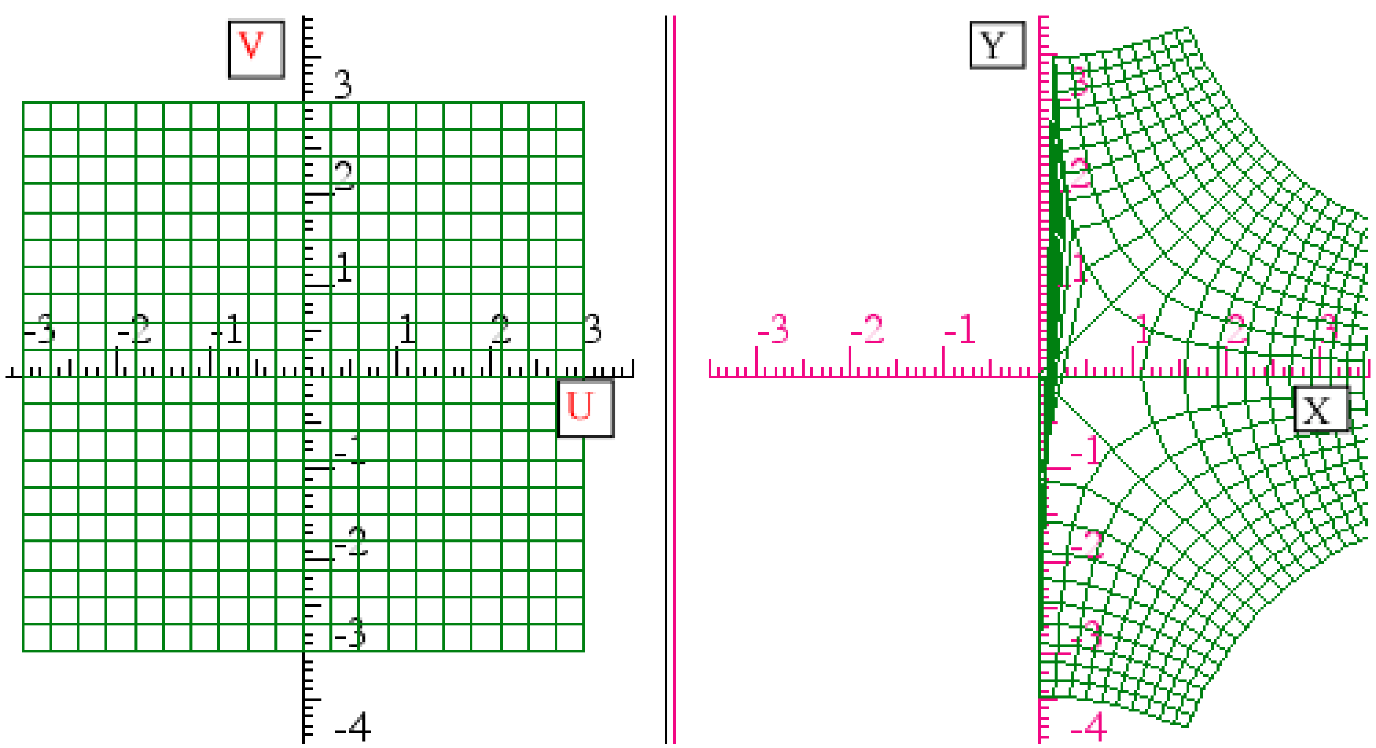
$$\left|\frac{dw}{dz}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \det|J|$$

...equals Jacobian Determinant

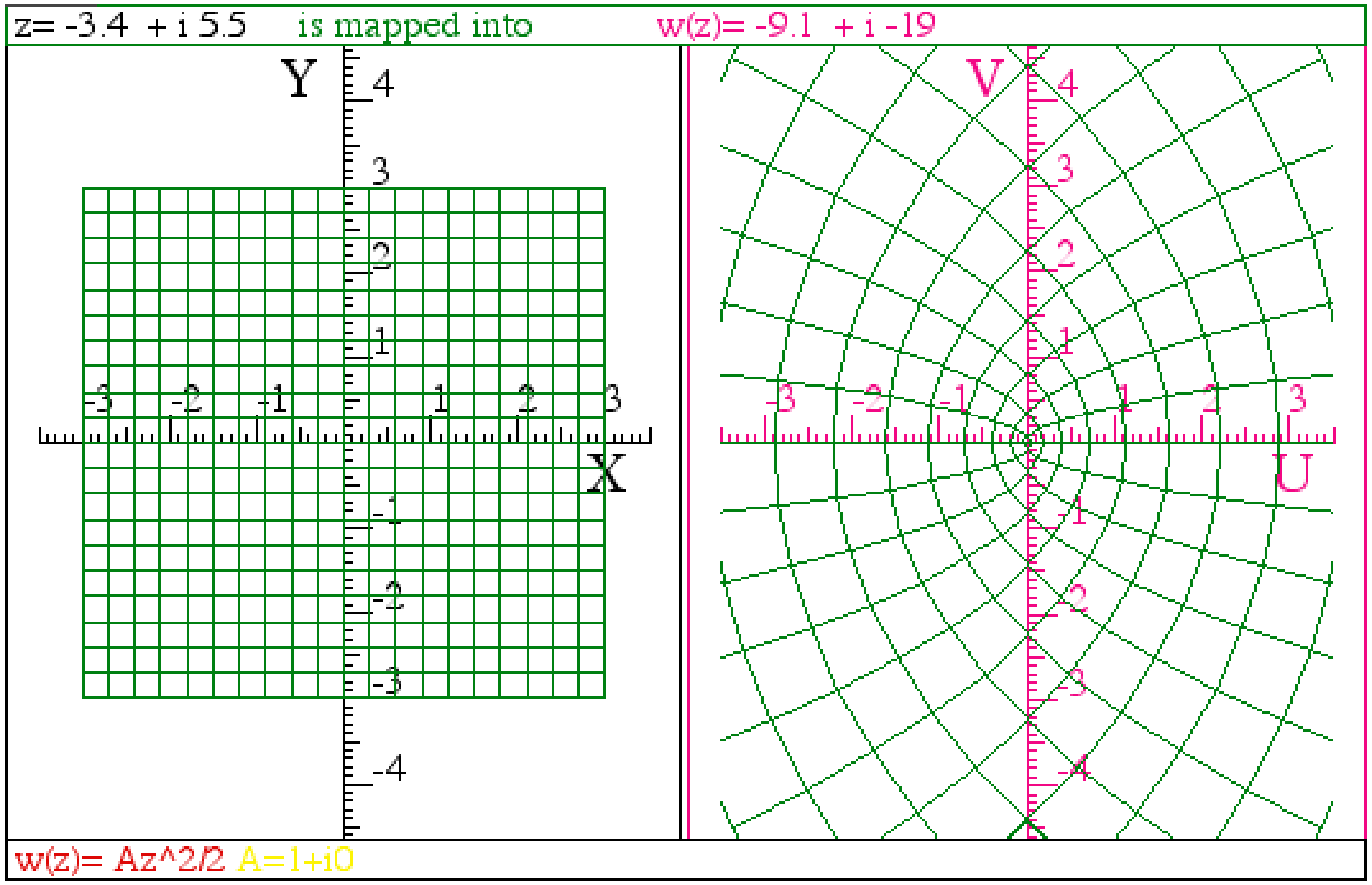
$w(z) = z^2$ gives parabolic OCC

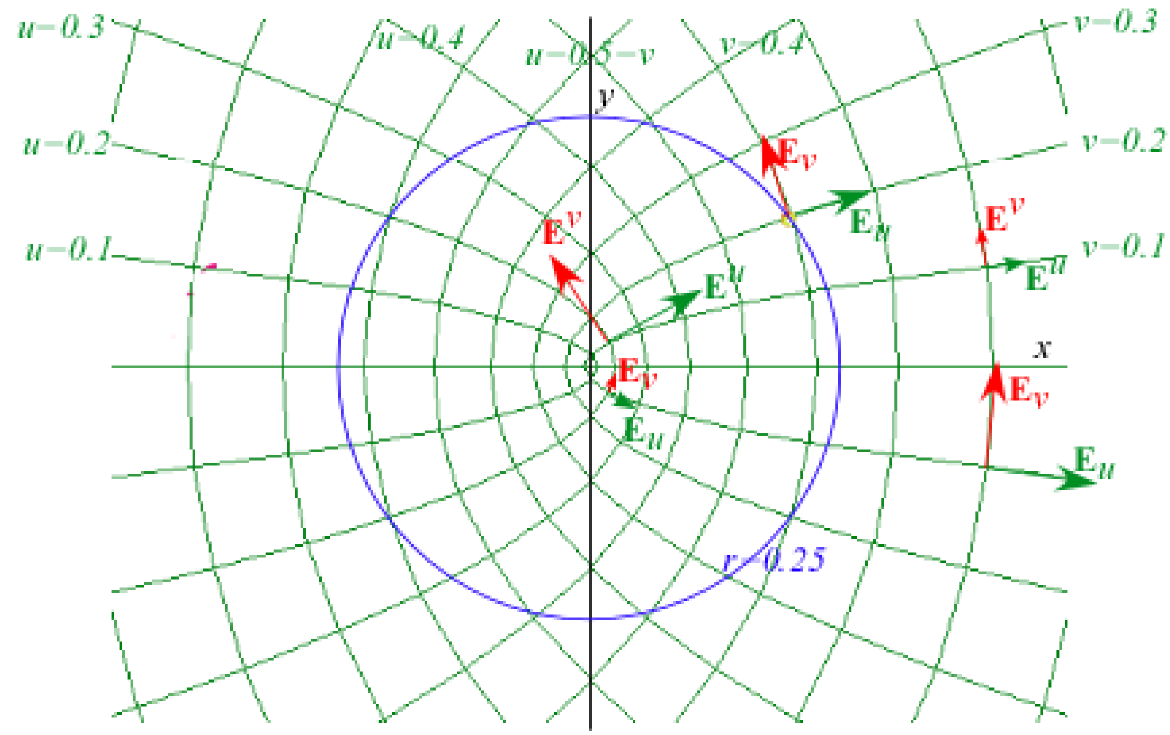




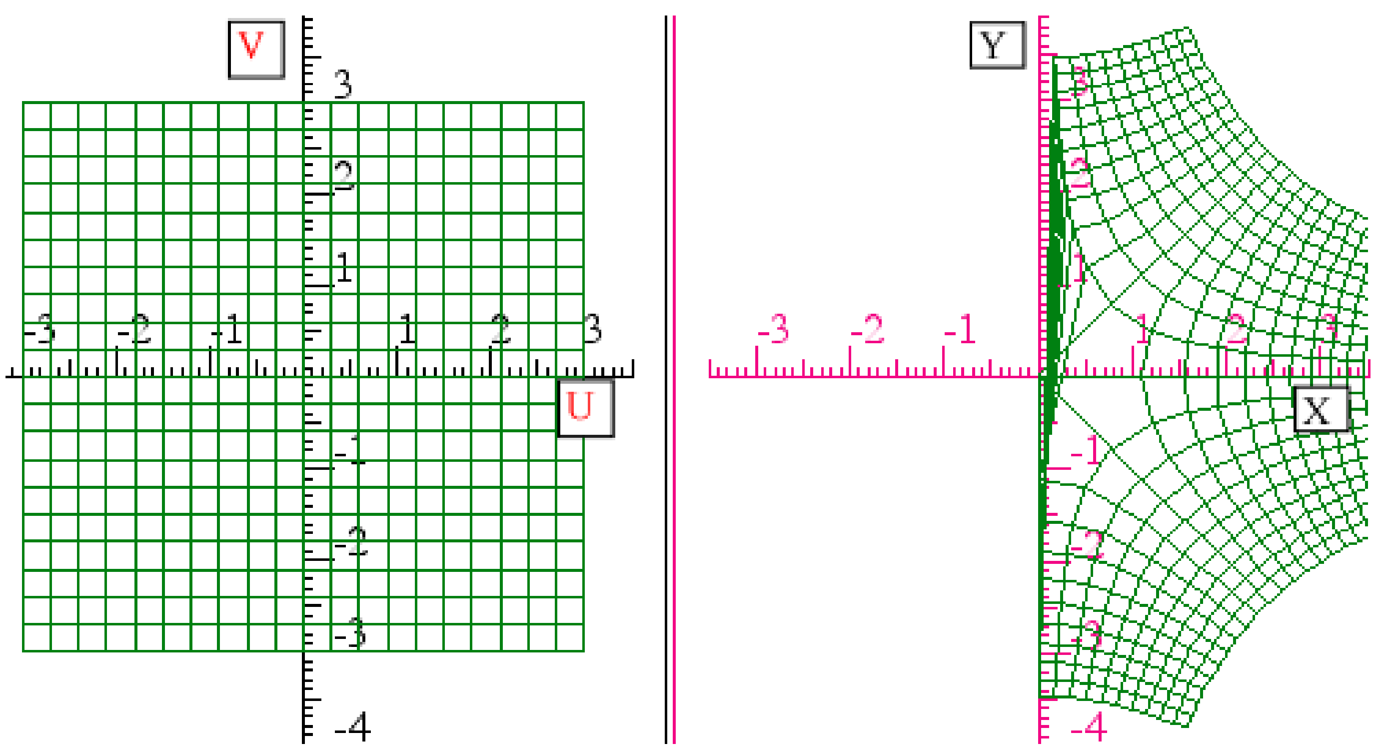


$w(z) = z^2$ gives parabolic OCC





Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



5. Mapping and Non-analytic 2D source field analysis

Non-analytic potential, force, and source field functions (Excerpts of Unit 1-Ch.10 and AnalyIt)

A general 2D complex field may have:

1. non-analytic *potential field function* $\phi(z,z^*)=\Phi(x,y)+iA(x,y)$,
2. non-analytic *force field function* $f(z,z^*) = fx(x,y) + ify(x,y)$,
3. non-analytic *source distribution function* $s(z,z^*) = \rho(x,y) + i\, I(x,y)$.

Source definitions generalize source-free fields ($\frac{df(z^*)}{dz}=0=\frac{df(z)}{dz^*}$) based on relations. $\frac{df}{dz}=\frac{1}{2}(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y})+\frac{i}{2}(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y})=\frac{1}{2}\nabla\bullet\mathbf{F}+\frac{i}{2}|\nabla\times\mathbf{F}|$

$$2\frac{df^*}{dz}=s^*(z,z^*) \qquad\qquad 2\frac{df}{dz^*}=s(z,z^*)$$

Field-f-from-potential-ϕ equations are like the older ($f(z)=\frac{d\phi}{dz}$ or $f^*(z*)=\frac{d\phi*}{dz*}$) but with an extra factor of *2*.

$$2\frac{d\phi}{dz}=f(z,z^*) \qquad\qquad 2\frac{d\phi^*}{dz^*}=f^*(z,z^*)$$

The new source equations expand into a real and imaginary parts that are divergence and curl terms, respectively.

$$s^*(z,z^*)=2\frac{df^*}{dz}=\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]\left[f_x^*(x,y)+if_y^*(x,y)\right] \quad =\rho-i\,I\ , \quad \text{where: } f_x^*=f_x\ ,\ \text{ and: } f_y^*=-f_y$$

$$=\left[\frac{\partial f_x^*}{\partial x}+\frac{\partial f_y^*}{\partial y}\right]+i\left[\frac{\partial f_y^*}{\partial x}-\frac{\partial f_x^*}{\partial y}\right] \quad =\left[\nabla\bullet\mathbf{f}^*\right]+i\left[\nabla\times\mathbf{f}^*\right]_Z$$

Real part: *Poisson scalar source equation* (*charge density* ρ). Imaginary part: *Biot-Savart vector source equation*(*current density I*)

$$\nabla\bullet\mathbf{f}^*=\rho \qquad\qquad \nabla\times\mathbf{f}^*=-I$$

Field-*vs*-potential has Re and Im parts that are x and y components of grad Φ and curlA_Z from potential $\phi=\Phi+iA$ or $\phi^*=\Phi-iA$.

$$f^*(z,z^*)=2\frac{d\phi^*}{dz^*}=\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right]\left(\Phi-iA\right)=f_x^*+if_y^*$$

$$=\left[\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right]+\left[\frac{\partial A}{\partial y}-i\frac{\partial A}{\partial x}\right]=\left[\nabla\Phi\right]+\left[\nabla\times\mathbf{A}_Z\right]$$

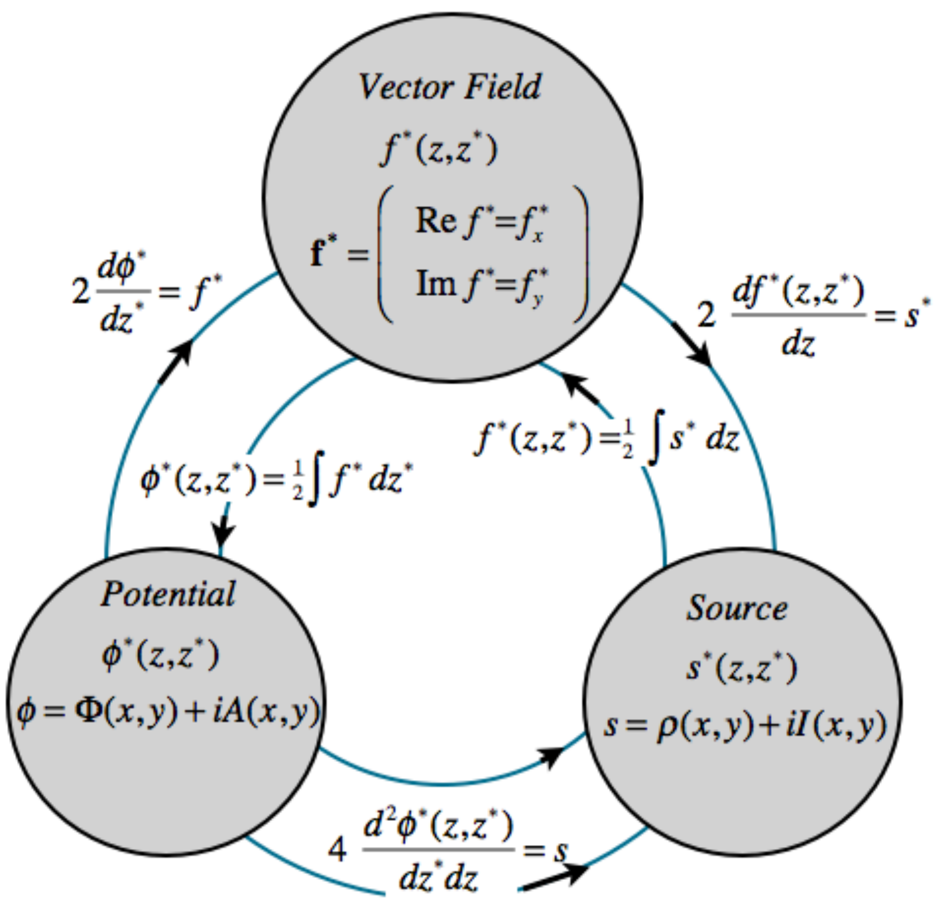
(For source-free analytic functions these two fields are identical.)

Gradient of scalar potential is the *longitudinal field* $\mathbf{f}_\mathbf{L}^*$ and curl of a vector potential is the *transverse field* $\mathbf{f}_\mathbf{T}^*$.

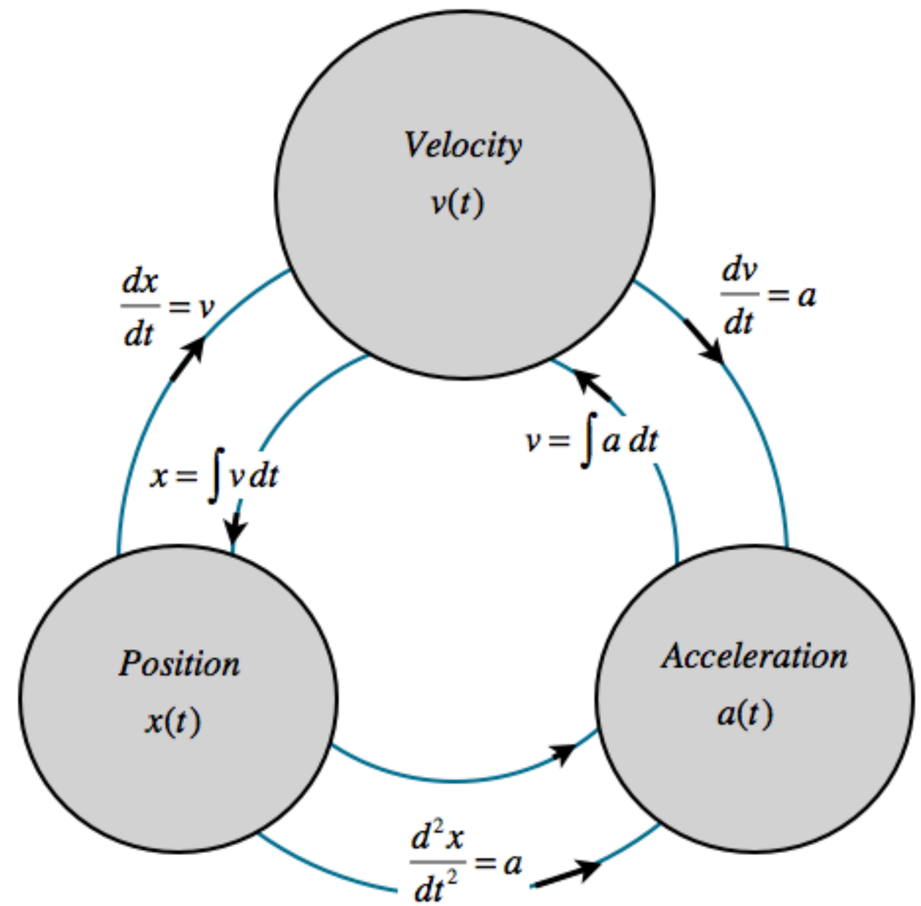
Total field is: $\mathbf{f}^*=\mathbf{f}_\mathbf{L}^*+\mathbf{f}_\mathbf{T}^*$ $\qquad \mathbf{f}_\mathbf{L}^*=\nabla\Phi$ $\qquad \mathbf{f}_\mathbf{T}^*=\nabla\times\mathbf{A}$

Potential, force, and source field equations vs. position, velocity, and acceleration equations

Field equations



Newton equations



Potential and source field theory reduced to sophomore mechanics of 1D-motion!

Example 1 Consider a non-analytic field $f(z) = (z^*)^2$ or $f^*(z) = z^2$.

Non-analytic source s^* is derivative of field f^*

$$s^*(z,z^*) = 2\frac{df^*}{dz} = 4z = 4x + i4y,$$

$$or: \quad \rho = 4x, \quad and: \quad I = -4y.$$

Non-analytic potential ϕ is integral of field f^*

$$\phi(z,z^*) = \frac{1}{2}\int f(z)\,dz = \frac{1}{2}\int (z^*)^2\,dz = \frac{z(z^*)^2}{2} = \frac{(x+iy)(x^2-y^2-i2xy)}{2},$$

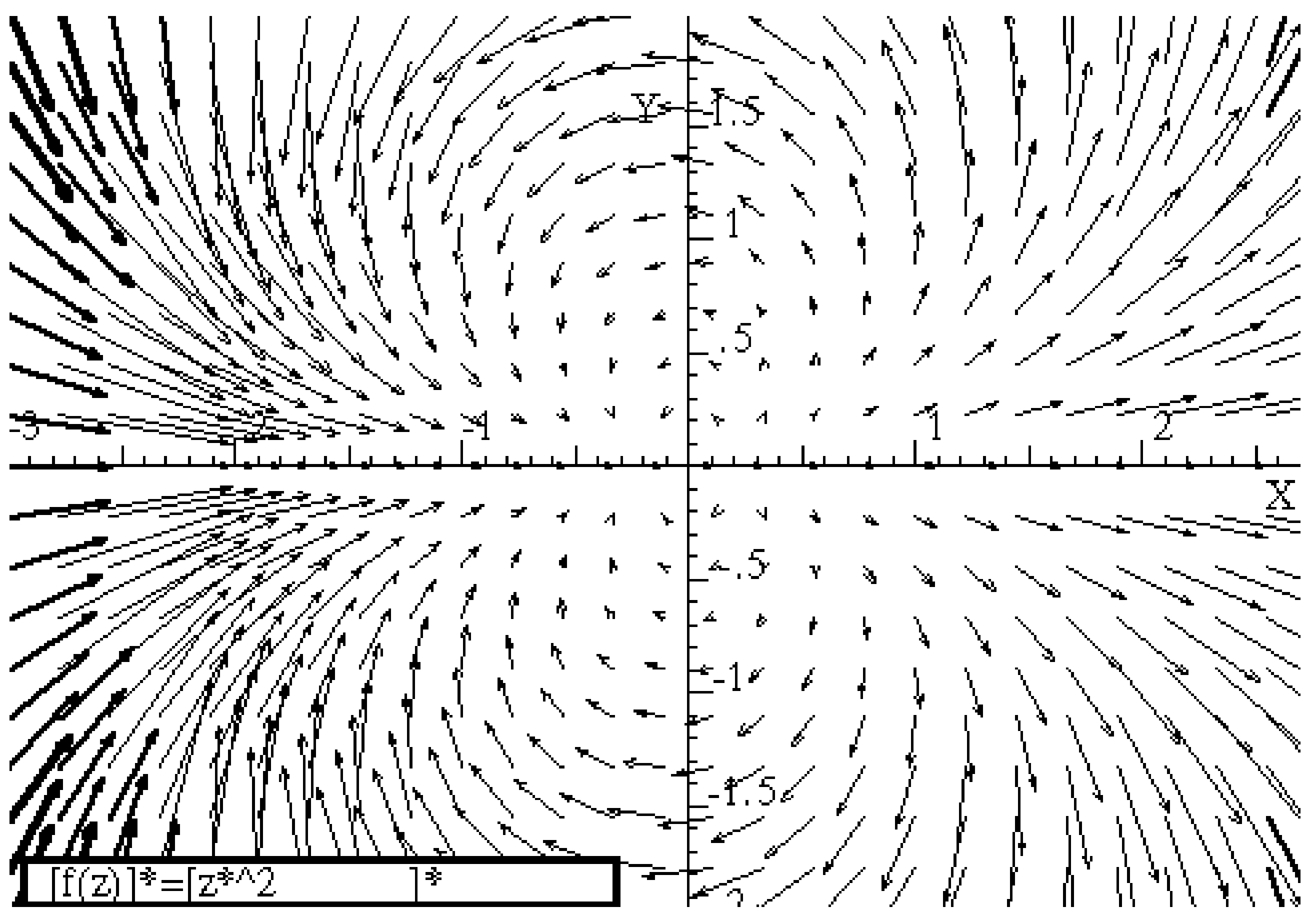
$$or: \quad \Phi = \frac{x^3+xy^2}{2}, \quad and: \ A = \frac{-y^3-yx^2}{2}.$$

The longitudinal field $\mathbf{f}_L^*$ is quite different from the transverse field $\mathbf{f}_L^*$

$$\mathbf{f}_L^* = \nabla\Phi = \nabla\left(\frac{x^3+xy^2}{2}\right) = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix}, \qquad \mathbf{f}_T^* = \nabla\times\mathbf{A} = \nabla\times\left(\frac{-y^3-yx^2}{2}\mathbf{e}_z\right) = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix}.$$

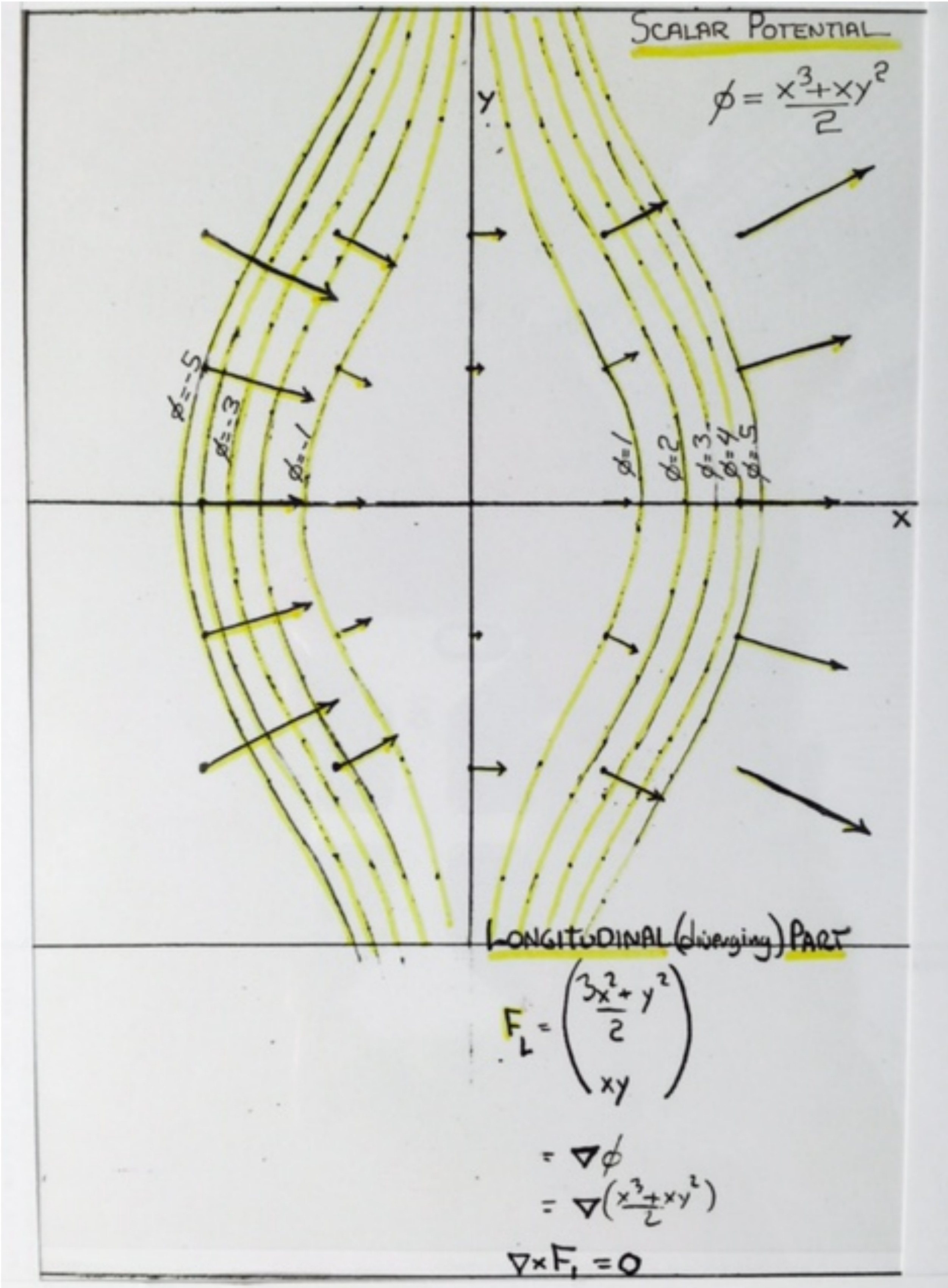
Longitudinal field $\mathbf{f}_L^*$ has no curl and the transverse field $\mathbf{f}_L^*$ has no divergence. Sum field has both.

$$\mathbf{f}^* = \mathbf{f}_L^* + \mathbf{f}_T^* = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix} + \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix} = \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}, \quad \nabla\bullet\mathbf{f}^* = \nabla\bullet\mathbf{f}_L^* = 4x = \rho, \quad \nabla\times\mathbf{f}^* = \nabla\times\mathbf{f}_T^* = 4y = -I.$$

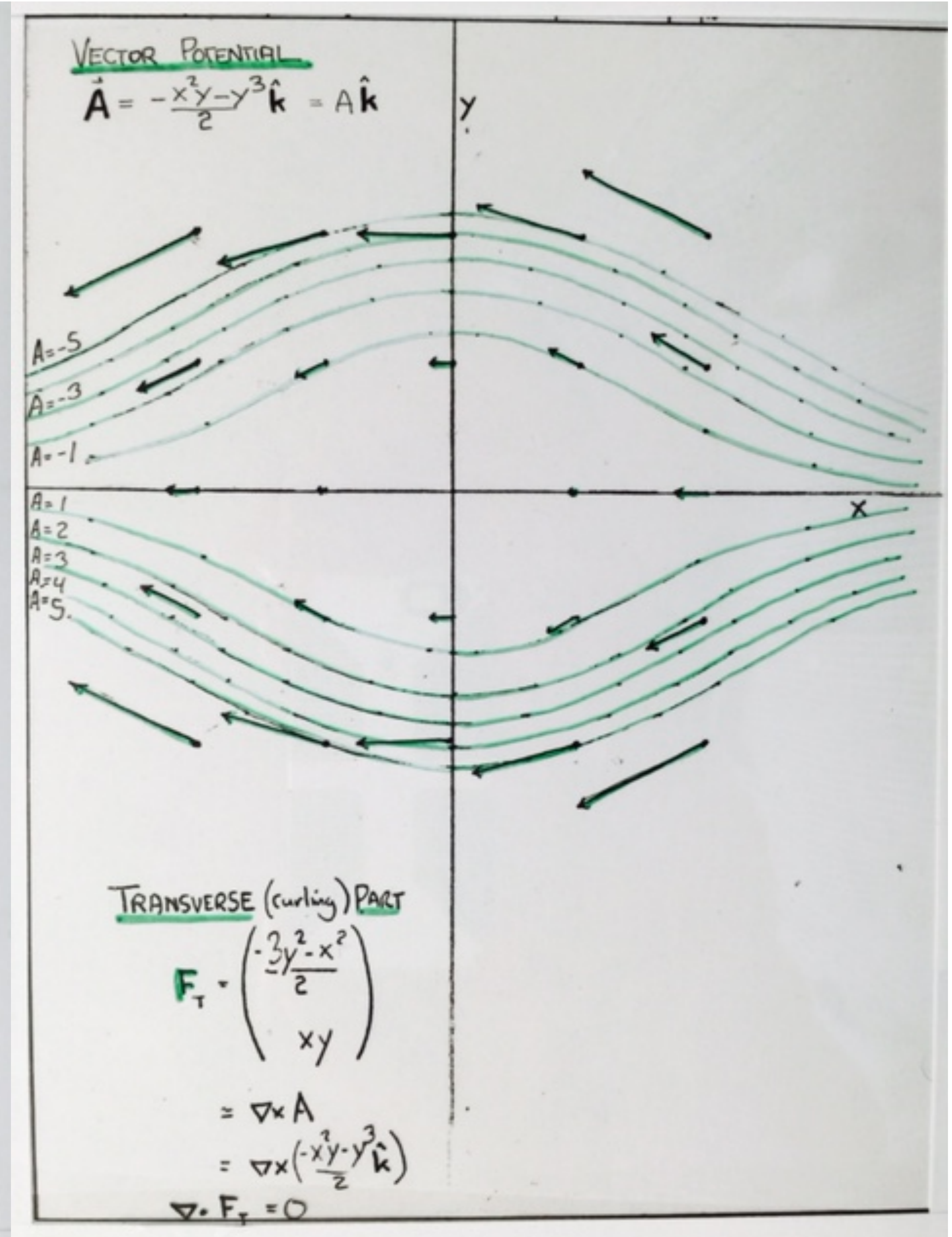


$\mathbf{f}_L^* + \mathbf{f}_T^*$

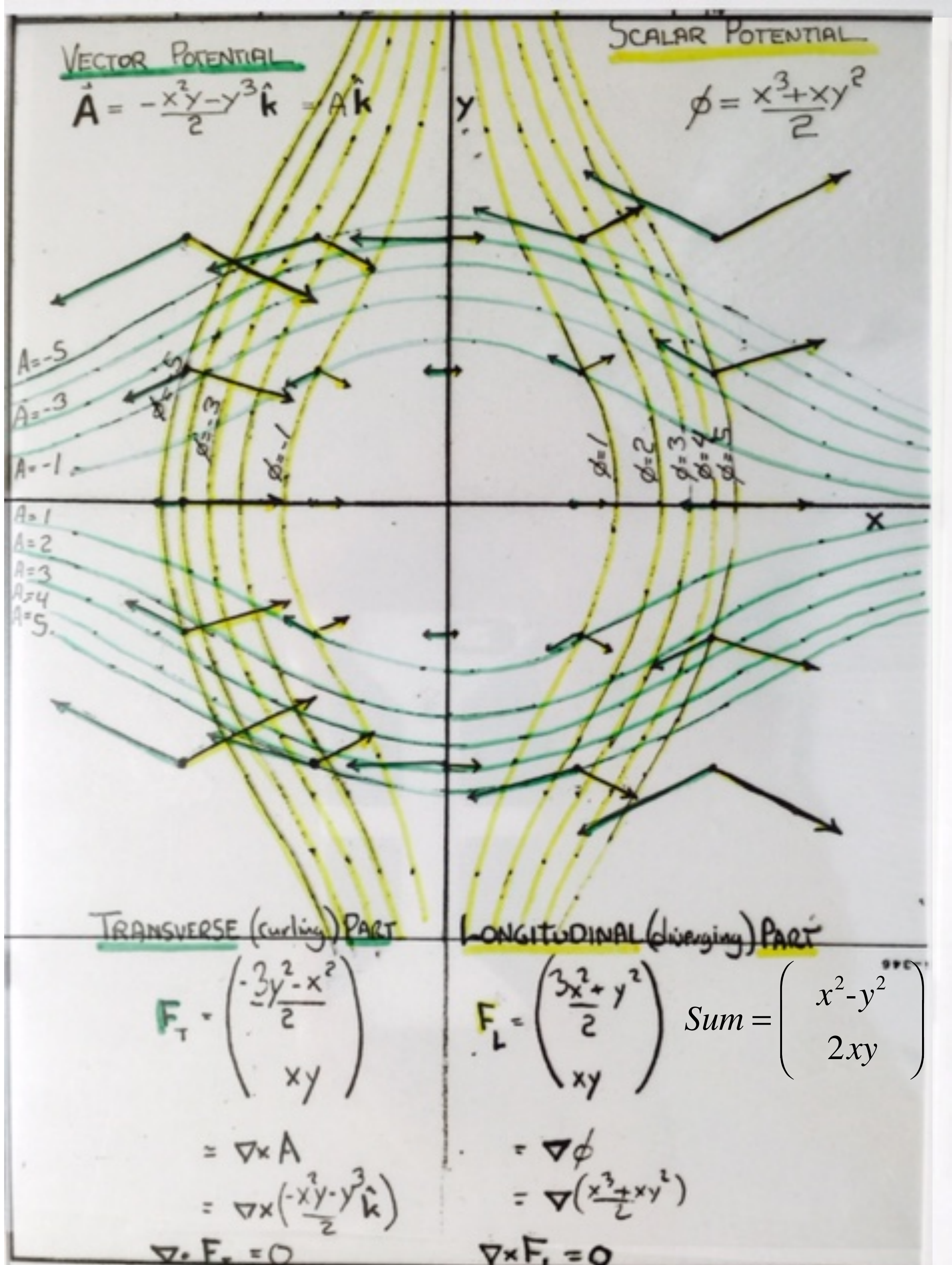
Fig.10.17 Force field vectors for non-analytic function $f(z) = (z^*)^2$

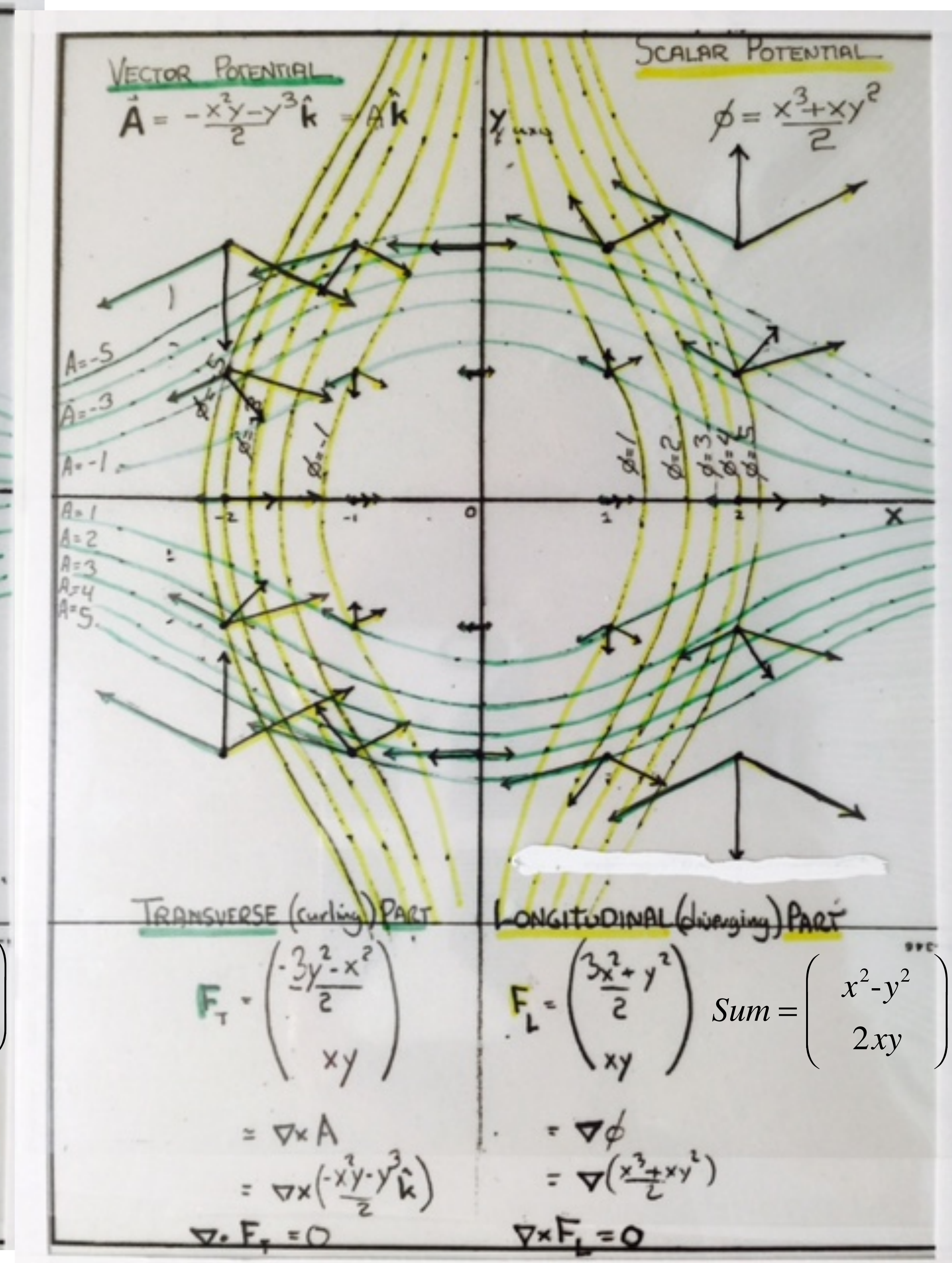


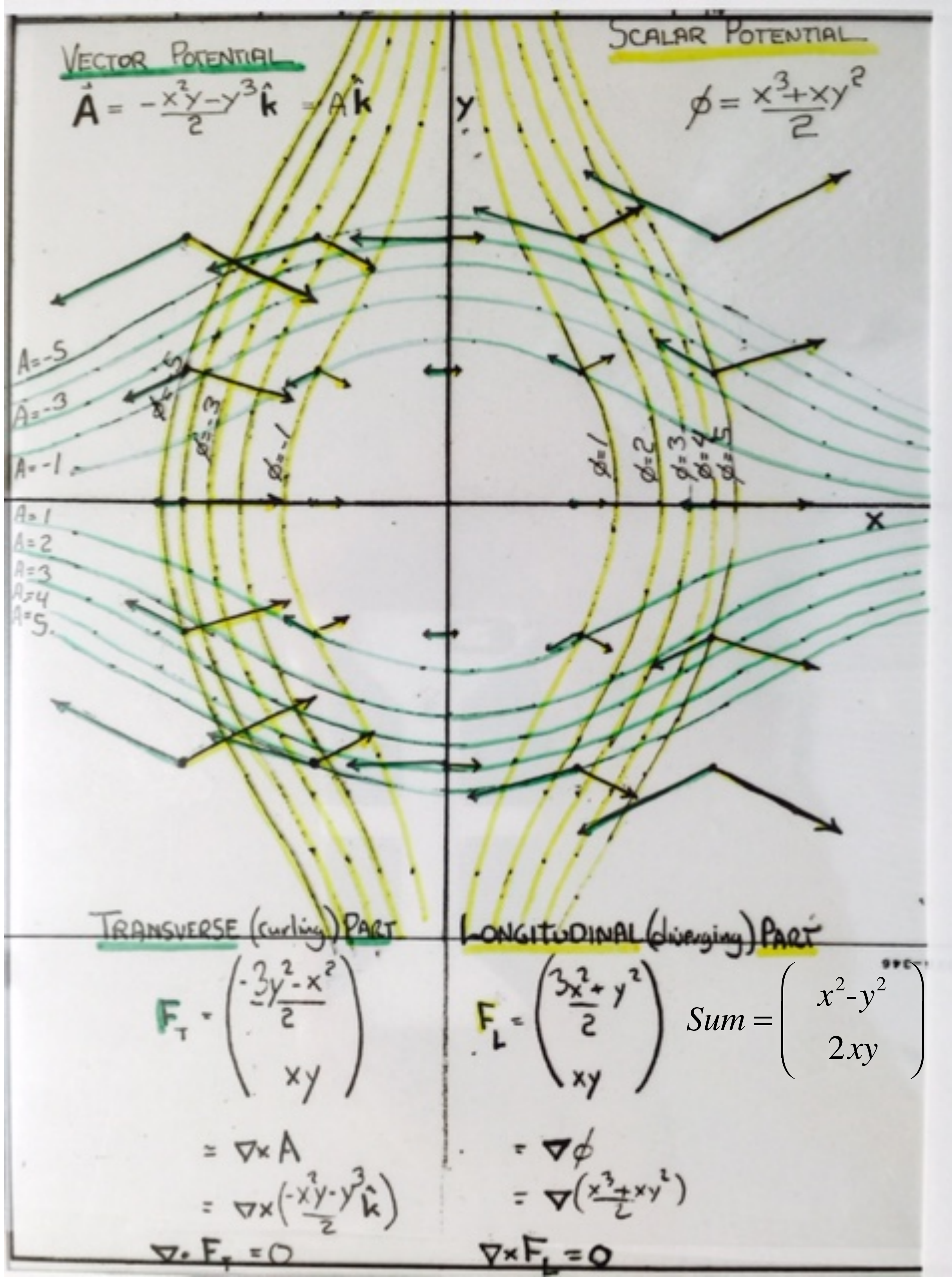
Scalar Potential
$\phi = \frac{x^3+xy^2}{2}$
y
x
$\phi=-5$
$\phi=-3$
$\phi=-1$
$\phi=1$
$\phi=2$
$\phi=3$
$\phi=4$
$\phi=5$
Longitudinal (diverging) Part
$F_L = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix}$
$= \nabla\phi$
$= \nabla\left(\frac{x^3+xy^2}{2}\right)$
$\nabla\times F_L = 0$



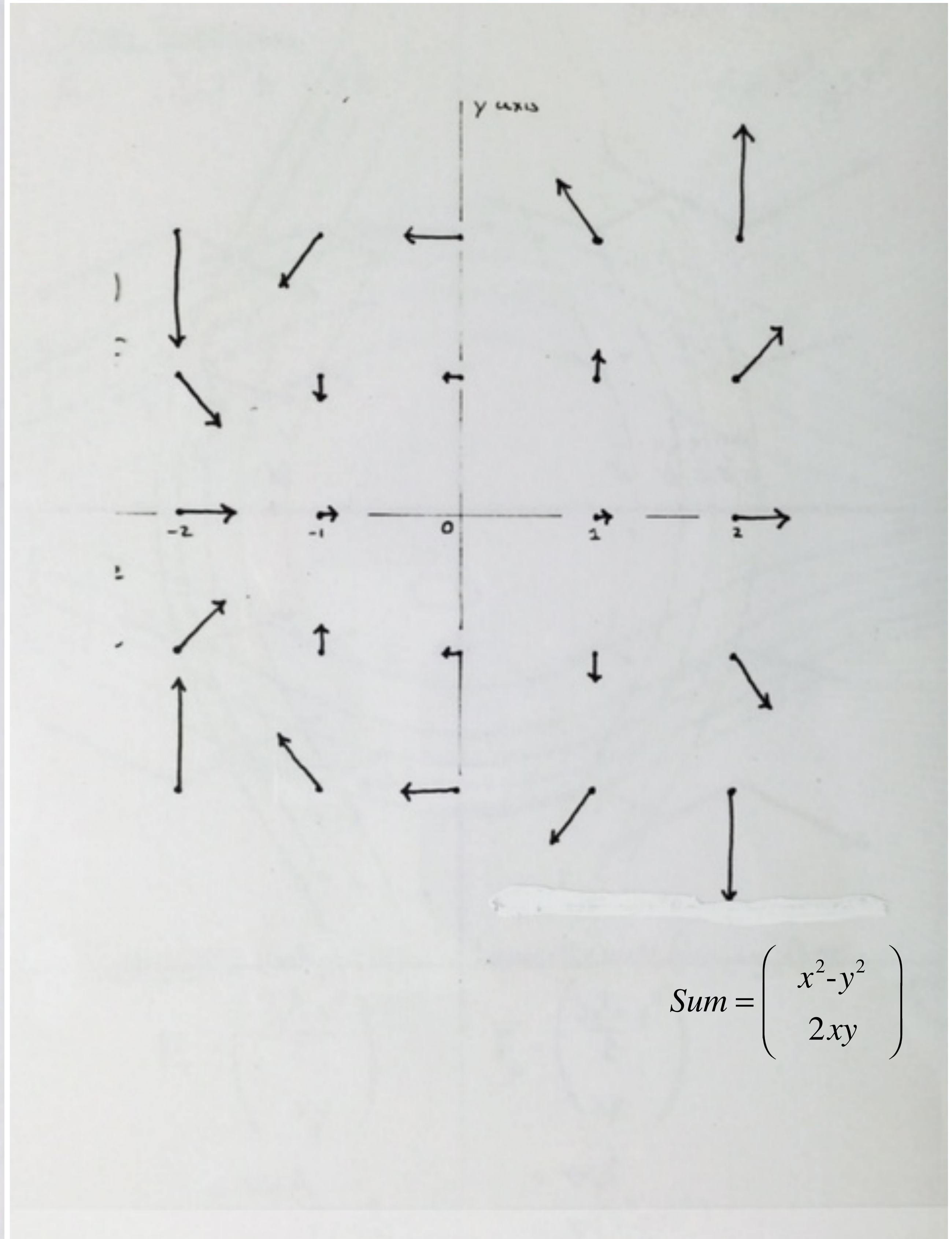
Vector Potential
$\hat{A} = -\frac{x^2y-y^3}{2}\hat{k} = A\hat{k}$
y
x
$A=-5$
$A=-3$
$A=-1$
$A=1$
$A=2$
$A=3$
$A=4$
$A=5$
Transverse (curling) Part
$F_T = \begin{pmatrix} \frac{3y^2-x^2}{2} \\ xy \end{pmatrix}$
$= \nabla\times A$
$= \nabla\times\left(-\frac{x^2y-y^3}{2}\hat{k}\right)$
$\nabla\cdot F_T = 0$


VECTOR POTENTIAL
$\vec{A} = -\frac{x^2y - y^3}{2}\hat{k} = A\hat{k}$
SCALAR POTENTIAL
$\phi = \frac{x^3 + xy^2}{2}$
y
x
A=-5
A=-3
A=-1
A=1
A=2
A=3
A=4
A=5
$\phi=-1$
$\phi=-3$
$\phi=1$
$\phi=2$
$\phi=3$
$\phi=4$
$\phi=5$
TRANSVERSE (curling) PART
LONGITUDINAL (diverging) PART
$F_T = \begin{pmatrix} -\frac{3y^2 - x^2}{2} \\ xy \end{pmatrix}$
$F_L = \begin{pmatrix} \frac{3x^2 + y^2}{2} \\ xy \end{pmatrix}$
$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$
$= \nabla \times A$
$= \nabla \times \left(-\frac{x^2y - y^3}{2}\hat{k}\right)$
$= \nabla \phi$
$= \nabla\left(\frac{x^3 + xy^2}{2}\right)$
$\nabla \cdot F_T = 0$
$\nabla \times F_L = 0$

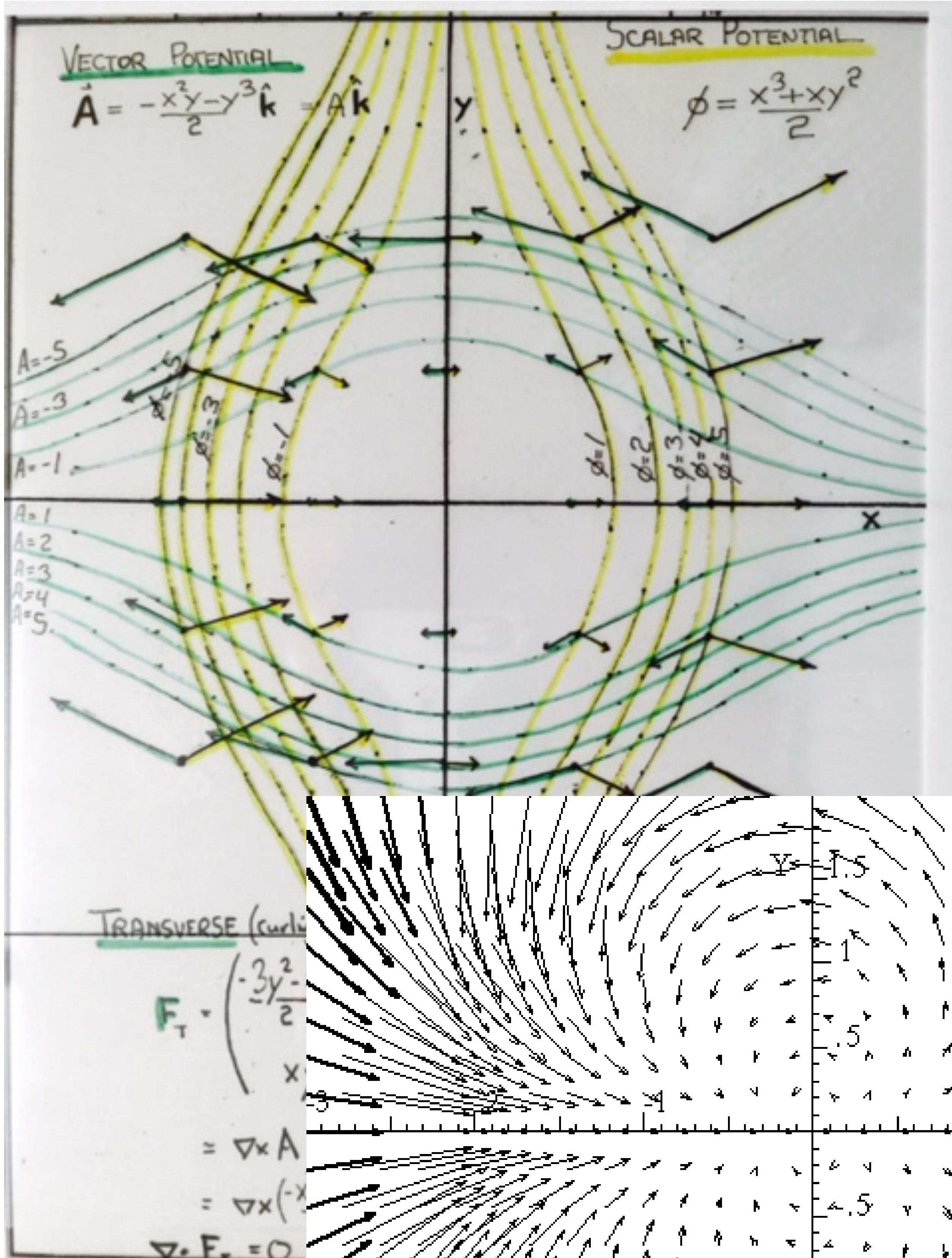

VECTOR POTENTIAL
$\vec{A} = -\frac{x^2y - y^3}{2}\hat{k} = A\hat{k}$
SCALAR POTENTIAL
$\phi = \frac{x^3 + xy^2}{2}$
y
x
-2
-1
0
1
2
A=-5
A=-3
A=-1
A=1
A=2
A=3
A=4
A=5
$\phi=-1$
$\phi=1$
$\phi=2$
$\phi=3$
$\phi=4$
$\phi=5$
TRANSVERSE (curling) PART
LONGITUDINAL (diverging) PART
$F_T = \begin{pmatrix} -\frac{3y^2 - x^2}{2} \\ xy \end{pmatrix}$
$F_L = \begin{pmatrix} \frac{3x^2 + y^2}{2} \\ xy \end{pmatrix}$
$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$
$= \nabla \times A$
$= \nabla \times \left(-\frac{x^2y - y^3}{2}\hat{k}\right)$
$= \nabla \phi$
$= \nabla\left(\frac{x^3 + xy^2}{2}\right)$
$\nabla \cdot F_T = 0$
$\nabla \times F_L = 0$

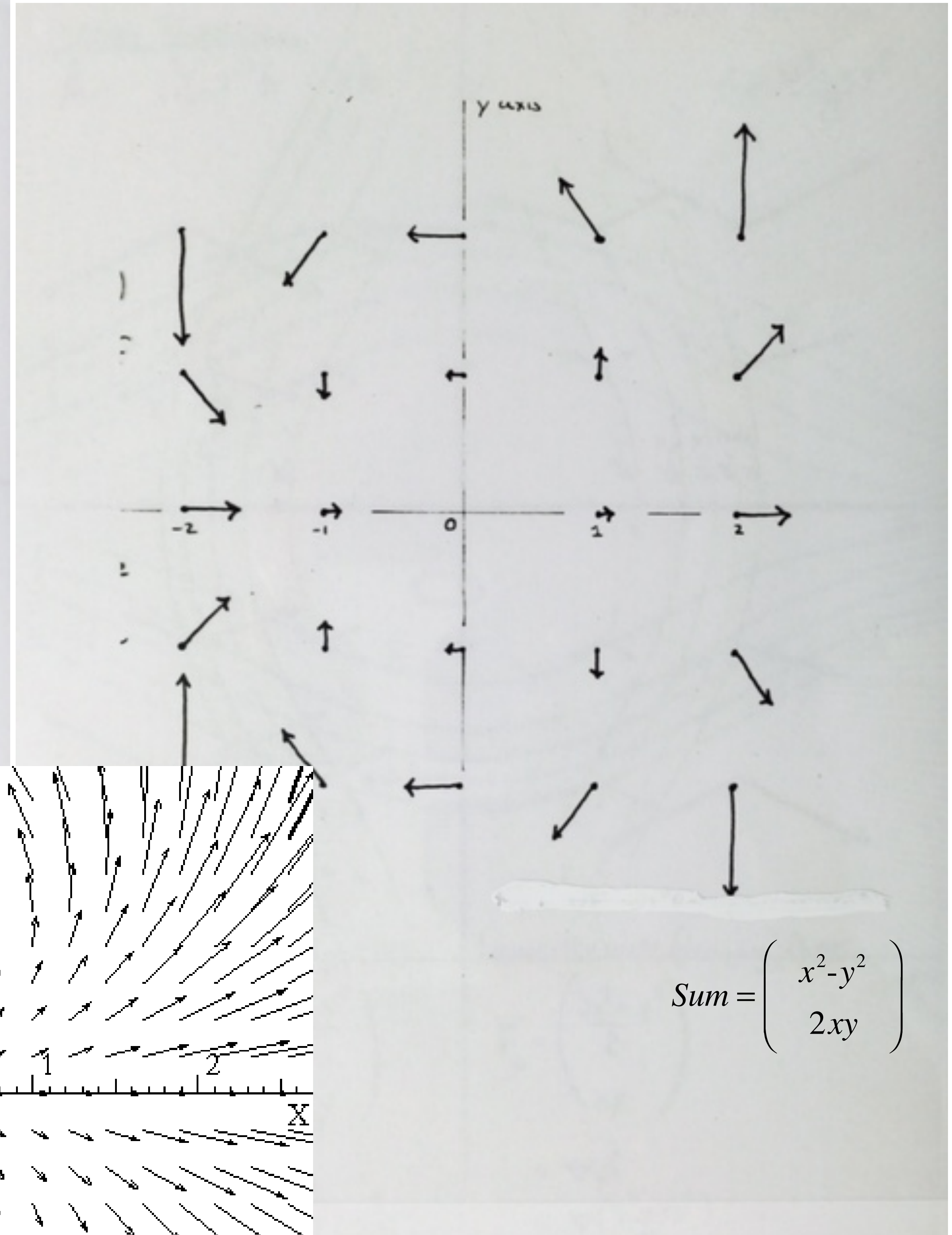


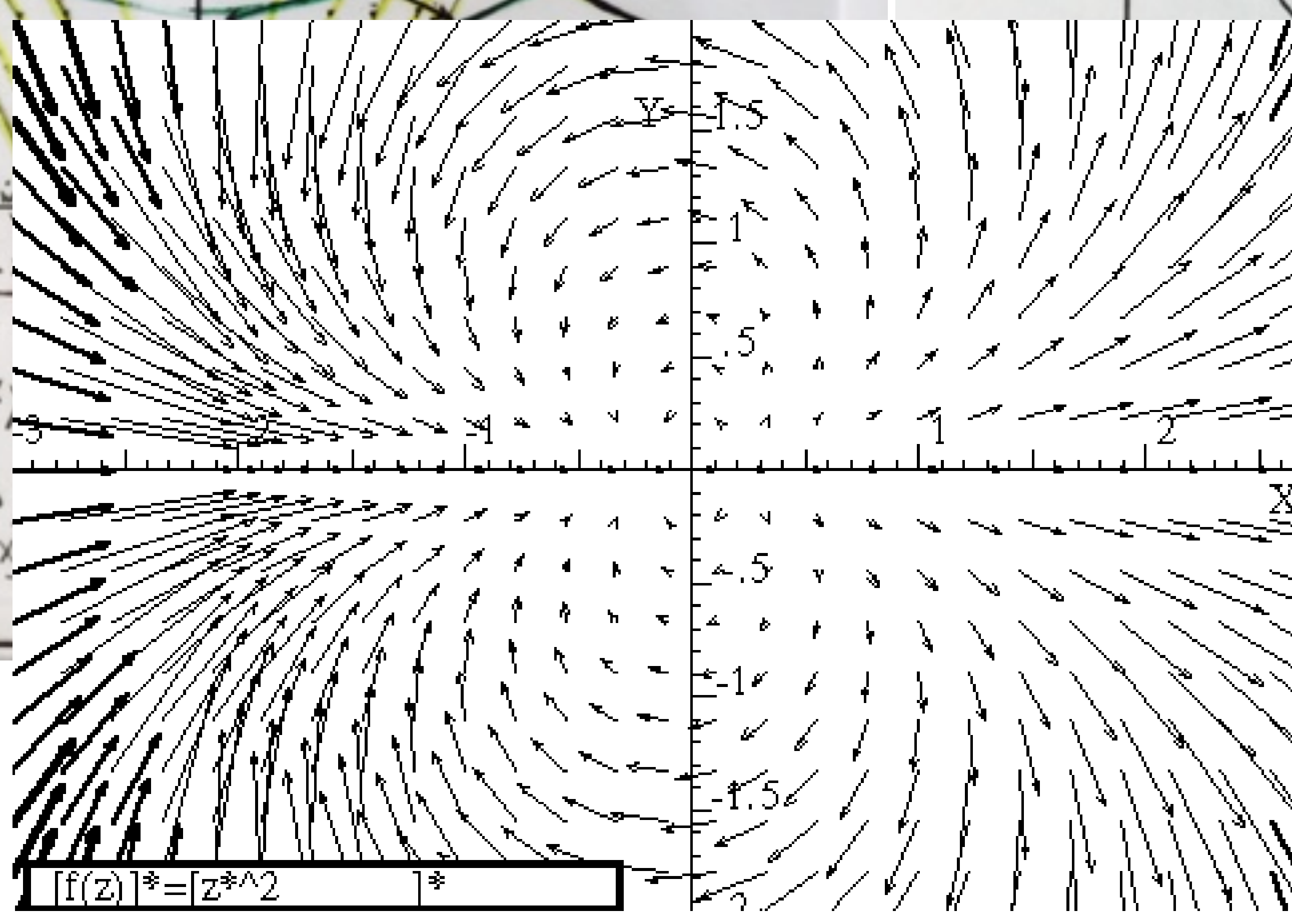
$$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$



$$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$


VECTOR POTENTIAL
$\vec{A} = -\frac{x^2y - y^3}{2}\hat{k} = A\hat{k}$
SCALAR POTENTIAL
$\phi = \frac{x^3 + xy^2}{2}$
y
x
A=-5
A=-3
A=-1
A=1
A=2
A=3
A=4
A=5
TRANSVERSE


y axis
-2
-1
0
1
2
$Sum = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$


Y
X
1.5
1
.5
-.5
-1
-1.5
-1
1
2
[f(z)]*=[z*^2]*