

Lecture 11

Tue. 9.26.2017

Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)

Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)

Clues for future assignments ([Web Simulation: CouIt](#))

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot ([Web Simulations: Pendulum](#), [Cycloidulum](#), [JerkIt \(Vert Driven Pendulum\)](#))

1D-HO phase-space control (Old Mac OS & [Web Simulations of "Catcher in the Eye"](#))

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

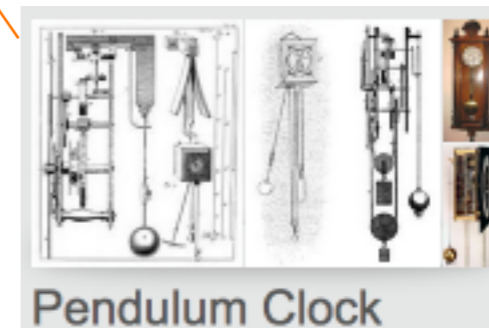
*Poincare identity and Action, **Jacobi**-Hamilton equations*

How Classicists might have "derived" quantum equations

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

(["Color-Quantization" simulations](#): Davis-Heller "Color-Quantization" or "Classical Chromodynamics")



Christiaan Huygens
(1629-1695)

→ Parabolic ~~1D~~ ~~III~~ orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)
Some clues for future assignments ([Web Simulation: CouIt](#))

Initial position $x(0) = 0$

Initial position $y(0) = 0.75$

Initial momentum $p_x(0) = 0$

Initial momentum $p_y(0) = 1$

Terminal time $t(\text{off}) = 3.45$

Maximum step size $dt = 0.01$

Start launch angle $\phi_1 = -180$

Start launch angle $\phi_2 = 180$

Number of burst paths = 182

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb $(k_{12}) = -1$

Core thickness $r = 0.000001$

x-Stark field $E_x = 0$

y-Stark field $E_y = -1$

Zeeman field $B_z = 0$

Diamagnetic strength $k = 0$

Plank constant $\hbar = 2$

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}) , $x = 8$

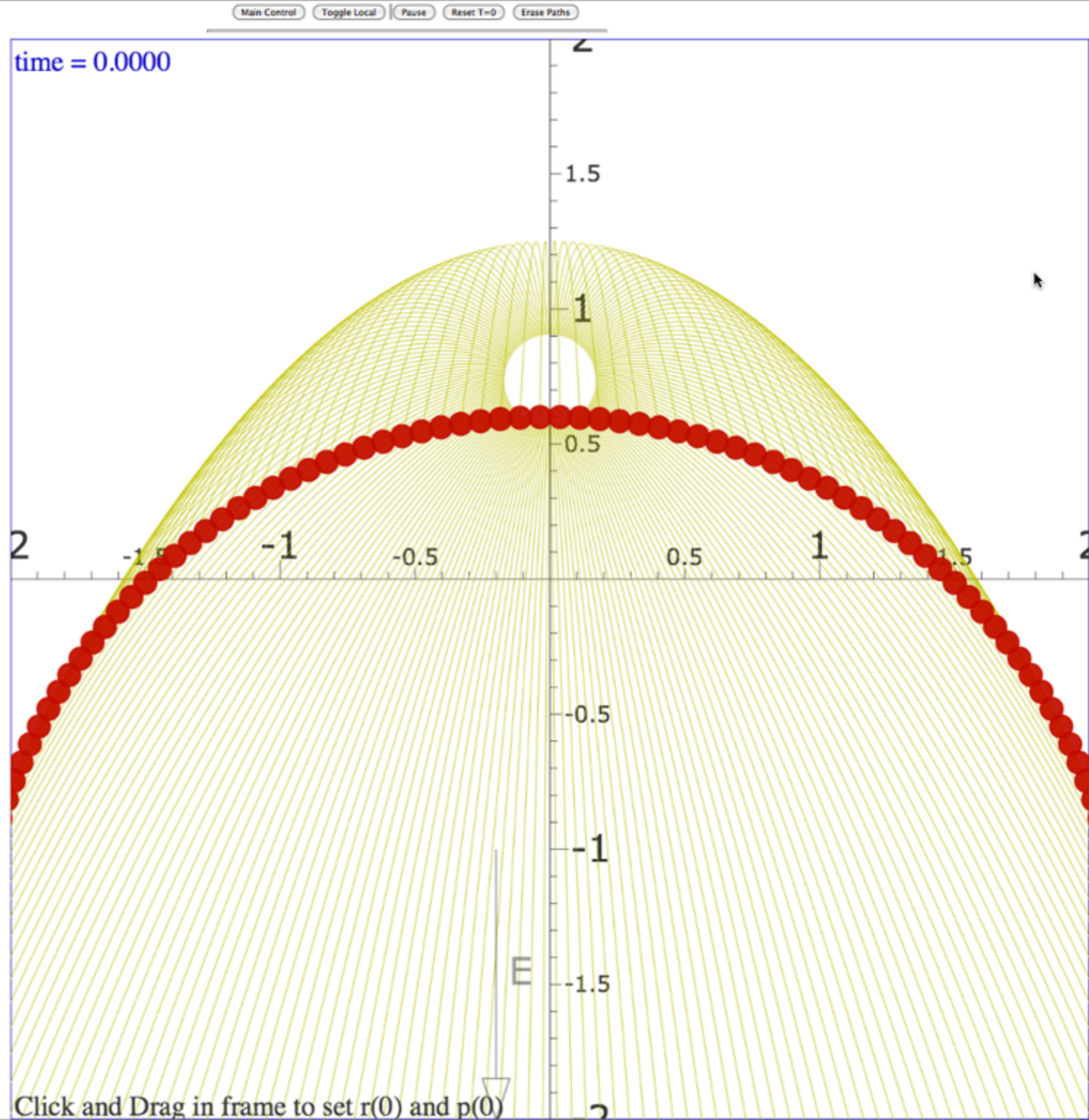
Plot $r(t)$ Plot $p(t)$ Fix $r(0)$ Fix $p(0)$

Do swarm Beam

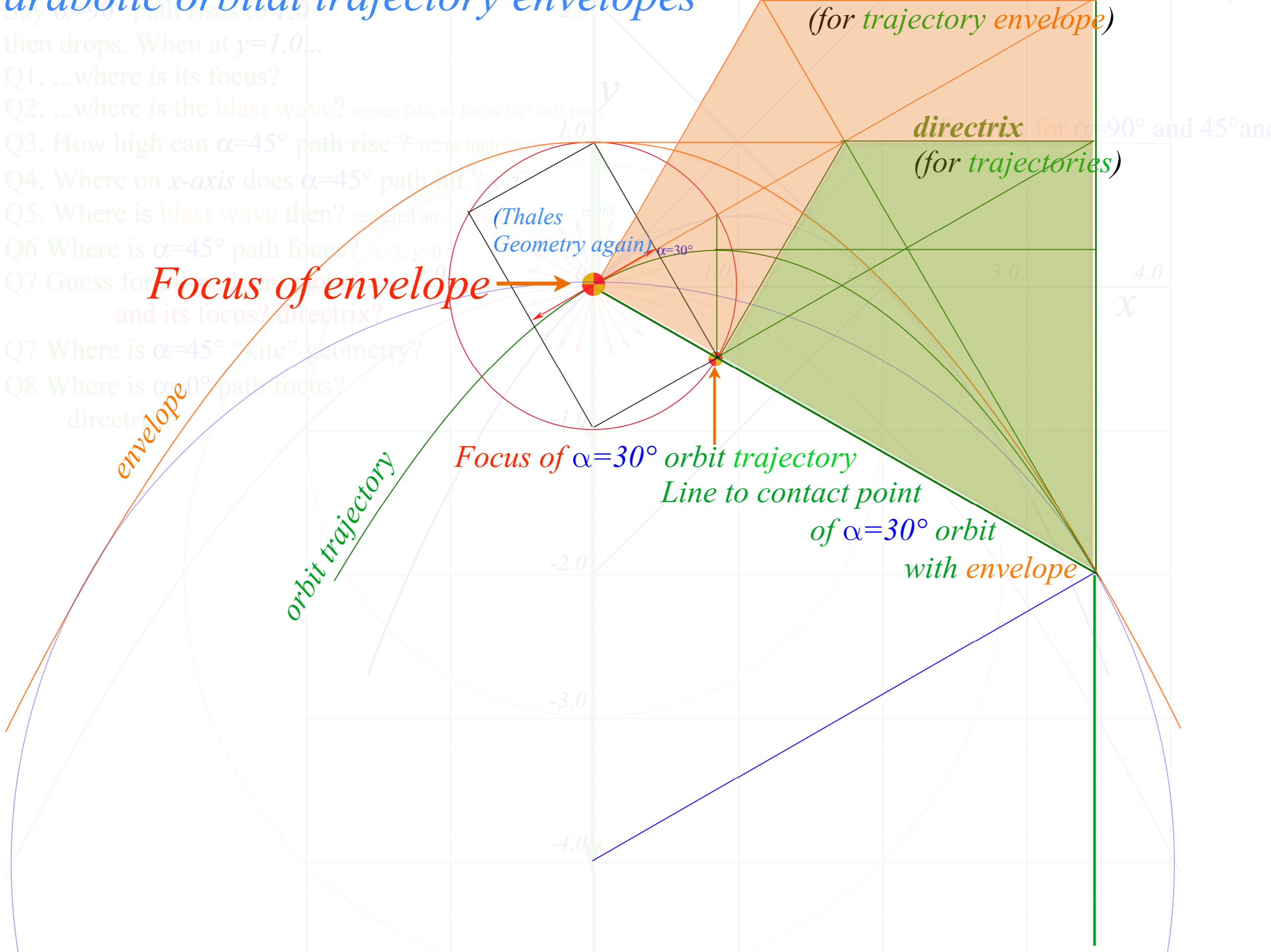
Color action No stops Field vectors Info

Draw masses Axes Coordinates Lenz

Set p by ϕ Elastic 2 Free



Parabolic orbital trajectory envelopes



Focus of envelope

Focus of $\alpha=30^\circ$ orbit trajectory

**Line to contact point
of $\alpha=30^\circ$ orbit
with envelope**

**directrix
(for trajectory envelope)**


**directrix
(for trajectories)**

envelope

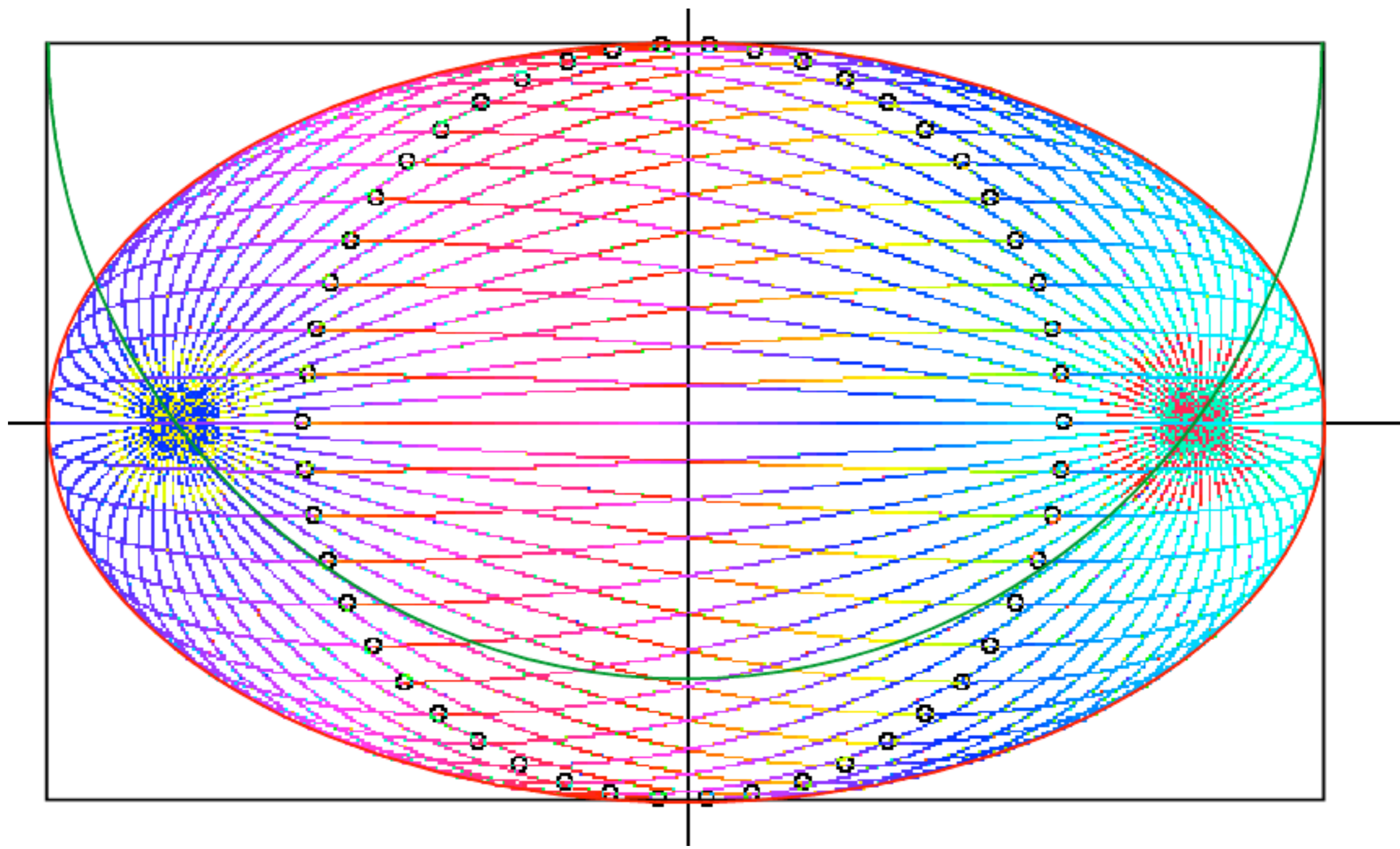
orbit trajectory

*(Thales
Geometry again)*

Say $\alpha=90^\circ$ path rises to 1.0... then drops. When at $y=1.0$...
 Q1. ...where is its focus?
 Q2. ...where is the blast wave?
 Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
 Q4. Where on x -axis does $\alpha=45^\circ$ path hit?
 Q5. Where is blast wave then?
 Q6 Where is $\alpha=45^\circ$ path focus?
 Q7 Guess for $\alpha=45^\circ$ path focus and its focus?
 Q7 Where is $\alpha=45^\circ$ "kite" geometry?
 Q8 Where is $\alpha=0^\circ$ path focus?

~~Parabolic~~  2D-IHO orbital envelopes (~~Figure 1 of Lecture 01, 56, 81, and 1~~ a generalization.)
Some clues for future assignments ([Web Simulation: CouItt](#))

Exploding-starlet elliptical envelope and contacting elliptical trajectories



*(Web Simulation: CouIt - Exploding*Starlet {IHO Potential})*

Initial position $x(0) = 1$

Initial position $y(0) = 0$

Initial momentum $p_x(0) = 0$

Initial momentum $p_y(0) = 1$

Terminal time $t(\text{off}) = 3.45$

Maximum step size $dt = 0.01$

Start launch angle $\phi_1 = -180$

Start launch angle $\phi_2 = 180$

Number of burst paths = 51

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k_{12}) = 0

Core thickness $r = 0.000001$

x-Stark field $E_x = 0$

y-Stark field $E_y = 0$

Zeeman field $B_z = 0$

Diamagnetic strength $k = -0.638$

Plank constant $\hbar = 2$

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), $x = 8$

Particle Size = 2

Fix $r(0)$ Fix $p(0)$ Do swarm Beam

Plot $r(t)$ Plot $p(t)$

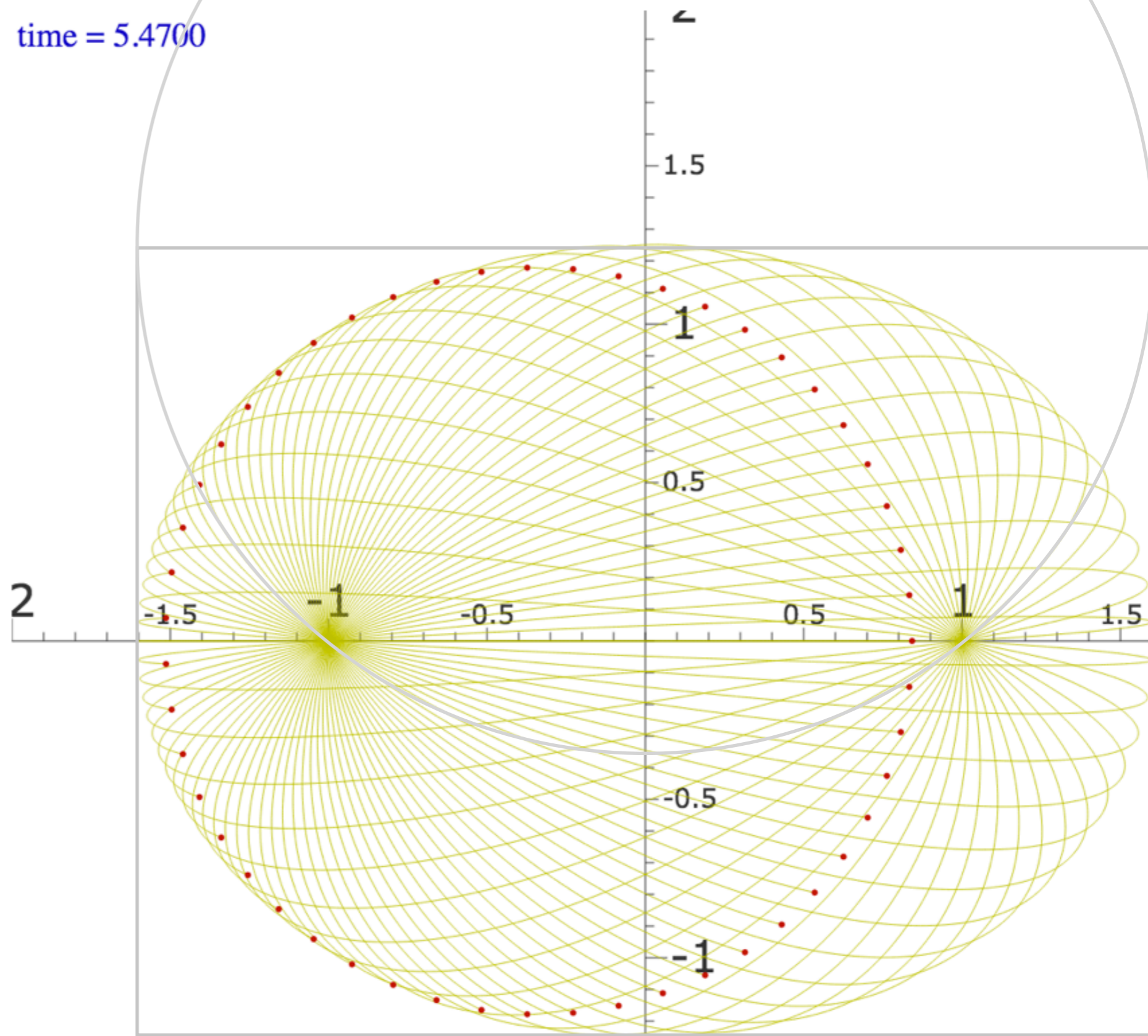
Color action No stops Field vectors Info

Draw masses Axes Coordinates Lenz

Set p by ϕ Elastic 2 Free

Save to GIF

time = 5.4700



(Web Simulation: CouIt - Exploding*Starlet {IHO Potential})

Examples of Hamiltonian mechanics in phase plots



1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt \(Vert Driven Pendulum\)](#))

Circular pendulum dynamics and elliptic functions

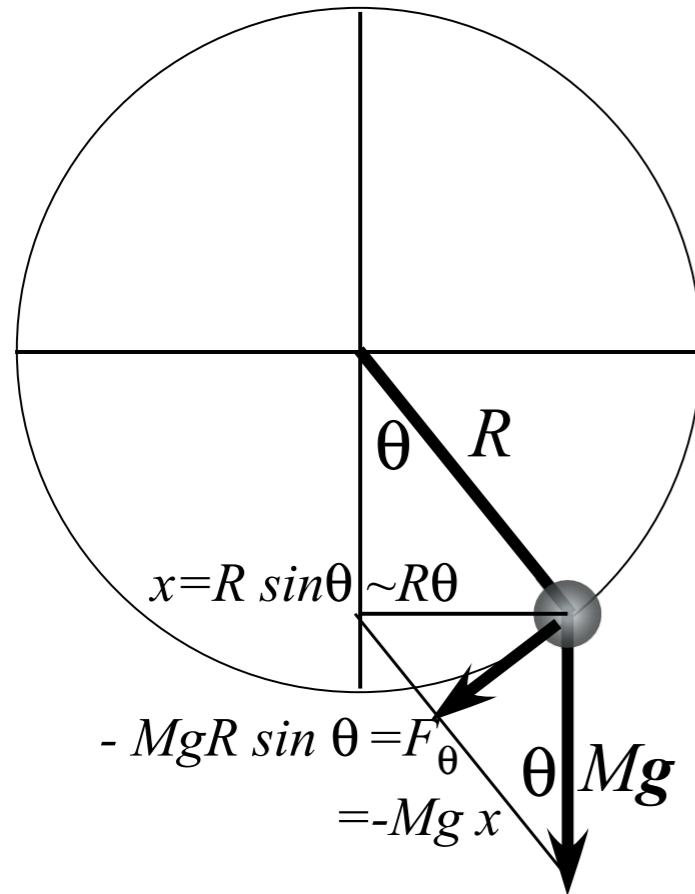
Cycloid pendulum dynamics and “sawtooth” functions

1D-HO phase-space control (Old Mac OS & [Web Simulations of “Catcher in the Eye”](#))

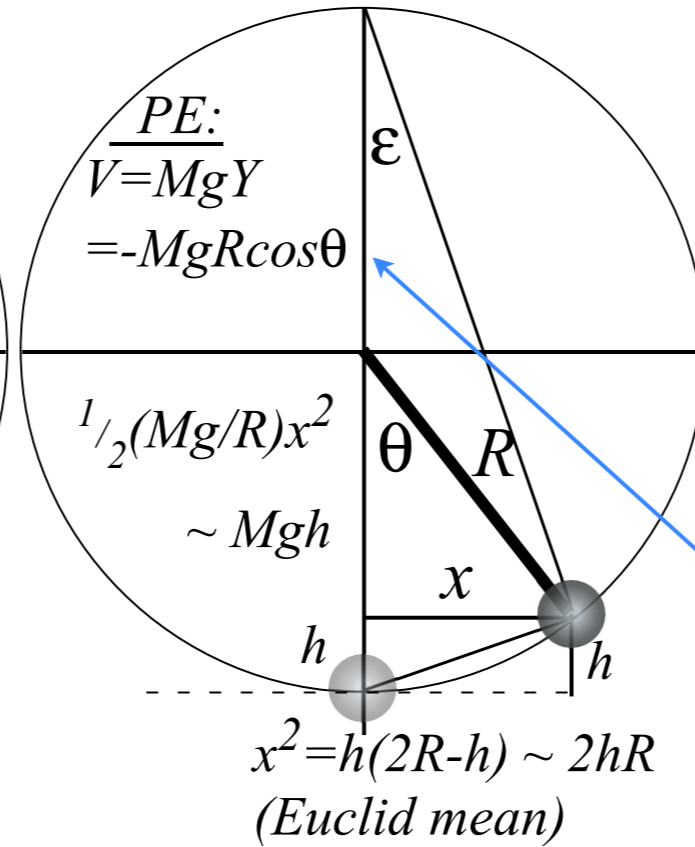
1D Pendulum and phase plot

(Unit 2 Chapter 7 Fig. 2)

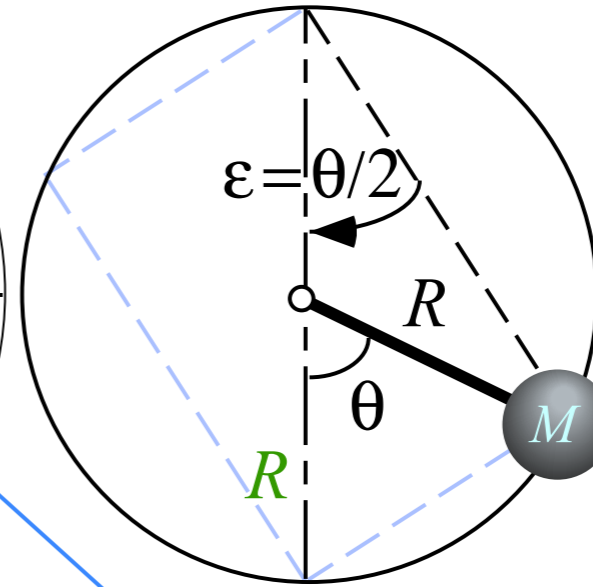
(a) Force geometry



(b) Energy geometry



(c) Time geometry



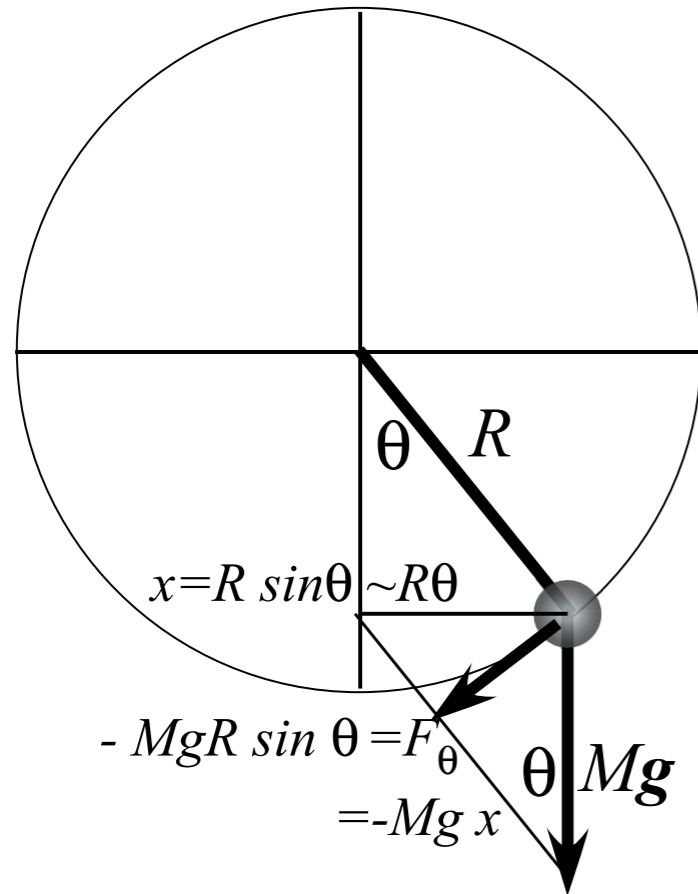
NOTE: Very common loci of \pm sign blunders

Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR \cos \theta$

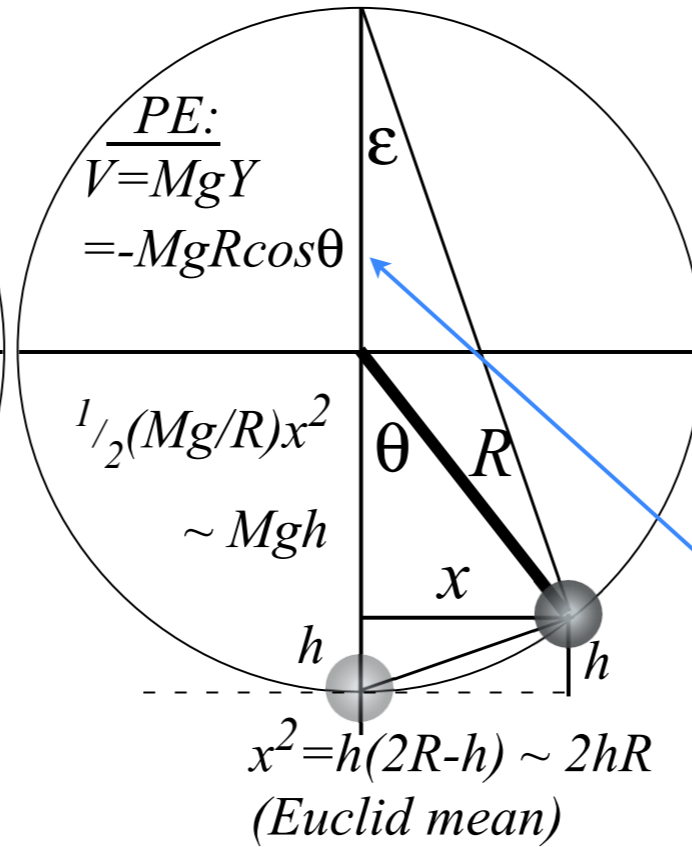
$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$

1D Pendulum and phase plot

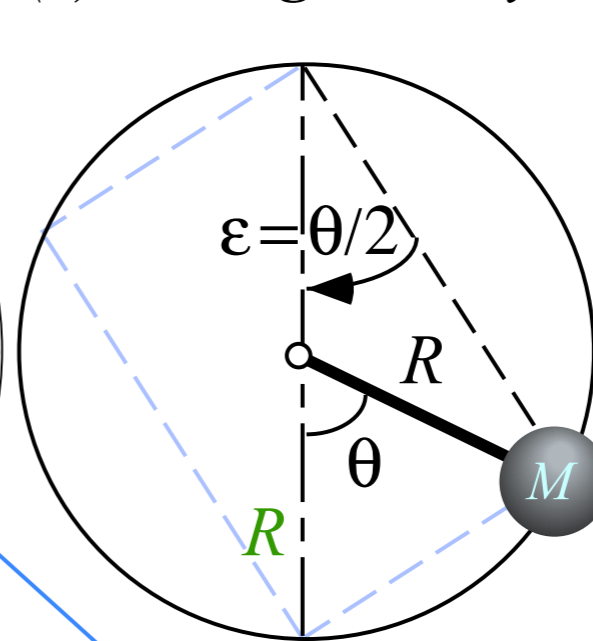
(a) Force geometry



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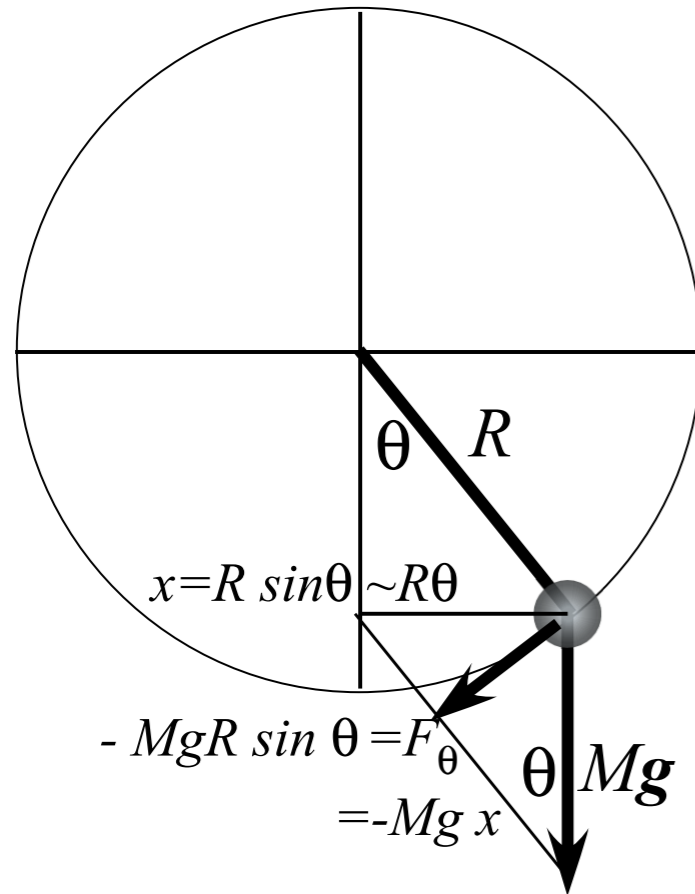
$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

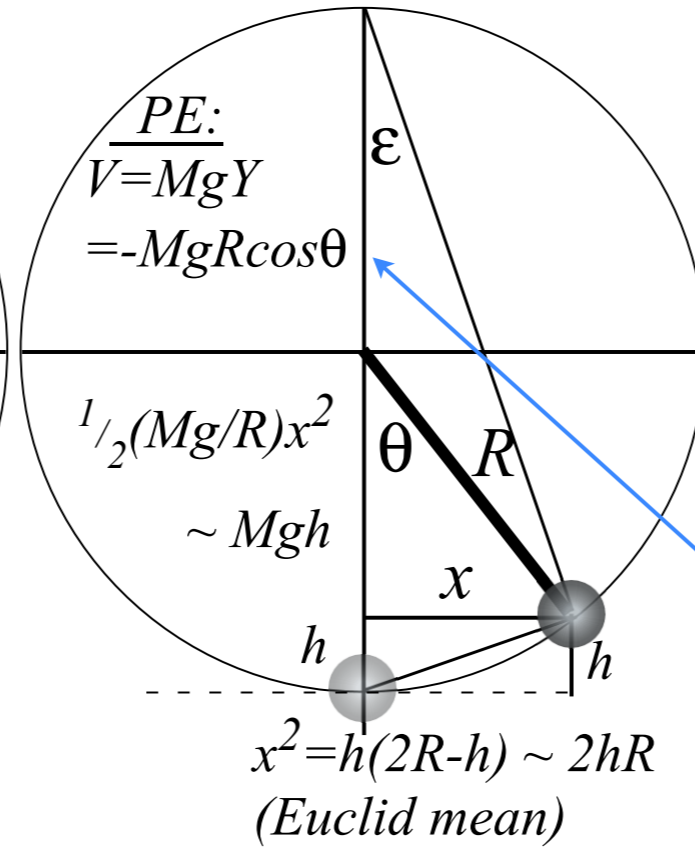
$$H(p_{\theta}, \theta) = \frac{1}{2I} p_{\theta}^2 + U(\theta) = \frac{1}{2I} p_{\theta}^2 - MgR \cos \theta = E = \text{const.}$$

1D Pendulum and phase plot

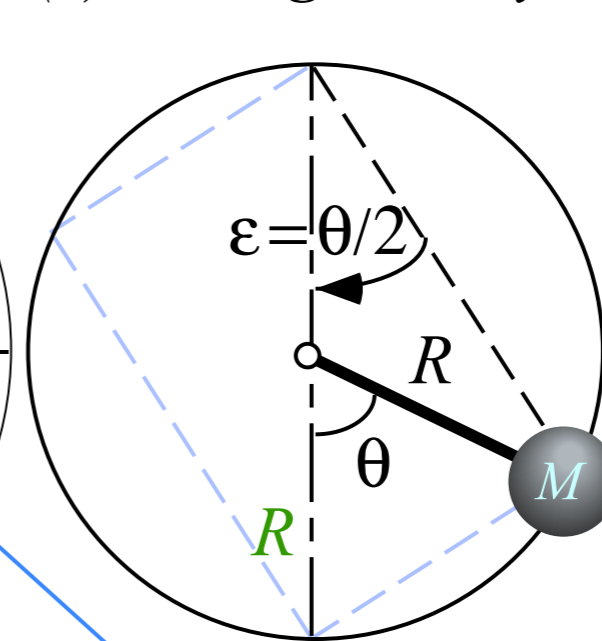
(a) Force geometry



(b) Energy geometry



(c) Time geometry



NOTE: Very common loci of \pm sign blunders

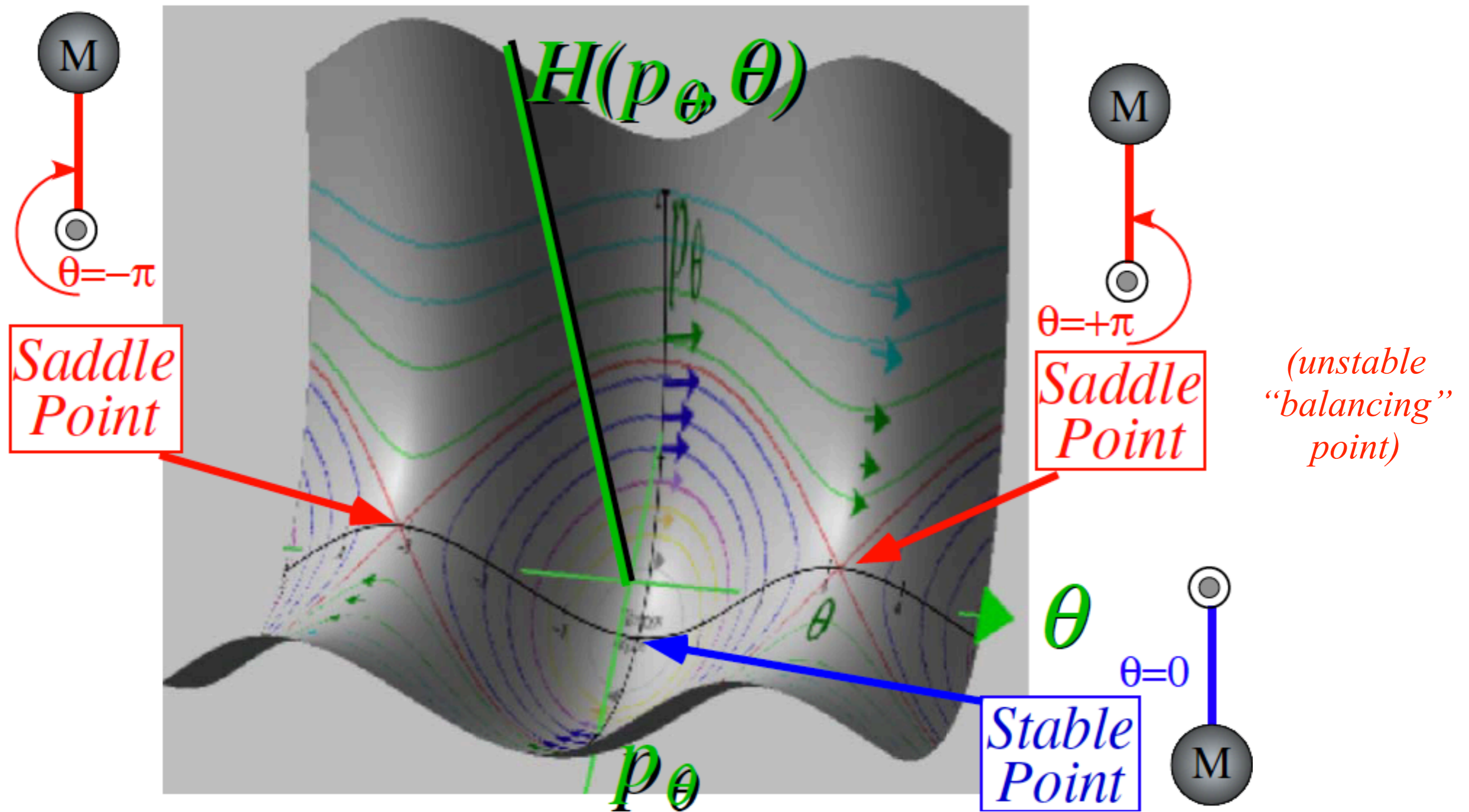
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Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

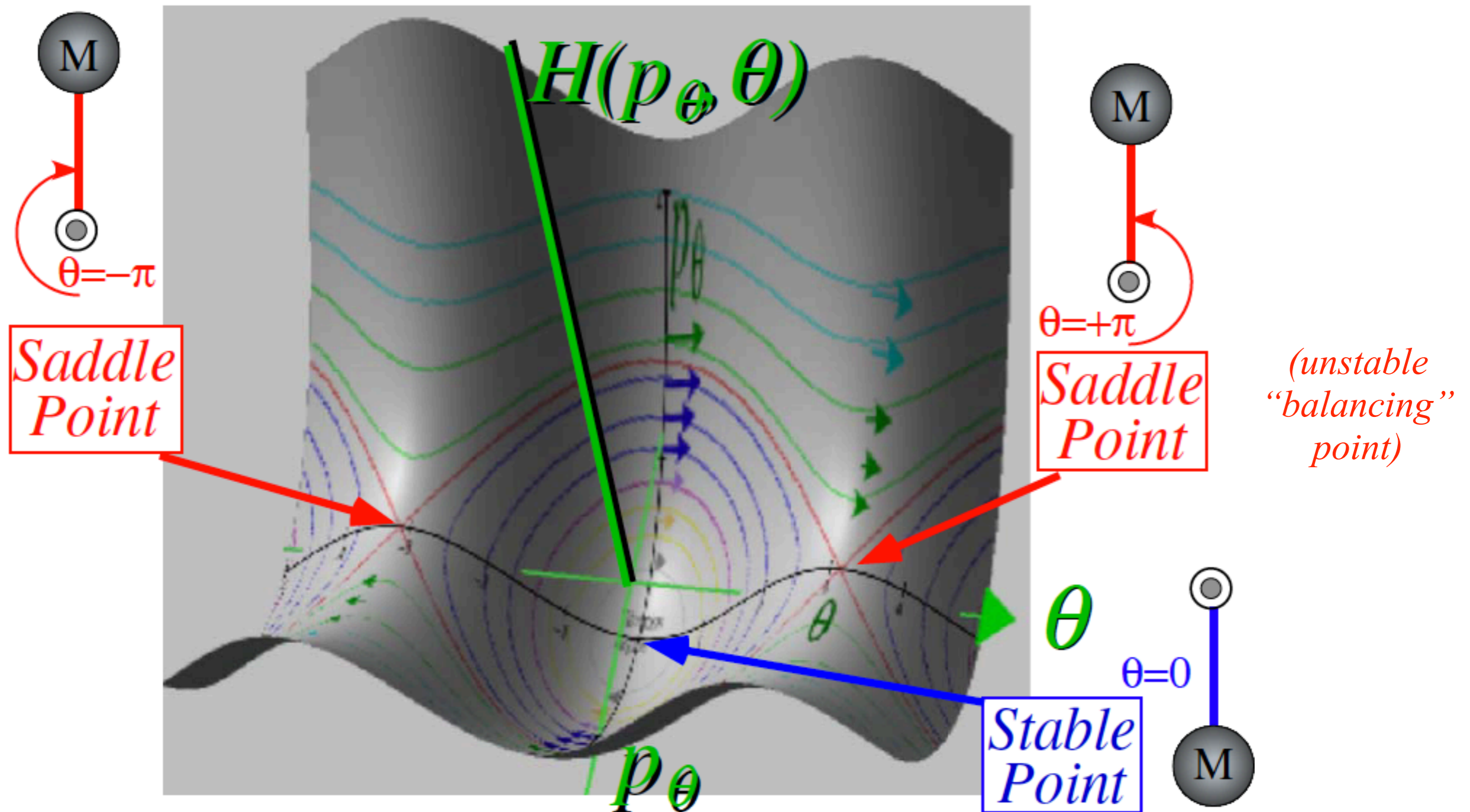
$$H(p_{\theta}, \theta) = \frac{1}{2I} p_{\theta}^2 + U(\theta) = \frac{1}{2I} p_{\theta}^2 - MgR \cos \theta = E = \text{const.}$$

implies: $p_{\theta} = \sqrt{2I(E + MgR \cos \theta)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \quad \text{or:} \quad p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \quad \text{or: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Funny way to look at Hamilton's equations:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \mathbf{e}_H \times (-\nabla H) = (\overrightarrow{\text{H-axis}}) \times (\overrightarrow{\text{fall line}}), \quad \text{where: } \begin{cases} (\overrightarrow{\text{H-axis}}) = \mathbf{e}_H = \mathbf{e}_q \times \mathbf{e}_p \\ (\overrightarrow{\text{fall line}}) = -\nabla H \end{cases}$$

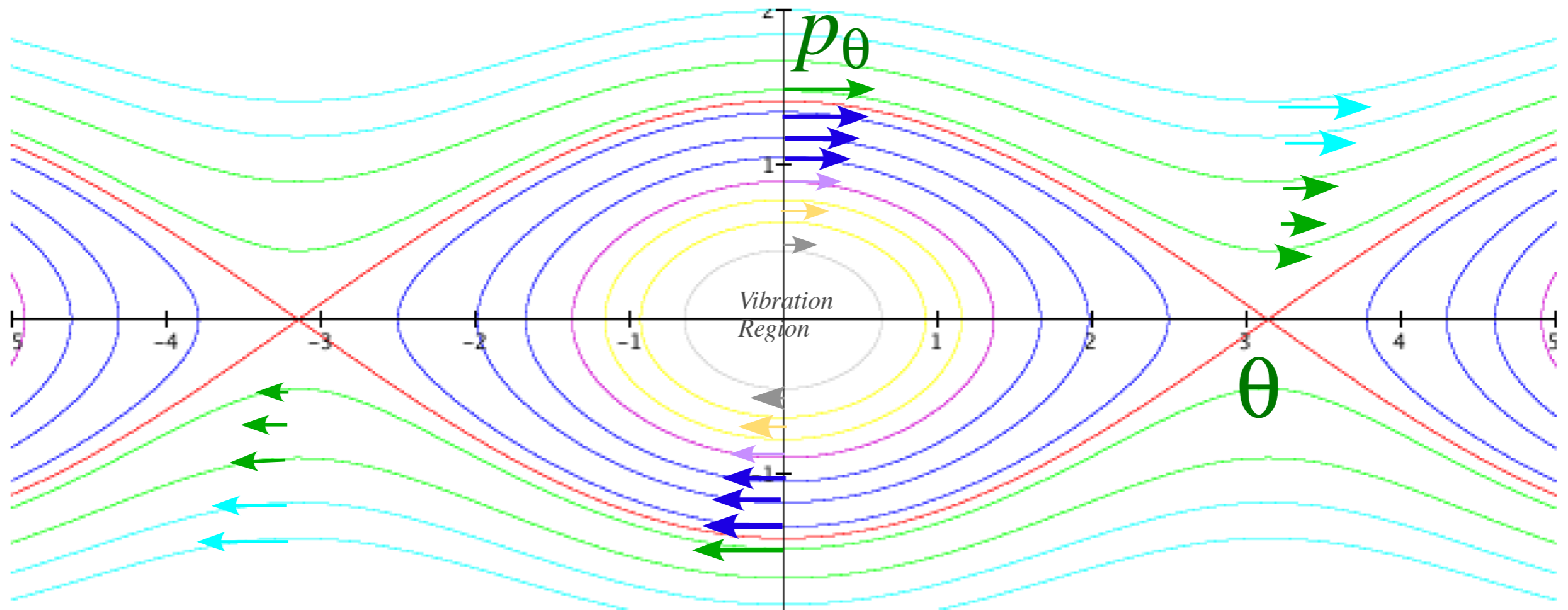


Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt \(Vert Driven Pendulum\)](#))



Circular pendulum dynamics and elliptic functions

Cycloid pendulum dynamics and “sawtooth” functions

1D-HO phase-space control (Old Mac OS & [Web Simulations of “Catcher in the Eye”](#))

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

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$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2$$

Circular pendulum dynamics and elliptic functions

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$E = MgY = -MgR \cos \theta_0$
↓

Circular pendulum dynamics and elliptic functions

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$E = MgY = -MgR \cos \theta_0$
↓

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time } 0 \text{ to } \theta_0) = \tau_{1/4}$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

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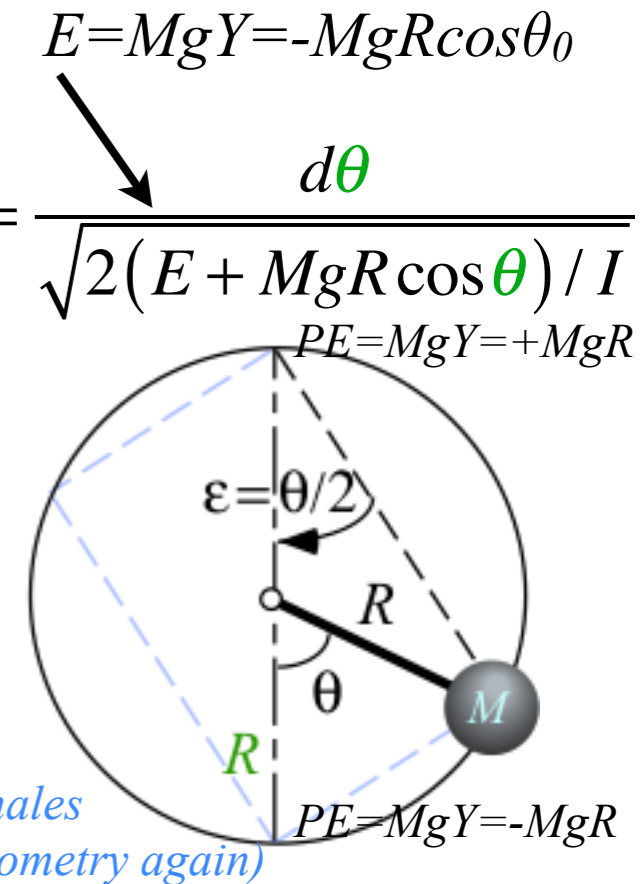
or: $dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time } 0 \text{ to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\epsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \epsilon,$$



Circular pendulum dynamics and elliptic functions

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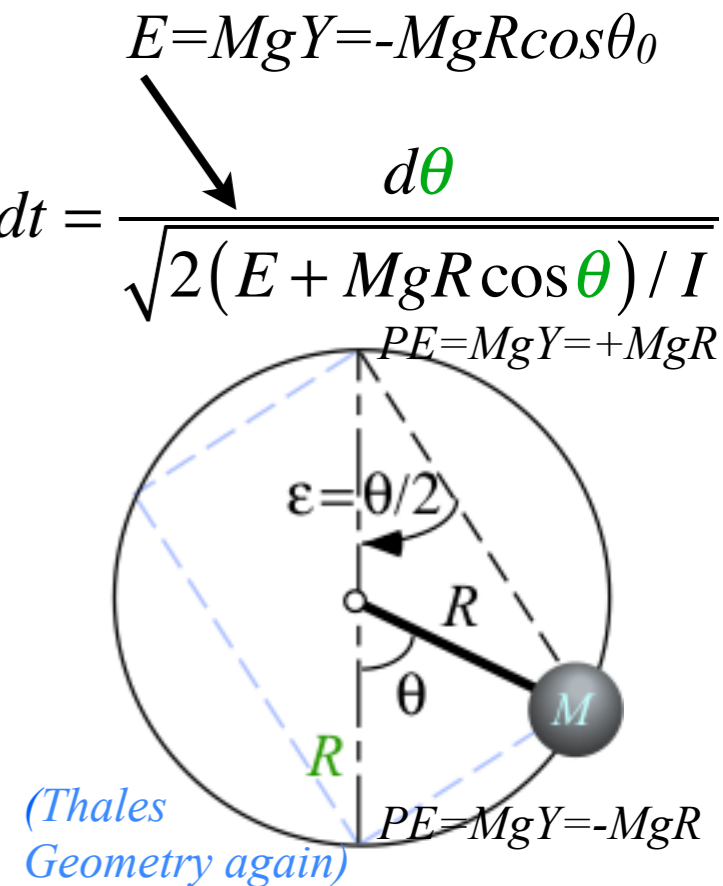
$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2 \quad \text{or: } dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time } 0 \text{ to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$



Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

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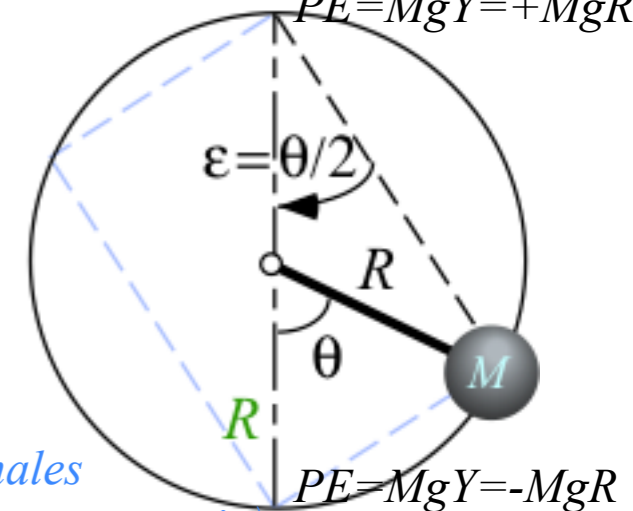
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$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}}$$

$$E = MgY = -MgR \cos \theta_0$$

$$\downarrow$$

$$\frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$



(Thales Geometry again)

Circular pendulum dynamics and elliptic functions

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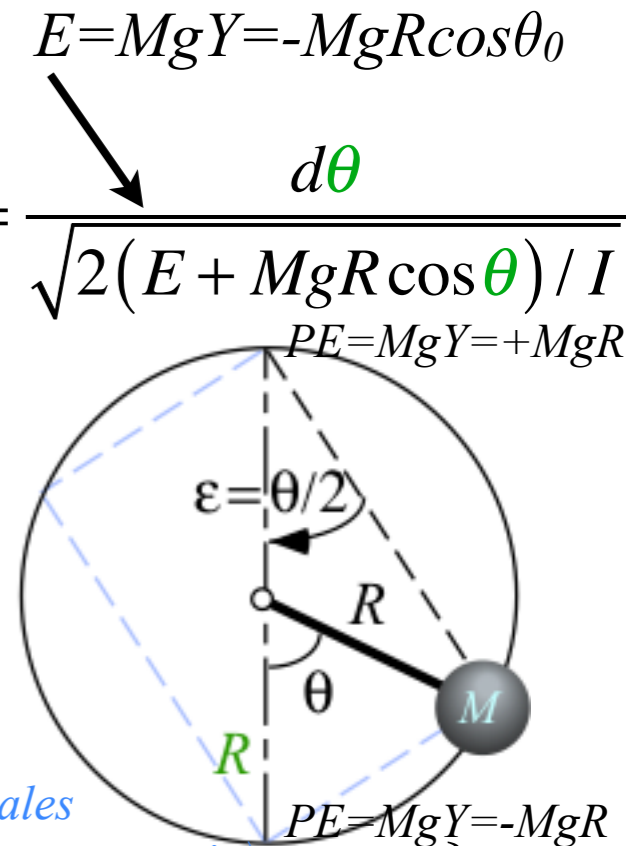
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Uses a half-angle coordinate $\varepsilon = \theta/2$

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$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{k d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}}, \quad \text{where: } \left\{ \begin{array}{l} 1/k = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\ I = MR^2 \end{array} \right.$$



(Thales Geometry again)

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

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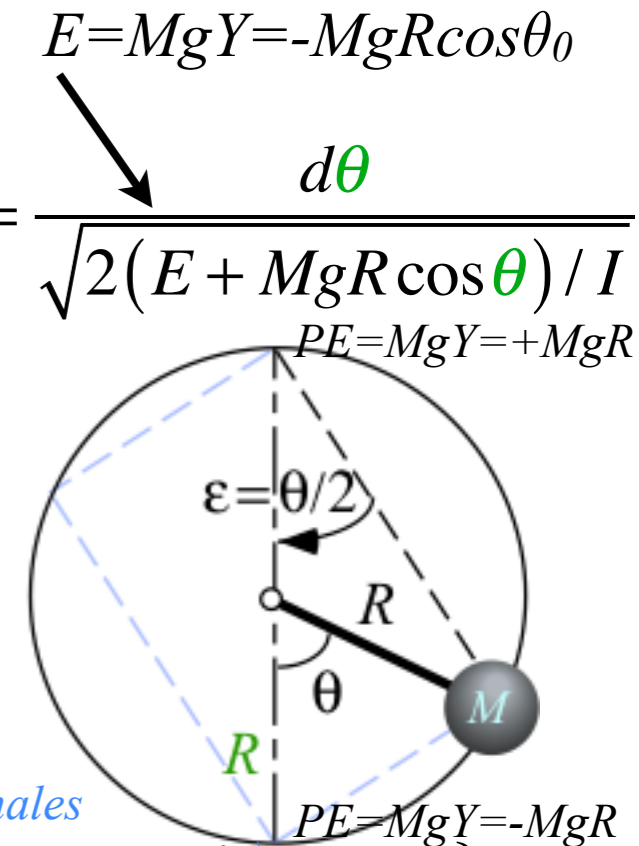
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(Thales Geometry again)

The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0)$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2 \quad \text{or: } dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

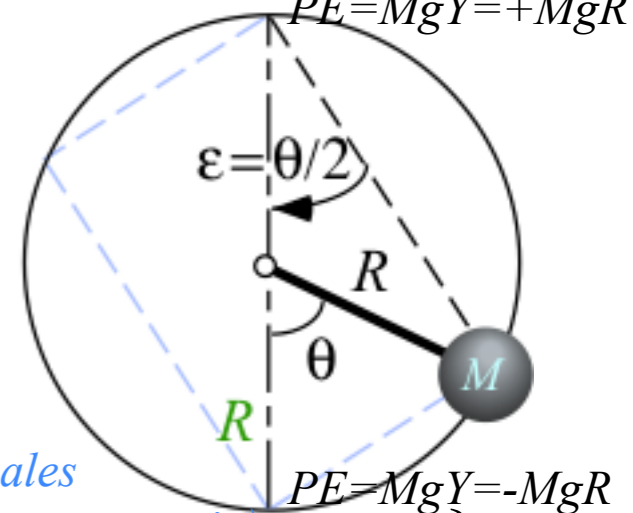
Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time } 0 \text{ to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$

$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{k d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}}, \quad \text{where: } \left\{ \begin{array}{l} 1/k = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\ I = MR^2 \end{array} \right.$$



(Thales Geometry again)

The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0) \quad \tau_{1/4} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon_0} = \sqrt{\frac{R}{g}} \frac{\pi}{2} = \tau \frac{2\pi}{4}$$

For low amplitude $\varepsilon \ll 1$: $\sin \varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau \frac{2\pi}{4}$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

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$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2 \quad \text{or: } dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

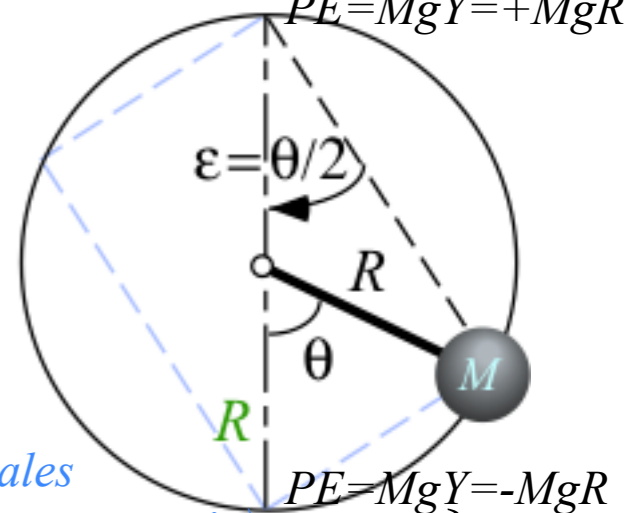
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Uses a half-angle coordinate $\varepsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$

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(Thales Geometry again)

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$$\text{low } \varepsilon \ll 1: t = \sqrt{\frac{R}{g}} \int_0^{\varepsilon(t)} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon(t)} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon(t)}{\varepsilon_0} \quad \text{For low amplitude } \varepsilon \ll 1: \sin \varepsilon_0 \simeq \varepsilon_0 \text{ reduces } \tau_{1/4} \text{ to } \tau \frac{2\pi}{4}$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2 \quad \text{or: } dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

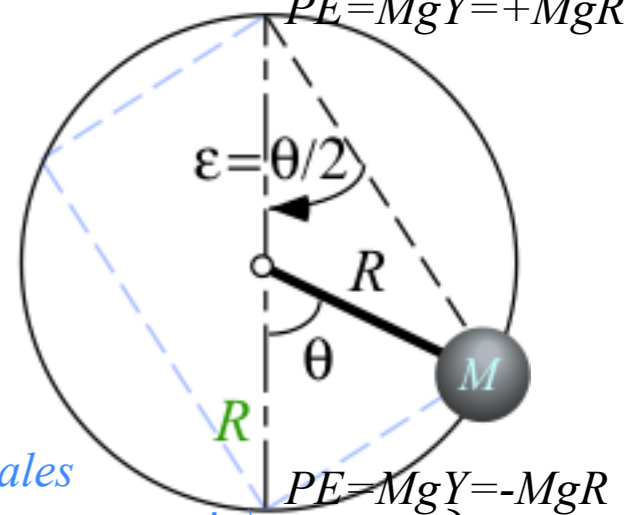
Quadrature integral gives quarter-period $\tau_{1/4}$:

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Uses a half-angle coordinate $\varepsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$

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(Thales Geometry again)

The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0) \quad \tau_{1/4} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon_0} = \sqrt{\frac{R}{g}} \frac{\pi}{2} = \tau \frac{2\pi}{4}$$

..reduces to sine...

$$\varepsilon(t) = \varepsilon_0 \sin \sqrt{\frac{g}{R}} t = \varepsilon_0 \sin \omega t, \quad \text{where: } \omega = \sqrt{\frac{g}{R}} \quad \text{For low amplitude } \varepsilon \ll 1: \sin \varepsilon_0 \simeq \varepsilon_0 \text{ reduces } \tau_{1/4} \text{ to } \tau \frac{2\pi}{4}$$

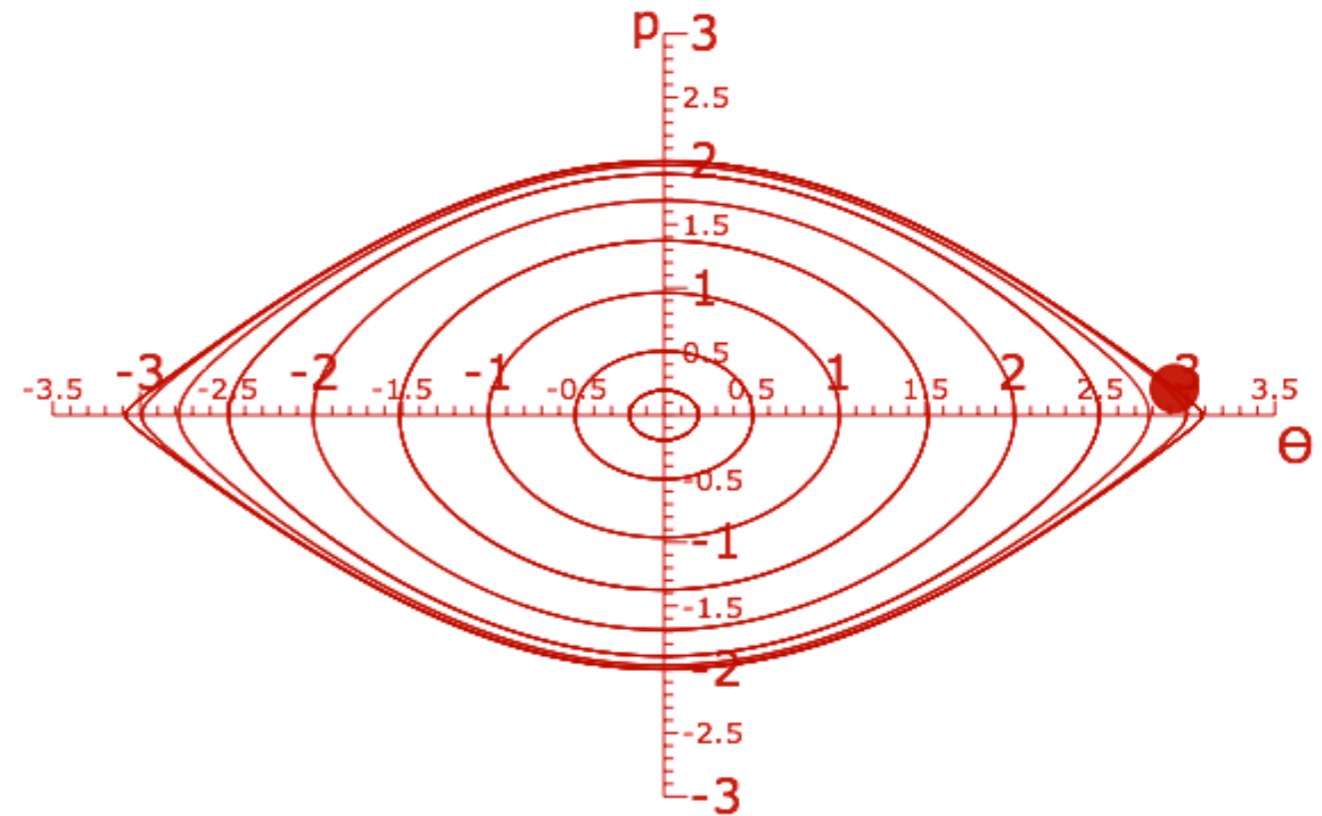
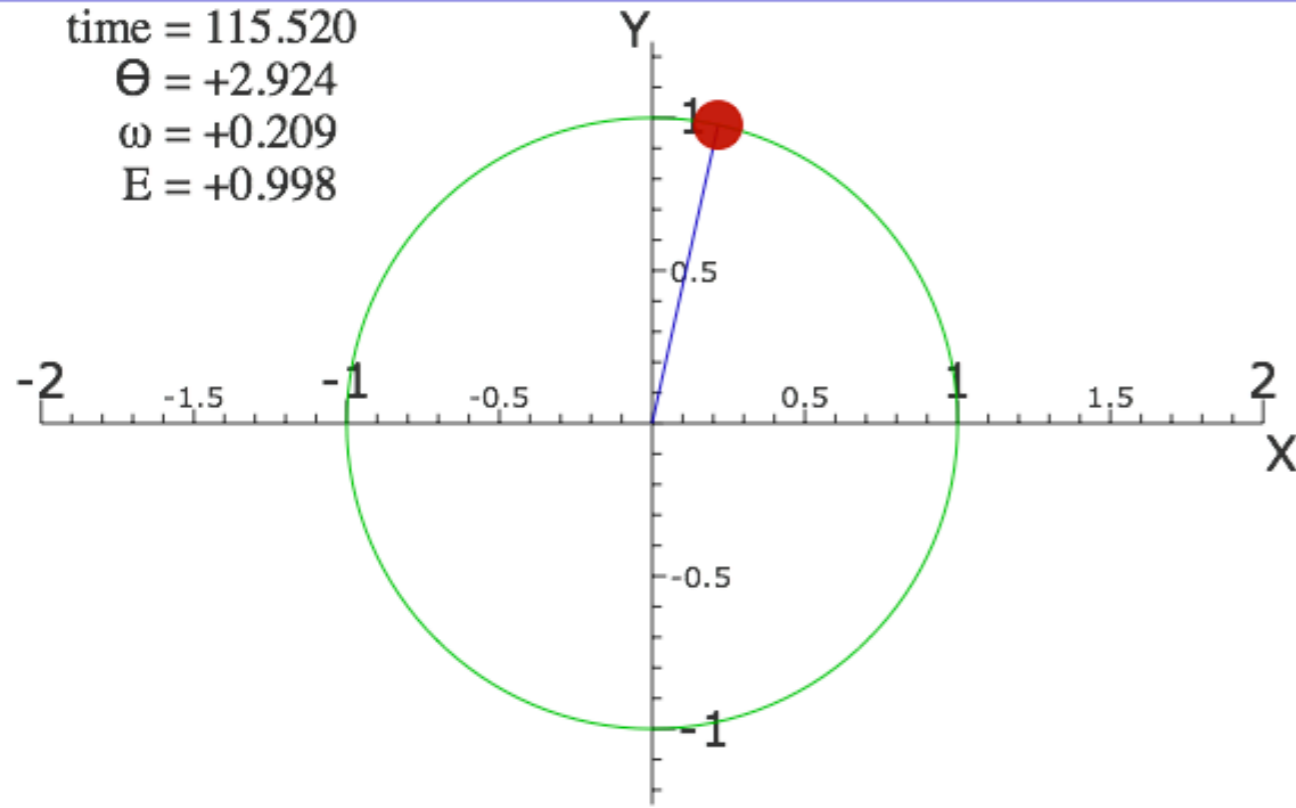
Circular pendulum dynamics and elliptic functions

time = 115.520

$\Theta = +2.924$

$\omega = +0.209$

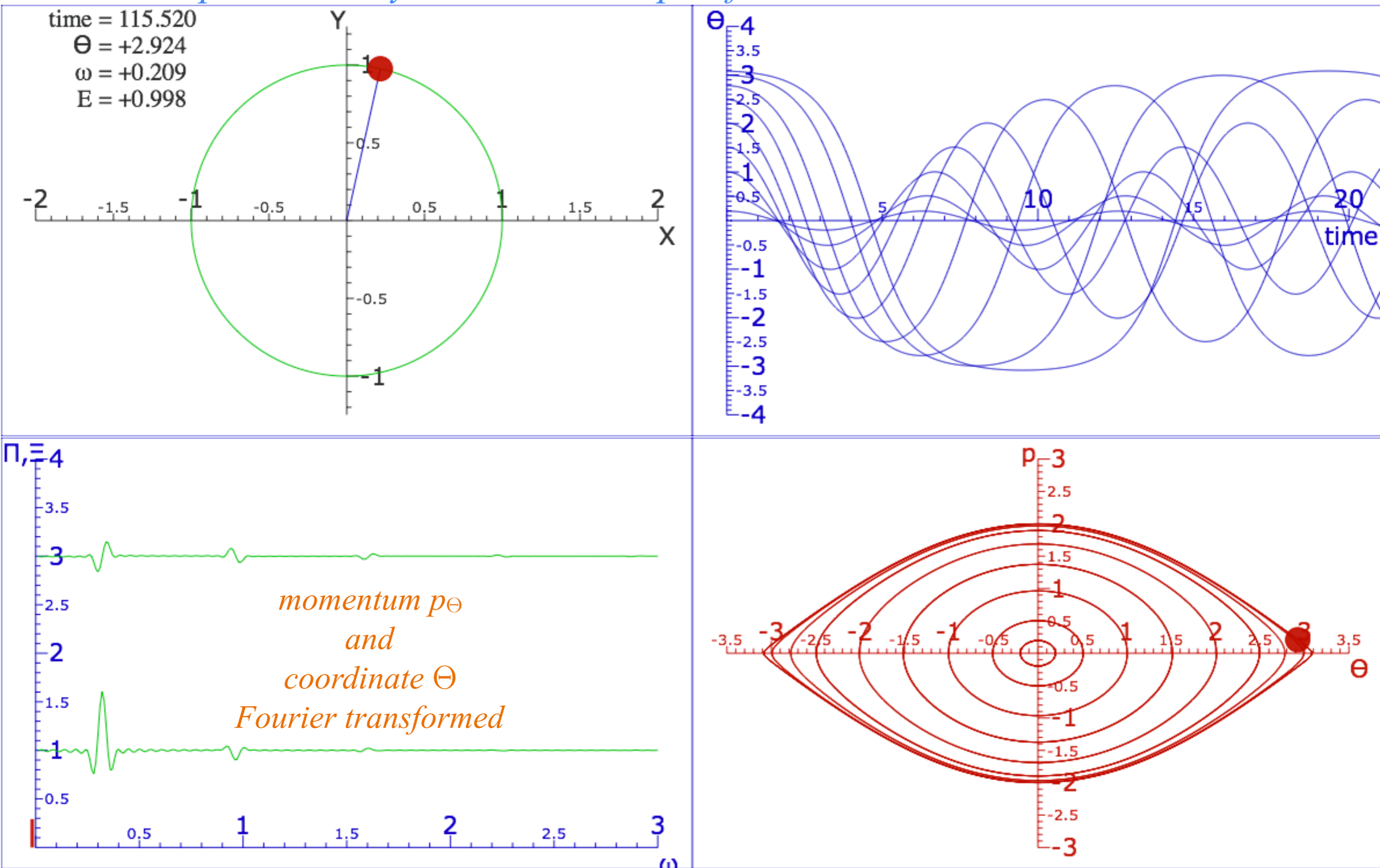
$E = +0.998$



(Simulations of pendulum)

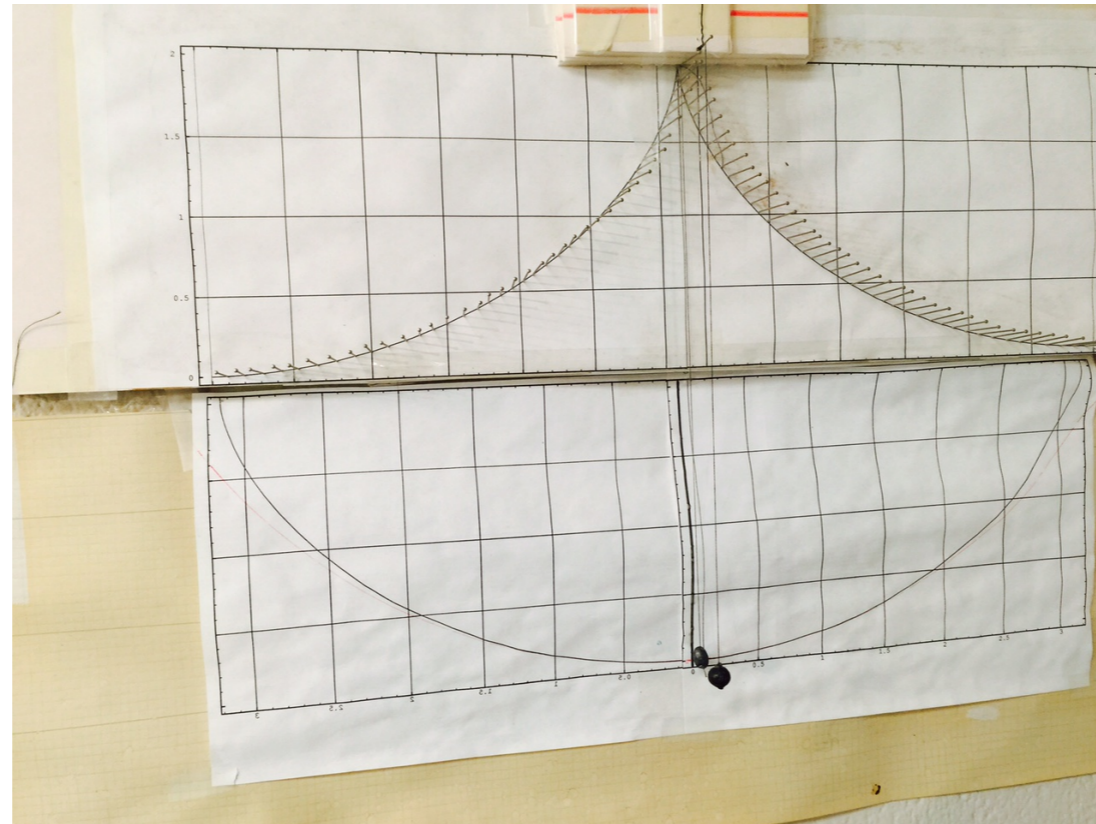
(See also: Simulation of cycloidally constrained pendulum)

Circular pendulum dynamics and elliptic functions



(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)



U of A (PHYS 241)
Cycloid pendulum

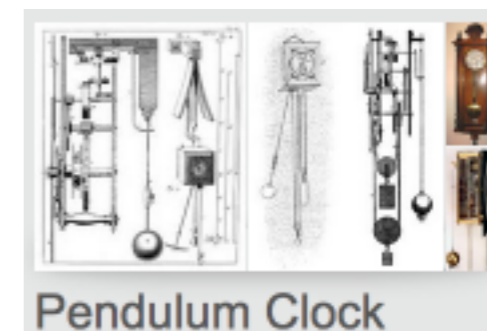
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt](#) (Vert Driven Pendulum))

Circular pendulum dynamics and elliptic functions

➔ *Cycloid pendulum dynamics and “sawtooth” functions*

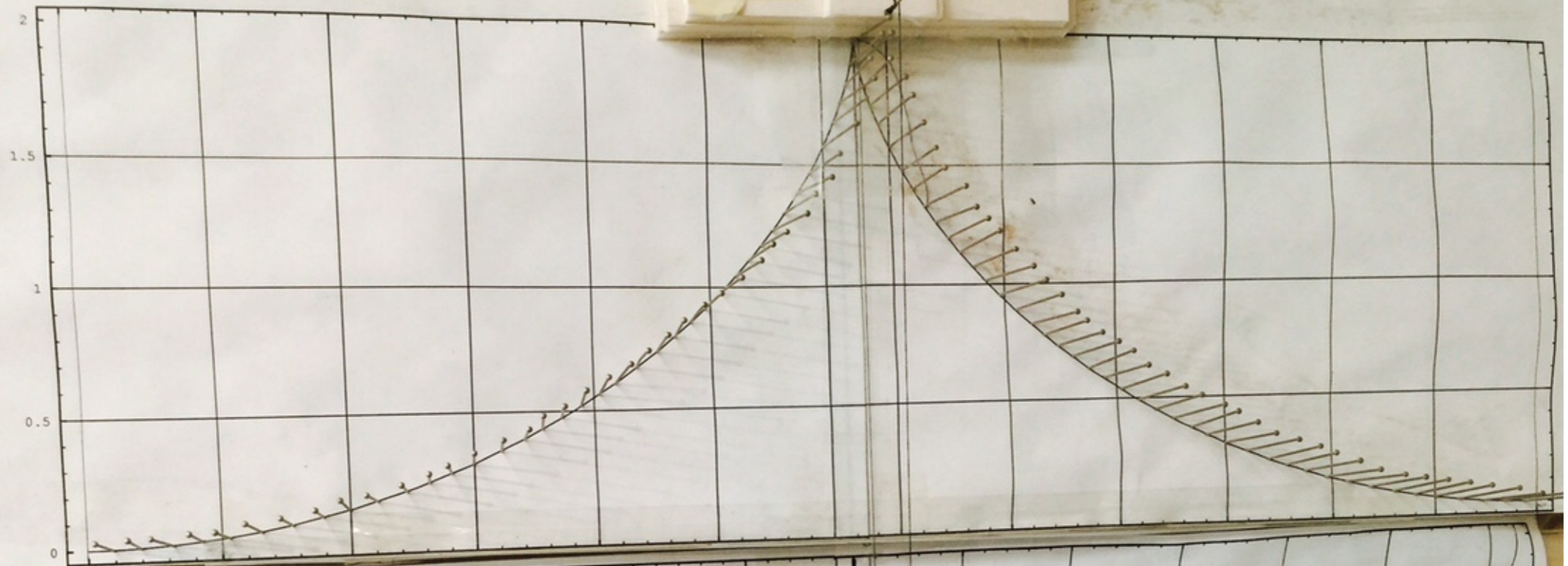
1D-HO phase-space control (Old Mac OS & [Web Simulations of “Catcher in the Eye”](#))



Pendulum Clock



Christiaan Huygens
 (1629-1695)



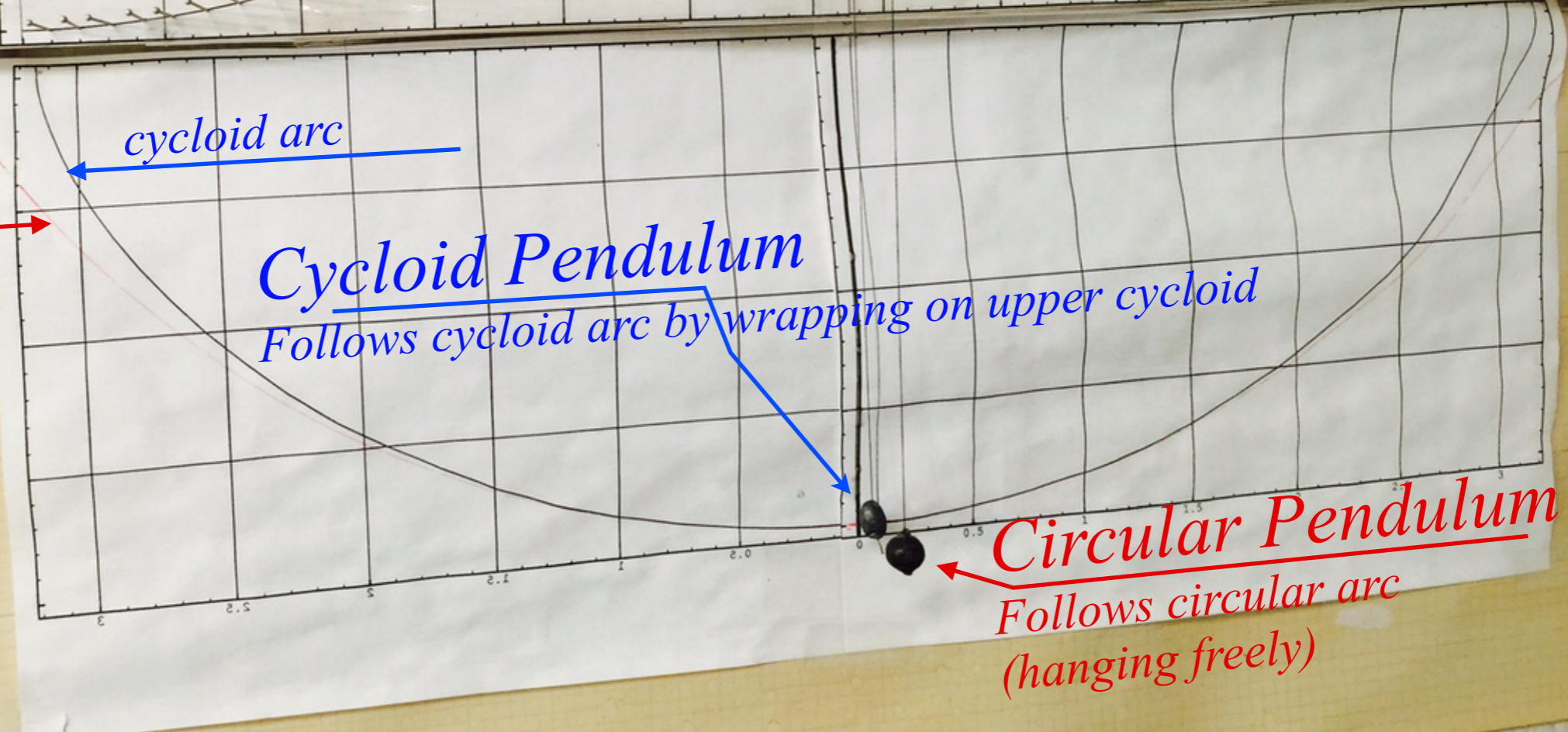
circular arc

cycloid arc

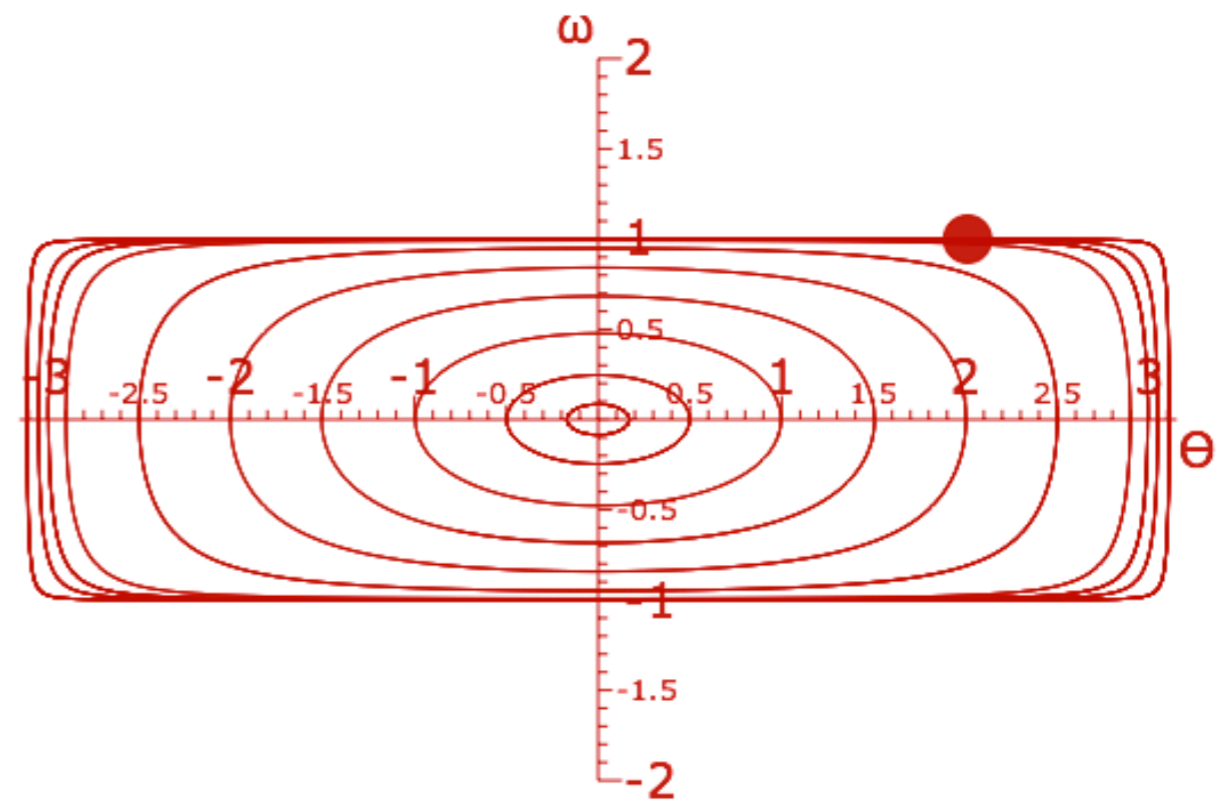
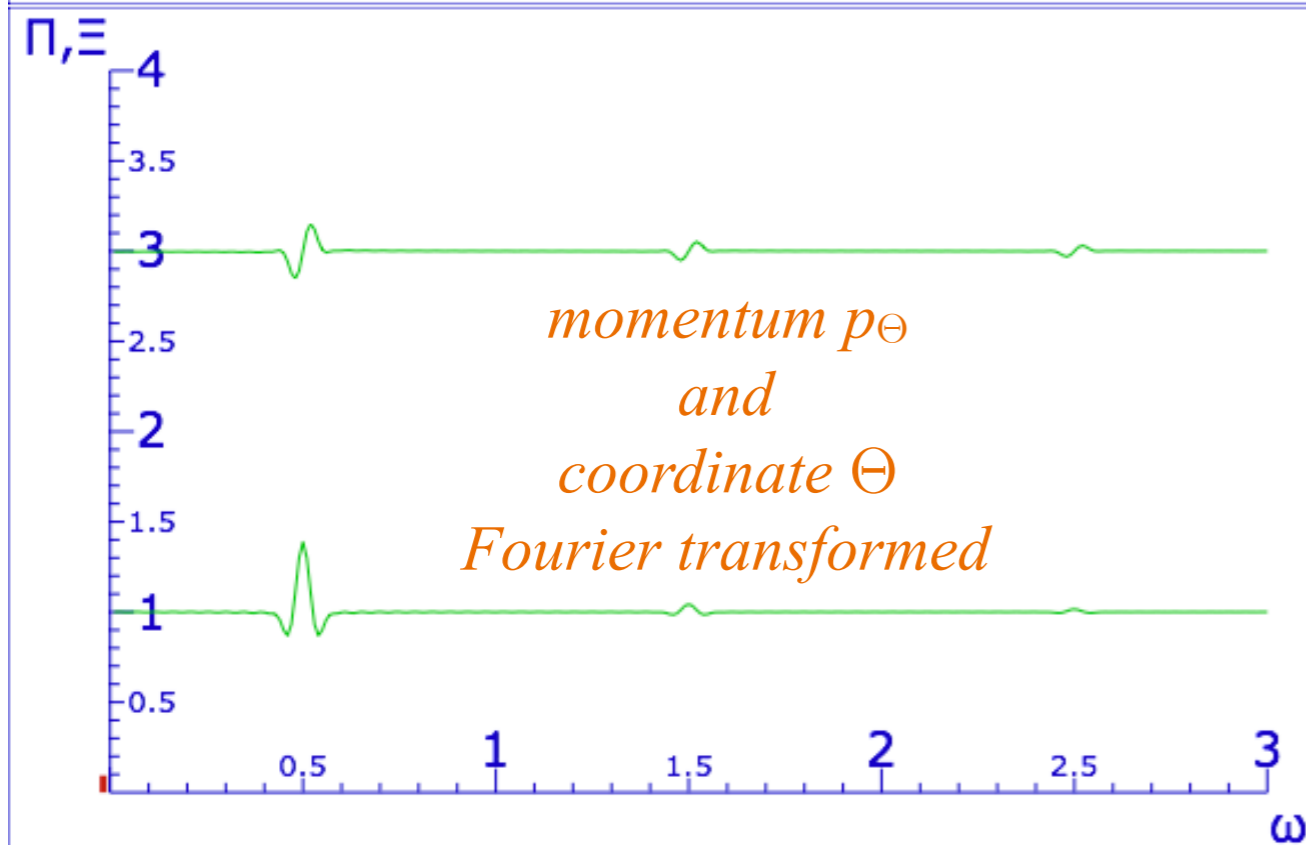
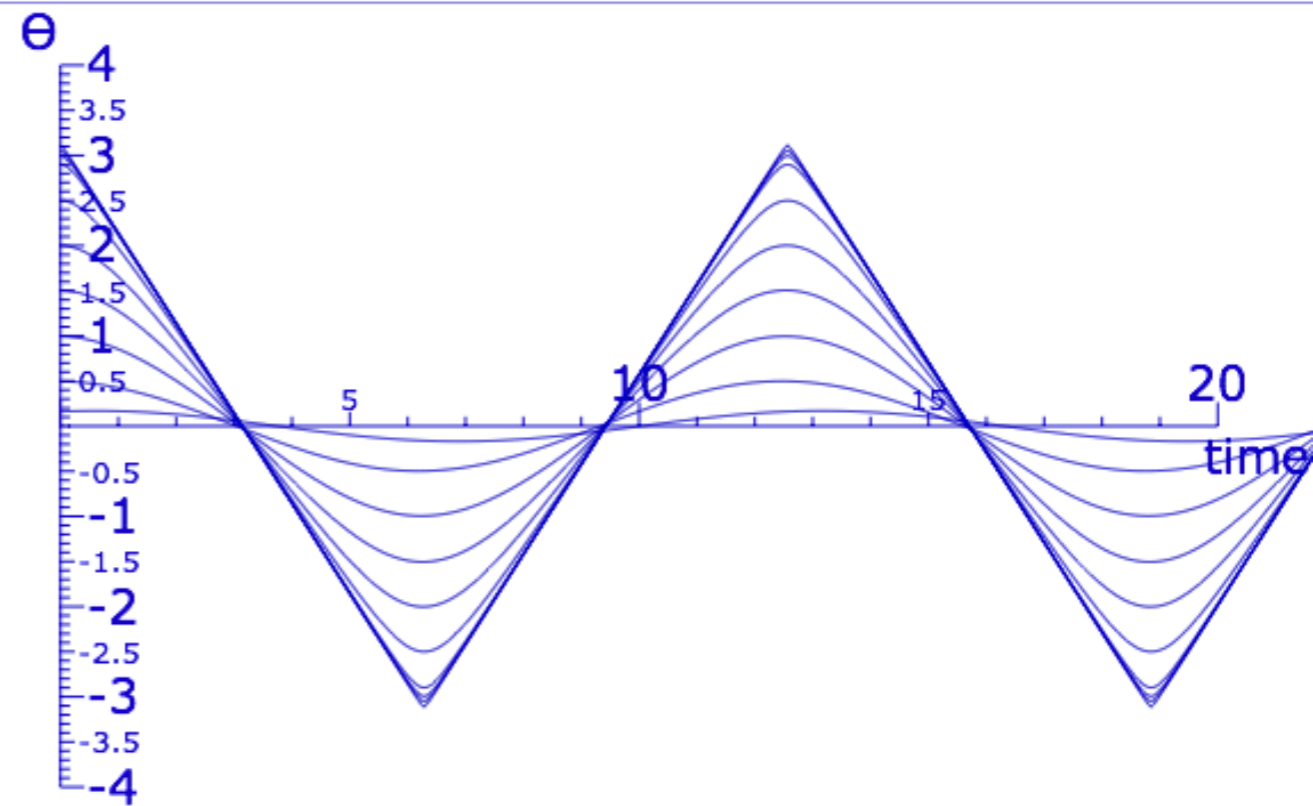
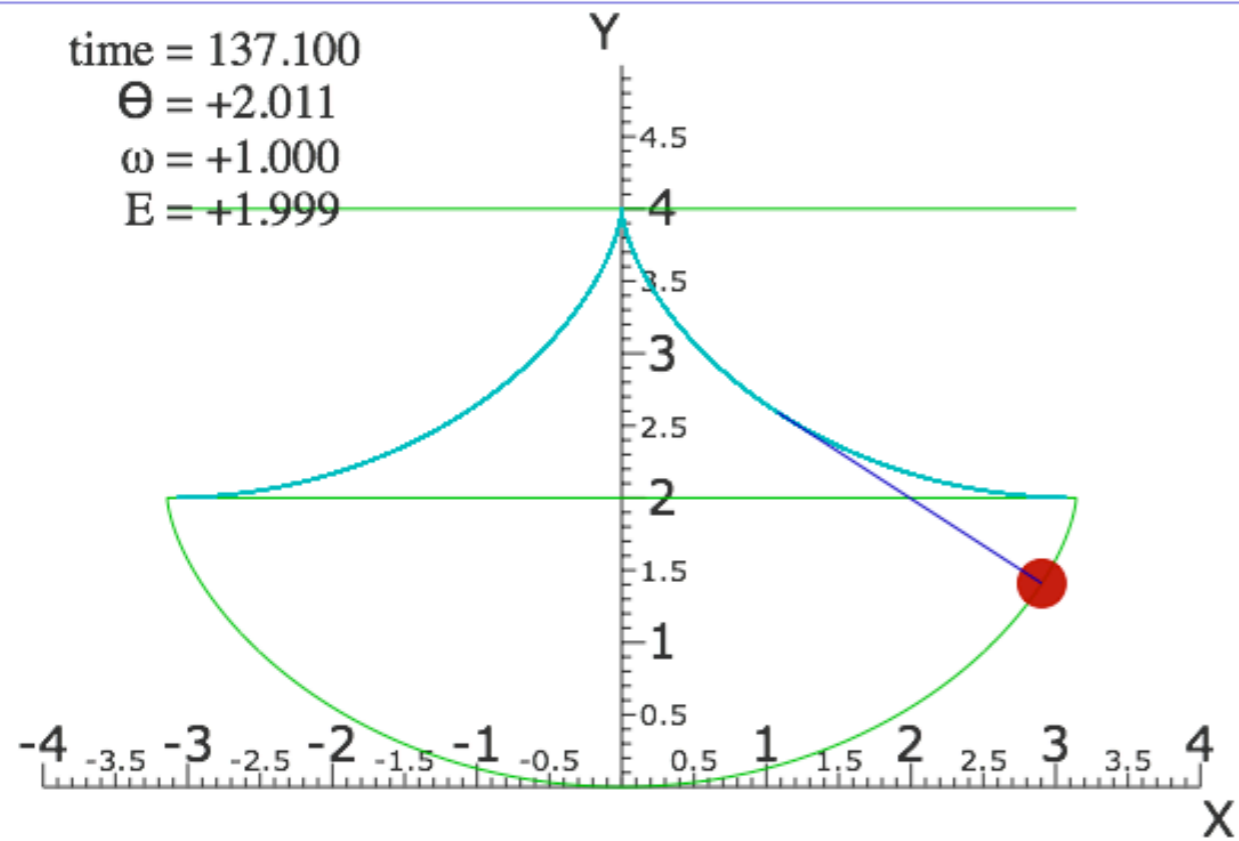
Cycloid Pendulum

Follows cycloid arc by wrapping on upper cycloid

Circular Pendulum
*Follows circular arc
 (hanging freely)*

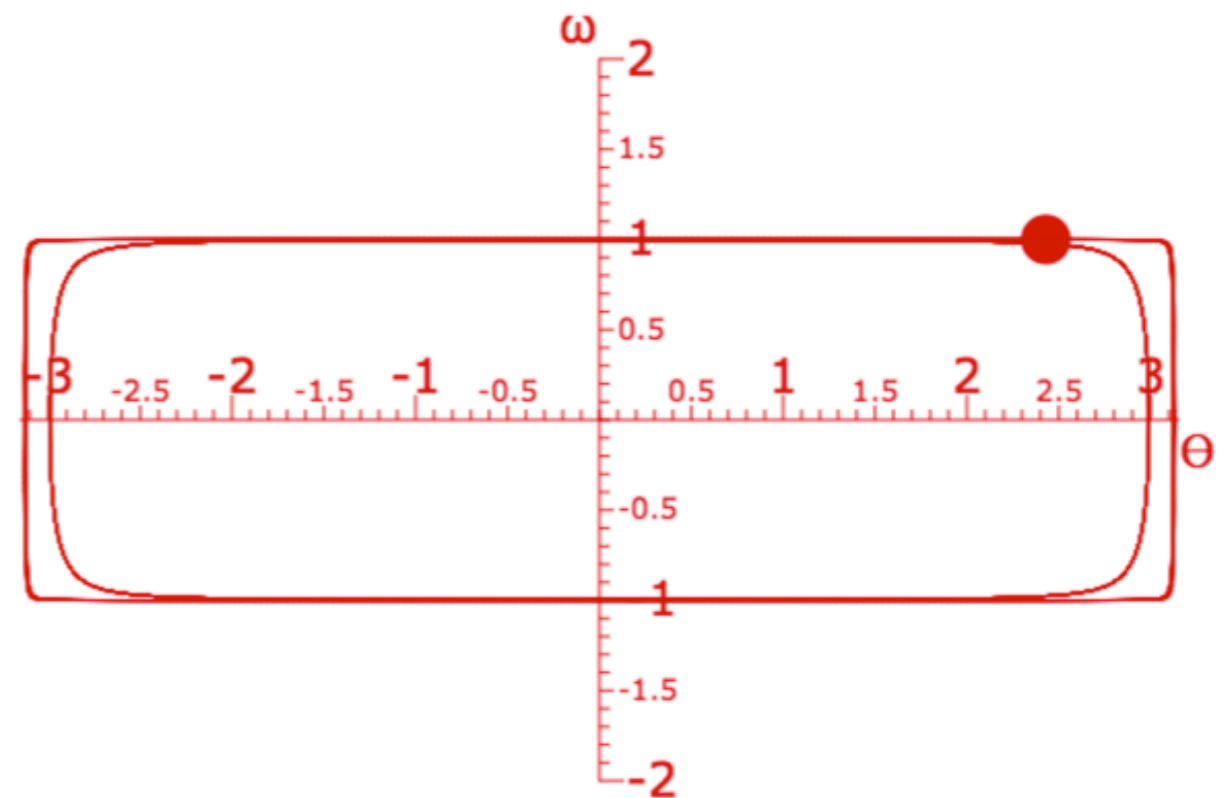
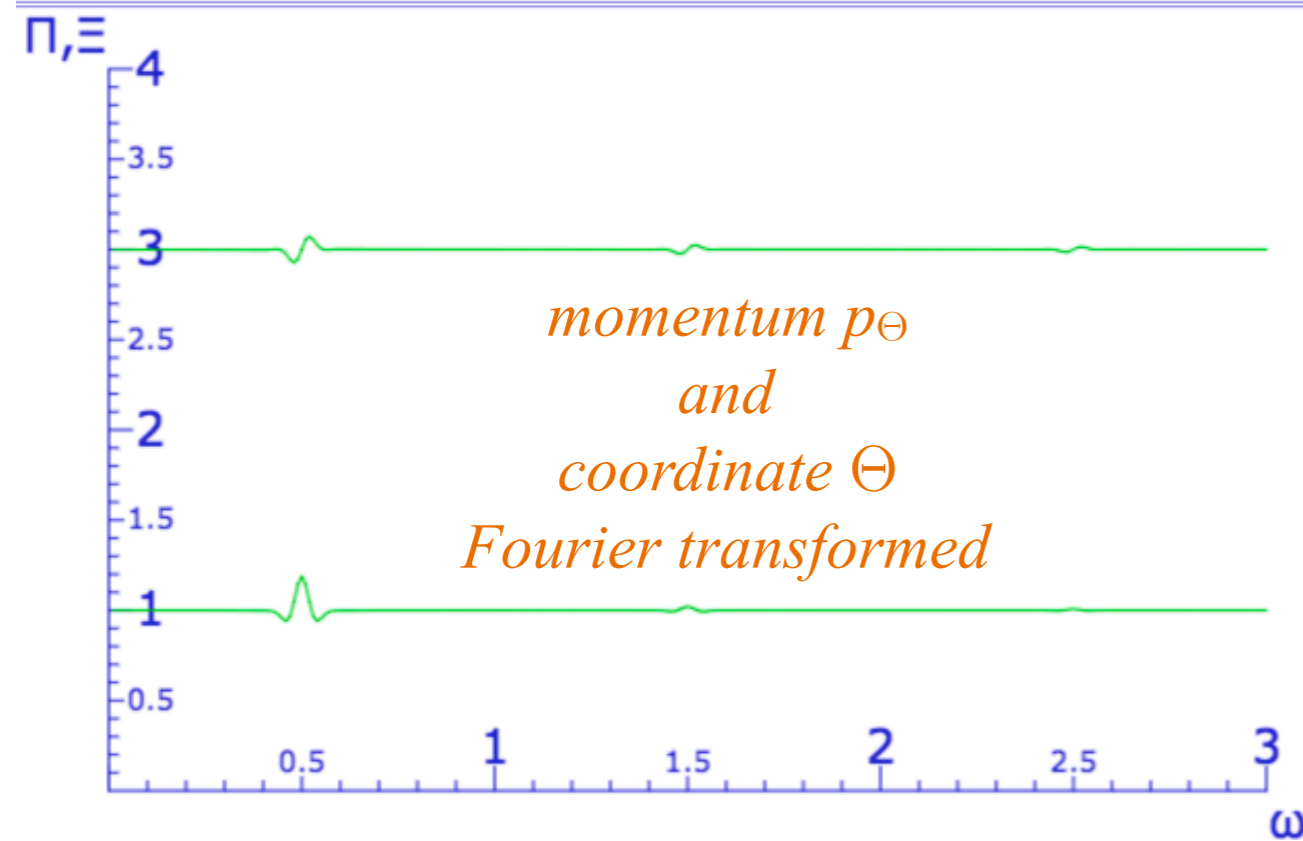
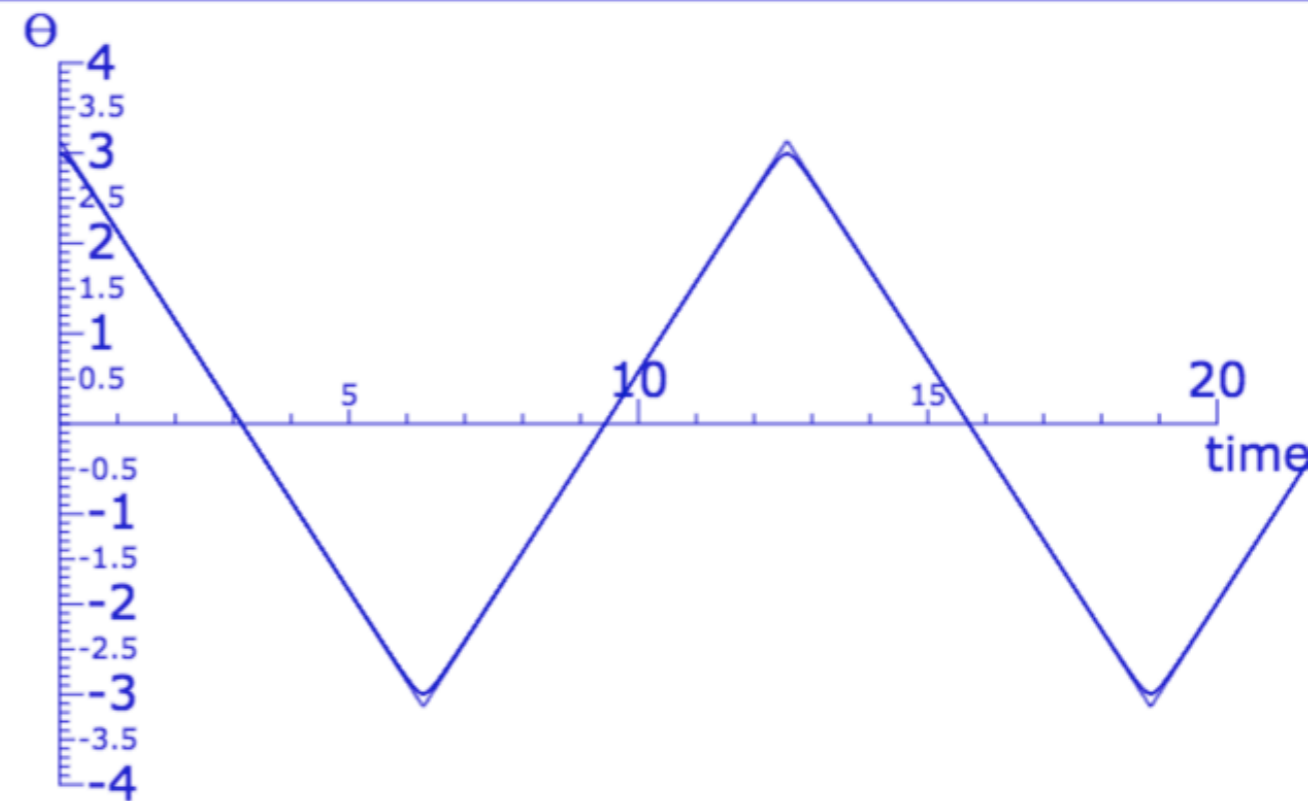
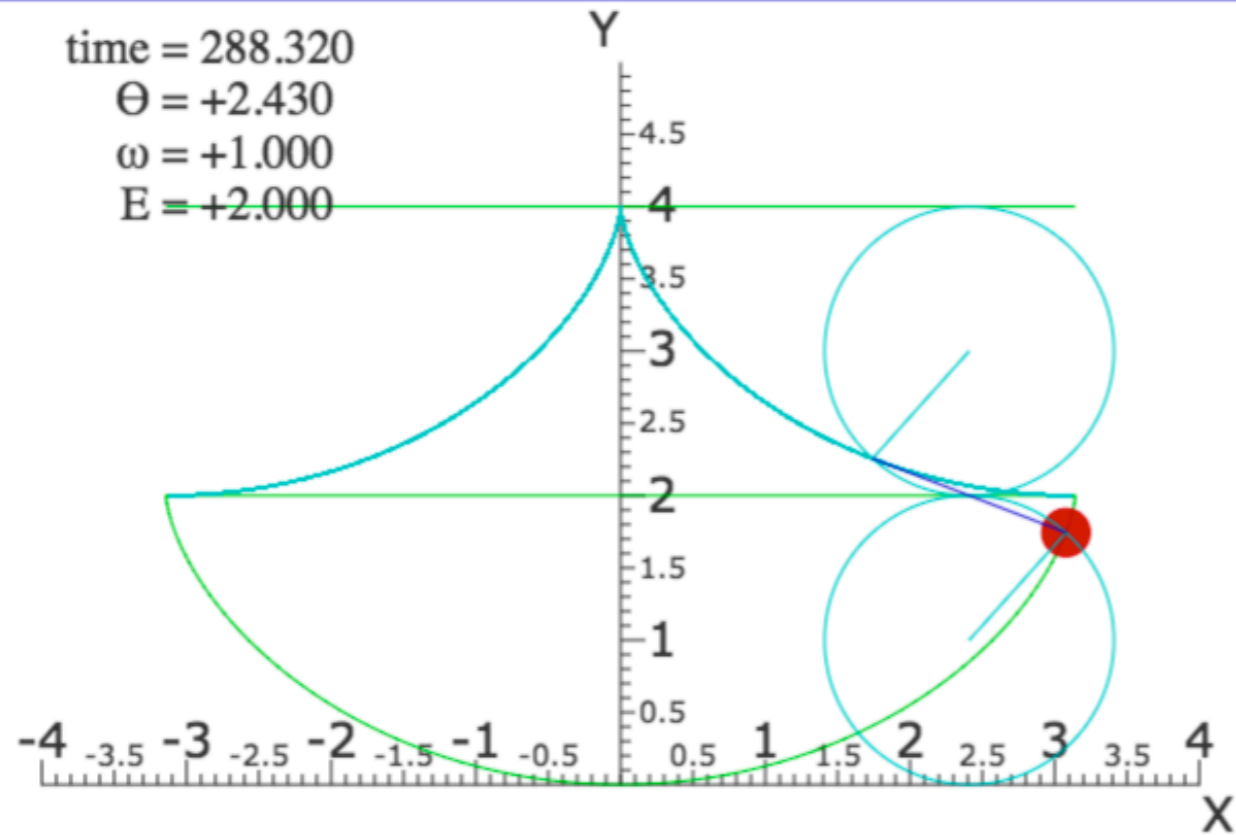


Cycloid pendulum dynamics and "sawtooth" functions



(Simulations of cycloidally constrained pendulum)

Cycloid pendulum dynamics and "sawtooth" functions



(Simulations of cycloidally constrained pendulum)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt \(Vert Driven Pendulum\)](#))

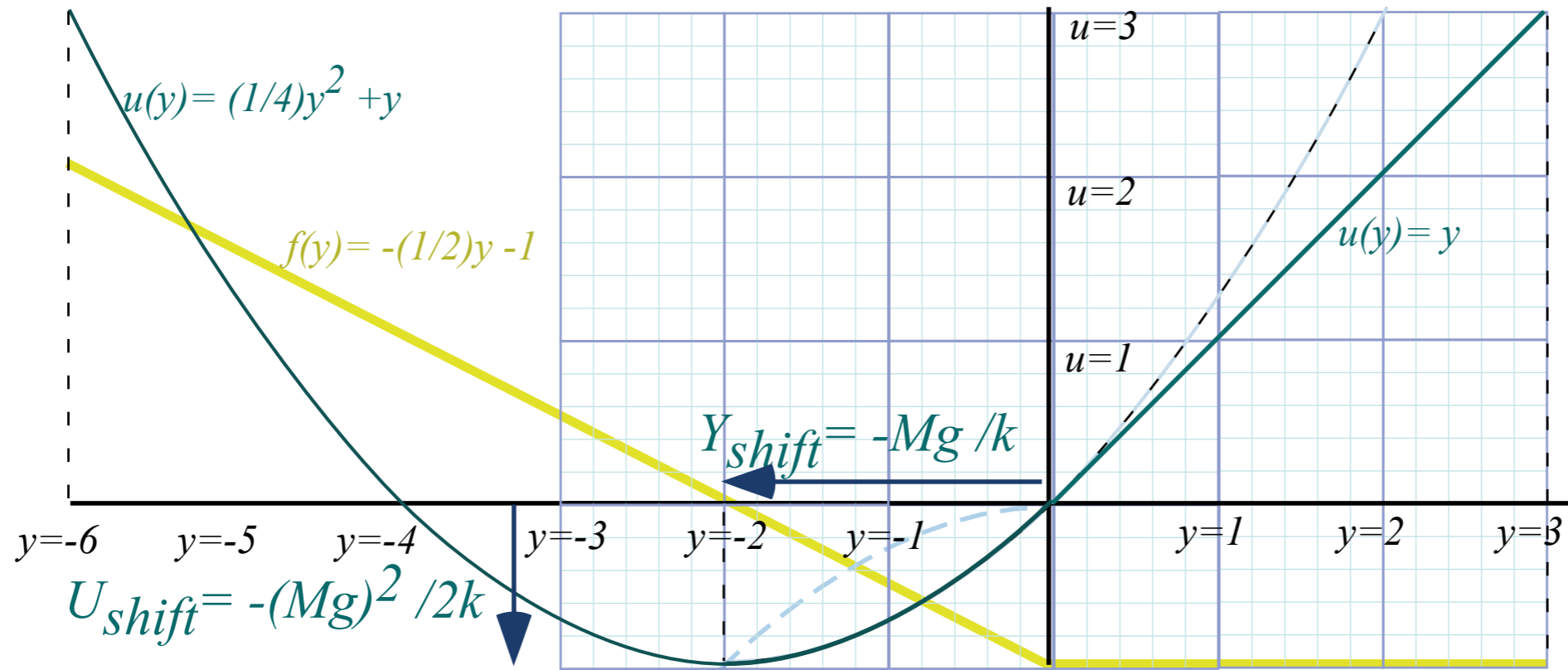
Circular pendulum dynamics and elliptic functions

Cycloid pendulum dynamics and “sawtooth” functions

 *1D-HO phase-space control (Old Mac OS & [Web Simulations of “Catcher in the Eye”](#))*

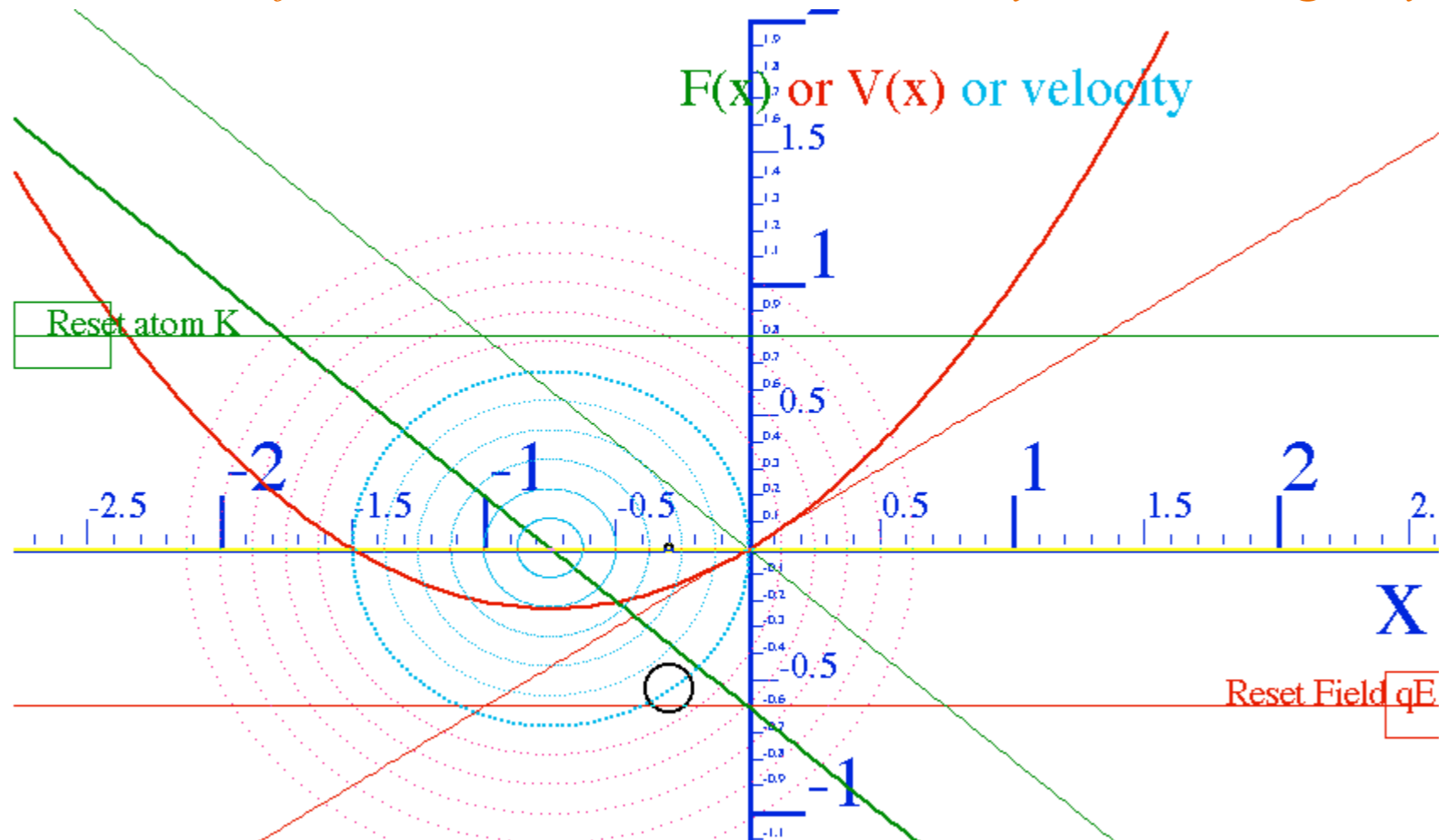
$$F(Y) = -kY - Mg$$

$$U(Y) = (1/2)kY^2 + MgY$$



Unit 1
Fig. 7.4

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control



Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations

Poincare identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations

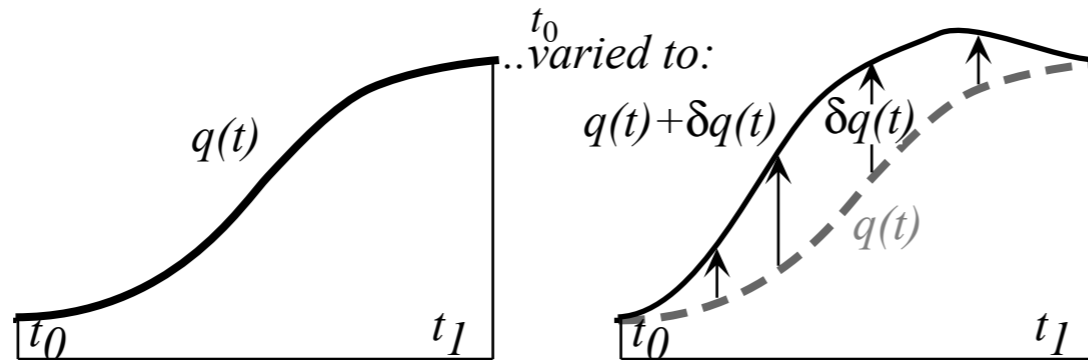
Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange "derivation" of Lagrange's equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): $S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t)$.



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

$$\delta q(t_0) = 0 = \delta q(t_1) \quad (1)$$

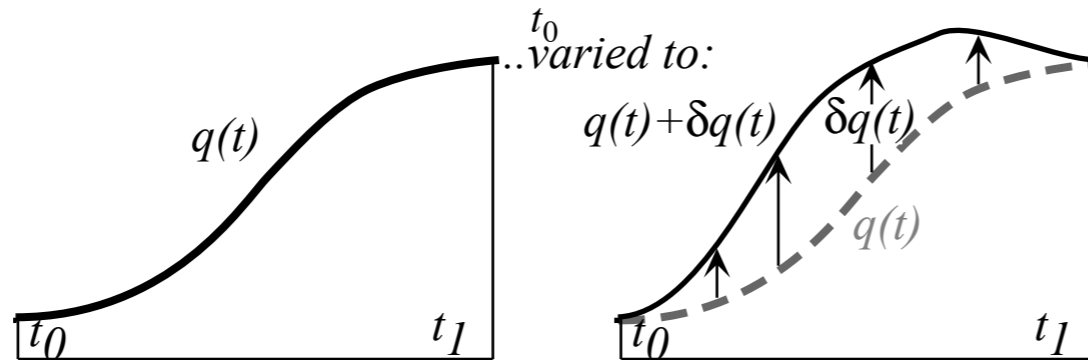
1st order $L(q + \delta q)$ approximate:

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \quad \text{where: } \delta \dot{q} = \frac{d}{dt} \delta q$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



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$$\delta q(t_0) = 0 = \delta q(t_1) \quad (1)$$

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt} \delta q$$

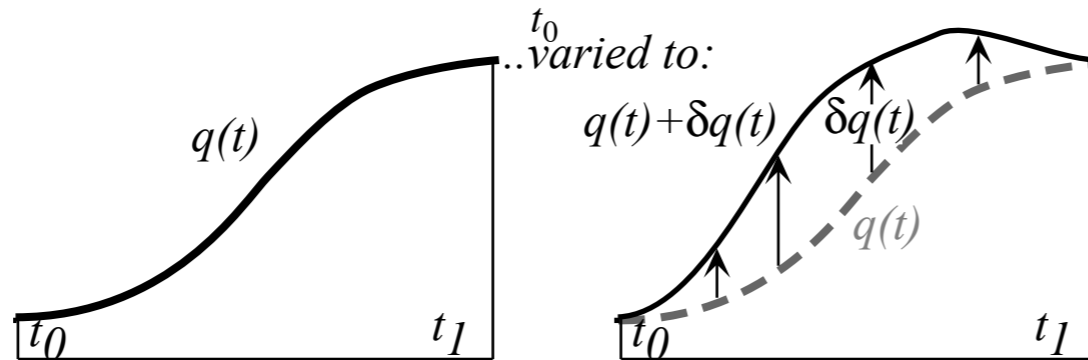
Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

Diagram showing the integration by parts formula: $u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt}v$. Arrows point from this formula to the terms in the replacement above.

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

1st order $L(q+\delta q)$ approximate: $\delta q(t_0) = 0 = \delta q(t_1)$ (1)

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt} \delta q$$

Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

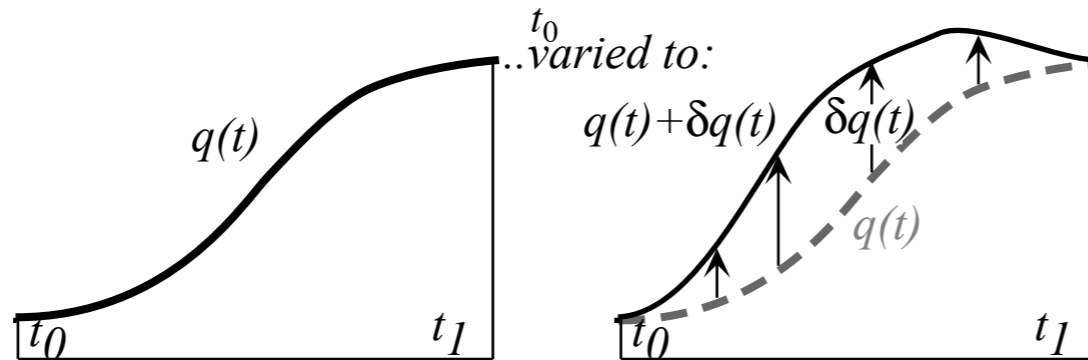
$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

Handwritten notes: $u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt}v$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

1st order $L(q+\delta q)$ approximate: $\delta q(t_0) = 0 = \delta q(t_1)$ (1)

Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

Integration by parts: $u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt} v$

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

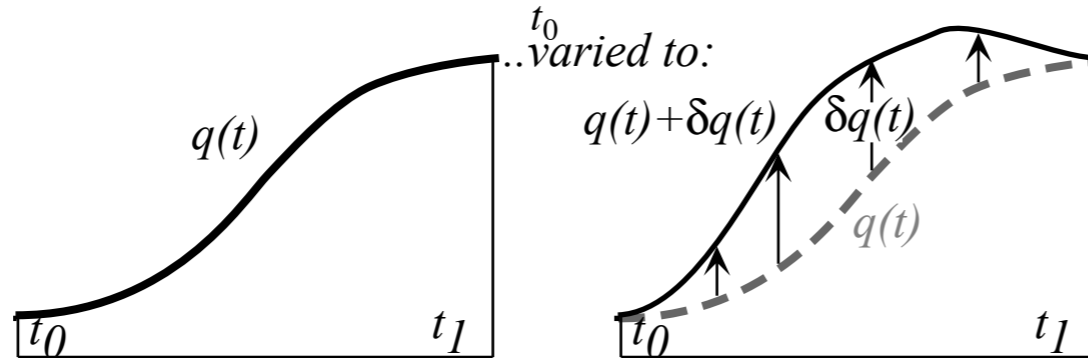
$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

$$= \int_{t_0}^{t_1} dt L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left. \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \right|_{t_0}^{t_1}$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

1st order $L(q+\delta q)$ approximate: $\delta q(t_0) = 0 = \delta q(t_1)$ (1)

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt} \delta q$$

Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

$$= \int_{t_0}^{t_1} dt L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \Big|_{t_0}^{t_1}$$

due to requiring (1)

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q$

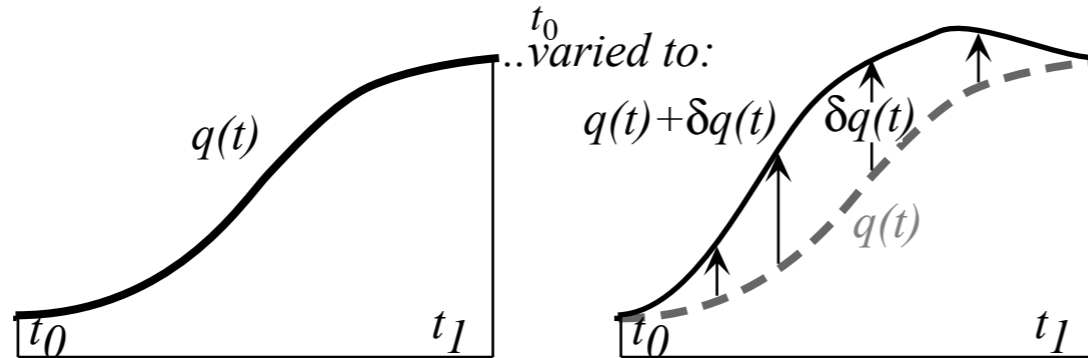
Extreme value (actually *minimum* value) of $S(q)$ occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

1st order $L(q+\delta q)$ approximate: $\delta q(t_0) = 0 = \delta q(t_1)$ (1)

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt} \delta q$$

Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

$$= \int_{t_0}^{t_1} dt L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left. \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \right|_{t_0}^{t_1}$$

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q$

Extreme value (actually *minimum* value) of $S(q)$ occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}$$

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian $L = T - U$???

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Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L \cdot dt = \mathbf{p} \cdot \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ implies: } \mathbf{v} \cdot dt = d\mathbf{r} \right)$$

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This is the time differential dS of *action* $S = \int L \cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where: } L = \frac{dS}{dt}$$

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Unit 8 shows *DeBroglie law* $\mathbf{p} = \hbar \mathbf{k}$ and *Planck law* $H = \hbar \omega$ make *quantum plane wave phase* Φ :

$$\Phi = S/\hbar = \int L \cdot dt / \hbar$$

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Q: When is the *Action*-differential dS integrable?

A: A differential $dW = f_x(x, y)dx + f_y(x, y)dy$ is *integrable* to a $W(x, y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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Similar to conditions for integrating work differential $dW = \mathbf{f} \cdot d\mathbf{r}$ to get potential $W(\mathbf{r})$. That condition is **no curl allowed**: $\nabla \times \mathbf{f} = \mathbf{0}$ or ∂ -symmetry of W :

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(Given “quantum wave”)

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Try 1st \mathbf{r} -derivative of wave ψ

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, t) = \frac{\partial}{\partial \mathbf{r}} e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial \mathbf{r}} e^{iS/\hbar} = (i/\hbar) \frac{\partial S}{\partial \mathbf{r}} \psi(\mathbf{r}, t)$$

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Momentum Operator
or \mathbf{p} -op in \mathbf{r} -basis
 $\mathbf{p} \Rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}}$

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$$= (i/\hbar)(-H) \psi(\mathbf{r}, t) \text{ or: } i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H \psi(\mathbf{r}, t)$$

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Schrodinger time equation
 $i\hbar \dot{\psi}(\mathbf{r}, t) = H \psi(\mathbf{r}, t)$

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Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics



Christaan Huygens
(1629-1695)

Huygen's contact transformations enforce minimum action

Each point \mathbf{r}_k on a wavefront "broadcasts" in all directions.

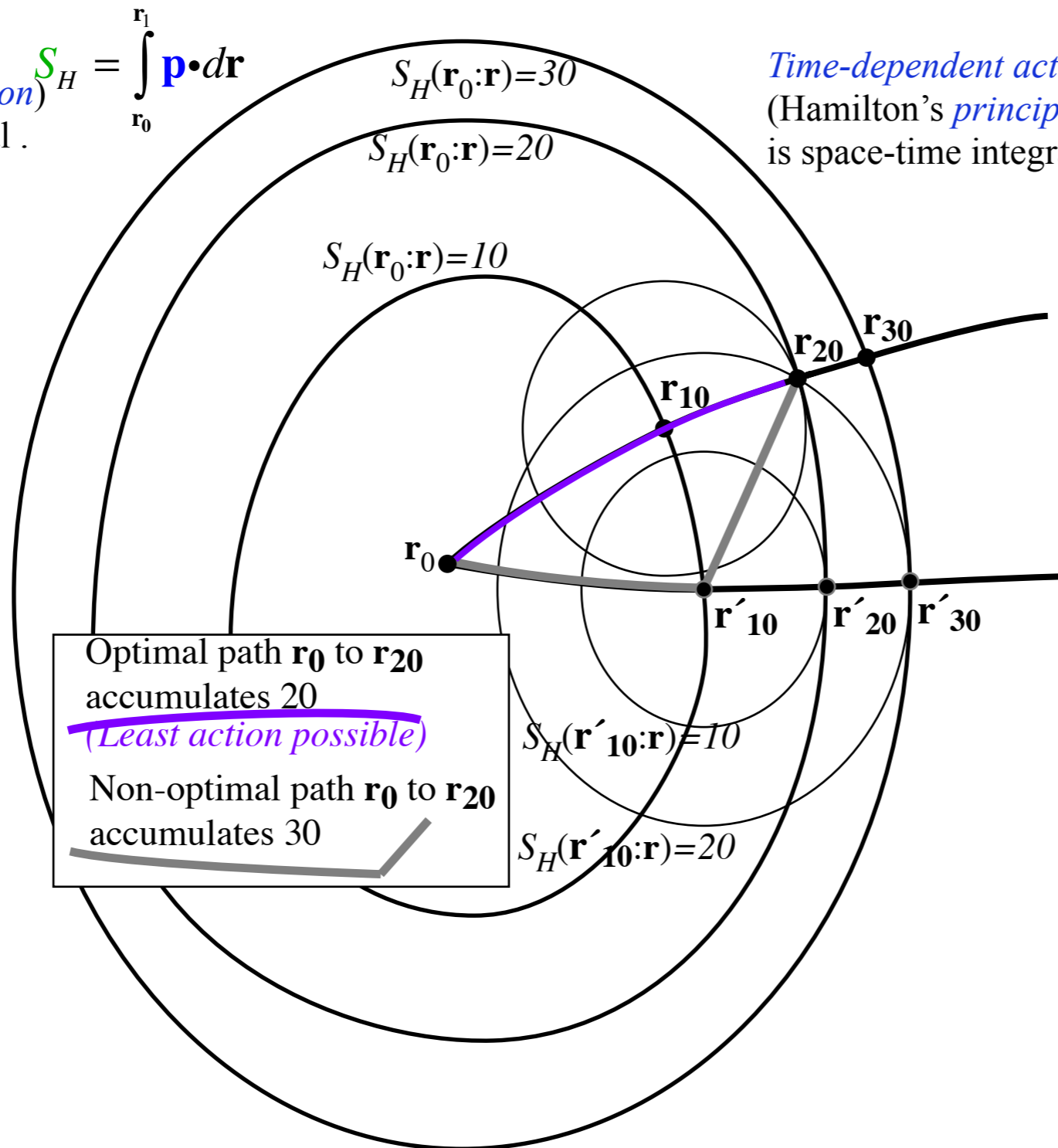
Only **minimum action** path interferes constructively

Time-independent action
(Hamilton's *reduced action*)
is a purely spatial integral .

$$S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \cdot d\mathbf{r}$$

Time-dependent action
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Optimal path \mathbf{r}_0 to \mathbf{r}_{20}
accumulates 20
(Least action possible)
Non-optimal path \mathbf{r}_0 to \mathbf{r}_{20}
accumulates 30

Unit 1
Fig. 12.12

Huygen's contact transformations enforce minimum action

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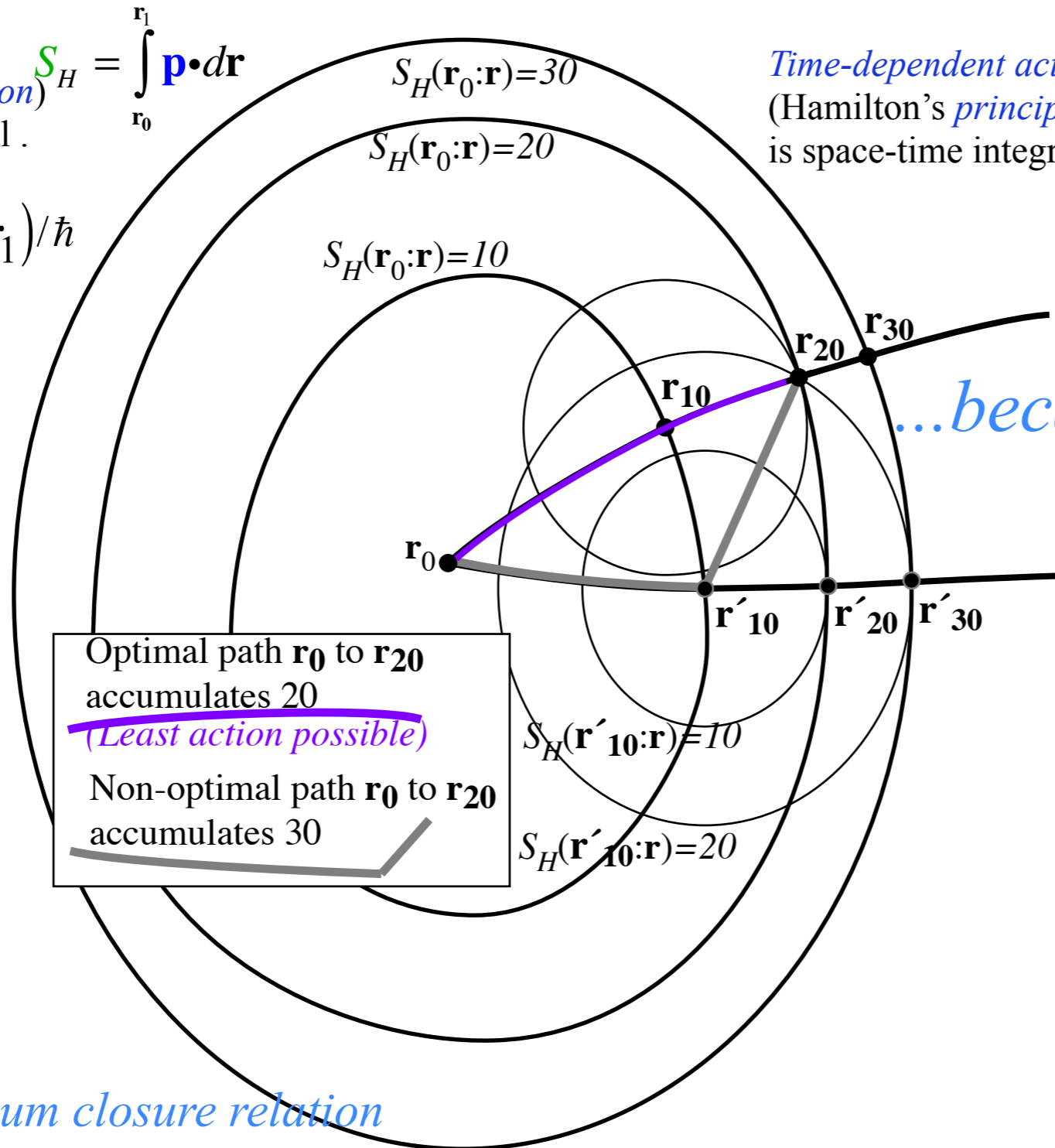
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$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1) / \hbar}$$

$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1) / \hbar}$$



Optimal path \mathbf{r}_0 to \mathbf{r}_{20}
 accumulates 20
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 Non-optimal path \mathbf{r}_0 to \mathbf{r}_{20}
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...because action is
 quantum wave phase

Unit 1
 Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i(S_H(\mathbf{r}_0 : \mathbf{r}') + S_H(\mathbf{r}' : \mathbf{r}_1)) / \hbar} = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1) / \hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

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Davis-Heller “Color-Quantization” or “Classical Chromodynamics”

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase S_H/\hbar in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \cdot d\mathbf{r}$. The integer n ($n = 0, 1, 2, \dots$) is a *quantum number*.

$$1 = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0:\mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi \hbar n = h n$$

Numerically integrate Hamilton's equations and Lagrangian L . Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L dt + Ht.$$

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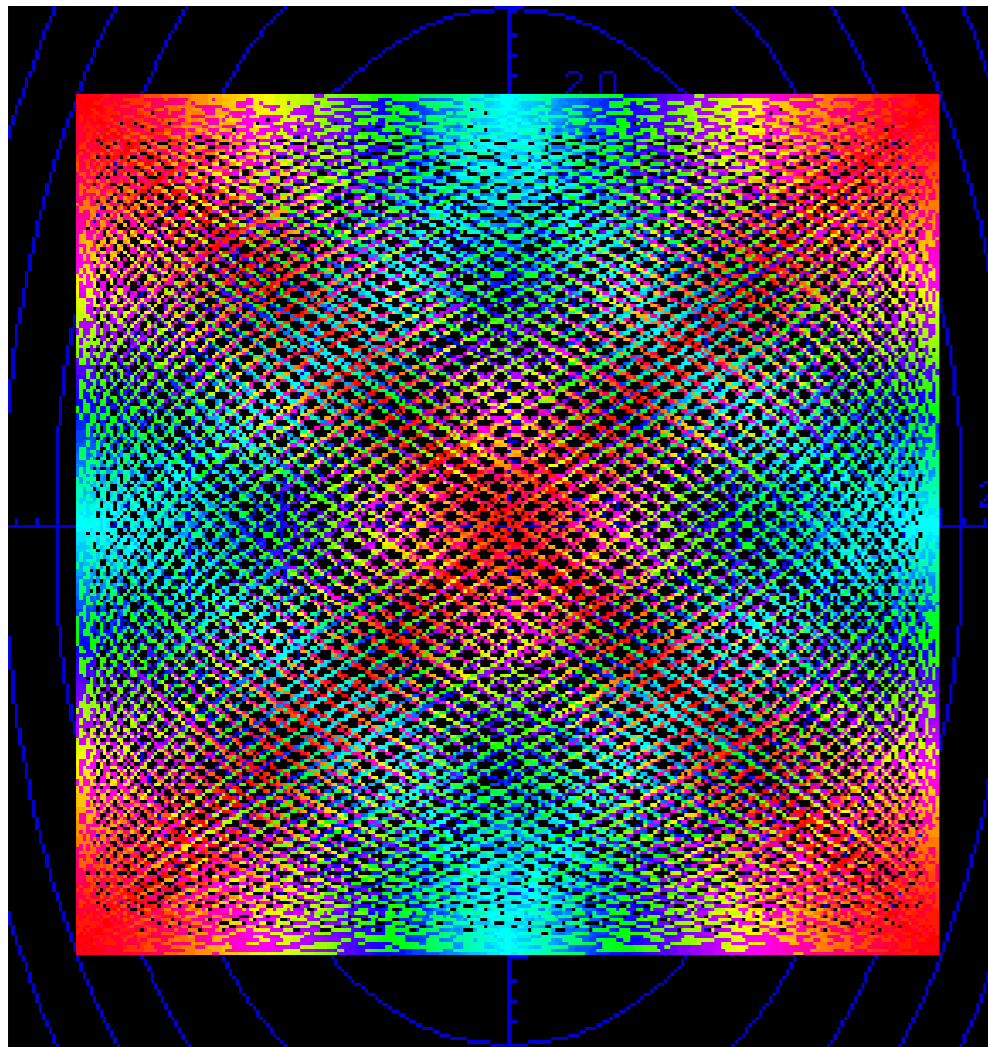
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The hue should represent the phase angle $S_H(\mathbf{0} : \mathbf{r})/\hbar \text{ modulo } 2\pi$ as, for example,

$0=\text{red}$, $\pi/4=\text{orange}$, $\pi/2=\text{yellow}$, $3\pi/4=\text{green}$, $\pi=\text{cyan}$ (opposite of red), $5\pi/4=\text{indigo}$, $3\pi/2=\text{blue}$, $7\pi/4=\text{purple}$, and $2\pi=\text{red}$ (full color circle). Interpolating action on a palette of 32 colors is enough precision for low quanta.



*simulation
by
"Color U(2)"*

Unit 1
Fig.
12.13

*closed system
has quantized E.
Standing wave has
only two phases(\pm)
cyan and *red*

Wavepacket and Color-quantization:
M. J. Davis and E. J. Heller, J. Chem. Phys. 75, 246 (1981)

[The Semiclassical Way to Molecular Spectroscopy:](#)
[Eric J. Heller, Acc. Chem. Res. 1981, 14, 368-375](#)

How to do quantum mechanics if you only know classical mechanics

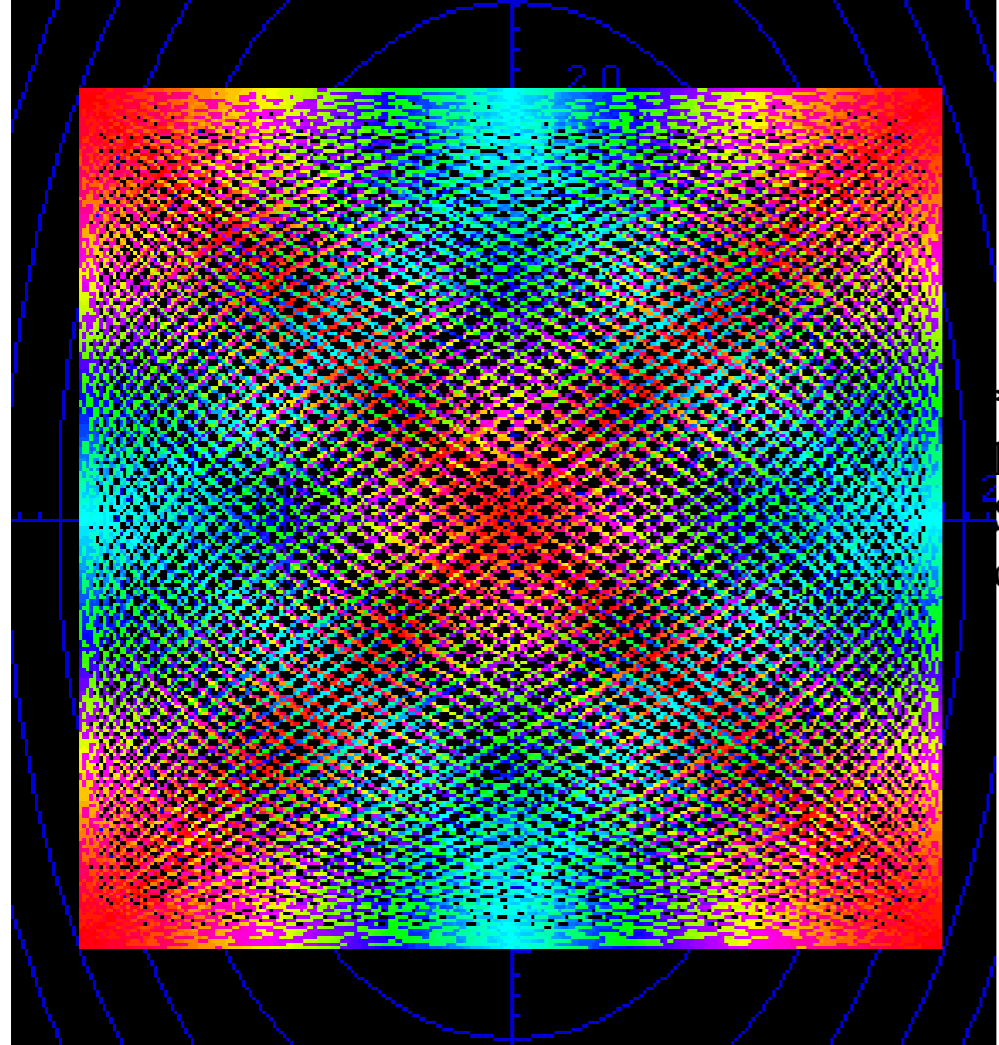
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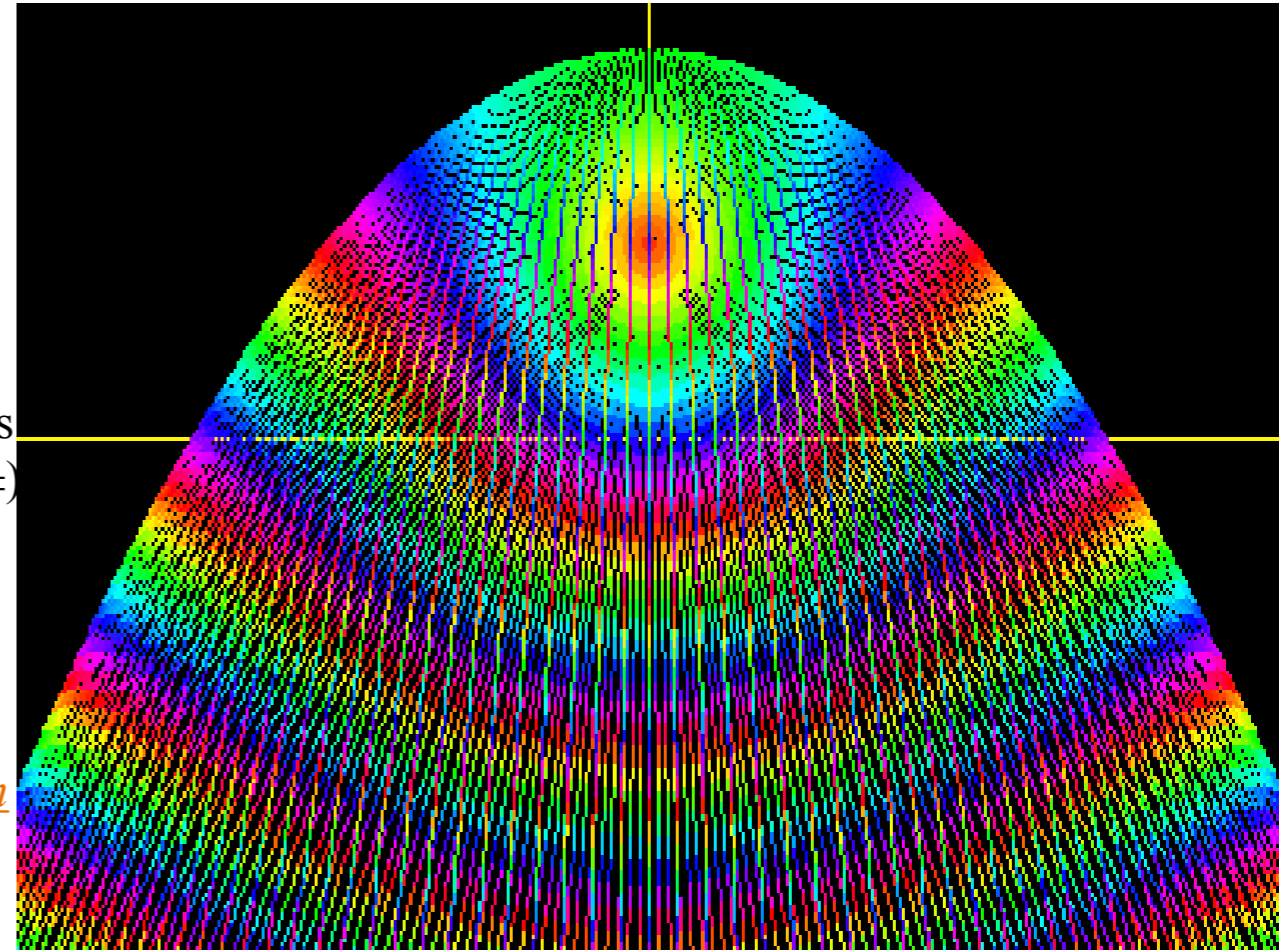
$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L dt + Ht.$$

The hue should represent the phase angle $S_H(\mathbf{0} : \mathbf{r})/\hbar \text{ modulo } 2\pi$ as, for example, $0=\text{red}$, $\pi/4=\text{orange}$, $\pi/2=\text{yellow}$, $3\pi/4=\text{green}$, $\pi=\text{cyan}$ (opposite of red), $5\pi/4=\text{indigo}$, $3\pi/2=\text{blue}$, $7\pi/4=\text{purple}$, and $2\pi=\text{red}$ (full color circle). Interpolating action on a palette of 32 colors is enough precision for low quanta.



Simulation by "Color U(2)"
 Unit 1 Fig. 12.13
 *closed system has quantized E
 Standing wave has only two phases(\pm) cyan and red
 Unit 1 Fig. 12.14
Web Simulation by "CoulIt"

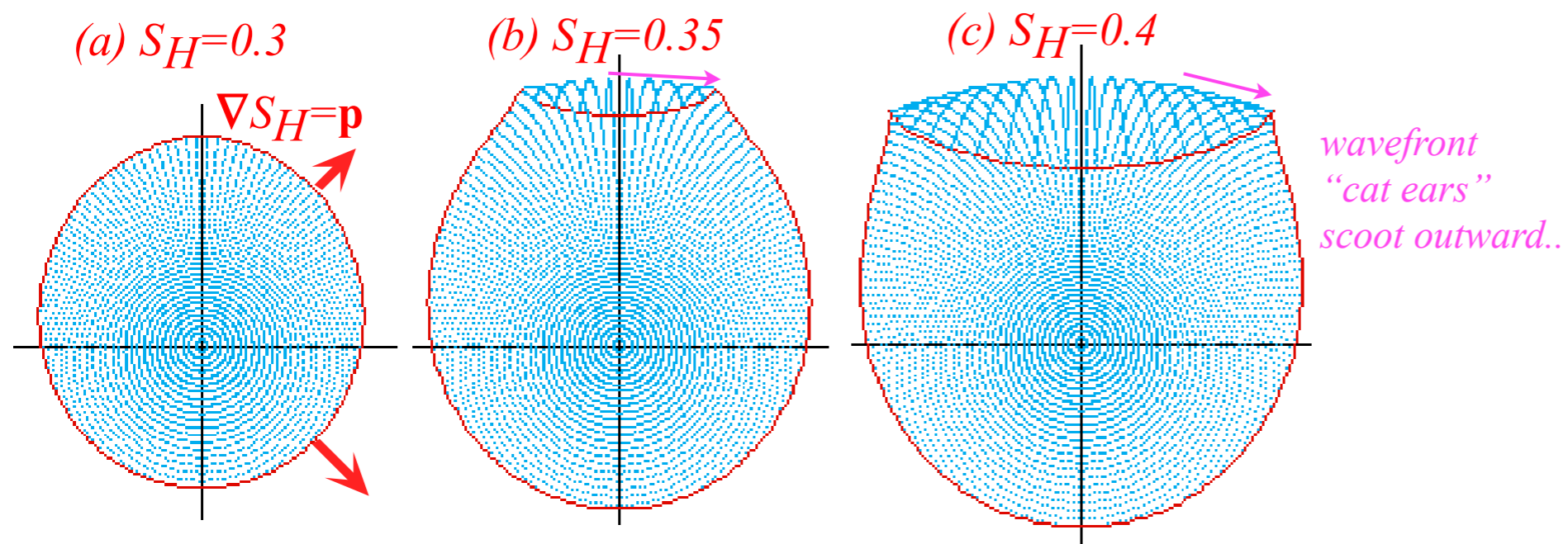
*open system has continuous energy



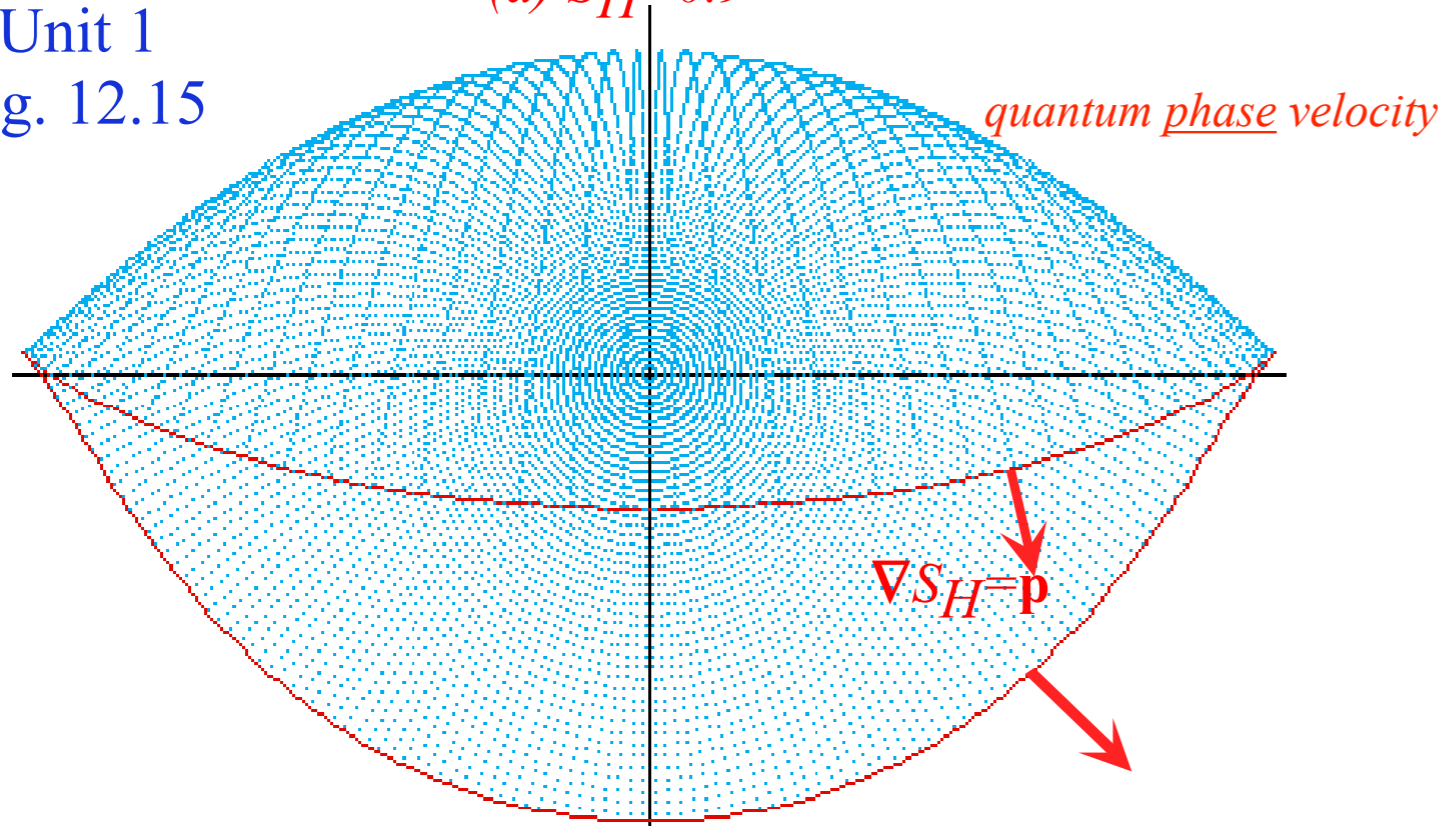
A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt.$

$$\mathbf{v}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

Quantum "phase wavefronts"



(d) $S_H=0.9$



Unit 1
Fig. 12.15

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt.$

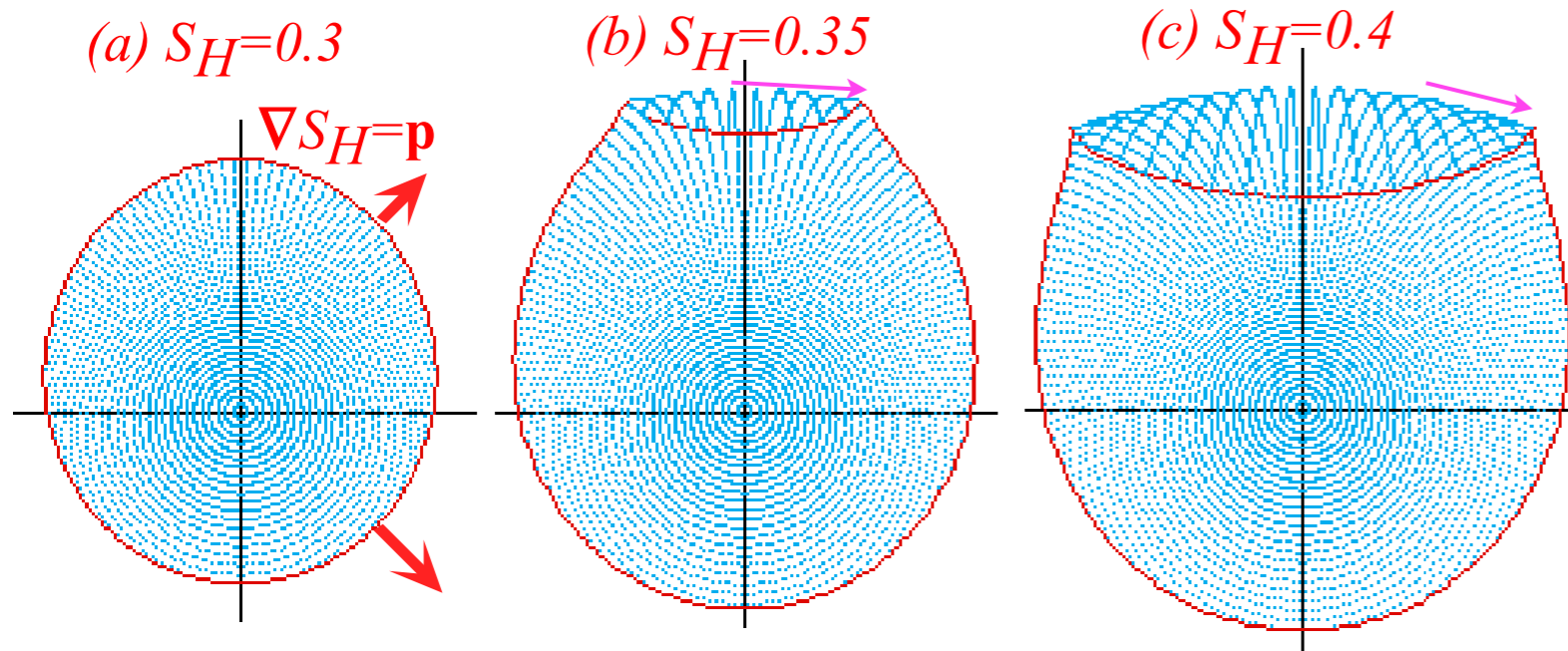
$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

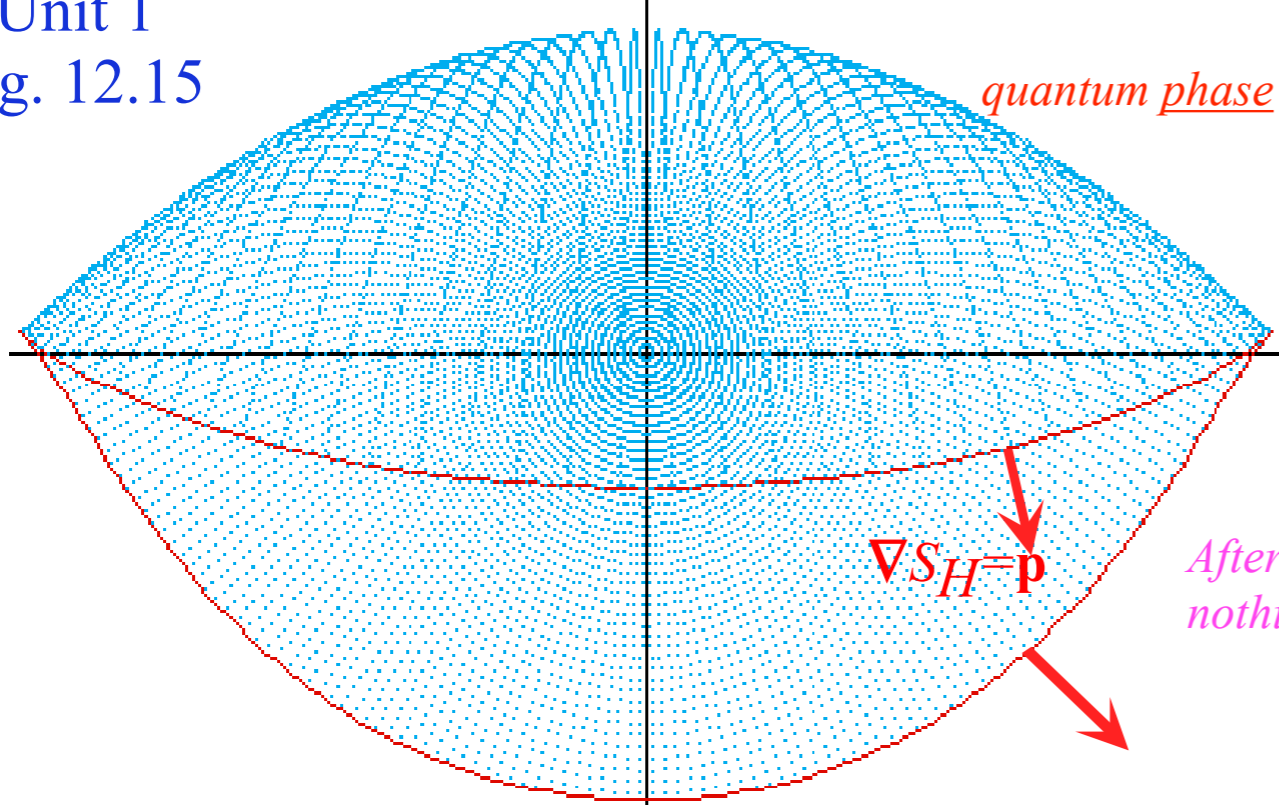
Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



wavefront
"cat ears"
scoot outward..

(d) $S_H=0.9$



quantum phase velocity

After a while ...
nothing left but a smile!

Unit 1
Fig. 12.15

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt.$

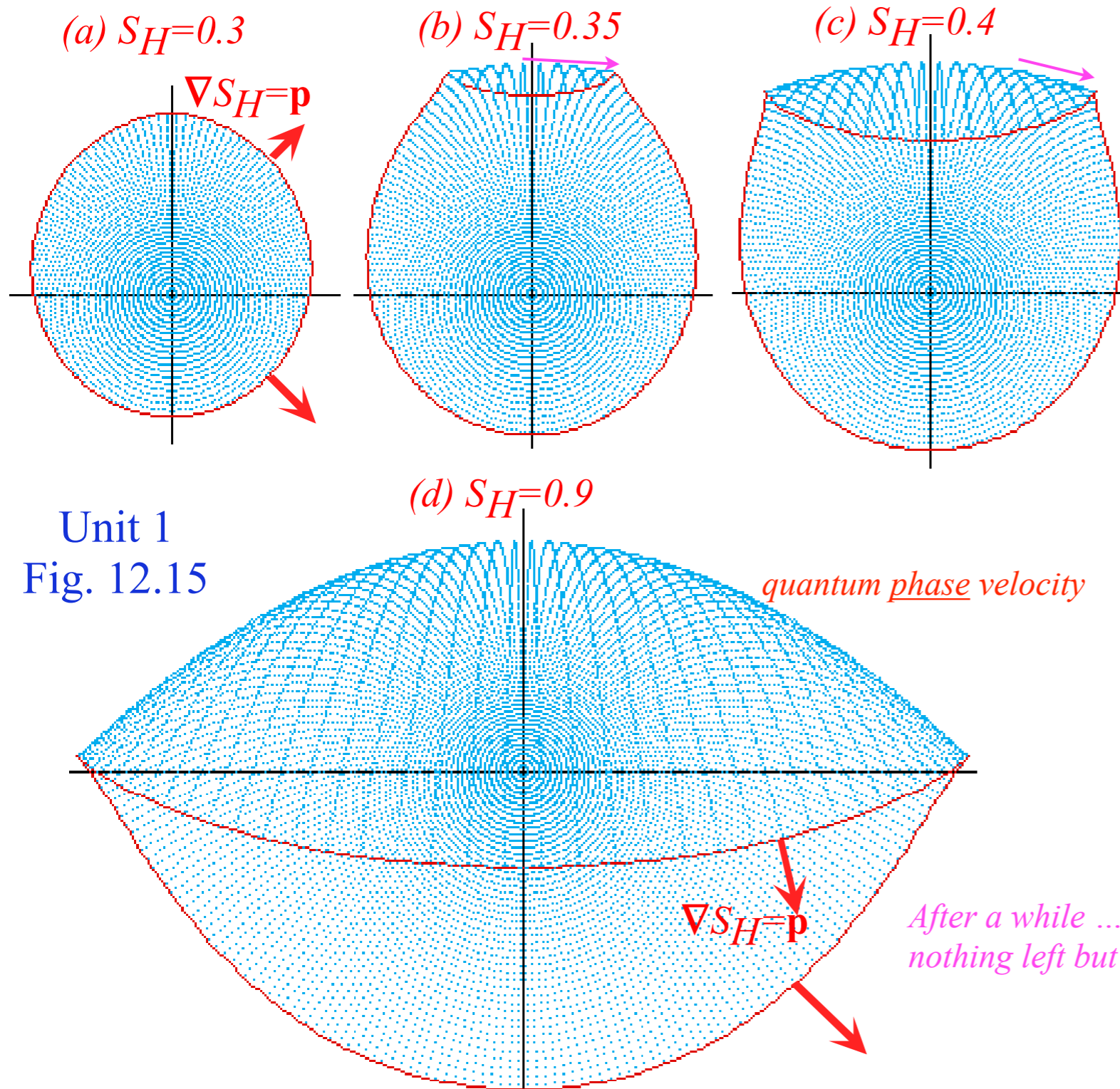
$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



wavefront
"cat ears"
scoot outward..

Unit 1
Fig. 12.15



16th Century carving on St. Wifred's in Grappenhall



...on St. Nicolas



After a while ...
nothing left but a smile!

From *Alice's Adventures in Wonderland* by Lewis Carrol (1865)

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt$.

$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

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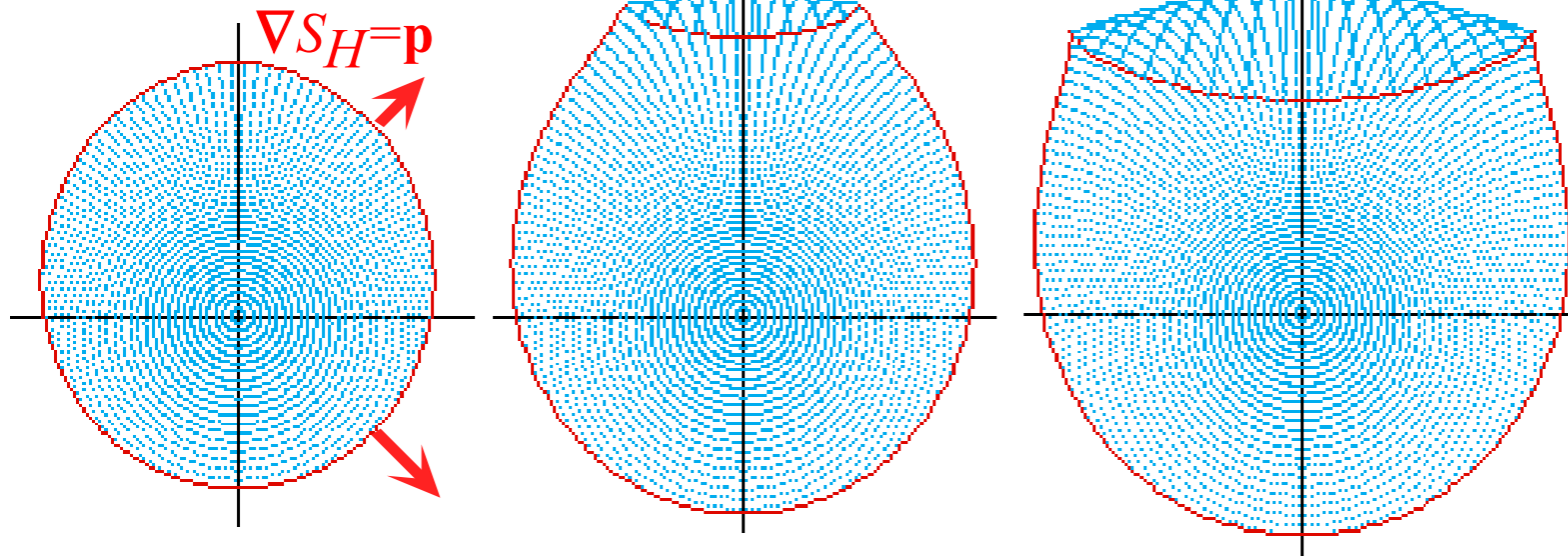
Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"

(a) $S_H=0.3$

(b) $S_H=0.35$

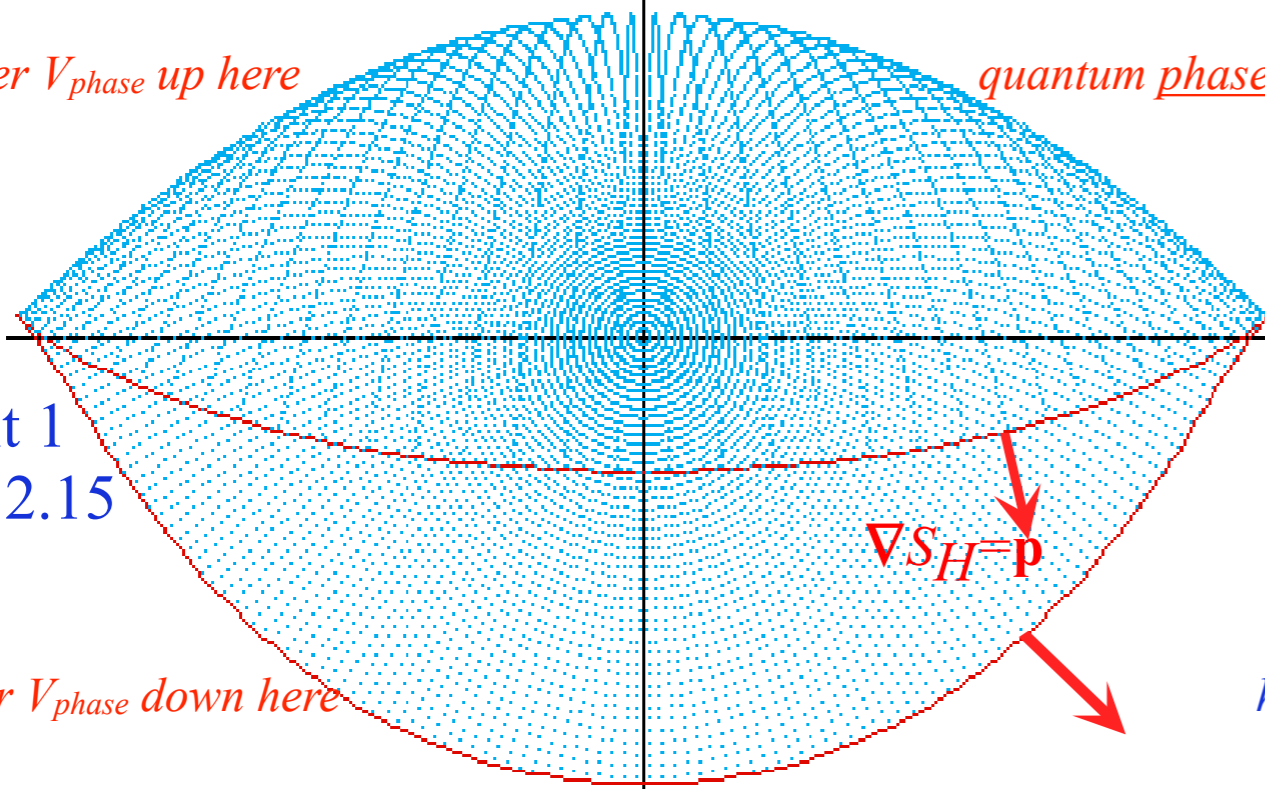
(c) $S_H=0.4$



(d) $S_H=0.9$

higher V_{phase} up here

quantum phase velocity

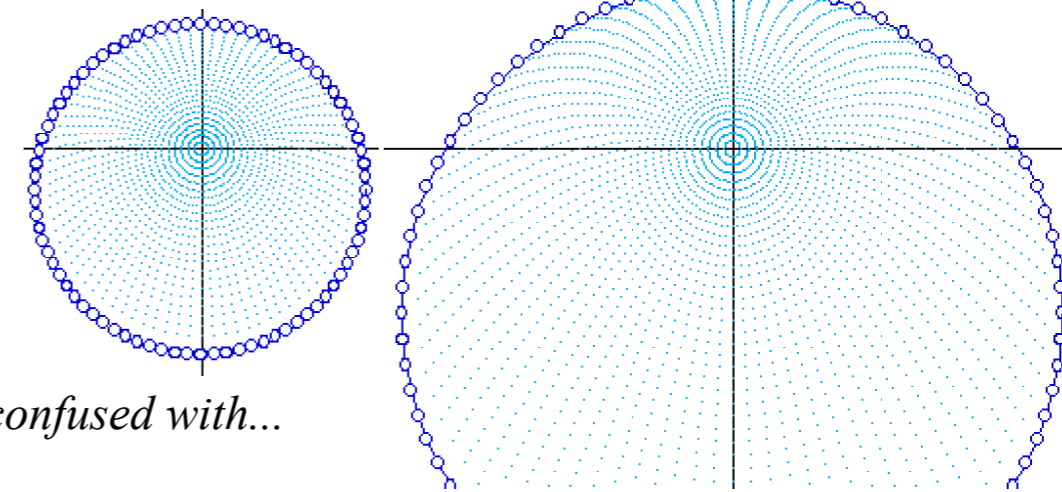


lower V_{phase} down here

Classical "blast wavefronts"

(a) $T=0.4$

(b) $T=1.0$



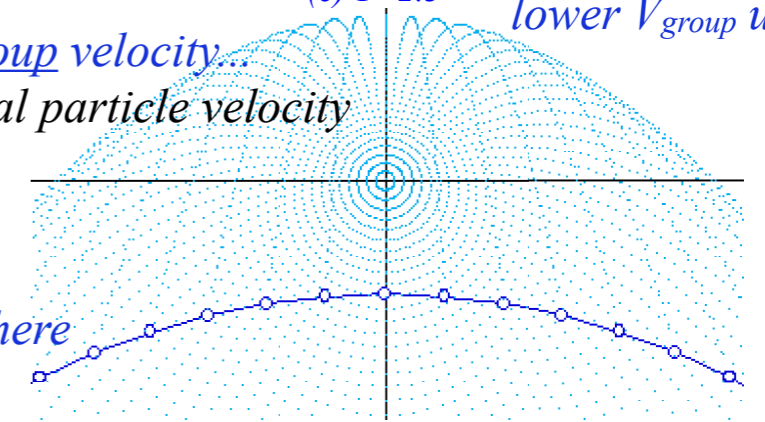
...not to be confused with...

...quantum group velocity...

that is classical particle velocity

(c) $T=2.3$

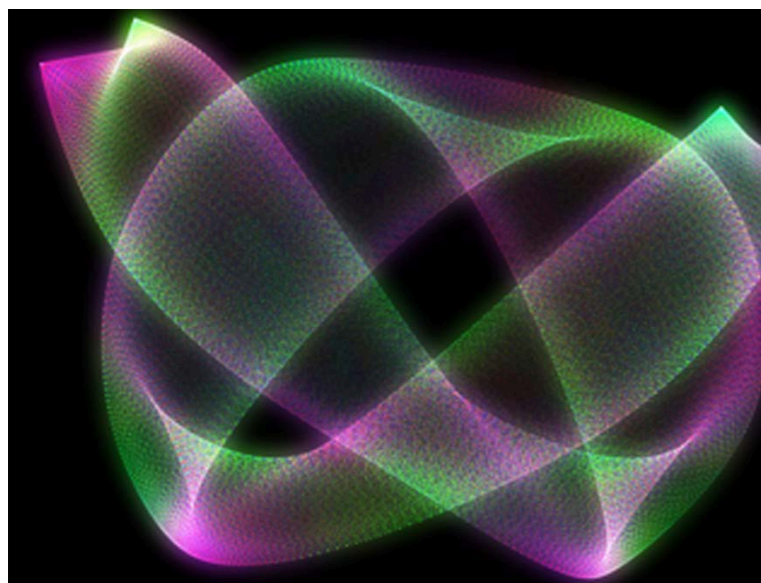
lower V_{group} up here



higher V_{group} down here

Check out the Heller Galleries

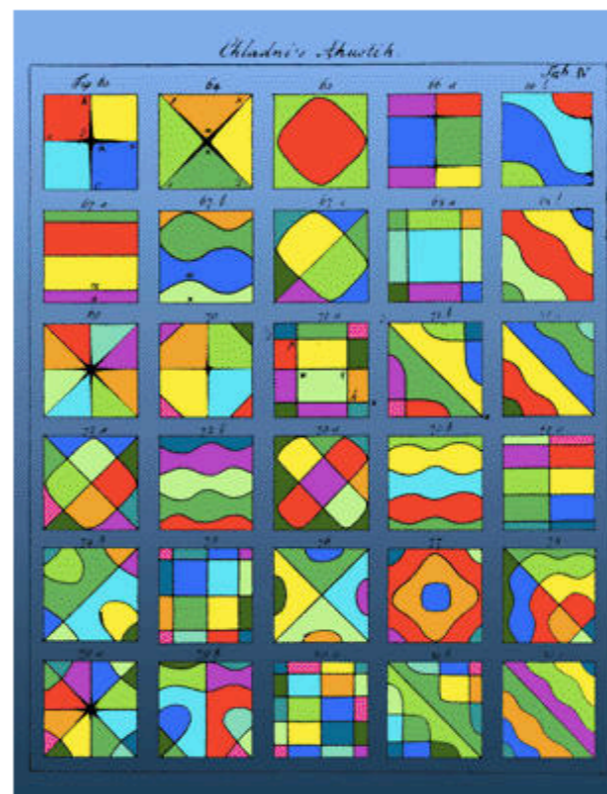
<http://jalbum.net/en/browse/user/album/1696720>



Resonance Fine Art

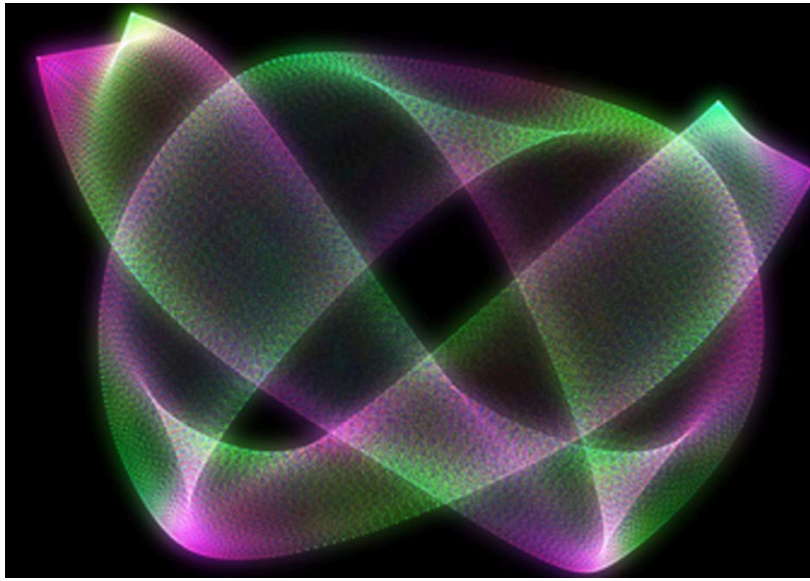
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[Chladni](#)



The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took the prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chladni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.

Check out the Heller Galleries



<http://jalbum.net/en/browse/user/album/1696720>

National Science Foundation (NSF)
Arlington, VA

September-November 2002

Selected images.

http://search.nsf.gov/search?ie=&site=nsf&output=xml_no_dtd&proxyreload=1&client=nsf&lr=&proxystylesheet=http%3A%2F%2Fwww.nsf.gov%2Fsearch%2Fnsf_new.xslt&oe=&btnG.x=0&btnG.y=0&q=eric+heller

University Museum, University of Arkansas, Fayetteville, AK

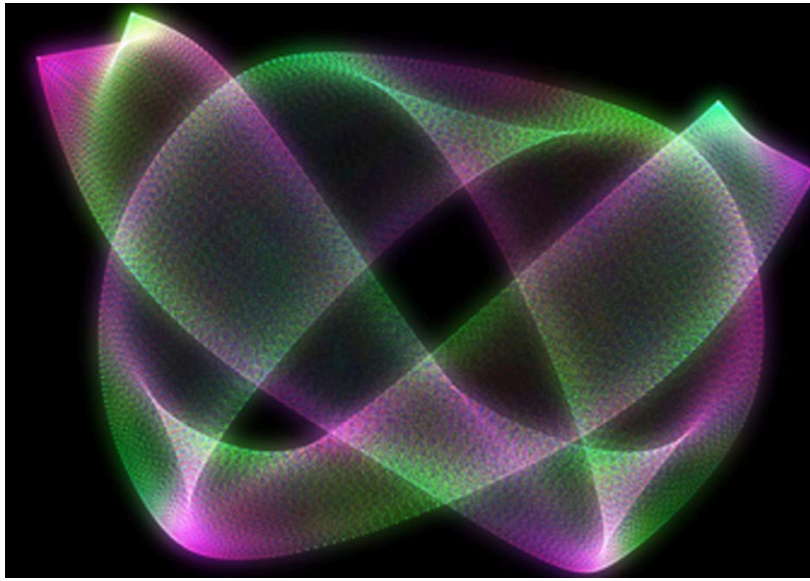
October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.



Check out the Heller Galleries



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University Museum, University of Arkansas, Fayetteville, AK**

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Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

**UAF Museum closed after this exhibit*



Lecture 11 ends here
Thu 9/26/2017