

Lecture 10

9.21.2017

Hamiltonian vs. Lagrange mechanics in Generalized Curvilinear Coordinates (GCC)

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lectures 8-9 procedures:

Lagrange prefers Covariant g_{mn} with Contravariant velocity \dot{q}^m

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m, q^n)$ using g^{mn} and covariant momentum p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

Hamilton's equations in Runge-Kutta (computer solution) form

Examples of Hamiltonian mechanics in effective potentials

Isotropic Harmonic Oscillator in polar coordinates and effective potential ([Web Simulation: OscillatorPE - IHO](#))

Coulomb orbits in polar coordinates and effective potential ([Web Simulation: OscillatorPE - Coulomb](#))

Examples of Hamiltonian mechanics in phase plots (Mostly for next Lecture 11)

1D Pendulum and phase plot ([Web Simulations: Pendulum, Cycloidulum](#).)

Quick Review of Lagrange Relations in Lectures 9-11

 *0th and 1st equations of Lagrange and Hamilton and their geometric relations*

Quick Review of Lagrange Relations in Lectures 8-9

0th and 1st equations of Lagrange and Hamilton

p. 25 of
Lecture 8

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and **Estrangian** have no explicit dependence on **momentum p**

$$\frac{\partial L}{\partial \mathbf{p}_k} \equiv 0 \equiv \frac{\partial E}{\partial \mathbf{p}_k}$$

Hamiltonian and **Estrangian** have no explicit dependence on **velocity v**

$$\frac{\partial H}{\partial \mathbf{v}_k} \equiv 0 \equiv \frac{\partial E}{\partial \mathbf{v}_k}$$

Lagrangian and **Hamiltonian** have no explicit dependence on **speedinum V**

$$\frac{\partial L}{\partial \mathbf{V}_k} \equiv 0 \equiv \frac{\partial H}{\partial \mathbf{V}_k}$$

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections

$$\begin{aligned} \nabla_{\mathbf{v}} L &= \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}}{2} \\ &= \mathbf{M} \cdot \mathbf{v} = \mathbf{p} \end{aligned}$$

$$\begin{aligned} \nabla_{\mathbf{p}} H &= \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p}}{2} \\ &= \mathbf{M}^{-1} \cdot \mathbf{p} = \mathbf{v} \end{aligned}$$

*Estrangian is neglected for now.
(It is related to dual ellipse geometry in Lecture 8 p. 71-79 and 99-101)*

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

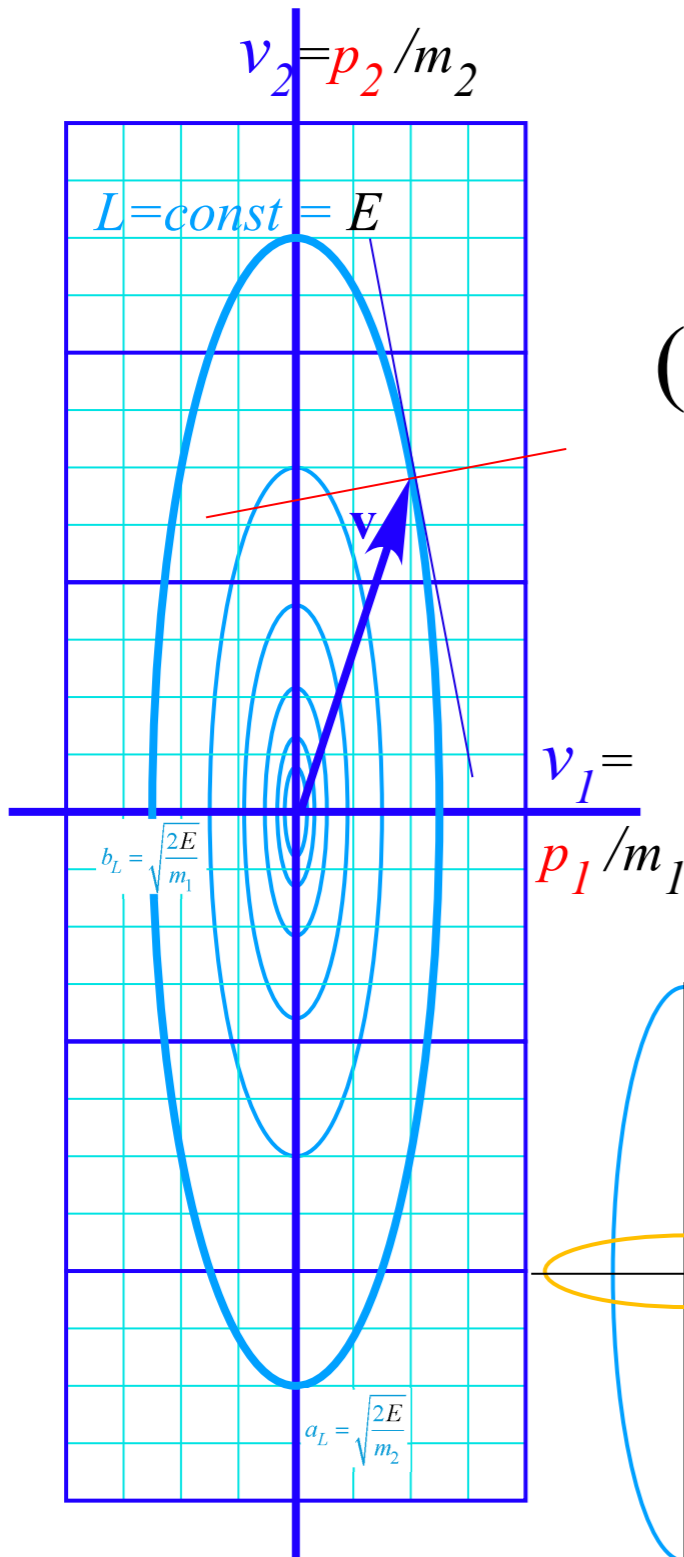
$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1st equation(s)

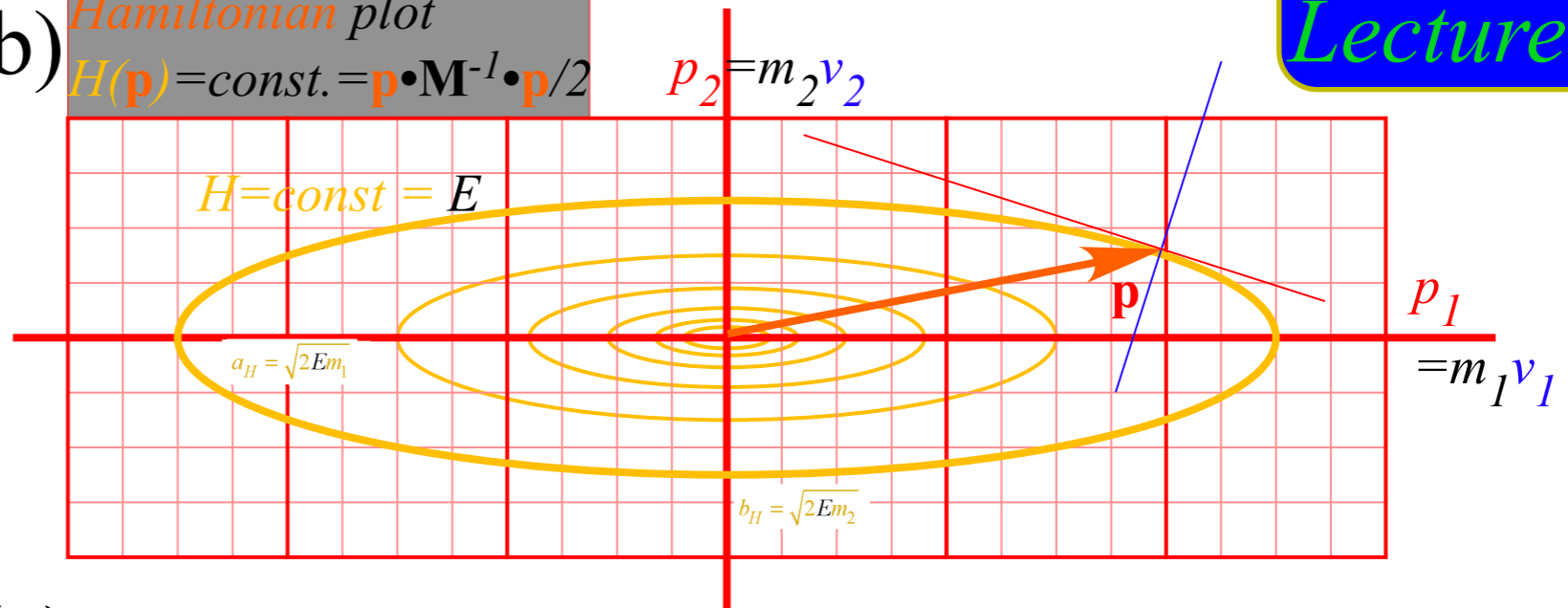
$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

† non-dependency due to stationary-value effects as shown on p. 28-31

(a) *Lagrangian plot*
 $L(\mathbf{v}) = \text{const.} = \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} / 2$



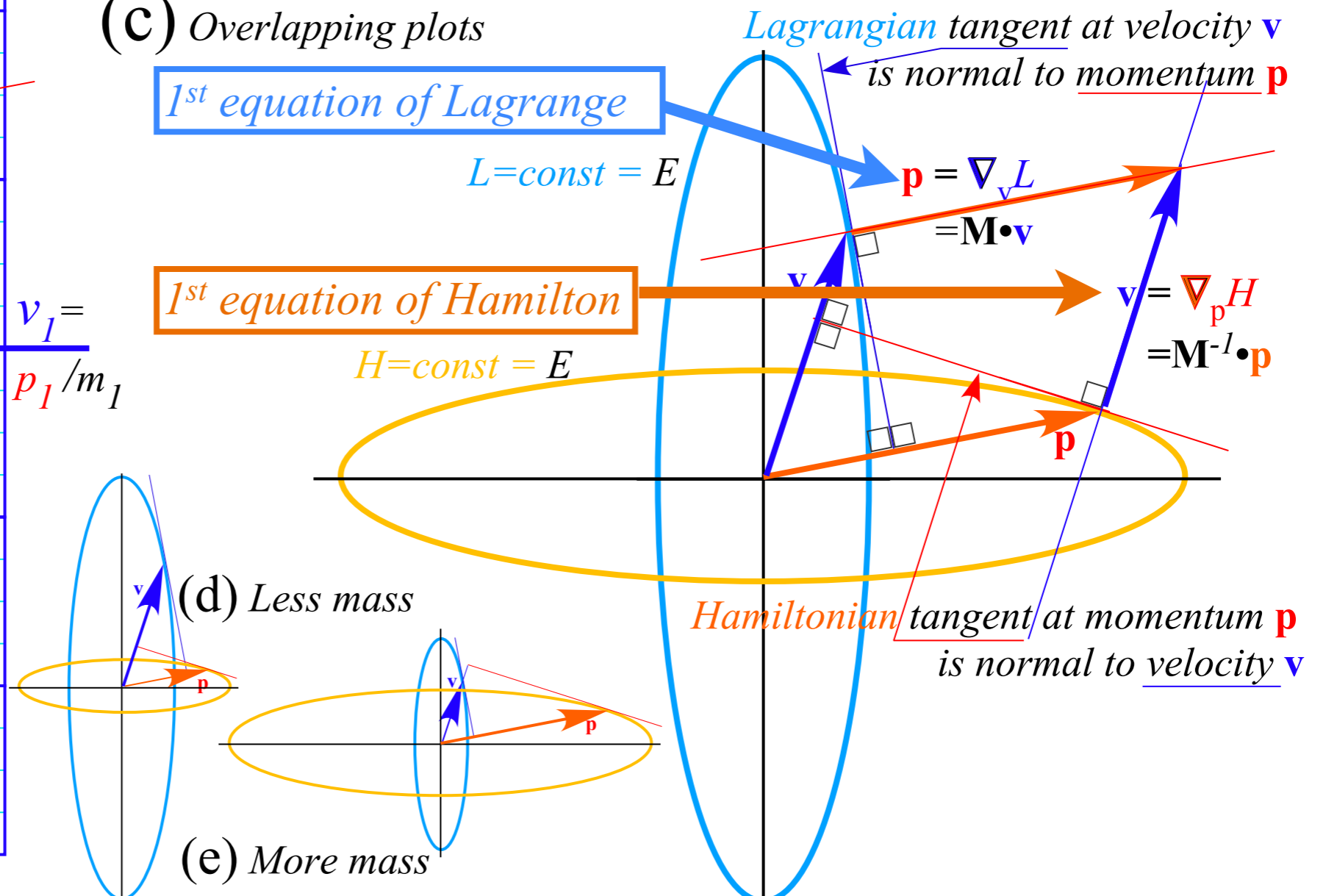
(b) *Hamiltonian plot*
 $H(\mathbf{p}) = \text{const.} = \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} / 2$



(c) *Overlapping plots*

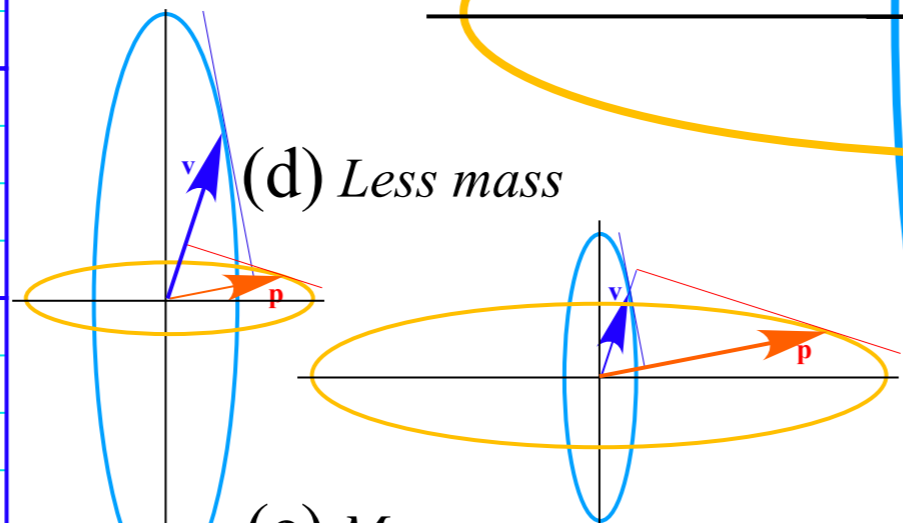
1st equation of Lagrange

1st equation of Hamilton



(d) *Less mass*

(e) *More mass*



Review of Lagrange Equations in Lecture 9

Lagrange prefers Covariant g_{mn} with Contravariant velocity \dot{q}^m

GCC Lagrangian definition

GCC “canonical” momentum p_m definition

 *GCC “canonical” force F_m definition*

Coriolis “fictitious” forces (... and weather effects)

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 9)**

$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (\mathbf{E}_m \dot{q}^m) \cdot (\mathbf{E}_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m :

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising;
radial momentum p_r has the
usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia
factor $M r^2$ automatically for the
angular momentum $p_\phi = M r^2 \omega$.

2nd L-equation involves total time derivative \dot{p}_m for each momentum p_m :

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal
force $M r \omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is **conserved** if
potential U has no explicit ϕ -dependence

Find \dot{p}_m directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (\dot{g}_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$ Equate it to \dot{p}_m in 2nd L-equation:

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M \ddot{r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$= M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2 M r \dot{r} \dot{\phi} + M r^2 \ddot{\phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is **conserved** if
potential U has no explicit ϕ -dependence

(Review of Lecture 9)

Rewriting GCC Lagrange equations :

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M \ddot{r}$$

Centrifugal (center-fleeing) force equals total

$$= M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centripetal (center-pulling) force

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Angular momentum p_ϕ is conserved if potential U has no explicit ϕ -dependence

Conventional forms

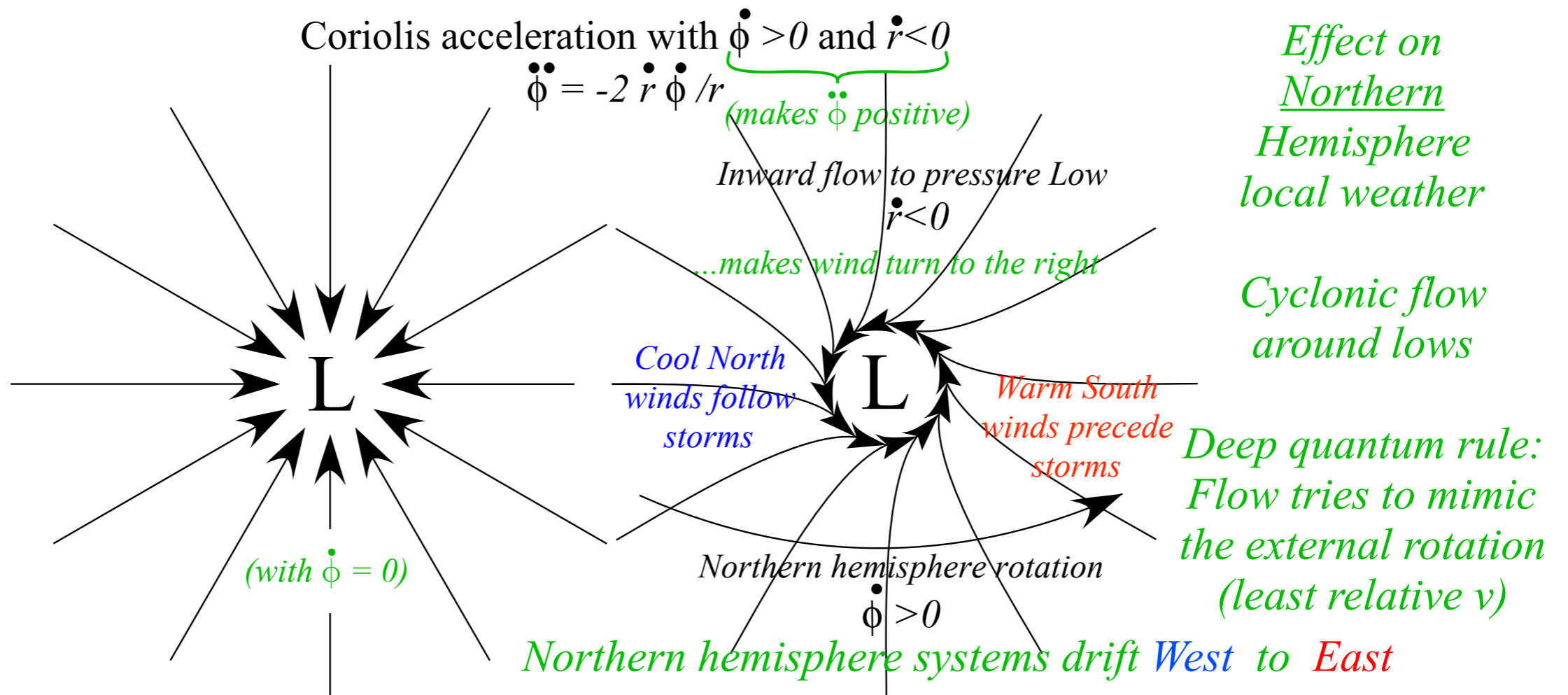
radial force: $M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free ($U=0$)

radial acceleration: $\ddot{r} = r \dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2 \frac{\dot{r}\dot{\phi}}{r}$



Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

→ Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m, q^n)$ using g^{mn} and covariant momentum p_m

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian $L=T-U$

that is explicit function of coordinates and **velocity** \dot{q} ...

$$\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}$$

Deriving Hamilton's equations from Lagrangian theory

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...of coordinates and **velocity** and **time**, too. (You can safely drop last chain-rule factor [$1=dt/dt$])

$$\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t} \frac{dt}{dt}$$

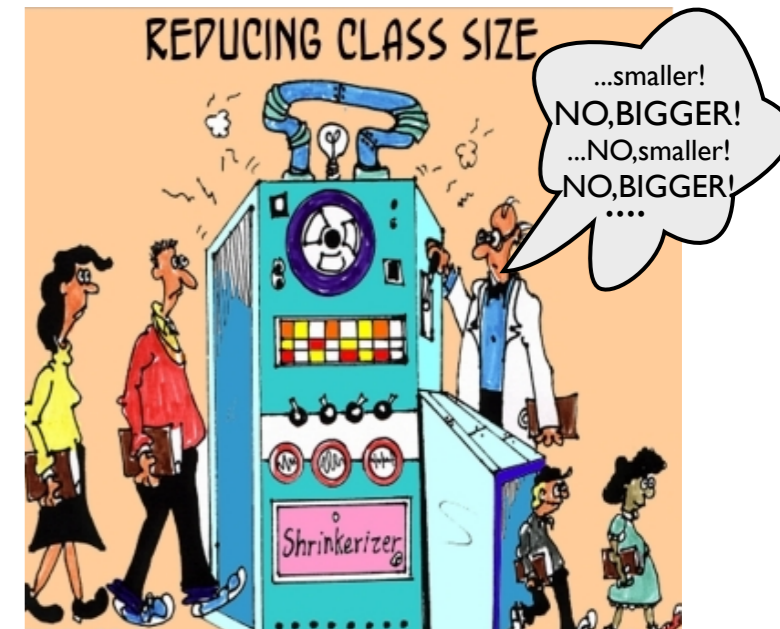
Deriving Hamilton's equations from Lagrangian theory

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...of coordinates and **velocity** and **time**, too. (Imagine Mad Scientist turning $U(t)$ -dial.)

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Cartoonish way to imagine
explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

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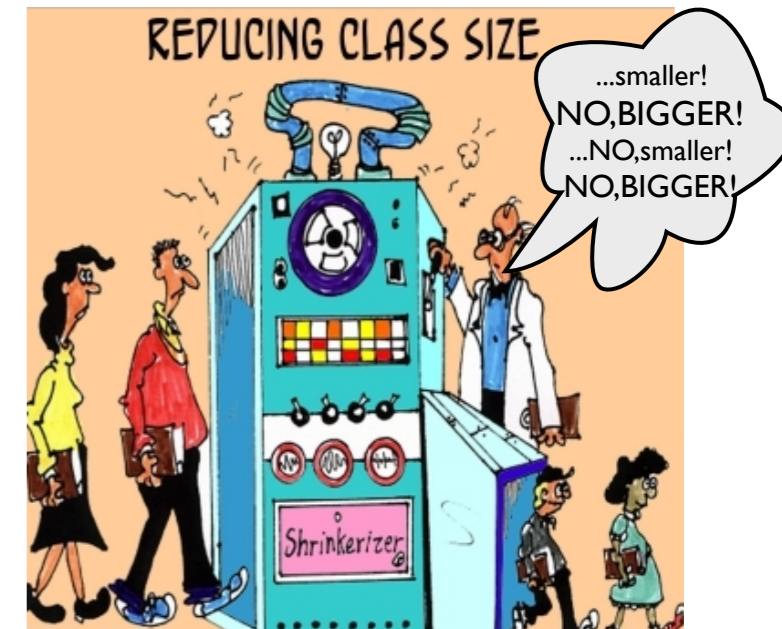
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Recall Lagrange equations:

$$\dot{p}_m = \frac{\partial L}{\partial q^m} \quad p_m = \frac{\partial L}{\partial \dot{q}^m}$$

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Cartoonish way to imagine explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

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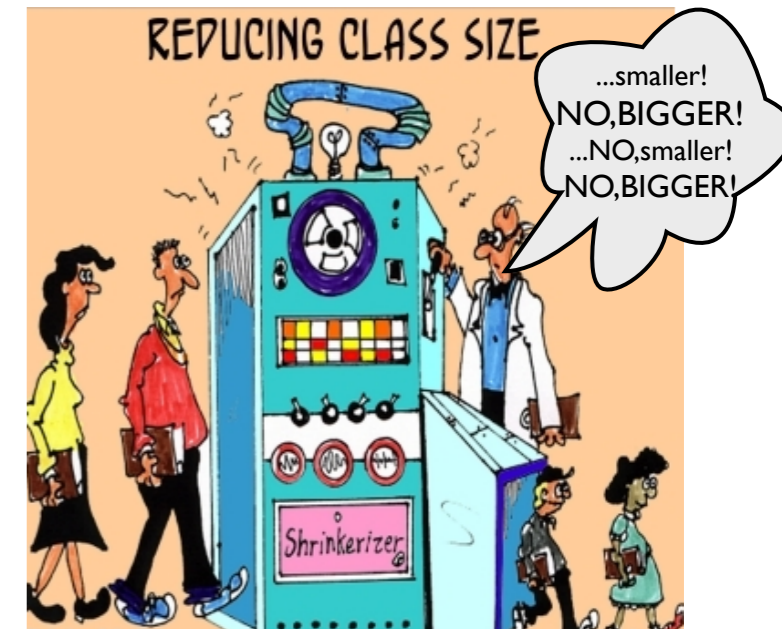
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Use product rule:

$$\dot{u} \frac{dv}{dt} + u \frac{d\dot{v}}{dt} = \frac{d}{dt} (u\dot{v})$$



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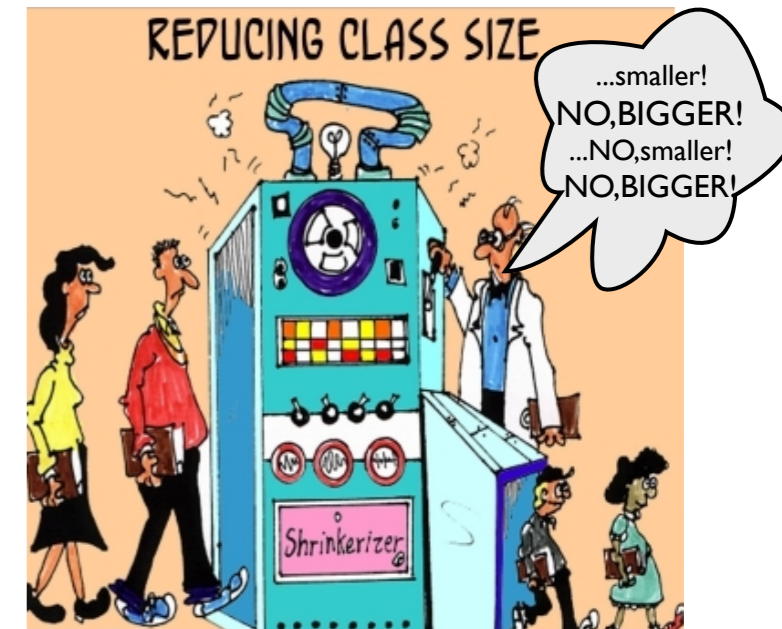
Use product rule:

$$\dot{u} \frac{dv}{dt} + u \frac{dv}{dt} = \frac{d}{dt}(uv)$$

$$= \frac{dL}{dt} = \frac{d}{dt} \left(p_m \dot{q}^m \right) + \frac{\partial L}{\partial t}$$

and switch the dL/dt and $\partial L/\partial t$ to define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})$

$$\frac{d}{dt} \left(p_m \dot{q}^m - L \right) = - \frac{\partial L}{\partial t} \quad \text{where: } H \equiv p_m \dot{q}^m - L$$



Cartoonish way to imagine explicit time dependence

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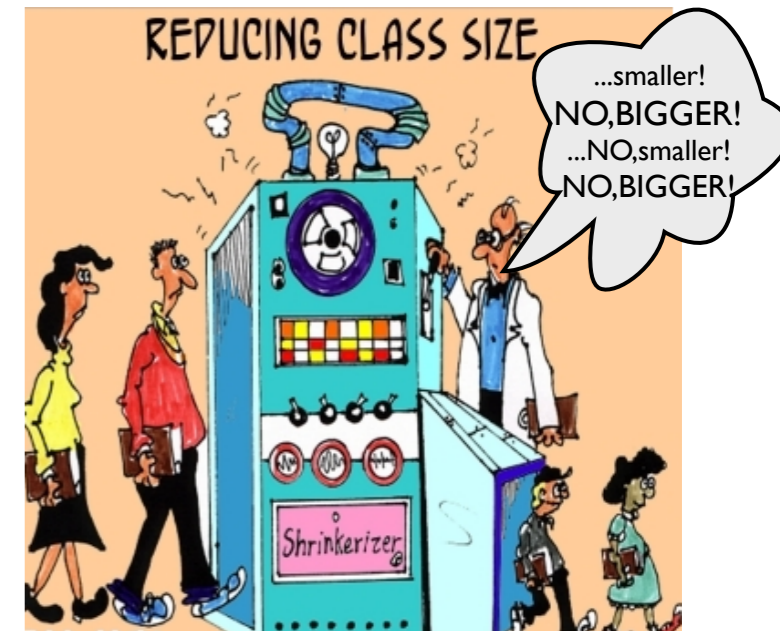
Use product rule:

$$\dot{u} \frac{dv}{dt} + u \frac{d\dot{v}}{dt} = \frac{d}{dt}(u\dot{v})$$

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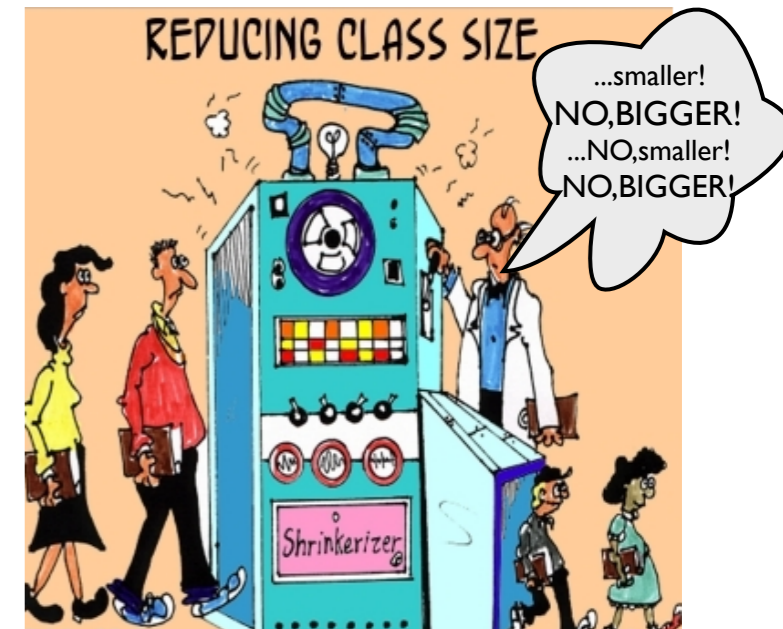
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$$= \frac{dL}{dt} = \frac{d}{dt} \left(p_m \dot{q}^m \right) + \frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})$

(That's the old Legendre transform)

$$\frac{d}{dt} \left(p_m \dot{q}^m - L \right) = - \frac{\partial L}{\partial t} \equiv \frac{dH}{dt} \quad \text{where: } H \equiv p_m \dot{q}^m - L$$



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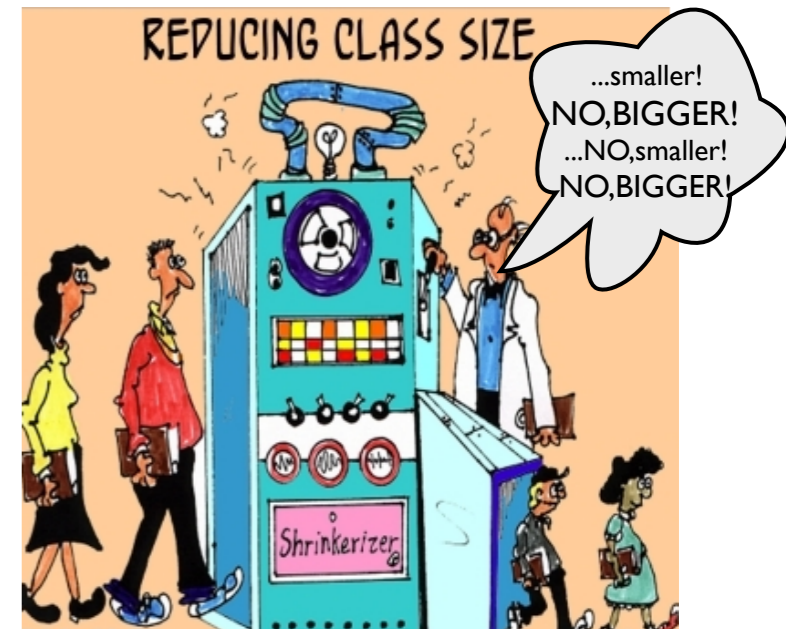
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where: $H \equiv p_m \dot{q}^m - L$

(That's the old Legendre transform)

(Recall: $\frac{\partial L}{\partial p_m} \equiv 0$ and: $\frac{\partial H}{\partial \dot{q}^m} \equiv 0$)



Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian $L=T-U$
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...of coordinates and **velocity** and time, too. (Imagine Mad Scientist turning U-dial.)

$$\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

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$$= \frac{dL}{dt} = \frac{d}{dt} \left(p_m \dot{q}^m \right) + \frac{\partial L}{\partial t}$$

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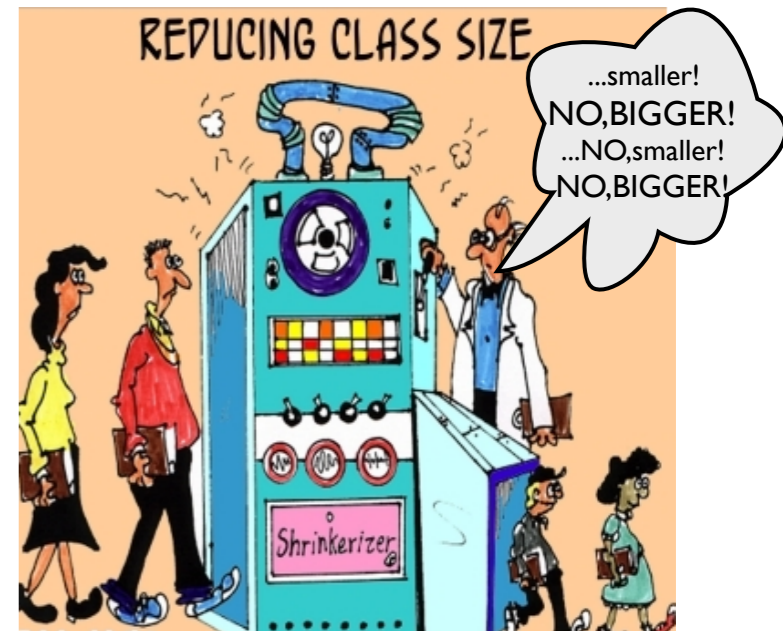
$$\frac{d}{dt} \left(p_m \dot{q}^m - L \right) = - \frac{\partial L}{\partial t} = \frac{dH}{dt}$$

where: $H = p_m \dot{q}^m - L$

(That's the old Legendre transform)

(Recall: $\frac{\partial L}{\partial p_m} \equiv 0$ and: $\frac{\partial H}{\partial \dot{q}^m} \equiv 0$)

so: $\frac{\partial H}{\partial p_m} = \frac{\partial p_m \dot{q}^m}{\partial p_m} - 0$



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Recall Lagrange equations:

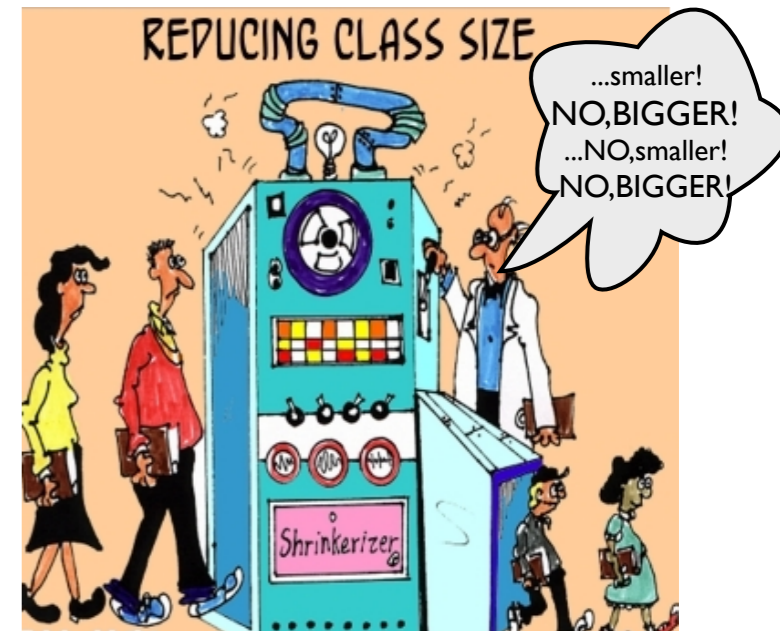
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Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m} = \dot{q}^m$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian $L=T-U$
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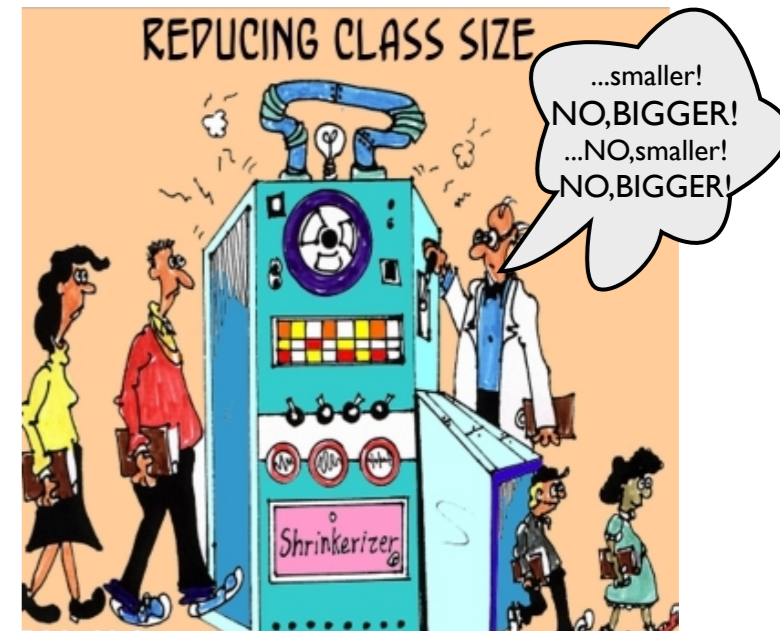
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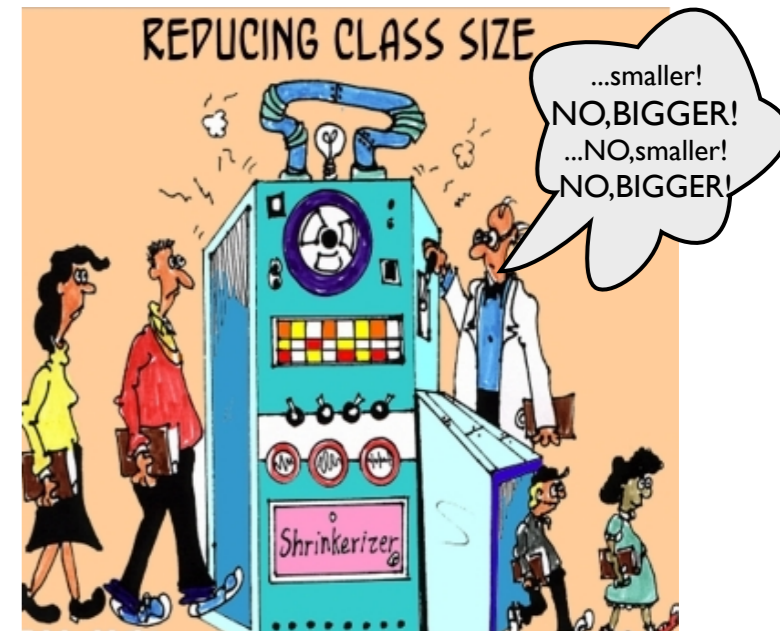
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Deriving Hamilton's equations from Lagrangian theory

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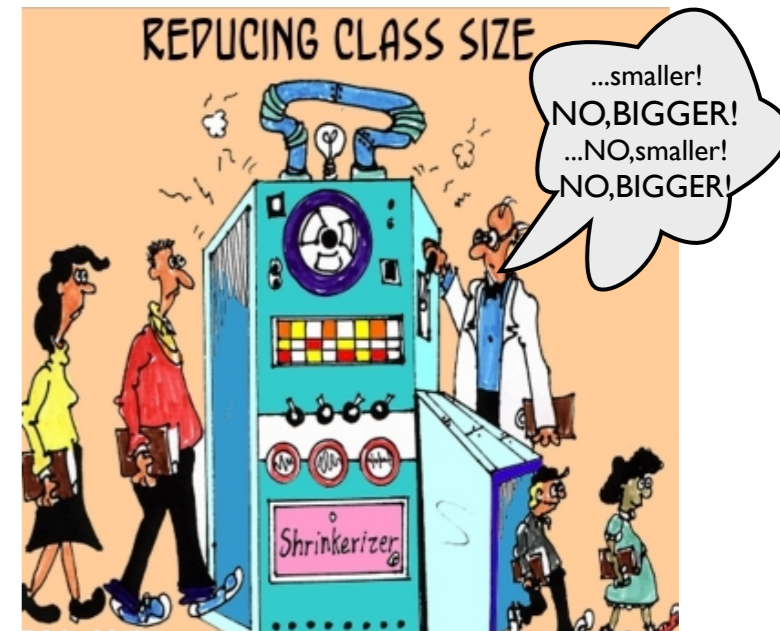
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Hamilton's 2nd GCC equation

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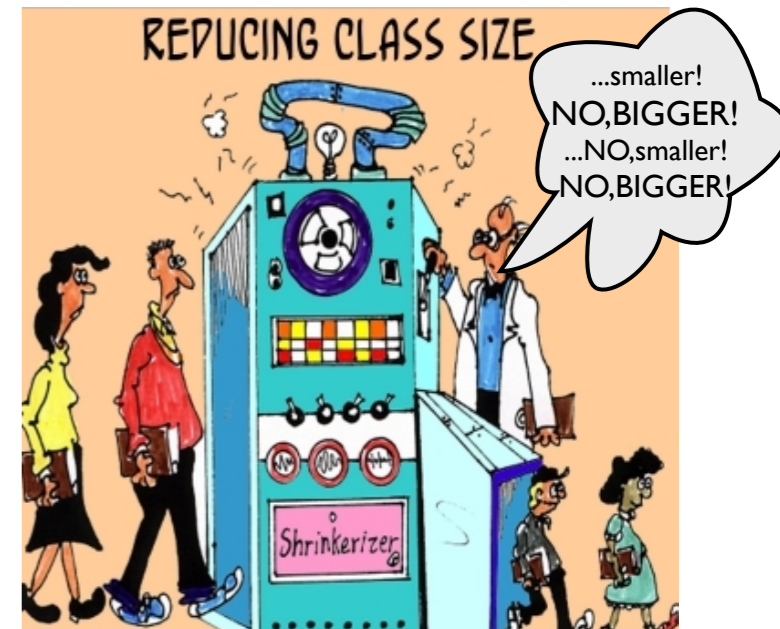
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a most peculiar relation involving partial vs total

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Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Deriving Hamilton's equations from Lagrange's equations

 *Expressing Hamiltonian $H(p_m, q^n)$ using g^{mn} and covariant momentum p_m*

Polar-coordinate example of Hamilton's equations compared to Lagrange's

Hamilton's equations in Runge-Kutta (computer solution) form

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Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m \dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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This gives an “illegal dependence” for the Hamiltonian (It musn't be “explicit” in velocity \dot{q}^m .)

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U \quad \left(\begin{array}{c} \text{Numerically} \\ \text{correct ONLY!} \end{array} \right)$$

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details on next pages

(Formally **and** Numerically)
correct

Details of metric tensor algebra:

Given: $H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$ Let: $\dot{q}^m = \frac{1}{M} g^{mn'} p_{n'}$

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Metric tensor symmetry:

$$g_{mn} = g_{nm}$$

$$g^{mn'} = g^{n'm}$$

(Always applies)

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(Formally **and** Numerically)
correct

Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi)$$

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Polar coordinate Lagrangian was given as: See covariant polar metric $g_{\mu\nu}$ on next page (p35)

$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

Covariant polar metric $g_{\mu\nu}$

[from p53 of Lecture 9]

Contravariant polar metric $g^{\mu\nu}$

Covariant g_{mn}

vs.

Invariant δ_m^n

vs.

Contravariant g^{mn}

$$\mathbf{E}_m \cdot \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \cdot \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \cdot \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \cdot \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor

g_{mn}

Invariant
Kronecker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor

g^{mn}

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\ \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix} \quad \langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix}$$

$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \quad \uparrow \mathbf{E}_r \quad \uparrow \mathbf{E}_\phi \quad \leftarrow \mathbf{E}^r = \mathbf{E}^1 \quad \leftarrow \mathbf{E}^\phi = \mathbf{E}^2$

Covariant g_{mn}

Invariant δ_m^n

Contravariant g^{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}^r & \mathbf{E}_r \cdot \mathbf{E}^\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}^r & \mathbf{E}_\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \cdot \mathbf{E}^r & \mathbf{E}^r \cdot \mathbf{E}^\phi \\ \mathbf{E}^\phi \cdot \mathbf{E}^r & \mathbf{E}^\phi \cdot \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m \dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:

$$H = p_m \dot{q}^m - L = \left(M g_{mn} \dot{q}^n \right) \dot{q}^m - \left(\frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U \right)$$

$$= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$$

This gives an “illegal dependence” for the Hamiltonian (It musn't be “explicit” in velocity \dot{q}^m .)

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U \quad \left(\begin{array}{l} \text{Numerically} \\ \text{correct ONLY!} \end{array} \right)$$

An inverse metric relation $\dot{q}^m = \frac{1}{M} g^{mn} p_n$ gives correct form that depends on momentum p_m .

$$H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E$$

(Formally **and** Numerically)
correct

Polar coordinate Lagrangian was given as: See covariant polar metric $g_{\mu\nu}$ on p39

$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

Polar coordinate Hamiltonian is given here:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi)$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m \dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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$$= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$$

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(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as: See covariant polar metric $g_{\mu\nu}$ on p39

$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

Polar coordinate Hamiltonian is given here: Contravariant polar metric $g^{\mu\nu}$ on p35

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m, q^n)$ using g^{mn} and covariant momentum p_m

 *Polar-coordinate example of Hamilton's equations compared to Lagrange's
Hamilton's equations in Runge-Kutta (computer solution) form*

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ *in 2D-polar coordinates satisfies:*

$$\text{Hamilton's 1st equations: } \frac{\partial H}{\partial p_m} = \dot{q}^m \quad \parallel \quad \text{Hamilton's 2nd equations: } \frac{\partial H}{\partial q^m} = -\dot{p}_m$$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ in 2D-polar coordinates satisfies:

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m} = \dot{q}^m$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{M}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

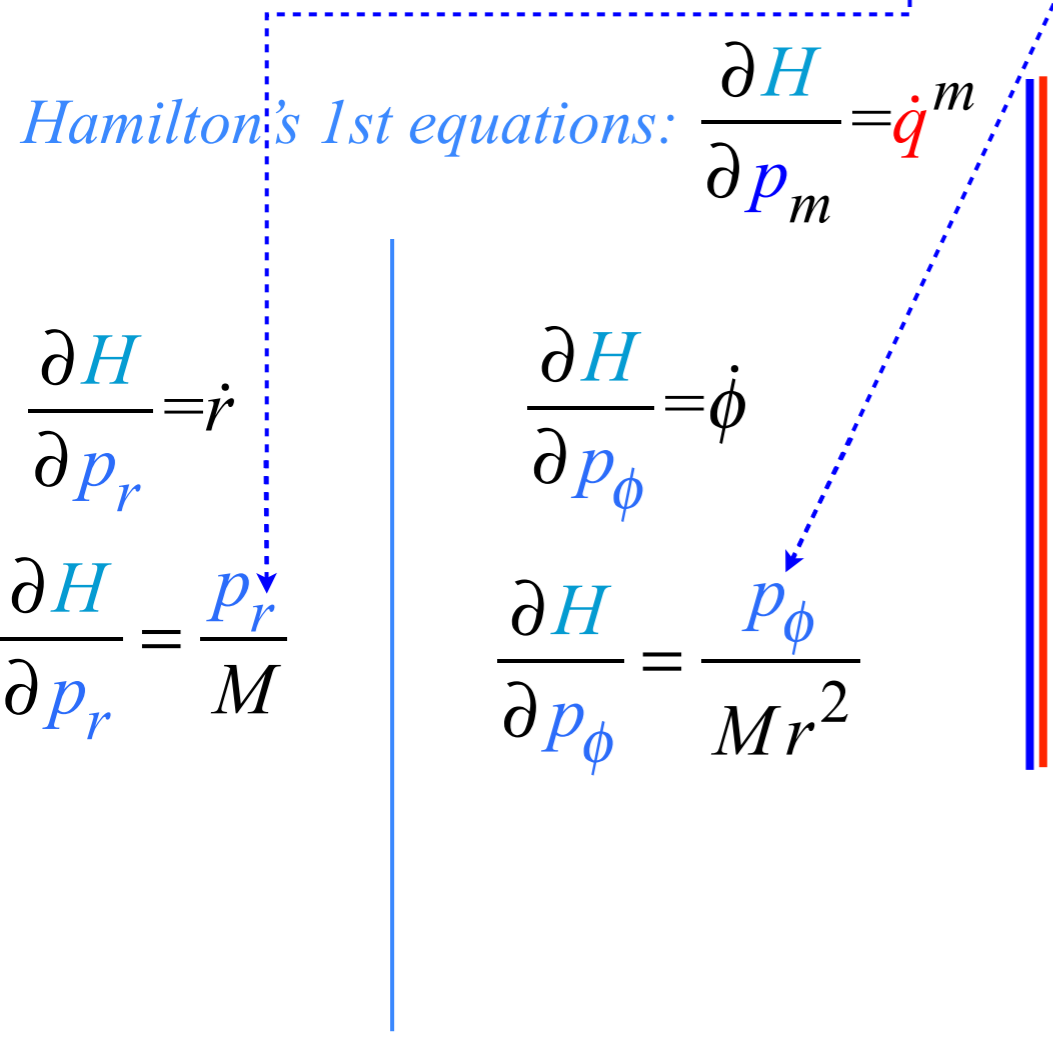
$$\frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{M}$$

$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{Mr^2}$$



Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

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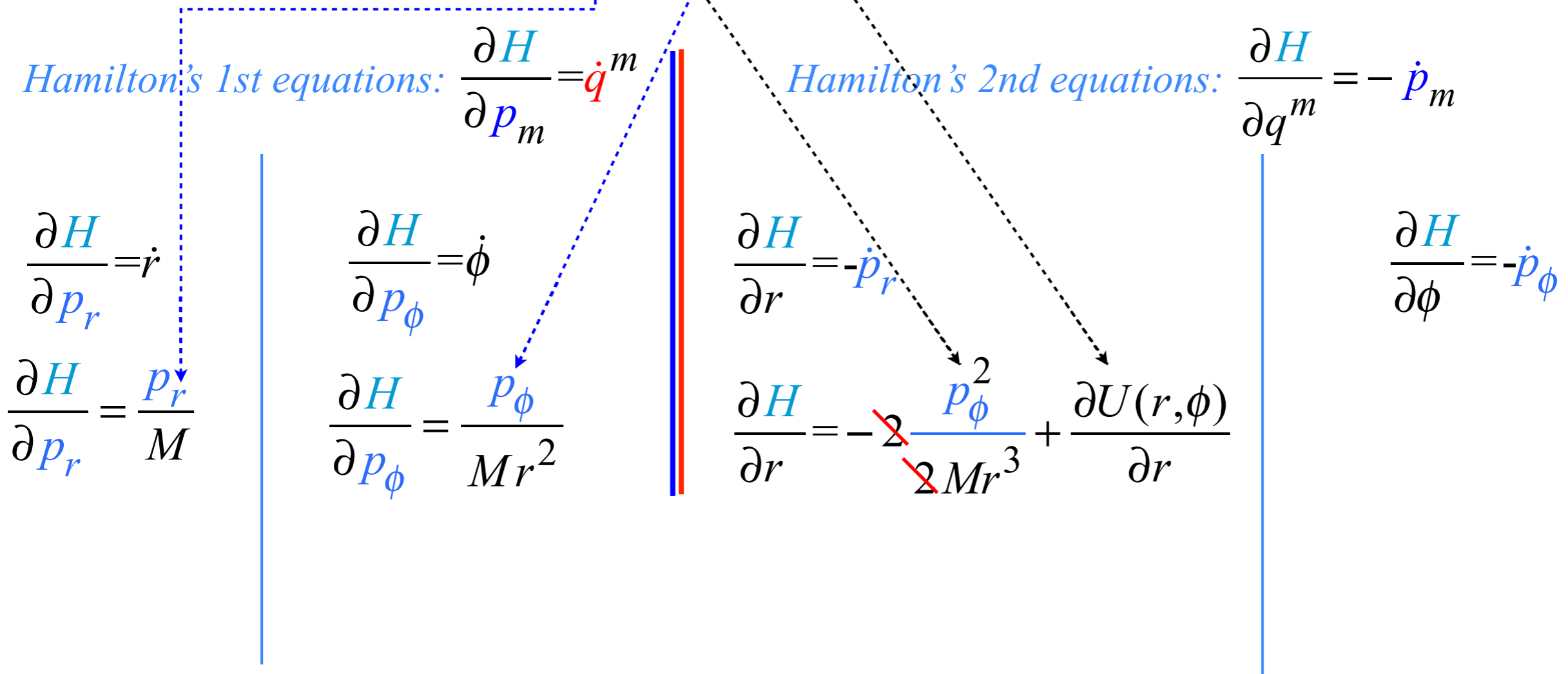
$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{M}$$

$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{Mr^2}$$

$$\frac{\partial H}{\partial r} = -2 \frac{p_\phi^2}{2Mr^3} + \frac{\partial U(r, \phi)}{\partial r}$$



Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ in 2D-polar coordinates satisfies:

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Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

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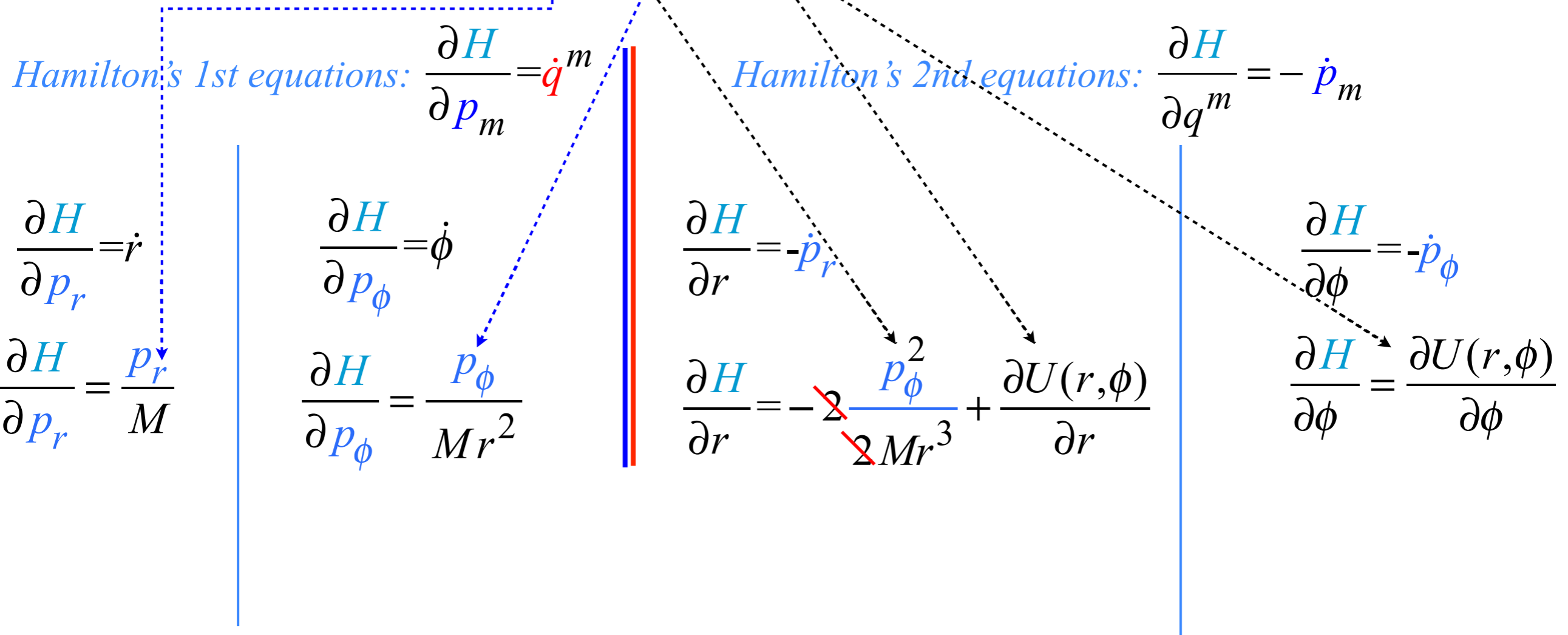
$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

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$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{Mr^2}$$

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$$\frac{\partial H}{\partial \phi} = \frac{\partial U(r, \phi)}{\partial \phi}$$



Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ in 2D-polar coordinates satisfies:

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$$p_r = M\dot{r}$$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{Mr^2}$$

$$p_\phi = Mr^2\dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial r} = -2 \frac{p_\phi^2}{2Mr^3} + \frac{\partial U(r, \phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

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Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ in 2D-polar coordinates satisfies:

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$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial r} = -2 \frac{p_\phi^2}{2Mr^3} + \frac{\partial U(r, \phi)}{\partial r}$$

$$\dot{p}_r = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

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Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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$$p_\phi = Mr^2\dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial r} = -2 \frac{p_\phi^2}{2Mr^3} + \frac{\partial U(r, \phi)}{\partial r}$$

$$\dot{p}_r = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi} = \frac{\partial U(r, \phi)}{\partial \phi}$$

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Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ in 2D-polar coordinates satisfies:

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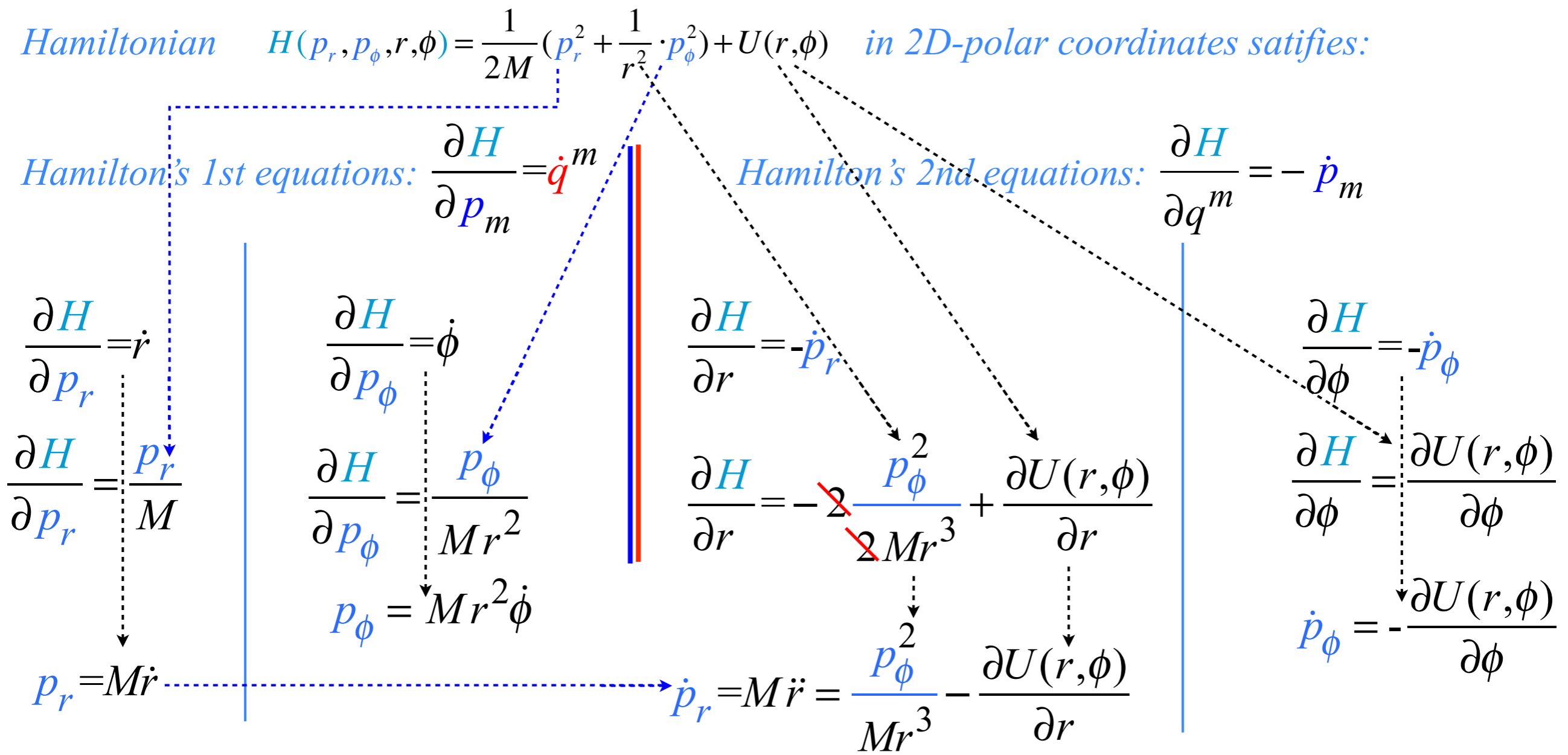
$$\frac{\partial H}{\partial r} = -2 \frac{p_\phi^2}{2Mr^3} + \frac{\partial U(r, \phi)}{\partial r}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi} = \frac{\partial U(r, \phi)}{\partial \phi}$$

$$\dot{p}_\phi = -\frac{\partial U(r, \phi)}{\partial \phi}$$



Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

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$$p_\phi = Mr^2\dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

$$\frac{\partial H}{\partial r} = -2 \frac{p_\phi^2}{2Mr^3} + \frac{\partial U(r, \phi)}{\partial r}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r}$$

$$\Rightarrow Mr\dot{\phi}^2 - \partial_r U(r, \phi)$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi} = \frac{\partial U(r, \phi)}{\partial \phi}$$

$$\dot{p}_\phi = -\frac{\partial U(r, \phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} p_\phi^2) + U(r, \phi)$ in 2D-polar coordinates satisfies:

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$$\frac{\partial H}{\partial \phi} = \frac{\partial U(r, \phi)}{\partial \phi}$$

$$p_r = M\dot{r}$$

$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r}$$

$$\dot{p}_\phi = -\frac{\partial U(r, \phi)}{\partial \phi}$$

$$\Rightarrow Mr\dot{\phi}^2 - \partial_r U(r, \phi)$$

$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r, \phi)$$

Compare these Hamilton's equations to Lagrange's on next page...

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 9)**

$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (\mathbf{E}_m \dot{q}^m) \cdot (\mathbf{E}_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m :

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising;
radial momentum p_r has the
usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia
factor $M r^2$ automatically for the
angular momentum $p_\phi = M r^2 \omega$.

2nd L-equation involves total time derivative \dot{p}_m for each momentum p_m :

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal
force $M r \omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is **conserved** if
potential U has no explicit ϕ -dependence

Find \dot{p}_m directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (\dot{g}_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$ Equate it to \dot{p}_m in 2nd L-equation:

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M \ddot{r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$= M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2 M r \dot{r} \dot{\phi} + M r^2 \ddot{\phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is **conserved** if
potential U has no explicit ϕ -dependence

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m, q^n)$ using g^{mn} and covariant momentum p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

 *Hamilton's equations in Runge-Kutta (computer solution) form*

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = Mr\dot{r}$$

$$\begin{aligned}\dot{p}_r = M\ddot{r} &= \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r} \\ &= Mr\dot{\phi}^2 - \partial_r U(r, \phi)\end{aligned}$$

$$p_\phi = Mr^2\dot{\phi}$$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r, \phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \dots)$$

$$\dot{x}_2 = \dot{x}_2(x_1, x_2, x_3, \dots)$$

$$\dot{x}_3 = \dot{x}_3(x_1, x_2, x_3, \dots)$$

⋮

Polar coordinate example: Hamilton's equations in Runge-Kutta form

$$p_r = M\dot{r}$$
$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r, \phi)}{\partial r}$$
$$= Mr\dot{\phi}^2 - \partial_r U(r, \phi)$$

$$p_\phi = Mr^2\dot{\phi}$$
$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r, \phi)$$

Hamiltonian eqs. in
Runge-Kutta form:

$$\dot{r} = \dot{r}(r, p_r, \phi, p_\phi) = \frac{p_r}{M}$$
$$\dot{p}_r = \dot{p}_r(r, p_r, \phi, p_\phi) = \frac{p_\phi^2}{Mr^3} - \partial_r U(r, \phi)$$
$$\dot{\phi} = \dot{\phi}(r, p_r, \phi, p_\phi) = \frac{p_\phi}{Mr^2}$$
$$\dot{p}_\phi = \dot{p}_\phi(r, p_r, \phi, p_\phi) = -\partial_\phi U(r, \phi)$$

Runge-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \dots)$$
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$$\vdots$$

Examples of Hamiltonian mechanics in effective potentials

→ *Isotropic Harmonic Oscillator in polar coordinates and effective potential* ([Web Simulation: OscillatorPE - IHO](#))
Coulomb orbits in polar coordinates and effective potential ([Web Simulation: OscillatorPE - Coulomb](#))

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m -conservation

Consider polar coordinate Hamiltonian for *I*_{sotropic} *H*_{armonic} *O*_{scillator} potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + k \cdot r^2 / 2 = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + \frac{k \cdot r^2}{2} = E = \text{const.}$$

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H is not explicit function of ϕ , and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = \text{const.}$

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Same applies to any radial potential $U(r)$

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$$E = \frac{p_r^2}{2M} + \underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} + \underbrace{U(r)}_{\text{"real" PE}}$$

"effective" PE

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}$$

Time vs r : $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}}$

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Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2} \cdot r^2$

Called the "quadrature" or 1/4-cycle solution if $r_{<} = 0$ and $r_{>} = \text{max amplitude}$

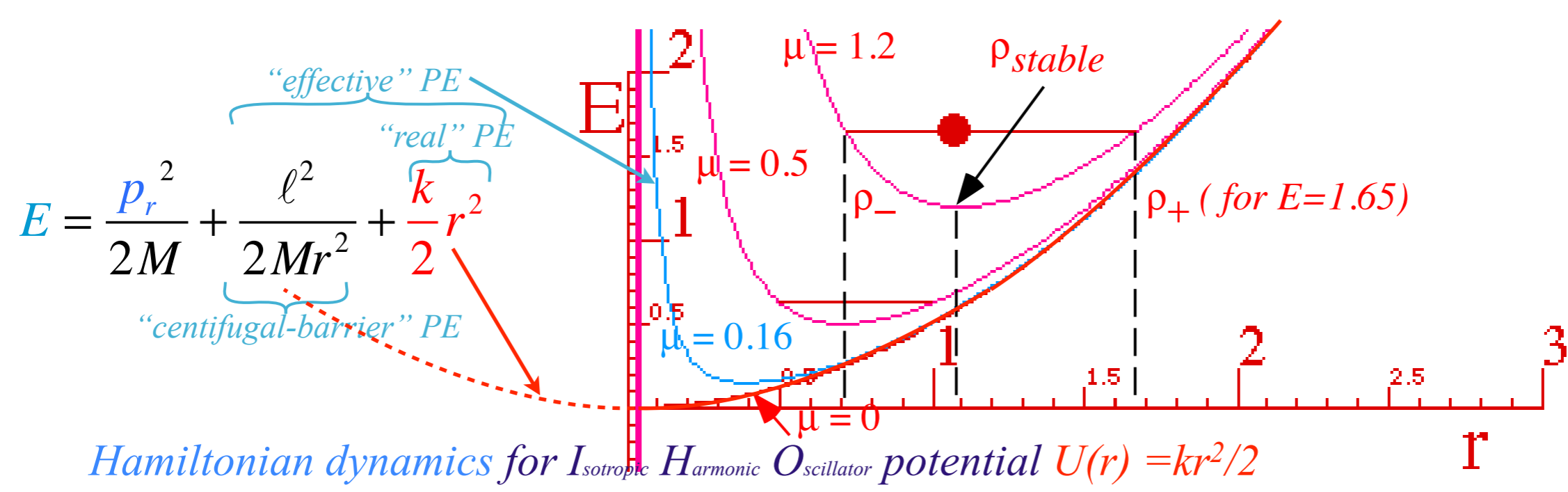
Radial velocity:

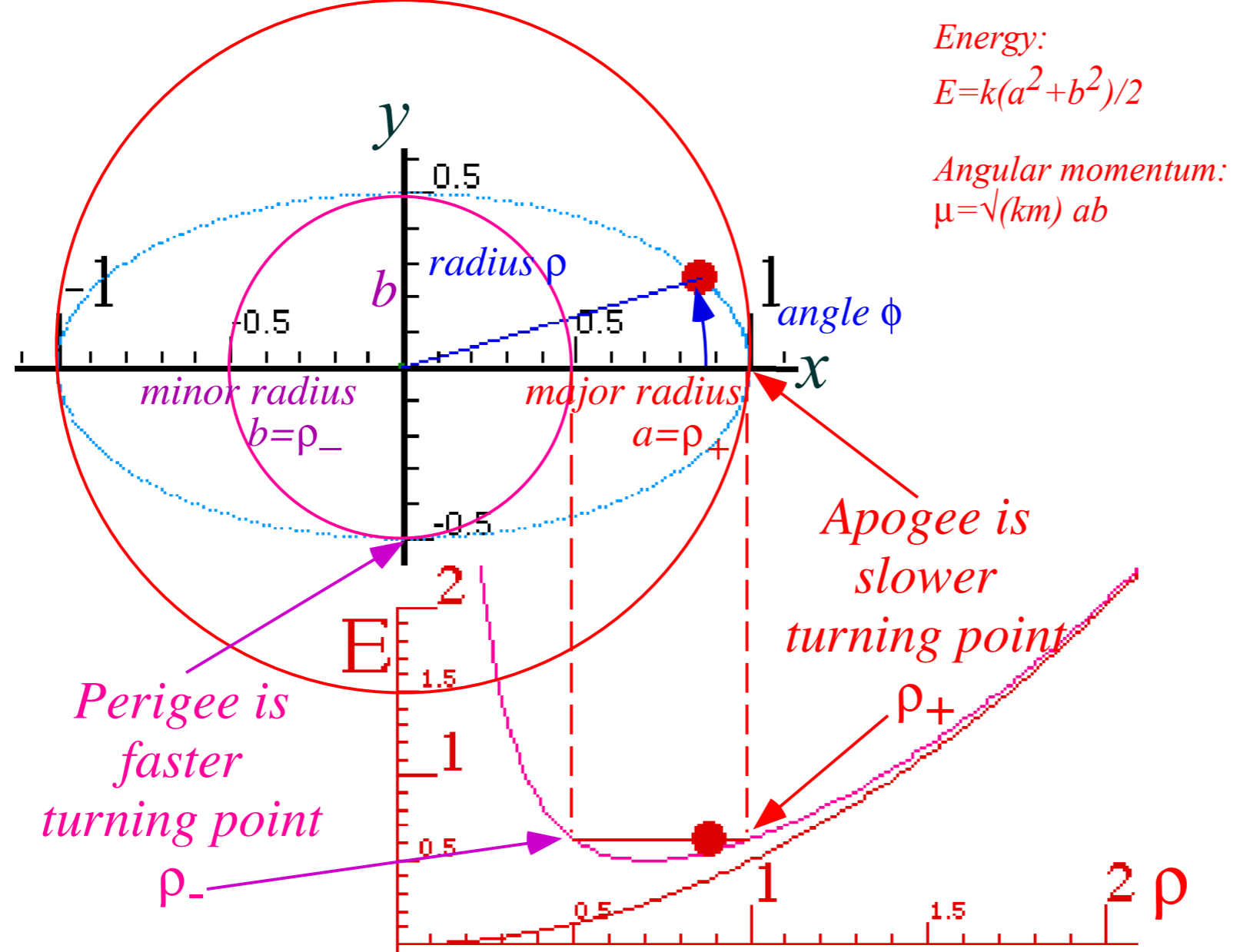
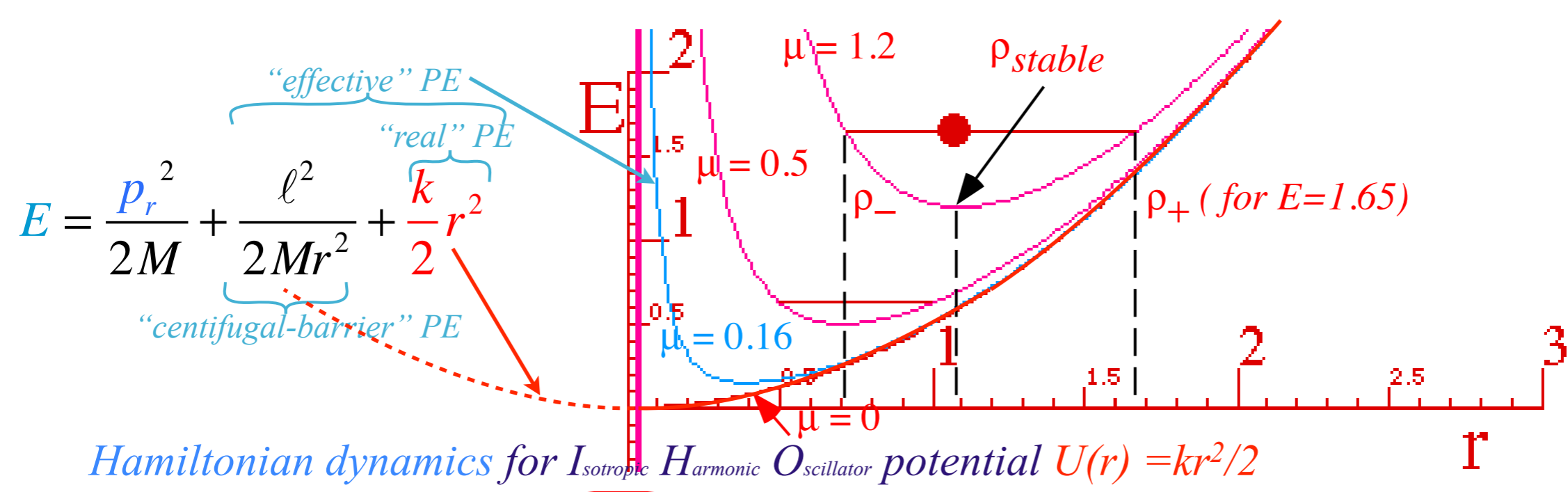
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Time vs r : $t = \int_{r_{<}}^{r_{>}} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}}$

Time vs r for any radial $U(r)$:

$$t = \int_{r_{<}}^{r_{>}} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{2U(r)}{M}}}$$

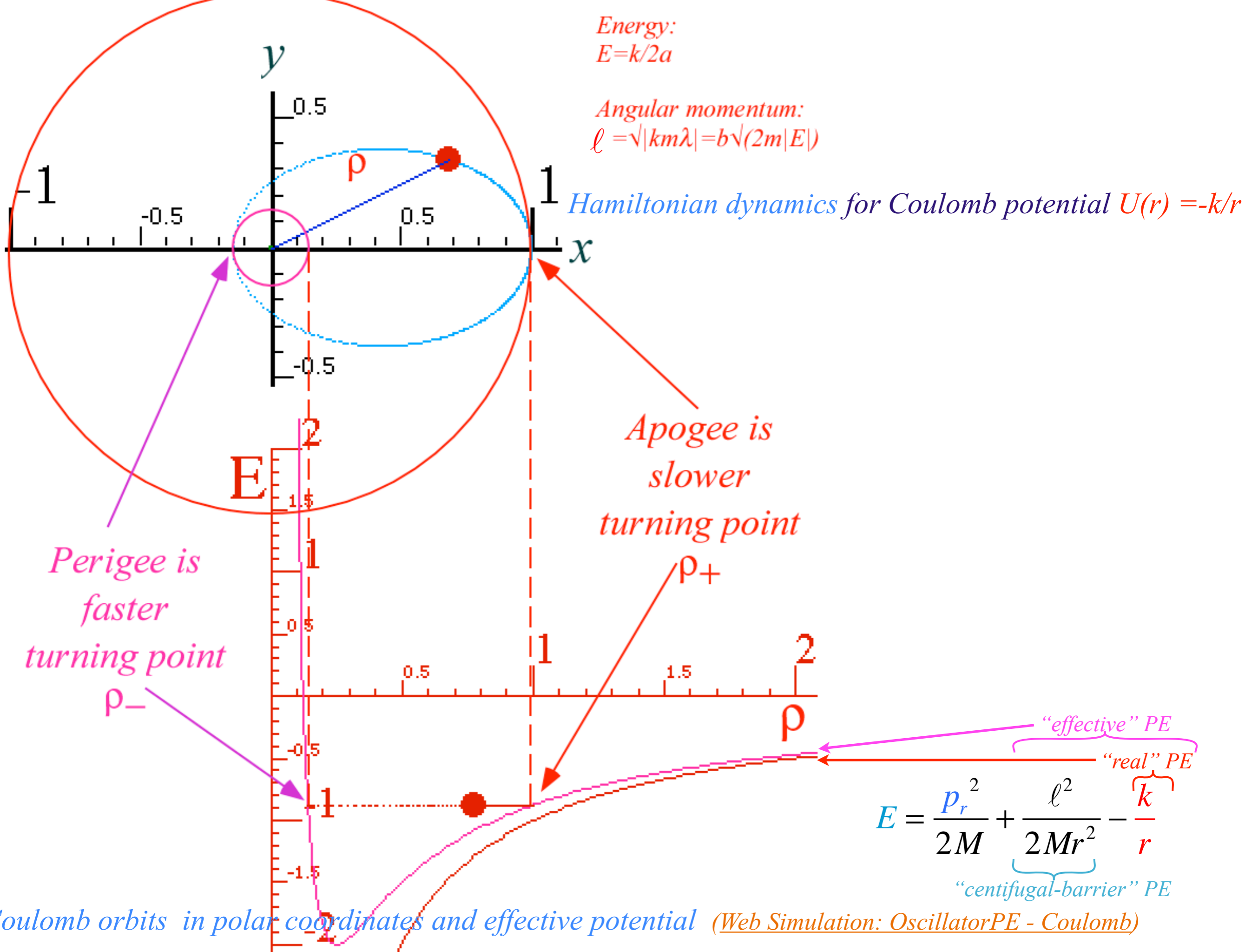


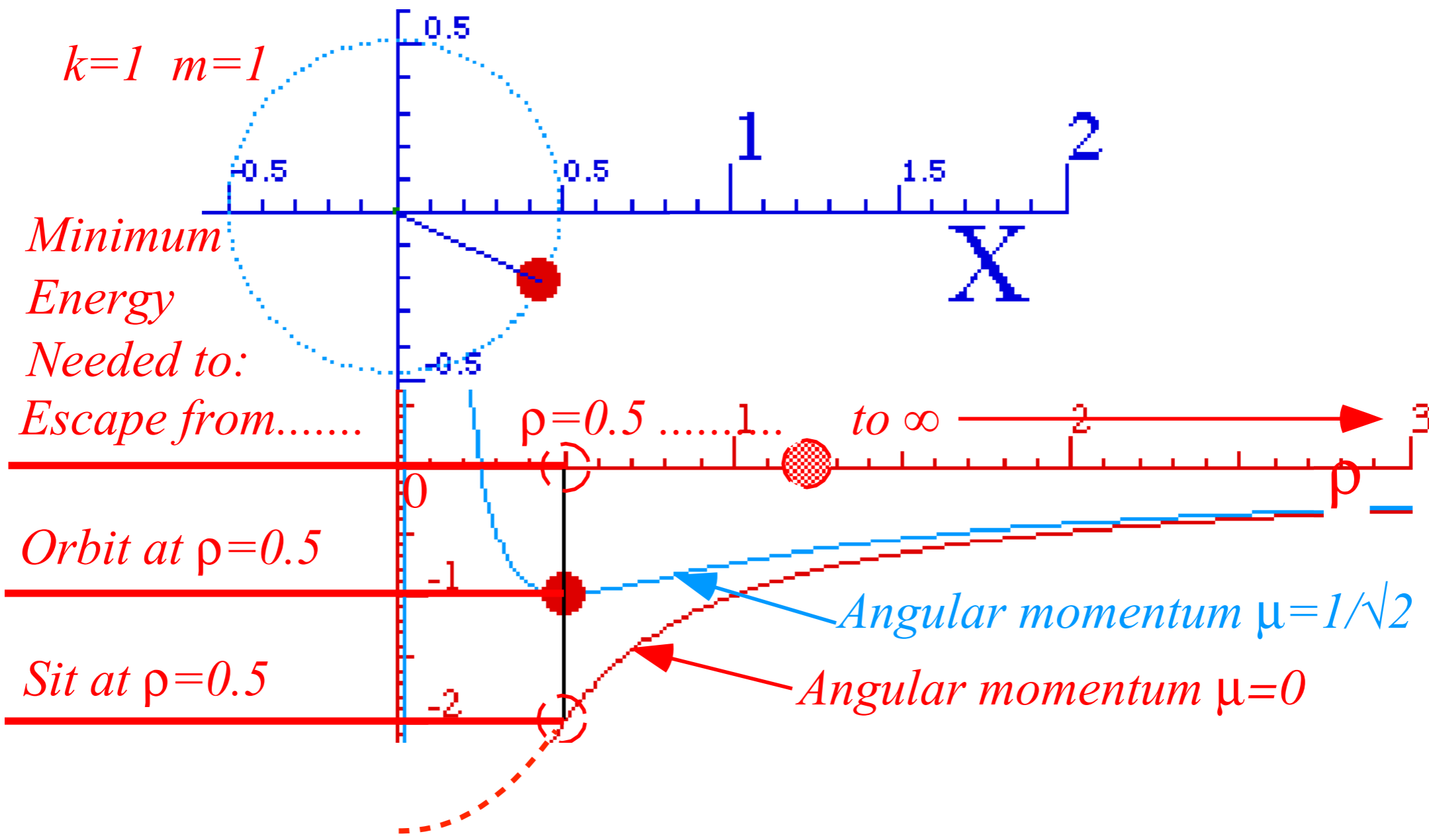


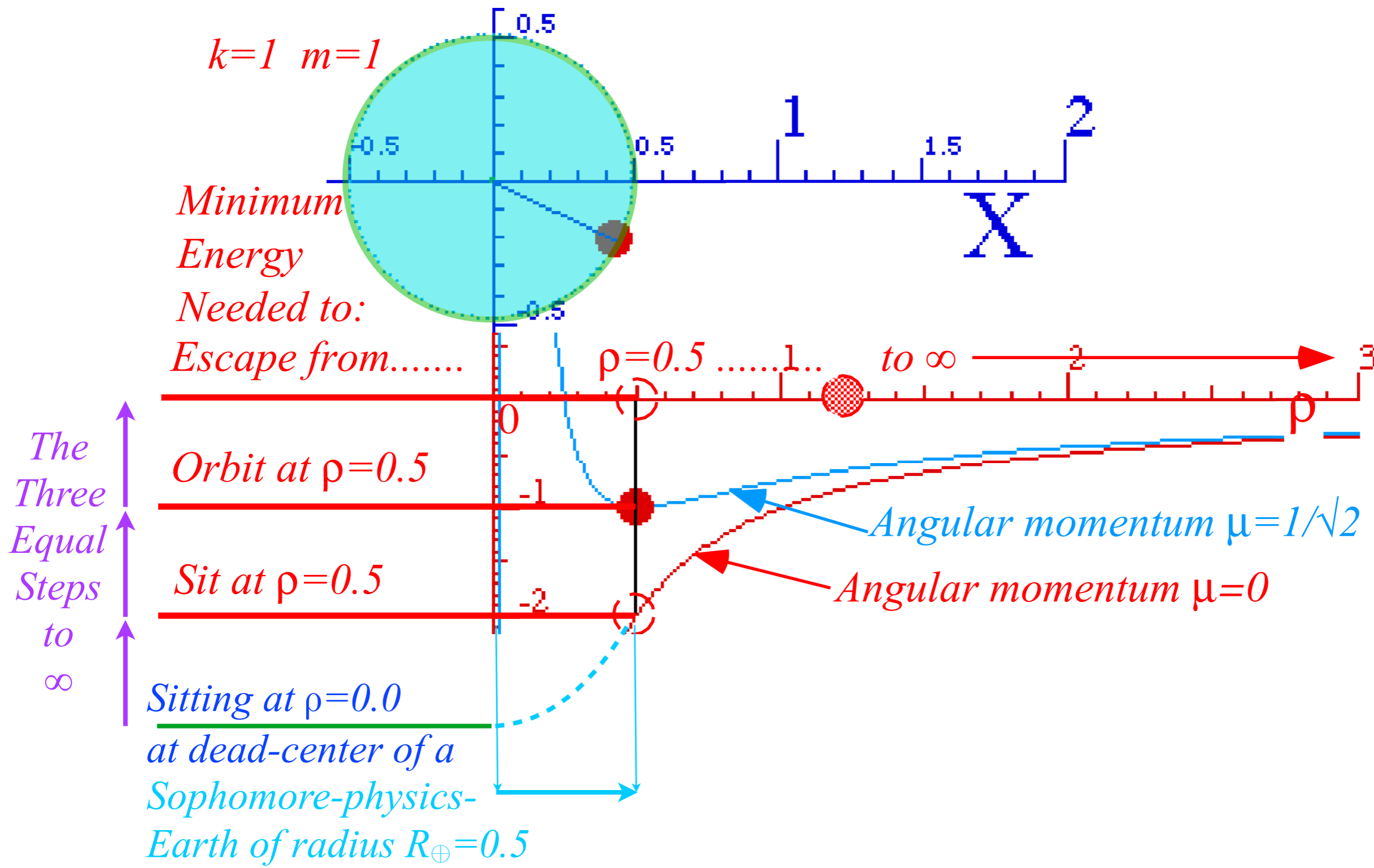
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From p. 74 Lect. 6 on next page

Sophomore-physics-Earth (inside and out): "3-steps out of (or into) Hell"

...and surface orbit at $r=R_{\oplus}$

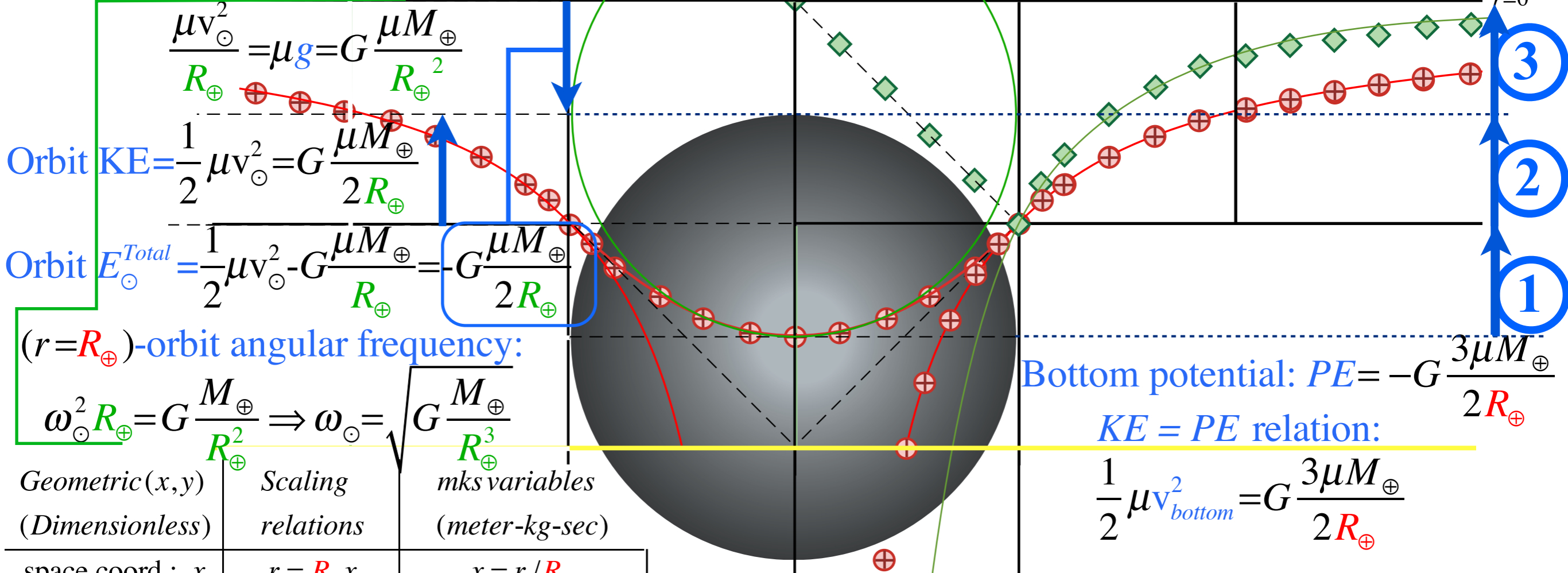
From p. 75 Lect. 6

Centrifugal force = surface gravity:

surface gravity: $g = -G \frac{M_{\oplus}}{R_{\oplus}^2}$

$r=R_{\oplus}$ Dissociation threshold : $PE=0$

escape from $r=0$



Geometric (x,y) (Dimensionless)	Scaling relations	mks variables (meter-kg-sec)			
space coord.: x	$r = R_{\oplus} x$	$x = r / R_{\oplus}$			
PE for $ x \geq 1$: $y^{PE} = \frac{-1}{x}$	$PE^{mks}(r) = \frac{GM\mu}{R_{\oplus}} y^{PE}$	$PE^{mks}(r) = -\frac{GM\mu}{r} = -\frac{GM\mu}{R_{\oplus}} \frac{1}{x}$	PE for $ x < 1$: $y^{PE} = \frac{x^2}{2} - \frac{3}{2}$	$PE^{mks}(r) = \frac{GM\mu}{R_{\oplus}} \left(\frac{r^2}{2R_{\oplus}^2} - \frac{3}{2} \right)$	$(r=0)$ -escape-velocity $v_{bottom} = \sqrt{3G \frac{M_{\oplus}}{R_{\oplus}}}$
$Force$ for $ x \geq 1$: $y^{Force} = \frac{-1}{x^2}$	$F^{mks}(r) = \frac{GM\mu}{R_{\oplus}^2} y^{Force}$	$F^{mks}(r) = -\frac{GM\mu}{r^2} = -\frac{GM\mu}{R_{\oplus}^2} \frac{1}{x^2}$	$Force$ for $ x < 1$: $y^{Force} = -x$	$F^{mks}(r) = -\frac{GM\mu}{R_{\oplus}^3} r$	$(r=R_{\oplus})$ -escape velocity: $v_{escape} = \sqrt{2G \frac{M_{\oplus}}{R_{\oplus}}}$

Sophomore-physics-Earth inside and out: "3-steps to Hell"

Suppose Earth radius crushed to 1/2: ($R_{\oplus} = 6.4 \cdot 10^6 m$ crushed to $R_{\oplus}/2 = 3.2 \cdot 10^6 m$)

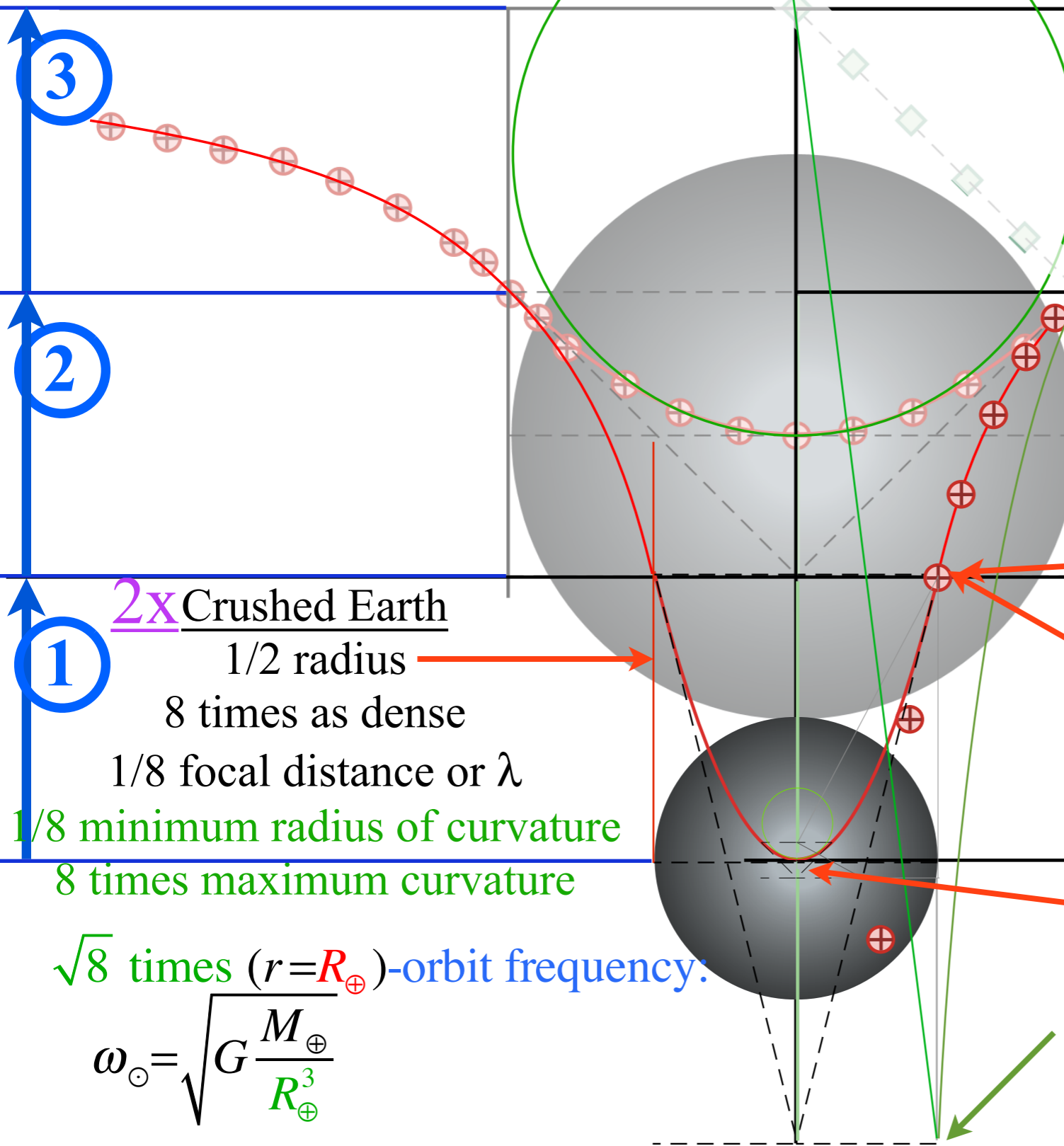
From p. 79 Lect. 6

R_{\oplus} to $R_{\oplus}/2$

Escape level : $PE=0$

All formulas identical to ones derived on p.15 to 27.

Imagine reducing R_{\oplus} to $R_{\oplus}/2$



Orbit at R_{\oplus} level : $PE = -G \frac{M_{\oplus}}{2R_{\oplus}}$

2 times \odot -orbit energy: $E_{\odot} = -G \frac{M_{\oplus}}{2R_{\oplus}}$

$\sqrt{2}$ times \odot -orbit speed: $v_{\odot} = \sqrt{G \frac{M_{\oplus}}{R_{\oplus}}}$

(Sit at R_{\oplus})-level : $PE = -G \frac{M_{\oplus}}{R_{\oplus}}$

2 times the surface potential: $PE = -G \frac{M_{\oplus}}{R_{\oplus}}$

$\sqrt{2}$ times surface escape speed: $v_e = \sqrt{G \frac{2M_{\oplus}}{R_{\oplus}}}$

(Sit at $r=0$)-level : $PE = -G \frac{3M_{\oplus}}{2R_{\oplus}}$

4 times the surface gravity: $g = -G \frac{M_{\oplus}}{R_{\oplus}^2}$

2x Crushed Earth

1/2 radius

8 times as dense

1/8 focal distance or λ

1/8 minimum radius of curvature

8 times maximum curvature

$\sqrt{8}$ times ($r=R_{\oplus}$)-orbit frequency:

$$\omega_{\odot} = \sqrt{G \frac{M_{\oplus}}{R_{\oplus}^3}}$$

Next Hamiltonian Lecture 11 ...

Examples of Hamiltonian mechanics in phase plots

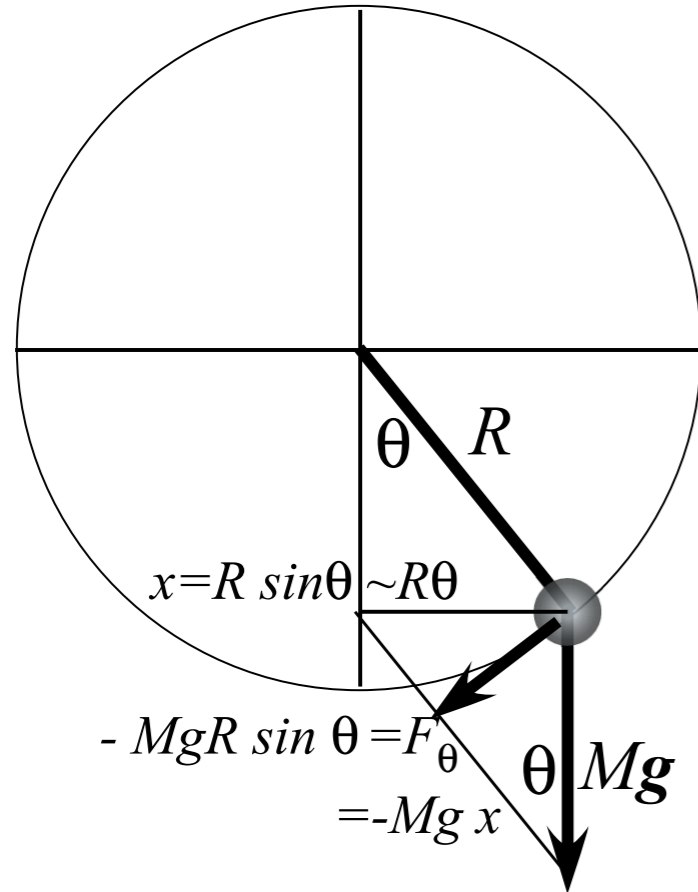
1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt](#) (Vertically Driven Pendulum))

1D-HO phase-space control (Classic Simulation of “Catcher in the Eye”, [Web Simulation:JerkIt](#))

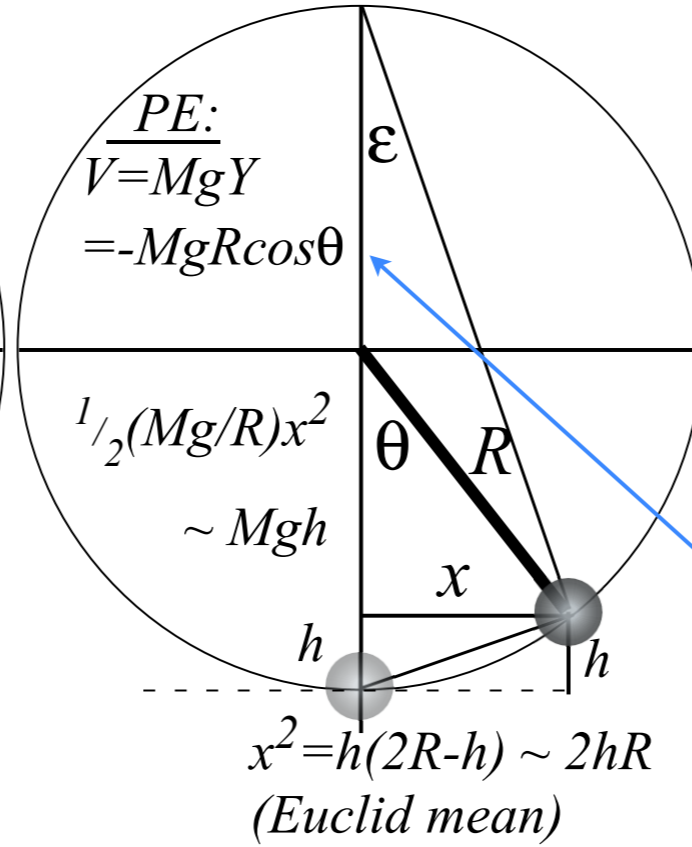
Lecture 10 ends here
Thur 9/21/2017

1D Pendulum and phase plot

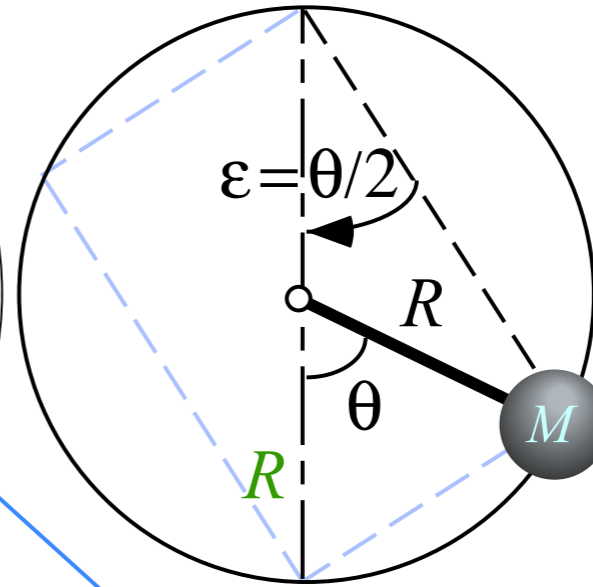
(a) Force geometry



(b) Energy geometry



(c) Time geometry



NOTE: Very common loci of \pm sign blunders

Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$