Equations of Lagrange and Hamilton mechanics in Generalized Curvilinear Coordinates (GCC)
(Ch. 12 of Unit 1 and Ch. 1-5 of Unit 2 and Ch. 1-5 of Unit 3)

Quick Review of Lagrange Relations in Lectures 7-8

Using differential chain-rules for coordinate transformations
- Polar coordinate example of Generalized Curvilinear Coordinates (GCC)
- Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
- Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2

How to say Newton’s “F=ma” in Generalized Curvilinear Coords.
- Use Cartesian KE quadratic form \( KE = T = \frac{1}{2} v \cdot M \cdot v \) and \( F = M \cdot a \) to get GCC force
- Lagrange GCC trickery gives Lagrange force equations
- Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)

GCC Cells, base vectors, and metric tensors
- Polar coordinate examples: \textbf{Covariant} \( E_m \) vs. \textbf{Contravariant} \( E^m \)
- \textbf{Covariant} \( g_{mn} \) vs. \textbf{In}variant \( \delta^m_n \) vs. \textbf{Contravariant} \( g^{mn} \)
- Lagrange prefers \textbf{Covariant} \( g_{mn} \) with \textbf{Contravariant} velocity

GCC Lagrangian definition
- GCC “canonical” momentum \( p_m \) definition
- GCC “canonical” force \( F_m \) definition
- Coriolis “fictitious” forces (… and weather effects)
Quick Review of Lagrange Relations in Lectures 7-8

$0^{th}$ and $1^{st}$ equations of Lagrange and Hamilton
Quick Review of Lagrange Relations in Lectures 7-8
0th and 1st equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and Estrangian have no explicit dependence on momentum \( p \)

\[
\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}
\]

Hamiltonian and Estrangian have no explicit dependence on velocity \( v \)

\[
\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}
\]

Lagrangian and Hamiltonian have no explicit dependence on speedinum \( V \)

\[
\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}
\]

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections

\[
\nabla_v L = \frac{\partial L}{\partial v} = \frac{\partial}{\partial v} \frac{v \cdot M \cdot v}{2} = M \cdot v = p
\]

\[
\begin{pmatrix}
\frac{\partial L}{\partial v_1} \\
\frac{\partial L}{\partial v_2}
\end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}
\]

Lagrange’s 1st equation(s)

\[
\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial v} = p
\]

\[
\nabla_p H = v = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \frac{p \cdot M^{-1} \cdot p}{2} = M^{-1} \cdot p = v
\]

(Forget Estrangian for now)

\[
\begin{pmatrix}
\frac{\partial H}{\partial p_1} \\
\frac{\partial H}{\partial p_2}
\end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]

Hamilton’s 1st equation(s)

\[
\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial p} = v
\]
(a) **Lagrangian plot**
\[ L(v) = \text{const.} = v \cdot M \cdot v / 2 \]

(b) **Hamiltonian plot**
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2 \]

(c) **Overlapping plots**

\[ v_2 = p_2 / m_2 \]
\[ v_1 = p_1 / m_1 \]

\[ H = \text{const} = E \]
\[ a_1 = \sqrt{2Em_1} \]
\[ b_1 = \sqrt{2Ep_1} \]

\[ a_2 = \sqrt{2Em_2} \]
\[ b_2 = \sqrt{2Ep_2} \]

(d) **Less mass**

(e) **More mass**

Lagrange tangent at velocity \( v \)

is normal to momentum \( p \)

\[ p = \nabla_L L = M \cdot v \]

\[ p = \nabla_p H = M^{-1} \cdot p \]

\[ v = \nabla_p H \]

\[ L = \text{const} = E \]

\[ H = \text{const} = E \]
(a) Lagrangian plot 
\[ L(v) = \text{const.} = v \cdot M \cdot \frac{v}{2} \]

(b) Hamiltonian plot 
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot \frac{p}{2} \]

(c) Overlapping plots

1st equation of Lagrange 
\[ L = \text{const.} = E \]

1st equation of Hamilton 
\[ H = \text{const.} = E \]

(d) Less mass

(e) More mass

Hamiltonian tangent at momentum \( p \) is normal to velocity \( v \)

Lagrangian tangent at velocity \( v \) is normal to momentum \( p \)
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Using differential chain-rules\textsuperscript{†} for coordinate transformations

A pair of 2-variable functions $f(x, y)$ and $g(x, y)$ can define a coordinate system on $(x, y)$-space

\begin{align*}
df(x, y) &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\
dg(x, y) &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy
\end{align*}

for example: polar coordinates

\begin{align*}
r^2(x, y) &= x^2 + y^2 \\
\theta(x, y) &= \text{atan2}(y, x)
\end{align*}

\begin{align*}
\frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
\frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
\end{align*}

(Not in text. Recall Lecture 8 p. 15-19)\textsuperscript{†}
Using differential chain-rules† for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \( (x,y) \)-space for example: polar coordinates

\[
df(x,y) = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]

\[
dg(x,y) = \frac{\partial g}{\partial x} \, dx + \frac{\partial g}{\partial y} \, dy
\]

\( r^2(x,y) = x^2 + y^2 \) and \( \theta(x,y) = \text{atan2}(y,x) \)

\[
dr(x,y) = \frac{\partial r}{\partial x} \, dx + \frac{\partial r}{\partial y} \, dy
\]

\[
d\theta(x,y) = \frac{\partial \theta}{\partial x} \, dx + \frac{\partial \theta}{\partial y} \, dy
\]

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
dx = \frac{\partial x}{\partial f} \, df + \frac{\partial y}{\partial g} \, dg
\]

\( x = r \cos \theta \)

\( y = r \sin \theta \)

\[
dy = \frac{\partial y}{\partial f} \, df + \frac{\partial y}{\partial g} \, dg
\]

\[
\begin{pmatrix}
dx \\
dy
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix}
\begin{pmatrix}
\frac{dx}{dr} \\
\frac{dx}{d\theta}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\begin{pmatrix}
dr \\
d\theta
\end{pmatrix}
\]
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space for example: polar coordinates

\[
df(x,y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\
\[
dg(x,y) = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \\
\[
(\text{Not in text. Recall Lecture 8 p. 15-19})^\dagger
\]

\[
dr(x,y) = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
\[
d\theta(x,y) = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
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dy = \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg \\
\]

\[
x = r \cos \theta \\
y = r \sin \theta
\]

\[
\begin{pmatrix}
dx \\
dy
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix} \begin{pmatrix}
dr \\
d\theta
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix} \begin{pmatrix}
dx \\
dy
\end{pmatrix}
\]

Notation for differential GCC (Generalized Curvilinear Coordinates \(\{q^1, q^2, q^3, \ldots\}\))

\[
dx^j = \frac{\partial x^j}{\partial q^m} dq^m \quad \equiv \quad \sum_{m=1}^{N} \frac{\partial x^j}{\partial q^m} dq^m \quad \{\text{Defining a shorthand dummy-index } m\text{-sum}\}
\]

These \(x^j\) are plain old CC (Cartesian Coordinates \(\{dx^1=dx, dx^2=dy, dx^3=dx, dx^4=dt\}\))

What does “\(q\)” stand for? One guess: “Queer” And they do get pretty queer!
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space for example: polar coordinates

\[
dr(x,y) = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy
\]

\[
d\theta(x,y) = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
\]

(Not in text. Recall Lecture 8 p. 15-19)

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
dx = \frac{\partial x}{\partial f} df + \frac{\partial y}{\partial g} dg
\]

\[
dy = \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg
\]

\[
x = r \cos \theta
\]

\[
y = r \sin \theta
\]

Notation for differential GCC (Generalized Curvilinear Coordinates \( \{q^1, q^2, q^3, \ldots\} \))

\[
dx^j = \frac{\partial x^j}{\partial q^m} dq^m \equiv \sum_{m=1}^{N} \frac{\partial x^j}{\partial q^m} dq^m \quad \text{Defining a shorthand dummy-index } m \text{-sum}
\]

Connection lines may help to indicate summation (OK on scratch paper...Difficult in text)

These \( x' \) are plain old CC (Cartesian Coordinates \( \{dx^1=dx, \, dx^2=dy, \, dx^3=dx, \, dx^4=dt\} \) )
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule

Same kind of linear relation exists between CC velocity \( \dot{v}^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( \dot{v}^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \)
Getting the GCC ready for mechanics:  
*Generalized velocity relation follows from GCC chain rule*

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv q^m \equiv \frac{dq^m}{dt} \)

\[
\dot{x}^j = \frac{\partial x^j}{\partial q^m} q^m \quad \text{or} \quad \frac{\partial \dot{x}^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \quad \text{(lemma-1)}
\]  

This is a key "lemma-1" for setting up mechanics:
Getting the GCC ready for mechanics:
Generalized velocity relation follows from GCC chain rule

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \)

**This is a key “lemma-1” for setting up mechanics:**

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

or:

\[ \frac{\partial \dot{x}^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \]

**Jacobian** \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ J^j_m \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial q^m} \]

Defining Jacobian matrix component

Recall polar coordinate transformation matrix:

\[ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]
Getting the GCC ready for mechanics: Generalized velocity relation follows from GCC chain rule

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \).

This is a key "lemma-1" for setting up mechanics:

Jacobian \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

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Defining Jacobian matrix component

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\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\]

Inverse (so-called) Kajobian \( K^m_j \) matrix is flipped partial derivatives of \( J^j_m \).

\[ K^m_j = \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial \dot{x}^j} \]

Defining "Kajobian" (inverse to Jacobian)

Polar coordinate inverse transformation matrix:

\[
\begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{pmatrix} =
\begin{pmatrix}
r \cos \theta & r \sin \theta \\
-sin \theta & \cos \theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{pmatrix}^{-1}
=\frac{1}{\det J} =
\begin{pmatrix}
\frac{\partial x}{\partial r} & -\frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & -\frac{\partial y}{\partial \theta}
\end{pmatrix}^{-1} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

Defining 2x2 matrix inverse: (always test inverse matrices!)

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} =
\begin{pmatrix}
D & -B \\
-C & A
\end{pmatrix}
\frac{1}{AD-BC}
\]

\[
\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}
\]

Lemma-1
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule:

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j = \dot{x}^j = \frac{dx^j}{dt} \) and GCC velocity \( \nu^m = \dot{q}^m = \frac{dq^m}{dt} \):

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

This is a key “lemma-1” for setting up mechanics:

**Jacobian** \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ J^j_m \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \] \[ \text{Defining Jacobian matrix component} \]

Recall polar coordinate transformation matrix:

\[ \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{bmatrix} \]

Inverse (so-called) **Jacobian** \( K^m_j \) matrix is flipped partial derivatives of \( J^j_m \).

\[ K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j} \] \[ \text{Defining "Jacobian" (inverse to Jacobian)} \]

Polar coordinate inverse transformation matrix:

\[ \begin{bmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\
\frac{\partial \theta}{\partial r} & \frac{\partial \theta}{\partial \theta}
\end{bmatrix} = \frac{1}{\det J} = \begin{bmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{bmatrix} \]

Defining 2x2 matrix inverse:

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{\det(AD-BC)} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} \frac{D}{AD-BC} & -B \\ \frac{-C}{AD-BC} & A \end{pmatrix} \]

Wednesday, September 21, 2016
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule:

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

This is a key "lemma-1" for setting up mechanics:

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \).

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

Jacobian \( J^m_j \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ \frac{\partial x^j}{\partial q^m} = \frac{\partial x^i}{\partial q^j} \]

Recall polar coordinate transformation matrix:

\[
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\]

Inverse (so-called) Kajobian \( K^m_j \) matrix is flipped partial derivatives of \( J^m_j \).

\[
K^m_j = \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial \dot{x}^j} \quad \text{Defining "Kajobian"}
\]

(inverse to Jacobian)

Product of matrix \( J^m_j \) and \( K^m_j \) is a unit matrix by definition of partial derivatives:

\[ K^m_j \cdot J^j_n = \frac{\partial q^m}{\partial x^j} \cdot \frac{\partial x^j}{\partial q^n} = \frac{\partial q^m}{\partial q^n} = \delta^m_n = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases} \]

always test inverse matrices!
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1

Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

\[ \ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \ddot{q}^m + \frac{\partial x^j}{\partial q^m} \dddot{q}^m \]

First apply \( \frac{d}{dt} \) to velocity \( \dot{x}^j \) and use product rule:

\[ \frac{d}{dt} (u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt} \]
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule:

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \ddot{q}^m + \frac{\partial x^j}{\partial q^m} \dddot{q}^m$$

Apply derivative chain sum to Jacobian.

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) dq^n \frac{dt}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) dq^n$$
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt} (u \cdot v) = \frac{du}{dt} v + u \cdot \frac{dv}{dt}$

\[
\dot{x}^j \equiv \frac{d}{dt} x^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m
\]

(Not in text. Recall Lecture 9 p. 15-19)

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\frac{\partial}{\partial q^m} \frac{\partial}{\partial q^n} = \frac{\partial}{\partial q^n} \frac{\partial}{\partial q^m}$

\[
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)
\]
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply \( \frac{d}{dt} \) to velocity \( \dot{x}^j \) and use product rule: \( \frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt} \)

\[
\dot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m
\]

(Not in text. Recall Lecture 9 p. 15-19)†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. \( \partial_m \partial_n = \partial_n \partial_m \)

\[
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^m \partial q^n} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)
\]

By chain-rule def. of CC velocity:

\[
= \frac{\partial}{\partial q^m} \left( \dot{x}^j \right)
\]
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\dot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \ddot{q}^m + \frac{\partial x^j}{\partial q^m} \dddot{q}^m$$

(Not in text. Recall Lecture 9 p. 15-19)†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)$$

By chain-rule def. of CC velocity:

$$= \frac{\partial}{\partial q^m} (\dot{x}^j)$$

This is the key “lemma-2” for setting up Lagrangian mechanics.

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m} \text{ lemma}^2$$


Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply \( \frac{d}{dt} \) to velocity \( \dot{x}^j \) and use product rule:

\[
\frac{d}{dt} (u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}
\]

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. \( \partial_m \partial_n = \partial_n \partial_m \)

\[
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)
\]

By chain-rule def. of CC velocity:

\[
= \frac{\partial}{\partial q^m} (\dot{x}^j)
\]

The “lemma-1” was in the GCC velocity analysis just before this one for acceleration.

This is the key “lemma-2” for setting up Lagrangian mechanics.

\[
\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \text{lemma 1}
\]

\[
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m} \quad \text{lemma 2}
\]
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE = T = \frac{1}{2}v \cdot M \cdot v$ and $F = M \cdot a$ to get GCC force.

Lagrange GCC trickery gives Lagrange force equations.

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2).
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II
Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \)

\[
T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \text{ where: } M_{jk} \text{ are CC inertia constants}
\]

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \)

\[
T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{where: } M_{jk} \text{ are inertia constants that are symmetric: } M_{jk} = M_{kj}
\]

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = \mathbf{F} \cdot d\mathbf{x} \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

(It’s time to bring in the queer \( q^m \) !)
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know... (sort of)

Multidimensional CC version of kinetic energy $\frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$

$$T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{where: } M_{jk} \text{ are inertia constants that are symmetric: } M_{jk} = M_{kj}$$

Multidimensional CC version of Newt-II ($\mathbf{F} = \mathbf{M} \cdot \mathbf{a}$) using $M_{jk}$ constants

$$f_j = M_{jk} a^k = M_{jk} \ddot{x}^k$$

Multidimensional CC version of work-energy differential ($dW = \mathbf{F} \cdot dx$). Insert GCC differentials $dq^m$

$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

(It’s time to bring in the queer $q^m$ !)

d$q^m$ are independent so $dq^m$-sum is true term-by-term.

$$dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m$$
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \)

\[
T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k
\]

where: \( M_{jk} \) are inertia constants

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = \mathbf{F} \cdot d\mathbf{x} \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

\( dq^m \) are independent so \( dq^m \)-sum is true term-by-term. (Still holds if all \( dq^m \) are zero but one.)

\[
dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \quad \Rightarrow \quad F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}
\]
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \)

\[
T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k
\]

where: \( M_{jk} \) are inertia constants

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

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f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = \mathbf{F} \cdot d\mathbf{x} \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

\( dq^m \) are independent so \( dq^m \)-sum is true term-by-term. (Still holds if all \( dq^m \) are zero but one.)

\[
dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \quad \Rightarrow \quad F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}
\]

Here generalized GCC force component \( F_m \) is defined:

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}
\]

where: \( F_m = f_j \frac{\partial x^j}{\partial q^m} \)
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form \( KE = T = \frac{1}{2}v \cdot M \cdot v \) and \( F = M \cdot a \) to get GCC force

Lagrange GCC trickery gives Lagrange force equations
Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)
Now Lagrange GCC trickery begins

Obvious stuff... (sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set \( A = M_{jk} \dot{x}^k \) and \( B = \frac{\partial x^j}{\partial q^m} \) with calc. formula:

\[
\ddot{A}B = \frac{d}{dt}( \dot{A}B ) - \dot{A} \dot{B}
\]

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)
\]
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\[
\ddot{A}B = \frac{d}{dt}(\dot{A}B) - \dot{A}\dot{B}
\]

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt}\left(M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m}\right) - M_{jk} \dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)
\]

Cartesian \( M_{jk} \)

must be constant

for this to work

(Bye, Bye relativistic mechanics or QM!)
Now Lagrange GCC trickery begins

Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set $A = M_{jk} \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\ddot{A}B = \frac{d}{dt}(\dot{A}B) - \dot{A}\dot{B}$

$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$

Then convert $\partial x^j$ to $\dot{x}^j$ by Lemma 1 and Lemma 2 on 2nd term.

$F_m = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \left( \frac{\partial \dot{x}^j}{\partial q^m} \right)$

Cartesian $M_{jk}$ must be constant for this to work

(Bye, Bye relativistic mechanics or QM!)
Now Lagrange GCC trickery begins

Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set \( A = M_{jk} \dot{x}^k \) and \( B = \frac{\partial x^j}{\partial q^m} \) with calc. formula:

\[
\ddot{AB} = \frac{d}{dt}(\dot{AB}) - \dot{A}\dot{B}
\]

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)
\]

Then convert \( \partial x^j \) to \( \partial \dot{x}^j \) by Lemma 1 and Lemma 2 on 2\textsuperscript{nd} term.

\[
F_m = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial q^m}
\]

Simplify using:

\[
[M_{ij} v^i \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial v^i v^j}{2}]
\]

where \( q \) may be \( q^m \) or \( q^m \)

\[
F_m = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right) - \frac{\partial}{\partial q^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right)
\]
Now Lagrange GCC trickery begins
Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set \( A = M_{jk} \ddot{x}^k \) and \( B = \frac{\partial x^j}{\partial q^m} \) with calc. formula:

\[
\ddot{AB} = \frac{d}{dt} (\dot{AB}) - \dot{AB}
\]

Then convert \( \partial x^j \) to \( \partial \dot{x}^j \) by Lemma 1 and Lemma 2 on 2\(^{nd} \) term.

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \ddot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)
\]

Simplify using:

\[
[M_{ij} v^i \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial}{\partial q} \frac{v^i v^j}{2}]
\]

where \( q \) may be \( q^m \) or \( q^m \)

\[
F_m = \frac{d}{dt} \frac{\partial}{\partial q^m} \left( M_{jk} \ddot{x}^k \dot{x}^j \right) - \frac{\partial}{\partial q^m} \left( \frac{M_{jk} \ddot{x}^k \dot{x}^j}{2} \right)
\]

The result is Lagrange’s GCC force equation in terms of kinetic energy

\[
T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k
\]

\[
F_m = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m} \quad \text{or:} \quad F = \frac{d}{dt} \frac{\partial T}{\partial \mathbf{v}} - \frac{\partial T}{\partial \mathbf{r}}
\]
How to say Newton’s “$F=ma$” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE=T=1/2v \cdot M \cdot v$ and $F=M \cdot a$ to get GCC force

Lagrange GCC trickery gives Lagrange force equations

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)

If the force is conservative it’s a gradient \( \mathbf{F} = -\nabla U \)

In GCC: \( F_m = -\frac{\partial U}{\partial q^m} \)

\[
F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial q^m} - \frac{\partial T}{\partial q^m}
\]
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)

If the force is conservative it’s a gradient \( \mathbf{F} = -\nabla U \)  

In GCC:  \( F_m = -\frac{\partial U}{\partial q^m} \)

\[
F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}
\]

Becomes \textit{Lagrange’s GCC potential equation} with a new definition for the \textit{Lagrangian}: \( L = T-U \).

\[
0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}
\]

\( L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m) \)

This trick requires:  \( \frac{\partial U}{\partial \dot{q}^m} \equiv 0 \)  

\( U(r) \) has \textit{NO explicit velocity dependence}!
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)
If the force is conservative it’s a gradient \( F = -\nabla U \)
In GCC: \( F_m = -\frac{\partial U}{\partial q^m} \)

\[
F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}
\]

Becomes *Lagrange’s GCC potential equation* with a new definition for the *Lagrangian*: \( L = T - U \).

\[
0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}
\]

\[
L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)
\]

This trick requires: \( \frac{\partial U}{\partial \dot{q}^m} \equiv 0 \)

\( U(r) \) has **NO** explicit velocity dependence!

---

**Lagrange’s 1st GCC equation**
*(Defining GCC momentum)*

\[
p_m = \frac{\partial L}{\partial \dot{q}^m}
\]

*Recall:*

\[
p = \frac{\partial L}{\partial v}
\]

**Lagrange’s 2nd GCC equation**
*(Change of GCC momentum)*

\[
\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}
\]
GCC Cells, base vectors, and metric tensors

Polar coordinate examples:
- **Covariant** $E_m$ vs. **Contravariant** $E^m$
- **Covariant** $g_{mn}$ vs. **Invariant** $\delta^m_n$ vs. **Contravariant** $g^{mn}$
A dual set of quasi-unit vectors show up in Jacobian J and Kajobian K. J-Columns are covariant vectors \{E_1=E_r, \ E_2=E_\phi\} K-Rows are contravariant vectors \{E^1=E^r, \ E^2=E^\phi\}.

\[
\langle J \rangle = \begin{pmatrix}
  \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
  \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
  \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
  \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix}
  \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
  \frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix}
\]

\[E^r = E_1, \quad E^\phi = E_2\]

Inverse polar definition:
\[r^2 = x^2 + y^2 \text{ and } \phi = \text{atan2}(y,x)\]

Derived from polar definition: \(x = r \cos \phi\) and \(y = r \sin \phi\)

(a) Polar coordinate bases

Unit 1
Fig. 12.10
A dual set of quasi-unit vectors show up in Jacobian J and Kajobian K. J-Columns are covariant vectors \( \{ \mathbf{E}_1 = \mathbf{E}_r, \mathbf{E}_2 = \mathbf{E}_\phi \} \) and K-Rows are contravariant vectors \( \{ \mathbf{E}^1 = \mathbf{E}^r, \mathbf{E}^2 = \mathbf{E}^\phi \} \).

\[
\begin{bmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \\
\frac{\partial y^1}{\partial q^1} & \frac{\partial y^1}{\partial q^2} \\
\frac{\partial y^2}{\partial q^1} & \frac{\partial y^2}{\partial q^2}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{bmatrix}
\]

\[
\langle J \rangle = \langle J^{-1} \rangle = \begin{bmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{bmatrix}
\leftarrow \mathbf{E}^r = \mathbf{E}^1
\]

\[
\langle K \rangle = \begin{bmatrix}
\frac{\partial r}{\partial x} = \cos \phi \quad & \frac{\partial r}{\partial \phi} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi \quad & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{bmatrix}
\leftarrow \mathbf{E}^\phi = \mathbf{E}^2
\]

\[\text{Inverse polar definition: } r^2 = x^2 + y^2 \text{ and } \phi = \text{atan2}(y, x)\]

Derived from polar definition: \( x = r \cos \phi \) and \( y = r \sin \phi \)

(a) Polar coordinate bases

(b) Covariant bases \( \{ \mathbf{E}_1 \mathbf{E}_2 \} \) (Tangent)

\[
d\mathbf{r} = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2
\]

(c) Contravariant bases \( \{ \mathbf{E}^1 \mathbf{E}^2 \} \) (Normal)

\[
\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2
\]

**Unit 1**

**Fig. 12.10**
Comparison: **Covariant** $E_m = \frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

Covariant bases $\{E_1, E_2\}$ match cell walls

$(\text{Tangent})$

$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$

*is based on chain rule:*

$$dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$$

$\Delta q^1 = 1.0$

$\Delta q^2 = 1.0$

$E_1 = \frac{\partial r}{\partial q^1}$

$E_2$

$q^1 = 100$

$q^2 = 200$

$q^2 = 201$

$q^1 = 101$

$\frac{\partial r}{\partial q^2}$

**NOTE:** These are 2D drawings!

*No 3D perspective*
Comparison: **Covariant** \( E_m = \frac{\partial r}{\partial q^m} \) vs. **Contravariant** \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

**Covariant bases** \( \{E_1, E_2\} \)** match cell walls

\[ \Delta r = E_1 \Delta q^1 + E_2 \Delta q^2 \]

is based on chain rule: \[ dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2 \]

\( E_1 \) follows tangent to \( q^2 = \text{const.} \) ... since only \( q^1 \) varies in \( \frac{\partial r}{\partial q^1} \) while \( q^2, q^3, \ldots \) remain constant

**NOTE:** These are 2D drawings! **No 3D perspective**
Comparison: **Covariant** \( \mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} \) vs. **Contravariant** \( \mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m \)

Covariant bases \( \{ \mathbf{E}_1, \mathbf{E}_2 \} \) match cell walls

\[ \Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2 \]

is based on chain rule:

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2 \]

\( E_1 \) follows tangent to \( q^2 = \text{const.} \) ... since only \( q^1 \) varies in \( \frac{\partial \mathbf{r}}{\partial q^1} \) while \( q^2, q^3, \ldots \) remain constant

\( \mathbf{E}_m \) are convenient bases for extensive quantities like distance and velocity.

\[ \mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2} \]

NOTE: These are 2D drawings!

No 3D perspective
Comparison: **Covariant** \( \mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} \) vs. **Contravariant** \( \mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m \)

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\[ \mathbf{\Delta r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2 \]

**is based on chain rule:**

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2 \]

\( \mathbf{E}_1 \) follows tangent to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial \mathbf{r}}{\partial q^1} \)

while \( q^2, q^3, ... \) remain constant

\( \mathbf{E}_m \) are convenient bases for extensive quantities like distance and velocity.

\[ \mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2} \]

**Contravariant** \( \{ \mathbf{E}^1 \mathbf{E}^2 \} \) match reciprocal cells

\[ \frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}^2 \]

\[ \mathbf{F} = F^1 \mathbf{E}^1 + F^2 \mathbf{E}^2 \]

\( \mathbf{E}^1 \) is normal to \( q^1 = \text{const.} \) since

**gradient** of \( q^1 \) is vector sum

\[ \nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix} \]

NOTE: These are 2D drawings!

No 3D perspective
**Comparison:** Covariant \( E_m = \frac{\partial r}{\partial q^m} \) vs. Contravariant \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

**Covariant bases** \( \{E_1, E_2\} \) match cell walls

\[ \Delta r = E_1 \Delta q^1 + E_2 \Delta q^2 \]

is based on chain rule:

\[ dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2 \]

\( E_1 \) follows tangent to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial r}{\partial q^1} \)

while \( q^2, q^3, \ldots \) remain constant

\( E_m \) are convenient bases for extensive quantities like distance and velocity.

\[ \mathbf{v} = v^1 E_1 + v^2 E_2 = v^1 \frac{\partial r}{\partial q^1} + v^2 \frac{\partial r}{\partial q^2} \]

**Contravariant** \( \{E^1, E^2\} \) match reciprocal cells

\( \frac{\partial q^2}{\partial r} = \nabla q^2 = E^2 \)

\[ \mathbf{F} = F_1 E^1 + F_2 E^2 \]

\( E^1 \) is normal to \( q^1 = \text{const.} \) since gradient of \( q^1 \) is vector sum \( \nabla q^1 = \left( \frac{\partial q^1}{\partial x}, \frac{\partial q^1}{\partial y} \right) \)

\( E^m \) are convenient bases for intensive quantities like force and momentum.

\[ \mathbf{F} = F_1 E^1 + F_2 E^2 = F_1 \frac{\partial q^1}{\partial r} + F_2 \frac{\partial q^2}{\partial r} = F_1 \nabla q^1 + F_2 \nabla q^2 \]

**NOTE:** These are 2D drawings! No 3D perspective
Comparison: **Covariant** $\mathbf{E}_m = -\frac{\partial \mathbf{r}}{\partial q^m}$ vs. **Contravariant** $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

**Covariant bases** $\{\mathbf{E}_1, \mathbf{E}_2\}$ match cell walls

$$\Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$

$\mathbf{E}_1$ follows **tangent** to $q^2 = \text{const.}$...

since only $q^1$ varies in $\frac{\partial \mathbf{r}}{\partial q^1}$

while $q^2$, $q^3$, ... remain constant

$\mathbf{E}_m$ are convenient bases for **extensive** quantities like distance and velocity.

$$\mathbf{v} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

**Contravariant** $\{\mathbf{E}^1, \mathbf{E}^2\}$ match reciprocal cells

$$\frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}^2$$

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$

$\mathbf{E}^1$ is **normal** to $q^1 = \text{const.}$ since

**gradient** of $q^1$ is vector sum $\nabla q^1 = \left(\frac{\partial q^1}{\partial x}, \frac{\partial q^1}{\partial y}\right)$

$\mathbf{E}^m$ are convenient bases for **intensive** quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

**Co-Contra dot products** $\mathbf{E}_m \cdot \mathbf{E}^n$ are **orthonormal**: $\mathbf{E}_m \cdot \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{q}^n}{\partial \mathbf{r}} = \delta^m_n$
GCC Cells, base vectors, and metric tensors

Polar coordinate examples: Covariant $E_m$ vs. Contravariant $E^m$
Covariant $g_{mn}$ vs. Invariant $\delta_{mn}$ vs. Contravariant $g^{mn}$
**Covariant** $g_{mn}$ **vs.** **Invariant** $\delta_{m}^{n}$ **vs.** **Contravariant** $g^{mn}

\begin{align*}
E_{m} \cdot E_{n} &= \frac{\partial r}{\partial q^{m}} \cdot \frac{\partial r}{\partial q^{n}} \equiv g_{mn} \\
E_{m} \cdot E^{n} &= \frac{\partial r}{\partial q^{m}} \cdot \frac{\partial q^{n}}{\partial r} = \delta^{n}_{m} \\
E^{m} \cdot E^{n} &= \frac{\partial q^{m}}{\partial r} \cdot \frac{\partial q^{n}}{\partial r} \equiv g^{mn}
\end{align*}

**Covariant metric tensor** $g_{mn}$  
**Invariant** Kronecker unit tensor $\delta_{m}^{n}$  
**Contravariant metric tensor** $g^{mn}$

\[
\delta^{n}_{m} \equiv \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases}
\]
**Covariant** $g_{mn}$ **vs.** **Invariant** $\delta^m_n$ **vs.** **Contravariant** $g^{mn}$

$E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} = g_{mn}$

$E^m \cdot E^n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta^m_n$

$E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} = g^{mn}$

**Covariant metric tensor** $g_{mn}$

**Invariant Kroneker unit tensor** $\delta^m_n$

$\delta^m_n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$

**Contravariant metric tensor** $g^{mn}$

**Polar coordinate examples (again):**

$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\ \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix}$

$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix} \leftarrow E^r = E^1$

$\uparrow E_1 \uparrow E_2 \uparrow E_r \uparrow E_\phi$
**Covariant** $g_{mn}$ vs. **Invariant** $\delta^m_n$ vs. **Contravariant** $g^{mn}$

\[
E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}
\]

\[
E^m \cdot E^n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta^m_n
\]

\[
E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}
\]

**Covariant metric tensor**

\[g_{mn}\]

**Invariant Kroneker unit tensor**

\[\delta^m_n \equiv \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases}\]

**Contravariant metric tensor**

\[g^{mn}\]

Polar coordinate examples (again):

\[
\left\langle J \right\rangle = \begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\left\langle K \right\rangle = \left\langle J^{-1} \right\rangle = \begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
& \begin{pmatrix}
r^2
\end{pmatrix}
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E^r \cdot E^r & E^r \cdot E^\phi \\
E^\phi \cdot E^r & E^\phi \cdot E^\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1/r^2
\end{pmatrix}
\]
Lagrange prefers **Covariant** $g_{mn}$ with **Contra**variant velocity $\dot{q}^m$  

GCC Lagrangian definition  
GCC “canonical” momentum $p_m$ definition  
GCC “canonical” force $F_m$ definition  
Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian $L=KE-U$ is supposed to be explicit function of velocity.

\[ L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q}) \]
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53) 

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi \phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi^2}) - U(r, \phi)$$
Lagrange prefers \textbf{Covariant} \( g_{mn} \) with \textbf{Contra}variant velocity \( q^m \)

- GCC Lagrangian definition
- GCC “canonical” momentum \( p_m \) definition
- GCC “canonical” force \( F_m \) definition
- Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian $KE-U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi \phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

*(From preceding page)*
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. **1st** $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}^r} = M \ g_{rr} \dot{r} = M \ \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-U is supposed to be explicit function of velocity.

$$ L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} - U = \frac{1}{2} M (E_m \mathbf{q}^m) \cdot (E_n \mathbf{\dot{q}}^n) - U = \frac{1}{2} M (g_{mn} \mathbf{\dot{q}}^m \mathbf{\dot{q}}^n) - U = L(\mathbf{\dot{q}}) $$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$
\begin{pmatrix}
  g_{rr} & g_{r\phi} \\
  g_{\phi r} & g_{\phi\phi}
\end{pmatrix} =
\begin{pmatrix}
  E_r \cdot E_r & E_r \cdot E_\phi \\
  E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  0 & r^2
\end{pmatrix}
$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$ L(\mathbf{\dot{r}}, \mathbf{\dot{\phi}}) = \frac{1}{2} M (g_{rr} \mathbf{\dot{r}}^2 + g_{\phi\phi} \mathbf{\dot{\phi}}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \mathbf{\dot{r}}^2 + r^2 \mathbf{\dot{\phi}}^2) - U(r, \phi) $$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$ p_r = \frac{\partial L}{\partial \mathbf{\dot{r}}} = M g_{rr} \mathbf{\dot{r}} = M \mathbf{\dot{r}} \quad \text{Nothing too surprising; radial momentum } p_r \text{ has the usual linear } M \cdot \mathbf{v} \text{ form} $$

$$ p_\phi = \frac{\partial L}{\partial \mathbf{\dot{\phi}}} = Mg_{\phi\phi} \mathbf{\dot{\phi}} = Mr^2 \mathbf{\dot{\phi}} \quad \text{Wow! } g_{\phi\phi} \text{ gives moment-of-inertia factor } Mr^2 \text{ automatically for the angular momentum } p_\phi = Mr^2 \mathbf{\omega}. $$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity $q^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (… and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (\mathbf{E}_m \dot{q}^m) \cdot (\mathbf{E}_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \ddot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = \dot{M} r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

*(From preceding page)*
Lagrange prefers **Covariant** \( g_{mn} \) with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

\[
L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M ( E_m \dot{q}^m ) \cdot ( E_n \dot{q}^n ) - U = \frac{1}{2} M ( g_{mn} \dot{q}^m \dot{q}^n ) - U = L(\dot{q})
\]

Use polar coordinate **Covariant** \( g_{mn} \) metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix} =
\begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M ( g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2 ) - U(r, \phi) = \frac{1}{2} M ( 1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2 ) - U(r, \phi)
\]

GCC Lagrange equations follow. **1st** \( L \)-equation is momentum \( p_m \) definition for each coordinate \( q^m \):

\[
p_r = \frac{\partial L}{\partial \dot{r}^r} = M g_{rr} \dot{r} = M \dot{r}
\]

Nothing too surprising; radial momentum \( p_r \) has the usual linear \( M \cdot v \) form

\[
p_\phi = \frac{\partial L}{\partial \dot{\phi}^\phi} = M g_{\phi \phi} \dot{\phi} = Mr^2 \dot{\phi}
\]

Wow! \( g_{\phi \phi} \) gives moment-of-inertia factor \( Mr^2 \) automatically for the angular momentum \( p_\phi = Mr^2 \omega \).

**2nd** \( L \)-equation involves total time derivative \( \dot{p}_m \) for each momentum \( p_m \):

\[
\dot{p}_r = \frac{\partial L}{\partial r} = \frac{1}{2} \frac{\partial g_{\phi \phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}
\]

Centrifugal force \( M r \omega^2 \)

\[
\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}
\]

Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence

\[
\frac{d}{dt} \frac{\partial L}{\partial q^m} = \frac{\partial L}{\partial q^m}
\]

Recall:

\[
p_m = \frac{\partial L}{\partial \dot{q}^m}
\]

\[
p = \frac{\partial L}{\partial v}
\]

Lagrange’s **1st** GCC equation
(Defining GCC momentum)

\[
p_m = \frac{\partial L}{\partial \dot{q}^m}
\]

Lagrange’s **2nd** GCC equation
(Change of GCC momentum)

\[
\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}
\]
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \ddot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_{\phi} \\
E_{\phi} \cdot E_r & E_{\phi} \cdot E_{\phi}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. **1st** $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $M r^2$ automatically for the angular momentum $p_{\phi} = M r^2 \omega$.

**2nd** $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $M r \omega^2$

$$\dot{p}_{\phi} = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_{\phi}$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from **1st** $L$-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \dot{q}^n + g_{nm} \ddot{q}^n)$ Equate it to $\dot{p}_m$ in **2nd** $L$-equation:
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity $q^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

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$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (E_n \dot{q}^n) (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$

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\dot{p}_r = \frac{\partial L}{\partial r} = M \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}
$$

Centrifugal force $Mr \omega^2$

$$
\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}
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Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from **1st** $L$-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$ Equate it to $\dot{p}_m$ in **2nd** $L$-equation:

(From preceding page)
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Equate it to $\dot{p}_m$ in 2nd $L$-equation:

$$\dot{p}_r = \frac{dp_r}{dt} = M \dot{r}$$

Centrifugal (center-fleeing) force

equals total

Centripetal (center-pulling) force
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** *velocity*

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Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from *1st* $L$-equation:

\[\dot{p}_r \equiv \frac{dp_r}{dt} = M \dot{r}\]

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

\[\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2 M r \dot{\phi} + M r^2 \ddot{\phi}\]

Torque relates to two distinct parts: Coriolis and angular acceleration

\[\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 0 - \frac{\partial U}{\partial \phi}\]

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence
Rewriting GCC Lagrange equations:

\[ \dot{p}_r = \frac{dp_r}{dt} = M \ddot{r} \]

Centrifugal (center-fleeing) force equals total
Centripetal (center-pulling) force

\[ = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r} \]

Conventional forms
radial force: \[ M \ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r} \]
Field-free (U=0)
radial acceleration: \[ \ddot{r} = r\dot{\phi}^2 \]

\[ \dot{p}_\phi = \frac{dp_\phi}{dt} = 2Mr\dot{\phi} + Mr^2\ddot{\phi} \]
Torque relates to two distinct parts:
Coriolis and angular acceleration
\[ = 0 - \frac{\partial U}{\partial \phi} \]
Angular momentum \(p_\phi\) is conserved if
potential \(U\) has no explicit \(\phi\)-dependence

\[ \ddot{\phi} = 0 - \frac{r}{r} \]

Coriolis acceleration with \(\dot{\phi} > 0\) and \(\ddot{r} < 0\)
\[ \ddot{\phi} = -2 \frac{\dot{r}\dot{\phi}}{r} \]
(makes \(\ddot{\phi}\) positive)

\[ \ddot{r} < 0 \]
Inward flow to pressure Low
...makes wind turn to the right

Effect on Northern Hemisphere local weather
Cyclonic flow around lows

\(L\)
Northern hemisphere rotation
\(\ddot{\phi} > 0\)
Rewriting GCC Lagrange equations:

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Centrifugal (center-fleeing) force equals total

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Centripetal (center-pulling) force

Field-free (U=0)

\[ \ddot{r} = r \dot{\phi}^2 \]

Effect on Northern Hemisphere local weather

\[ \text{Cool North winds follow storms} \]

\[ \text{Warm South winds precede storms} \]

\[ \text{Inward flow to pressure Low} \]

\[ \text{...makes wind turn to the right} \]

\[ \text{Coriolis acceleration with} \ \dot{\phi} > 0 \ \text{and} \ \dot{r} < 0 \]

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\[ \text{Effect on} \]

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Torque relates to two distinct parts:

- Coriolis and angular acceleration
- Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence

\[ = 0 - \frac{\partial U}{\partial \phi} \]

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Effect on Northern Hemisphere local weather

- Cyclonic flow around lows
- Warm South winds precede storms
- Cool North winds follow storms

Deep quantum rule:
Flow tries to mimic the external rotation (least relative \( v \))