Lecture 7
Tue. 9.13.2016

Kepler Geometry of IHO (Isotropic Harmonic Oscillator) Elliptical Orbits

(Ch. 9 and Ch. 11 of Unit 1)

Constructing 2D IHO orbital phasor "clock" dynamics in uniform-body

Constructing 2D IHO orbits using Kepler anomaly plots
- *Mean-anomaly and eccentric-anomaly geometry*
- *Calculus and vector geometry of IHO orbits*
- *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better in Ch. 12)*

Some Kepler's "laws" for all central (isotropic) force F(r) fields
- *Angular momentum invariance of IHO:* $F(r)=-k\cdot r$ *with* $U(r)=k\cdot r^2/2$ *(Derived here)*
- *Angular momentum invariance of Coulomb:* $F(r)=-GMm/r^2$ *with* $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*
- *Total energy E=KE+PE invariance of IHO:* $F(r)=-k\cdot r$ *(Derived here)*
- *Total energy E=KE+PE invariance of Coulomb:* $F(r)=-GMm/r^2$ *(Derived in Unit 5)*

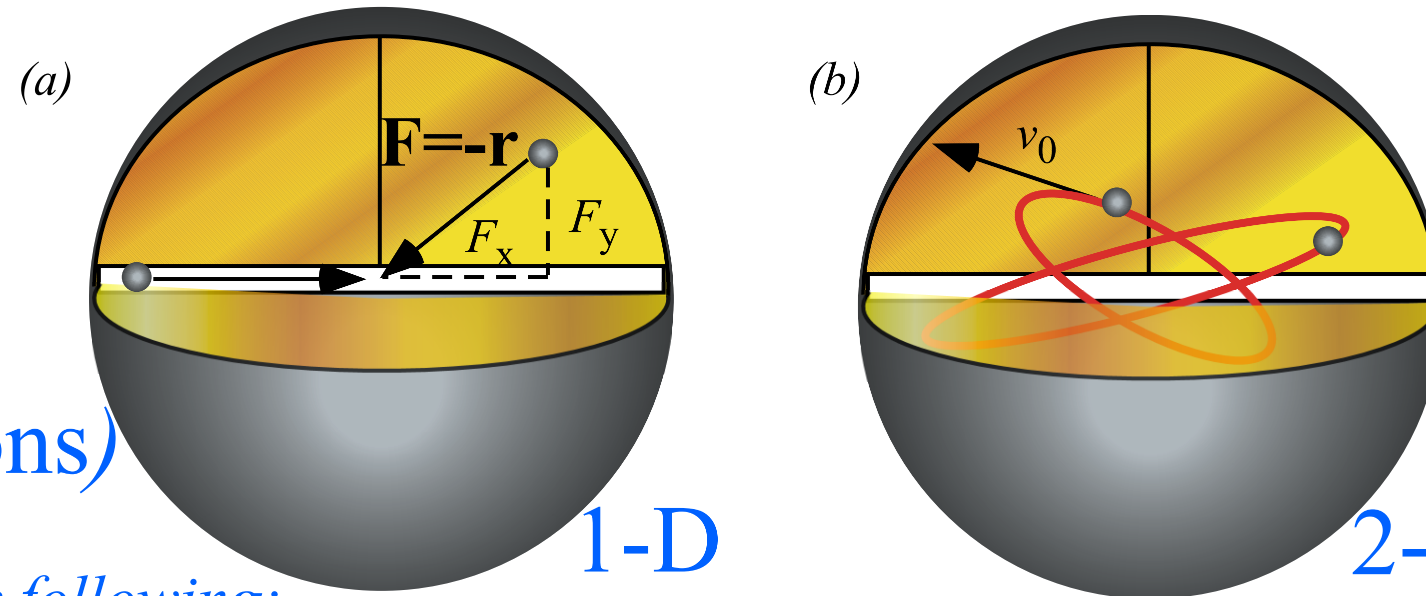
Introduction to dual matrix operator contact geometry (based on IHO orbits)
- *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$
  - *Duality norm relations (* $\mathbf{r}\bullet\mathbf{p}=1$*)*
  - *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*

Q:Where is this headed? A: Lagrangian-Hamiltonian duality

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

→ *Review of IHO orbital phasor "clock" dynamics in uniform-body with two "movie" examples*

Review of IHO orbital phase dynamics in uniform-body



Unit 1
Fig. 9.10

I.H.O. Force law

$F = -x$ (1-Dimension)

$\mathbf{F} = -\mathbf{r}$ (2 or 3-Dimensions)

1-D

2-D or 3-D

(Paths are *always* 2-D ellipses if viewed right!)

Each dimension x, y, or z obeys the following:

$$Total\ E = KE + PE = \frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = const.$$

Equations for x-motion [x(t) and v_x=v(t)] are given first. They apply as well to dimensions [y(t) and v_y=v(t)] and [z(t) and v_z=v(t)] in the ideal isotropic case.

$$1 = \frac{mv^2}{2E} + \frac{kx^2}{2E} = \left(\frac{v}{\sqrt{2E/m}}\right)^2 + \left(\frac{x}{\sqrt{2E/k}}\right)^2$$

$$1 = \frac{mv^2}{2E} + \frac{kx^2}{2E} = (\cos\theta)^2 + (\sin\theta)^2$$

Another example of the old "scale-a-circle" trick...

velocity: *Let :* **(1)** $v = \sqrt{2E/m}\cos\theta,$ *and :* *position:* **(2)** $x = \sqrt{2E/k}\sin\theta$

angular velocity: *def.* **(3)** $\omega = \frac{d\theta}{dt}$

velocity:

$$\sqrt{\frac{2E}{m}}\cos\theta = v = \frac{dx}{dt} = \frac{d\theta}{dt}\frac{dx}{d\theta} = \omega\frac{dx}{d\theta} = \omega\sqrt{\frac{2E}{k}}\cos\theta$$

by **(1)** — *by def.* **(3)** — *by* **(2)**

by def. **(3)**

$$\omega = \frac{d\theta}{dt} = \sqrt{\frac{k}{m}}$$

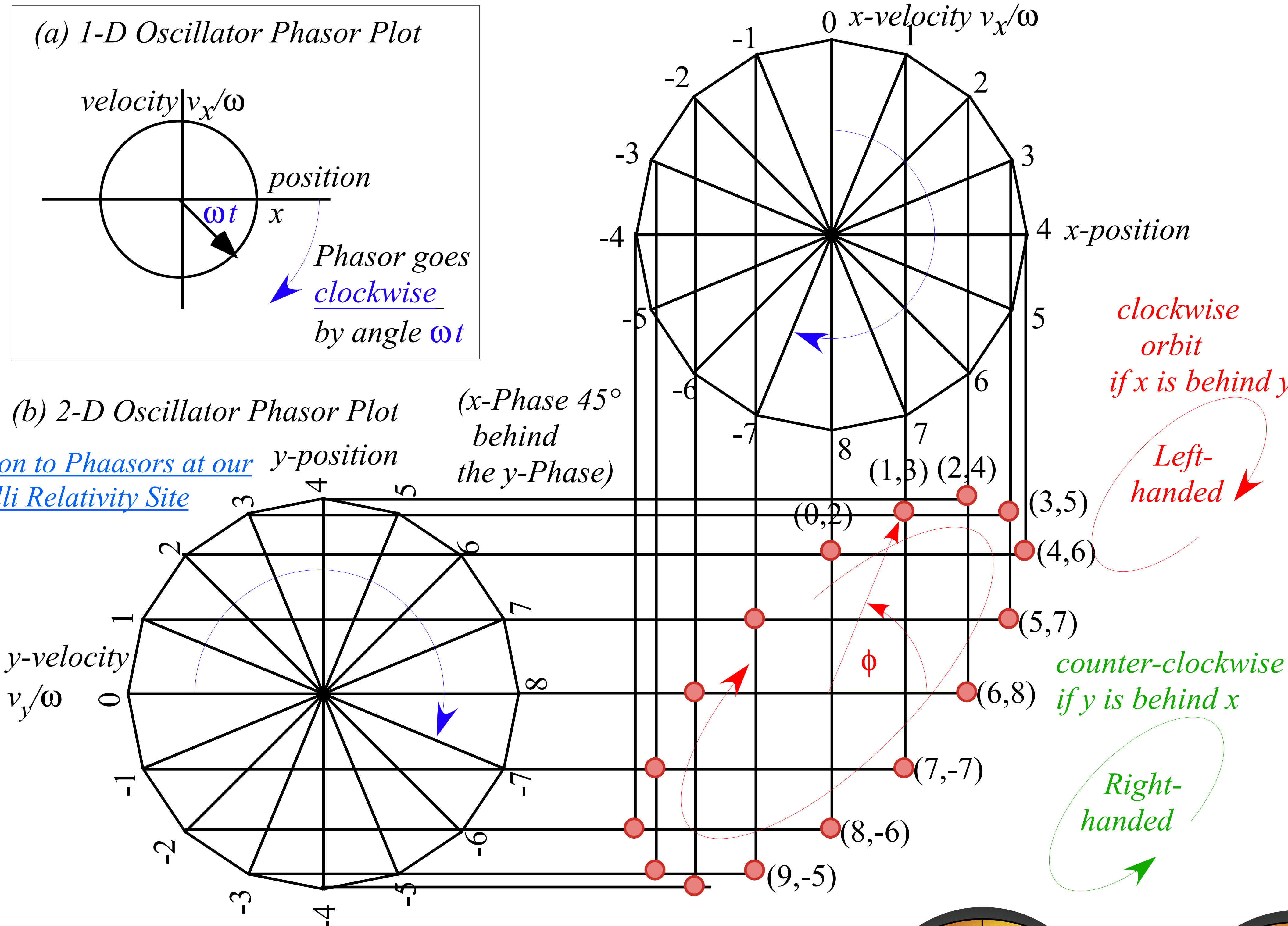
divide this by **(1)**

by integration given constant ω:

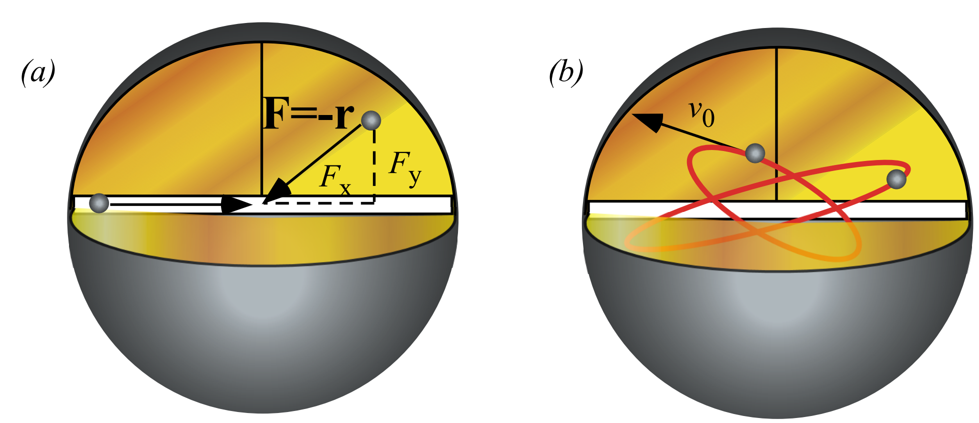
$$\theta = \int \omega \cdot dt = \omega \cdot t + \alpha$$

Review of IHO orbital phase dynamics in uniform-body

Unit 1
Fig. 9.10



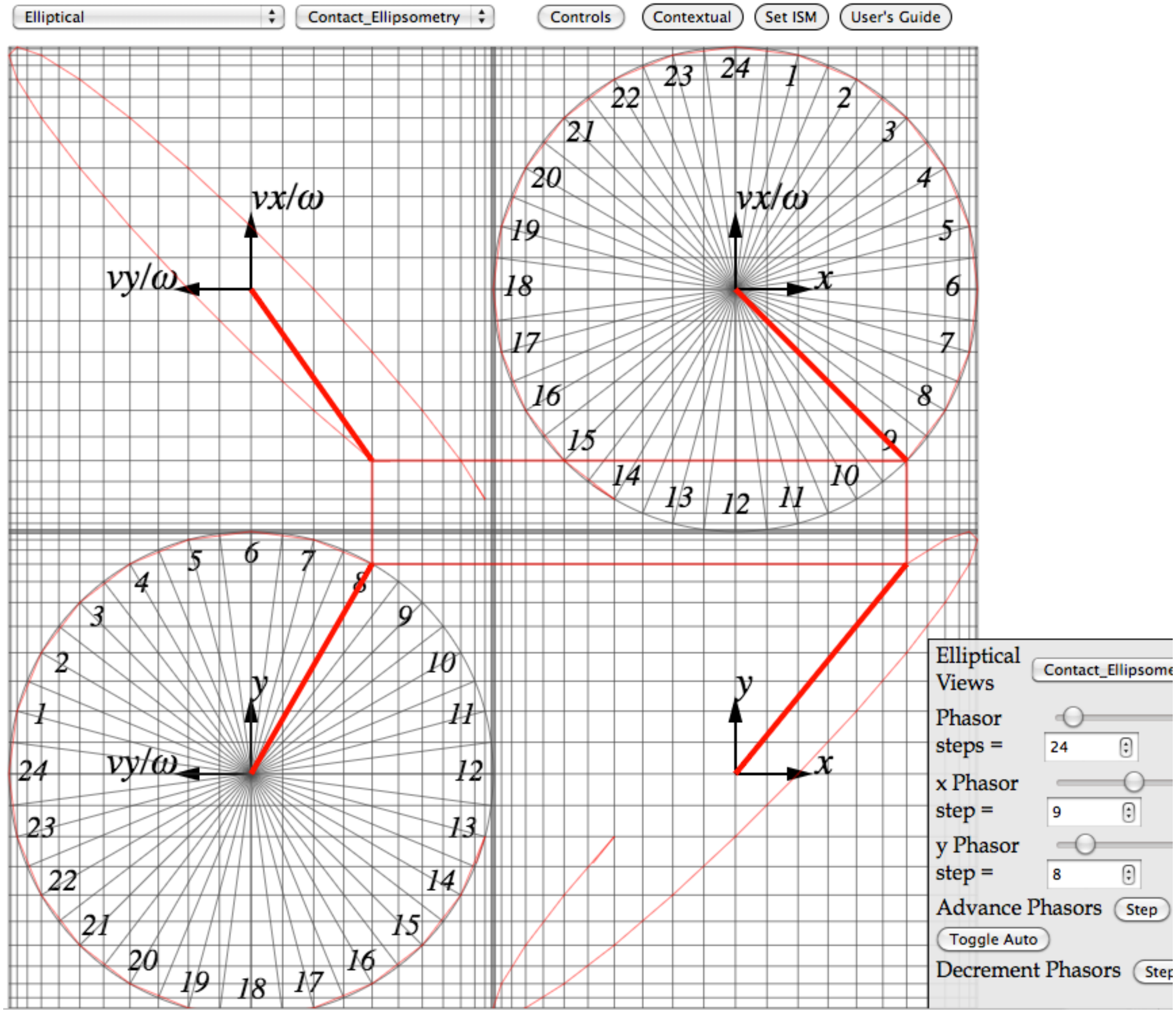
[Introduction to Phaasors at our Pirelli Relativity Site]



[BoxIt web simulation - *With y-Phasor is on other side of xy plot*]

[RelaWavity web simulation - Contact ellipsometry]

RelaWavity ellipsometry web-app

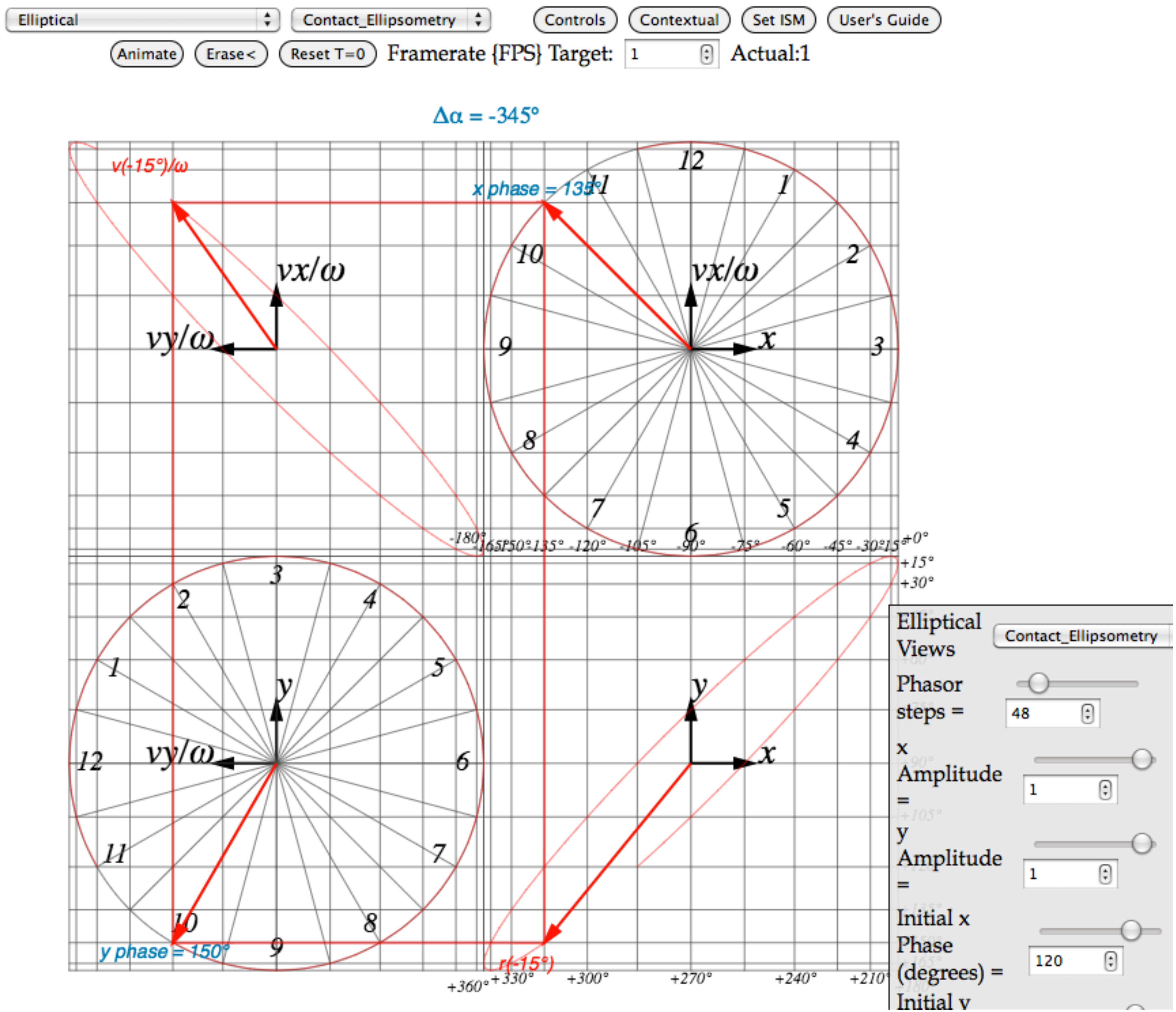


[RelaWavity Web Simulation *Ellipsometry*]

Geometry of Kepler anomalies for vectors [$\mathbf{r}(\phi)$, $\mathbf{v}(\phi)$] *in coordinate (x,y) space rendered by animation web-apps BoxIt and RelaWavity described below after p.70.*

http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?phaserSteps=24&use12Steps=0&plotType=1l2

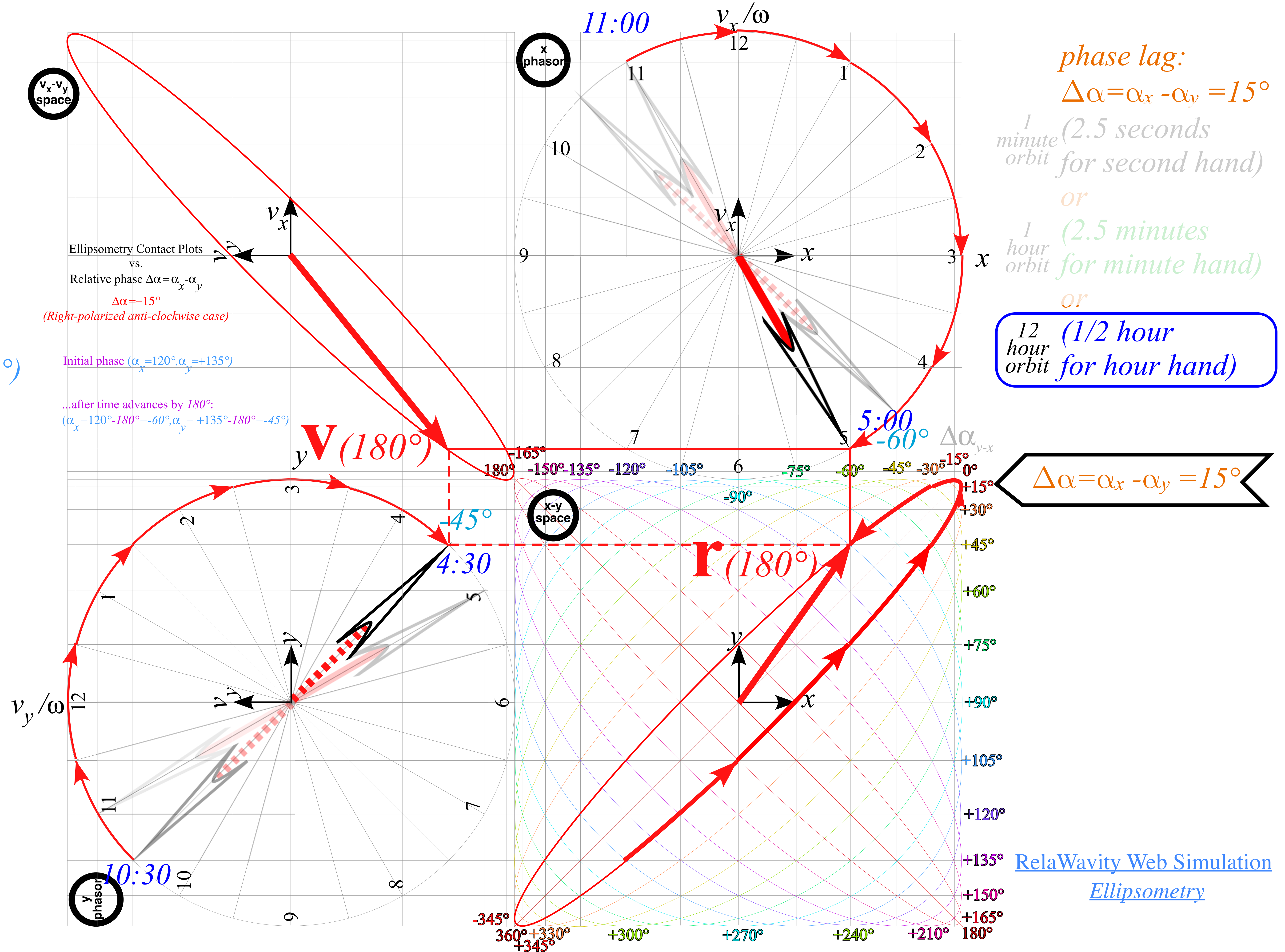
RelaWavity ellipsometry web-app



[RelaWavity Web Simulation *Ellipsometry*](http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?plotType=1l2)

Geometry of Kepler anomalies for vectors [$\mathbf{r}(\phi)$, $\mathbf{v}(\phi)$] *in coordinate (x,y) space rendered by animation web-apps BoxIt and RelaWavity described below after p.7 and p.17.*

http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?plotType=1l2



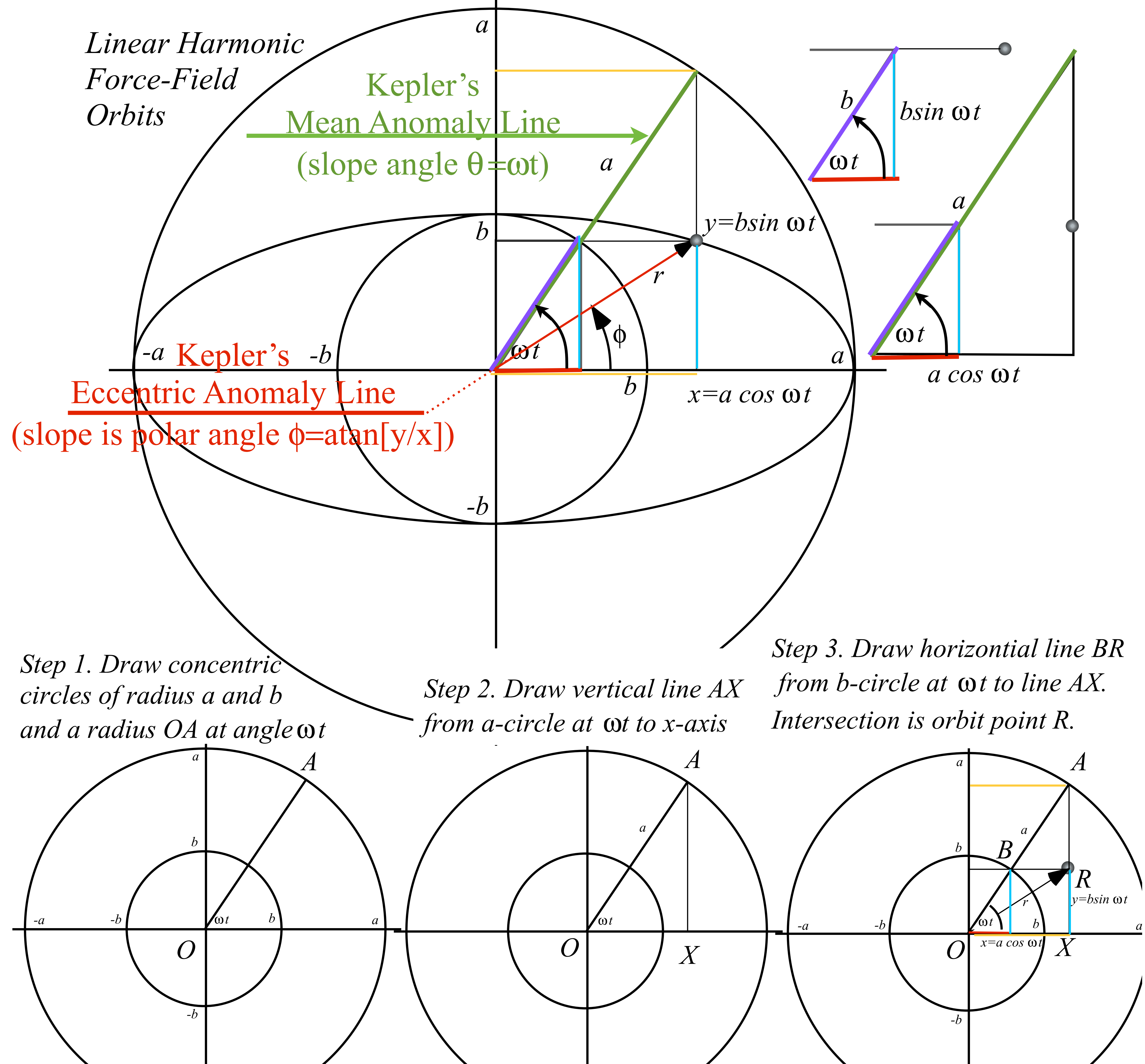
Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=−15°
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 180°:
(α_x=120°-180°=-60°,α_y = +135°-180°=-45°)
phase lag:
Δα=αx -αy =15°
1 minute orbit
(2.5 seconds for second hand)
or
1 hour orbit
(2.5 minutes for minute hand)
or
12 hour orbit
(1/2 hour for hour hand)
Δα=αx -αy =15°
vx-vy space
x phasor
y phasor
x-y space
vx/ω
vy/ω
11:00
5:00
4:30
10:30
V(180°)
r(180°)
-60°
-45°
Δαy-x
RelaWavity Web Simulation
Ellipsometry

Constructing 2D IHO orbits using Kepler anomaly plots

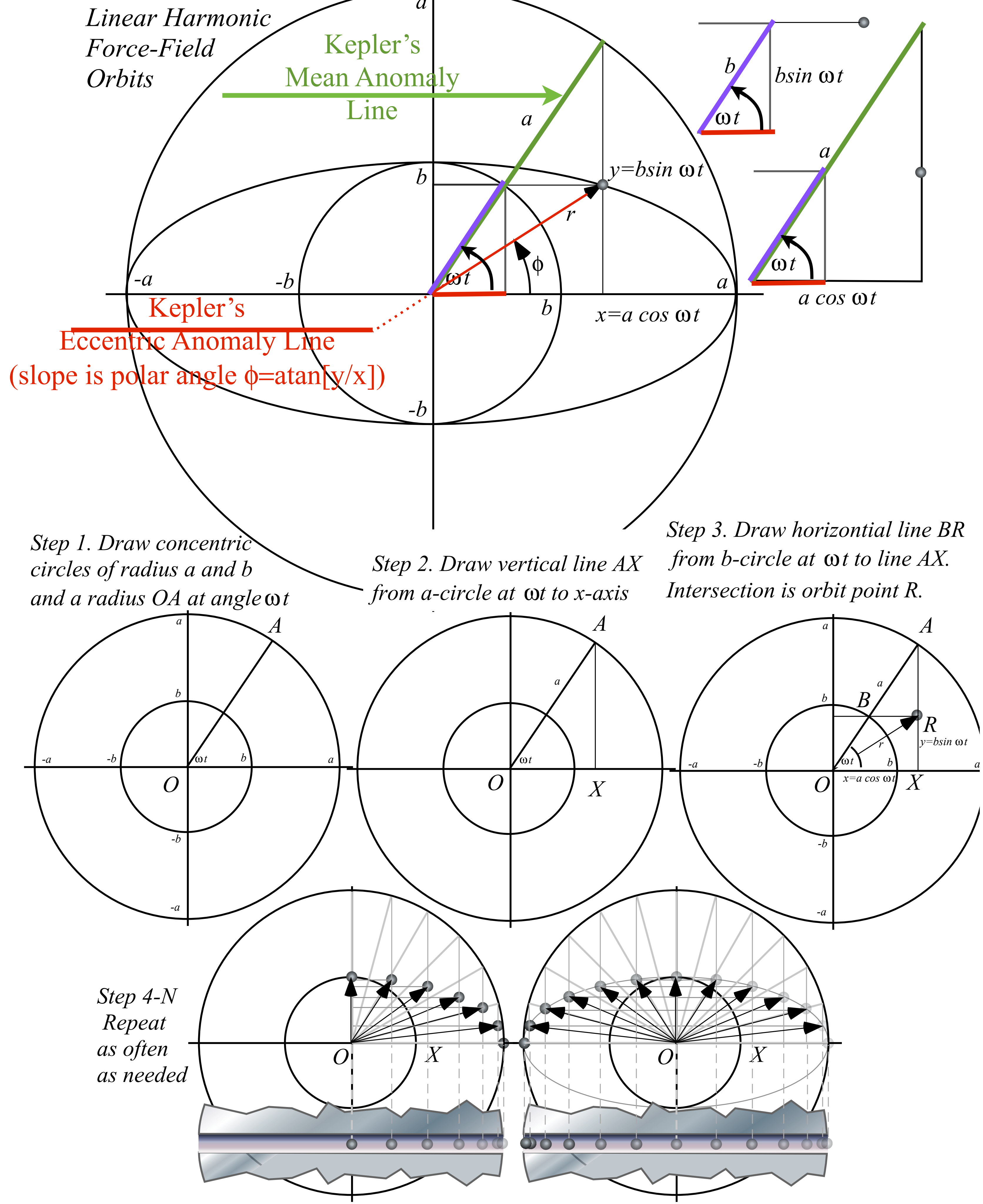
→ *Mean-anomaly and eccentric-anomaly geometry*

Calculus and vector geometry of IHO orbits

A confusing introduction to Coriolis-centrifugal force geometry *(Derived better in Ch. 12)*



Unit 1
Fig. 11.1
(top 2/3's)



Unit 1
Fig. 11.1

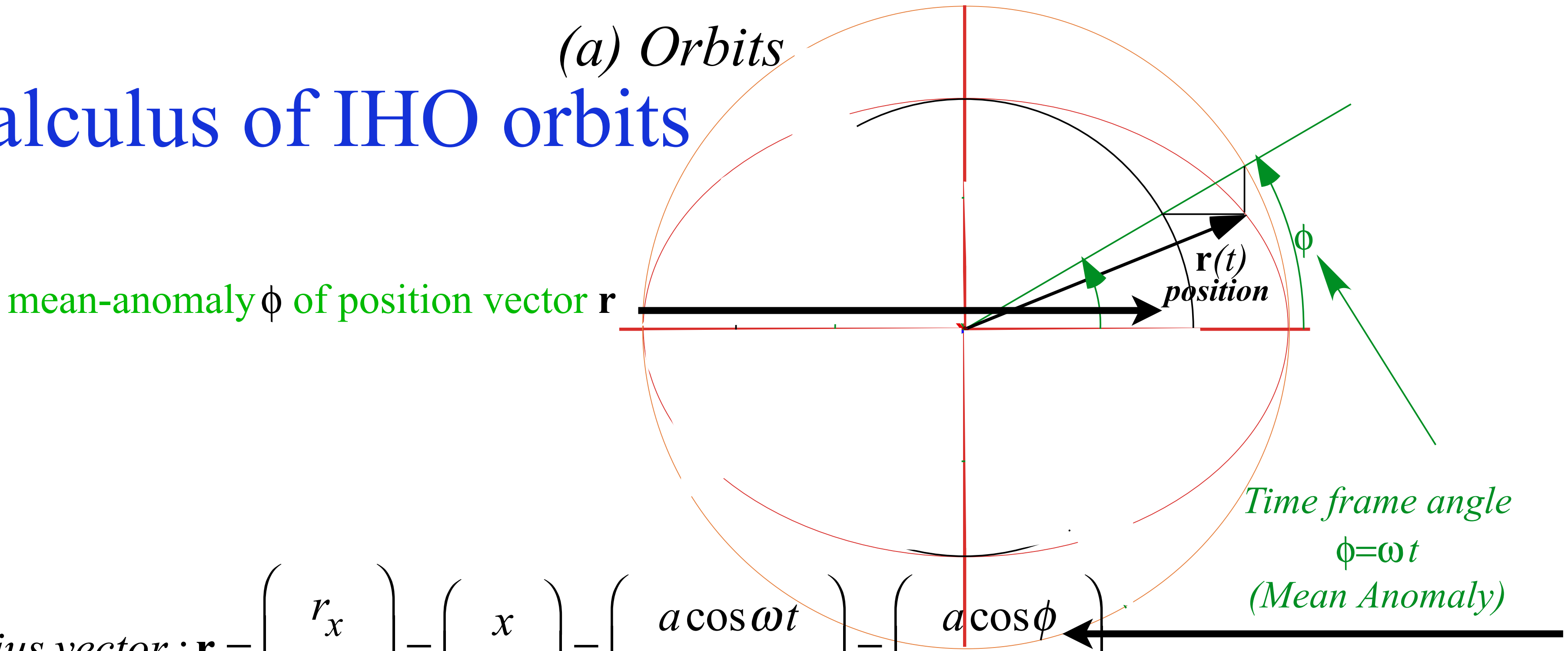
Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry

→ *Calculus and vector geometry of IHO orbits*

A confusing introduction to Coriolis-centrifugal force geometry *(Derived better in Ch. 12)*

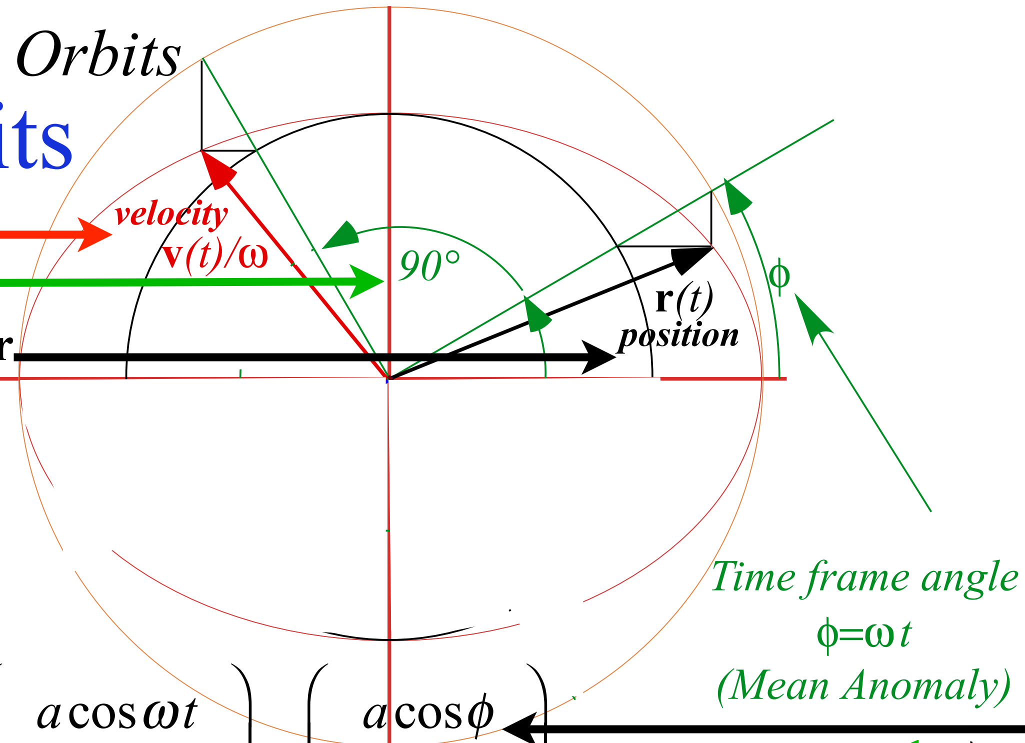
Calculus of IHO orbits



$$radius\ vector: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

Unit 1
Fig. 11.5

Calculus of IHO orbits



To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**

Unit 1
Fig. 11.5

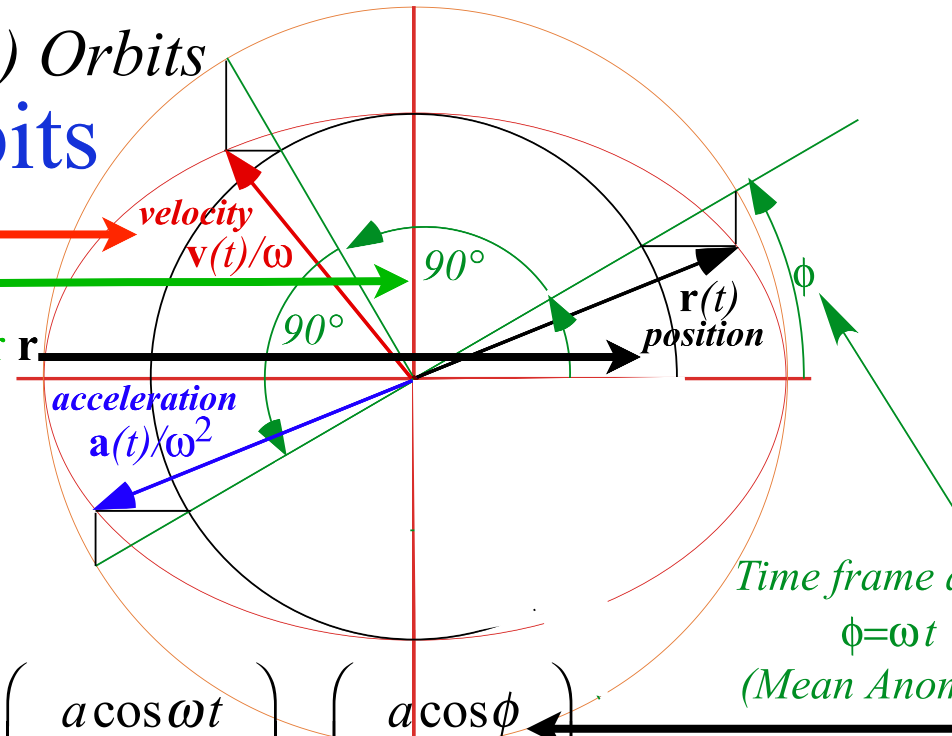
$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

Calculus of IHO orbits

To make velocity vector **v** just rotate by π/2 or 90° the mean-anomaly ϕ of position vector **r**



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Unit 1
Fig. 11.5

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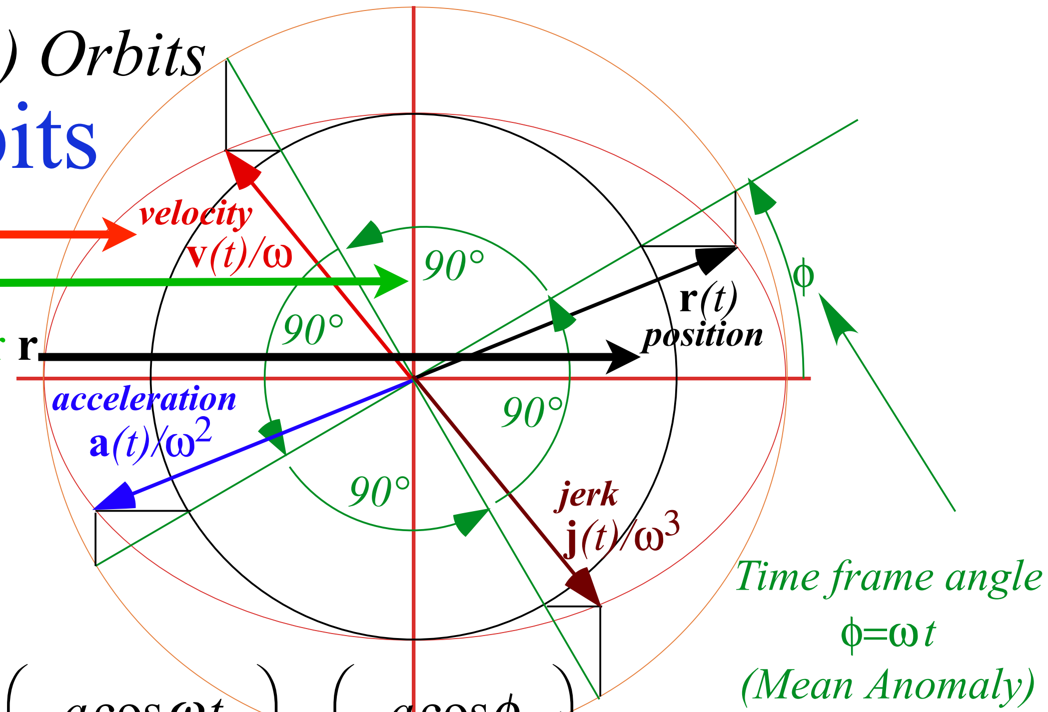
m.a. ϕ+π/2 of vector **v** rotated by another π/2 is *m.a.* of vector **a**

$$\text{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

Calculus of IHO orbits

(a) Orbits



To make velocity vector **v**
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Unit 1
Fig. 11.5

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or change of velocity

$$\text{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi + \frac{3\pi}{2}\right) \\ b\sin\left(\phi + \frac{3\pi}{2}\right) \end{pmatrix}$$

...and so forth...

Calculus of IHO orbits

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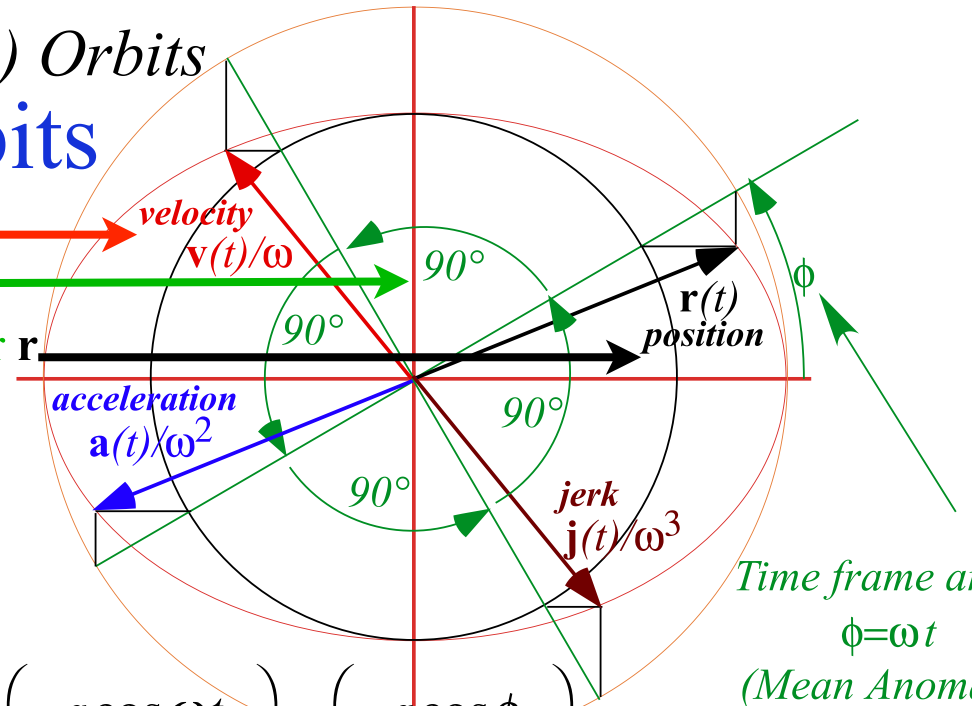

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Unit 1
Fig. 11.5

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m.a. ϕ+π/2 of vector **v** rotated by
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...and so forth...

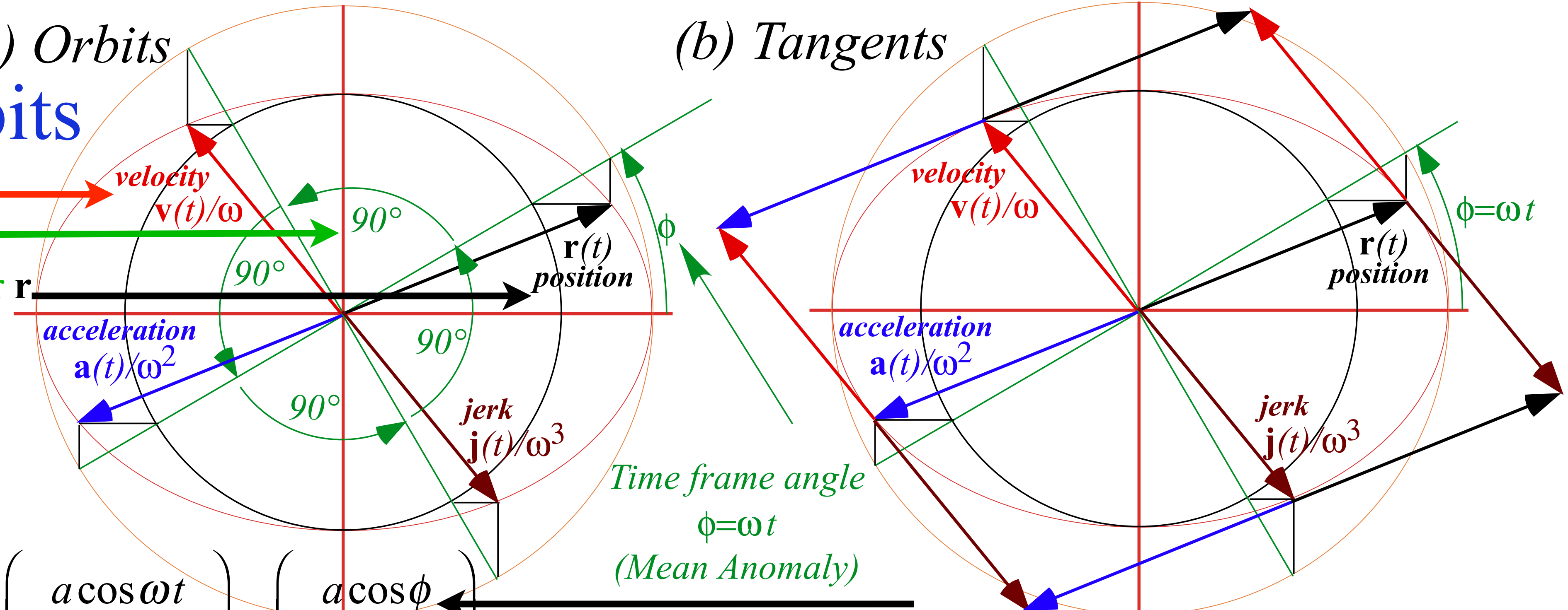
$$\textit{inauguration or change of jerk}: \mathbf{i} = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix} = \frac{d\mathbf{j}}{dt} = \dot{\mathbf{j}} = \ddot{\mathbf{a}} = \dddot{\mathbf{v}} = \ddddot{\mathbf{r}} = \frac{d^4\mathbf{r}}{dt^4} = \begin{pmatrix} a\cos\left(\phi+\frac{4\pi}{2}\right) \\ b\sin\left(\phi+\frac{4\pi}{2}\right) \end{pmatrix}$$

...and so on...
...But, now it repeats after 4 *t*-derivatives

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**

Link ⇒ BoxIt simulation of IHO orbits



mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

Link → IHO Exegesis Plot

Link → IHO orbital time rates of change

$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

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m.a. φ+π/2 of vector **v** rotated by
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or change of velocity

...and so forth...

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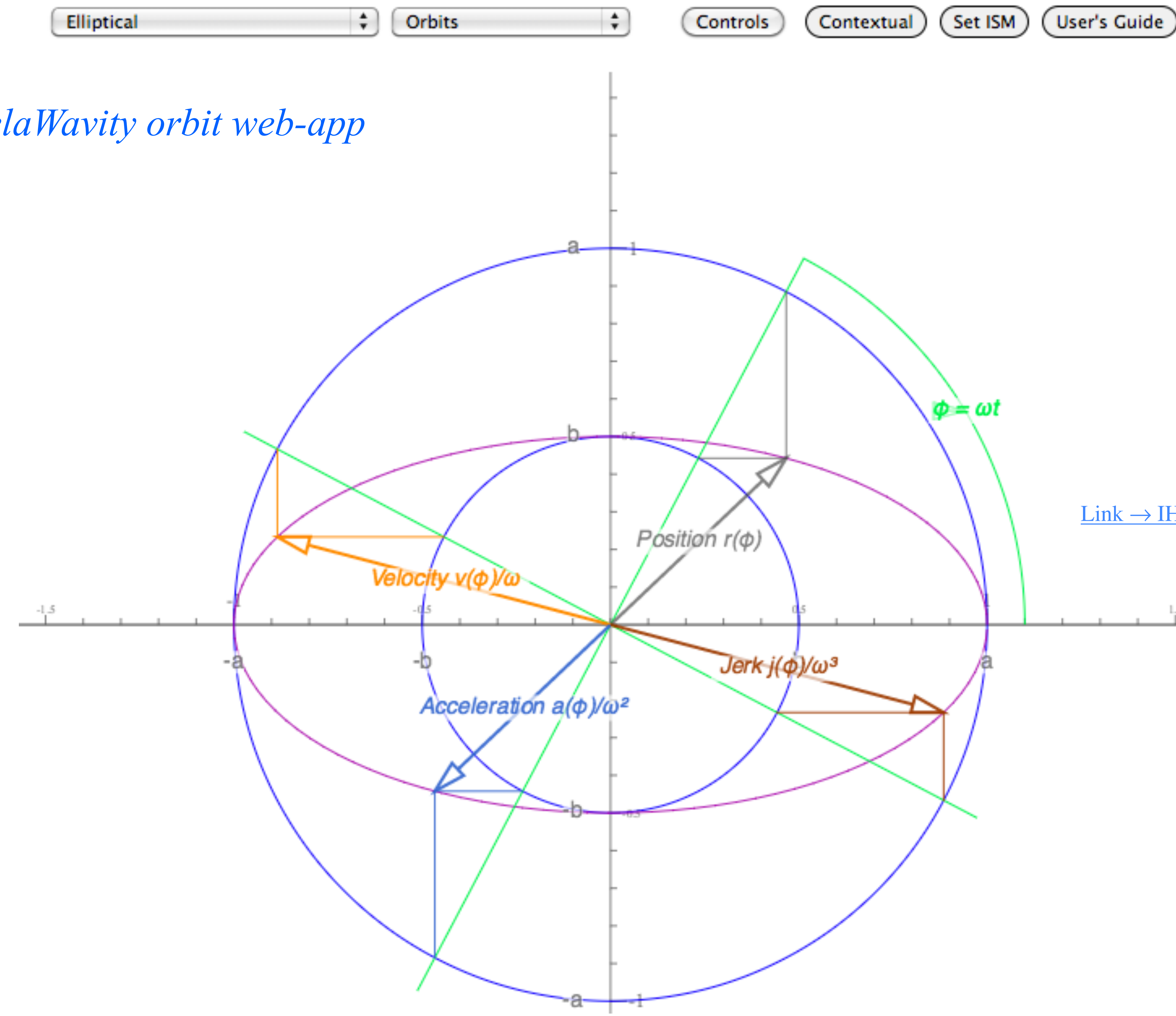
$$\textit{inauguration or change of jerk}: \mathbf{i} = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix} = \frac{d\mathbf{j}}{dt} = \dot{\mathbf{j}} = \ddot{\mathbf{a}} = \dddot{\mathbf{v}} = \ddddot{\mathbf{r}} = \frac{d^4\mathbf{r}}{dt^4} = \begin{pmatrix} a\cos\left(\phi + \frac{4\pi}{2}\right) \\ b\sin\left(\phi + \frac{4\pi}{2}\right) \end{pmatrix}$$

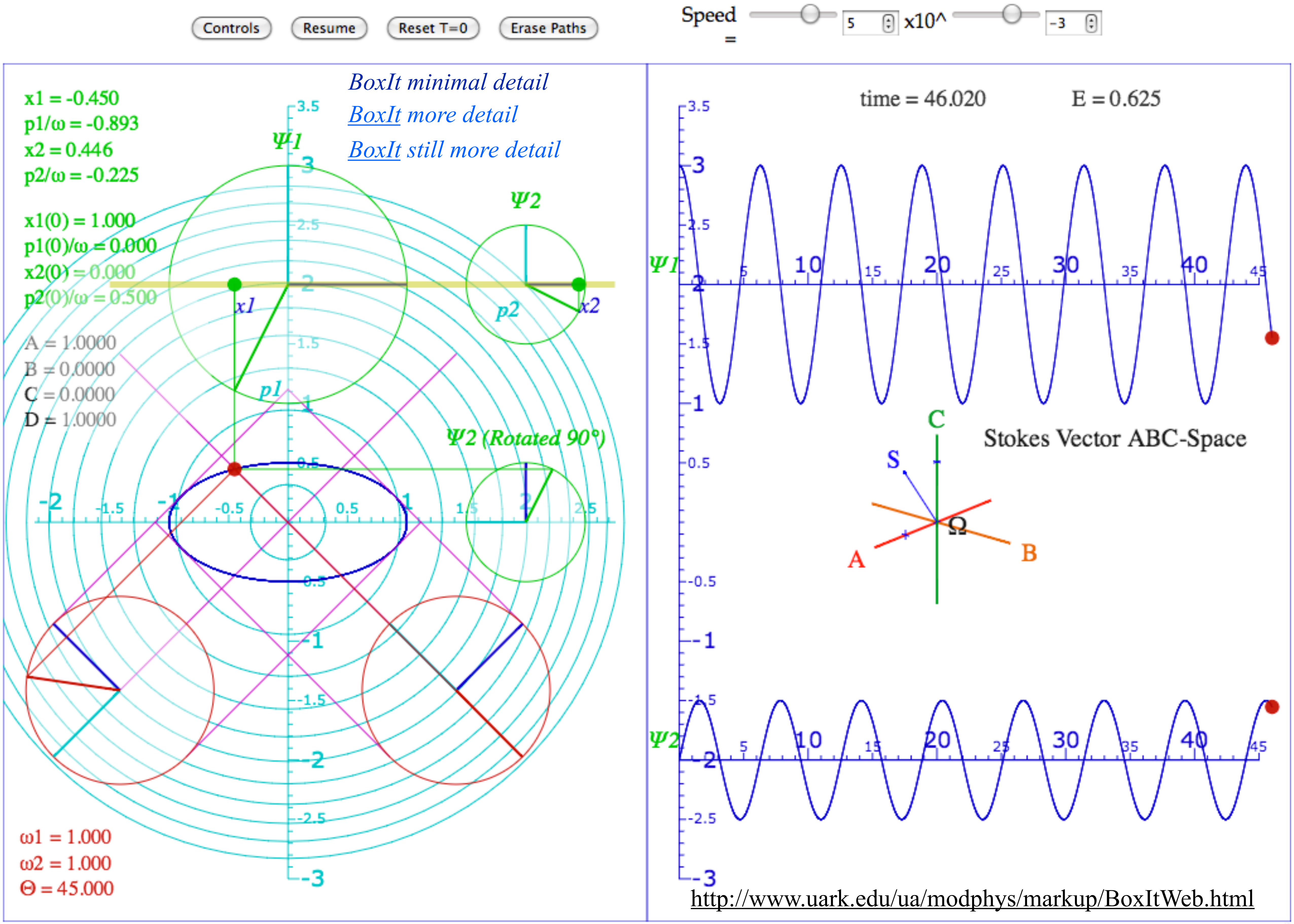
...and so on...
...But, now it repeats after 4 *t*-derivatives

RelaWavity orbit web-app



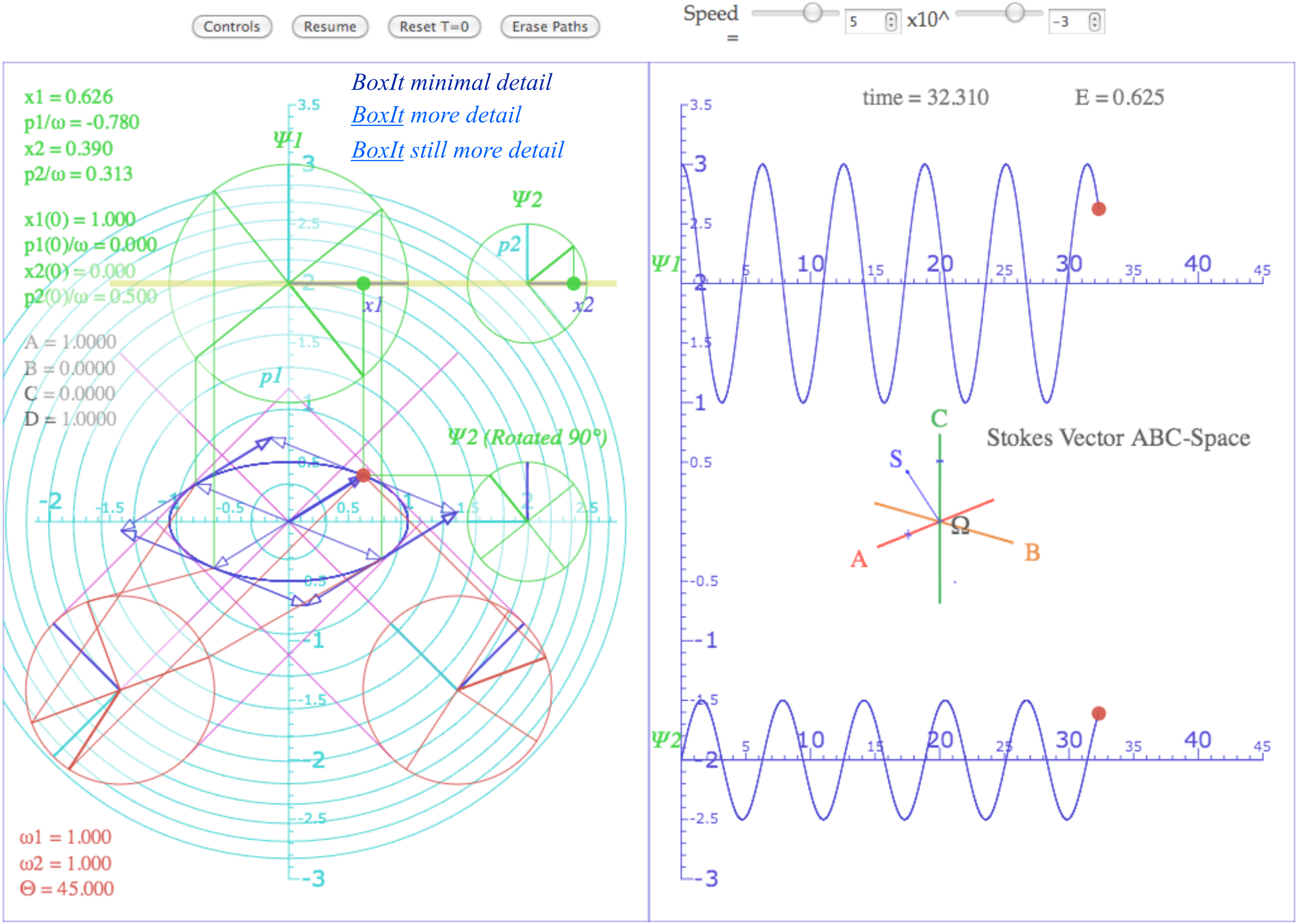
[Link → IHO orbital time rates of change]()

Geometry of Kepler anomalies for vectors [$\mathbf{r}(\phi)$, $\mathbf{v}(\phi)$, $\mathbf{a}(\phi)$, $\mathbf{j}(\phi)$,] in coordinate (x,y) space rendered by animation web-apps BoxIt and RelaWavity.



Geometry of Kepler anomalies for vectors [**r**(ϕ)] *in coordinate (x,y) space and 2-particle* (x_1,x_2) *space rendered by animation web-apps BoxIt.*

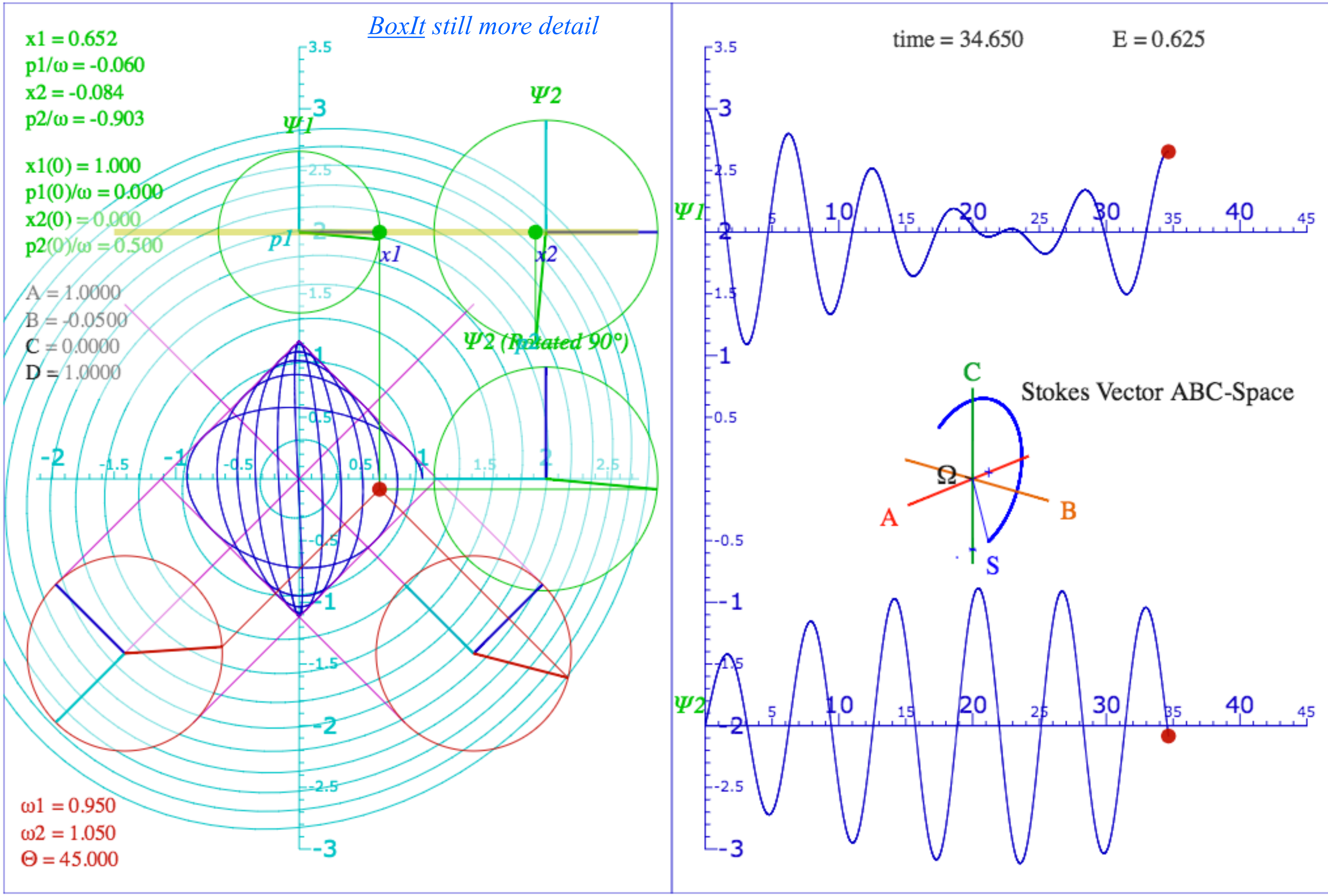
[*BoxIt Web Stokes Simulation*]



*Geometry of Kepler anomalies for vectors [**r**(φ), **v**(φ), **a**(φ), **j**(φ),] in coordinate (x,y) space and 2-particle (x_1,x_2) space rendered by animation web-apps BoxIt.*

[BoxIt Web Simulation - *w/Derivatives*]

BoxIt minimal detail
BoxIt more detail
BoxIt still more detail



Geometry of vectors [**r**(ϕ), **p**(ϕ)] *and quantum spin S-space and 2-particle* (x_1,x_2) *space rendered by animation web-apps BoxIt.*

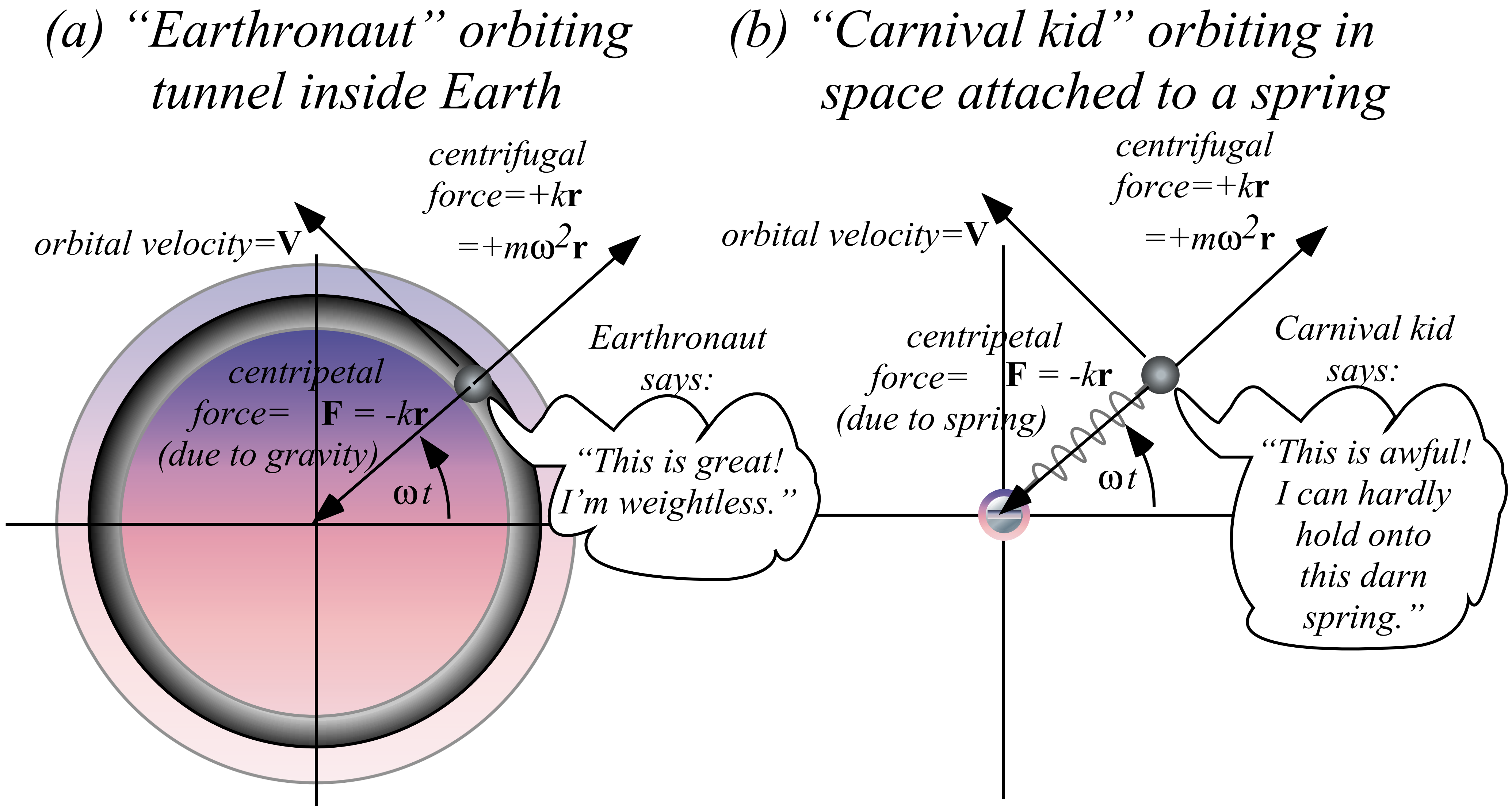
BoxIt Web Simulation - *B-Type Motion*

Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry

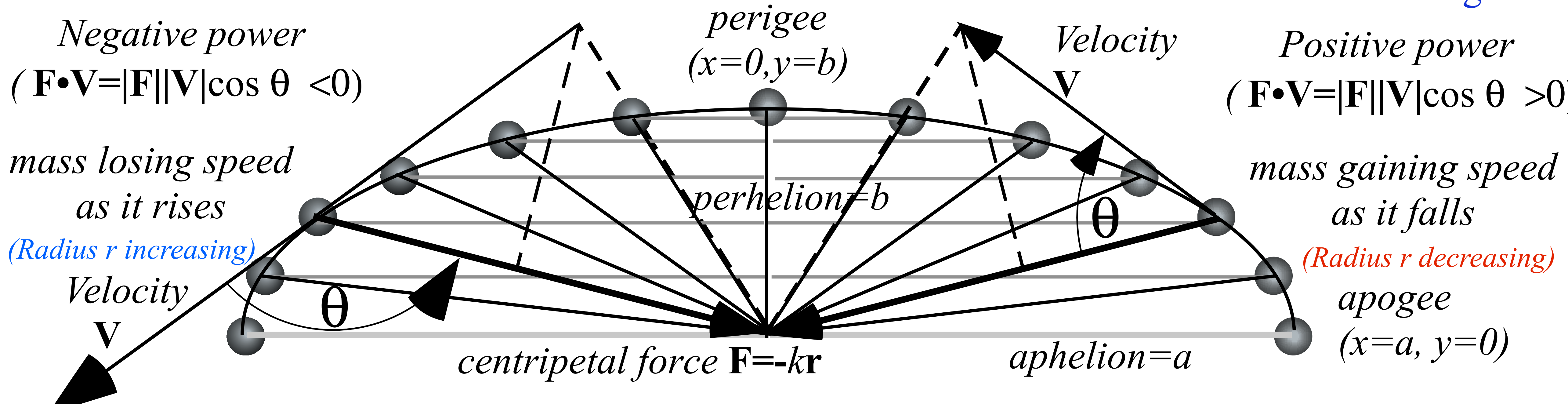
Calculus and vector geometry of IHO orbits

→ *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better in Ch. 12)*

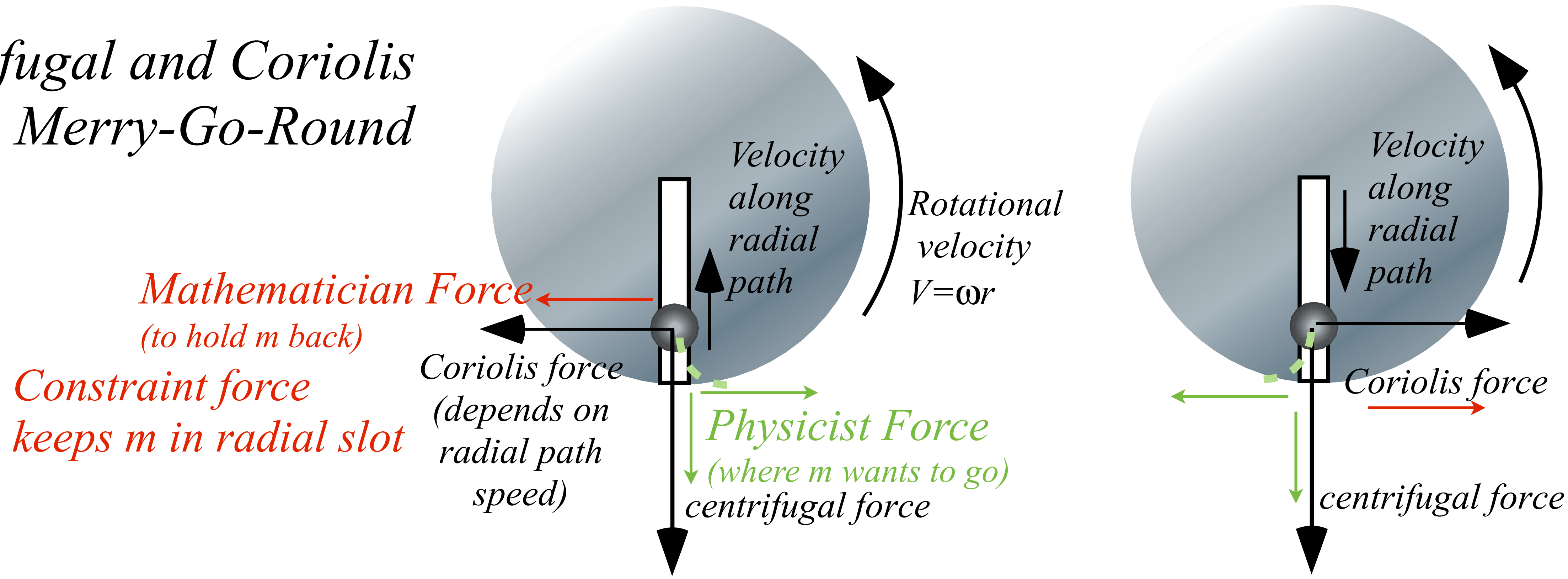


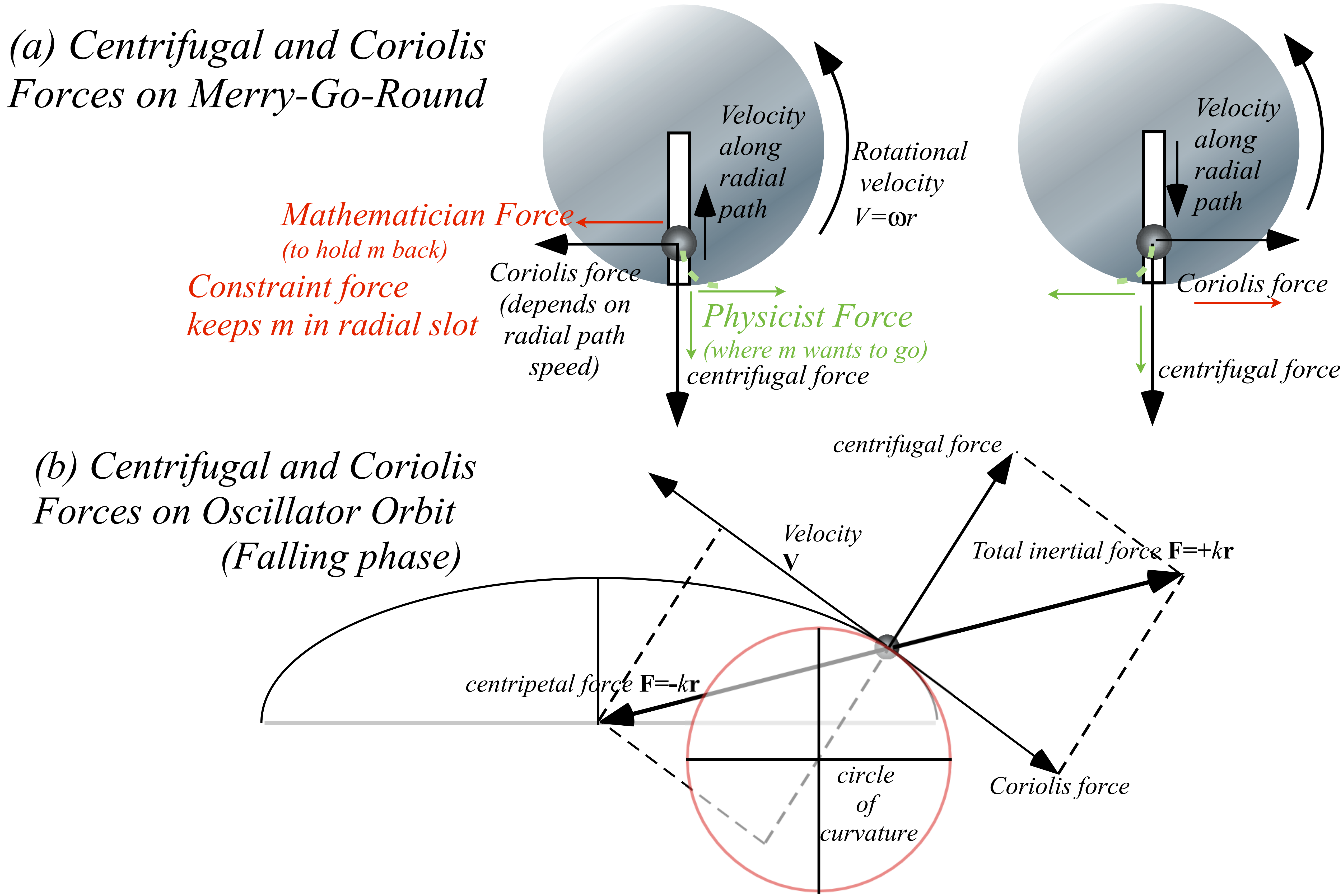
Unit 1
Fig. 11.2

Unit 1
Fig. 11.3

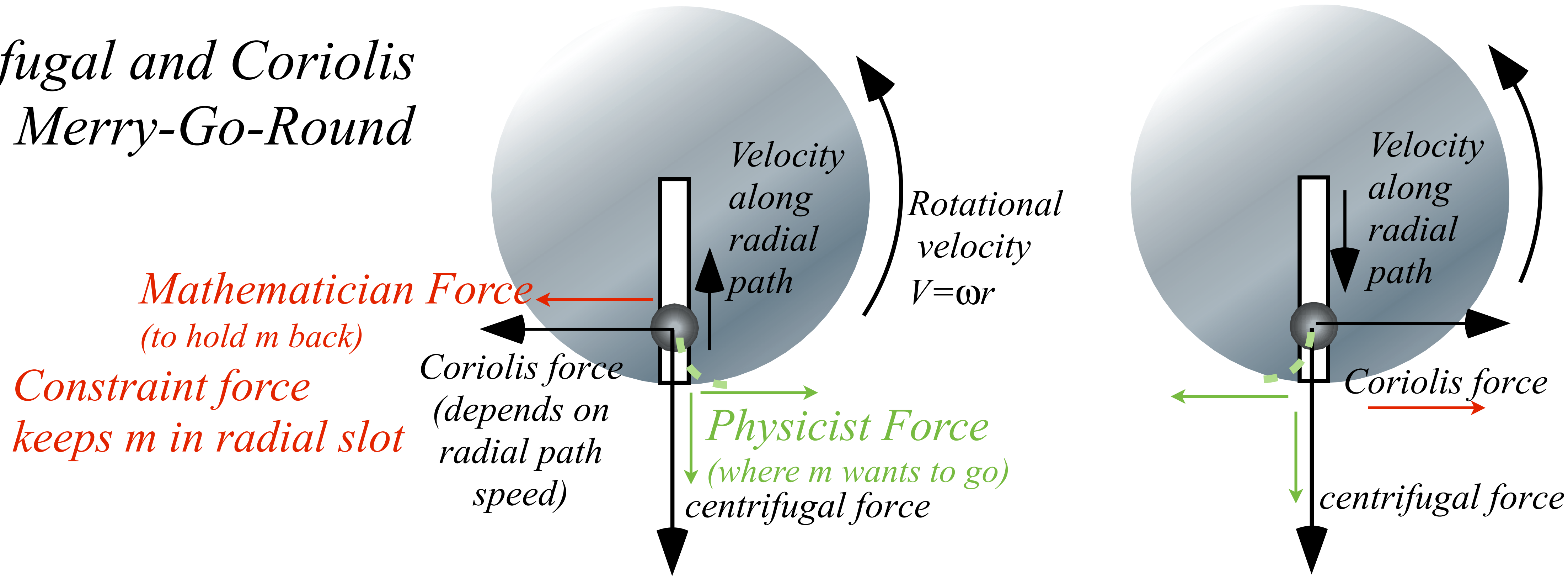


(a) Centrifugal and Coriolis Forces on Merry-Go-Round

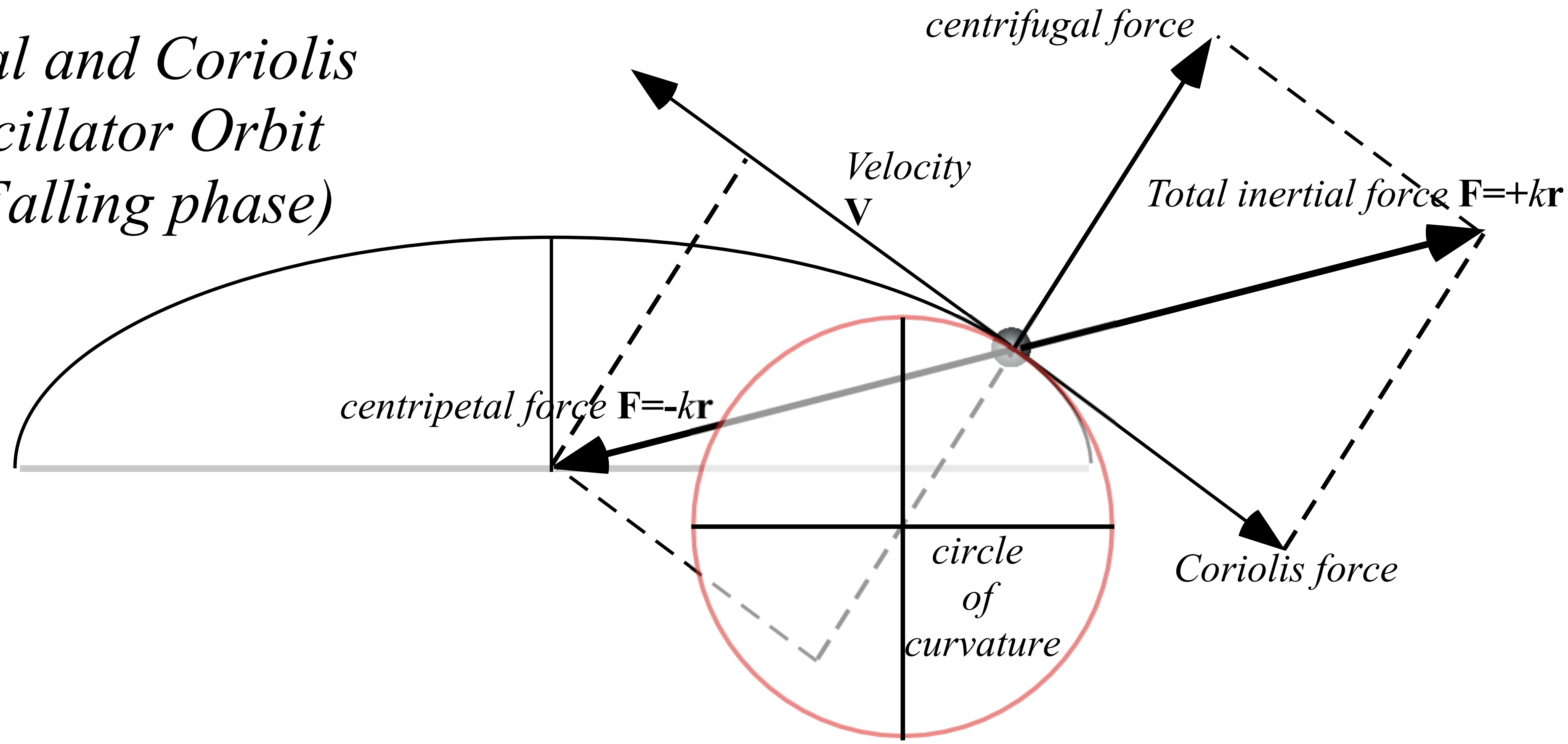



(a) Centrifugal and Coriolis
Forces on Merry-Go-Round
Velocity
along
radial
path
Rotational
velocity
V=ωr
Mathematician Force
(to hold m back)
Constraint force
keeps m in radial slot
Coriolis force
(depends on
radial path
speed)
Physicist Force
(where m wants to go)
centrifugal force
Velocity
along
radial
path
Coriolis force
centrifugal force
(b) Centrifugal and Coriolis
Forces on Oscillator Orbit
(Falling phase)
centrifugal force
Velocity
V
Total inertial force F=+kr
centripetal force F=-kr
circle
of
curvature
Coriolis force

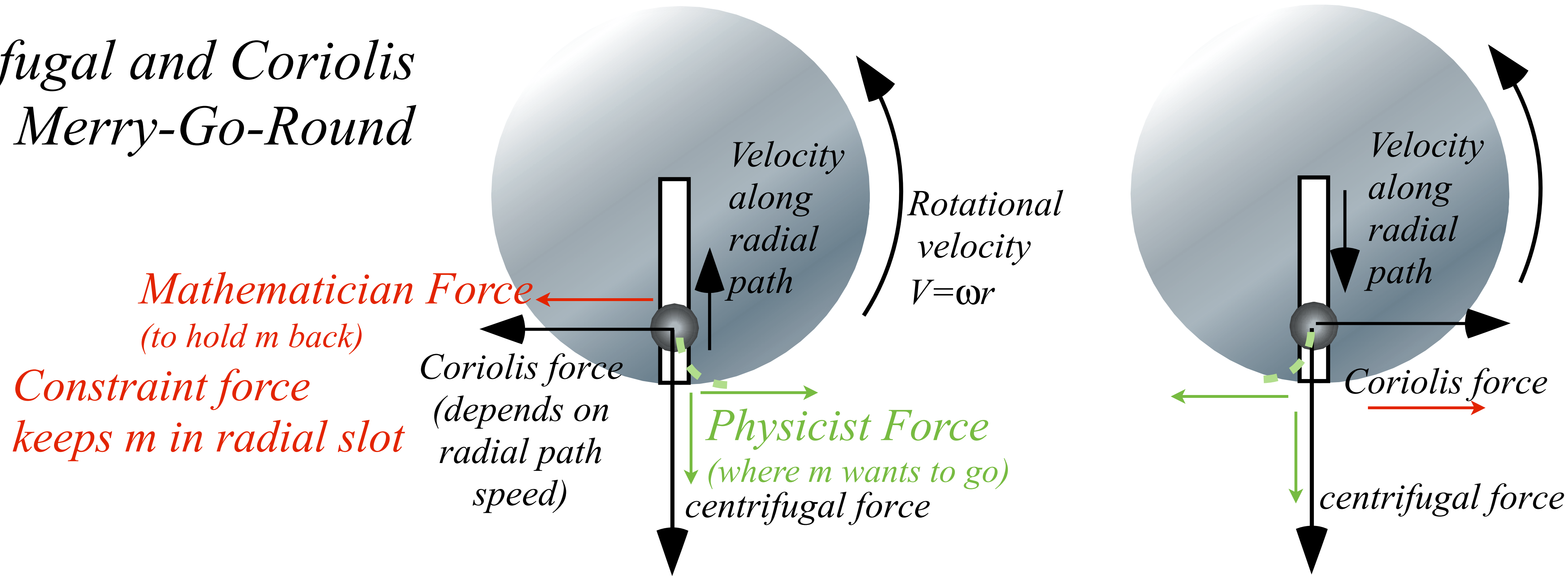
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



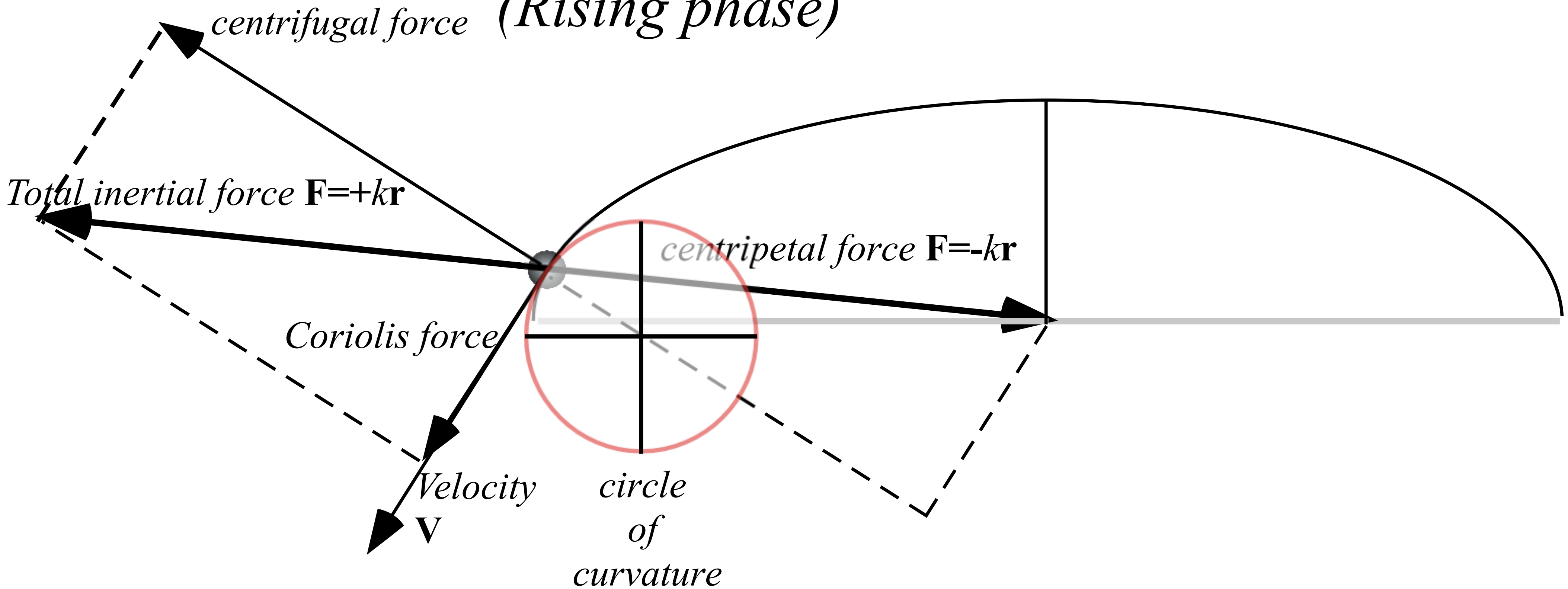
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)

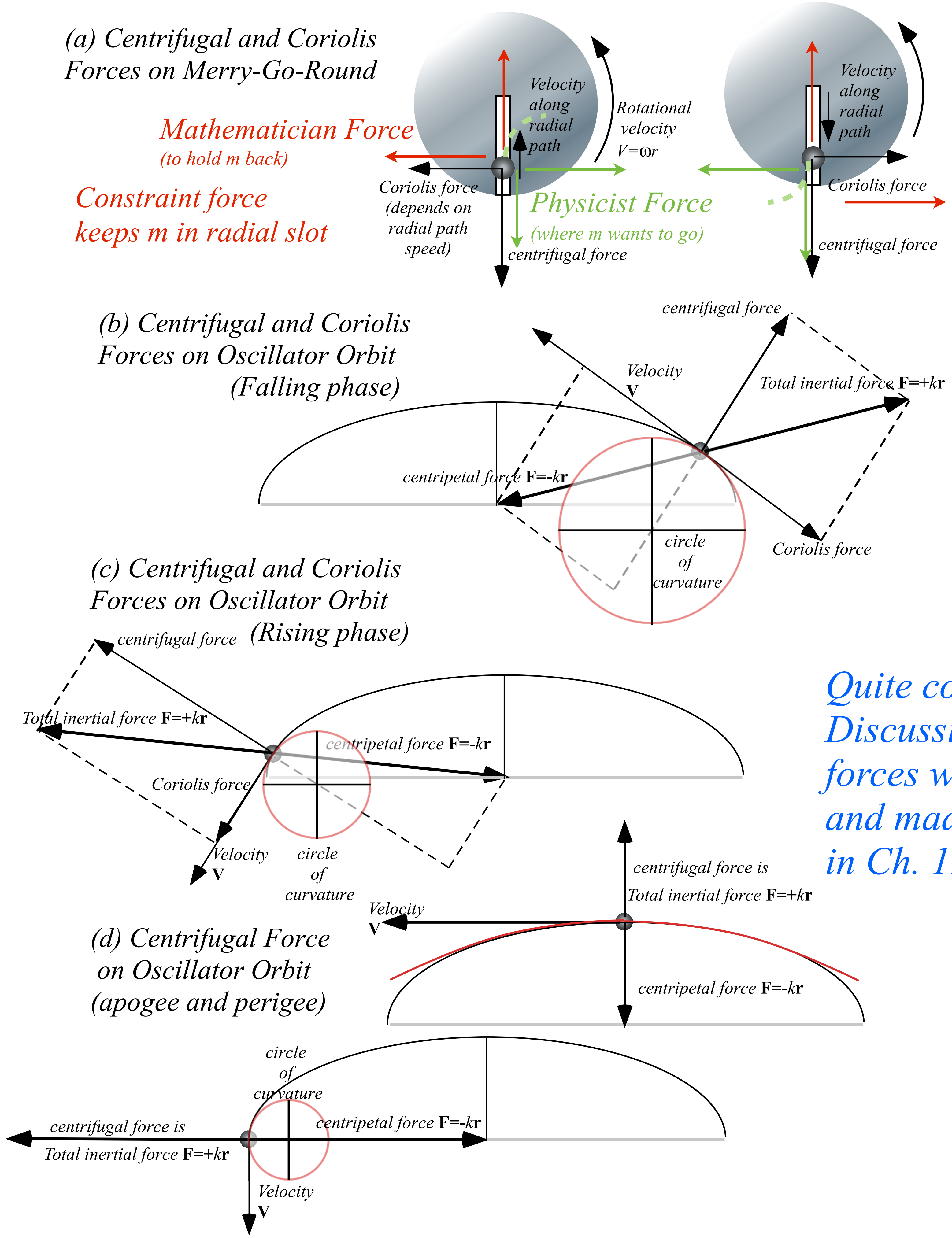


(a) Centrifugal and Coriolis Forces on Merry-Go-Round



(c) Centrifugal and Coriolis Forces on Oscillator Orbit (Rising phase)





Unit 1
Fig. 11.4
a-d

Quite confusing?
Discussion of Coriolis forces will be done more elegantly and made more physically intuitive in Ch. 12 of Unit1 and in Unit 6.

Some Kepler's "laws" for <u>all</u> central (isotropic) force F(r) fields

→ *Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$* *(Derived here)*

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*

Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ *(Derived here)*

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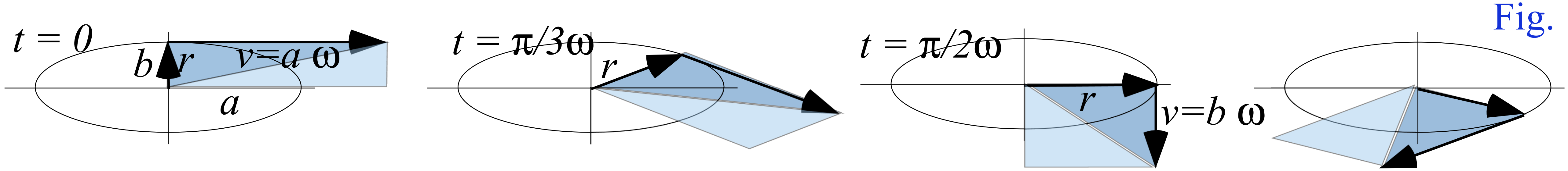
Some Kepler's "laws" for central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2　　(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - b\sin\omega t\cdot(-a\omega\sin\omega t) = ab\cdot\omega\left(\cos^2\omega t+\sin^2\omega t\right)$$

✓ *for IHO*

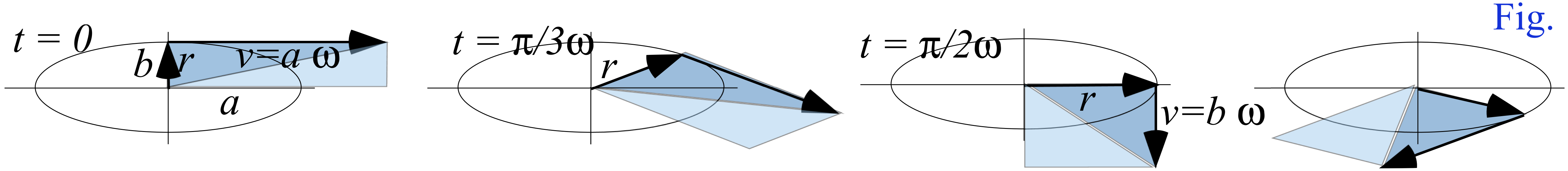
$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} \qquad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2

(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



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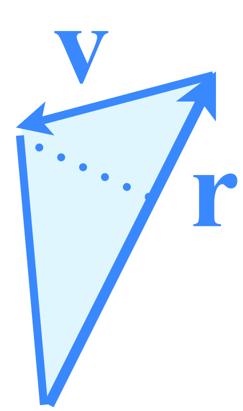
✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m(r_x v_y - r_y v_x) = m\cdot ab\cdot\omega$$

✓ *for IHO*

$|\mathbf{r}\times\mathbf{v}| = r\cdot v\cdot \sin\measuredangle^{v}_{r}$



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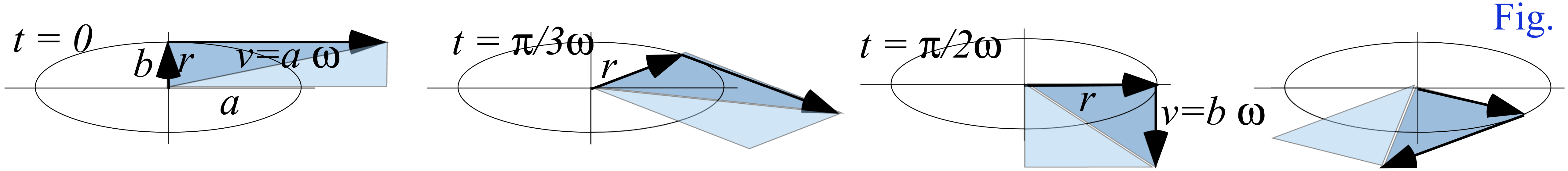
Unit 1
Fig. 11.8



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✓ *for IHO*

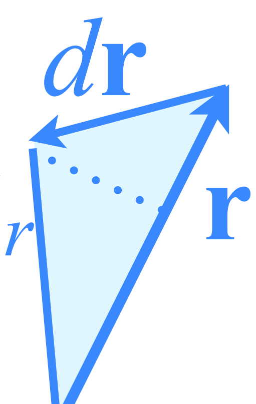
3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T$$

by 2.

✓ *for IHO*

$$|\mathbf{r}\times d\mathbf{r}| = r\cdot dr\cdot \sin\measuredangle_{r}^{dr}$$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

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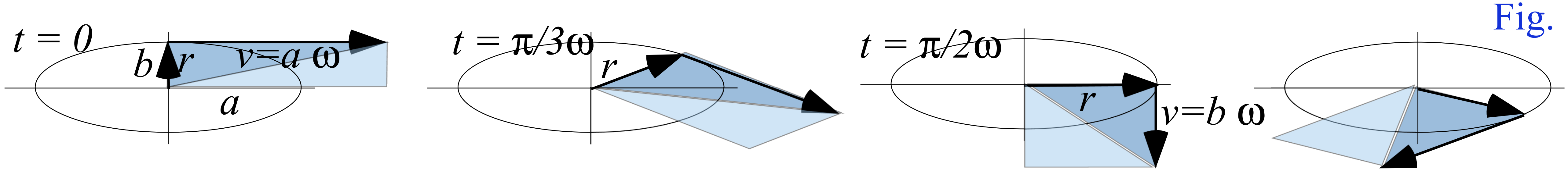
Unit 1
Fig. 11.8



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✓ *for IHO*

2. *Angular momentum* $L = m\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\mathbf{r}\times\mathbf{v} = m(r_x v_y - r_y v_x) = m\cdot ab\cdot\omega = m\cdot ab\cdot\frac{2\pi}{\tau}$$

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✓ *for IHO*

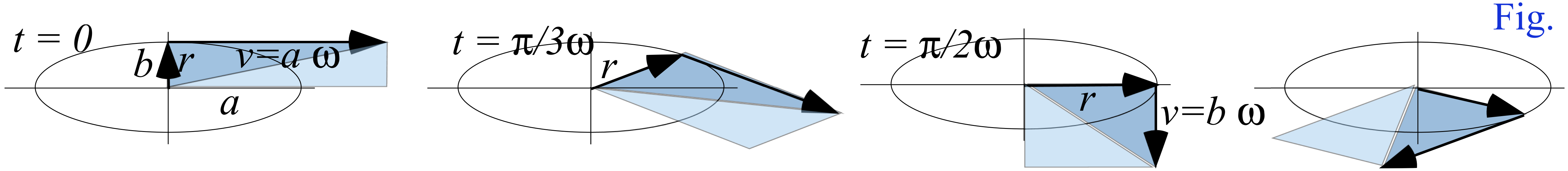
In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

(Recall from Lecture 6: $k = Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



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(Recall from Lecture 6: $\omega = \sqrt{k/m} = \sqrt{G\rho_\oplus 4\pi/3}$)

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" for <u>all</u> central (isotropic) force F(r) fields

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*

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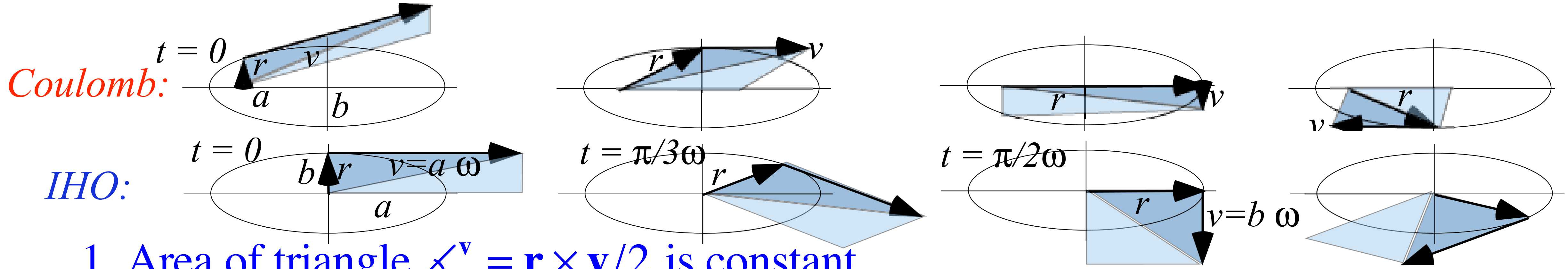
Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = \begin{cases} ab\cdot\sqrt{G\rho_{\oplus}4\pi/3} & \textit{for IHO} \\ a^{-1/2}b\sqrt{GM_{\oplus}} & \textit{for Coul.} \end{cases}$$

✓ *for IHO*

(Derived in Unit 5) ✓ *for Coul.*

(G IHO formulas from Lect. 6 p.70-79)

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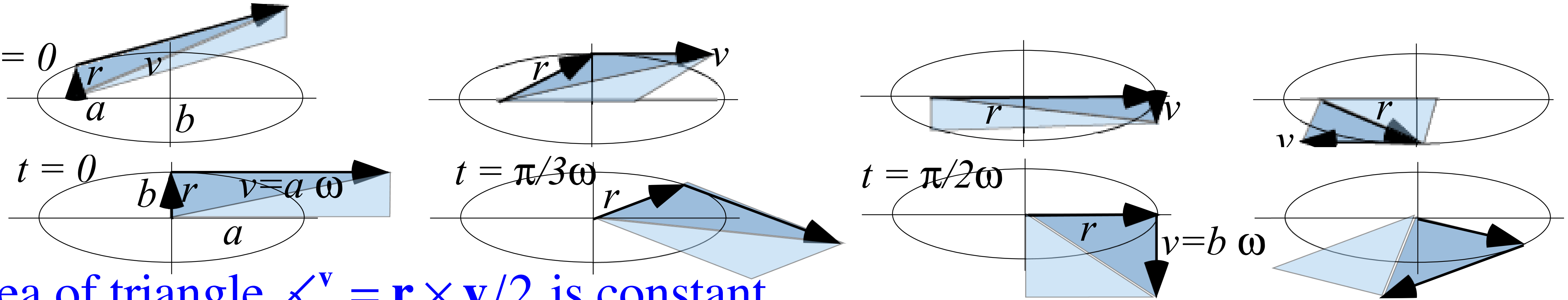
Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$

Coulomb:

IHO:



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(Derived in Unit 5) ✓ for Coul.

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(... in Unit 5) ✓ for Coul.

✓ for IHO

(G IHO formulas from Lect. 6 p.70-79)

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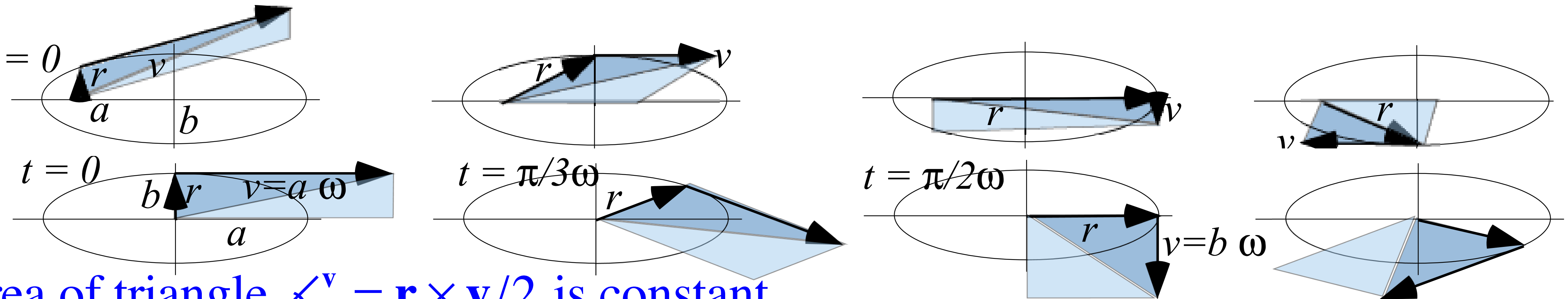

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(... in Unit 5)

✓ *for IHO*

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In one period:

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Applies to any central F(r)

Applies to IHO and Coulomb

(G IHO formulas from Lect. 6 p.70-79)

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Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$

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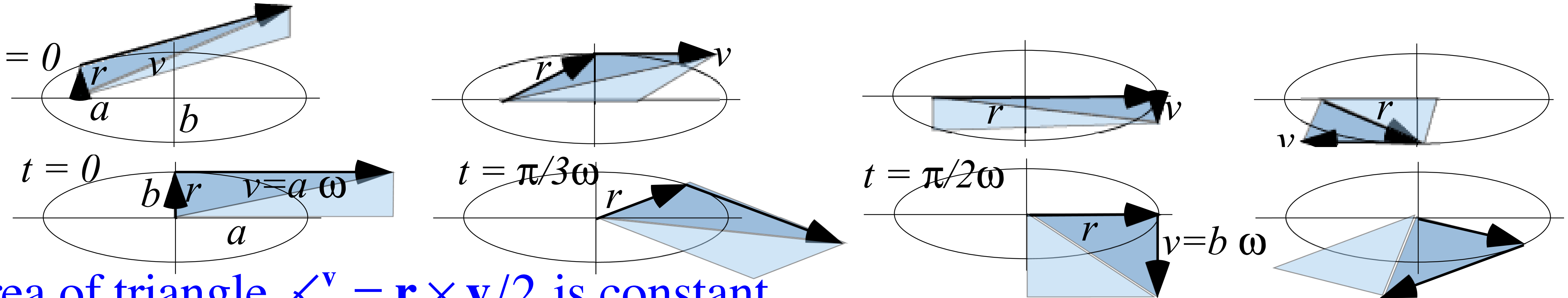
IHO:



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Applies to any central F(r)

Applies to IHO and Coulomb

(not a function of *a* or *b*)

that is ω_{IHO}

that is ω_{Coul}

(not a function of *b*)

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Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}=\begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix}v_x & v_y\end{pmatrix}\bullet\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}\bullet\begin{pmatrix}v_x\\ v_y\end{pmatrix}+\begin{pmatrix}r_x & r_y\end{pmatrix}\bullet\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\bullet\begin{pmatrix}r_x\\ r_y\end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2=k$

(G IHO formulas from Lect. 6 p.70-79)

Some Kepler's "laws" for <u>all</u> central (isotropic) force F(r) fields

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived here)

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived in Unit 5)

Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ (Derived here)

→ *Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$* (Derived in Unit 5)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)\qquad Given: k=m\omega^2$$

$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_{\oplus}4\pi/3}$ or: $m\omega^2=k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

(G IHO formulas from Lect. 6 p.70-79)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

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$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

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$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)\qquad Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)\quad\text{since: }\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}\quad\text{or: }m\omega^2=k$$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

$$E=KE+PE=\frac{1}{2}mv^2{}_x+\frac{1}{2}mv^2{}_y-\frac{k}{r}=\frac{1}{2}mv^2{}_x+\frac{1}{2}mv^2{}_y-\frac{GM_\oplus m}{r}=-\frac{GM_\oplus m}{a}$$

→ *Introduction to dual matrix operator contact geometry (based on IHO orbits)*

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\overbrace{\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}\;=1} = 1 = \overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}} \bullet \overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}\;=1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}} = a^2 p_x^2 + b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\underbrace{\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}\;=1} = 1 = \overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}} \bullet \overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}=\mathbf{p}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Defined mapping between ellipses

A inverse matrix Q^{-1} *generates an ellipse by* **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

$$\underbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}_{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}\;=1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}=\mathbf{r}} = a^2 p_x^2 + b^2 p_y^2$$

Introduction to dual matrix operator contact geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

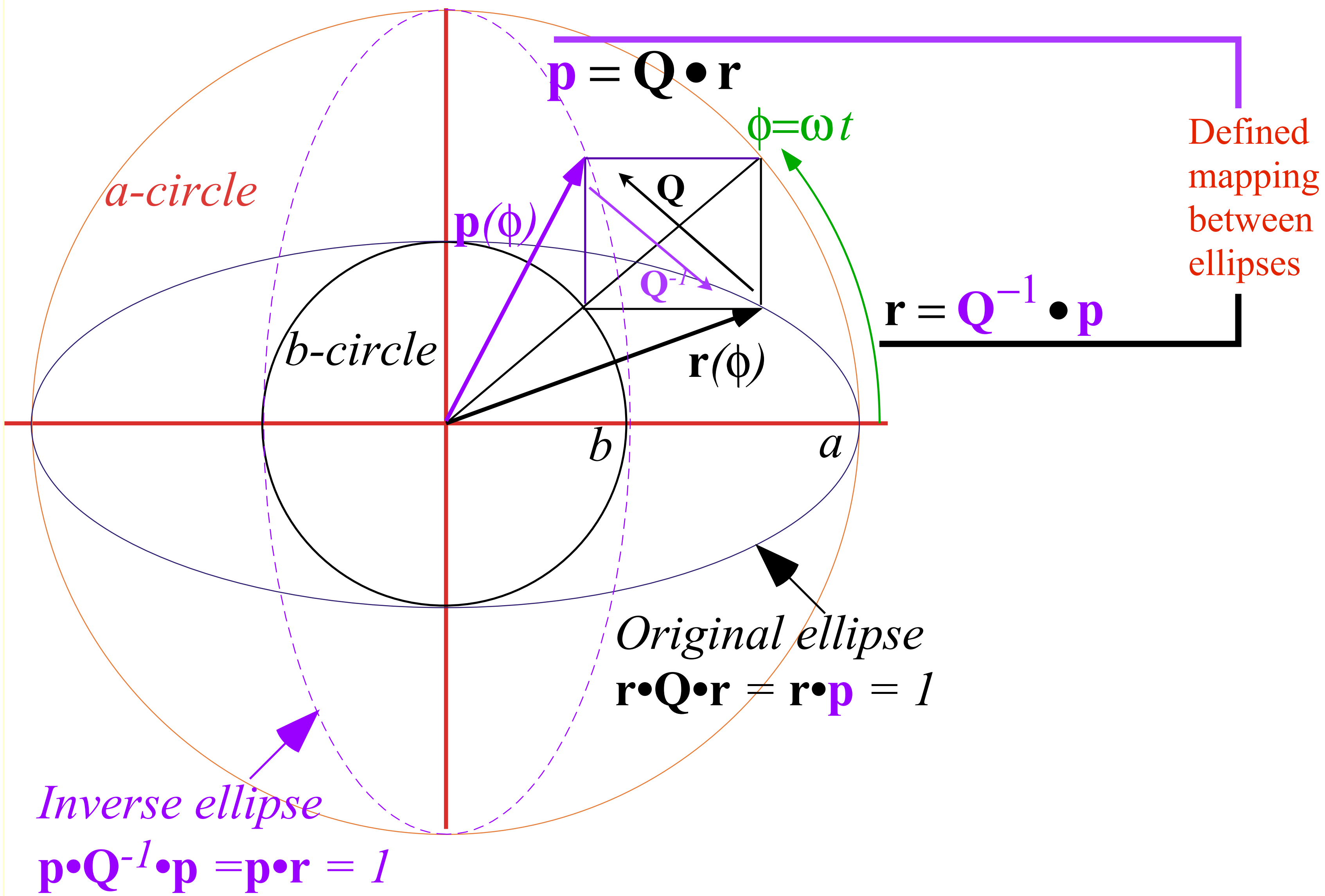
Operator geometric sequences and eigenvectors

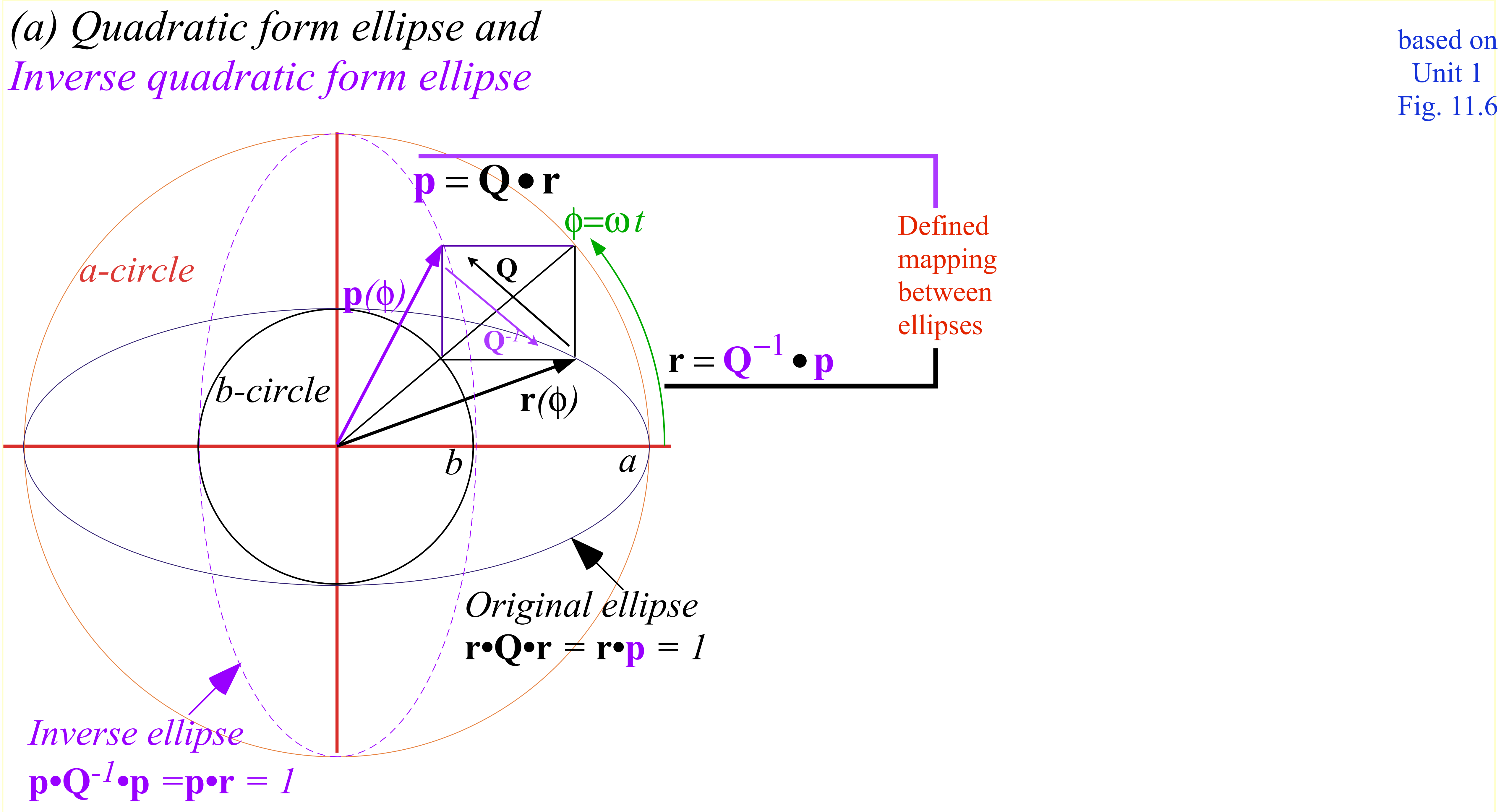
Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6





Here plot of **p**-ellipse is re-scaled by scalefactor $S=a \cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

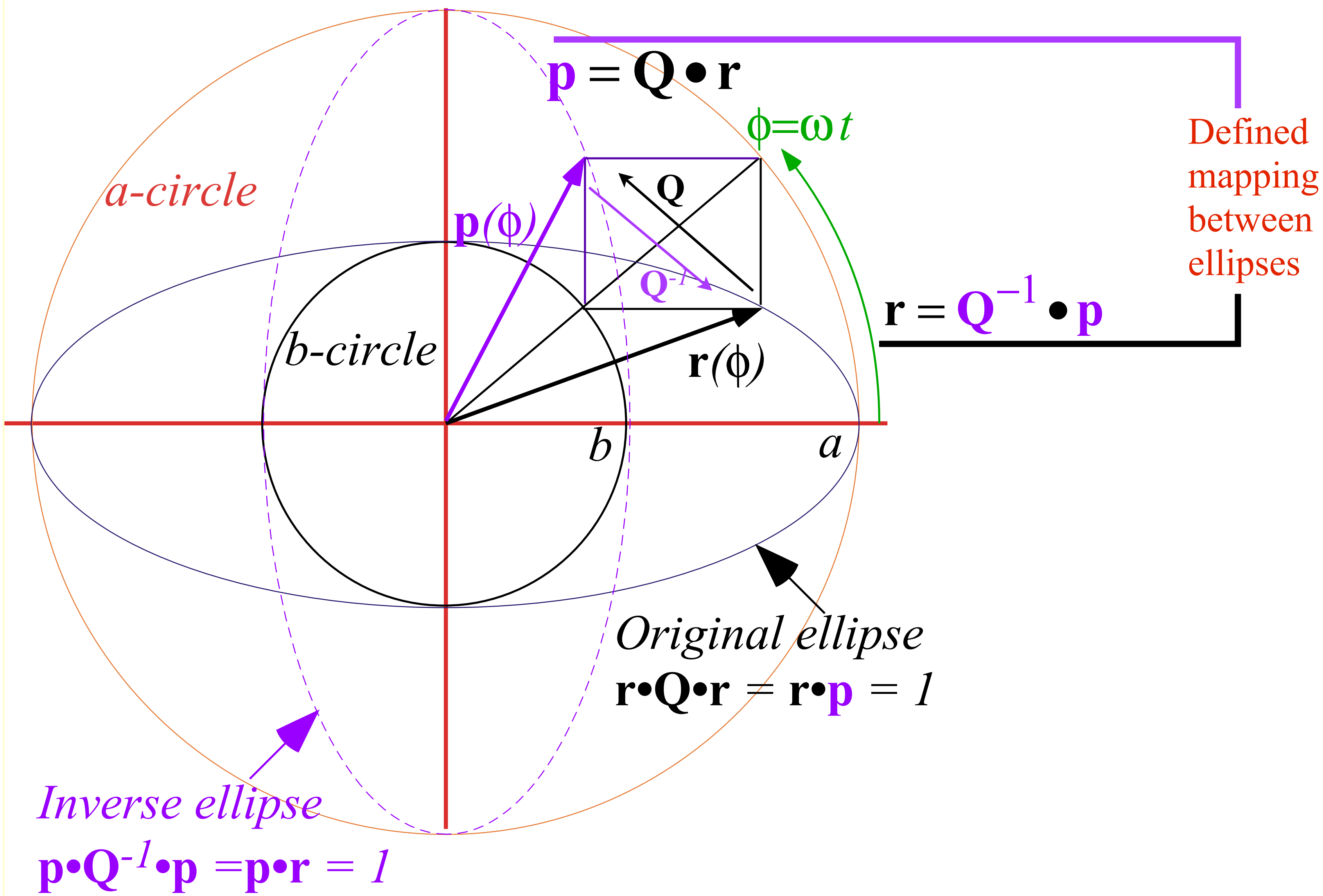
Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

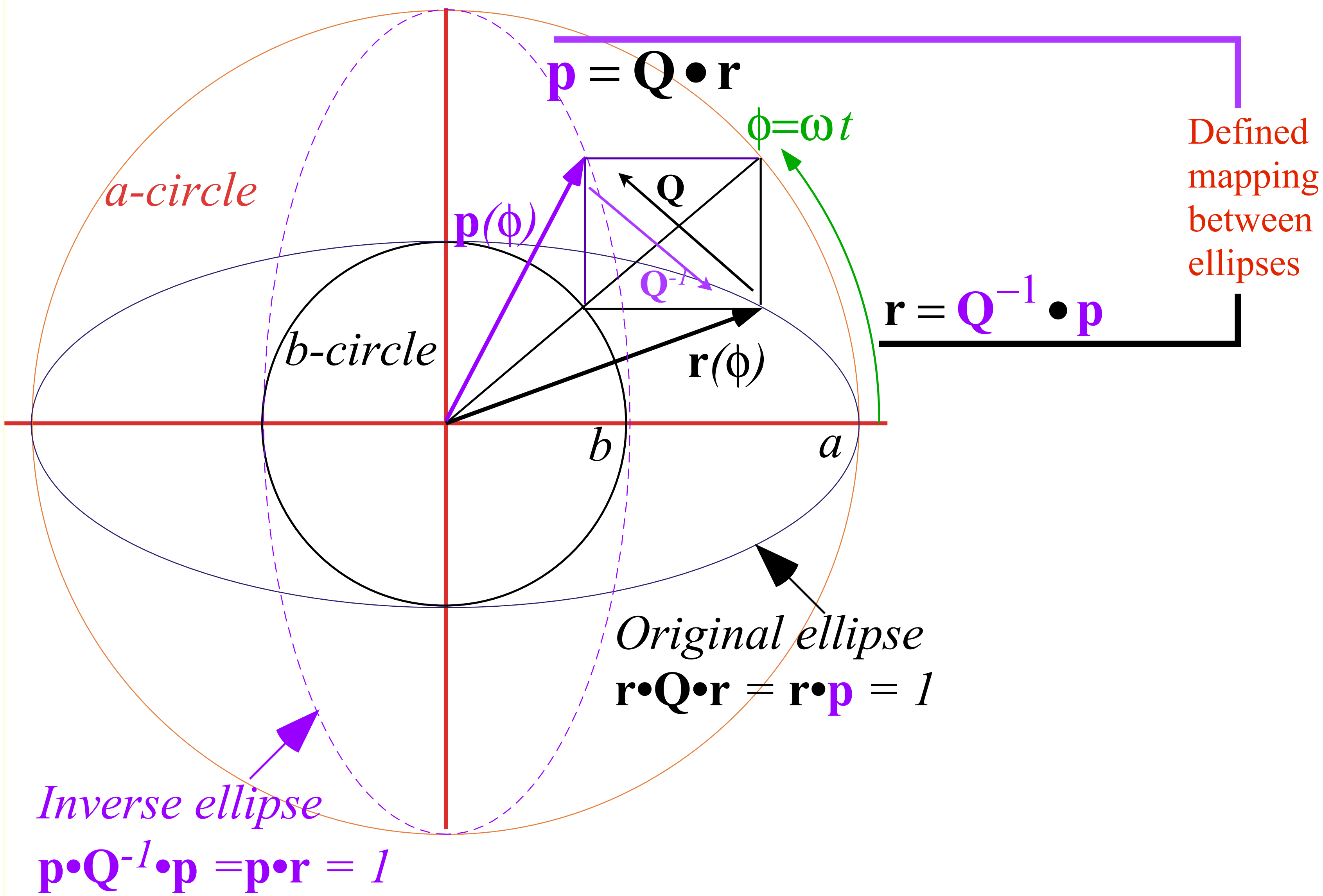
Here plot of **p**-ellipse is re-scaled by scalefactor $S = a \cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Quadratic form **r•Q•r** $=1$ *has mutual duality relations with inverse form* **p•Q^{-1}•p** $=1=$ **p•r**

$$\mathbf{p}=\mathbf{Q}\cdot\mathbf{r}=\overbrace{\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}}^{\mathbf{Q}}\bullet\overbrace{\begin{pmatrix}x\\ y\end{pmatrix}}^{\mathbf{r}}=\overbrace{\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}}^{\mathbf{p}}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $x=r_x=a\cos\phi=a\cos\omega t$, $y=r_y=b\sin\phi=b\sin\omega t$ so: $\mathbf{p}\cdot\mathbf{r}=1$

Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

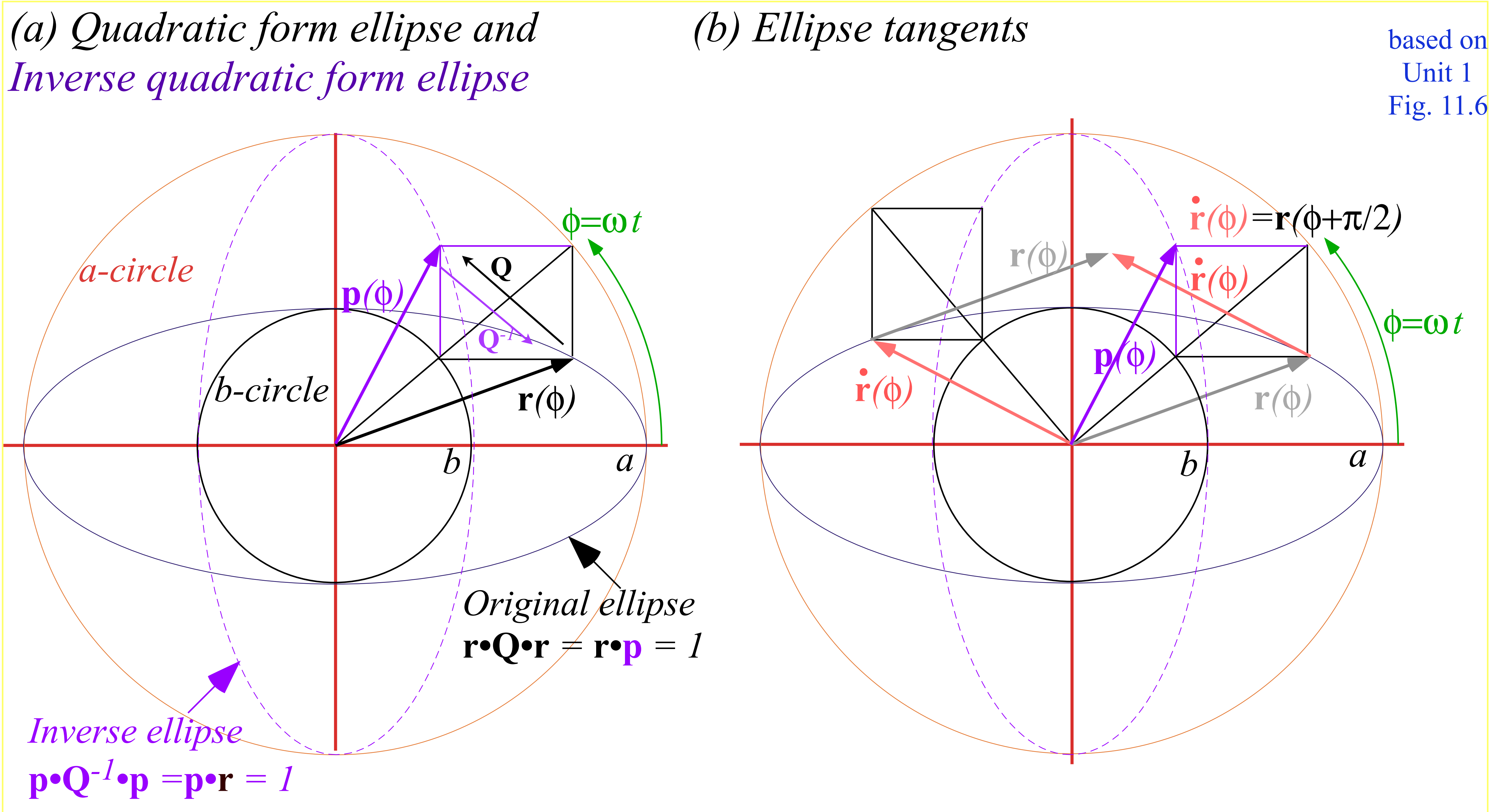
Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

→ *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation



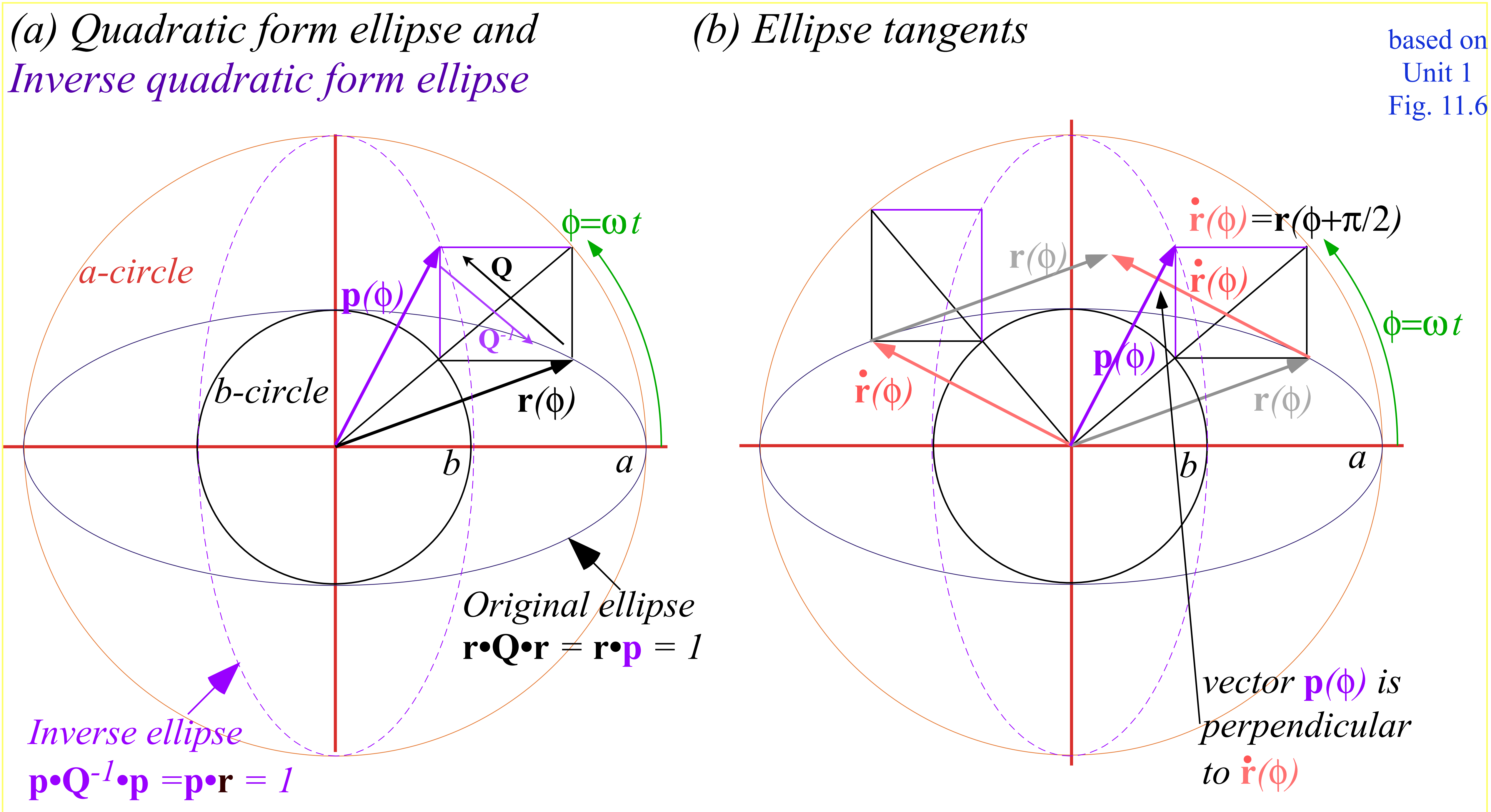
Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q^{-1}•p** *=1=* **p•r**

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ =b\sin\omega t\end{matrix}\quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

p-ellipse x-radius=1/a plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

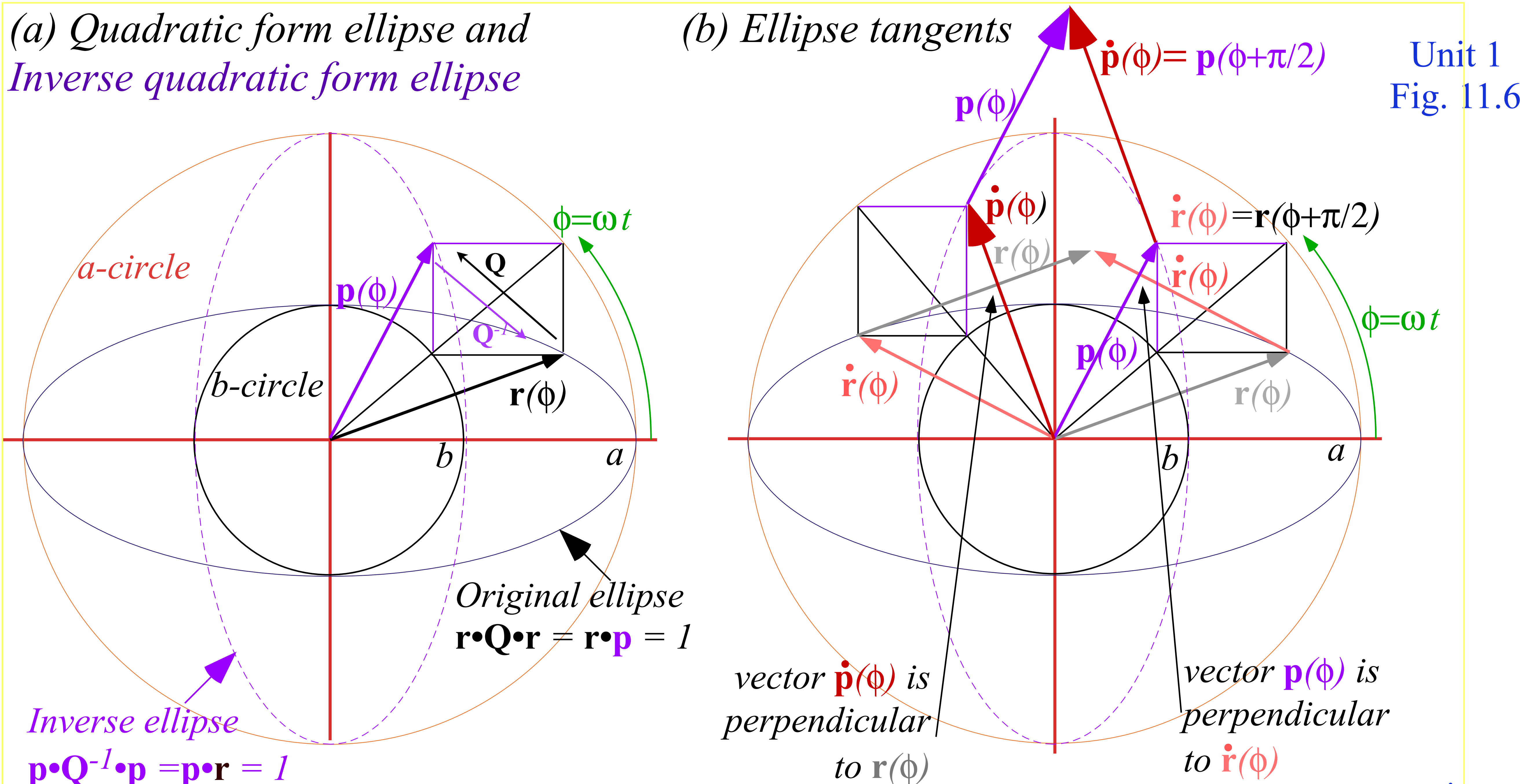


Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p} = \mathbf{Q}\bullet\mathbf{r} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{matrix} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi = b\sin\omega t \end{matrix} \quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r} = 1}$$

$\mathbf{p}$ *is perpendicular to velocity* $\mathbf{v} = \dot{\mathbf{r}}$ *, a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p} = 0} = \begin{pmatrix} \dot{r}_x & \dot{r}_y \end{pmatrix}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} -a\sin\phi & b\cos\phi \end{pmatrix}\bullet\begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{matrix} \dot{r}_x = -a\sin\phi \\ \dot{r}_y = b\cos\phi \end{matrix} \quad \text{and: } \begin{matrix} p_x = (1/a)\cos\phi \\ p_y = (1/b)\sin\phi \end{matrix}$$



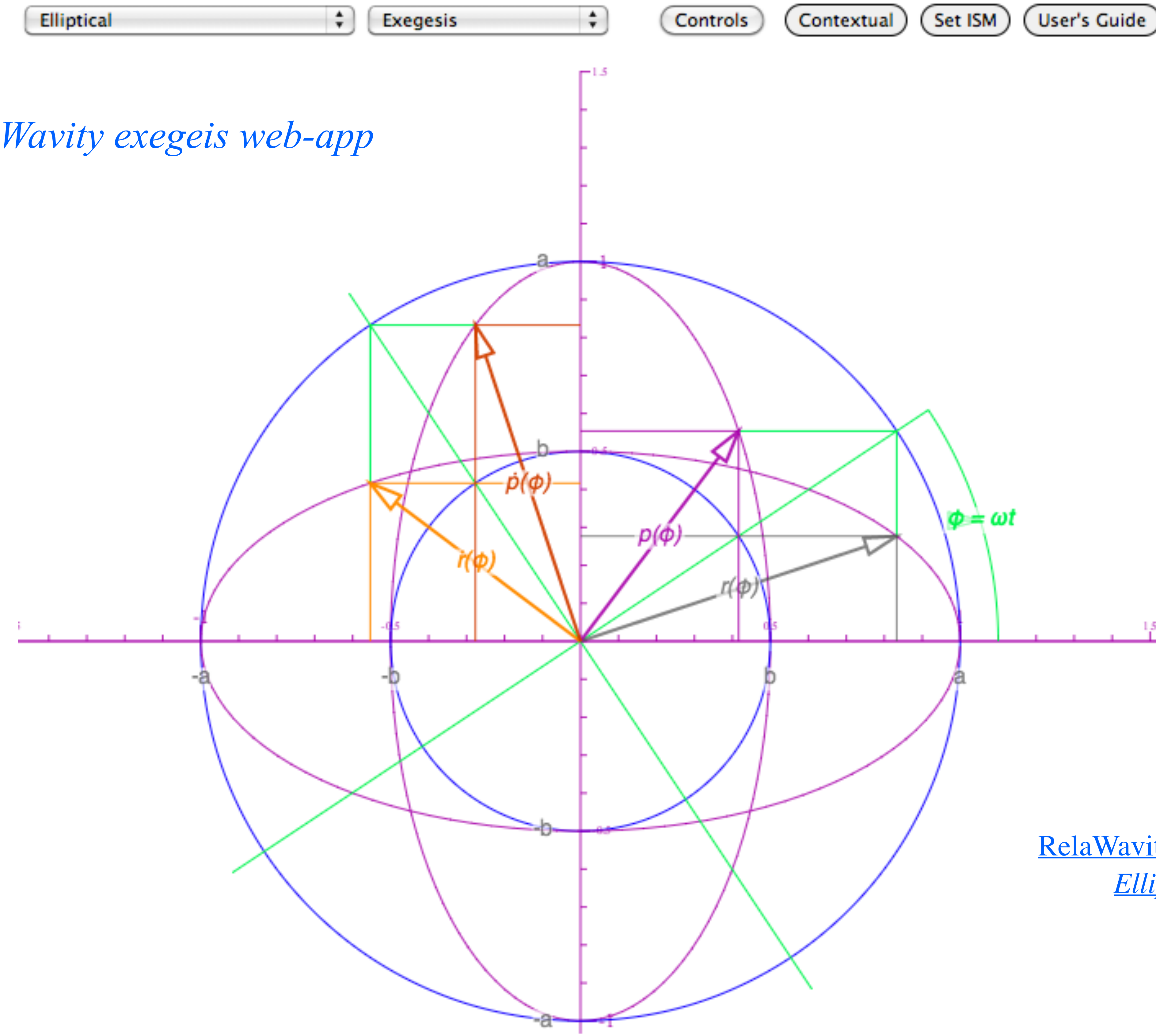
Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q⁻¹•p** *=1* — *unit mutual projection*

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ =b\sin\omega t\end{matrix}\quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

p *is perpendicular to velocity* **v**=**ṙ** *, a mutual orthogonality. So is* **r** *perpendicular to* **ṗ**: $\boxed{\dot{\mathbf{p}}\bullet\mathbf{r}=0}$

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot r_x & \dot r_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}\dot r_x=-a\sin\phi\\ \dot r_y=b\cos\phi\end{matrix}\quad \text{and: } \begin{matrix}p_x=(1/a)\cos\phi\\ p_y=(1/b)\sin\phi\end{matrix}$$

RelaWavity exegeis web-app



[RelaWavity Web Simulation](#)
[*Ellipse/Exegesis*](#)

Geometry of dual ellipse Kepler anomalies for vectors $[\mathbf{r}(\phi), \mathbf{p}(\phi)]$ and $^{d}/_{dt}[\mathbf{r}(\phi), \mathbf{p}(\phi),]$ *in coordinate (x,y) space rendered by animation web-app in RelaWavity and described in Lect. 12-advanced.*

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

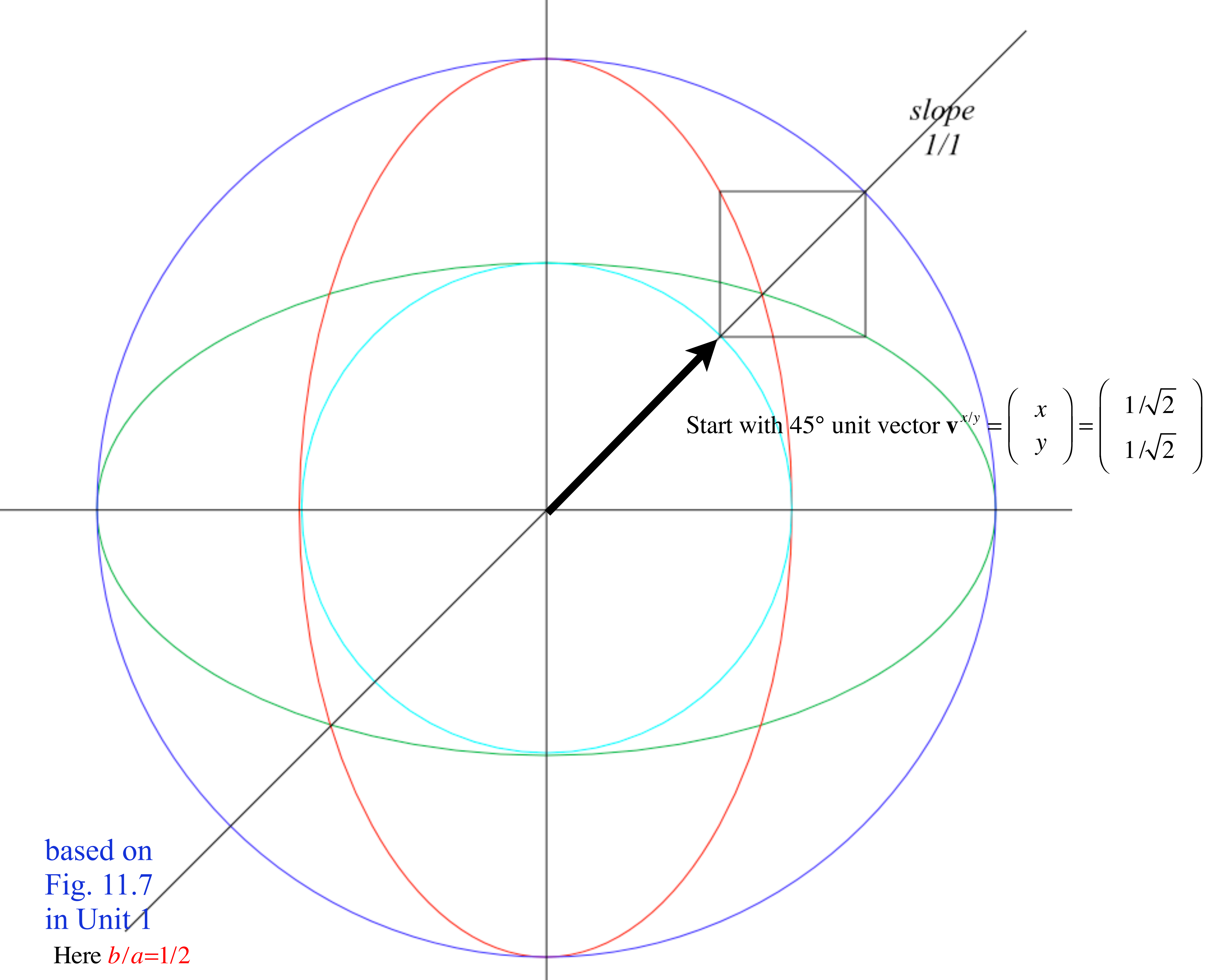
Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

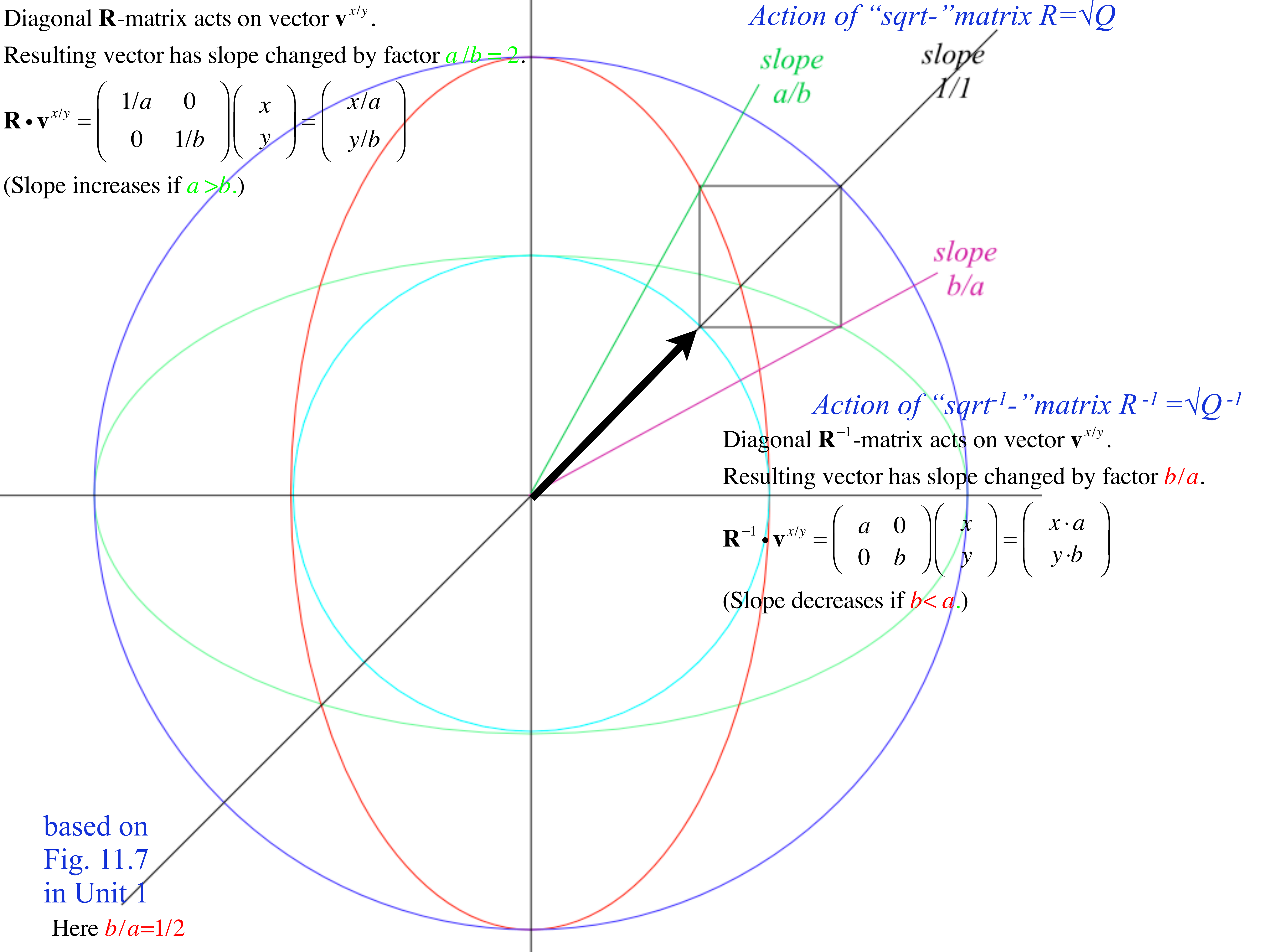
Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

→ *Operator geometric sequences and eigenvectors*

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation


slope
1/1
Start with 45° unit vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.
based on
Fig. 11.7
in Unit 1
Here $b/a=1/2$



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.
Resulting vector has slope changed by factor $a/b = 2$.
$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$
(Slope increases if $a > b$.)
Action of "sqrt-"matrix $R=\sqrt{Q}$
slope
a/b
slope
1/1
slope
b/a
Action of "sqrt^{-1}-"matrix $R^{-1} = \sqrt{Q}^{-1}$
Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.
Resulting vector has slope changed by factor b/a.
$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a \\ y\cdot b \end{pmatrix}$$
(Slope decreases if $b < a$.)
based on
Fig. 11.7
in Unit 1
Here $b/a = 1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

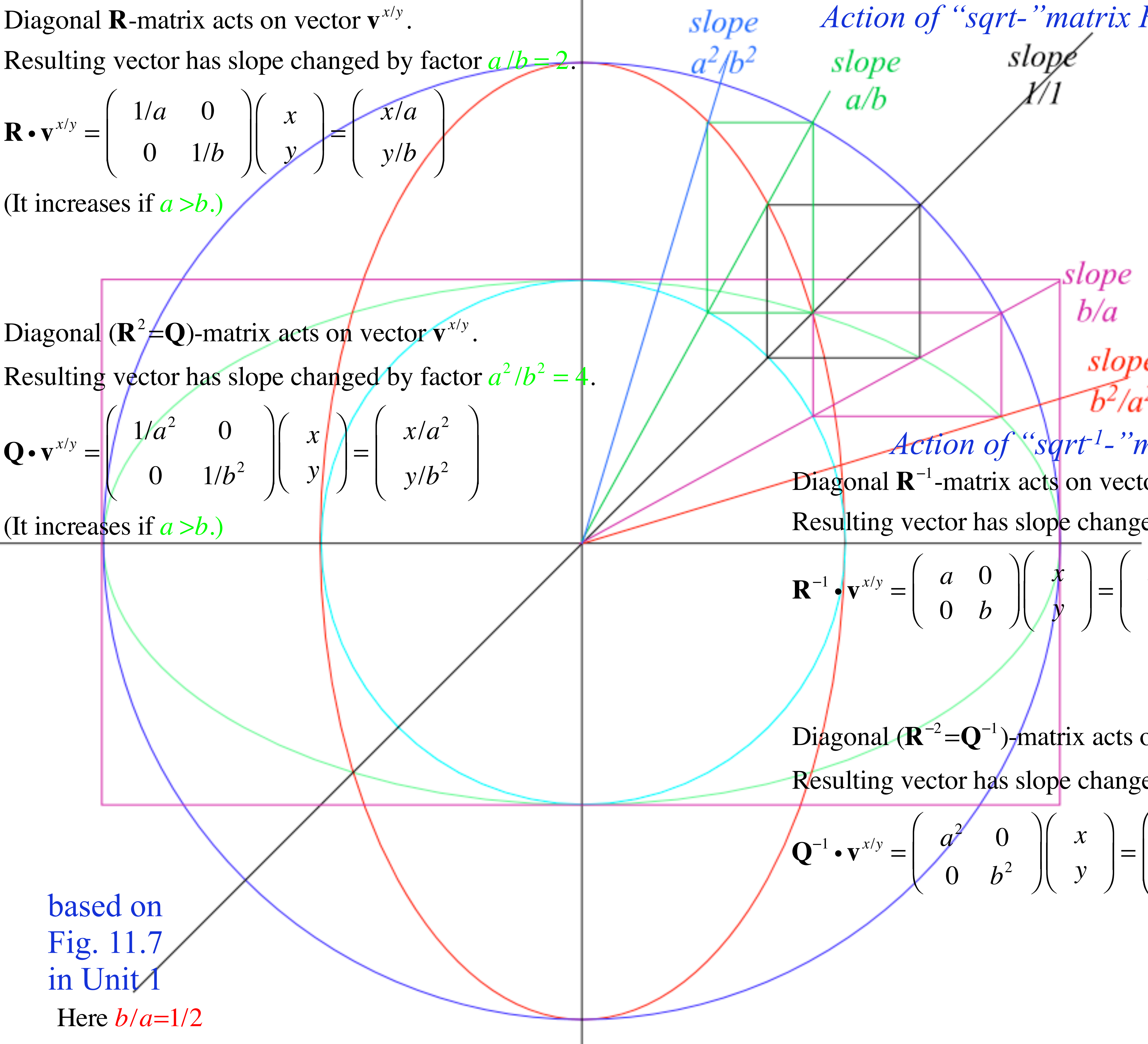
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b/a = 1/2$.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a \\ y\cdot b \end{pmatrix}$$

Diagonal ($\mathbf{R}^{-2} = \mathbf{Q}^{-1}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^2/a^2 = 1/4$.

$$\mathbf{Q}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a^2 \\ y\cdot b^2 \end{pmatrix}$$

based on Fig. 11.7 in Unit 1

Here $b/a = 1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

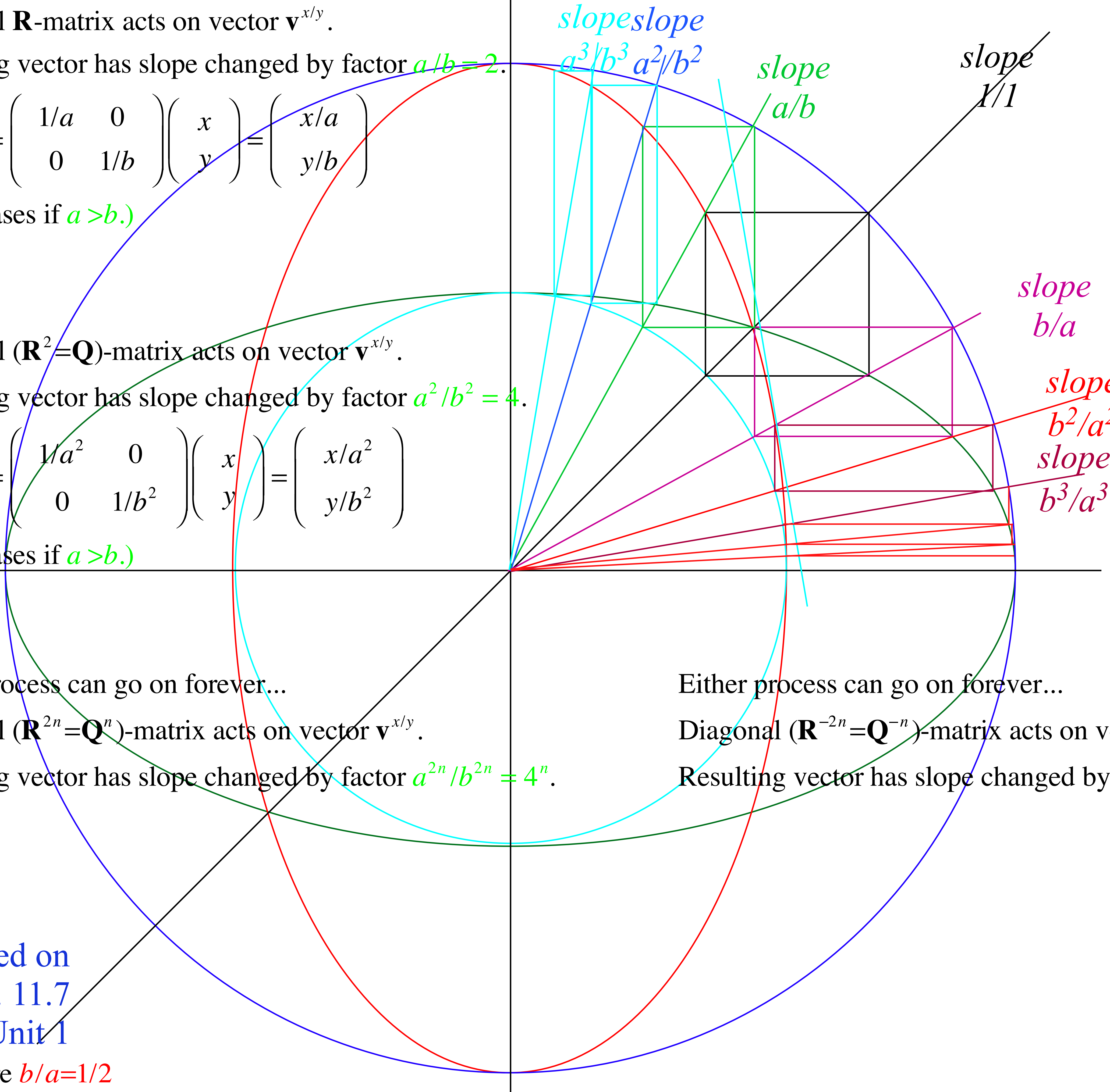
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

based on
Fig. 11.7
in Unit 1

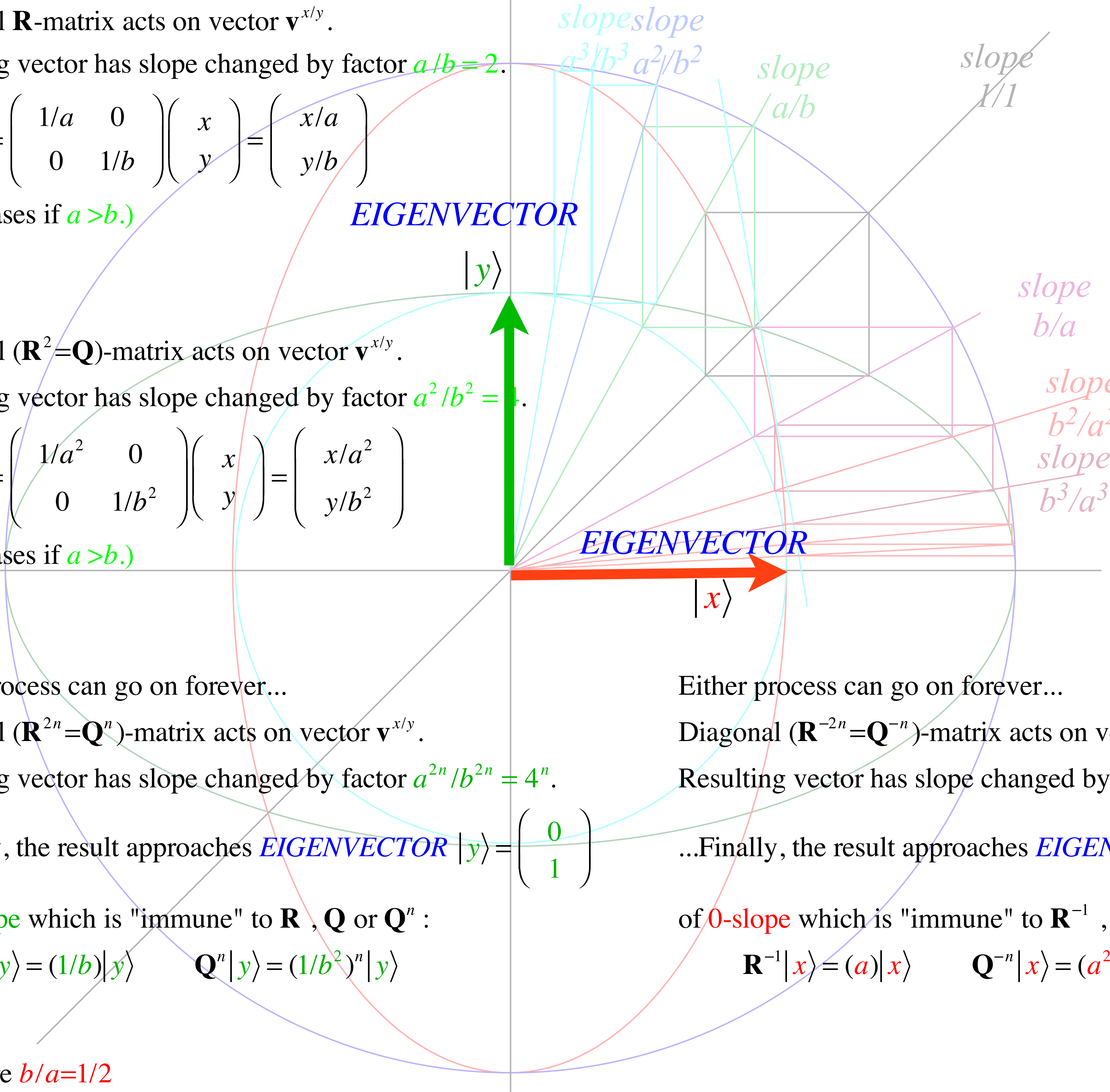
Here b/a=1/2

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R} \bullet \mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)



Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q} \bullet \mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

$$\mathbf{R}|y\rangle = (1/b)|y\rangle \qquad \mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$$

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle \qquad \mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$$

Here $b/a = 1/2$

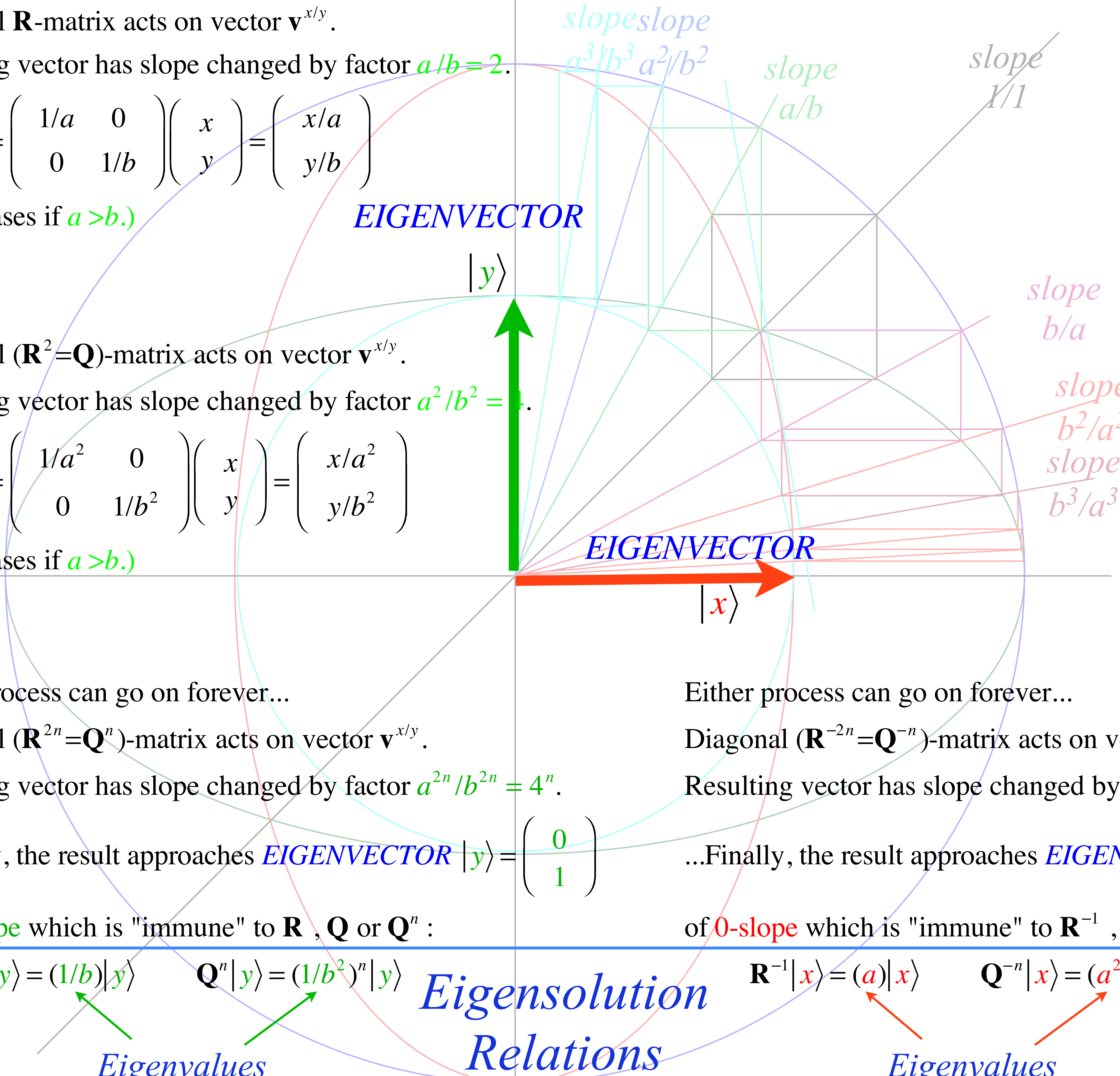
Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R} \bullet \mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)

Here $b/a = 1/2$



Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q} \bullet \mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$\mathbf{R}|y\rangle = (1/b)|y\rangle$ $\quad$ $\mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$

Eigenvalues

Eigensolution Relations

$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle$ $\quad$ $\mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$

Eigenvalues

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

Operator geometric sequences and eigenvectors

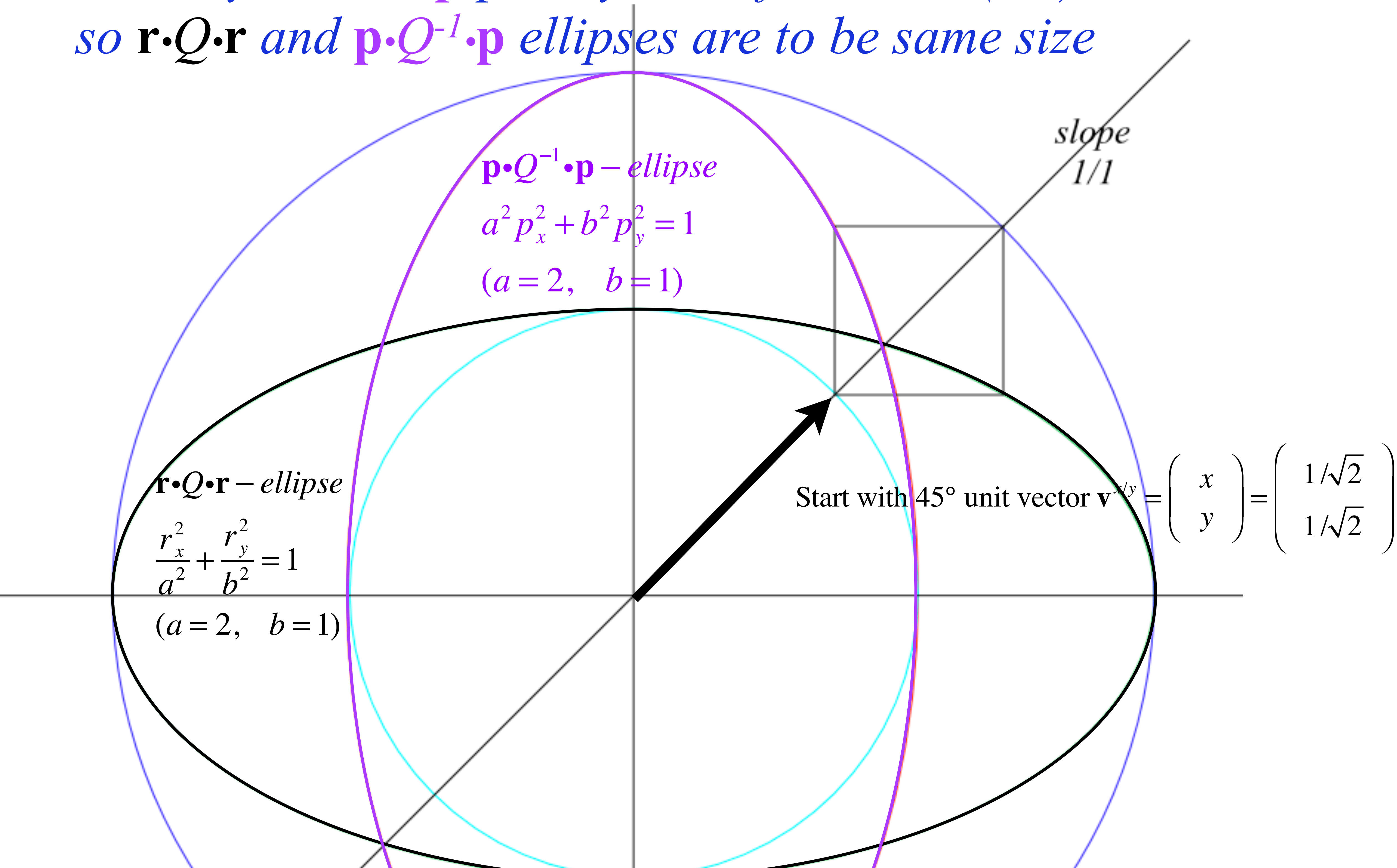
→ *Alternative scaling of matrix operator geometry*

Vector calculus of tensor operation

You may rescale **p***-plot by scale factor $S=(a\cdot b)$*

so $\mathbf{r}\cdot Q\cdot\mathbf{r}$ *and* $\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ *ellipses are to be same size*

Here $b/a=1/2$



Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

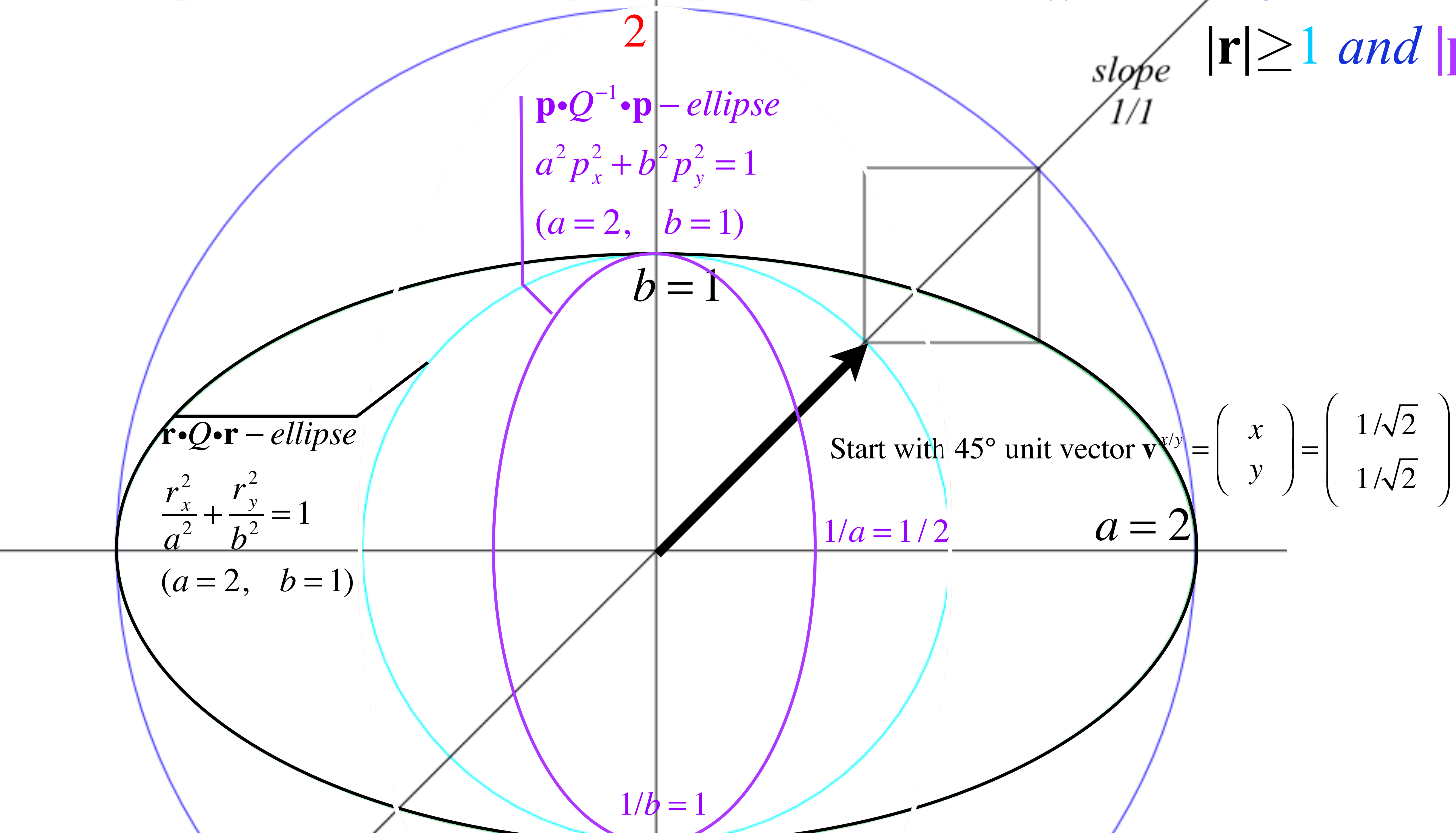
p-ellipse x-radius$=1/a$ plotted at: $S(1/a)=b$ (=1 for $a=2$, $b=1$)

p-ellipse y-radius$=1/b$ plotted at: $S(1/b)=a$ (=2 for $a=2$, $b=1$)

..or else rescale **p***-plot by scale factor* $S=b$

to separate **r**$\cdot Q \cdot$**r** *and* **p**$\cdot Q^{-1} \cdot$**p** *ellipses into different regions*

Here $b/a=1/2$



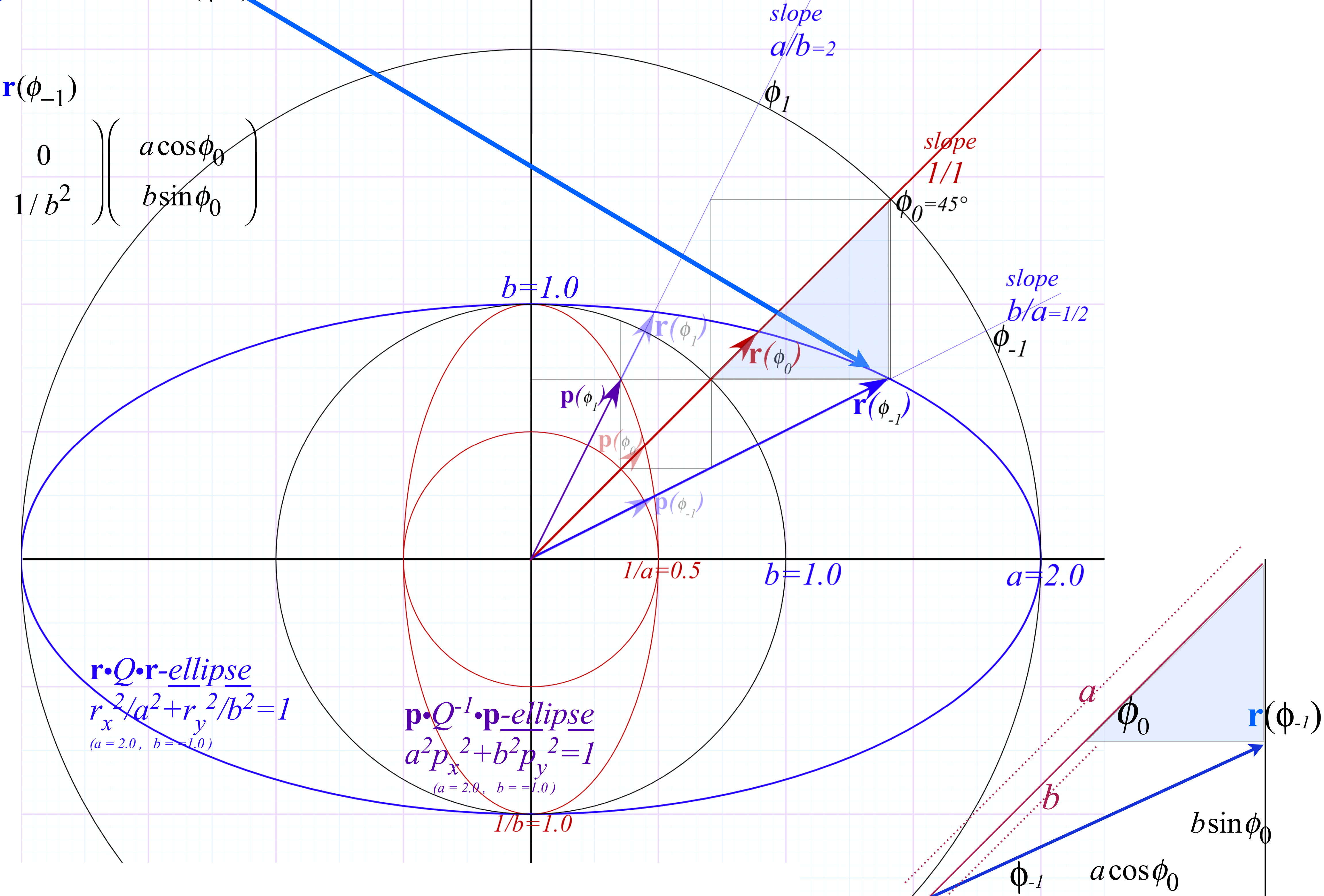
Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius$=1/a$ plotted at: $S(1/a)=b/a$ ($=1/2$ for $a=2$, $b=1$)

p-ellipse y-radius$=1/b$ plotted at: $S(1/b)=1$

Action of matrix Q that generates an **r**-*ellipse* (**r**•Q• **r** =1) *on a single* **r**-*vector* **r**(ϕ_{-1})...

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1}) = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$



Variation of Fig. 11.7 in Unit 1

Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius=1/a plotted at: $S(1/a)=b/a$ (=1/2 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)$=1

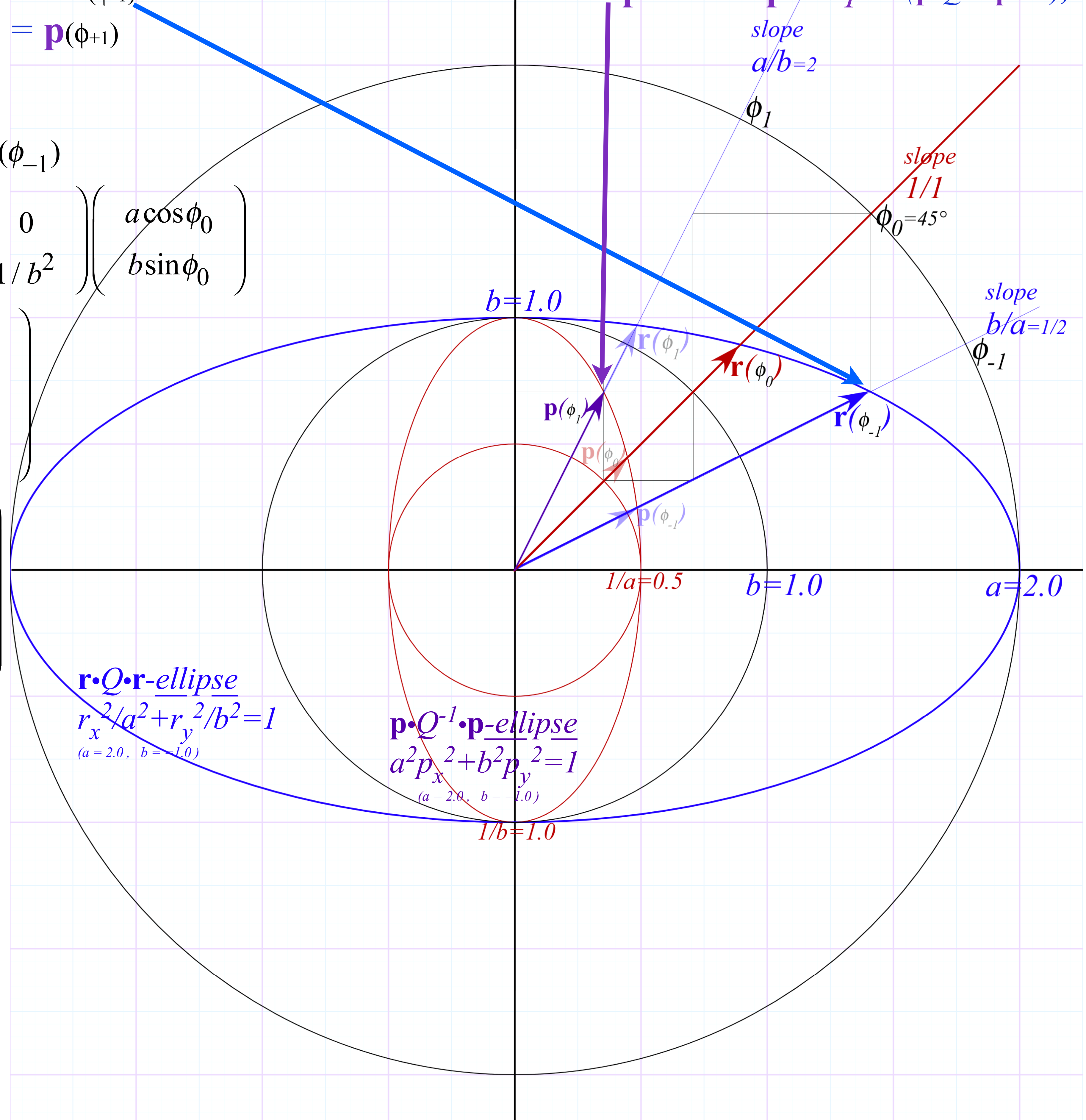
Action of matrix Q that generates an **r**-*ellipse* ($\mathbf{r}\bullet Q\bullet\mathbf{r} = 1$)
on a single **r**-*vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to a new vector* **p** *on the* **p**-*ellipse* ($\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p} = 1$),
that is, $Q\bullet\mathbf{r}(\phi_{-1}) = \mathbf{p}(\phi_{+1})$

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

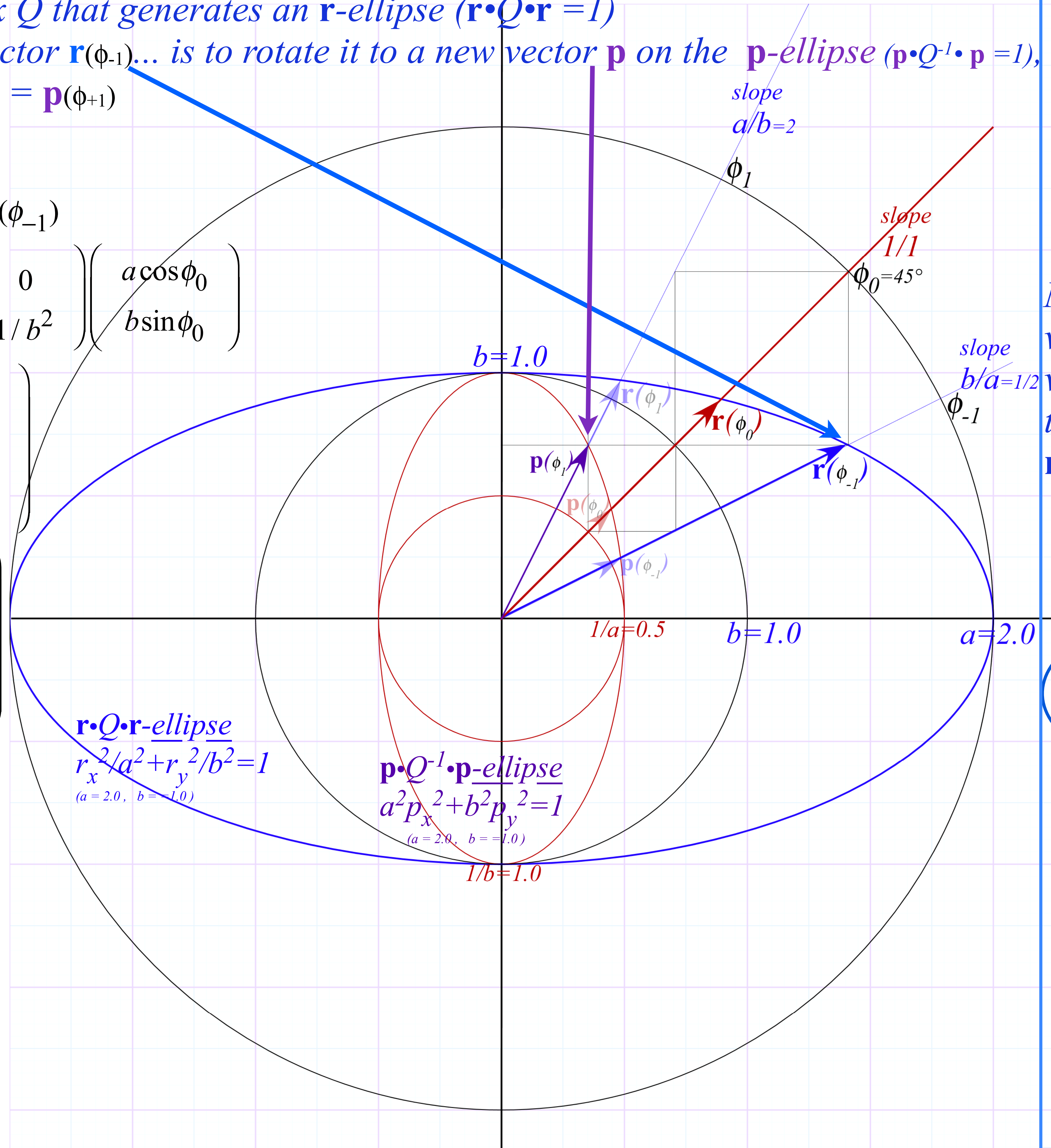
Action of matrix Q that generates an **r**-*ellipse* (**r**•*Q*•**r** *=1*) *on a single* **r**-*vector* **r**(ϕ_{-1})... *is to rotate it to a new vector* **p** *on the* **p**-*ellipse* (**p**•Q^{-1}• **p** *=1*), *that is,* *Q*•**r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

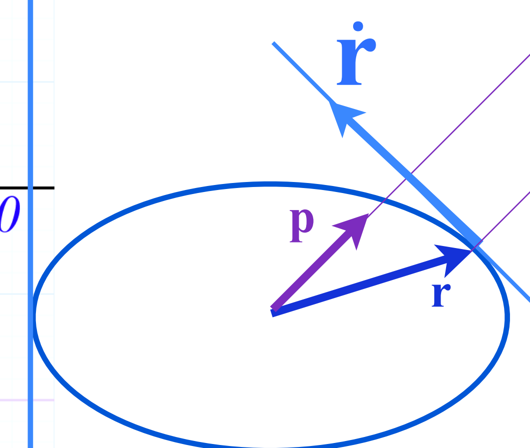
$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* $\dot{\mathbf{r}}$ *to its* **r**•*Q*•**r**-*ellipse.*



Variation of Fig. 11.7 in Unit 1

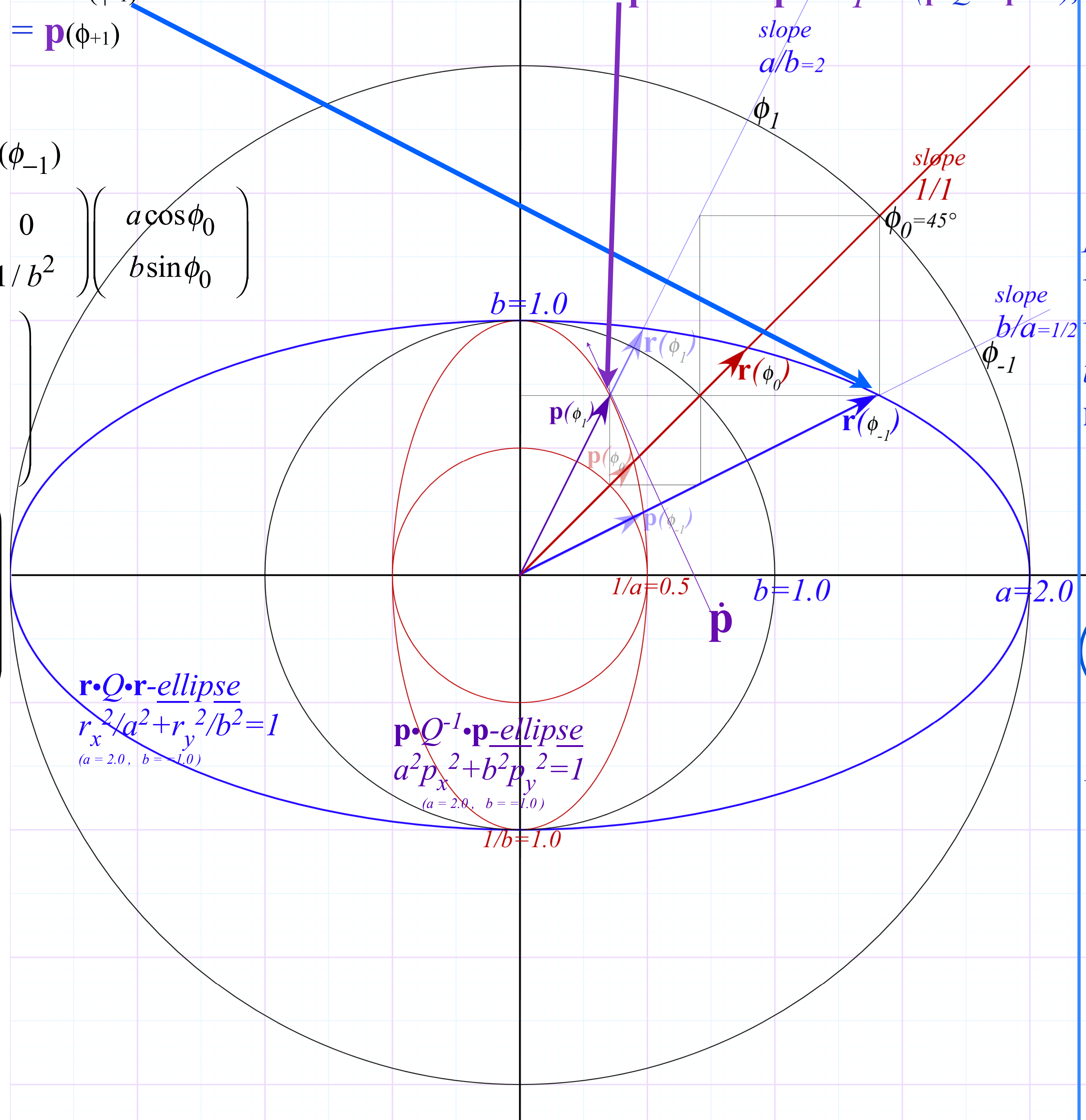
Action of matrix Q that generates an **r**-*ellipse* (**r**•*Q*•**r** =1) *on a single* **r**-*vector* **r**(ϕ_{-1})... *is to rotate it to a new vector* **p** *on the* **p**-*ellipse* (**p**•Q^{-1}• **p** =1), *that is,* Q•**r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

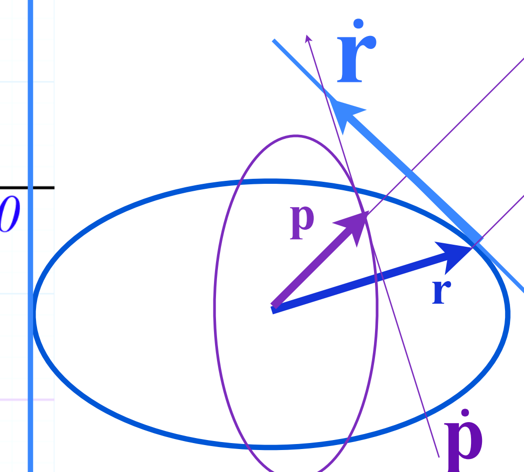
$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of Fig. 11.7 in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* **ṙ** *to its* **r**•*Q*•**r**-*ellipse.*



Matrix Q^{-1} *maps* **p** *back to* **r** *that is normal to the tangent* **ṗ** *to its* **p**• Q^{-1}• **p**-*ellipse.*

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

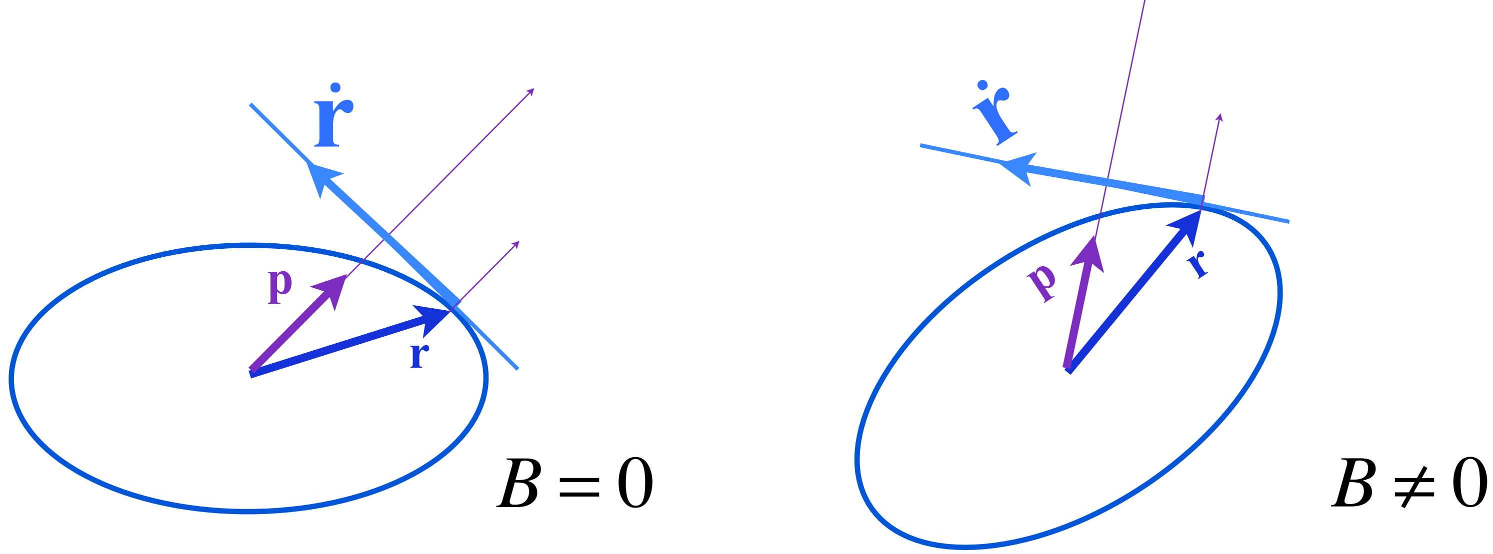
Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

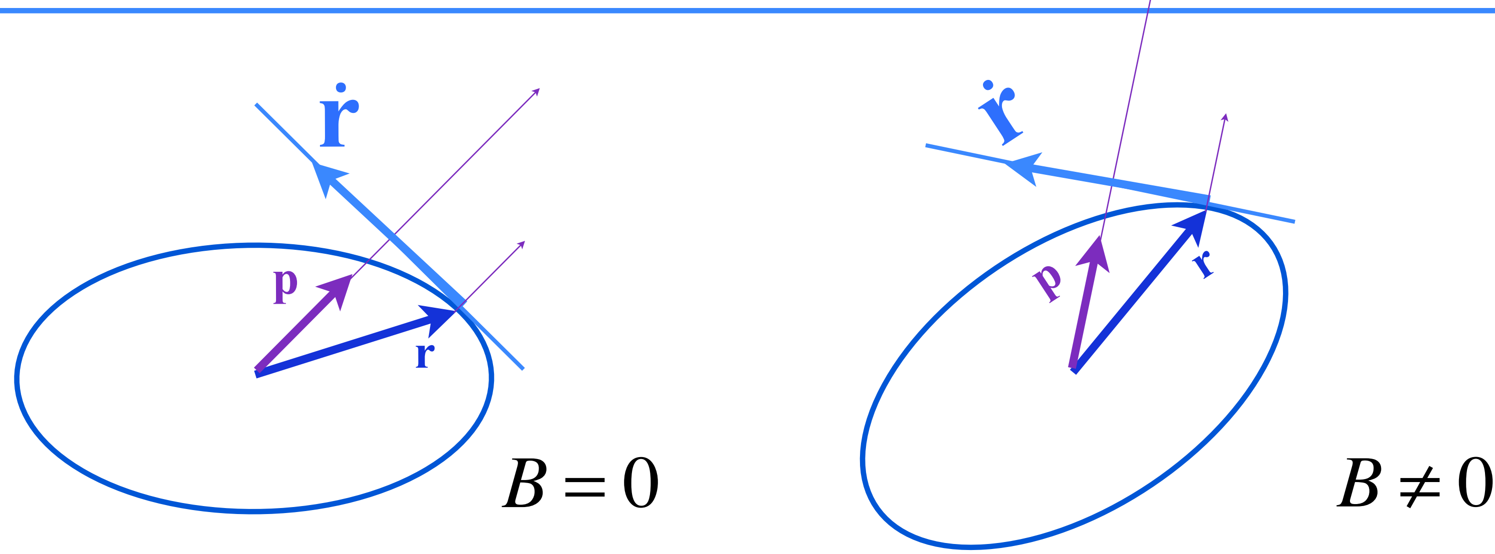
→ *Vector calculus of tensor operation*



Derive matrix "normal-to-ellipse"geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1 = \mathbf{r} \bullet Q \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$



Derive matrix "normal-to-ellipse"geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

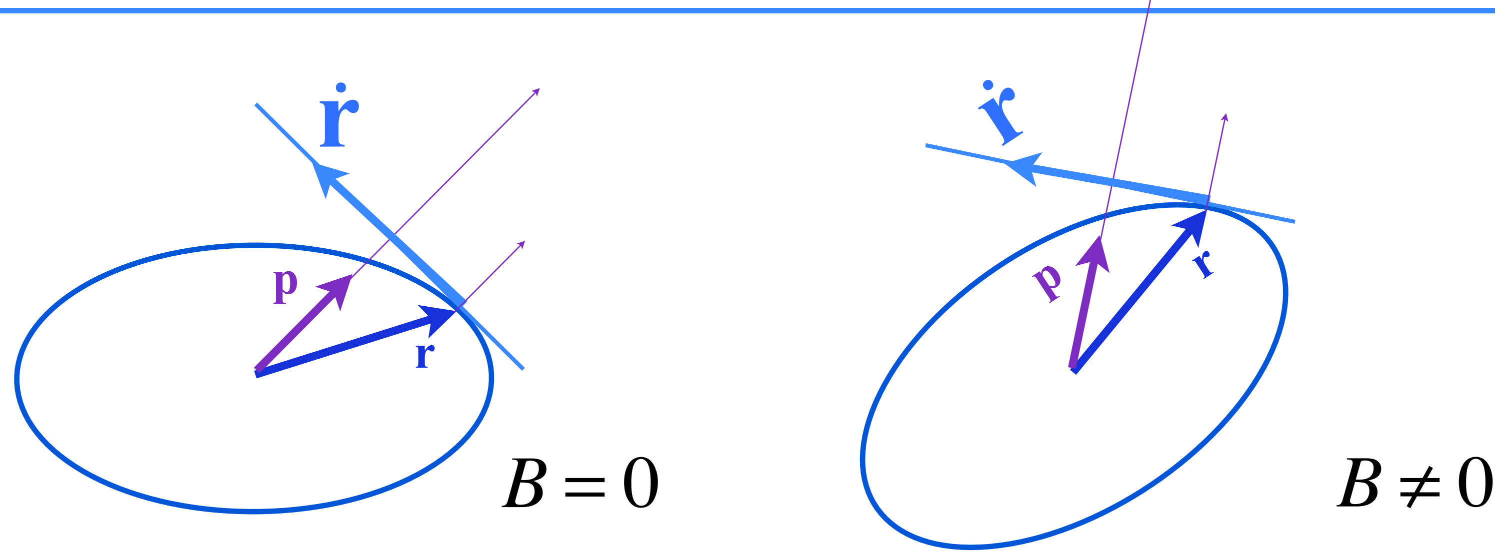
define the ellipse $1 = \mathbf{r} \bullet Q \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$

Compare operation by Q *on vector* $\mathbf{r}$ *with vector derivative or gradient of* $\mathbf{r} \bullet Q \bullet \mathbf{r}$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial \mathbf{r}}(\mathbf{r} \bullet Q \bullet \mathbf{r}) = \nabla(\mathbf{r} \bullet Q \bullet \mathbf{r})$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \left(A \cdot x^2 + 2B \cdot xy + D \cdot y^2\right) = \begin{pmatrix} 2A \cdot x + 2B \cdot y \\ 2B \cdot x + 2D \cdot y \end{pmatrix}$$



Derive matrix "normal-to-ellipse"geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1 = \mathbf{r} \bullet Q \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$

Compare operation by Q on vector **r** *with* *vector derivative or gradient of* **r**•*Q*•**r**

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial \mathbf{r}} (\mathbf{r} \bullet Q \bullet \mathbf{r}) = \nabla (\mathbf{r} \bullet Q \bullet \mathbf{r})$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \left(A \cdot x^2 + 2B \cdot xy + D \cdot y^2 \right) = \begin{pmatrix} 2A \cdot x + 2B \cdot y \\ 2B \cdot x + 2D \cdot y \end{pmatrix}$$

Very simple result:

$$\frac{\partial}{\partial \mathbf{r}} \left(\frac{\mathbf{r} \bullet Q \bullet \mathbf{r}}{2} \right) = \nabla \left(\frac{\mathbf{r} \bullet Q \bullet \mathbf{r}}{2} \right) = Q \bullet \mathbf{r}$$

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

(Still more) Operator geometric sequences and eigenvectors

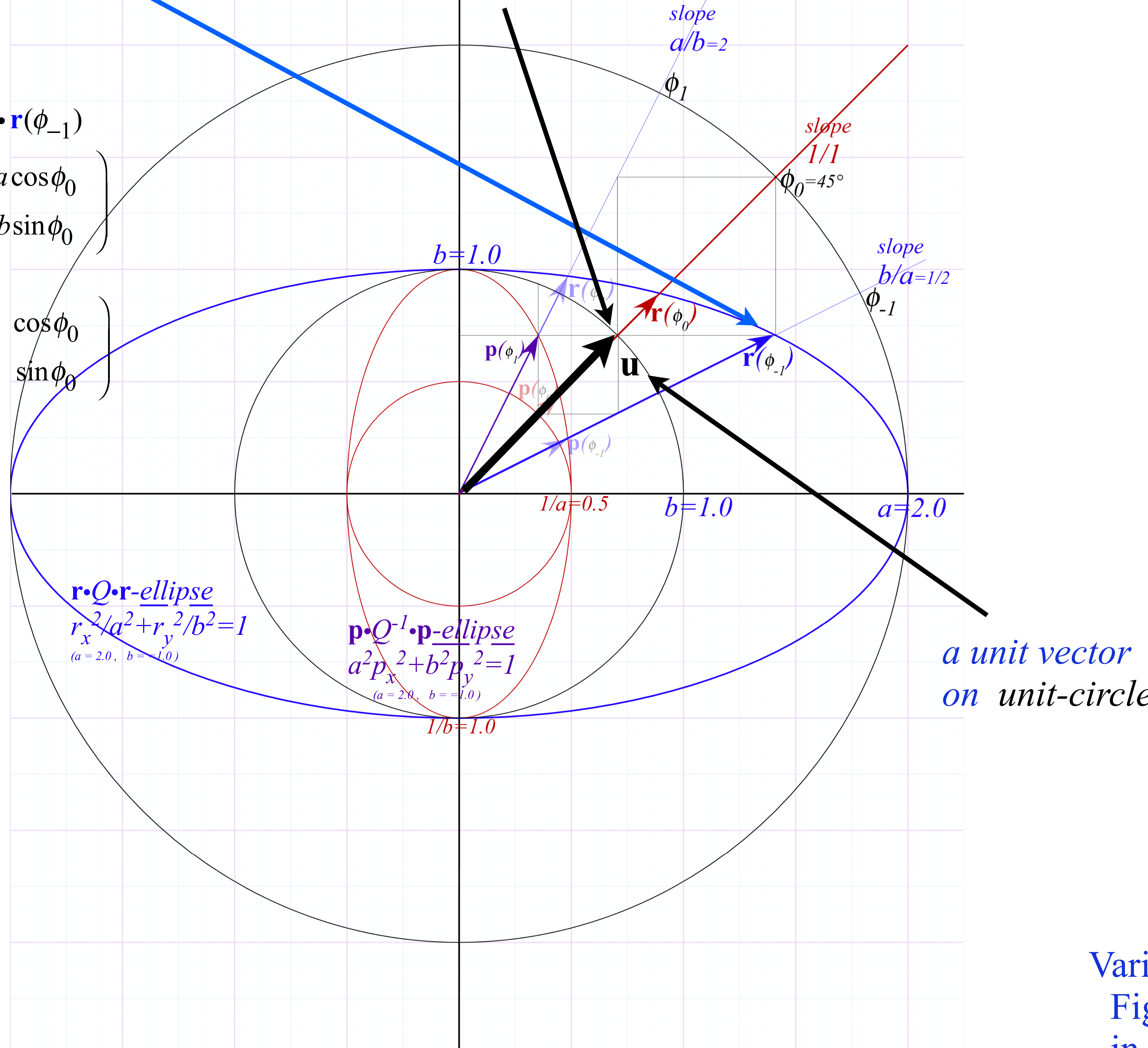
Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

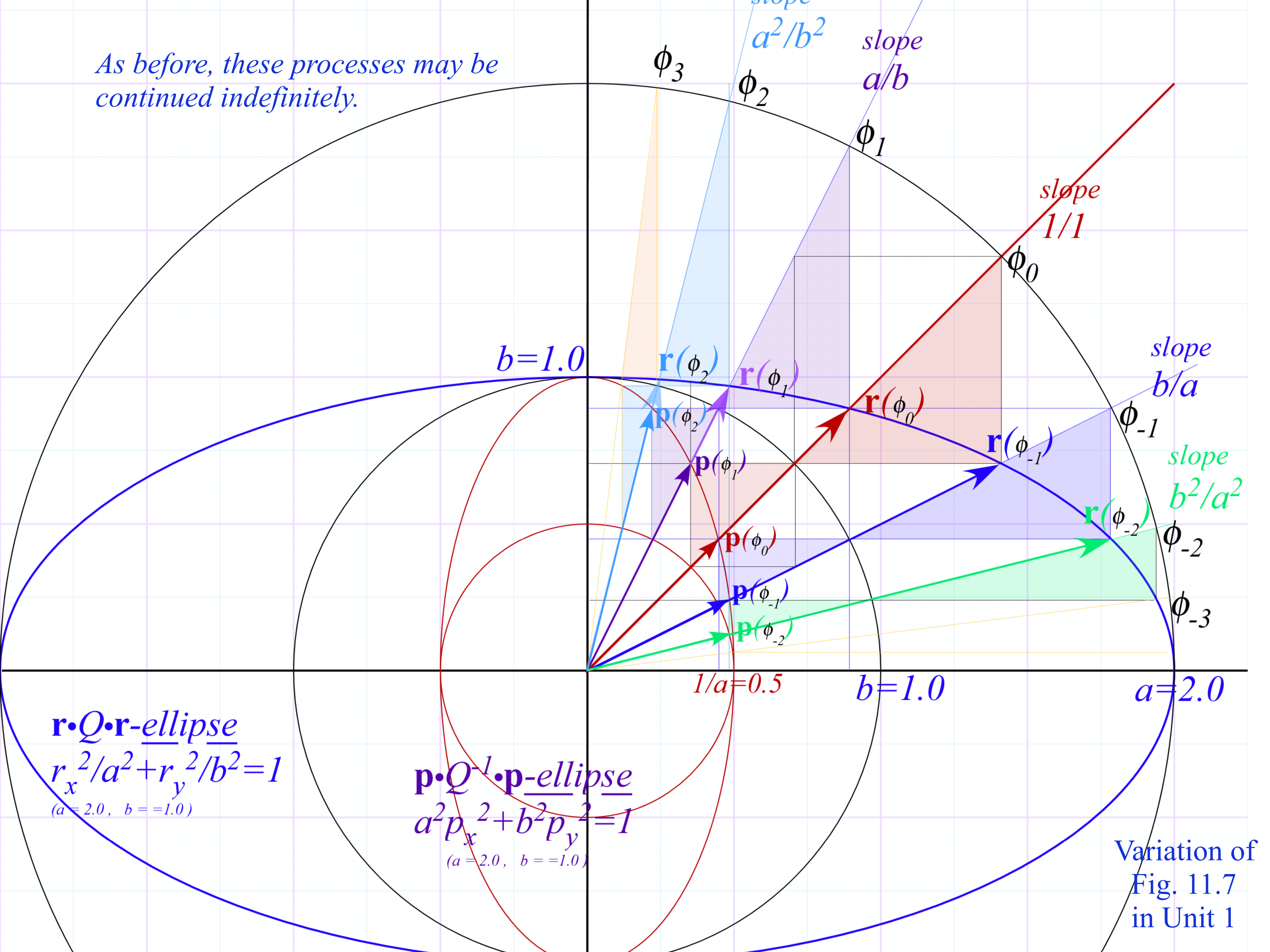
Action of "sqrt-"matrix $R=\sqrt{Q}$ on a single **r***-vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to* **u***-circle (*$\mathbf{u}\bullet\mathbf{u}=1$*), that is,* $R\bullet\mathbf{r}(\phi_{-1}) = \mathbf{u} = (const.)\mathbf{r}(\phi_0)$

(R generates another ellipse $\mathbf{r}\bullet R\bullet\mathbf{r}=1$ *not shown)*

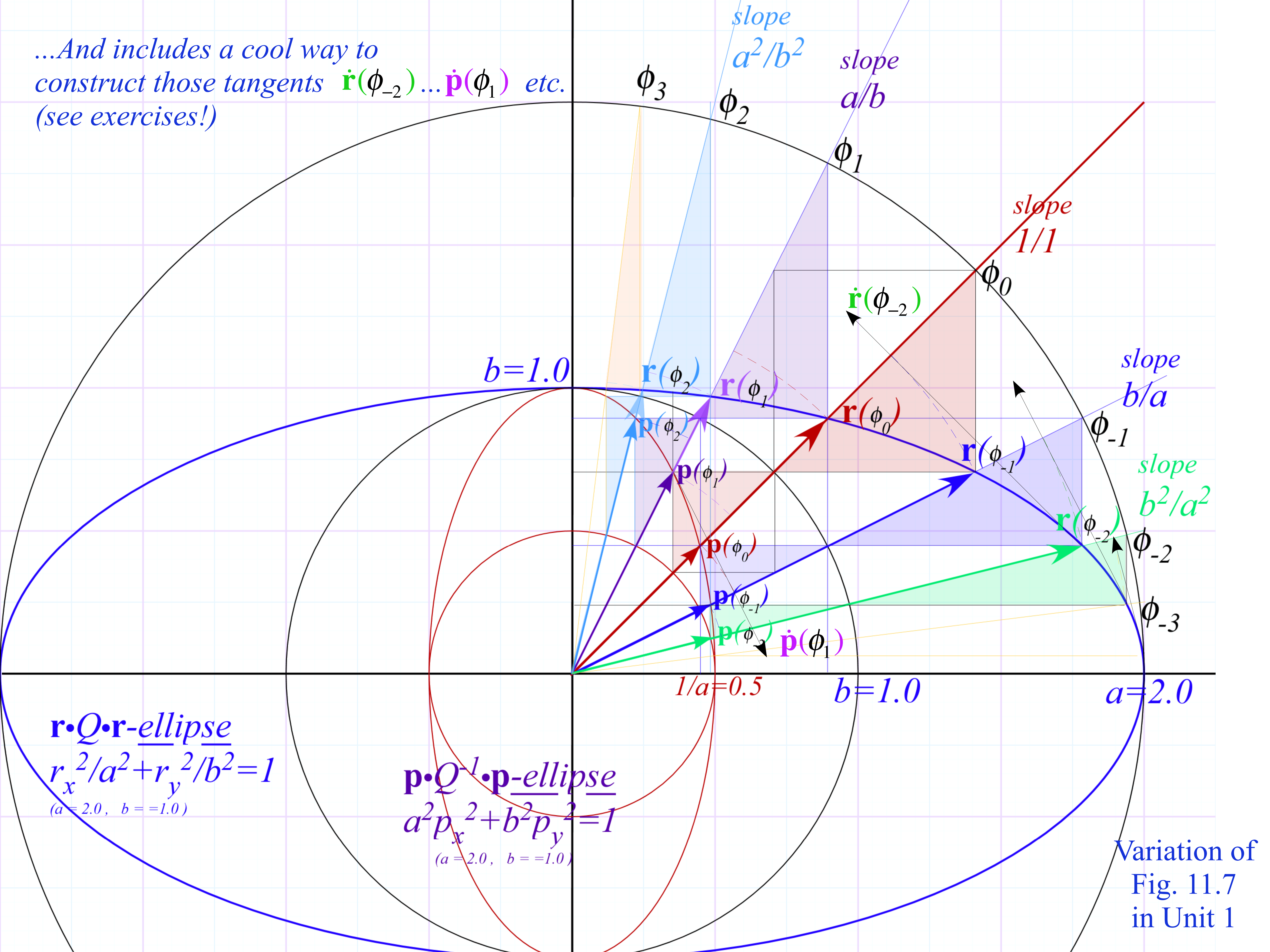
$$\mathbf{u} = \sqrt{\mathbf{Q}}\bullet\mathbf{r}(\phi_{-1}) = \mathbf{R}\bullet\mathbf{r}(\phi_{-1})$$
$$= \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{a}a\cos\phi_0 \\ \frac{1}{b}b\sin\phi_0 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1



Variation of Fig. 11.7 in Unit 1


...And includes a cool way to
construct those tangents ṙ(ϕ₋₂) ... ṗ(ϕ₁) etc.
(see exercises!)
slope a²/b²
slope a/b
slope 1/1
slope b/a
slope b²/a²
ϕ3 ϕ2 ϕ1 ϕ0 ϕ-1 ϕ-2 ϕ-3
b=1.0
r(ϕ2) r(ϕ1) r(ϕ0) r(ϕ-1) r(ϕ-2)
p(ϕ2) p(ϕ1) p(ϕ0) p(ϕ-1) p(ϕ-2)
ṙ(ϕ₋₂)
ṗ(ϕ₁)
1/a=0.5
b=1.0
a=2.0
r•Q•r-ellipse
rx²/a²+ry²/b²=1
(a = 2.0 , b = =1.0)
p•Q⁻¹•p-ellipse
a²px²+b²py²=1
(a = 2.0 , b = =1.0)
Variation of
Fig. 11.7
in Unit 1

Q:Where is this headed? *A: Lagrangian-Hamiltonian duality*

Preview of Lecture 8

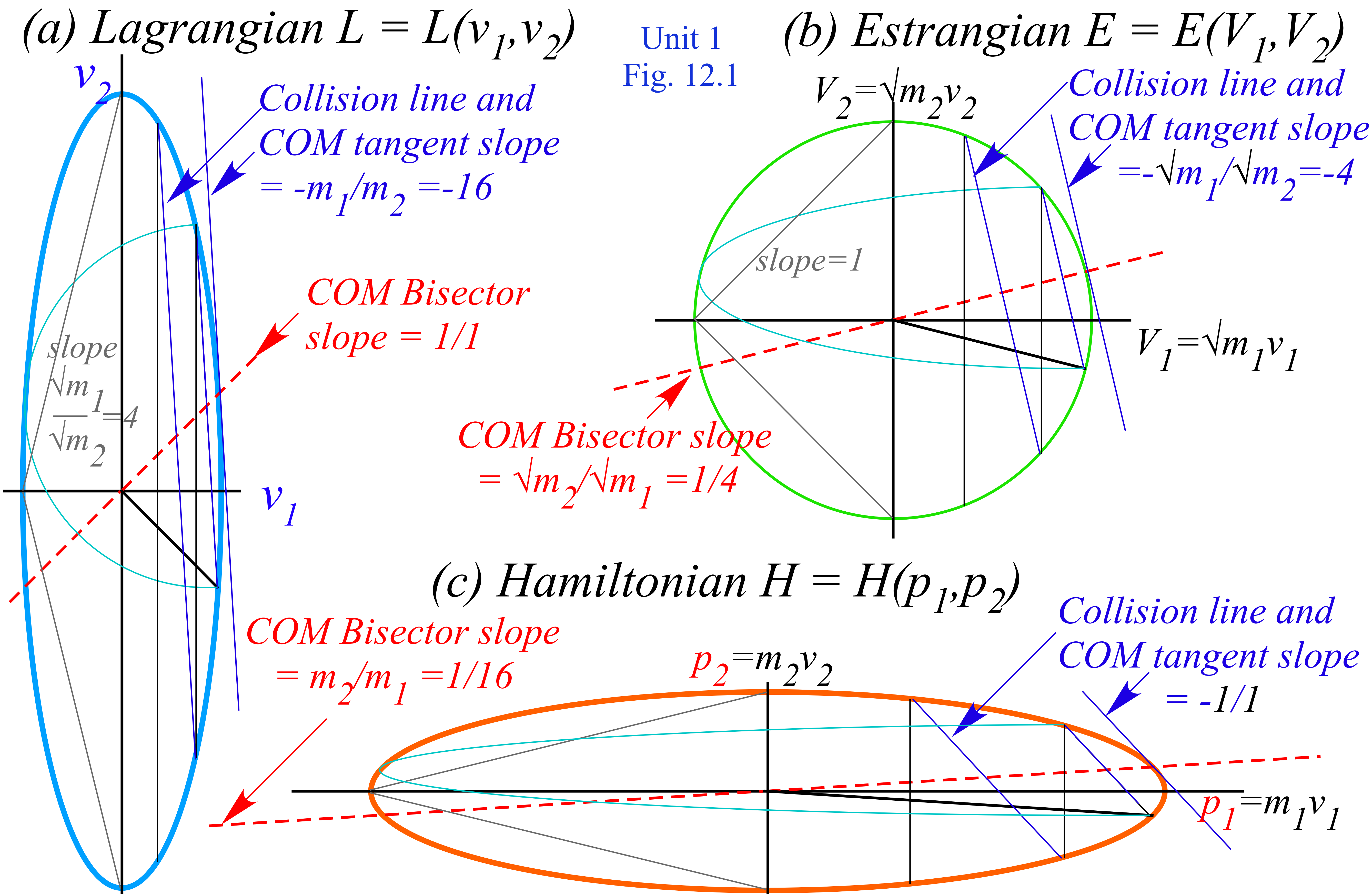
The R and Q matrix transformations are like the mechanics rescaling matrices $\sqrt{\mathbf{M}}$ *and* $\mathbf{M}$*:*

Like $Q=R^2$*:* $$\mathbf{M}=\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}=\mathbf{R}^2$$

Like $\sqrt{Q}=R$*:* $$\sqrt{\mathbf{M}}=\begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}=\mathbf{R}$$

Like $Q^{-1}=R^{-2}$*:* $$\mathbf{M}^{-1}=\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}=\mathbf{R}^{-2}$$

Unit 1
Fig. 12.1



Unit 1

Fig. 12.2