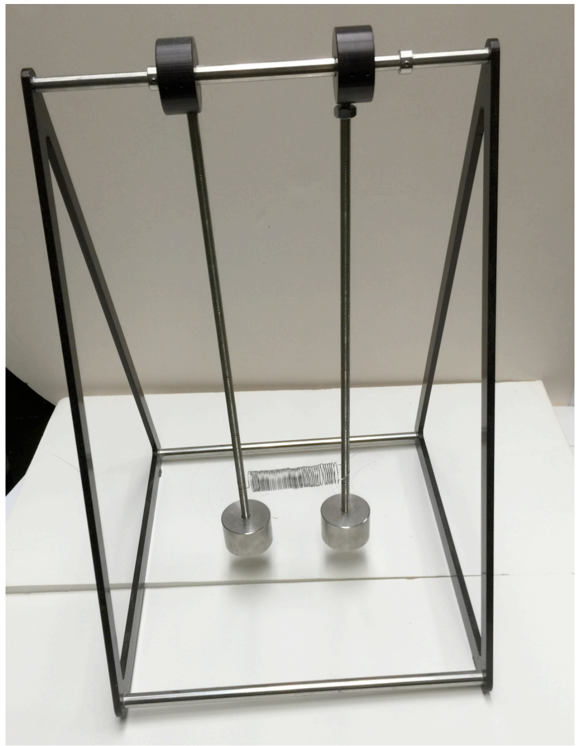
Lecture 21
Fri. 11.04.2016

Introduction to coupled oscillation and eigenmodes

(Ch. 2-4 of Unit 4 11.12.15)



2D harmonic oscillator equations
- *Lagrangian and matrix forms and Reciprocity symmetry*

2D harmonic oscillator equation eigensolutions
- *Geometric method*

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
- *Secular equation*
- *Hamilton-Cayley equation and projectors*
- *Idempotent projectors (how eigenvalues⇒eigenvectors)*
- *Operator orthonormality and Completeness (Idempotent means:* **P·P=P***)*

Spectral Decompositions
- *Functional spectral decomposition*
- *Orthonormality vs. Completeness vis-a`-vis Operator vs. State*
- *Lagrange functional interpolation formula*
  - *Diagonalizing Transformations (D-Ttran) from projectors*

2D-HO eigensolution example with bilateral (B-Type) symmetry
- *Mixed mode beat dynamics and fixed π/2 phase*

2D-HO eigensolution example with asymmetric (A-Type) symmetry
- *Initial state projection, mixed mode beat dynamics with variable phase*

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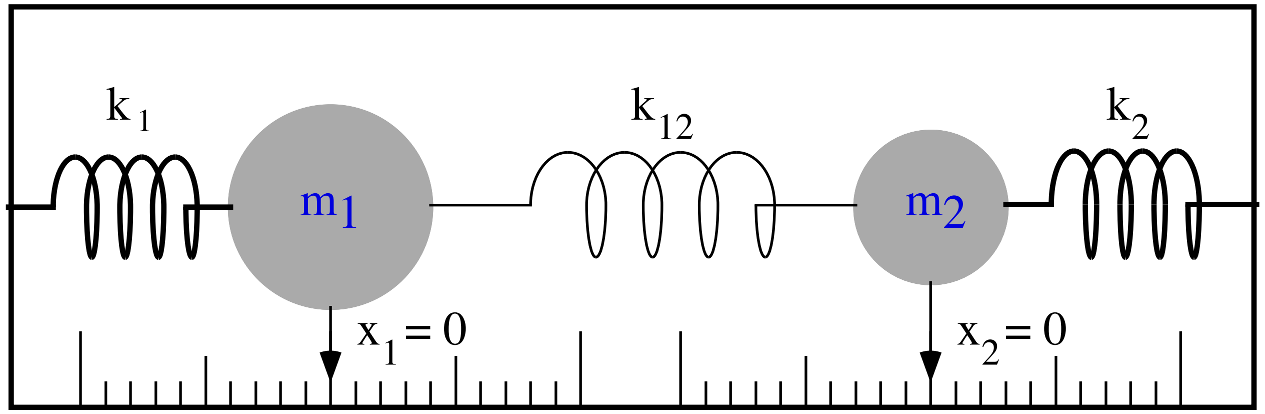
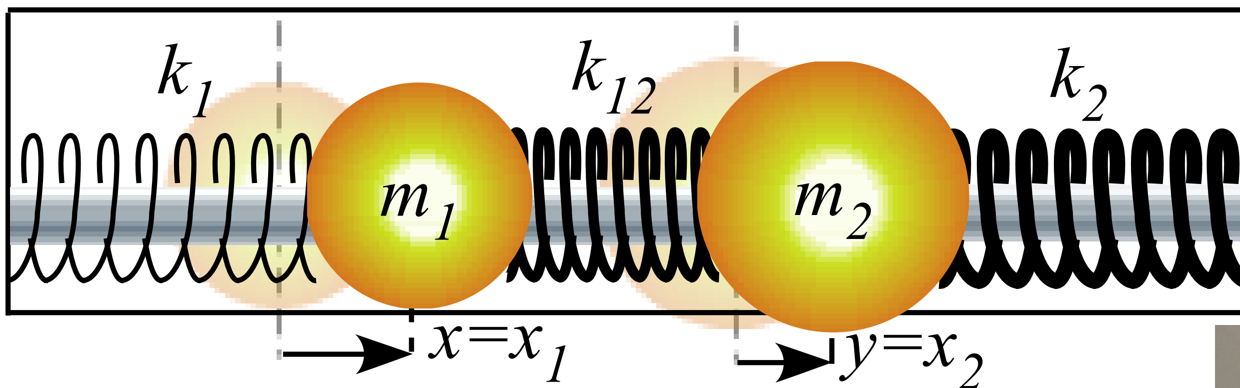


Fig. 3.3.1 Two 1-dimensional coupled oscillators



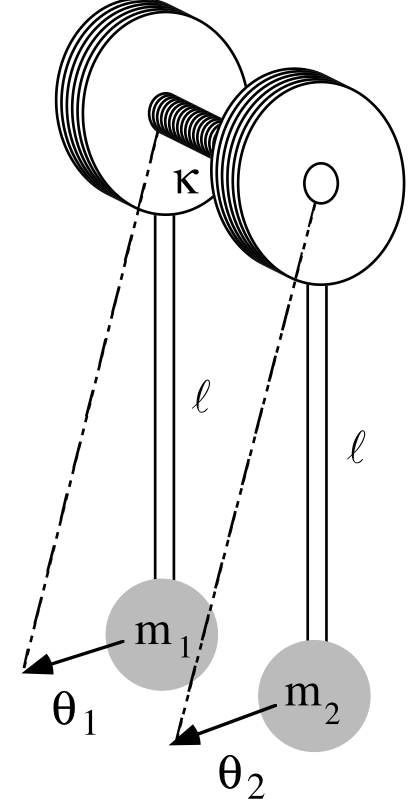


Fig. 3.3.2 Coupled pendulums

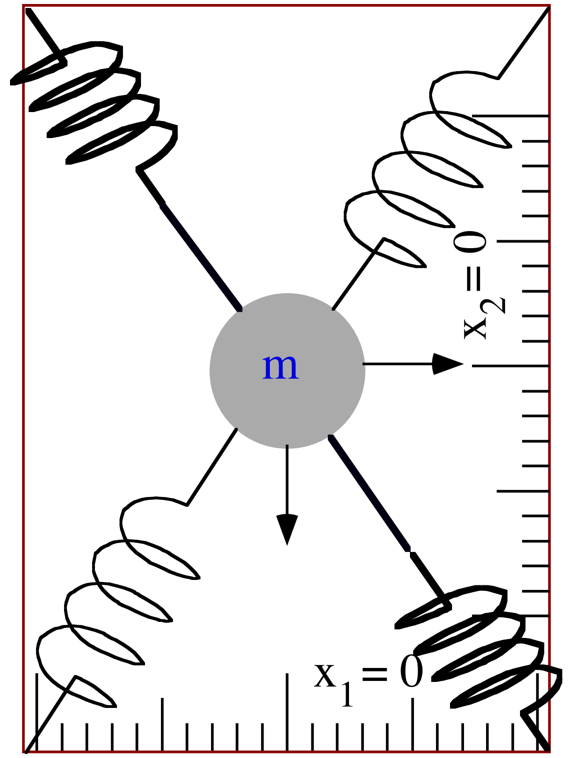
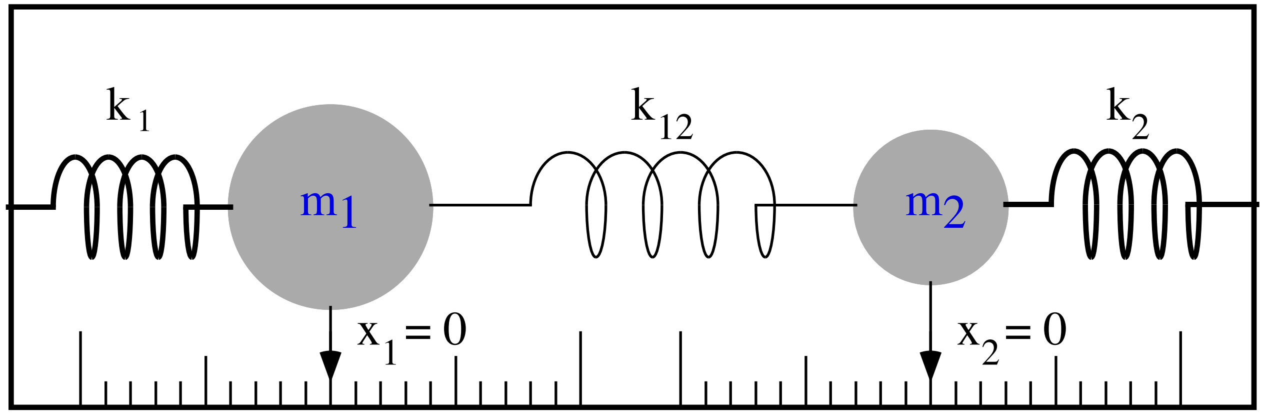


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D harmonic oscillator energy



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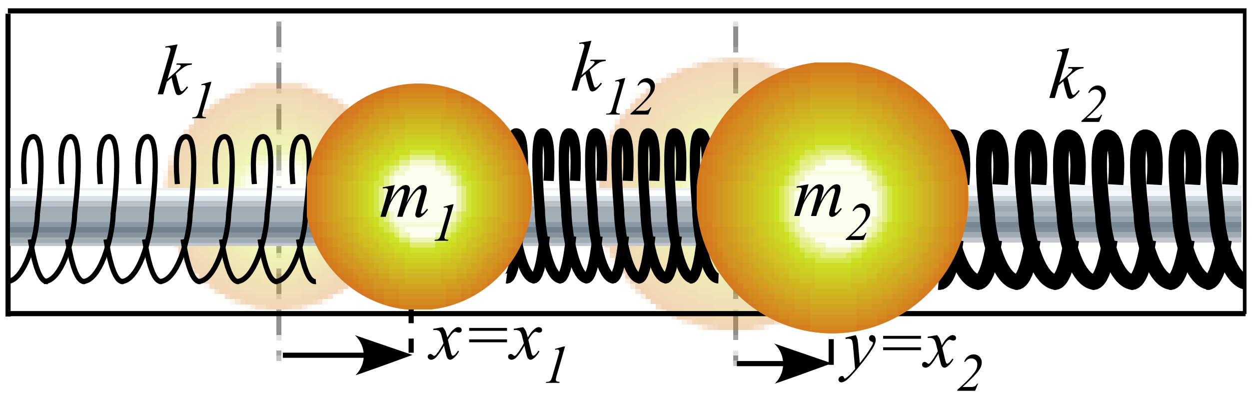

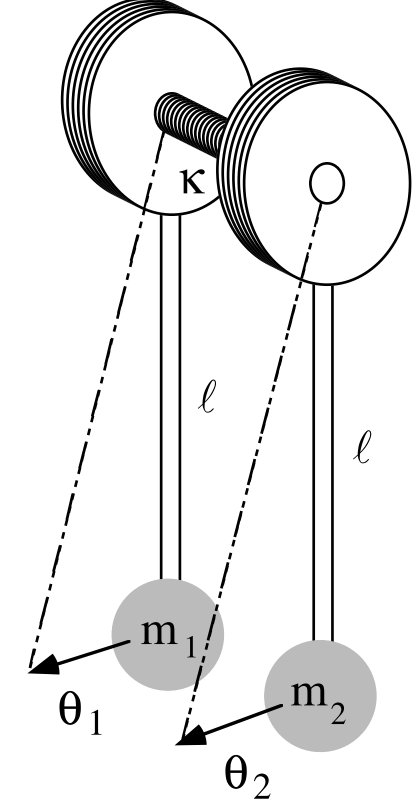

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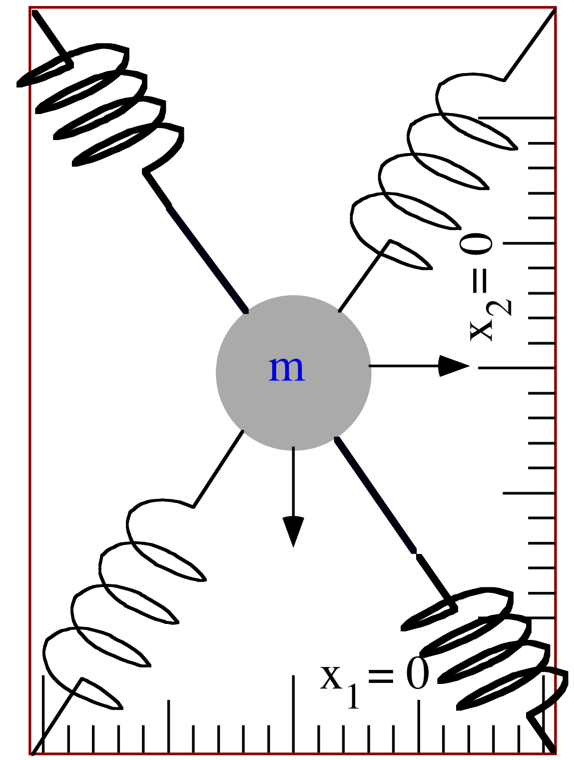

Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

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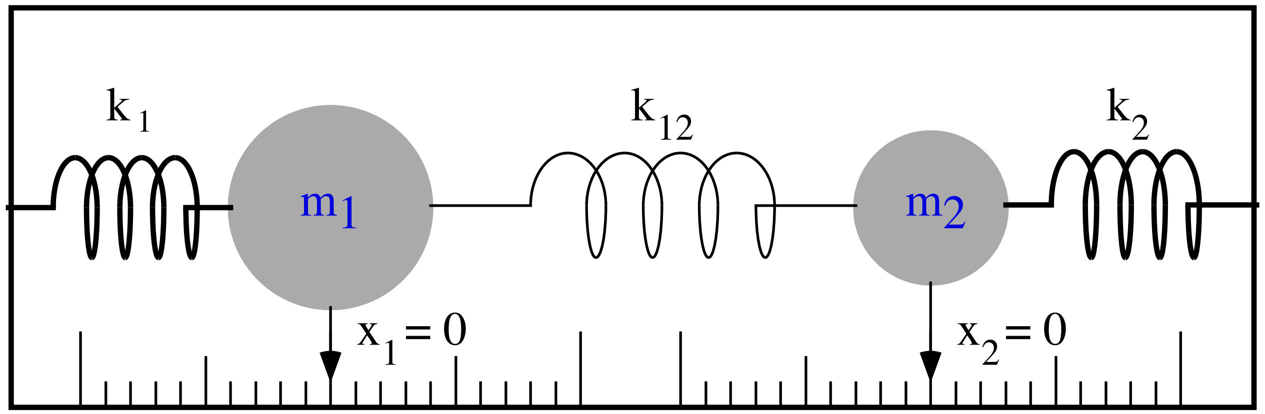
2D harmonic oscillator energy



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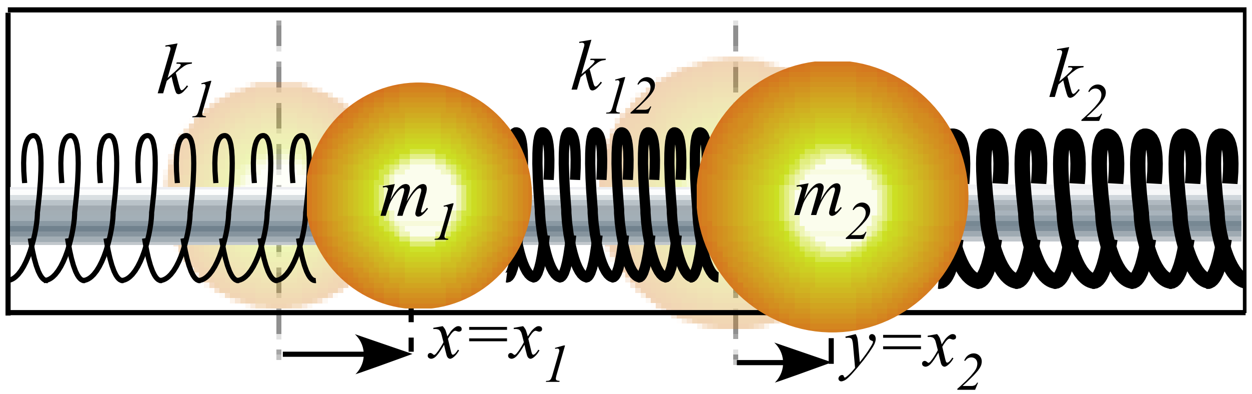

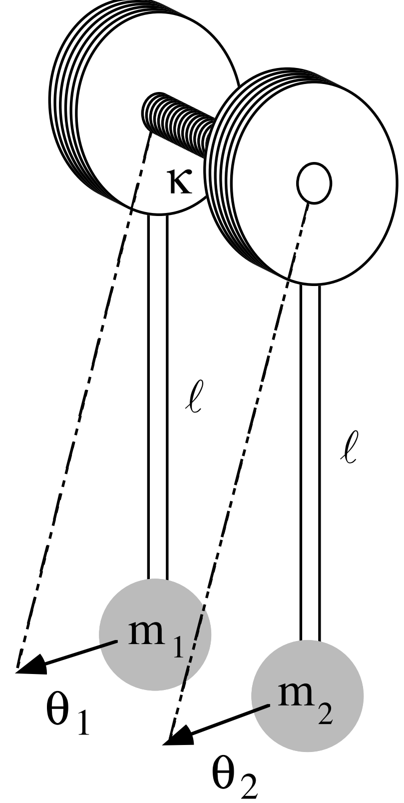

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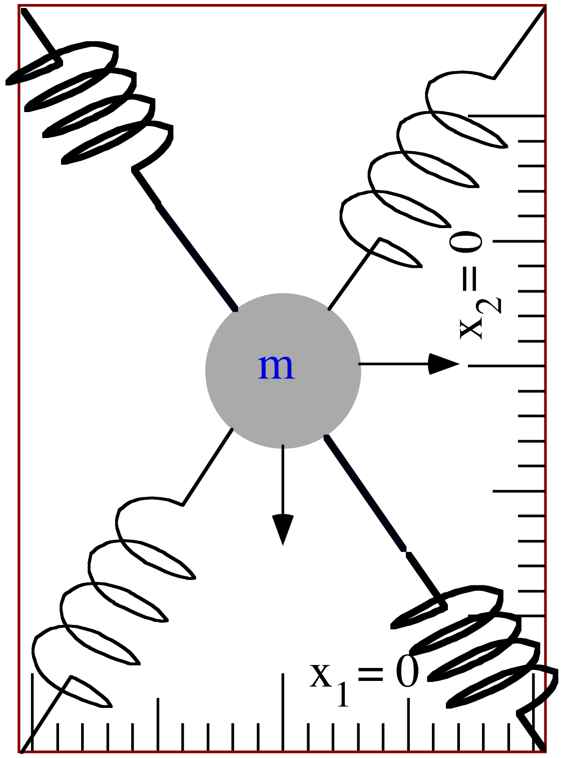

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$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2$$

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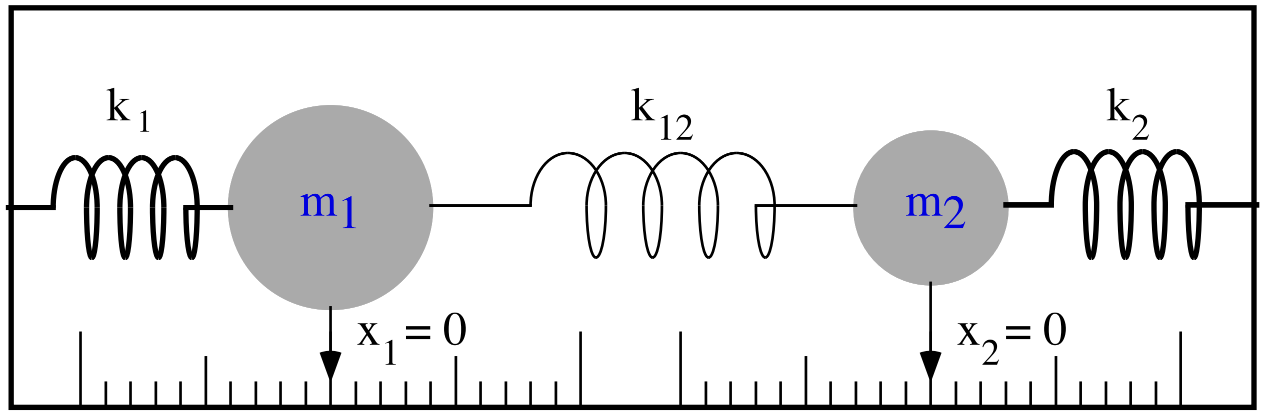

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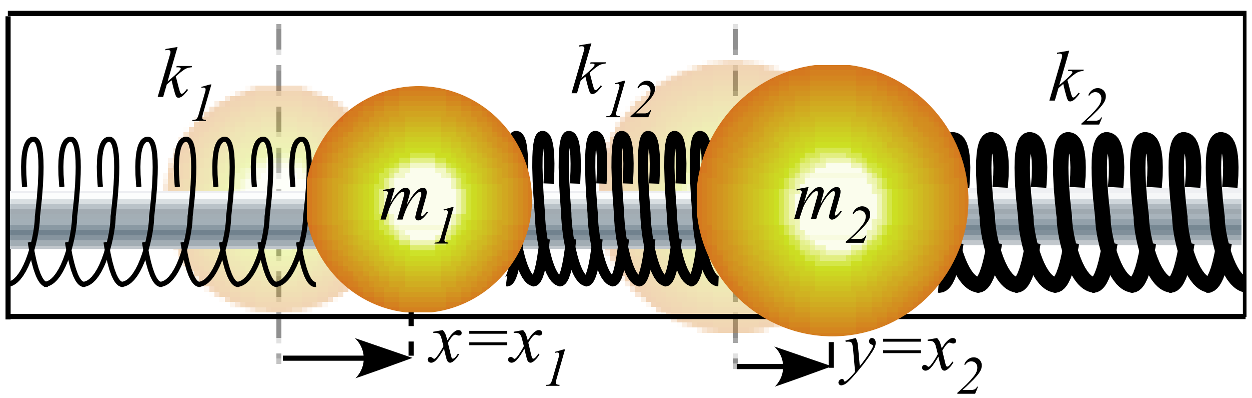

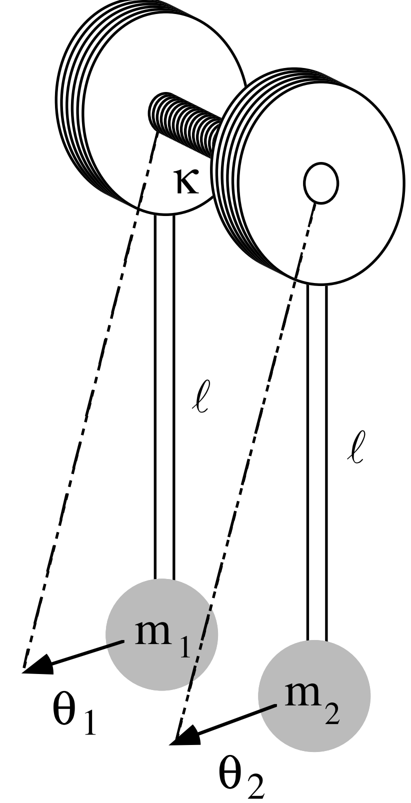

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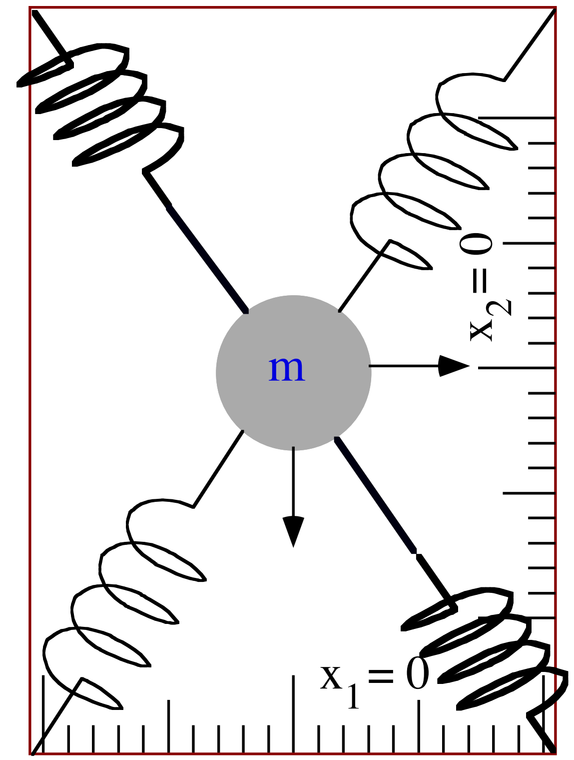

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Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12} \right) x_1 + k_{12} x_2$$

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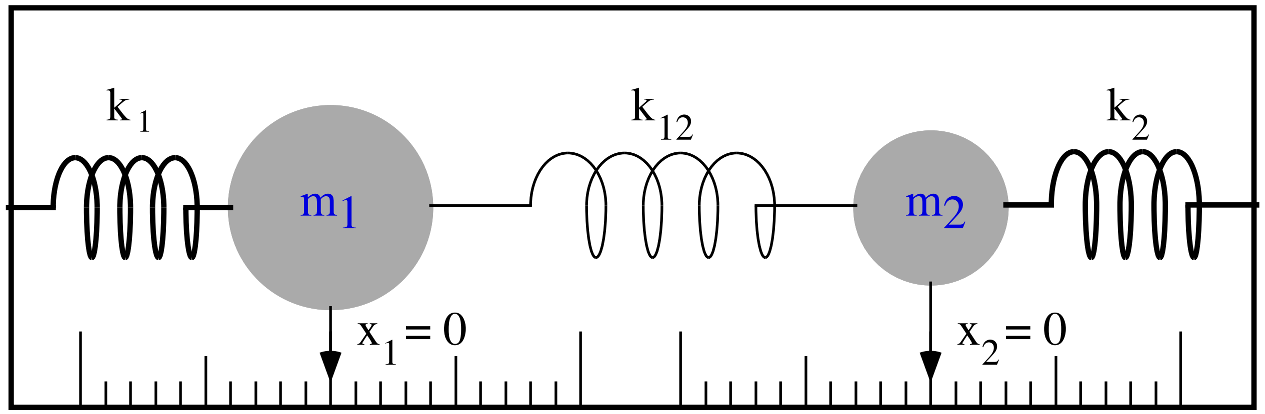
2D harmonic oscillator equations



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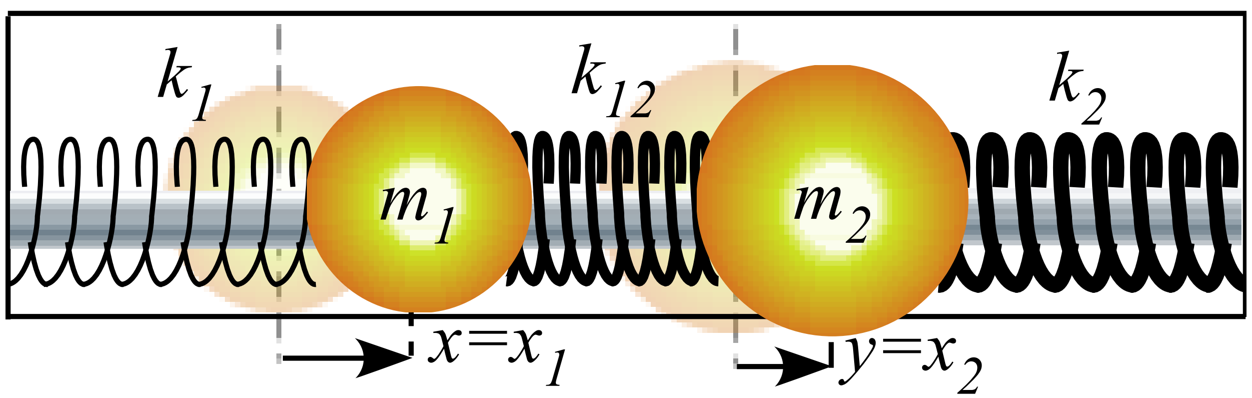

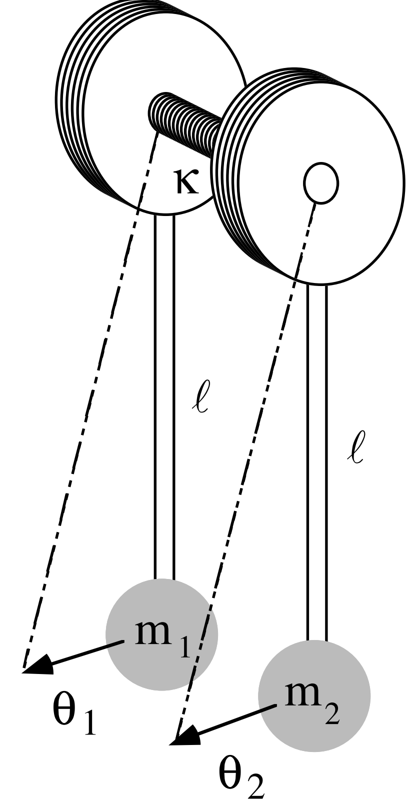

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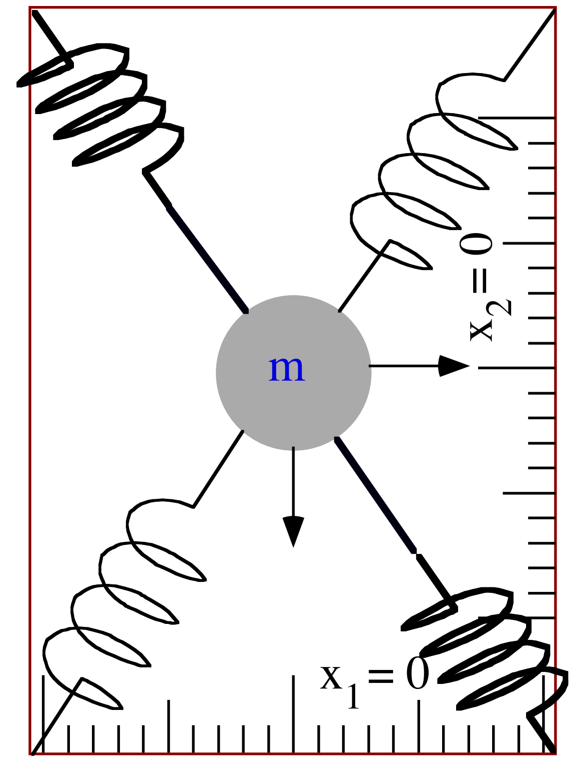

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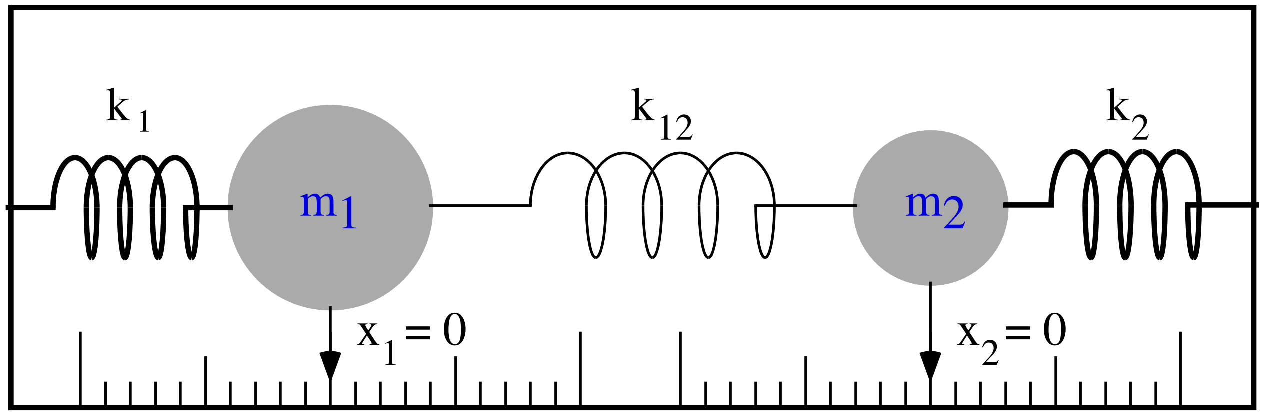

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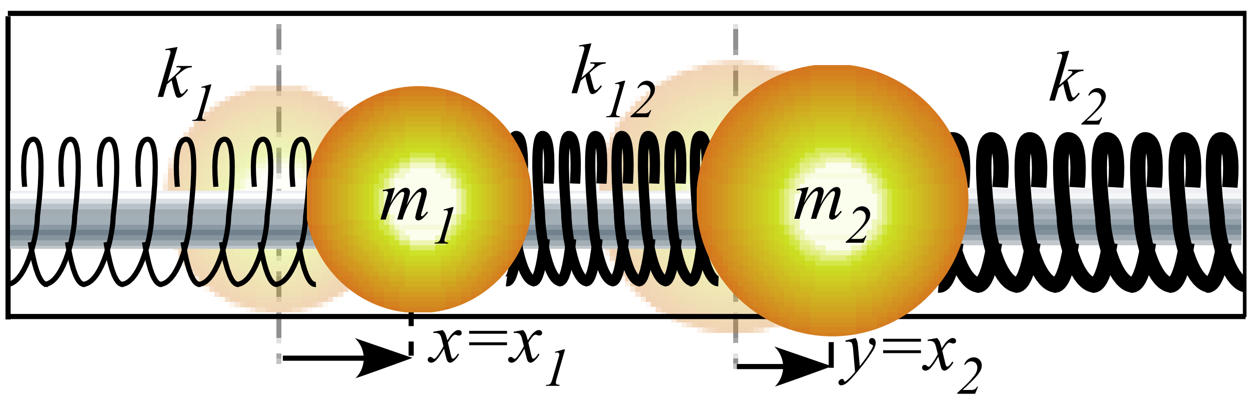

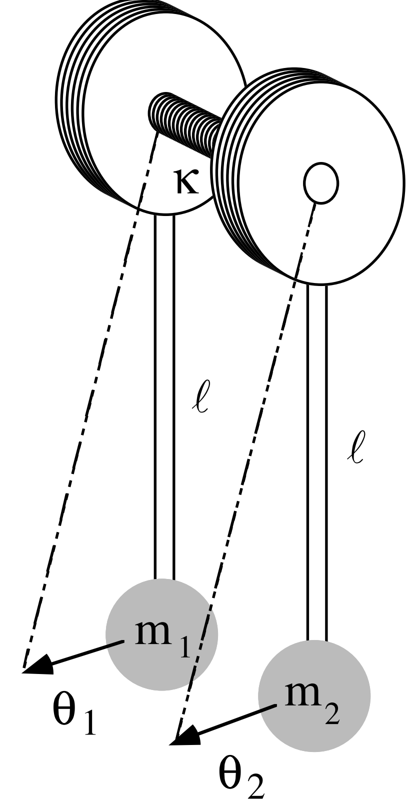

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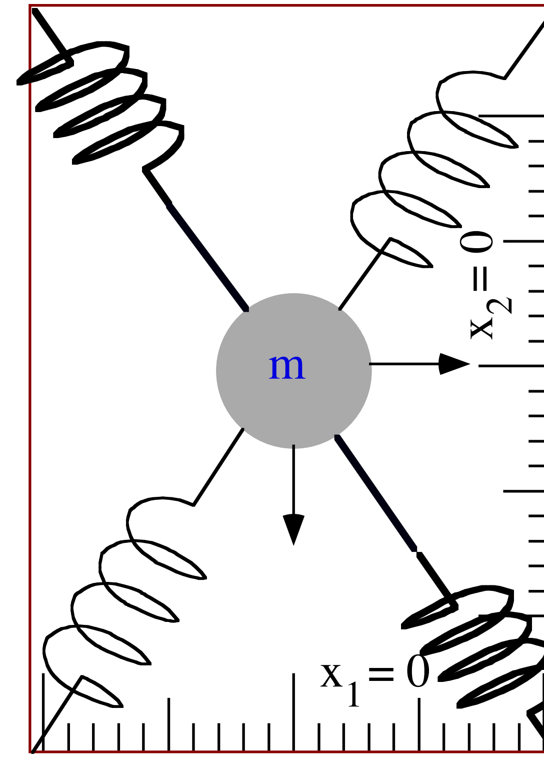

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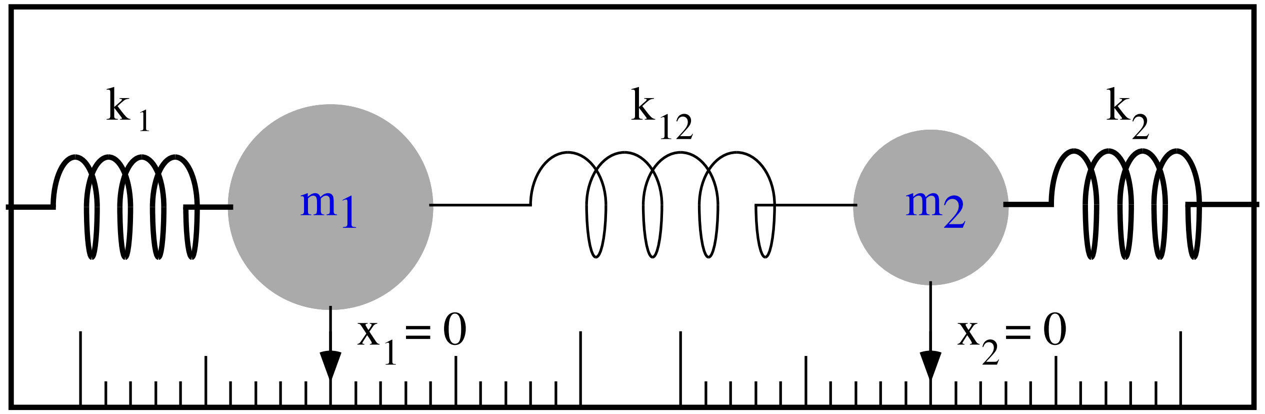
Matrix operator notation:

$$\mathbf{M} \bullet \left|\ddot{\mathbf{x}}\right\rangle \quad = \quad - \quad \mathbf{K} \bullet \left|\mathbf{x}\right\rangle$$

2D harmonic oscillator equations



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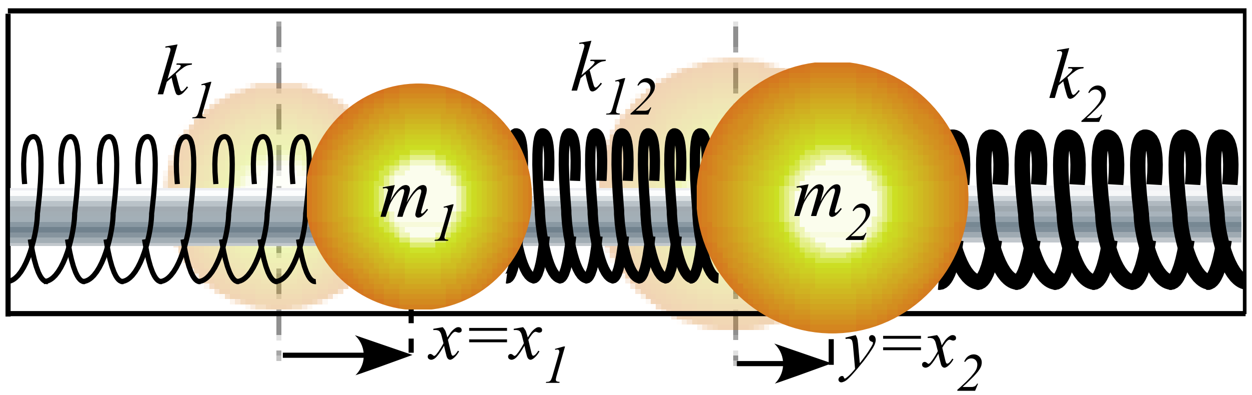

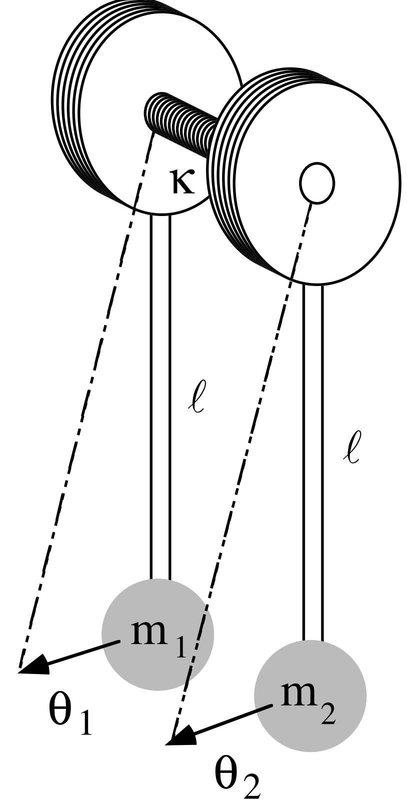

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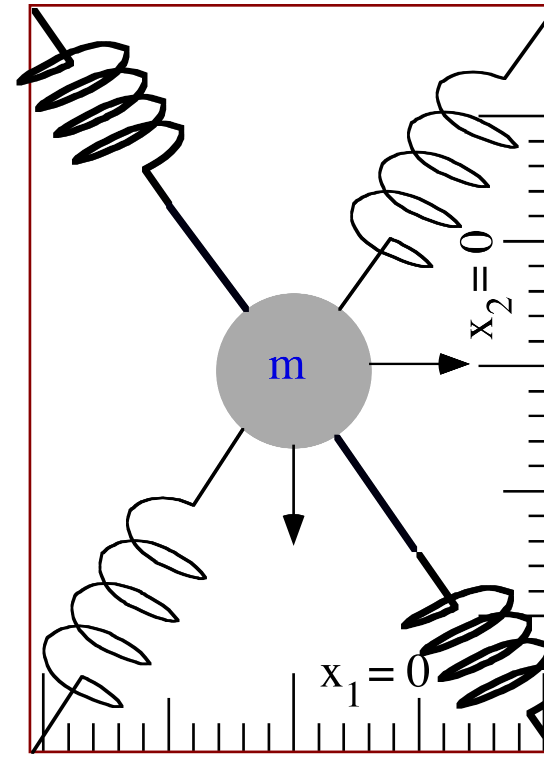

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$$= \frac{1}{2}\langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

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Matrix equations and reciprocity symmetry

General form of 2D-HO equation of motion has force matrix components: $\kappa_{11} = k_1 + k_{11}, \quad \kappa_{22} = k_2 + k_{22}$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} m_1 \ddot{x}_1 \\ m_2 \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Off-diagonal force constants satisfy *Reciprocity Relations:* $-\kappa_{12} = k_{12} = \frac{\partial F_1}{\partial x_2} = -\frac{\partial^2 V}{\partial x_2 \partial x_1} = -\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial F_2}{\partial x_1} = k_{21} = -\kappa_{21}$

Rescaling and symmetrization

Each coordinate (x_1, x_2) is rescaled $(q_1 = s_1 x_1, q_2 = s_2 x_2)$ to symmetrize mass factors on $\ddot{q}_j$-terms.

$$-\frac{m_1}{s_1}\ddot{q}_1 = \kappa_{11}\frac{q_1}{s_1} + \kappa_{12}\frac{q_2}{s_2} \qquad -\ddot{q}_1 = \frac{\kappa_{11}}{m_1} q_1 + \frac{\kappa_{12} s_1}{m_1 s_2} q_2 \equiv \mathrm{K}_{11} q_1 + \mathrm{K}_{12} q_2$$

$$-\frac{m_2}{s_2}\ddot{q}_2 = \kappa_{12}\frac{q_1}{s_1} + \kappa_{22}\frac{q_2}{s_2} \qquad -\ddot{q}_2 = \frac{\kappa_{12} s_2}{m_2 s_1} q_1 + \frac{\kappa_{22}}{m_2} q_2 \equiv \mathrm{K}_{21} q_1 + \mathrm{K}_{22} q_2$$

New constants K_{ij} have pseudo-reciprocity symmetry for a special scale factor ratio: $\frac{s_2}{s_1} = \sqrt{\frac{m_2}{m_1}}$

$$\mathrm{K}_{21} = \frac{\kappa_{12} s_2}{m_2 s_1} = \mathrm{K}_{12} = \frac{\kappa_{12} s_1}{m_1 s_2} = \frac{-k_{12}}{\sqrt{m_1 m_2}}$$

Diagonal constants K_{jj} are not affected by scaling. To be equal requires: $\frac{\kappa_{11}}{m_1} = \frac{\kappa_{22}}{m_2}$ or: $\frac{\kappa_{11}}{\kappa_{22}} = \frac{m_1}{m_2}$

$$\mathrm{K}_{11} = \frac{\kappa_{11}}{m_1} = \frac{k_1 + k_{12}}{m_1} \qquad \mathrm{K}_{22} = \frac{\kappa_{22}}{m_2} = \frac{k_2 + k_{12}}{m_2}$$

Caution is advised since such forced symmetry may give modes with imaginary frequency.

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

➡ ***2D harmonic oscillator equation eigensolutions*** ⬅
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2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

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and ω_n is an *eigenfrequency*

To introduce eigensolutions we take a simple case of unit masses *(m_1=1=m_2)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

➔ *Geometric method* ←

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

2D-HO eigensolution example with bilateral (B-Type) symmetry

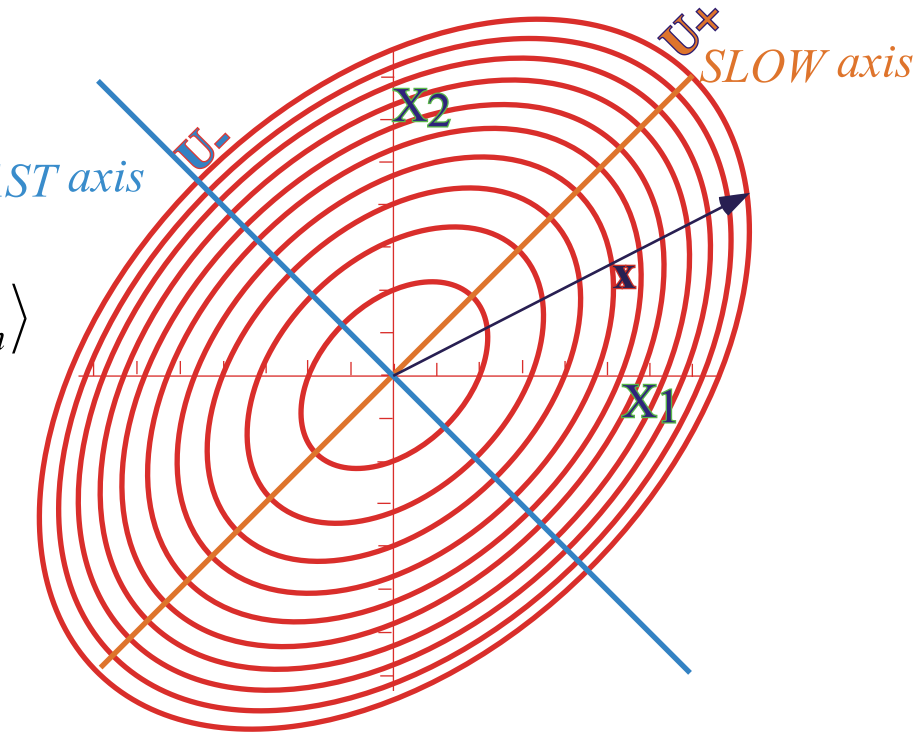
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

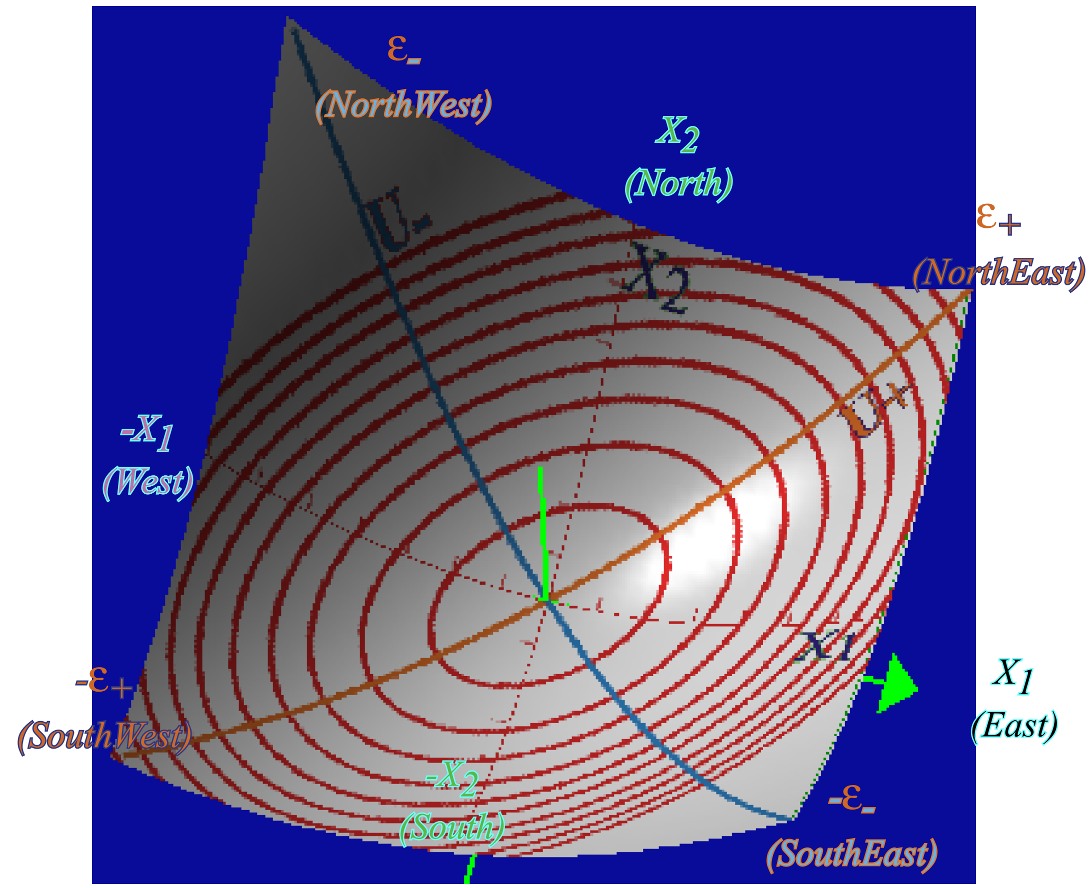
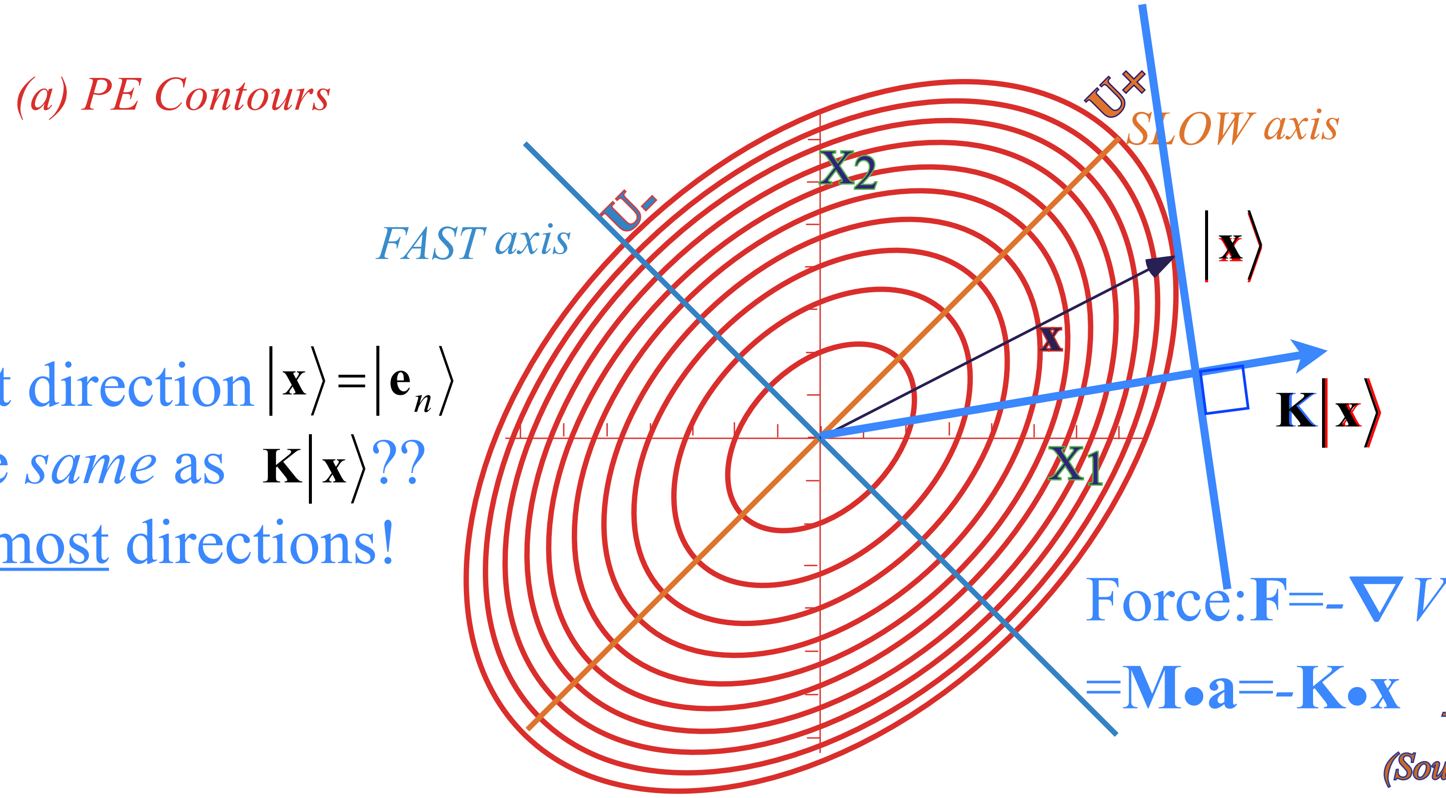


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

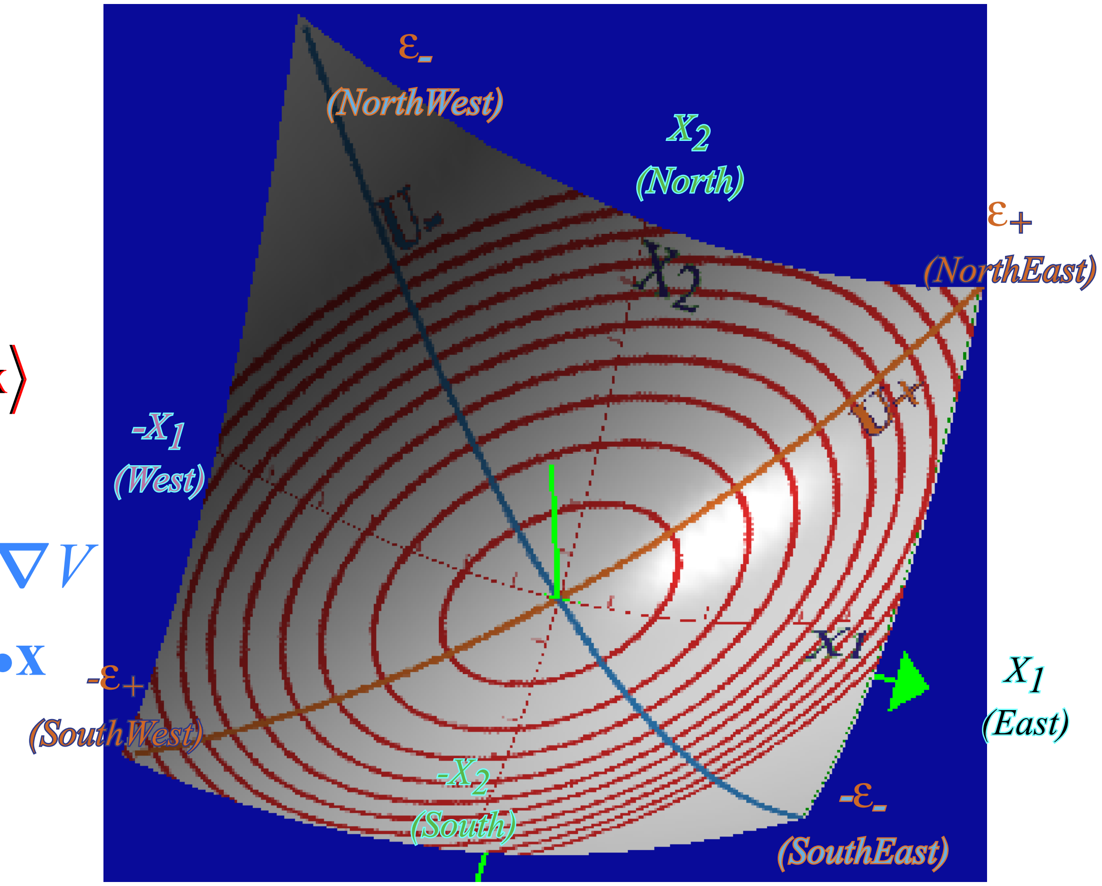
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V=\frac{1}{2}(k_1+k_{12})x_1^2-k_{12}x_1x_2+\frac{1}{2}(k_2+k_{12})x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



What direction $|\mathbf{x}\rangle=|\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!

Force: $\mathbf{F}=-\nabla V$
$=\mathbf{M}\bullet\mathbf{a}=-\mathbf{K}\bullet\mathbf{x}$



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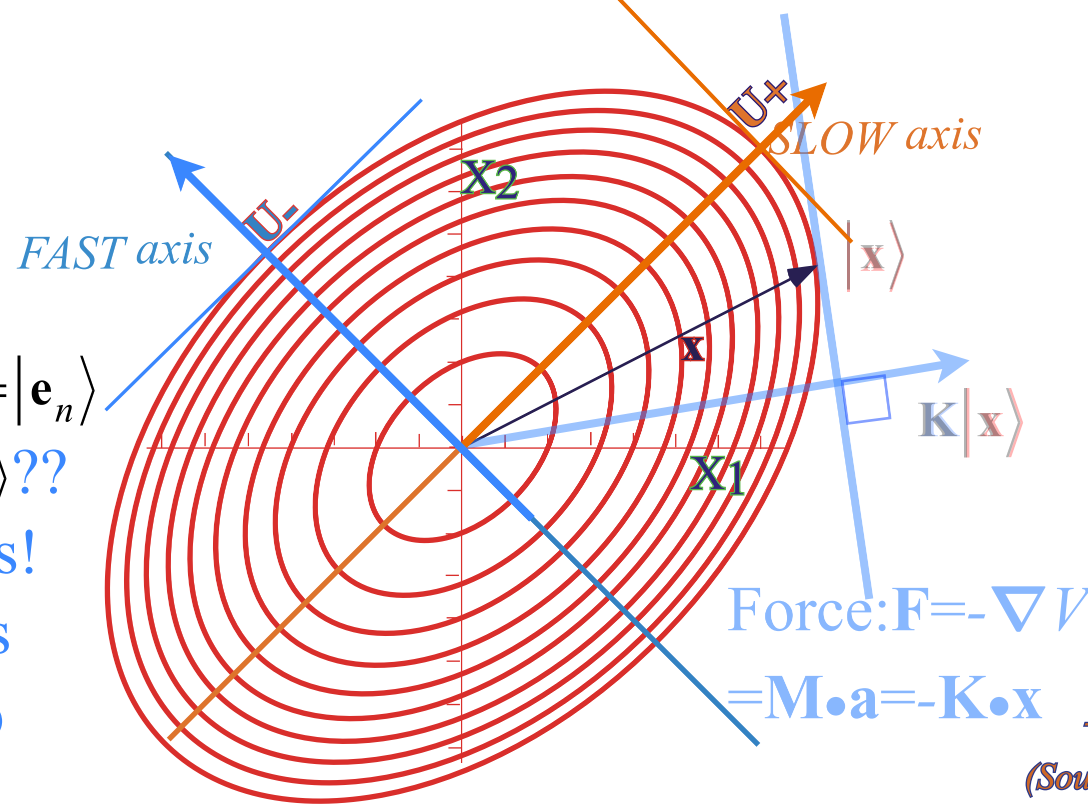
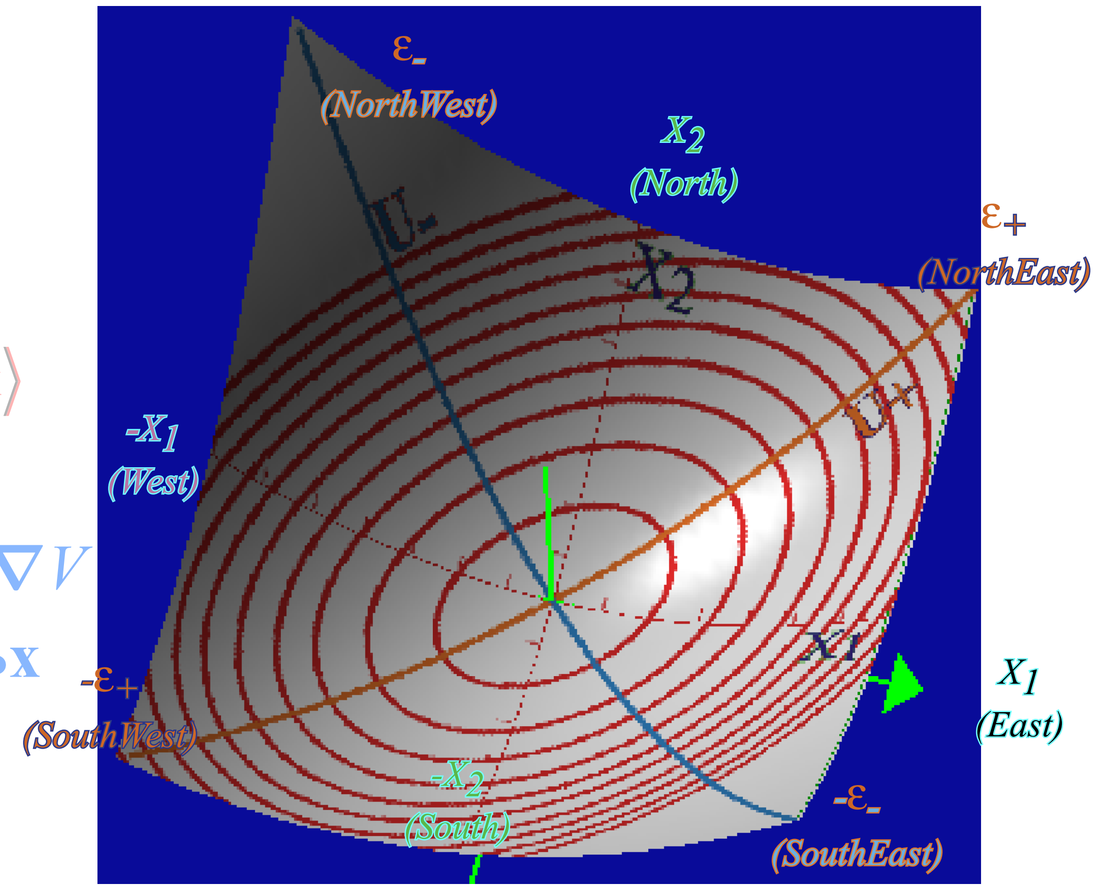
(a) PE Contours

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$
is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not <u>most</u> directions!
Only extremal axes
work. (major or minor axes)





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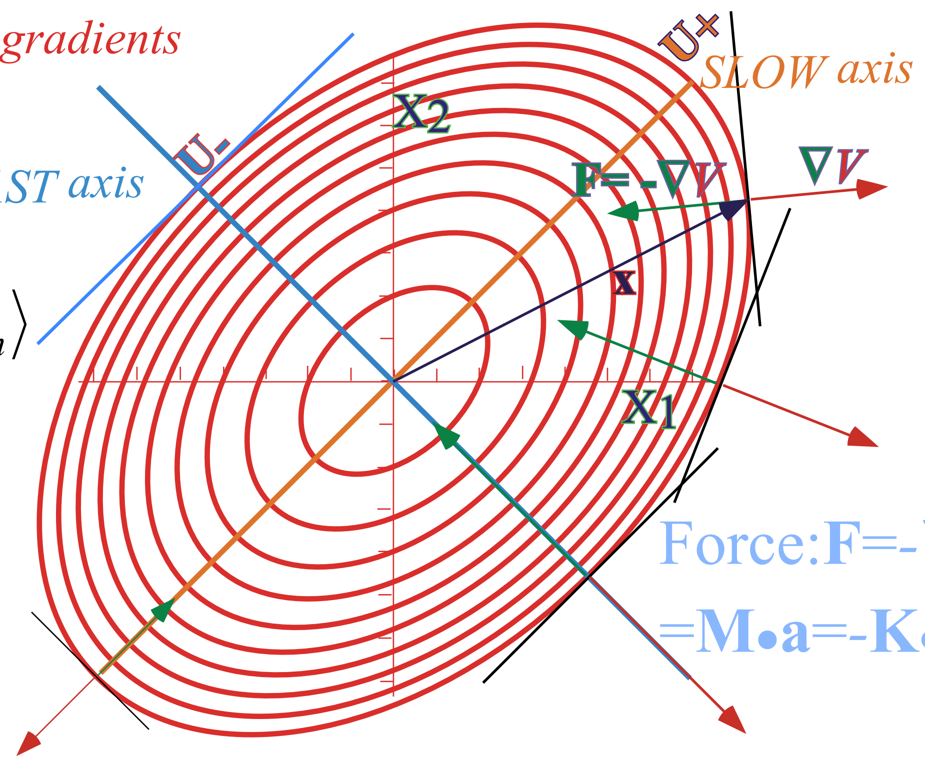
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k+k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the same as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!
Only extremal axes work. (major or minor axes)



Force: $\mathbf{F} = -\nabla V$
$= \mathbf{M} \bullet \mathbf{a} = -\mathbf{K} \bullet \mathbf{x}$

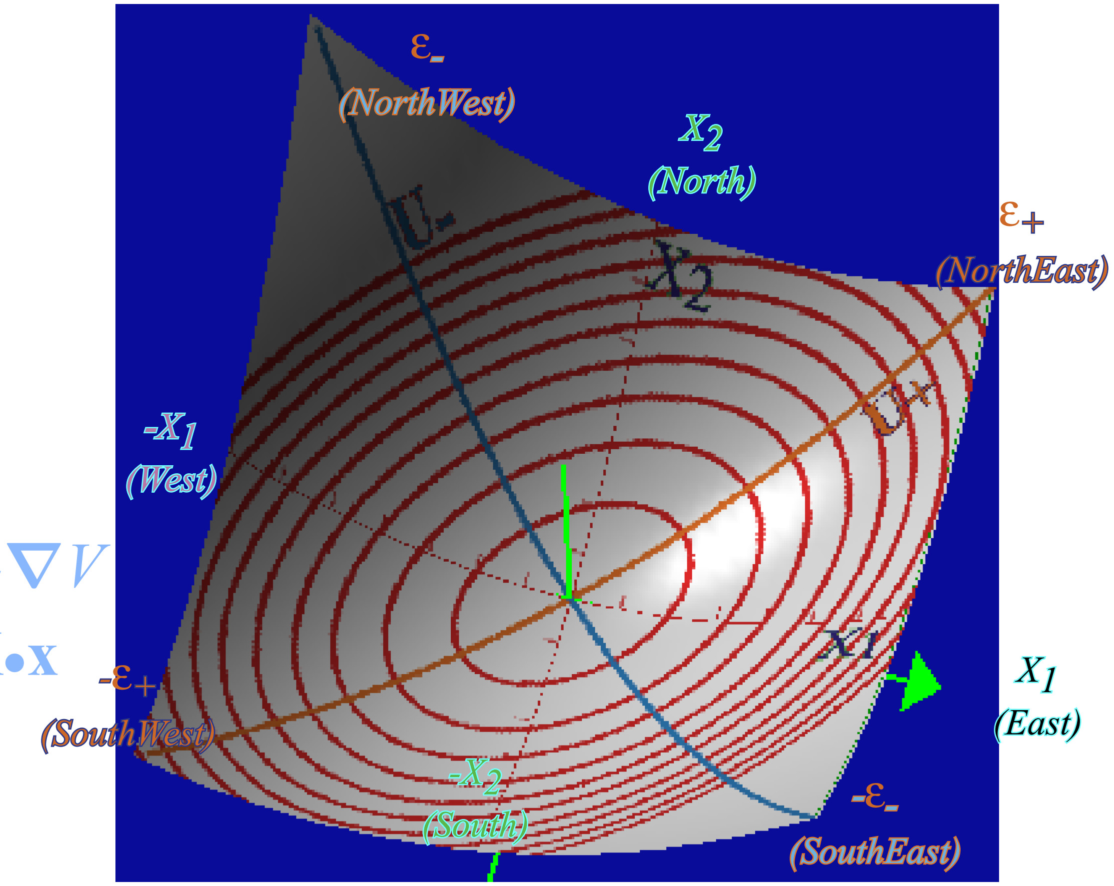
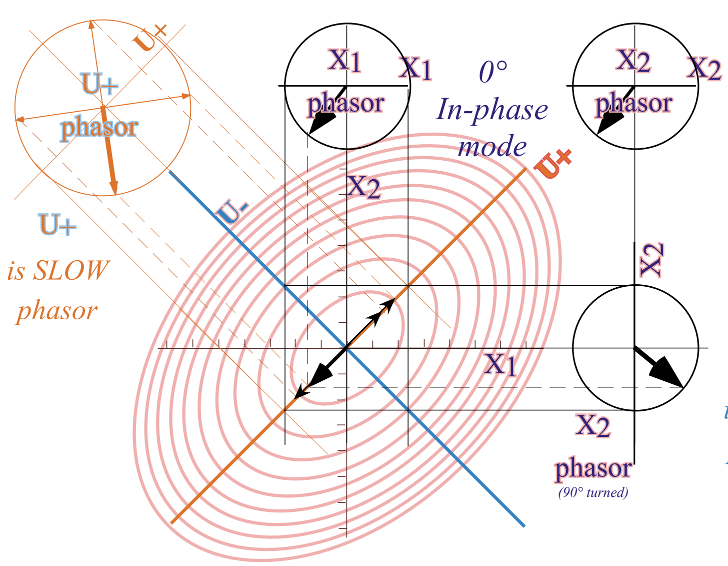
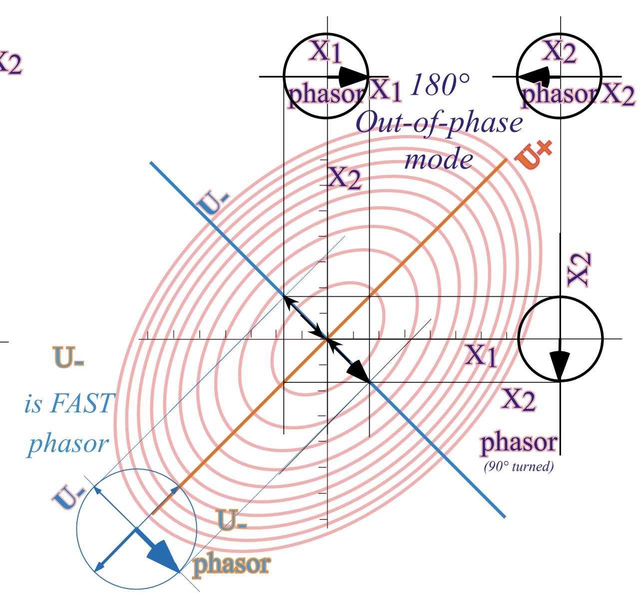


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode



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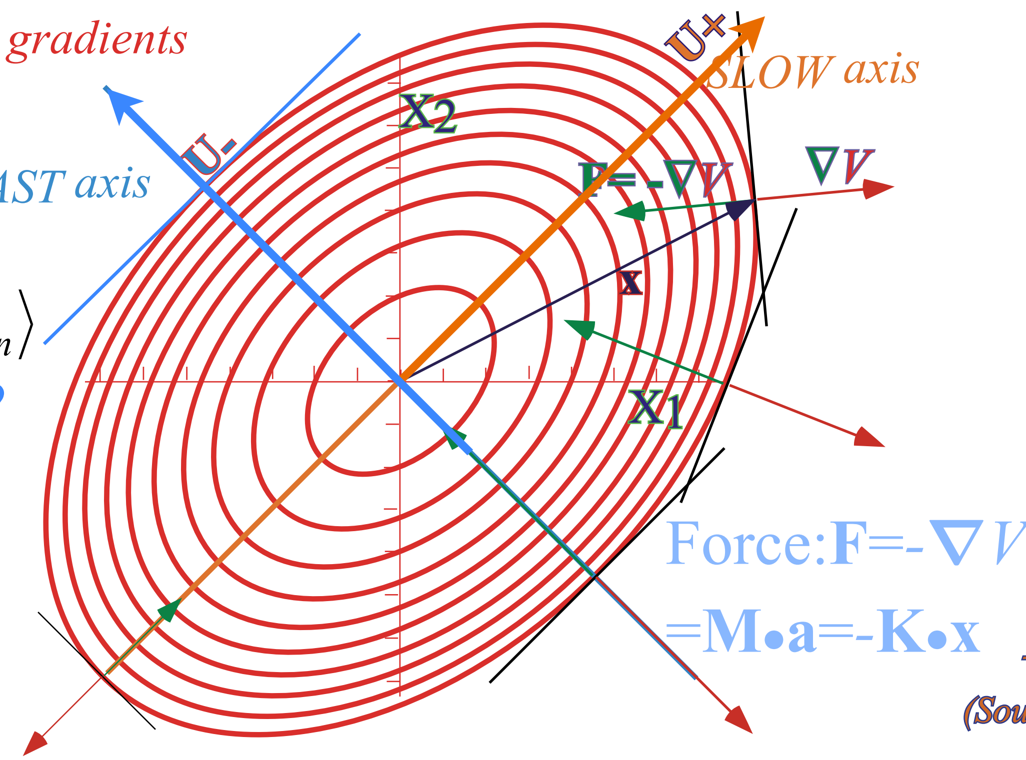
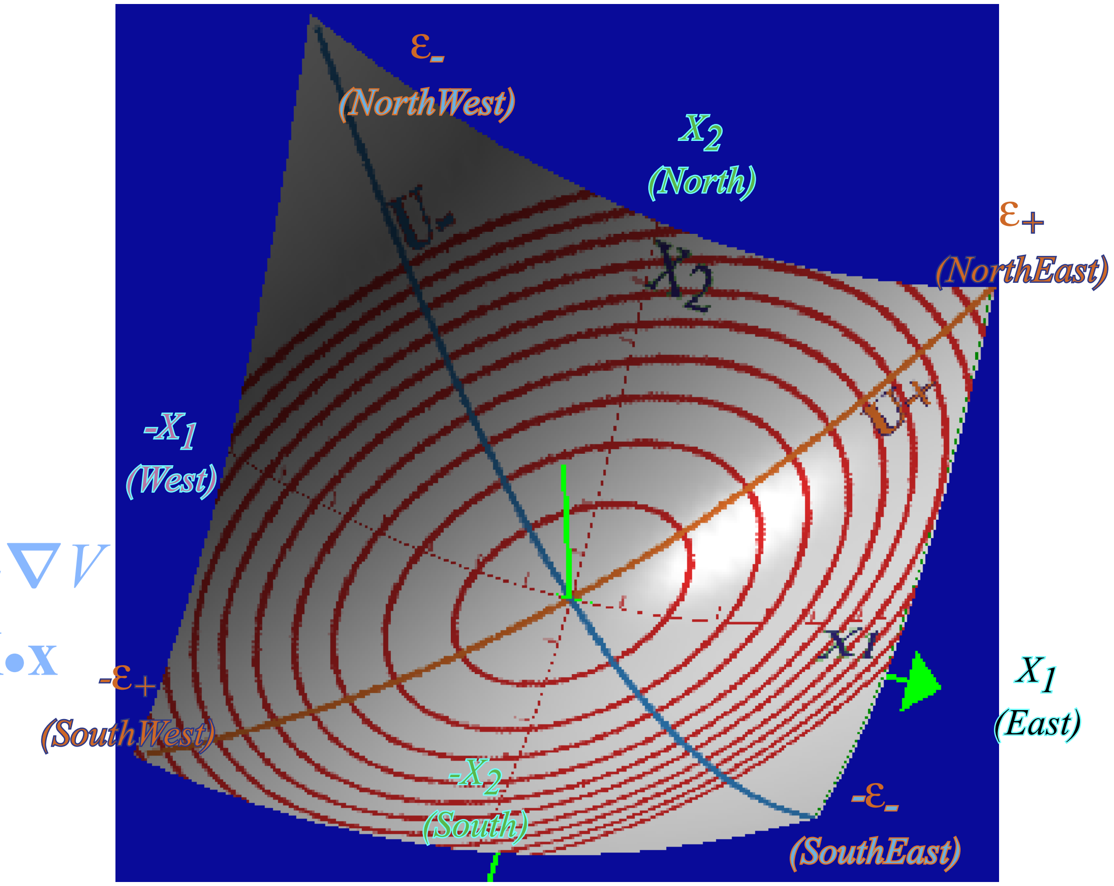


What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$?? Not most directions! Only extremal axes work. (major or minor axes)



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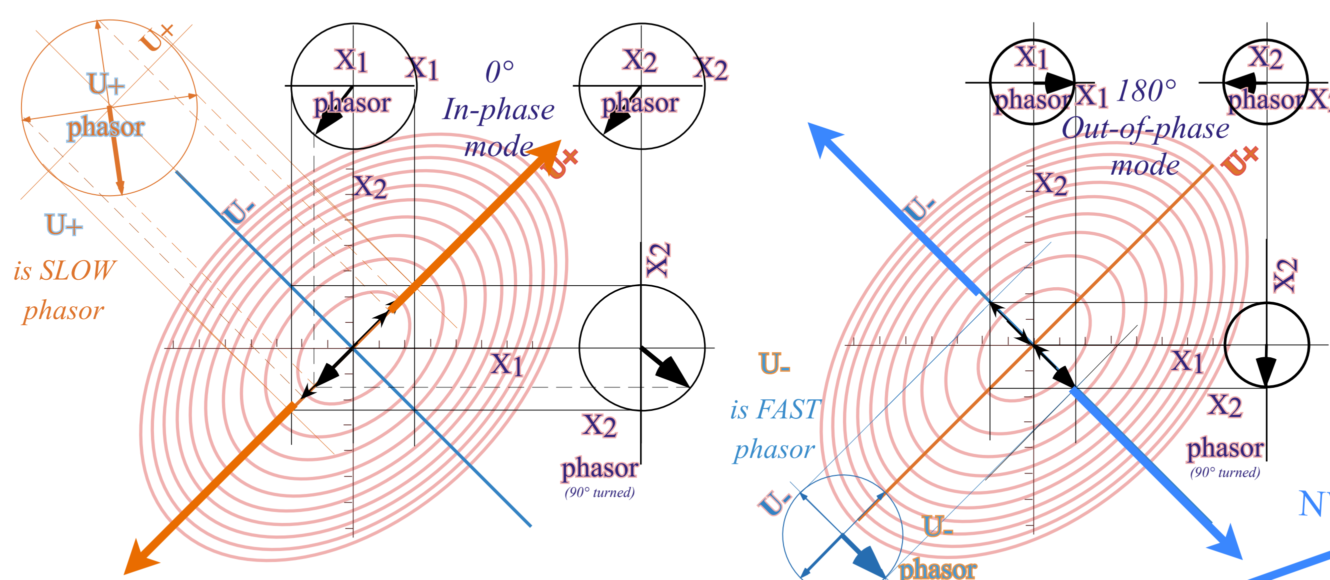

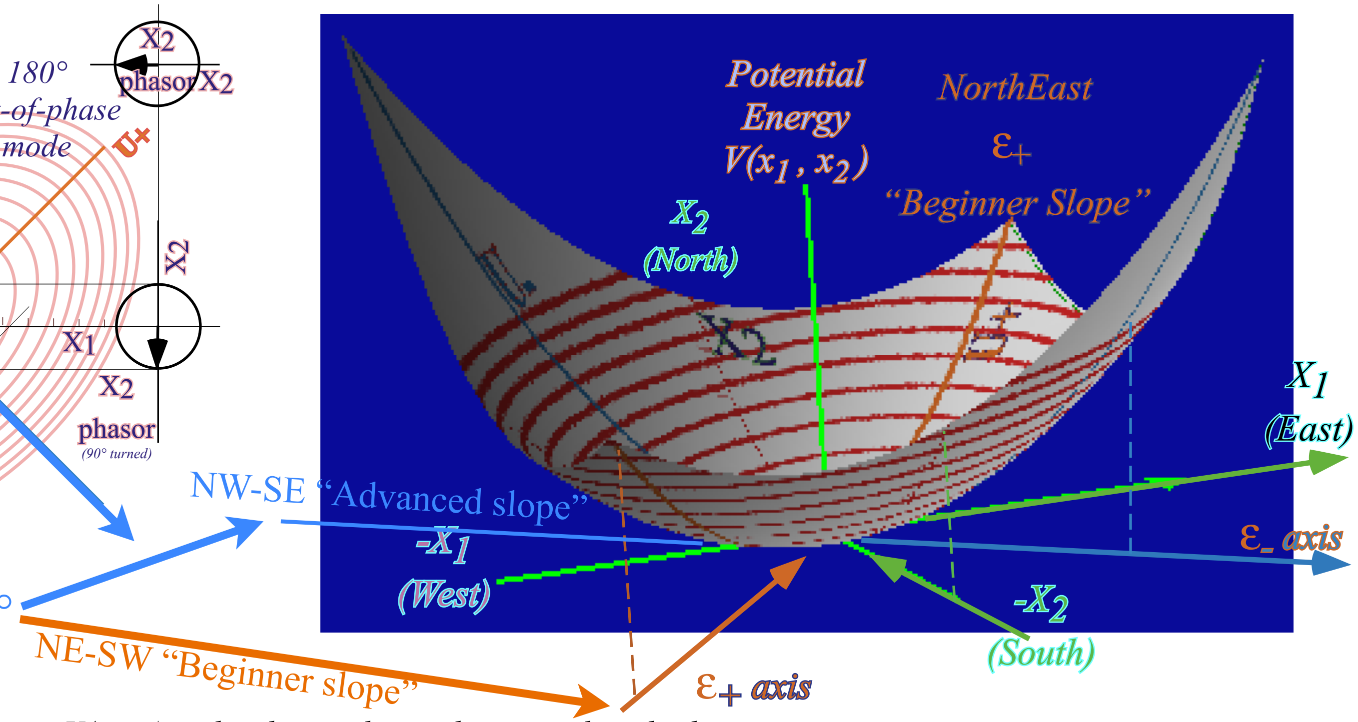

With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

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An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle, \text{ or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

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ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix}=\begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

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Trying to solve by Kramer's inversion:

$$x = \frac{\det\left|\begin{pmatrix} 0 & 1 \\ 0 & 2-\varepsilon \end{pmatrix}\right|}{\det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|} \quad \text{and} \quad y = \frac{\det\left|\begin{pmatrix} 4-\varepsilon & 0 \\ 3 & 0 \end{pmatrix}\right|}{\det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|}$$

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First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum \text{diagonal k-by-k minors of } \mathbf{M},\cdots,\ a_n=(-1)^n\det|\mathbf{M}|$$

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}-\varepsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|=\det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})$$

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where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{or:}\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{so let:}\ \ \varepsilon_1=1\ \ \text{and:}\ \ \varepsilon_2=5$$

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Notice $\mathbf{p}_k$ *commutes with* **M**,..
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$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}\left(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}\right) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}\left(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j\right) = \mathbf{p}_j\prod_{m\neq k}\left(\varepsilon_j-\varepsilon_m\right) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right) & if: j=k \end{cases}$$

With example matrix

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

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Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

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$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

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With example matrix

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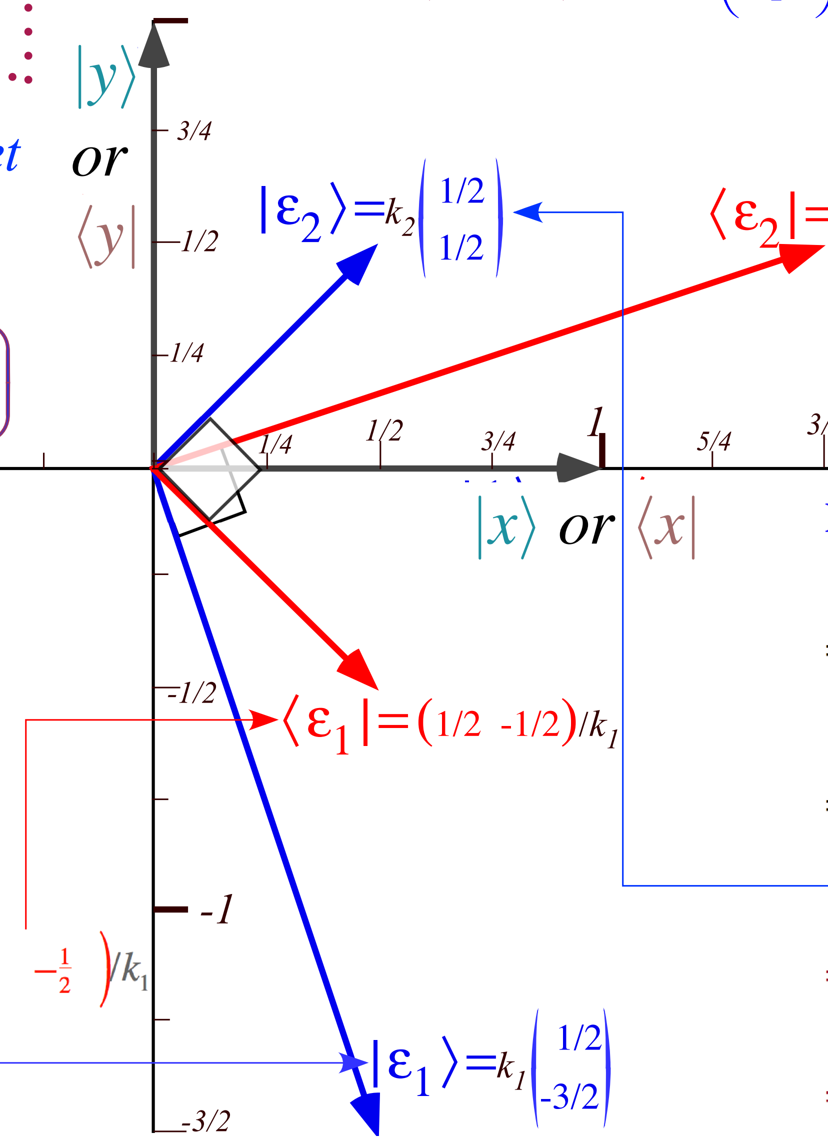

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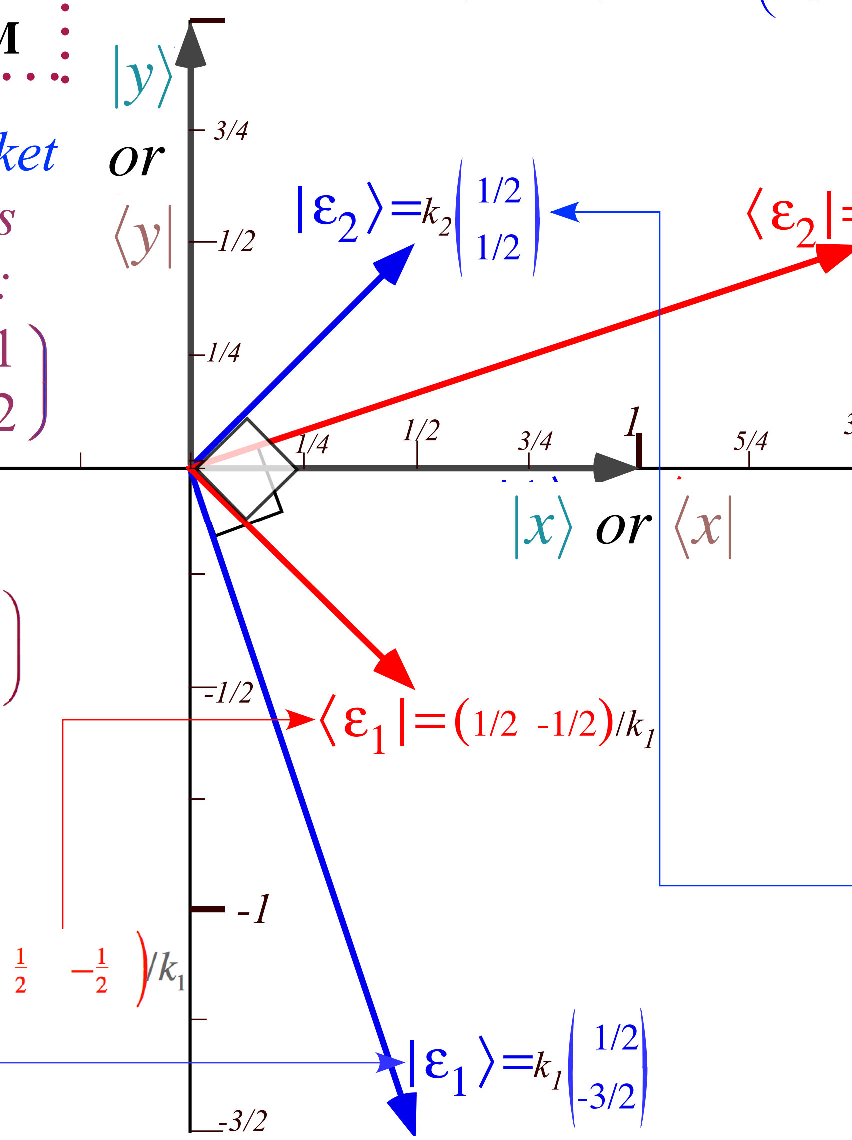

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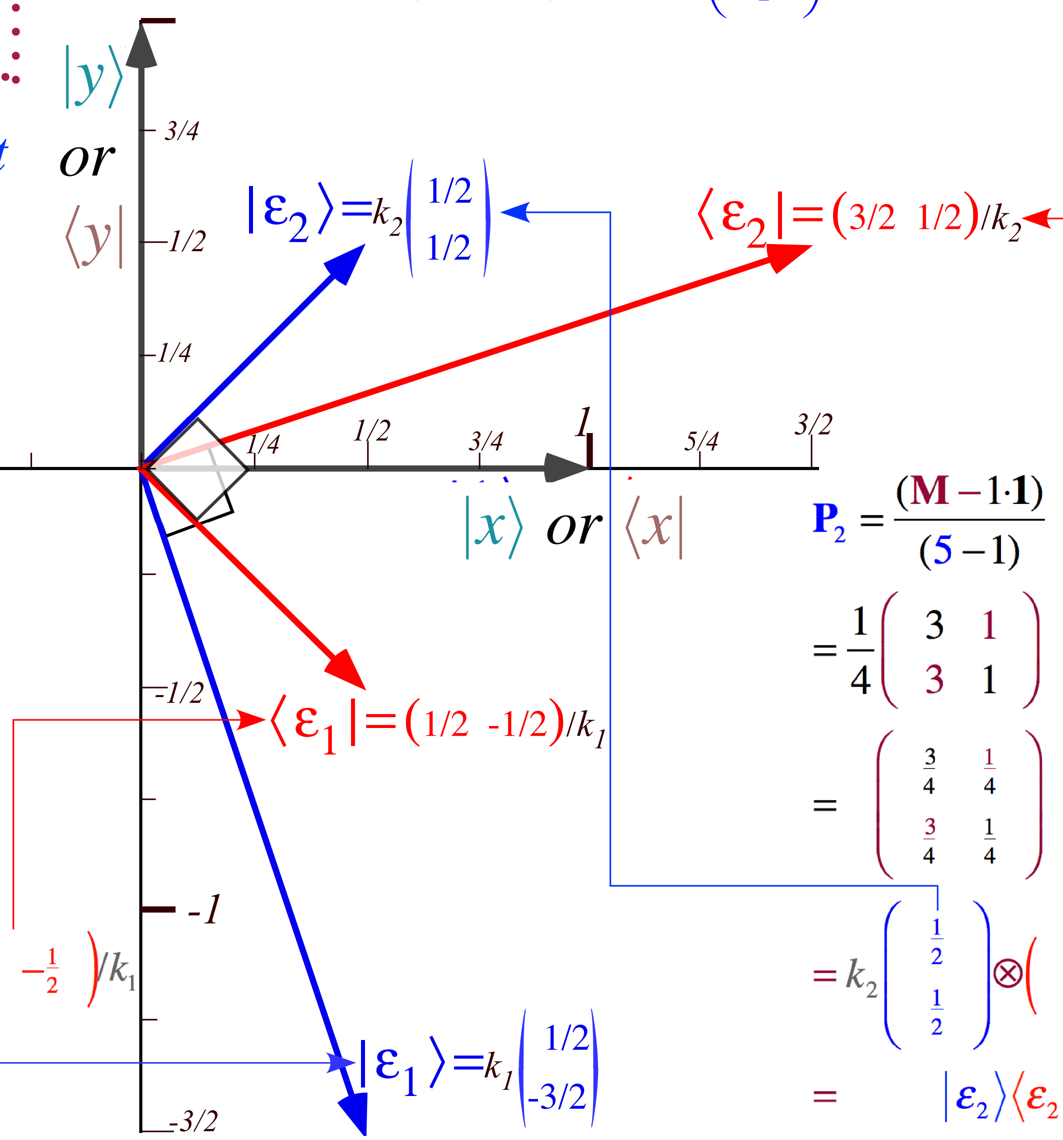

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Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

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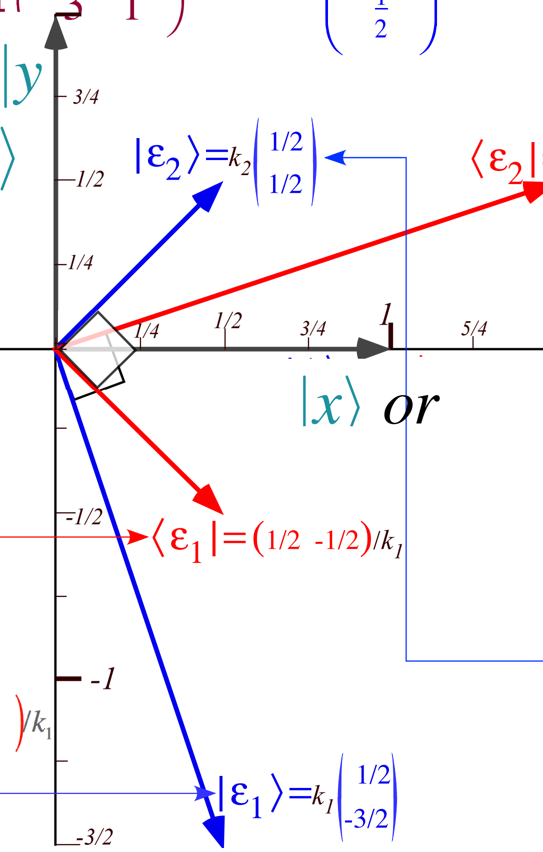


$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}/k_1=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}/k_2=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & \text{if : } j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & \text{if : } j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if : } j\neq k \\ \mathbf{P}_k & \text{if : } j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$ *implies :* $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

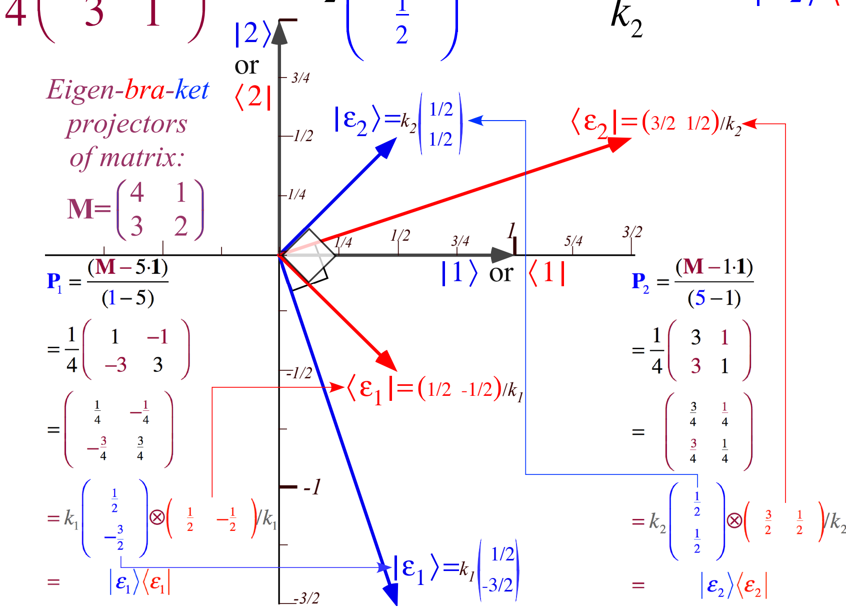
$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$



The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
➔ *Functional spectral decomposition* ←
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase
2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:
make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$
implies:
$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle&\langle\varepsilon_1|\varepsilon_2\rangle\\\langle\varepsilon_2|\varepsilon_1\rangle&\langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1}= \mathbf{P}_1 + \mathbf{P}_2 +...+ \mathbf{P}_n$
$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+...+|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle1|+5|2\rangle\langle2|=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+...+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+...+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+...+f(\varepsilon_n)\mathbf{P}_n$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k=\prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j)=\mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m)=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k=\dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

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$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$ *implies*: $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac12\\-\frac32\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac12&-\frac12\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac12\\\frac12\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac32&\frac12\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle&\langle\varepsilon_1|\varepsilon_2\rangle\\\langle\varepsilon_2|\varepsilon_1\rangle&\langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n$

$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+\ldots+|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle1|+5|2\rangle\langle2|=1\begin{pmatrix}\frac14&-\frac14\\-\frac34&\frac34\end{pmatrix}+5\begin{pmatrix}\frac34&\frac14\\\frac34&\frac14\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n$$

Example:

$$\mathbf{M}^{50}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1^{50}\begin{pmatrix}\frac14&-\frac14\\-\frac34&\frac34\end{pmatrix}+5^{50}\begin{pmatrix}\frac34&\frac14\\\frac34&\frac14\end{pmatrix}=\frac14\begin{pmatrix}1+3\cdot5^{50}&5^{50}-1\\3\cdot5^{50}-3&5^{50}+3\end{pmatrix}$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$ *implies:* $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle&\langle\varepsilon_1|\varepsilon_2\rangle\\\langle\varepsilon_2|\varepsilon_1\rangle&\langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$$\mathbf{1}= \mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n$$
$$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+\ldots+|\varepsilon_n\rangle\langle\varepsilon_n|$$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle1|+5|2\rangle\langle2|=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n$$

Examples:

$$\mathbf{M}^{50}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1^{50}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5^{50}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1+3\cdot5^{50}&5^{50}-1\\3\cdot5^{50}-3&5^{50}+3\end{pmatrix}$$

$$\sqrt{\mathbf{M}}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=\pm\sqrt{1}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}\pm\sqrt{5}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
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Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
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Orthonormality vs. Completeness

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if : j=k \end{cases}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P***)*

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{P}_k & if : j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

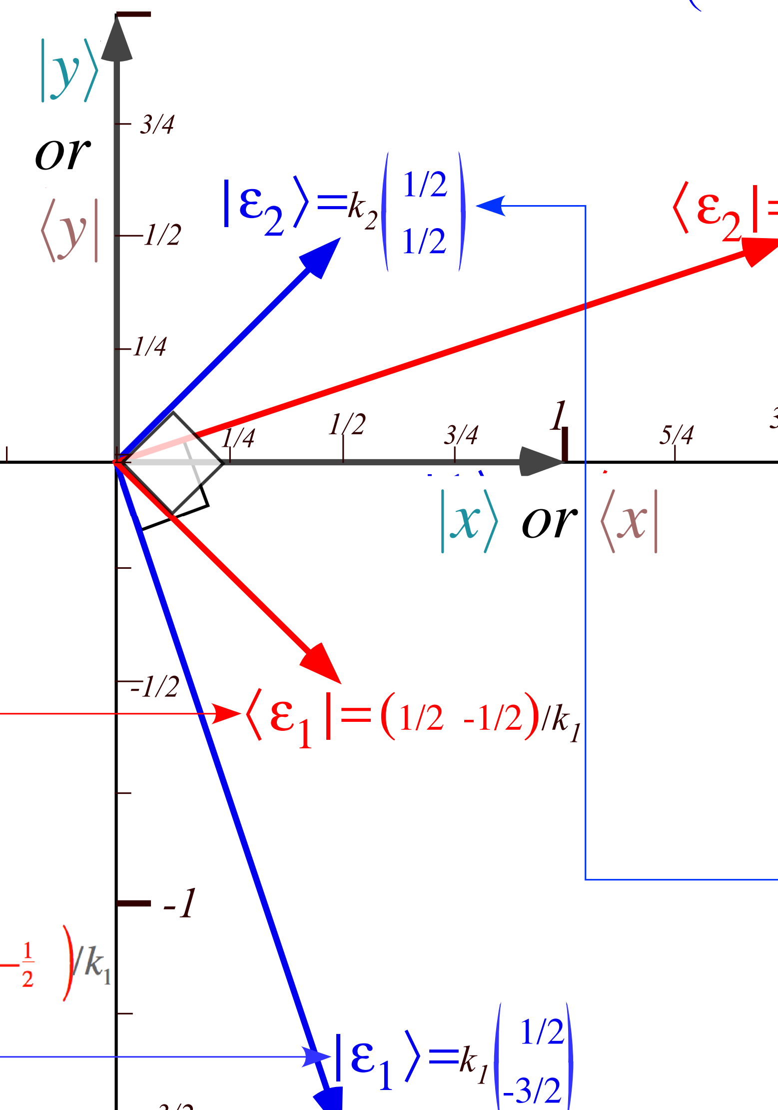
"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigen-bra-ket projectors of matrix: $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$



...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1}= \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}/k_1 = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}/k_2 = |\varepsilon_2\rangle\langle\varepsilon_2|$$

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

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$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

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*State vector representations of orthonormality are quite **similar** to representations of completeness.*
Like 2-sides of the same coin.

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$$\langle x|y\rangle = \delta(x,y) = \quad \psi_1(x)\psi_1^*(y) + \psi_2(x)\psi_2^*(y) + ..$$

Dirac δ-function

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However Schrodinger wavefunction notation $\psi(x) = \langle x|\psi\rangle$ *shows quite a difference...*

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However Schrodinger wavefunction notation $\psi(x) = \langle x|\psi\rangle$ shows quite a difference…

…particularly in the orthonormality integral.

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

➤ *Lagrange functional interpolation formula* ⮜

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

A Proof of Projector Completeness (Truer-than-true by Lagrange interpolation)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$$f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

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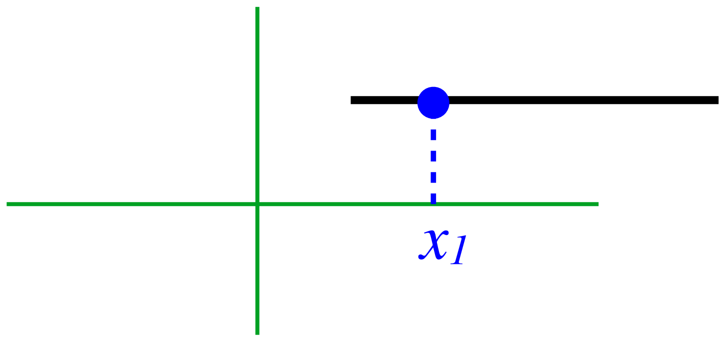

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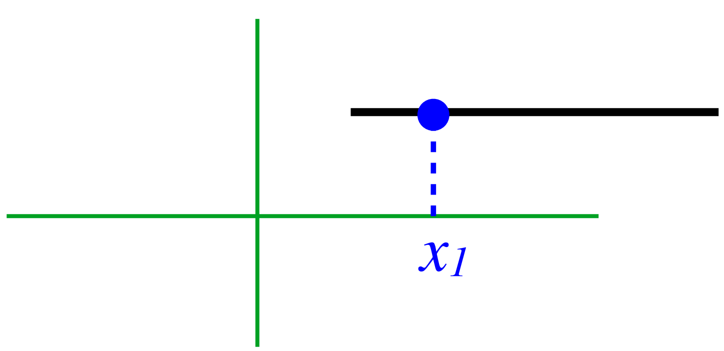

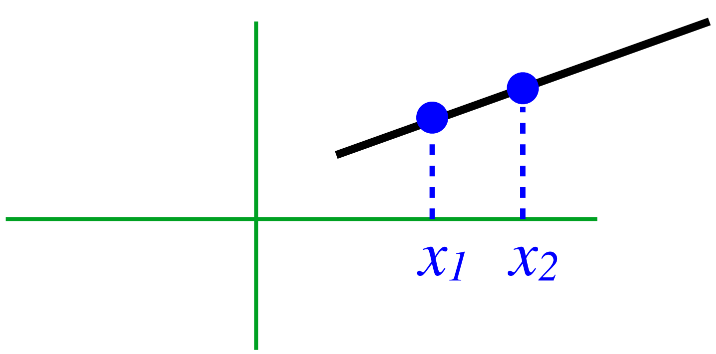

A Proof of Projector Completeness *(Truer-than-true)*

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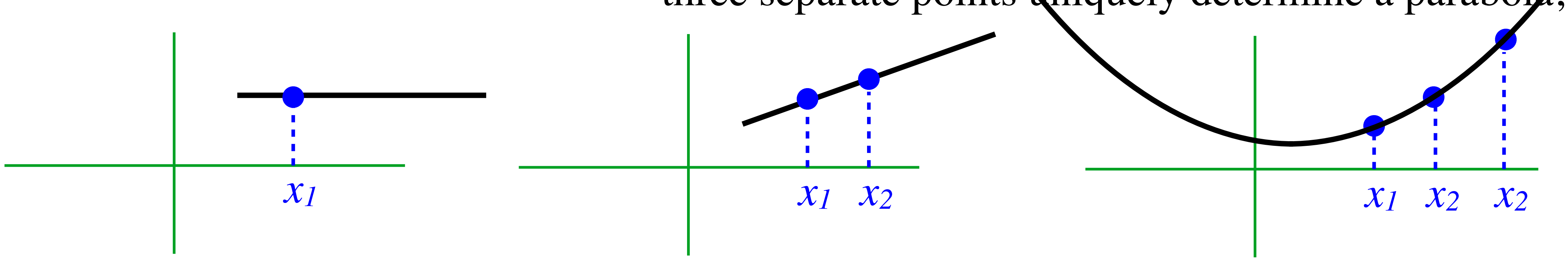

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Lagrange interpolation formula→*Completeness formula* as $x\to\mathbf{M}$ and as $x_k\to\varepsilon_k$ and as $P_k(x_k)\to\mathbf{P}_k$

A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}\qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points $x_1, x_2,\ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N}f(x_k)P_k(x)\qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N\text{-}1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

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$$\mathbf{P}_1+\mathbf{P}_2=\frac{\prod_{j\neq1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq1}\left(\varepsilon_1-\varepsilon_j\right)}+\frac{\prod_{j\neq1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq1}\left(\varepsilon_2-\varepsilon_j\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}+\frac{\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_2-\varepsilon_1\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)-\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}=\frac{-\varepsilon_2\mathbf{1}+\varepsilon_1\mathbf{1}}{\left(\varepsilon_1-\varepsilon_2\right)}=\mathbf{1}\ (\text{for } all\ \varepsilon_j)$$

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However, only *select* values ε_k work for eigen-forms $\mathbf{MP}_k=\varepsilon_k\mathbf{P}_k$ or orthonormality $\mathbf{P}_j\mathbf{P}_k=\delta_{jk}\mathbf{P}_k$.

2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

➤ *Diagonalizing Transformations (D-Ttran) from projectors* ◀

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

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Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

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$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad$ $(1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \quad , \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

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$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Diagonalizing Transformations (D-Ttran) from projectors

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$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix} \langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle \\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle \\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

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$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

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$$\begin{pmatrix}\langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle\\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix},\quad\begin{pmatrix}\langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle\\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}$$

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$$\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are "easy"

$$\begin{pmatrix}\langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle\\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle\\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}\langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle\\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

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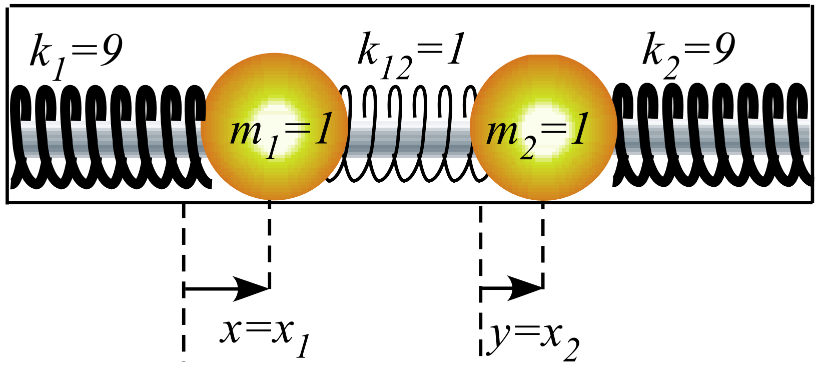
Initial state projection, mixed mode beat dynamics with variable phase



Analyzing 2D-HO beats and mixed mode eigen-solutions

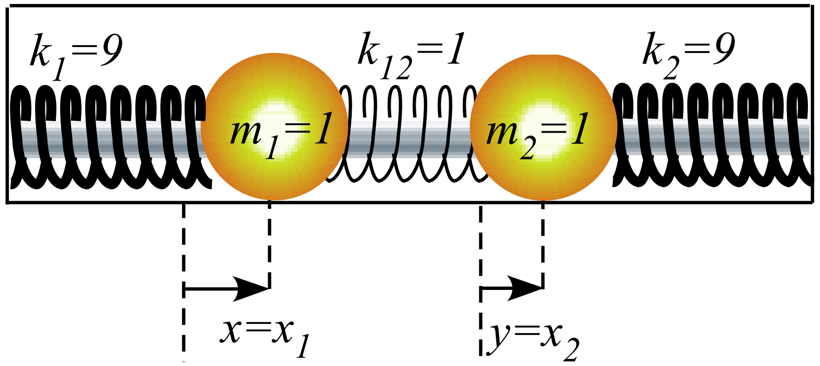
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$

$Trace(\mathbf{K}) = 10 + 10 = 20$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$



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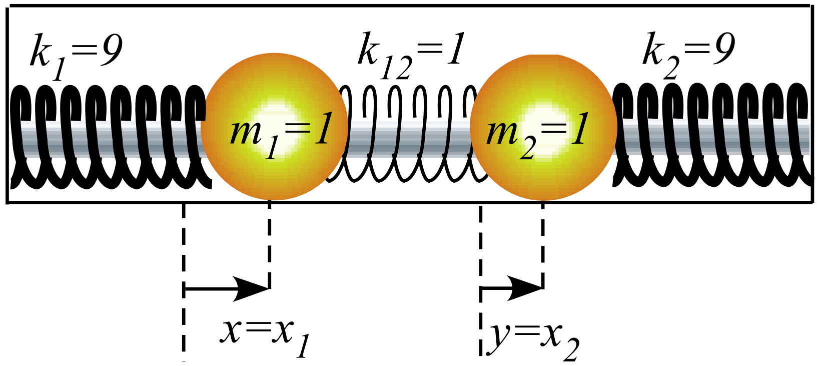

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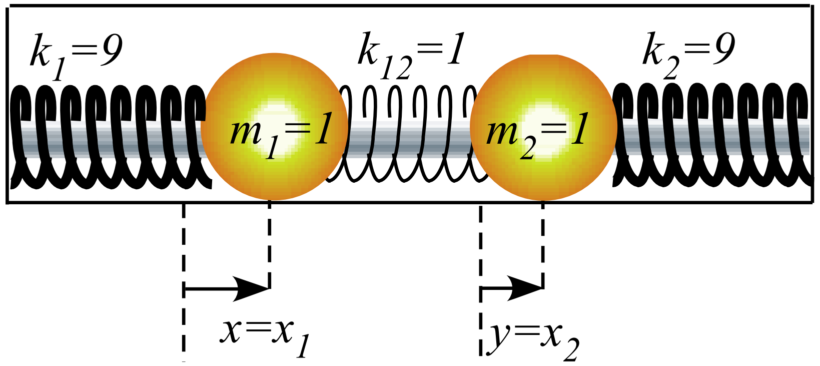

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Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

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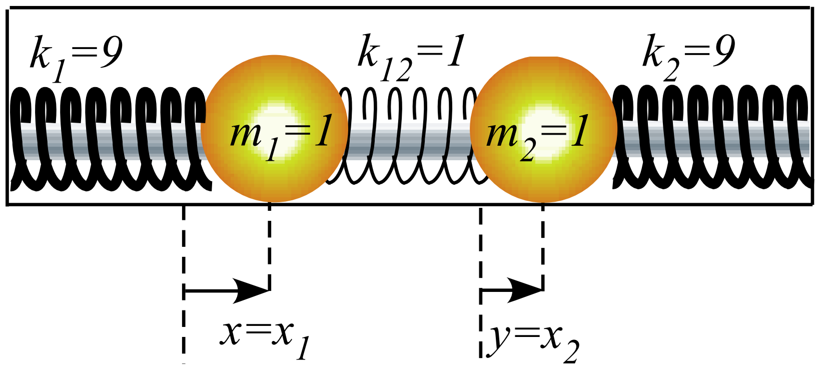
Analyzing 2D-HO beats and mixed mode eigen-solutions



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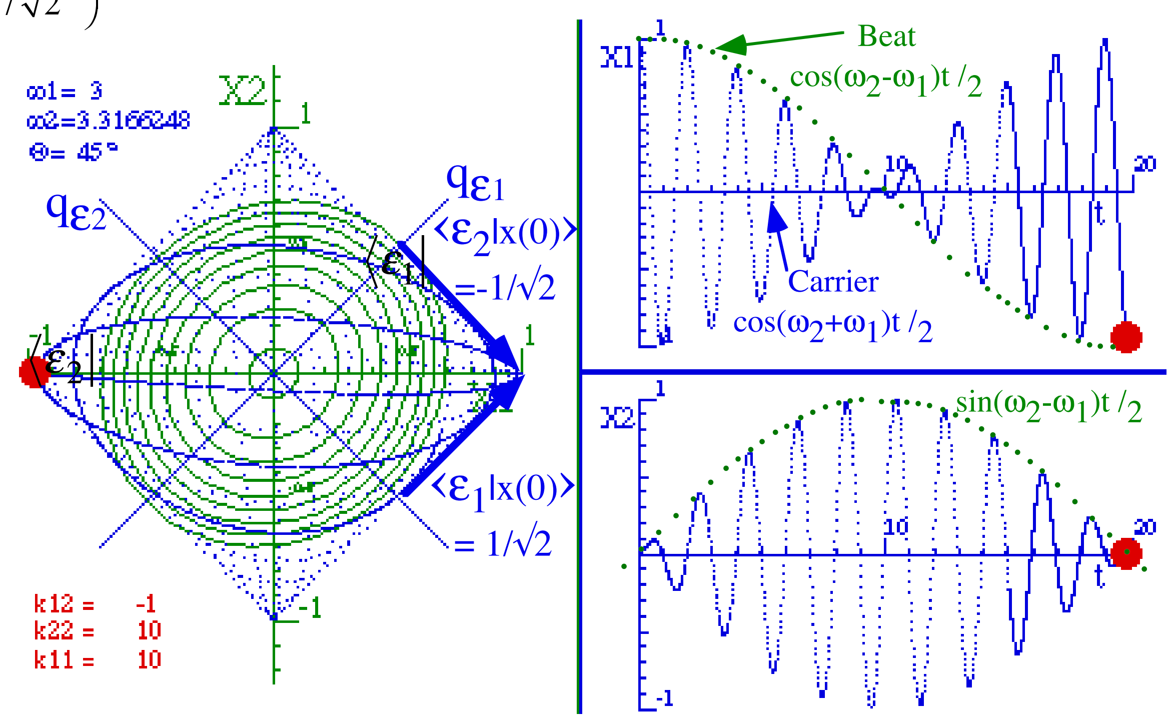
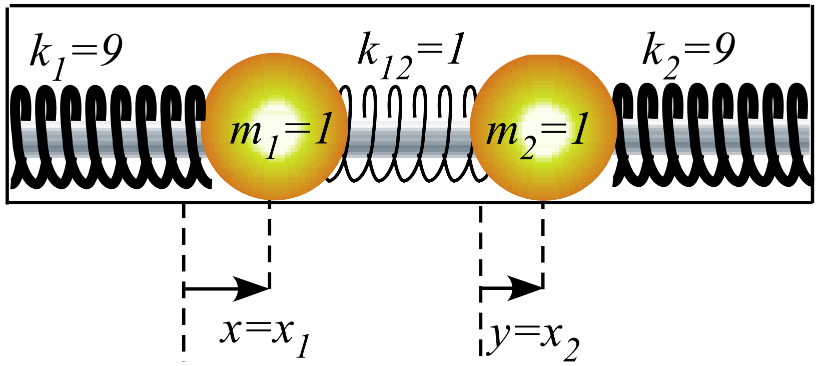
100% modulation (SWR=0) $\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} \frac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}$$



BoxIt (Beating) Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



Analyzing 2D-HO beats and mixed mode eigen-solutions

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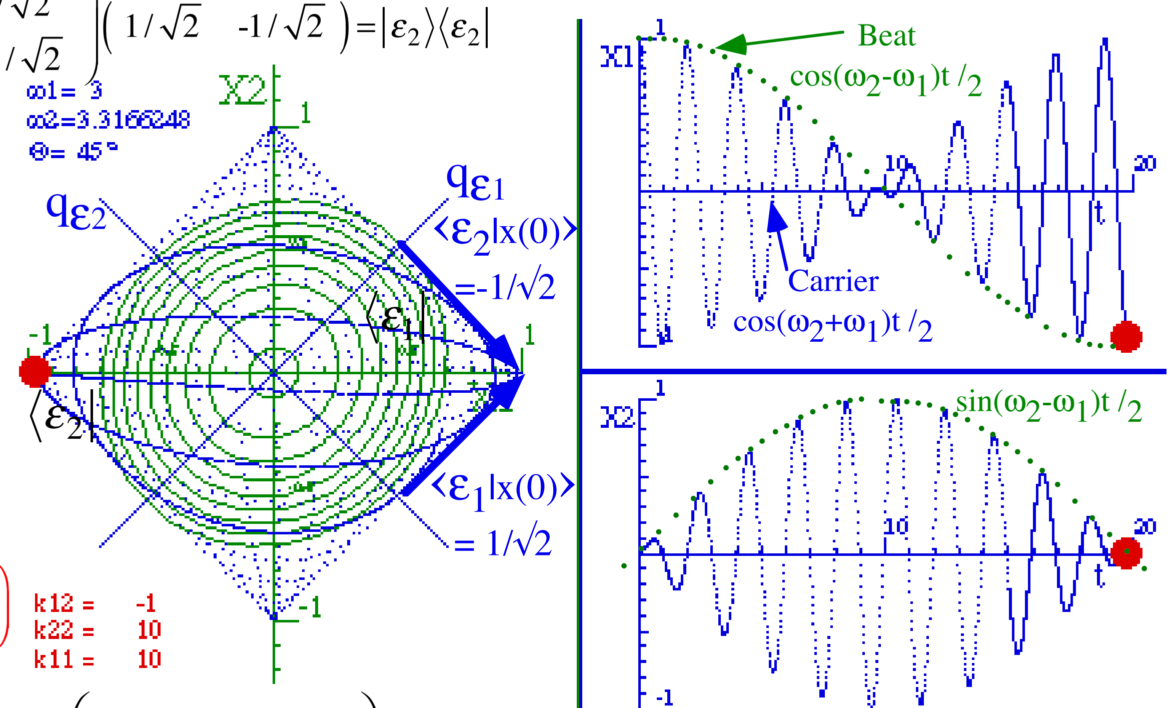
100% modulation (SWR=0)

$$\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}=e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$



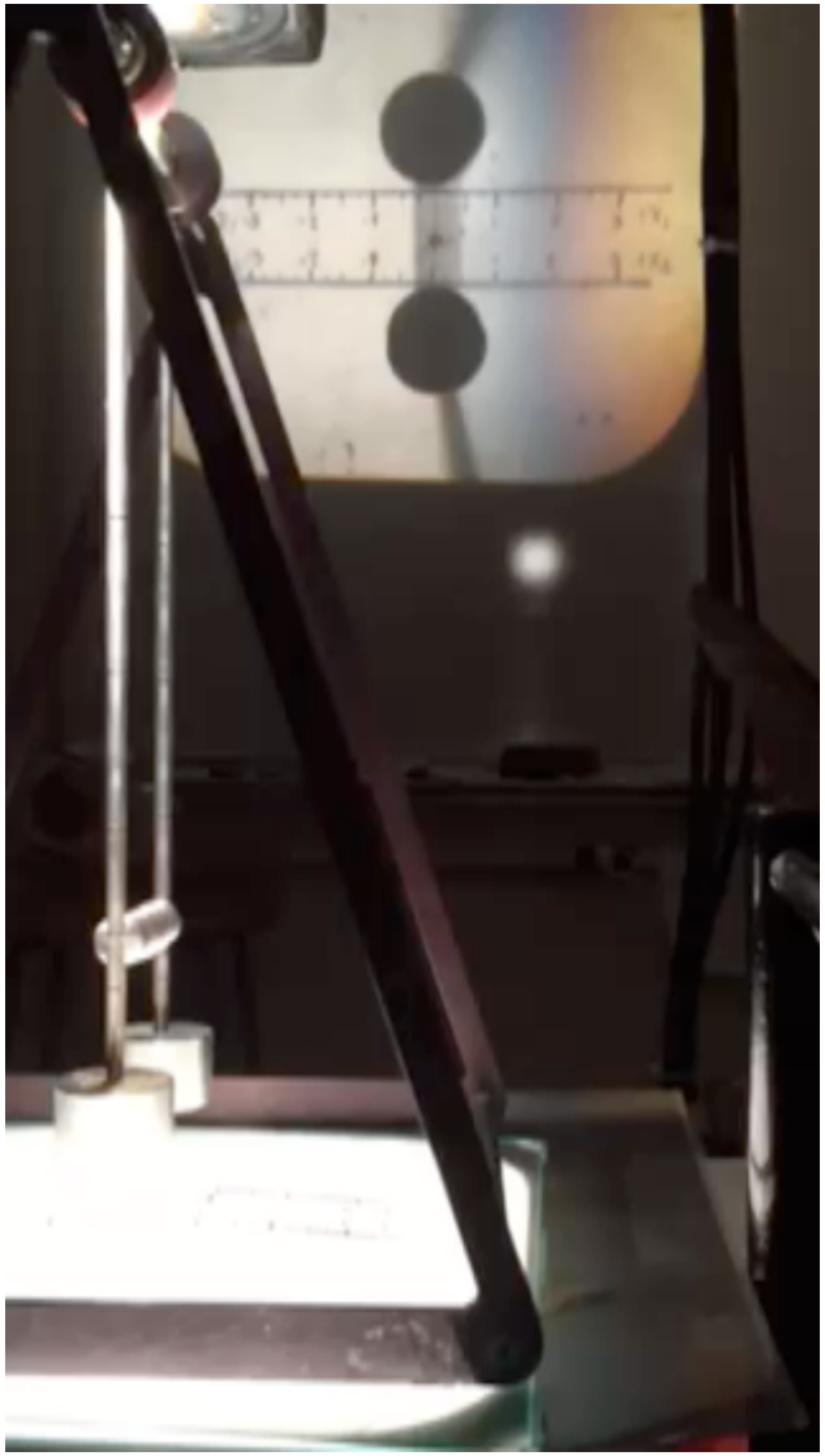
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} \dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2} \end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t} \end{pmatrix}=e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\dfrac{(\omega_2-\omega_1)t}{2} \\ i\sin\dfrac{(\omega_2-\omega_1)t}{2} \end{pmatrix}$$

Note the i phase

BoxIt (Beating) Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

Videos of Coupled Pendula aided by Overhead Projector



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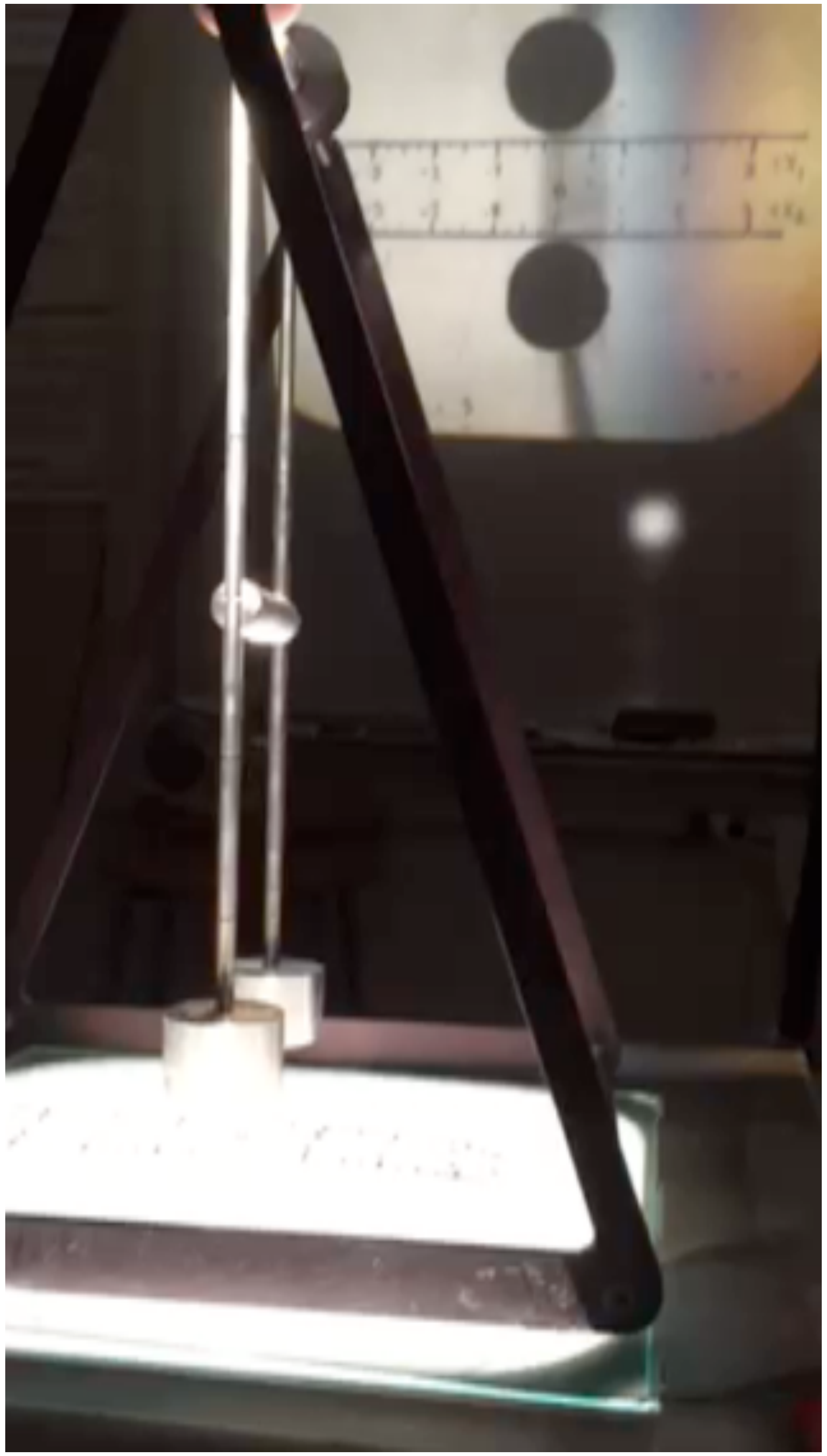
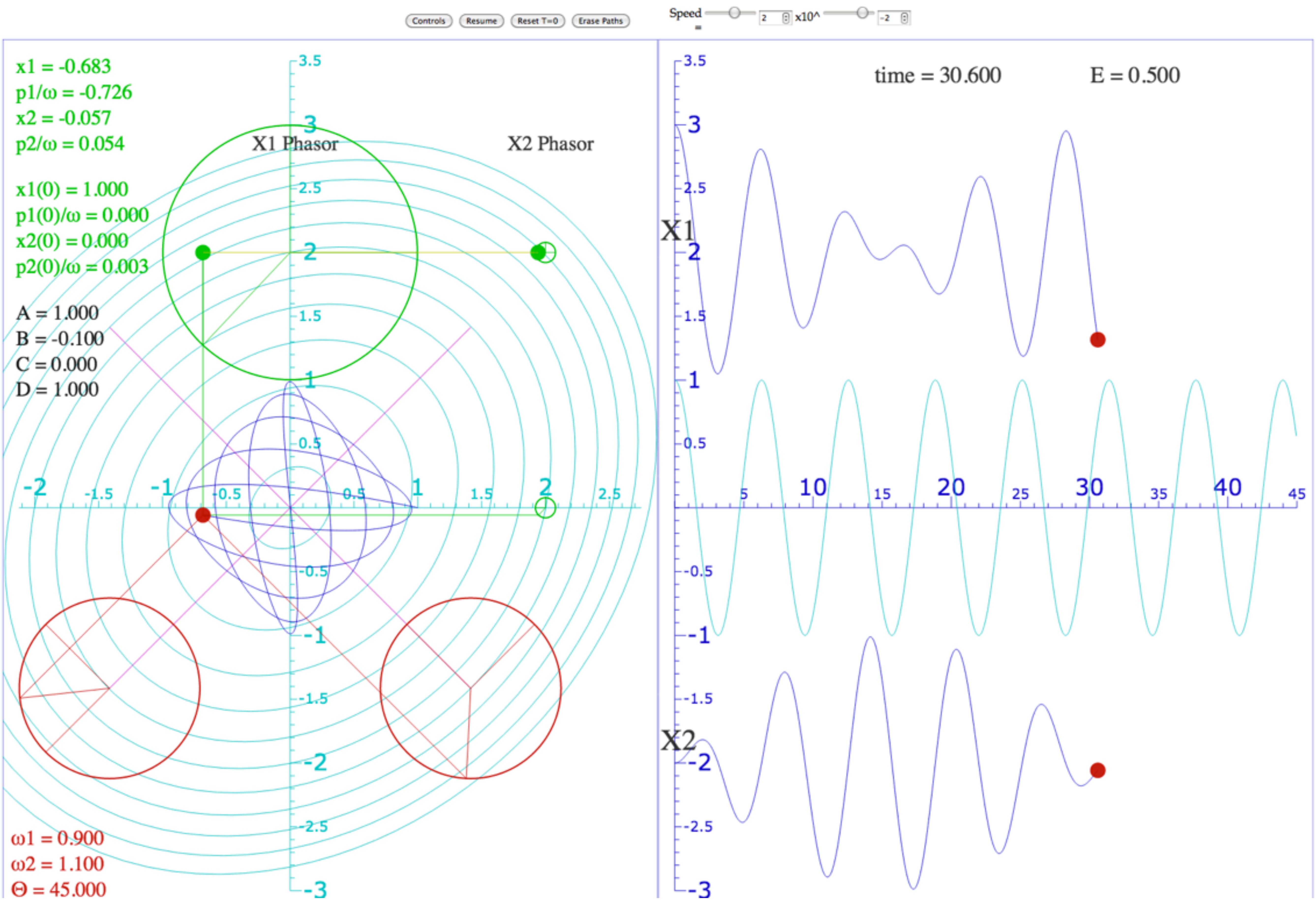
or
⇐ view on YouTube ⇒

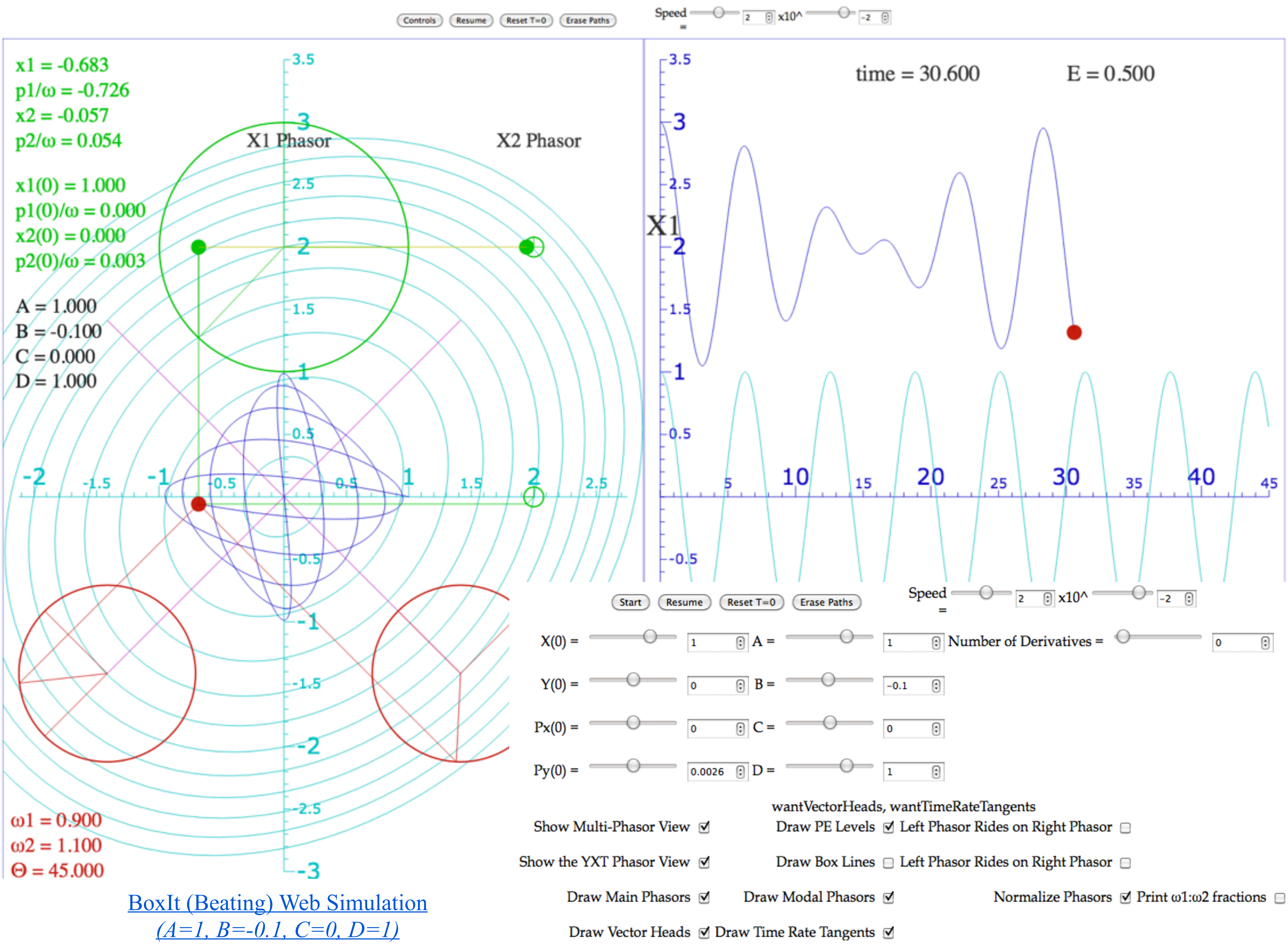


[View on YouTube](#) YouTube

Stronger coupling on the right, illustrated indirectly by a darker looking spring on screen



[BoxIt (Beating) Web Simulation](#)
(A=1, B=-0.1, C=0, D=1)



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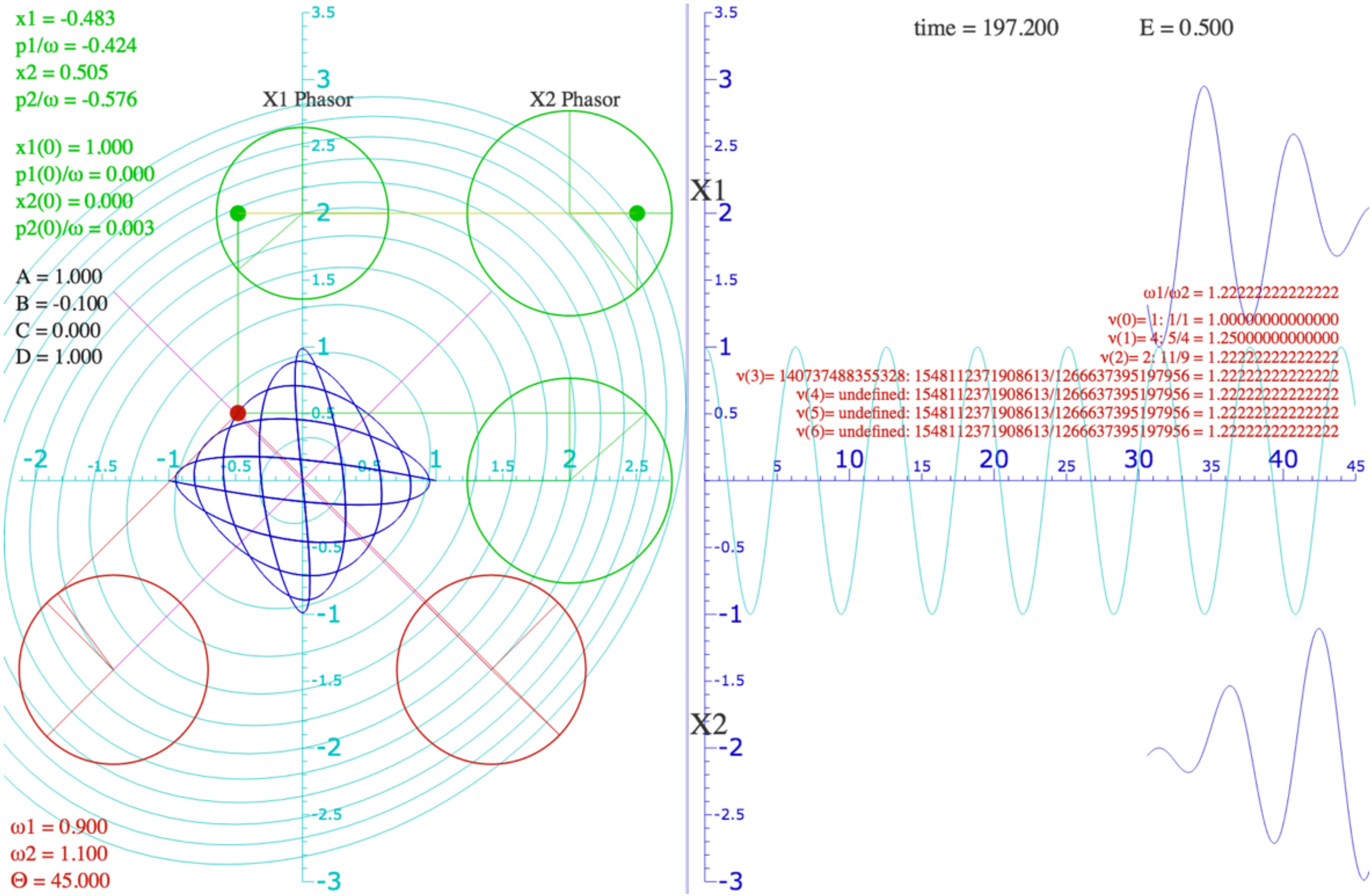

[BoxIt (Beating) Web Simulation *(A=1, B=-0.1, C=0, D=1) with frequency ratios*]

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

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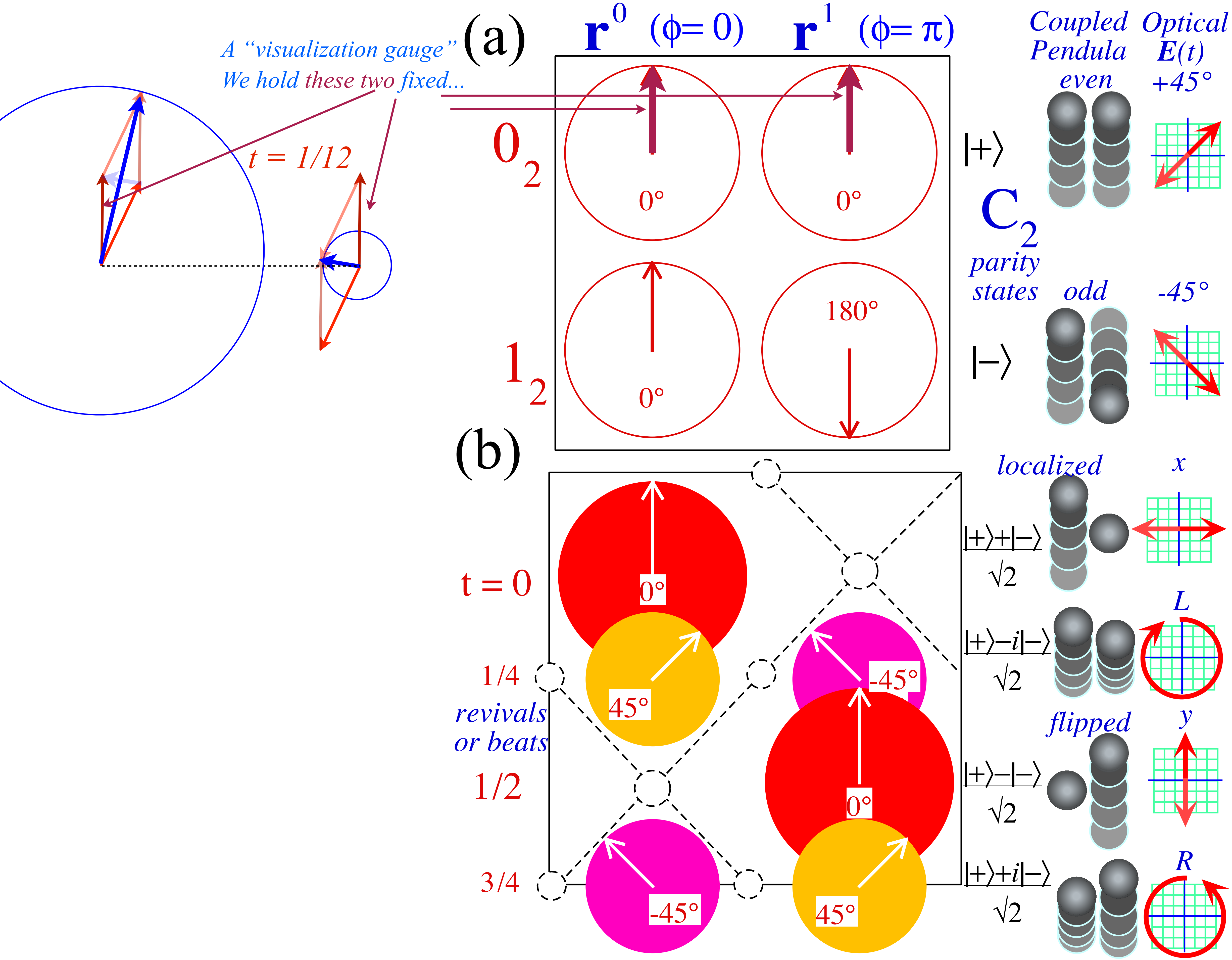
2D-HO beats and mixed mode geometry



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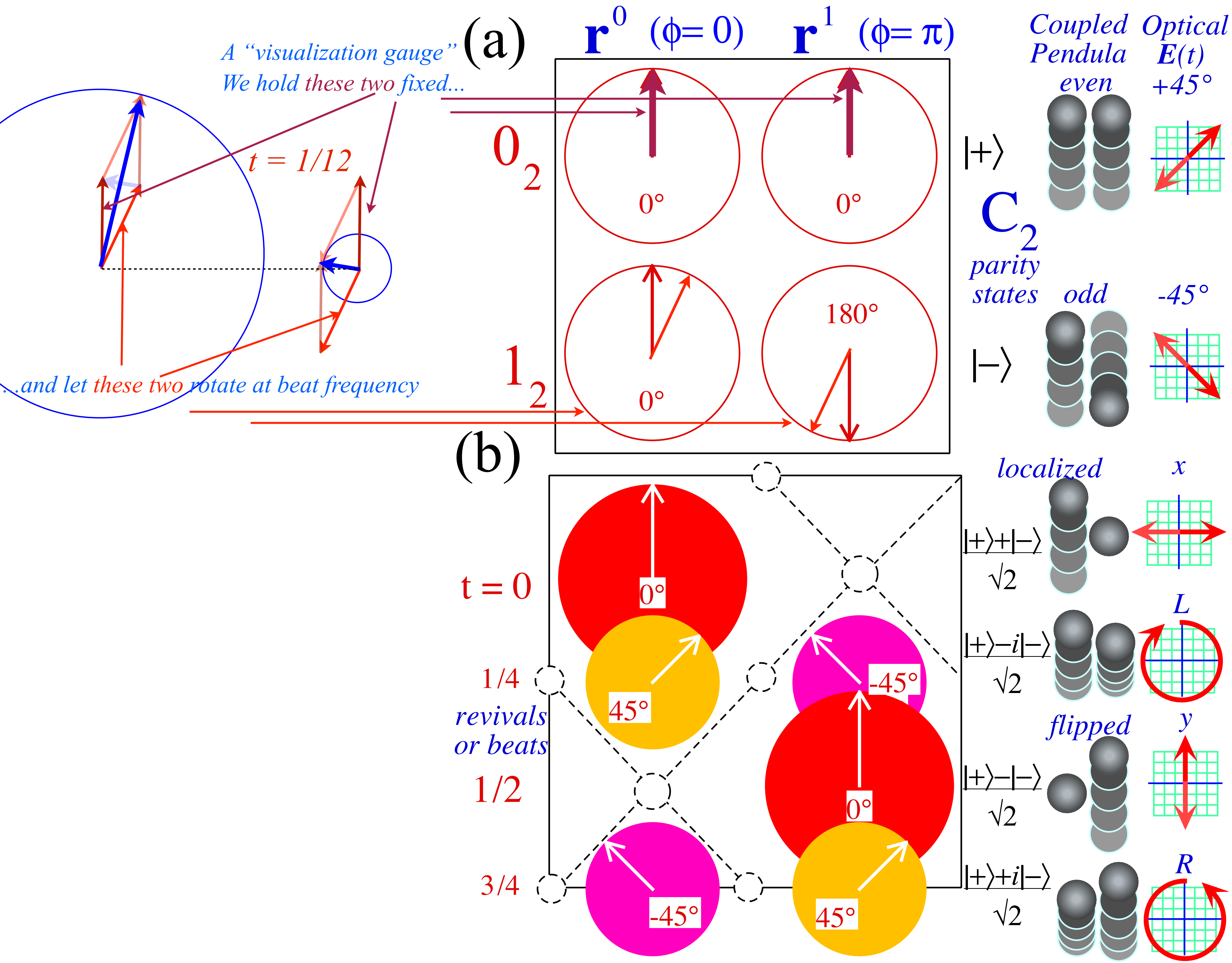

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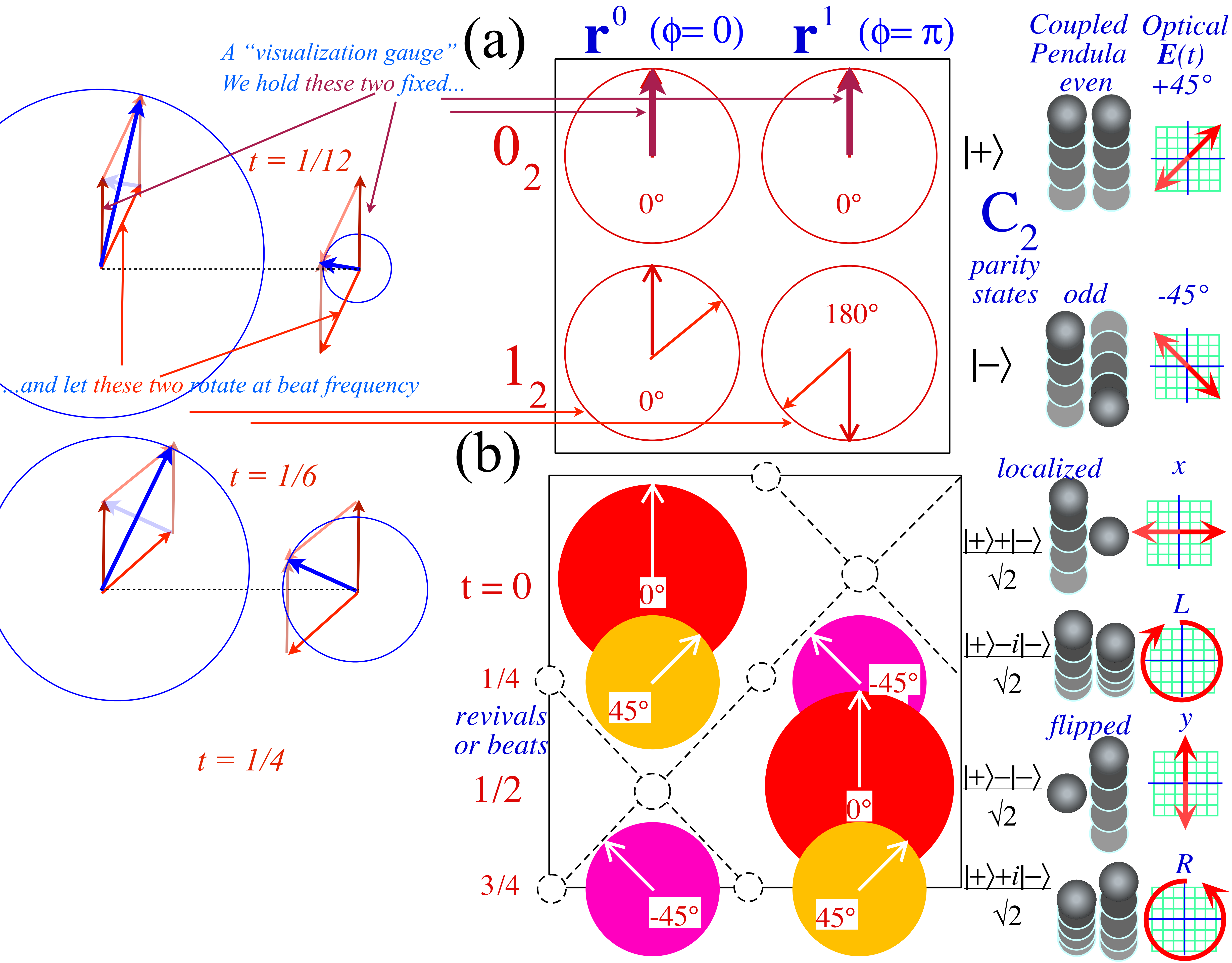

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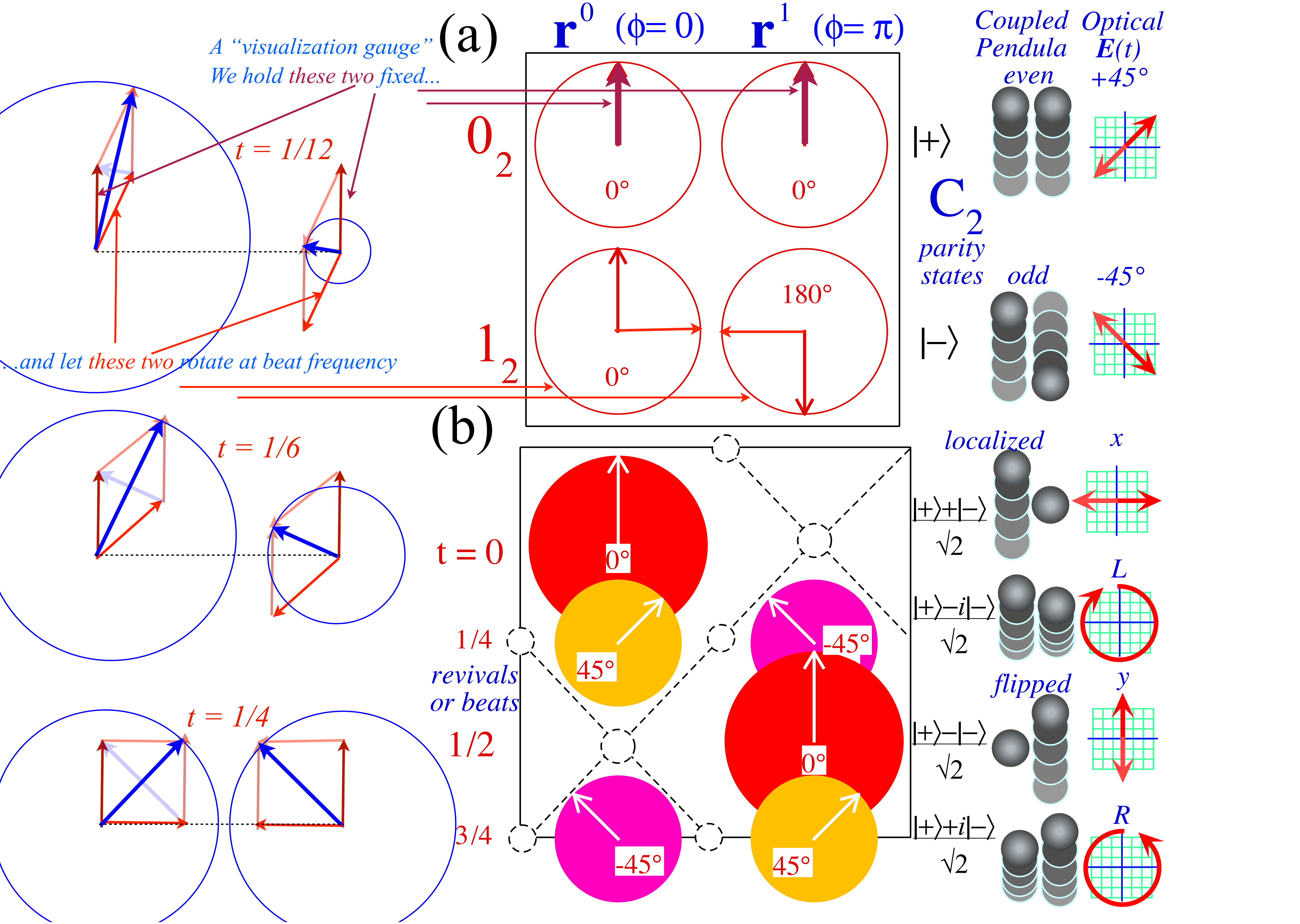

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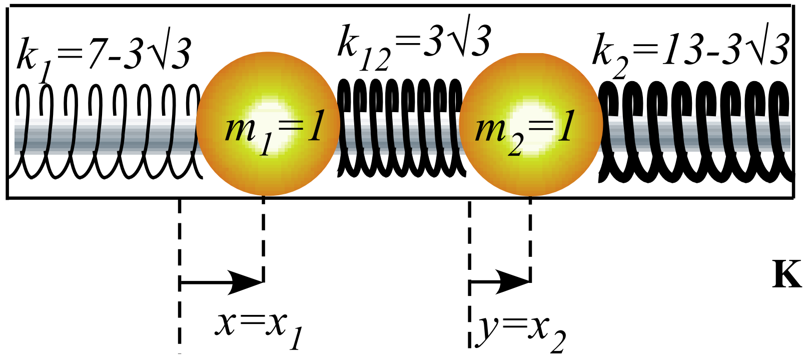
Initial state projection, mixed mode beat dynamics with variable phase

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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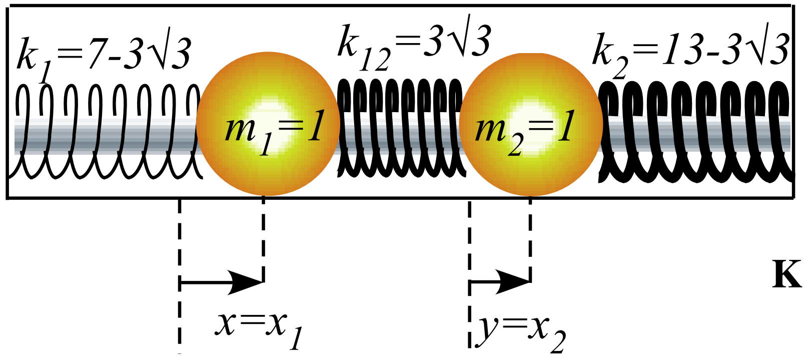

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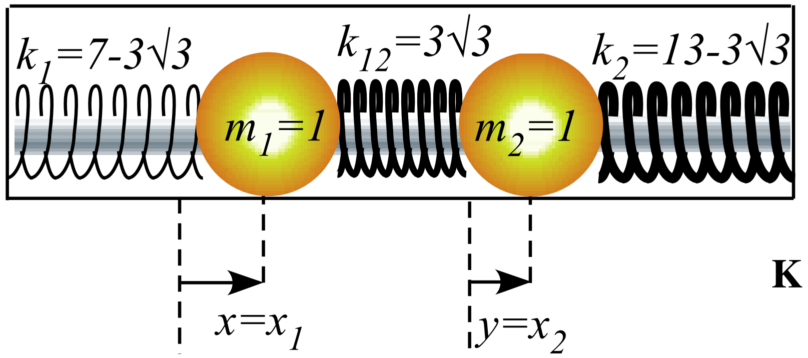

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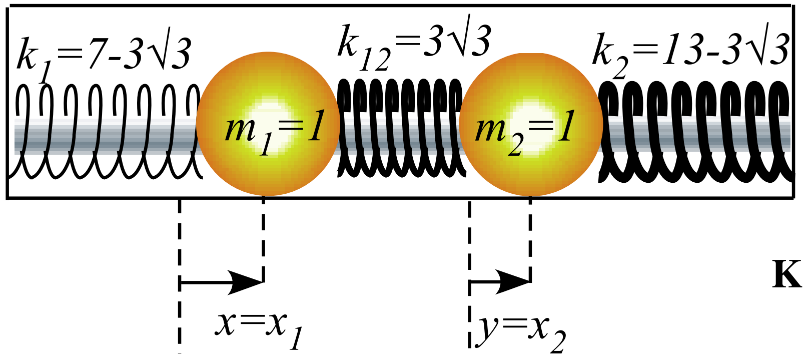

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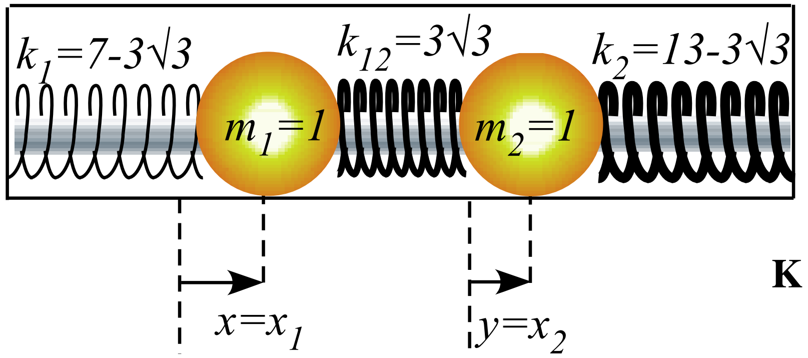

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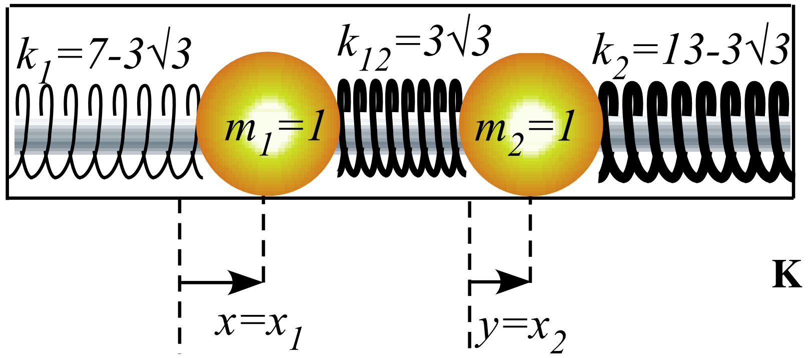

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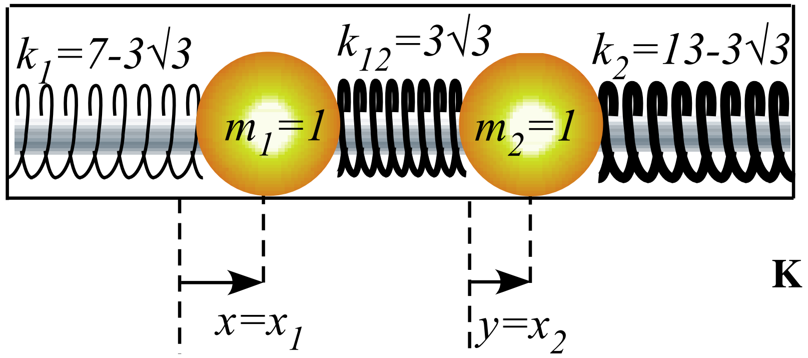

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2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

➔ *Initial state projection, mixed mode beat dynamics with variable phase* ←

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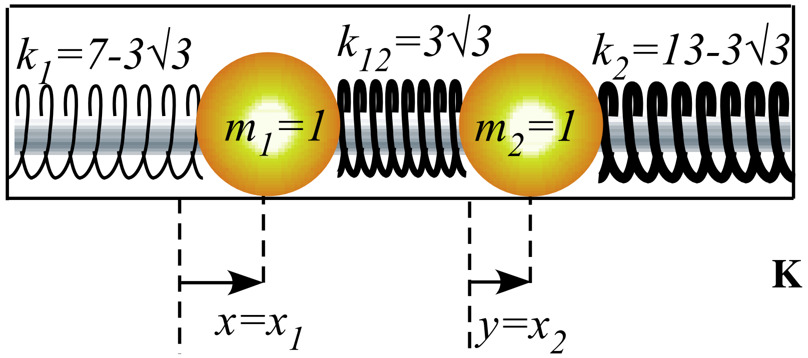

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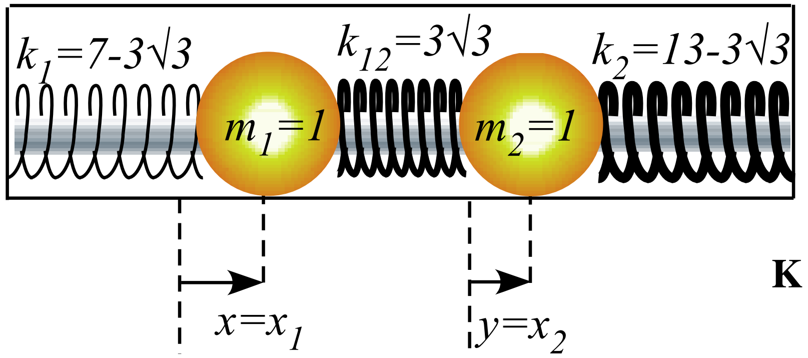

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(Note projection onto eigen-axes)

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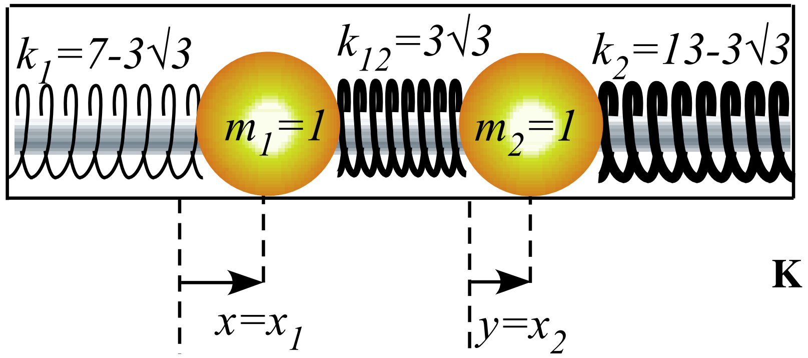

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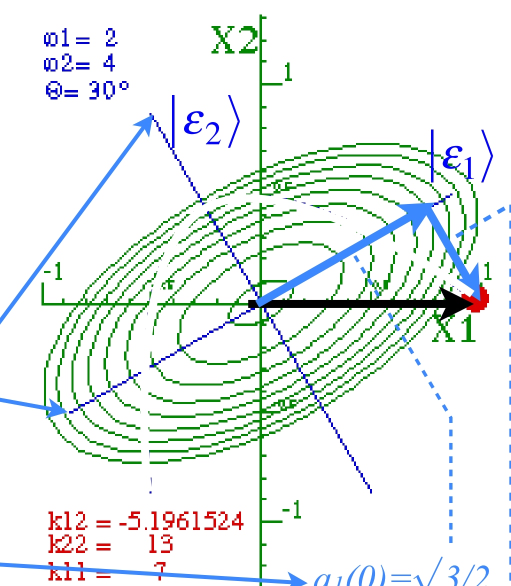
$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\frac{\sqrt{3}}{2}\right)+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\frac{1}{2}\right)$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

$$\mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$



$q_1(0)=\sqrt{3}/2$

$q_2(0)=-1/2$

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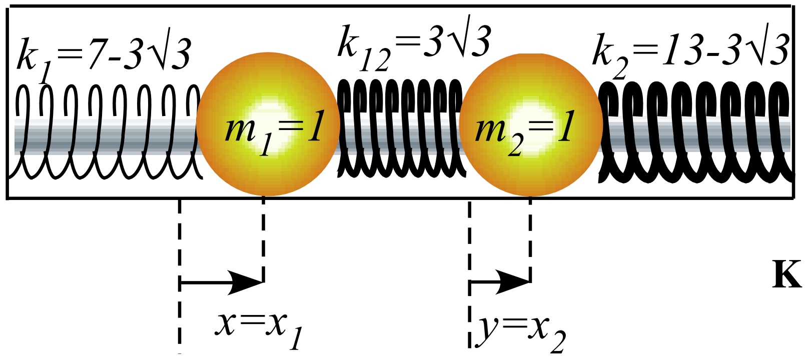

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$$\mathbf{P}_1=\frac{\begin{pmatrix}K_{11}-K_2 & K_{12}\\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix}7-16 & -3\sqrt{3}\\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix}9 & +3\sqrt{3}\\ +3\sqrt{3} & 3\end{pmatrix}}{12}=\frac{\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix}K_{11}-K_1 & K_{12}\\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix}7-4 & -3\sqrt{3}\\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix}3 & -3\sqrt{3}\\ -3\sqrt{3} & 9\end{pmatrix}}{12}=\frac{\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t\right)\qquad \mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$

Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2(t)=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}\left[q_1(t)\right]^2+\frac{1}{2}$$

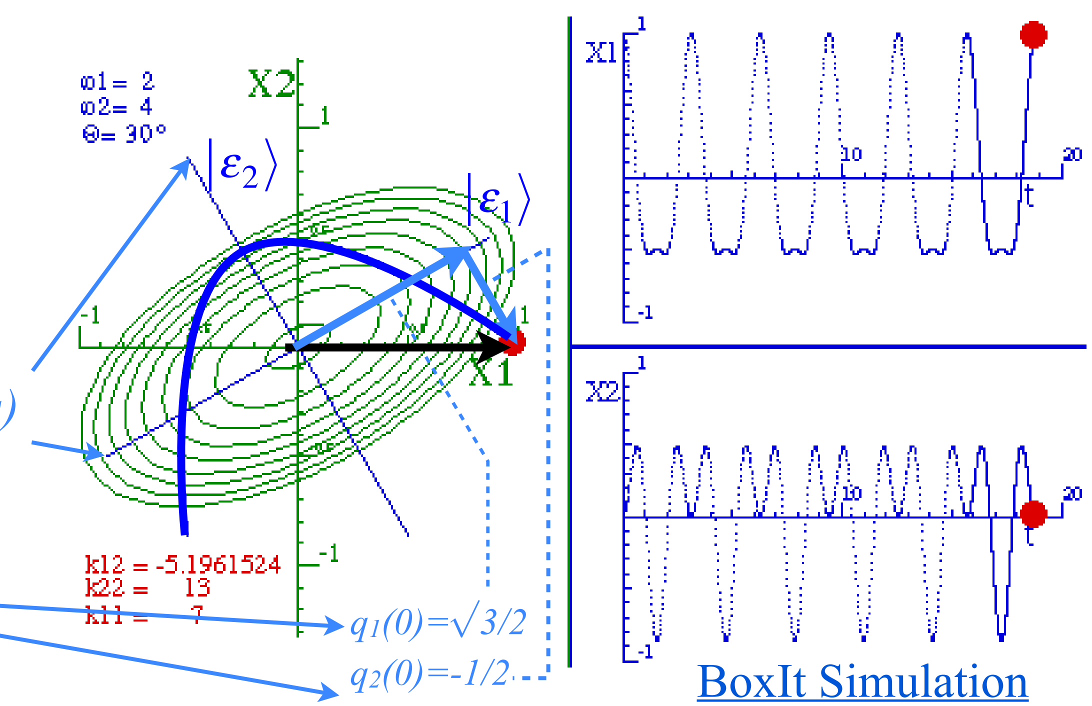
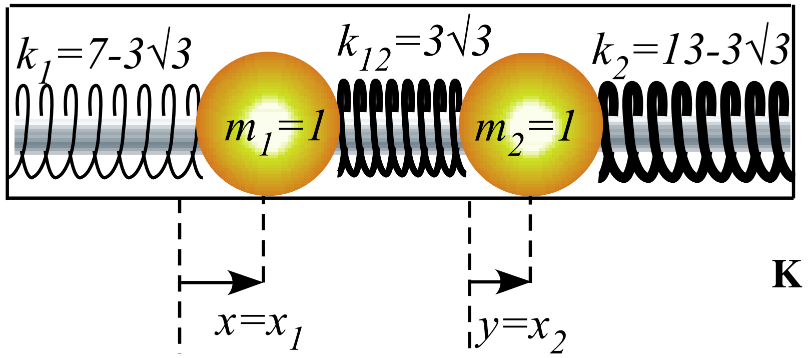


Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix}K_{11}&K_{12}\\K_{12}&K_{22}\end{pmatrix}=\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_2+k_{12}\end{pmatrix}=\begin{pmatrix}7&-3\sqrt{3}\\-3\sqrt{3}&13\end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

$Det(\mathbf{K})=7\cdot13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\qquad K_1=\omega_0^2(\varepsilon_1)=4,\quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix}K_{11}-K_2&K_{12}\\K_{12}&K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix}7-16&-3\sqrt{3}\\-3\sqrt{3}&13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix}9&+3\sqrt{3}\\+3\sqrt{3}&3\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix}3&\sqrt{3}\\\sqrt{3}&1\end{pmatrix}}{4}=\begin{pmatrix}\sqrt{3}/2\\1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2&1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix}K_{11}-K_1&K_{12}\\K_{12}&K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix}7-4&-3\sqrt{3}\\-3\sqrt{3}&13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix}3&-3\sqrt{3}\\-3\sqrt{3}&9\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix}1&-\sqrt{3}\\-\sqrt{3}&3\end{pmatrix}}{4}=\begin{pmatrix}-1/2\\\sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1/2&\sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

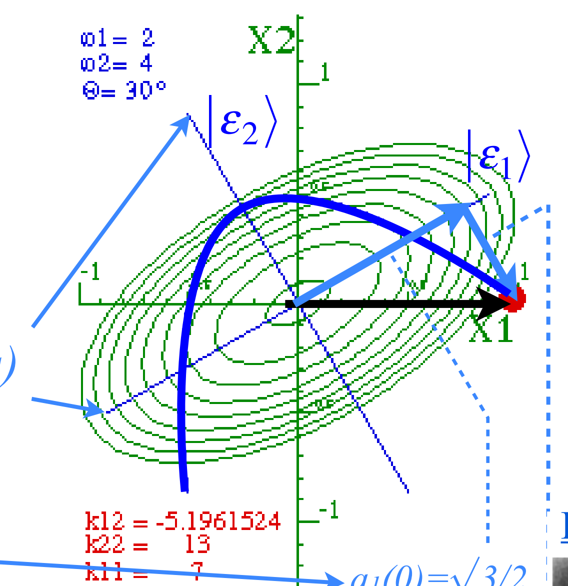
Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix}\sqrt{3}/2&1/2\end{pmatrix},\quad\langle\varepsilon_2|=\begin{pmatrix}-1/2&\sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*



[BoxIt Simulation]

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

$$\mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$

$q_1(0)=\sqrt{3/2}$

$q_2(0)=-1/2$

Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2(t)=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}\left[q_1(t)\right]^2+\frac{1}{2}$$

Example of a Tschebycheff Polynomial order 2

[Pafnuty Chebyshev]

Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshov, Tchebychev or Tchebycheff, or Tschebyschev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk

Died: December 8, 1894, Saint Petersburg