Quadratic form geometry and development of mechanics of Lagrange and Hamilton
(Ch. 12 of Unit 1 and Ch. 4-5 of Unit 7)

Review of partial differential calculus
Chain rule and order \( \frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x} \) symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
- Introducing 0\(^{th}\) Lagrange and 0\(^{th}\) Hamilton differential equations of mechanics
- Introducing 1\(^{st}\) Lagrange and 1\(^{st}\) Hamilton differential equations of mechanics

Introducing the Poincaré' and Legendre contact transformations
- Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
- Example from thermodynamics
- Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics
- Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
- Intuitive-geometric development of “” and “"

Link ⇒ CoulIt - Simulation of the Volcanoes of Io
Link ⇒ RelaWavity - Physical Terms \( H(p) \) & \( L(u) \)
Review of partial differential calculus

*Chain rule and order* $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

*Scaling transformation between Lagrangian and Hamiltonian views of KE*
  *Introducing* $0^{th}$ Lagrange and $0^{th}$ Hamilton differential equations of mechanics
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*Introducing the Poincare’ and Legendre contact transformations*
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*An elementary contact transformation from sophomore physics*
  *Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”*
  *Intuitive-geometric development of “ ” and “ ”*
Begin with a function $z = f(z)$ of 2-dimensions $(x, y)$ and plotted in 3-D (Then approximate by cells and tiles.)
Begin with a function \( z = f(z) \) of 2-dimensions \((x, y)\) and plotted in 3-D (Then approximate by cells and tiles.)

\[
f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x
\]

Slope: \( \frac{\partial f}{\partial x}(x_0, y_0) \)

\[
\begin{align*}
(x_0, y_0) &= (x_0, y_0) \\
(x_0, y_0 + \Delta y) &= (x_0, y_0 + \Delta y) \\
(x_0 + \Delta x, y_0) &= (x_0 + \Delta x, y_0) \\
(x_0 + 2\Delta x, y_0) &= (x_0 + 2\Delta x, y_0)
\end{align*}
\]
Begin with a function $z=f(x,y)$ of 2-dimensions $(x,y)$ and plotted in 3-D (Then approximate by cells and tiles.)
Begin with a function $z = f(z)$ of 2-dimensions $(x, y)$ and plotted in 3-D (Then approximate by cells and tiles.)

$$f(x_0, y_1) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y$$

$$\Delta x$$

$$y-axis$$

$$\Delta y$$

$$\Delta x$$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x$$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y$$

$$\Delta x$$

$$\Delta y$$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x$$

slop: $\frac{\partial f}{\partial y}(x_1, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta x$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x$$

$$\Delta x$$

$$\Delta y$$

$$\Delta x$$

$$\Delta y$$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y$$

$$\Delta x$$

$$\Delta y$$

$$\Delta x$$

$$\Delta y$$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x$$

slop: $\frac{\partial f}{\partial y}(x_1, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta x$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x$$

$$\Delta x$$

$$\Delta y$$

$$\Delta x$$

$$\Delta y$$

$$f(x_1, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x$$

slop: $\frac{\partial f}{\partial y}(x_1, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta x$
Begin with a function \( z = f(x, y) \) of 2-dimensions \((x, y)\) and plotted in 3-D (Then approximate by cells and tiles.)

\[
\begin{align*}
\frac{\partial f}{\partial x}(x_0, y_1) &= \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \\
\frac{\partial f}{\partial y}(x_0, y_0) &= \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x}(x_0, y_0) \right) \Delta x \\

\end{align*}
\]

\[
\begin{align*}
f(x_0, y_1) &= f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \\
f(x_1, y_0) &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x \\

\end{align*}
\]
\[ z = f(x, y) \]

**axis**

\[ f(x_1, y_1) = f(x_0, y_1) + \frac{\partial f}{\partial x}(x_0, y_1) \Delta x \]

\[ f(x_0, y_1) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \]

**slope:**

\[ \frac{\partial f}{\partial x}(x_0, y_1) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \Delta y \]

\[ \frac{\partial f}{\partial y}(x_0, y_1) = \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta x \]

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\[ \Delta x \]

\[ \Delta y \]

\[ (x_0, y_0) \]

\[ (x_0, y_0 + \Delta y) \]

\[ (x_0, y_0 + 2\Delta y) \]

\[ (x_0, y_0 + \Delta x, y_0) \]

\[ (x_0 + 2\Delta x, y_0) \]

\[ (x_1, y_1) \]

\[ (x_2, y_1) \]

\[ (x_2, y_2) \]

\[ (x_1, y_2) \]
\[ f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x \]

\[ + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \]

\[ z = f(x, y) \]

- **x-axis**
  - \( x_0, y_0 \)
  - \( x_0 + \Delta x, y_0 \)

- **y-axis**
  - \( x_0, y_0 + \Delta y \)
  - \( x_0, y_0 + 2\Delta y \)

- **slope**: \( \frac{\partial f}{\partial x}(x_1, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta x \)

- **slope**:
  - \( \frac{\partial f}{\partial y}(x_0, y_0) \)

\[ \frac{\partial f}{\partial y}(x_0, y_0) \]

\[ \frac{\partial f}{\partial x}(x_0, y_0) \]
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\[ = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \left( \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \right) \Delta x \]

\[ = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \Delta y \Delta x \]

\[ \frac{\partial f}{\partial x}(x_0, y_1) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \Delta y \]

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\begin{align*}
    f(x_1, y_1) &= f(x_0, y_1) + \frac{\partial f}{\partial x}(x_0, y_1) \Delta x \\
    &= f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \left( \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \right) \Delta x \\
    &= f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \Delta x
\end{align*}
\]
\[
f(x_1, y_1) = f(x_0, y_1) + \frac{\partial f}{\partial x}(x_0, y_1) \Delta x + \frac{\partial f}{\partial y}(x_0, y_1) \Delta y
\]

\[
f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y
\]

\[
d(\Delta x) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y
\]

\[
slope: \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta x
\]

\[
slope: \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)
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slope: \[ \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \Delta y \]

\[ z = f(x, y) \]

\[ f(x_0, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x \]

slope: \[ \frac{\partial f}{\partial y}(x_1, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta x \]
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What the geometry indicates....(Two important results)

\[ f(x_1, y_1) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \Delta x \Delta y \]

\[ = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \Delta x \]
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\[
f(x_0, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \Delta x
\]

If \( f(x, y) \) is continuous around \((x_0, y_0)\) and \((x_1, y_1)\) then \( \frac{\partial}{\partial y} \frac{\partial f}{\partial x} \) equals \( \frac{\partial}{\partial x} \frac{\partial f}{\partial y} \).
What the geometry indicates....(Two important results)

\[ f(x_1, y_1) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \Delta x \Delta y \]

\[ = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \Delta x \]

1. Chain rules

\[ [f(x_1, y_1) - f(x_0, y_0)] = df = \frac{\partial f}{\partial x}(x_0, y_0) dx + \frac{\partial f}{\partial y}(x_0, y_0) dy \ldots \text{(keep } 1^{st}\text{-order terms only!)} \]

\[ \frac{df}{dt} = \frac{\partial f}{\partial x}(x_0, y_0) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0, y_0) \frac{dy}{dt} \]

\[ \dot{f} = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \quad \text{(shorthand notation)} \]

If \( f(x, y) \) is continuous around \((x_0, y_0)\) and \((x_1, y_1)\), then \( \frac{\partial}{\partial y} \frac{\partial f}{\partial x} \) equals \( \frac{\partial}{\partial x} \frac{\partial f}{\partial y} \)
What the geometry indicates....(Two important results)

\[ f(x_1, y_1) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y \]

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1. Chain rules

\[ [f(x_1, y_1) - f(x_0, y_0)] = df = \frac{\partial f}{\partial x}(x_0, y_0) dx + \frac{\partial f}{\partial y}(x_0, y_0) dy \text{ (keep 1st-order terms only!)} \]

\[ \frac{df}{dt} = \frac{\partial f}{\partial x}(x_0, y_0) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0, y_0) \frac{dy}{dt} \]

\[ \dot{f} = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \text{ (shorthand notation)} = \partial_x f \dot{x} + \partial_y f \dot{y} \]

2. Symmetry of partial deriv. ordering

\[ \frac{\partial \partial f}{\partial y \partial x} = \frac{\partial \partial f}{\partial x \partial y} \quad \text{or:} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or:} \quad \partial_y \partial_x f = \partial_x \partial_y f \]

(shorthand notation)

Tuesday, September 22, 2015
What the geometry indicates....(Two important results)

\[ f(x_1,y_1) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0) \Delta x + \frac{\partial f}{\partial y}(x_0,y_0) \Delta y + \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0,y_0) \Delta x \Delta y \]

\[ = f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0) \Delta y + \frac{\partial f}{\partial x}(x_0,y_0) \Delta x + \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0,y_0) \Delta y \Delta x \]

1. Chain rules

\[ \left[ f(x_1,y_1) - f(x_0,y_0) \right] = df = \frac{\partial f}{\partial x}(x_0,y_0) \Delta x + \frac{\partial f}{\partial y}(x_0,y_0) \Delta y \ldots \quad \text{(keep 1st-order terms only!)} \]

\[ \frac{df}{dt} = \frac{\partial f}{\partial x}(x_0,y_0) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0,y_0) \frac{dy}{dt} \]

\[ \dot{f} = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \quad \text{(shorthand notation)} \]

2. Symmetry of partial deriv. ordering

\[ \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} \quad \text{or:} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or:} \quad \partial_y \partial_x f = \partial_x \partial_y f \]

(Shorthand notation)

Let: \( \vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \) so: \( \vec{\nabla} f \cdot \text{dr} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left( \begin{array}{c} dx \\ dy \end{array} \right) = \partial_x f \; dx + \partial_y f \; dy = df \)
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Intuitive-geometric development of ” ” ” ” and ” ” ” ”
Three ways to express energy: Consider kinetic energy (KE) first

1. **Lagrangian** is explicit function of velocity: \[ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

\[
L(v_k ... ) = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2 + ...) = L(v...) = \frac{1}{2} v \cdot M \cdot v + ... = \frac{1}{2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + ...
\]

2. **“Estrangian”** is explicit function of \( R \)-rescaled velocity: 
(or l’Estrangian)

[** speedinum **] \( v = R \cdot v \)

or: \[
E(V_k ... ) = \frac{1}{2} (V_1^2 + V_2^2 + ...) = E(V...) = \frac{1}{2} V \cdot 1 \cdot V + ... = \frac{1}{2} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + ...
\]

3. **Hamiltonian** is explicit function of \( M=R^2 \)-rescaled velocity:

(or **momentum** \( p \)) \[ p = M \cdot v \]

\[
H(p_k ... ) = \frac{1}{2} \left( \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + ... \right) = H(p...) = \frac{1}{2} p \cdot M^{-1} \cdot p + ... = \frac{1}{2} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + ...
\]
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Introducing the (partial $\frac{\partial}{\partial t}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and Estrangian have no explicit dependence on momentum $p = M \cdot v$

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian and Estrangian have no explicit dependence on velocity $v = M^{-1} \cdot p$

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian and Hamiltonian have no explicit dependence on speedinum $v = M^{1/2} \cdot v$

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$
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Hamiltonian and Estrangian have no explicit dependence on velocity $v = M^{-1} \cdot p$

\[
\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}
\]

Lagrangian and Hamiltonian have no explicit dependence on speedinum $V = M^{1/2} \cdot v$

\[
\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}
\]

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections†

\[
\nabla_v L = \frac{\partial L}{\partial v} = \frac{\partial}{\partial v} \left( v \cdot M \cdot v \right) = \frac{1}{2} \left( \begin{array}{c} m_1 & 0 \\ 0 & m_2 \end{array} \right) \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) = \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right)
\]

Lagrange’s 1st equation(s)

\[
\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial v} = p
\]

\[
\nabla_p H = v = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \left( p \cdot M^{-1} \cdot p \right) = \frac{1}{2} \left( \begin{array}{c} m_1 & 0 \\ 0 & m_2 \end{array} \right) \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) = \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)
\]

Hamilton’s 1st equation(s)

\[
\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial p} = v
\]

Estrangian is neglected for now. (It is related to dual ellipse geometry in Lecture 8 p. 71-79 and 99-101 )

†non-dependency due to stationary-value effects as shown on p. 28-31
Unit 1
Fig. 12.2

(a) **Lagrangian plot**
\[ L(\mathbf{v}) = \text{const.} = \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} / 2 \]

(b) **Hamiltonian plot**
\[ H(\mathbf{p}) = \text{const.} = \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} / 2 \]

\[ p_2 = m_2 v_2 \]

\[ p_1 = m_1 v_1 \]

\[ H = \text{const} = E \]

\[ a_x = \sqrt{2E/m_2} \]

\[ a_y = \sqrt{2E/m_1} \]
Fig. 12.2

(a) Lagrangian plot
\[ L(\mathbf{v}) = \text{const.} = \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}/2 \]

(b) Hamiltonian plot
\[ H(p) = \text{const.} = p \cdot \mathbf{M}^{-1} \cdot p/2 \]

(c) Overlapping plots
- Lagrangian tangent at velocity \( \mathbf{v} \)
  - is normal to momentum \( \mathbf{p} \)

- Hamiltonian tangent at momentum \( \mathbf{p} \)
  - is normal to velocity \( \mathbf{v} \)

\[ \mathbf{v}_2 = \frac{p_2}{m_2} \]

\[ \mathbf{v}_1 = \frac{p_1}{m_1} \]

\[ p_2 = m_2 v_2 \]

\[ p_1 = m_1 v_1 \]

\[ H = \text{const.} = E \]

\[ L = \text{const.} = E \]

\[ a_1 = \sqrt{2Em_1} \]

\[ b_1 = \sqrt{2Em_1} \]

\[ a_2 = \sqrt{2Em_2} \]

\[ b_2 = \sqrt{2Em_2} \]
Unit 1
Fig. 12.2

(a) Lagrangian plot
\[ L(v) = \text{const.} = v \cdot M \cdot v / 2 \]

(b) Hamiltonian plot
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2 \]

(c) Overlapping plots

1st equation of Lagrange
\[ L = \text{const.} = E \]

1st equation of Hamilton
\[ H = \text{const.} = E \]

Lagrangian tangent at velocity \( v \) is normal to momentum \( p \)
\[ p = \nabla_v L = M \cdot v \]

Hamiltonian tangent at momentum \( p \) is normal to velocity \( v \)
\[ v = \nabla_p H = M^{-1} \cdot p \]

(d) Less mass

(e) More mass
Review of partial differential calculus
Chain rule and order $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
Introducing $0^{th}$ Lagrange and $0^{th}$ Hamilton differential equations of mechanics
Introducing $1^{st}$ Lagrange and $1^{st}$ Hamilton differential equations of mechanics

Introducing the Poincaré and Legendre contact transformations
Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of “ ” and “ ”
Introducing the Poincaré’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\[ Q\text{-forms } L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \quad \text{or} \quad H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \]

to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).
Introducing the Poincaré' and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\[ Q\text{-forms } L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \quad \text{or} \quad H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \]

to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).

Numerically-CORRECT, but Differentially-WRONG!
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\[ Q \text{-forms} \quad L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \quad \text{or} \quad H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \]

to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).

Numerically-CORRECT, but Differentially-WRONG! (In classical physics \( \mathbf{p} \cdot \mathbf{v} \) and \( \mathbf{v} \cdot \mathbf{p} \) are identical)

Instead try: \( H(\mathbf{p}..) = \mathbf{p} \cdot \mathbf{v} - (1/2) \mathbf{v} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}..) \) or else: \( L(\mathbf{v}..) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}..) \)
Introducing the Poincaré’ and Legendre contact transformations

Given matrix relation: $\mathbf{p} = \mathbf{M} \cdot \mathbf{v}$ or its inverse: $\mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p}$ you might be tempted to rewrite

$Q$-forms $L(\mathbf{v}..) = \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$ or $H(\mathbf{p}..) = \frac{1}{2} \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p}$ to be $H = \frac{1}{2} \mathbf{p} \cdot \mathbf{v}$ or equivalently $L = \frac{1}{2} \mathbf{v} \cdot \mathbf{p}$.

Numerically-CORRECT, but Differentially-WRONG!

Instead try: $H(\mathbf{p}..) = \mathbf{p} \cdot \mathbf{v} - \frac{1}{2} \mathbf{v} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}..)$ or else: $L(\mathbf{v}..) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}..)$

That is ... the Legendre contact transformation

$L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ or: $H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})$
Introducing the Poincaré' and Legendre contact transformations

Given matrix relation: $p = M \cdot v$ or its inverse: $v = M^{-1} \cdot p$ you might be tempted to rewrite

$Q$-forms $L(v.) = (1/2)v \cdot M \cdot v$ or $H(p.) = (1/2)p \cdot M^{-1} \cdot p$ to be $H = (1/2)p \cdot v$ or equivalently $L = (1/2)v \cdot p$.

Numerically-CORRECT, but Differentially-WRONG!

Instead try: $H(p.) = p \cdot v - (1/2)v \cdot p = p \cdot v - L(v.)$ or else: $L(v.) = p \cdot v - H(p.)$

That is ... the Legendre contact transformation

$L(v) = p \cdot v - H(p)$ or: $H(p) = p \cdot v - L(v)$

Now explicit dependency (non)-relations give the right derivatives

$$\frac{\partial L(v)}{\partial p} = \frac{\partial}{\partial p} p \cdot v - \frac{\partial H(p)}{\partial p}$$

$$0 = v - \frac{\partial H(p)}{\partial p}$$

$$\frac{\partial H(p)}{\partial v} = \frac{\partial}{\partial v} p \cdot v - \frac{\partial L(v)}{\partial v}$$

$$0 = p - \frac{\partial L(v)}{\partial v}$$
Introducing the Poincaré’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

Q-forms \( L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \) or \( H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \) to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).

Numerically-CORRECT, but Differentially-WRONG!

Instead try: \( H(\mathbf{p}..) = \mathbf{p} \cdot \mathbf{v} - (1/2) \mathbf{v} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}..) \) or else: \( L(\mathbf{v}..) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}..) \)

That is ... the Legendre contact transformation

\[
L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}) \quad \text{or:} \quad H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})
\]

Now explicit dependency (non)-relations give the right derivatives

\[
\frac{\partial L(\mathbf{v})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \mathbf{p} \cdot \mathbf{v} - \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \quad \frac{\partial H(\mathbf{p})}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \mathbf{p} \cdot \mathbf{v} - \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}
\]

\[
0 = \mathbf{v} - \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \quad 0 = \mathbf{p} - \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}
\]

That is Hamilton’s 1st equation(s) and Lagrange’s 1st equation(s)

\[
\mathbf{v} = \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \quad \mathbf{p} = \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}
\]
Review of partial differential calculus
    Chain rule and order $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
    Introducing 0th Lagrange and 0th Hamilton differential equations of mechanics
    Introducing 1st Lagrange and 1st Hamilton differential equations of mechanics

Introducing the Poincare’ and Legendre contact transformations
    Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
    Example from thermodynamics
    Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics
    Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
    Intuitive-geometric development of “ ” and “ ”
(a) Lagrangian plot
\[ L(v) = v \cdot p - H(p) \]

Unit 1
Fig. 12.3

(b) Hamiltonian plot
\[ H(p) = p \cdot v - L(v) \]
Preview of Unit 8:
Geometry of Legendre contact transformation persists in relativistic quantum mechanics!

(In fact it is due to the wave mechanics and phase invariance principles.)
How Legendre contact transformations work... (to make $\frac{\partial H}{\partial v} = 0$ or $\frac{\partial L}{\partial p} = 0$)

Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$ and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > ...$ for increasing velocity $v_{-2} > v_{-1} > ... > v_0$ lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has max $H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$

(a) Secant lines: $L(v) = p \cdot v - H$

for fixed slope $p$ and varying $H$

Tangent line points to extreme value $-H(v_0)$ of intercept $-H$ thus:

$\frac{dH(v)}{dv} = 0$

Unit 1

Fig. 12.4

Tuesday, September 22, 2015
How Legendre contact transformations work... (to make $\frac{\partial H}{\partial v} = 0$ or $\frac{\partial L}{\partial p} = 0$)

Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$ and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > ...$ for increasing velocity $v_{-2} > v_{-1} > ... > v_0$ lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has $\max H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$
**How Legendre contact transformations work...** (to make $\frac{\partial H}{\partial v} = 0$ or $\frac{\partial L}{\partial p} = 0$)

Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$ and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > ...$

for increasing velocity $v_{-2} > v_{-1} > ... > v_0$

lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has $\max H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$

(a) Secant lines: $L(v) = p \cdot v - H$

for fixed slope $p$ and varying $H$

Tangent line points to extreme value $-H(v_0)$ of intercept $-H$ thus:

$dH(v)/dv = 0$

$\frac{\partial H}{\partial v} = 0$ at each point $v = \frac{\partial H}{\partial p}$ of $L(v)$ with slope $p = \frac{\partial L}{\partial v}$
Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$ and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > ...$ for increasing velocity $v_{-2} > v_{-1} > ... > v_0$ lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has $\max H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$

(Similarly...)

Unit 1
Fig. 12.4
How Legendre contact transformations work... (to make \( \frac{\partial H}{\partial v} = 0 \) or \( \frac{\partial L}{\partial p} = 0 \))

Secant lines \( L(v) = p \cdot v - H \) of fixed slope \( p = \frac{\partial L}{\partial v} \) and decreasing intercept \(-H(v_{-2}) > -H(v_{-1}) > \ldots \)

for increasing velocity \( v_{-2} > v_{-1} > \ldots > v_0 \)

Thus \( \frac{\partial H}{\partial v} = 0 \)

Thus tangent contact point \( v = v_0 \) that has \( \max H(p, v_0) \)

(Similarly...)

Unit 1
Fig. 12.4

(a) Secant lines: \( L(v) = p \cdot v - H \) for fixed slope \( p \) and varying \( H \)

Tangent line points to extreme value \(-H(v_0)\) of intercept \(-H\) thus:
\( \frac{dH(v)}{dv} = 0 \)

(b) Secant lines: \( H(p) = p \cdot v - L(v) \) for fixed slope \( v \) and varying \( L \)

Tangent line points to extreme value \(-L(p_0)\) of intercept \(-L\) thus:
\( \frac{dL(p)}{dp} = 0 \)
Review of partial differential calculus
  Chain rule and order $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
  Introducing $0^{th}$ Lagrange and $0^{th}$ Hamilton differential equations of mechanics
  Introducing $1^{st}$ Lagrange and $1^{st}$ Hamilton differential equations of mechanics

**Introducing the Poincaré’ and Legendre contact transformations**

*Geometry of Legendre contact transformation* (Preview of Unit 8 relativistic quantum mechanics)

*Example from thermodynamics*

Legendre transform: special case of General Contact Transformation *(lights, camera, ACTION!)*

An elementary contact transformation from sophomore physics
  Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
  Intuitive-geometric development of ” ” ” ” and ” ” ” ”
**Example of Legendre contact transformation in thermodynamics**

*Internal energy* $U(S,V)$ is defined as a function of entropy $S$ and volume $V$.

A new function *enthalpy* $H(S,P)$ depends on entropy and *pressure* $P$.

It is a Legendre transform $H(S,P)=P \cdot V + U$ of energy $U(S,V)$ to new variable $P = -\left( \frac{\partial U}{\partial V} \right)_S$. 
Example of Legendre contact transformation in thermodynamics

Internal energy $U(S,V)$ is defined as a function of entropy $S$ and volume $V$.

A new function enthalpy $H(S,P)$ depends on entropy and pressure $P$.

It is a Legendre transform $H(S,P)=P\cdot V+U$ of energy $U(S,V)$ to new variable $P = -(\frac{\partial U}{\partial V})_S$.
Internal energy \( U(S,V) \) is defined as a function of entropy \( S \) and volume \( V \).

A new function enthalpy \( H(S,P) \) depends on entropy and pressure \( P \).

It is a Legendre transform \( H(S,P)=P \cdot V+U \) of energy \( U(S,V) \) to new variable \( P \).

Except for ± signs, it’s our Hamiltonian \( H(p)=p \cdot v-L(v) \) going from Lagrangian \( L(v) \) to use new variable momentum \( p = \left( \frac{\partial L}{\partial v} \right)_x \).
Review of partial differential calculus
Chain rule and order $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
Introducing $0^{th}$ Lagrange and $0^{th}$ Hamilton differential equations of mechanics
Introducing $1^{st}$ Lagrange and $1^{st}$ Hamilton differential equations of mechanics

Introducing the Poincaré’ and Legendre contact transformations
Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of ” ” ” ” ” ” ” ” ” ”
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function: \( S(x,y,X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S = \text{const.} \) curves.

Unit 1
Fig. 12.7
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function: \( S(x, y; X, Y) = \text{const.} \) does mapping.

Y(X) is mapped from y(x) as an envelope of contacting \( S = \text{const.} \) curves.

The Legendre transformation does it with contacting straight line tangents.
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function:

\[ S(x, y; X, Y) = \text{const.} \text{ does mapping.} \]

\[ Y(X) \text{ is mapped from } y(x) \text{ as an envelope of contacting } S = \text{const.} \text{ curves.} \]

...And, Visa-Versa !...

The Legendre transformation does it with contacting straight line tangents.

Unit 1
Fig. 12.7

Unit 1
Fig. 12.9

Legendre transform: special case of General Contact Transformation
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function: \( S(x,y,X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S = \text{const.} \) curves.

...And, Visa-Versa !...

The Legendre transformation does it with contacting straight line tangents.

Poincare’s differential action

\[
\begin{align*}
\text{d}S &= L \text{d}t = p \cdot \dot{q} \text{d}t - H \cdot \text{d}t \\
&= p \cdot dq - H \cdot \text{d}t
\end{align*}
\]

Unit 1
Fig. 12.7

Unit 1
Fig. 12.9
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y,X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S = \text{const.} \) curves.

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**

Legendre transformation: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y,X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S = \text{const.} \) curves.

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**

Legendre transformation: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y,X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S = \text{const.} \) curves.

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: $S(x,y:X,Y) = \text{const.}$ does mapping.

$Y(X)$ is mapped from $y(x)$ as an envelope of contacting $S = \text{const.}$ curves.

...And, Visa-Visa !...

The Legendre transformation does it with contacting straight line tangents.

Poincare’s differential action

$dS = L\,dt = p \cdot \dot{q} \, dt - H \cdot dt$

$= p \cdot dq - H \cdot dt$

$dS = L\,dt = \hbar k \cdot dr - \hbar \omega \cdot dt$

(Quantum phase differential)

This extraordinary claim needs extraordinary proof!

(...given by later lectures for Ch. 12 Unit 1 and Unit 8.)
Review of partial differential calculus
Chain rule and order $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
Introducing 0th Lagrange and 0th Hamilton differential equations of mechanics
Introducing 1st Lagrange and 1st Hamilton differential equations of mechanics

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Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of ” ” and ” ”
(b) Atomic clock controls expanding balls of Cesium atoms rising and falling in Earth gravity

(NIST Boulder Labs)

(c) Trajectory family for fixed $g$ and $v_0$.

Atom ball expands at constant rate $v_0$ as center falls at constantly increasing rate $g^t$ and it maintains two contact points with the envelope after reaching its highest point.
Link ⇒ CoulIt - Simulation of the Volcanoes of Io
Review of partial differential calculus

Chain rule and order \( \partial^2 \Psi / \partial x \partial y = \partial^2 \Psi / \partial y \partial x \) symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE

Introducing 0\(^{th}\) Lagrange and 0\(^{th}\) Hamilton differential equations of mechanics

Introducing 1\(^{st}\) Lagrange and 1\(^{st}\) Hamilton differential equations of mechanics

Introducing the Poincaré and Legendre contact transformations

Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)

Example from thermodynamics

Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics

Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”

Intuitive-geometric development of ” and …” and ” … ”
UP-1 formulas for trajectories in constant gravity $g$

$$x(t) = (v_0 \cos \alpha) t$$

$$y(t) = \left(v_0 \sin \alpha \right) t - \frac{1}{2} gt^2$$

$$\dot{x}(0) = v_x(0) = v_0 \cos \alpha$$

$$\dot{y}(0) = v_y(0) = v_0 \sin \alpha$$

Substitute time $t = x/(v_0 \cos \alpha)$ into $y(t)$

$$y(x) = \frac{v_0 \sin \alpha}{v_0 \cos \alpha} x - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$
Convert $y(x)$ solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$ becomes:

$S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0$
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha: x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha: x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

\( v_0 \)

\( \alpha = 45^\circ \)

\( 0 = S(v_0, \alpha, : x, y) \)

**Envelopes** of the \( v_0 \)-trajectory region contain extremal **contact points** with each trajectory where:

\[
\frac{\partial S(v_0, \alpha: x, y)}{\partial \alpha} = 0
\]
Convert $y(x)$ solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$

becomes:

$$S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0$$

Envelopes of the $v_0$-trajectory region contain extremal contact points with each trajectory where:

$$\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0$$

$$x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2 \sin \alpha}{\cos^3 \alpha}$$
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{becomes:} \quad S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

**Envelopes** of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{gives:} \quad \tan \alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g \tan \alpha}.
\]
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

\( v_0 \)

\( \alpha = 45^\circ \)

\( \alpha = 45^\circ \)

\( \alpha = 45^\circ \)

\( x \)

\( y \)

\( S(v_0, \alpha : x, y) = 0 \)

\( \alpha = 45^\circ \)

\( \alpha = 45^\circ \)

\( \alpha = 45^\circ \)

\( x \)

\( y \)

\( \text{Contact points} \)

\( \text{Envelopes} \) of the \( v_0 \)-trajectory region contain extremal \textit{contact points} with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2 \cos^3 \alpha} - \frac{2 \sin \alpha}{2v_0^2 \cos^3 \alpha} \Rightarrow \tan \alpha = \frac{v_0^2}{gx} \text{ or } x = \frac{v_0^2}{g \tan \alpha}.
\]

\[
y_{env}(x) = x \tan \alpha - \frac{gx^2}{2v_0^2} \left( 1 + \tan^2 \alpha \right) \Rightarrow y_{env}(x) = x \frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2} \left( 1 + \frac{v_0^4}{g^2 x^2} \right)
\]
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

**Envelopes** of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^2 \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2 \sin \alpha}{\cos^3 \alpha}
\]

\[
\tan \alpha = \frac{v_0^2}{gx} \quad \text{or} \quad x = \frac{v_0^2}{g \tan \alpha}
\]

\[
y_{env}(x) = x \tan \alpha - \frac{gx^2}{2v_0^2} \left(1 + \tan^2 \alpha\right) \Rightarrow y_{env}(x) = x \frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2} \left(1 + \frac{v_0^4}{g^2 x^4}\right)
\]

\[
y_{env}(x) = \frac{v_0^2}{g} - \frac{gx^2}{2v_0^2} - \frac{g^2}{2v_0^2} \frac{v_0^4}{g^2 x^2} = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}
\]
Review of partial differential calculus
Chain rule and order $\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
Introducing $0^{\text{th}}$ Lagrange and $0^{\text{th}}$ Hamilton differential equations of mechanics
Introducing $1^{\text{st}}$ Lagrange and $1^{\text{st}}$ Hamilton differential equations of mechanics

Introducing the Poincare’ and Legendre contact transformations
Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)

An elementary contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of ” ” ” ” and ” ” ” ”
The Plumes of Prometheus
NASA-Galileo Project
Io fly-by on August 18, 1997

Scientists are eager for a closer look at the solar system's strangest and most active volcanoes when Galileo flies by Io on October 11.

October 4, 1999: Thirty years ago, before the Voyager probes visited Jupiter, if you had described Io to a literary critic it would have been declared overwrought science fiction. Jupiter's strange moon is literally bursting with volcanoes. Dozens of active vents pepper the landscape which also includes gigantic frosty plains, towering mountains and volcanic rings the size of California. The volcanoes themselves are the hottest spots in the solar system with temperatures exceeding 1800 K (1527 °C). The plumes which rise 300 km into space are so large they can be seen from Earth by the Hubble Space Telescope. Confounding common sense, these high-rising ejecta seem to be made up of, not blisteringly hot lava, but frozen sulfur dioxide. And to top it all off, Io bears a striking resemblance to a pepperoni pizza. Simply unbelievable.

Right: Digital Radiance simulation of Pillan Patera just before the Galileo flyby. click for animation →
...conventional parabolic geometry...carried to extremes...

Recall Lecture 6 p.26 and p. 48-49 for kite geometry and application

Unit 1
Fig. 9.4
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises.
Q3. How high can $\alpha=45^\circ$ path rise ?
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit ?
Say $\alpha=90^\circ$ path rises to 1.0
then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. How high can $\alpha=45^\circ$ path rise?
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. How high can $\alpha=45^\circ$ path rise?
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit?
Say \( \alpha = 90^\circ \) path rises to 1.0 then drops. When at \( y = 1.0 \)...

Q1. ...where is its focus?

Q2. ...where is the blast wave? center falls as far as \( 90^\circ \) ball rises.

Q3. How high can \( \alpha = 45^\circ \) path rise? \( \frac{1}{2} \) as high.

Q4. Where on \( x \)-axis does \( \alpha = 45^\circ \) path hit? \( x = 2 \).

Q5. Where is blast wave then?

Q6 Where is \( \alpha = 45^\circ \) path focus?

Q7 Guess for all-path envelope? and its focus? directrix?

\[
\begin{align*}
\alpha &= 90^\circ \\
\sin \alpha &= \sin 90^\circ = 1 \\
\text{Right at the tippy-tip} \\
\sin \alpha &= \sin 45^\circ = \frac{1}{\sqrt{2}} \\
\text{implies: } v_0^2 \sin^2 \alpha &= v_0^2 / 2 \\
\text{so y-coord. KE is } 1/2 \text{ for } \alpha = 45^\circ \\
\text{So: y-peak PE is } 1/2 \text{ for } \alpha = 45^\circ \\
\text{This sets } \alpha = 45^\circ \text{ parabolic "kite" and focus and range, etc.}
\end{align*}
\]
Say $\alpha=90^\circ$ path rises to 1.0
then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises.
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on x-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is $\alpha=45^\circ$ path focus?
Q7 Guess for all-path envelope? and its focus? directrix?

Right at the tippy-tip $v_0 \sin \alpha = v_0 \sin 45^\circ = v_0/\sqrt{2}$ implies: $v_0^2 \sin^2 \alpha = v_0^2/2$ so y-coord. KE is 1/2 for $\alpha=45^\circ$

So: y-peak PE is 1/2 for $\alpha=45^\circ$

This sets $\alpha=45^\circ$ parabolic "kite" and focus and range, etc.

$\alpha=45^\circ$ Envelope CONTACT POINT
That is maximum horizontal range so must be tangent to blast circle and must be tangent to envelope
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises.
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high.
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$.
Q5. Where is blast wave then? centered on 45° normal.
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$.
Q7 Guess for all-path envelope? and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45$^\circ$ normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?

Where is $\alpha=30^\circ$ path?

Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45$^\circ$ normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?

Where is $\alpha=30^\circ$ path?
Say $\alpha=90^\circ$ path rises to $1.0$
then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises
Q3. How high can $\alpha=45^\circ$ path rise? $1/2$ as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on $45^\circ$ normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope?
and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus?
directrix?

Where is $\alpha=30^\circ$ path? 
...and kite structure?
For $\alpha=60^\circ$ parabolic trajectory, contact-parabolic envelope, timing ($\alpha=0^\circ$)-parabola, ($\alpha=90^\circ$)-blast-wave-circle, ($\alpha=60^\circ$)-blast-wave-circle.

Where is $\alpha=60^\circ$ path? ...and kite structure?
Given elevation $\alpha=30^\circ$ construct contact-parabola, blast-wave-circle, and time.

**Step 1:** Extend elevation $\alpha=30^\circ$ line $OD$ (polar $\beta=60^\circ$) to All-$\alpha$ directrix pt. $D$ to envelope directrix $F$ (focal radius $OF$ at the contact pt. $C$).

**Step 2:** Extend double-$\beta(2\beta=120^\circ)$-focal radius past focus-locus pt. $F$ to (eventually) intersect contact pt. $C$.

**Step 3:** Extend Thales-rectangle segment $TF$ past focus pt. $F$ to All-$\alpha$ directrix pt. $D'$.

**Step 4:** Drop vertical line $DC$ to intersect focal radius of the parabola tangent at contact pt. $C$.

**Step 5:** Parabola kite-axis line $DEC$ is parabola tangent at contact pt. $C$.

**Step 6:** Drop parabola kite-cross-axis line $TFED'$ by vertical line $DC$ to make contact-circle radius line $OC$. The $(\alpha=30^\circ)$-contact-circle is blast-wave-circle at the moment that $(\alpha=30^\circ)$-parabola contacts envelope, too.

**Step 7:** Draw timing-parabola $OT'$ (elevation $\alpha=0^\circ$-parabola). Where timing-parabola hits a blast circle (for example at $T'$ for $t_{\alpha=90^\circ}=1$ and at $T''$ for $t_{\alpha=30^\circ}=2$) marks the time (in “blast units” $v_0/g$ by $x$ value) for that circle and its contacting parabola.

$t_{\alpha=0}$  
$t_{\alpha=1}$  
$t_{\alpha=2}$  

“blast time units” $v_0/g$  

Note large kite for envelope that contacts $\alpha=30^\circ$ trajectory smaller kite that contacts $\alpha=30^\circ$ trajectory and the $\alpha=30^\circ$ blast wave circle.