Lecture 3
Thur. 9.01.2015

Analysis of 1D 2-Body Collisions

(Ch. 3, Ch. 4, and Ch. 5 of Unit 1)

Review of (V_1,V_2) and (y_1,y_2) geometry and X2 launcher in box

Integration of (V_1,V_2) data to space-time plots $(y_1(t),t)$ and $(y_2(t),t)$ plots
Integration of (V_1,V_2) data to space-space plots (y_1, y_2) (Lect. 2 topic not covered)
→ Example of (V_1,V_2) and (y_1, y_2) data for high mass ratios: m_1/m_2=49, 100,... ←

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions (and quiz question about linear solutions)
"Mass-bang" matrix **M**, "Floor-bang" matrix **F**, "Ceiling-bang" matrix **C**.
Algebra and Geometry of "ellipse-Rotation" group product: **R**= **C**•**M**

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle
How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12
Reflections in the clothing store: "It's all done with mirrors!"
Introducing hexagonal symmetry D_6~C_{6v} (Resulting for m_1/m_2=3)
Group multiplication and product table
Classical collision paths with D_6~C_{6v} (Resulting from m_1/m_2=3)
Solutions to Exercises 1.4.1 and 1.4.2

Geometry of X2 launcher bouncing in box (Review)

Integration of (V_1,V_2) data to space-time plots ($y_1(t)$,t) and ($y_2(t)$,t) plots

→ *Integration of (V_1,V_2) data to space-space plots (y_1, y_2)* ←

Example of (V_1,V_2) and (y_1, y_2) data for high mass ratios: m_1/m_2=49, 100,...

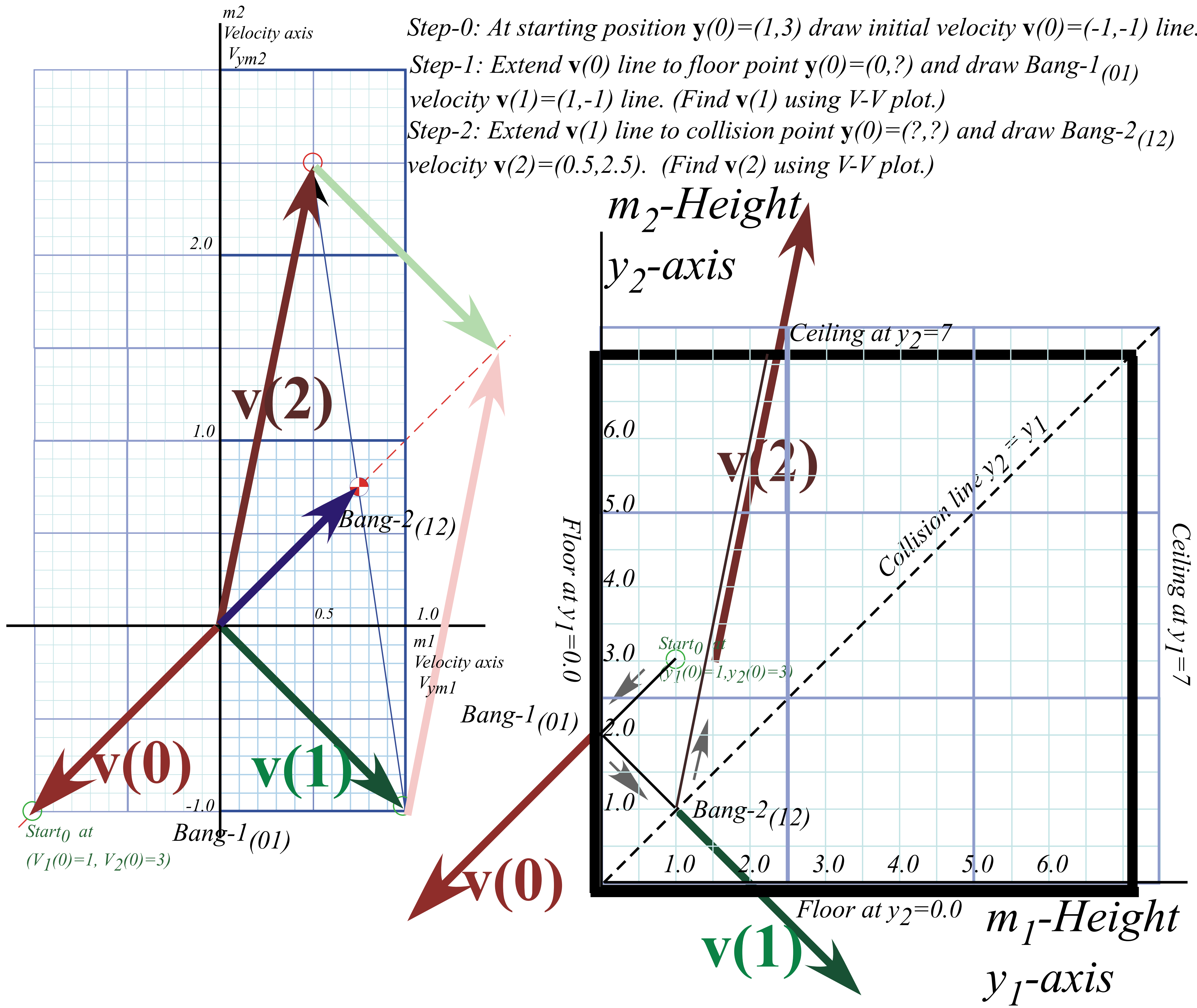
Geometric "Integration" (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

Step-0: At starting position **y***(0)=(1,3) draw initial velocity* **v***(0)=(-1,-1) line.*

Step-1: Extend **v***(0) line to floor point* **y***(0)=(0,?) and draw Bang-1$_{(01)}$ velocity* **v***(1)=(1,-1) line. (Find* **v***(1) using V-V plot.)*

Step-2: Extend **v***(1) line to collision point* **y***(0)=(?,?) and draw Bang-2$_{(12)}$ velocity* **v***(2)=(0.5,2.5). (Find* **v***(2) using V-V plot.)*



Geometric "Integration" (Converting Velocity data to Space-space trajectory)

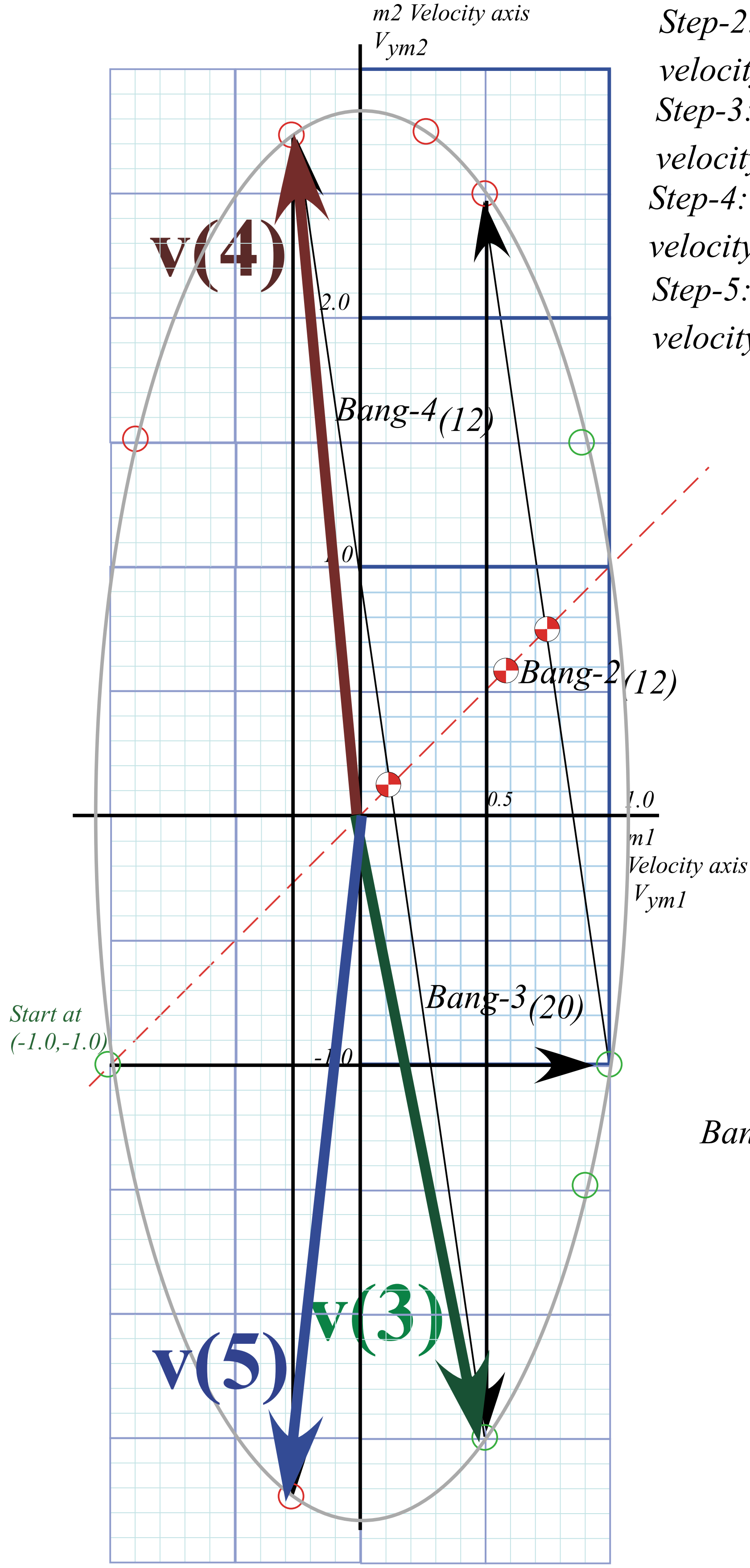
Fig. 4.11
in Unit 1

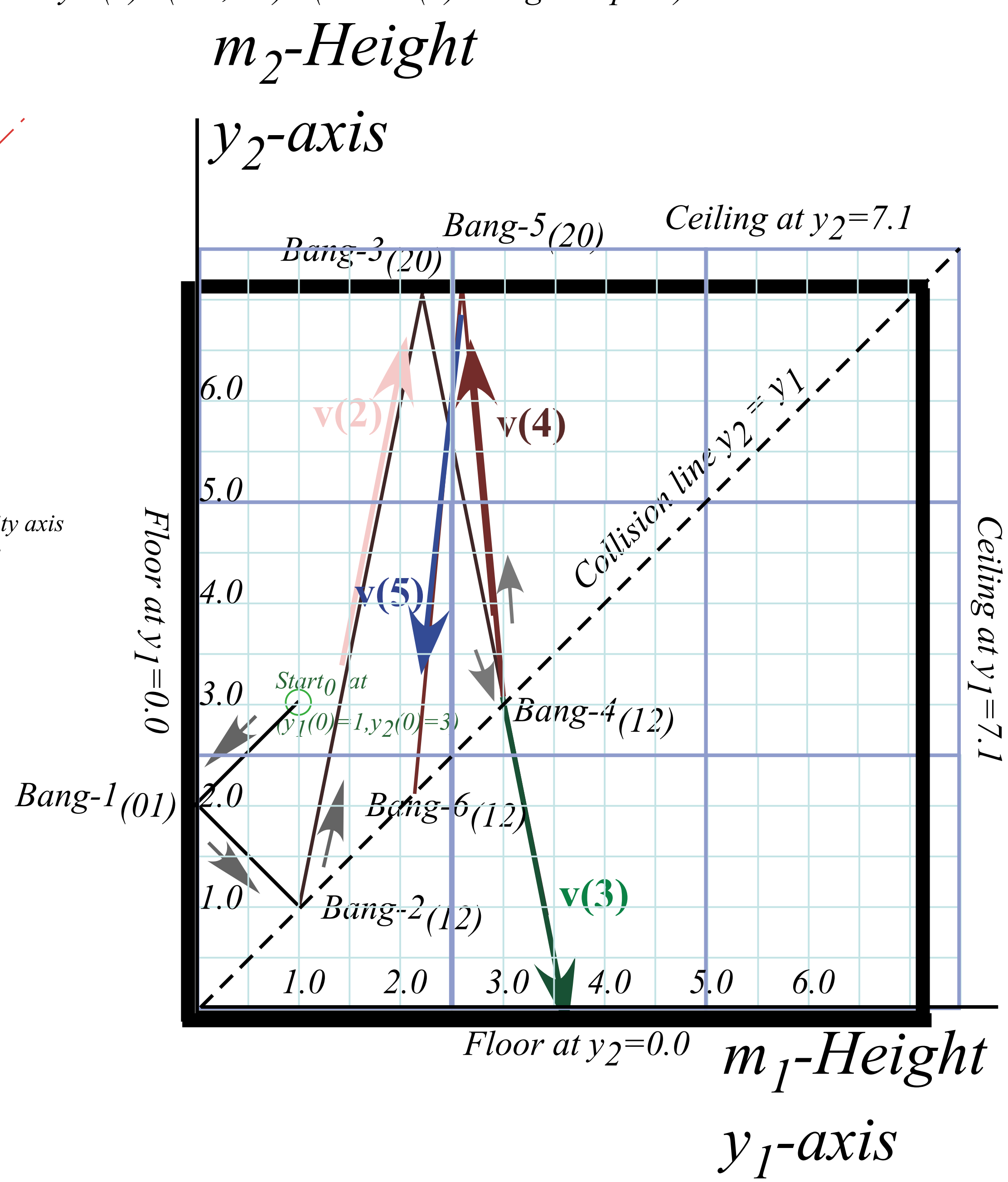
Step-2: Extend **v**(2) *line to ceiling point* **y**(3)=(?,7.1) *and draw Bang-3*$_{(20)}$ *velocity* **v**(3)=(1,-1) *line. (Find* **v**(3) *using V-V plot.)*

Step-3: Extend **v**(3) *line to collision point* **y**(4)=(?,?) *and draw Bang-4*$_{(12)}$ *velocity* **v**(4)=(0.5,2.5). *(Find* **v**(4) *using V-V plot.)*

Step-4: Extend **v**(4) *line to ceiling point* **y**(4)=(?,7.1) *and draw Bang-5*$_{(20)}$ *velocity* **v**(5)=(1,-1) *line. (Find* **v**(5) *using V-V plot.)*

Step-5: Extend **v**(5) *line to collision point* **y**(6)=(?,?) *and draw Bang-6*$_{(12)}$ *velocity* **v**(6)=(0.5,2.5). *(Find* **v**(6) *using V-V plot.)*





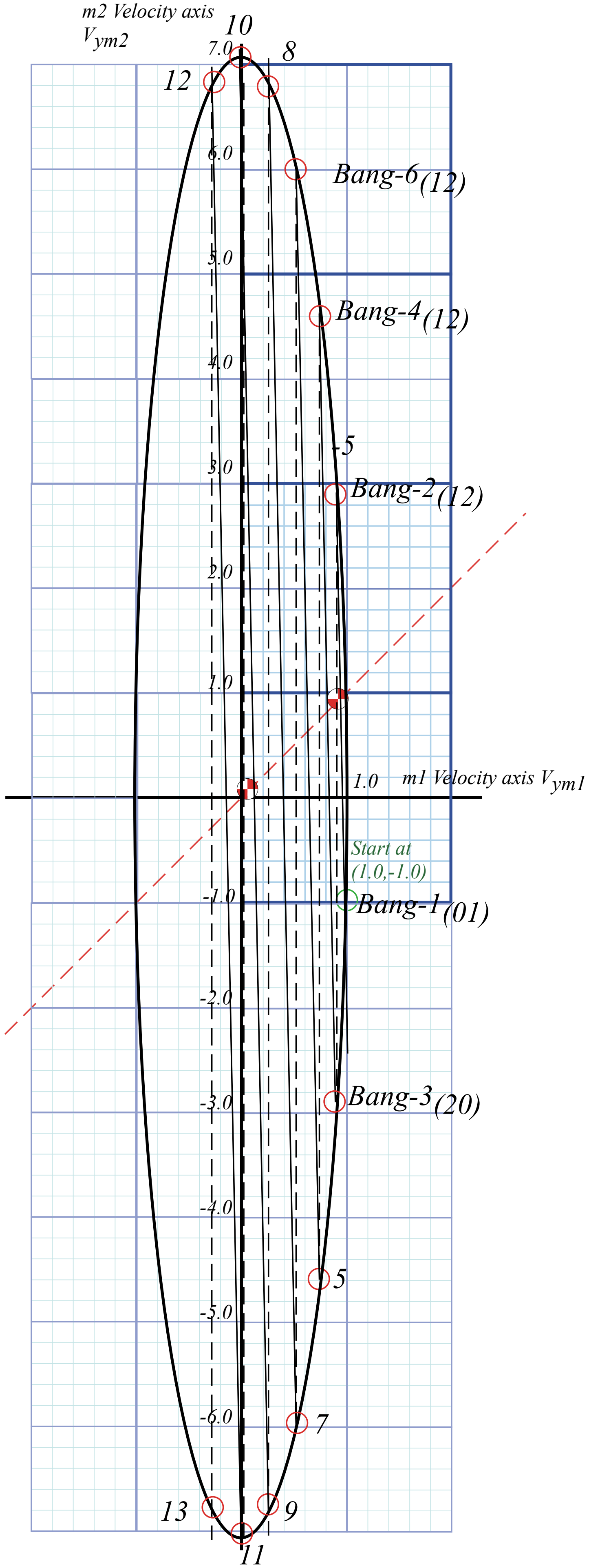
Geometry of X2 launcher bouncing in box (Review)

Integration of (V_1,V_2) data to space-time plots $(y_1(t),t)$ and $(y_2(t),t)$ plots

Integration of (V_1,V_2) data to space-space plots (y_1, y_2) *(Lect. 2 topic not covered)*

→ *Example of (V_1,V_2) and (y_1, y_2) data for high mass ratios: m_1/m_2=49, 100,...* ←

Geometric "Integration" (Converting Velocity data to Space-time trajectory)



Example with masses: m_1=49 and m_2=1

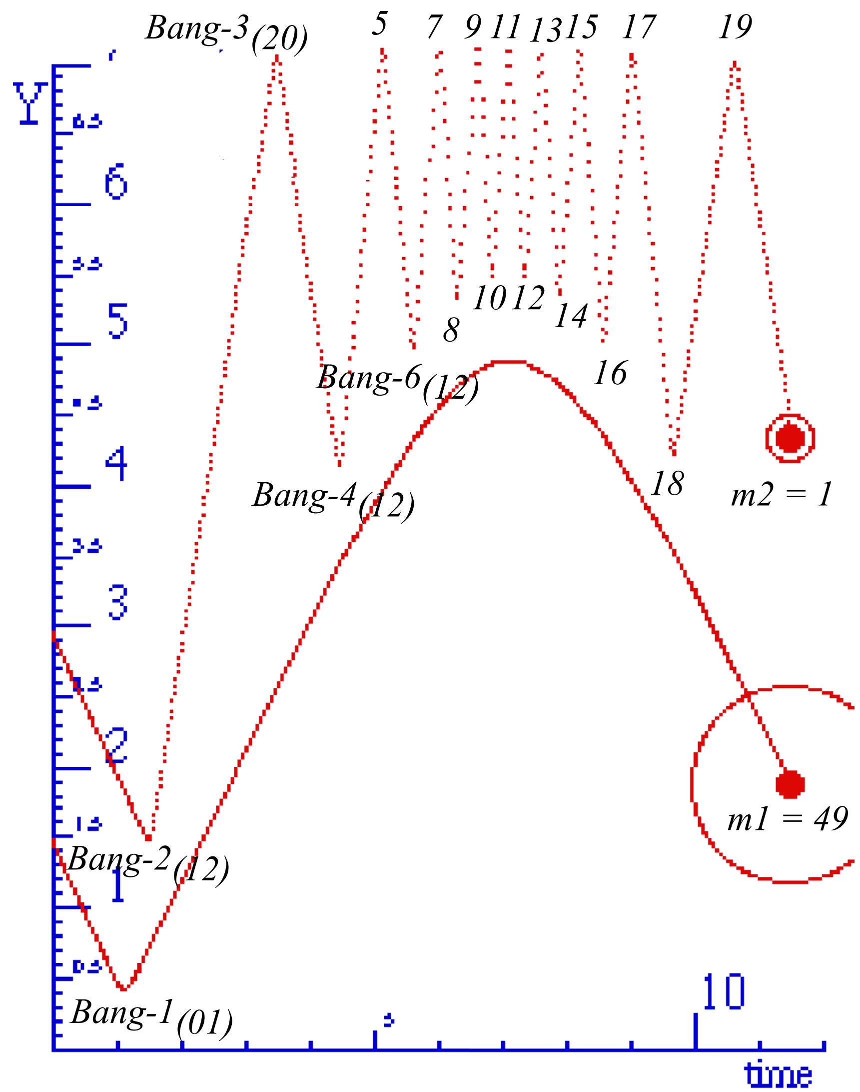
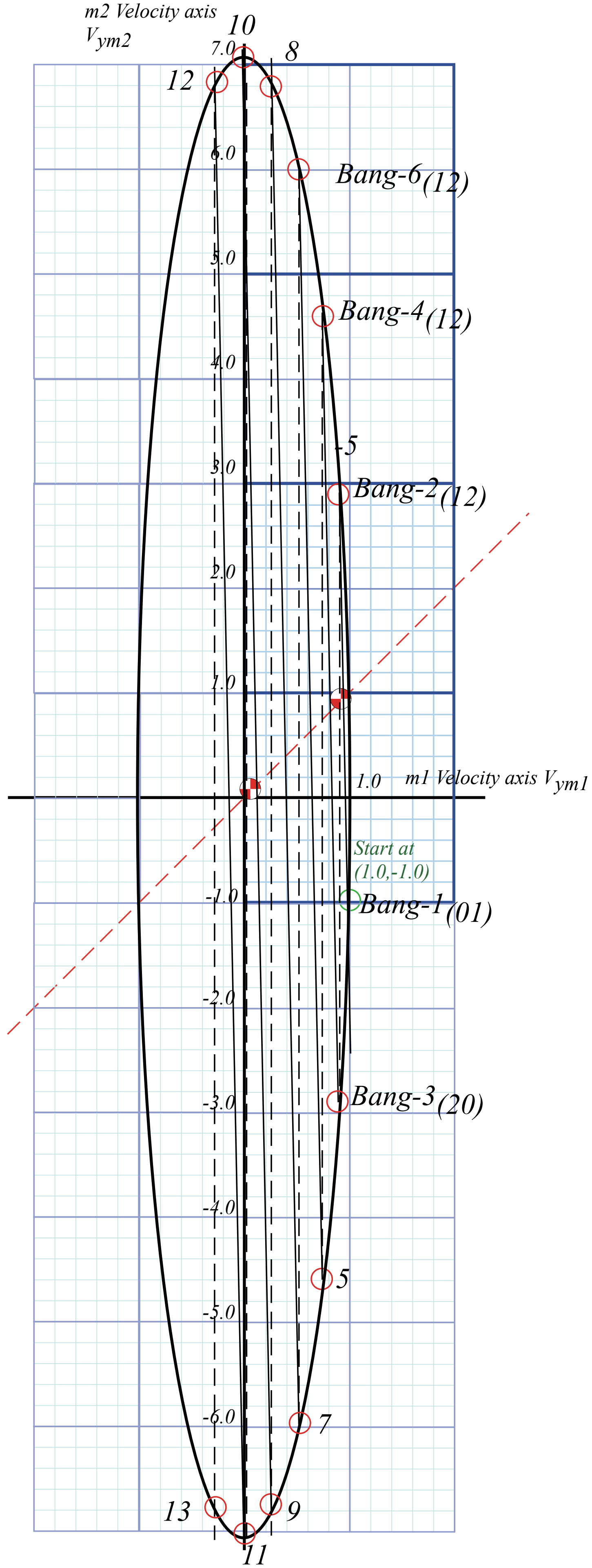


Fig. 5.1
in Unit 1

Geometric "Integration" (Converting Velocity data to Space-time trajectory)



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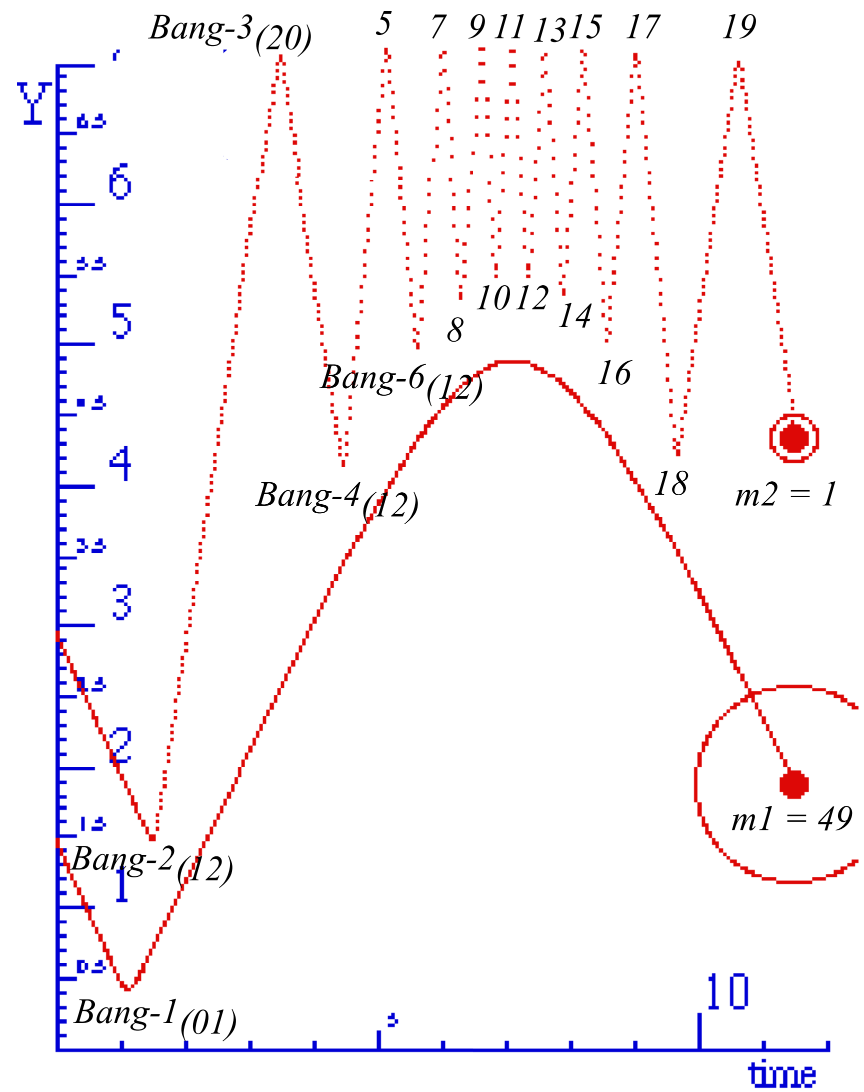

Fig. 5.1
in Unit 1

Kinetic Energy Ellipse

$$KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{49}{2} + \frac{1}{2} = 25$$

$$1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE/M_1} = \sqrt{2KE/49} = \sqrt{50/49} = 1.01$$

Ellipse radius 2

$$a_2 = \sqrt{2KE/m_2} = \sqrt{2KE/1} = \sqrt{50/1} = 7.07$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions *(and quiz question about linear solutions)*

"Mass-bang" matrix **M**, *"Floor-bang" matrix* **F**, *"Ceiling-bang" matrix* **C**.

Geometry and algebra of "ellipse-Rotation" group product: **R**= **C**•**M**

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \dfrac{V^{FIN} + V^{IN}}{2} = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} =$$

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Multiple Collisions by Matrix Operator Products

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Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \bullet \mathbf{v}^{IN}$...

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Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions *(and quiz question about linear solutions)*

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Quiz question about linear solution

Linear formula $\mathbf{v}^{FIN} = \mathbf{M} \bullet \mathbf{v}^{IN}$ gives just *one* solution to *quadratic* collision equations.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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Q: What is the *second* solution and to what simple process would it correspond?

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions *(and quiz question about linear solutions)*

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Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

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Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang **M** of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Floor-bang **F** of m_1:

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Mass-bang **M** of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ *and* $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions *(and quiz question about linear solutions)*

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Geometry and algebra of "ellipse-Rotation" group product: **R**= **C•M**

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Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang **M** of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1 = 49$ *and* $m_2 = 1$ $\quad \mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$

Define *"ellipse-Rotation"* **R** as *group product*: $\mathbf{R} = \mathbf{C} \bullet \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}$

$$\left|FIN^9\right\rangle = \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} \left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(\text{INITIAL } (0))}$$

$$|FIN^9\rangle = \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} \; |IN^0\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{\text{(INITIAL (0))}}$$

$$|FIN^9\rangle = \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{\text{(after Bang-1)}}$$

"ellipse-Rotation" group product: **R= C•M**

$$|FIN^9\rangle = \mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{F}\,|IN^0\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{\text{(INITIAL (0))}}$$

$$|FIN^9\rangle = \mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{F}\,|IN^0\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{\text{(after Bang-1)}}$$

$$= \begin{pmatrix} v_1 = 0.2925 \\ v_2 = -6.768 \end{pmatrix}_{\text{(after Bang-9)}}$$

"ellipse-Rotation" group product: **R= C•M**

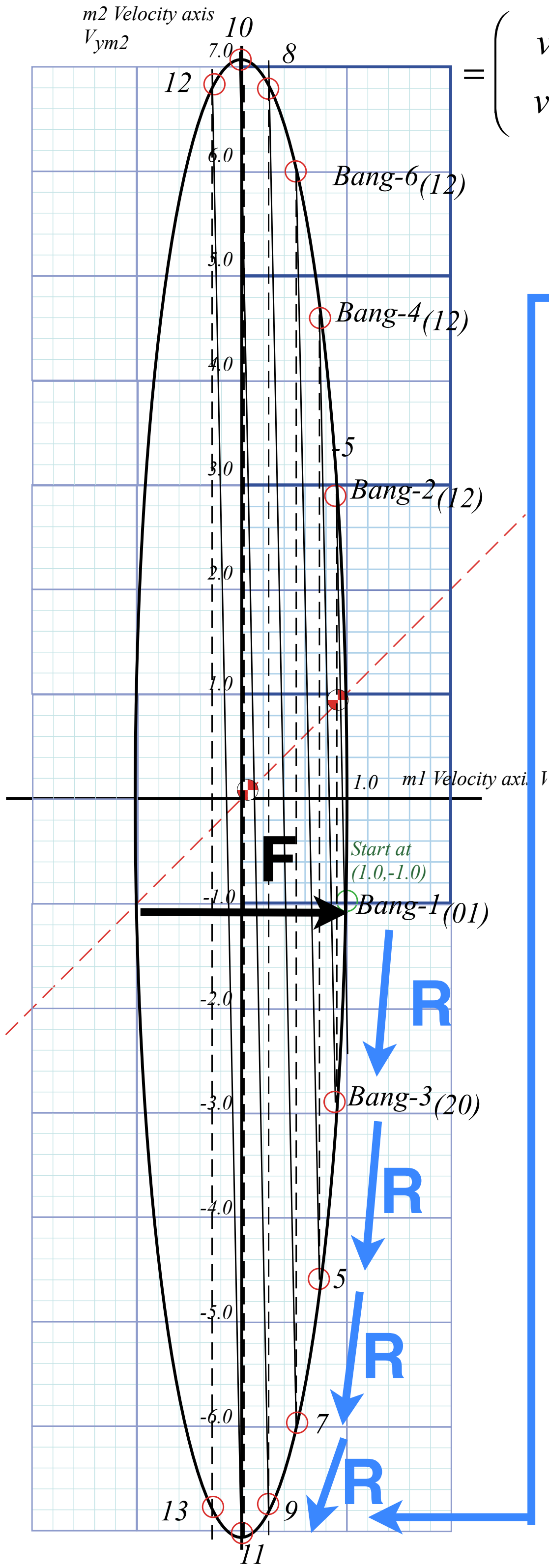
$$
\left|FIN^9\right\rangle = \mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{F}\left|IN^0\right\rangle
$$

$$
\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} v_1^{IN}=-1 \\ v_2^{IN}=-1 \end{pmatrix}_{\text{(INITIAL (0))}}
$$

$$
\left|FIN^9\right\rangle = \mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{F}\left|IN^0\right\rangle
$$

$$
\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} v_1=1 \\ v_2=-1 \end{pmatrix}_{\text{(after Bang-1)}}
$$

$$
= \begin{pmatrix} v_1 = 0.2925 \\ v_2 = -6.768 \end{pmatrix}_{\text{(after Bang-9)}}
$$



"ellipse-Rotation" group product: **R= C•M**

$$|FIN^9\rangle = \mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{F}\,|IN^0\rangle$$

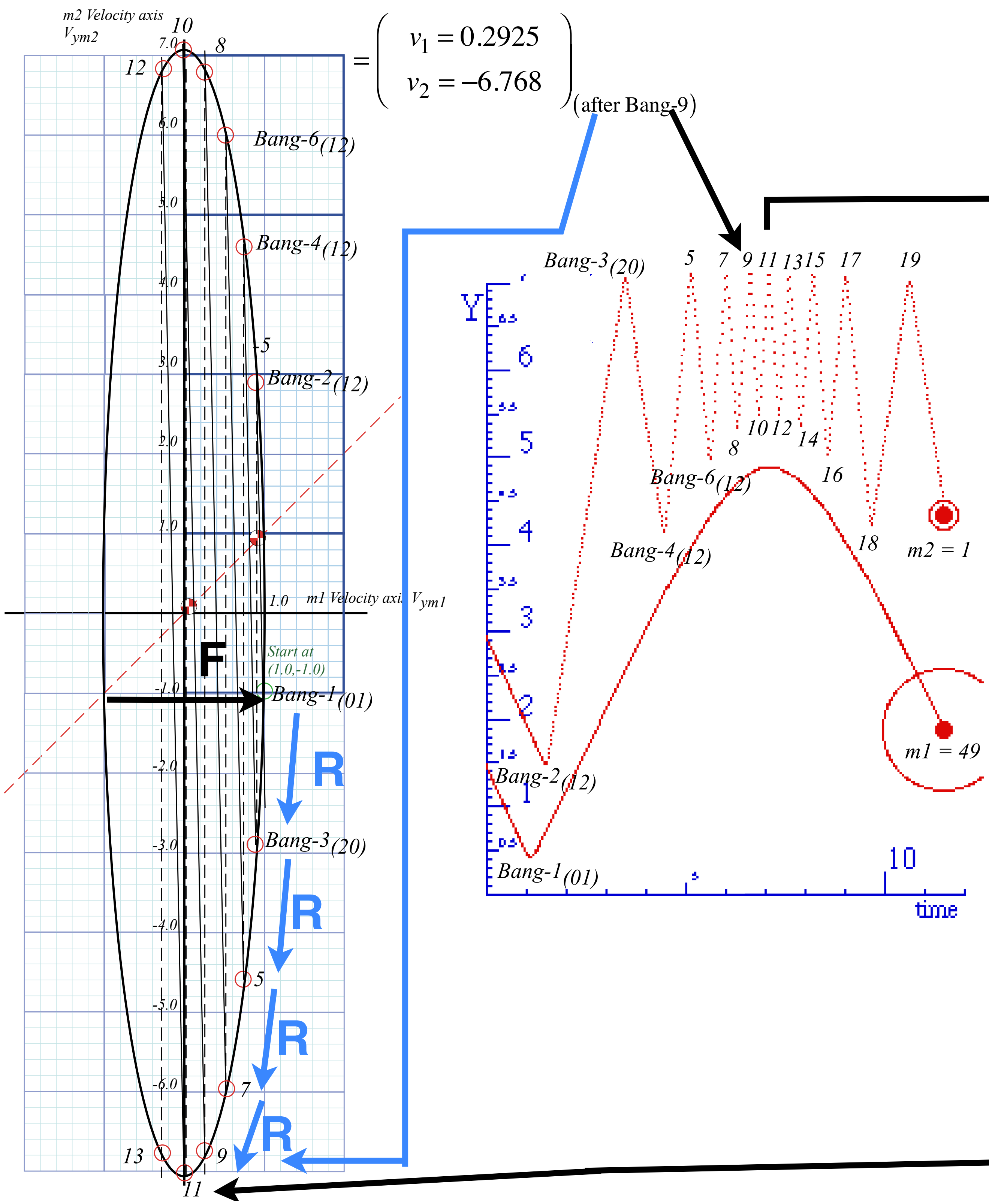
$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}\cdot\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{\text{(INITIAL (0))}}$$

$$|FIN^9\rangle = \mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{F}\,|IN^0\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{\text{(after Bang-1)}}$$

$$= \begin{pmatrix} v_1 = 0.2925 \\ v_2 = -6.768 \end{pmatrix}_{\text{(after Bang-9)}}$$

"ellipse-Rotation" group product: **R** = **C**·**M**



$$\begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}\cdot\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix}$$

$$= \begin{pmatrix} v_1 = 0.0100 \\ v_2 = -7.071 \end{pmatrix}_{\text{(after Bang-11)}}$$

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

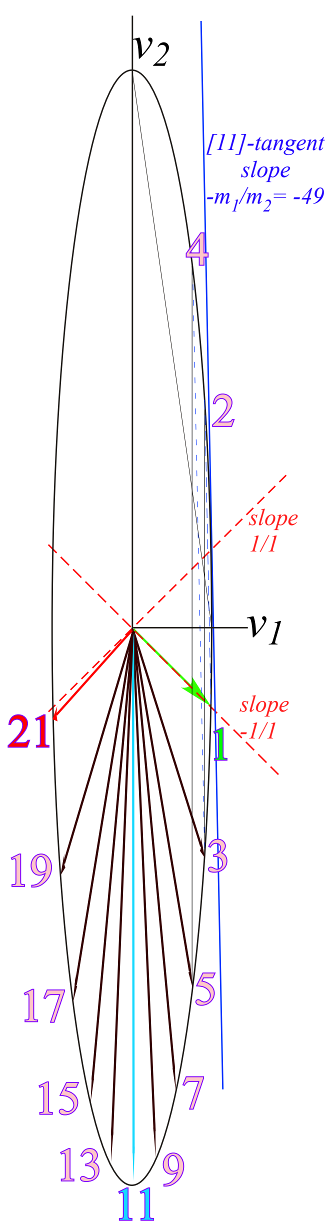
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$



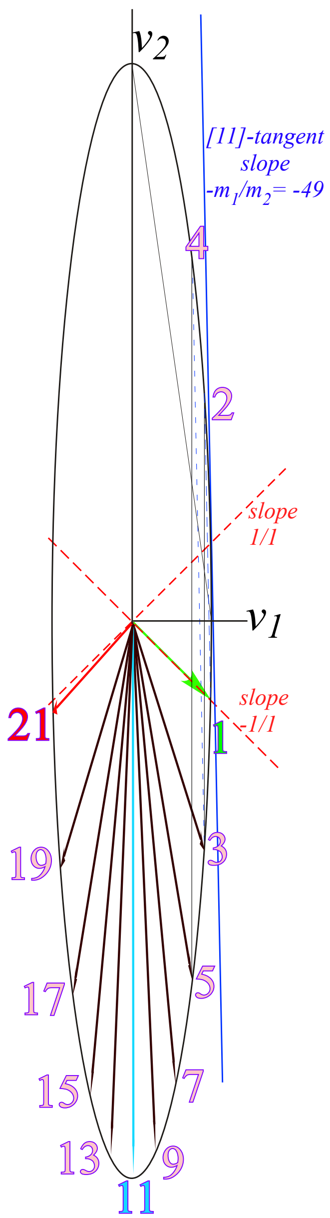
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \mathsf{V}_1^2 + \frac{1}{2} \mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} / \sqrt{m_1} \\ \mathsf{V}_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 / \sqrt{m_1} \\ \mathsf{V}_2 / \sqrt{m_2} \end{pmatrix}$$



Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \mathsf{V}_1^2 + \frac{1}{2} \mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

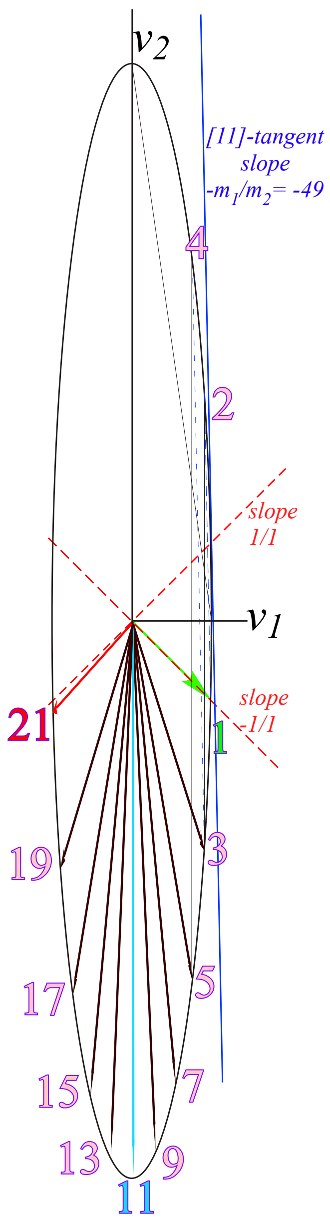
$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} / \sqrt{m_1} \\ \mathsf{V}_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 / \sqrt{m_1} \\ \mathsf{V}_2 / \sqrt{m_2} \end{pmatrix}$$

or:

$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{M} \bullet \vec{\mathsf{V}}$$

or:

$$\begin{pmatrix} \mathsf{V}_1^{FIN_2} \\ \mathsf{V}_2^{FIN_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{C} \bullet \mathsf{M} \bullet \vec{\mathsf{V}}$$



Ellipse rescaling geometry and reflection symmetry analysis

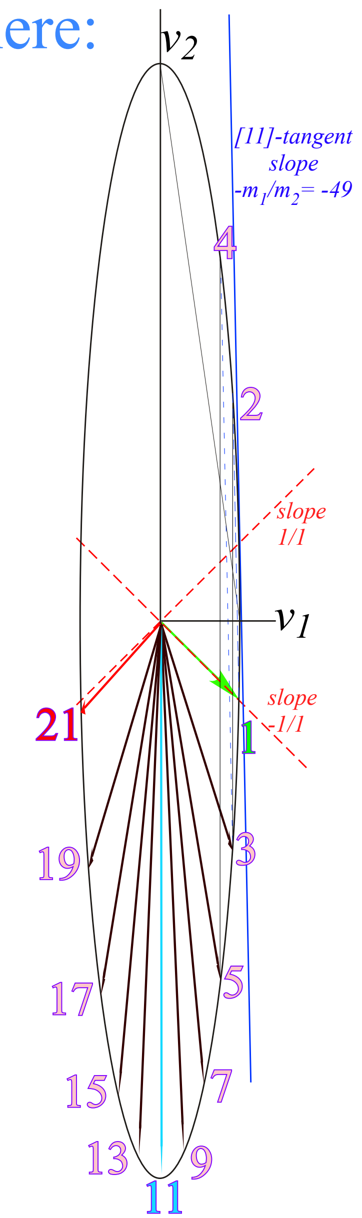
Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}\mathsf{V}_1^2 + \frac{1}{2}\mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$ becomes: $$\begin{pmatrix} \mathsf{V}_1^{FIN_1}/\sqrt{m_1} \\ \mathsf{V}_2^{FIN_1}/\sqrt{m_2} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1/\sqrt{m_1} \\ \mathsf{V}_2/\sqrt{m_2} \end{pmatrix}$$

or: $$\begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{M}\bullet\vec{\mathsf{V}}$$, or: $$\begin{pmatrix} \mathsf{V}_1^{FIN_2} \\ \mathsf{V}_2^{FIN_2} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{C}\bullet\mathsf{M}\bullet\vec{\mathsf{V}}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$



Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}\mathsf{V}_1^2 + \frac{1}{2}\mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

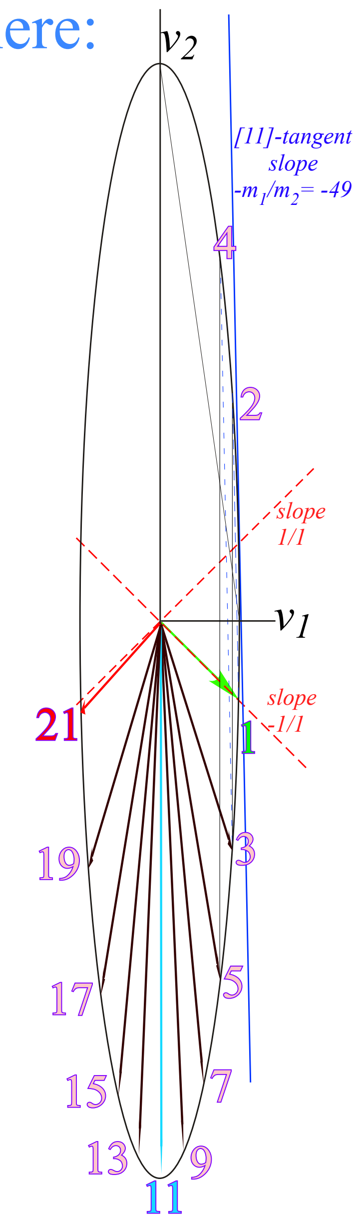
becomes:

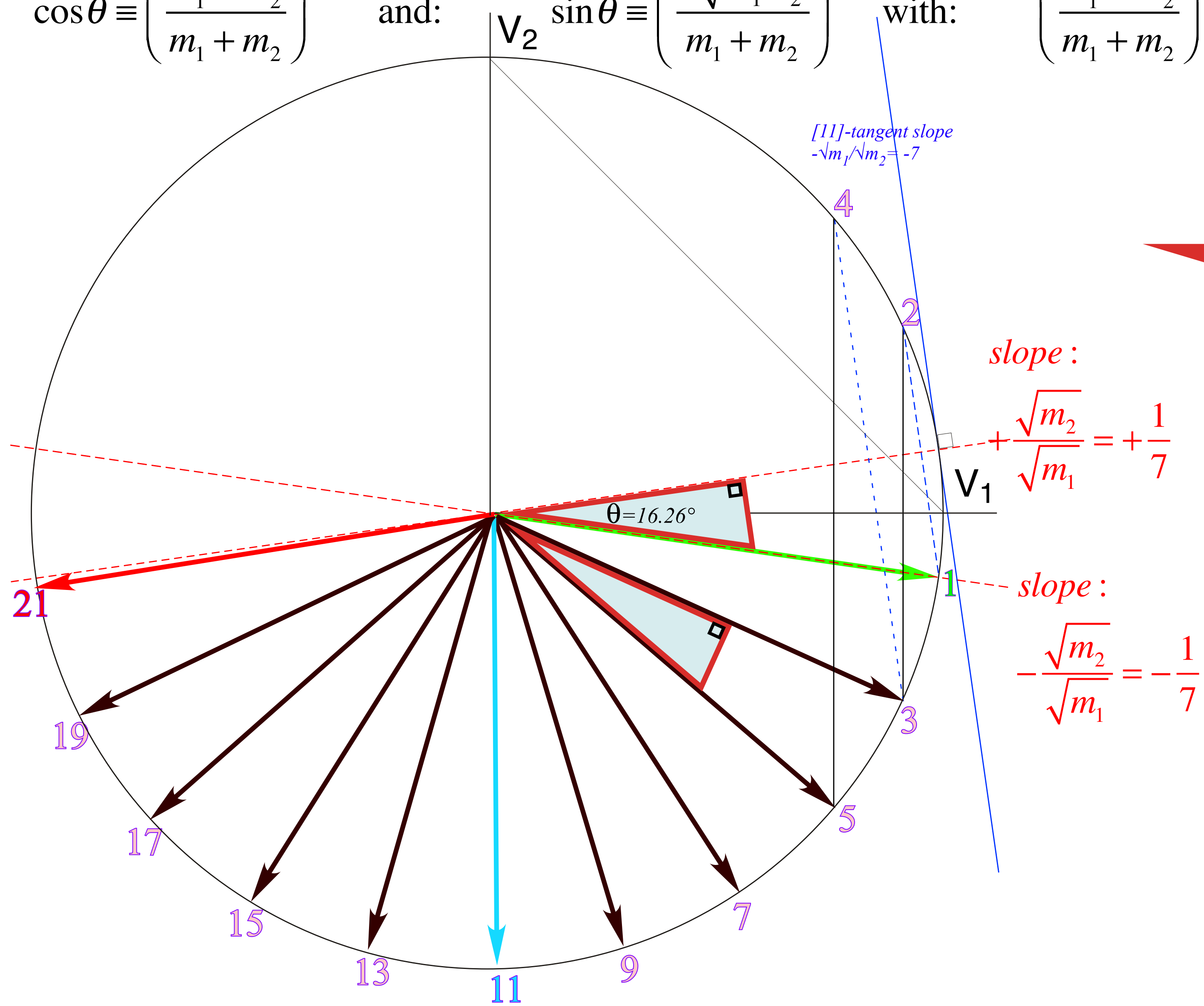
$$\begin{pmatrix} \mathsf{V}_1^{FIN_1}/\sqrt{m_1} \\ \mathsf{V}_2^{FIN_1}/\sqrt{m_2} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1/\sqrt{m_1} \\ \mathsf{V}_2/\sqrt{m_2} \end{pmatrix}$$

or: $\begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathbf{M}\bullet\vec{\mathsf{V}}$, or: $\begin{pmatrix} \mathsf{V}_1^{FIN_2} \\ \mathsf{V}_2^{FIN_2} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathbf{C}\bullet\mathbf{M}\bullet\vec{\mathsf{V}}$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$





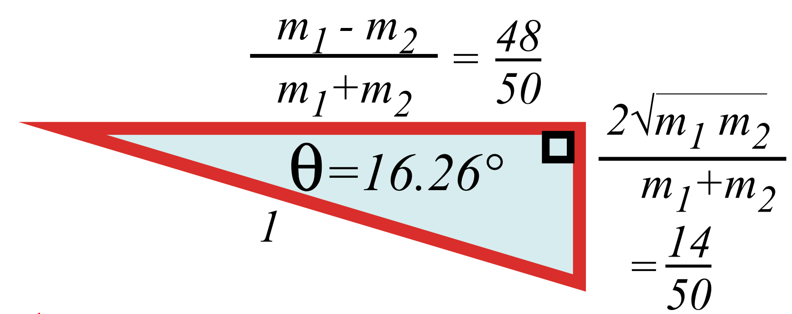


Fig. 5.2a-c

(revised)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

$$\begin{pmatrix} V_1^{FIN_1} / \sqrt{m_1} \\ V_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

or:

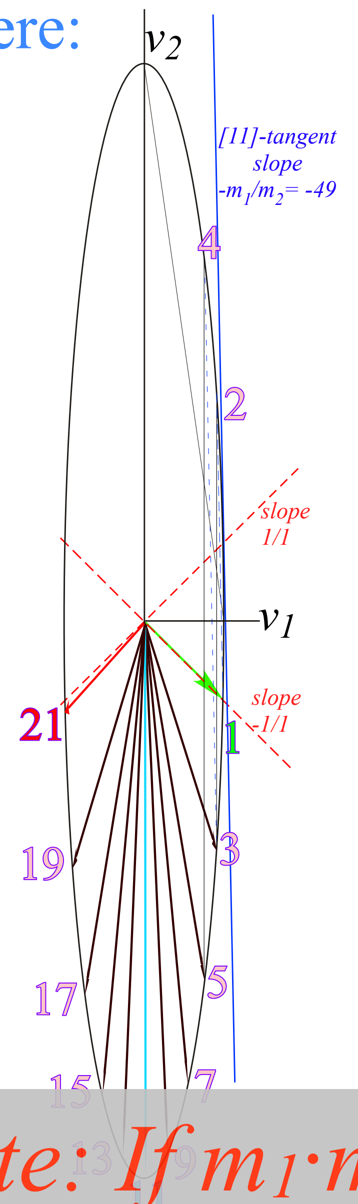
$$\begin{pmatrix} V_1^{FIN_1} \\ V_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \bullet \vec{V}$$

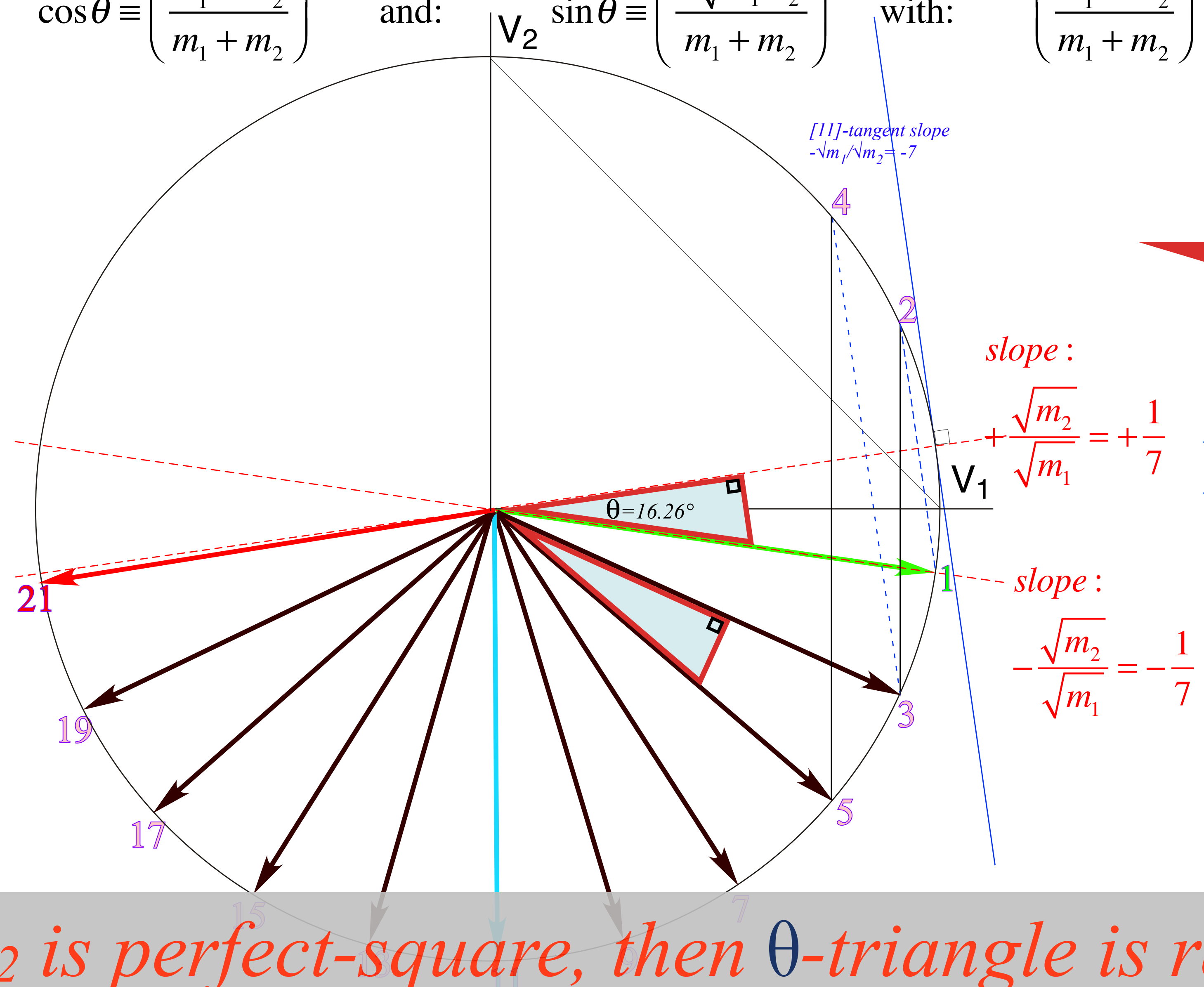
or:

$$\begin{pmatrix} V_1^{FIN_2} \\ V_2^{FIN_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \bullet \mathbf{M} \bullet \vec{V}$$

Then collisions become ***reflections*** $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become ***rotations*** $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$





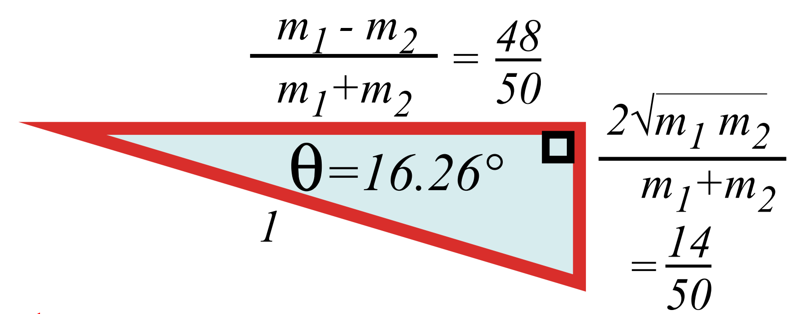


Fig. 5.2a-c

(revised)

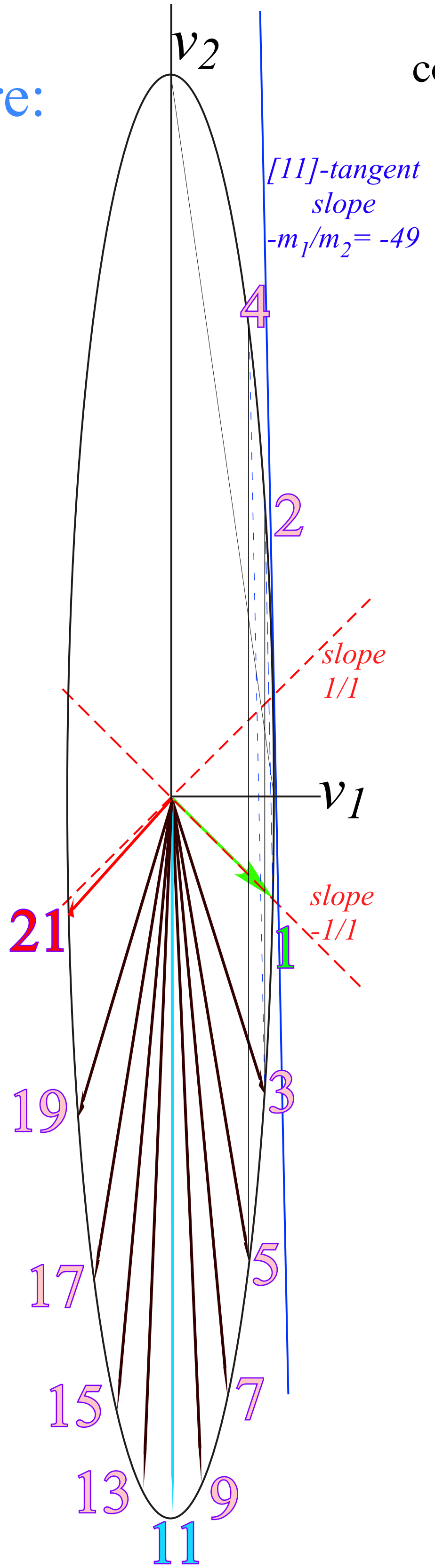
Note: If $m_1 \cdot m_2$ is perfect-square, then θ-triangle is rational *($3^2+4^2=5^2$, etc.)*

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}\mathsf{V}_1^2 + \frac{1}{2}\mathsf{V}_2^2$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations*

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$



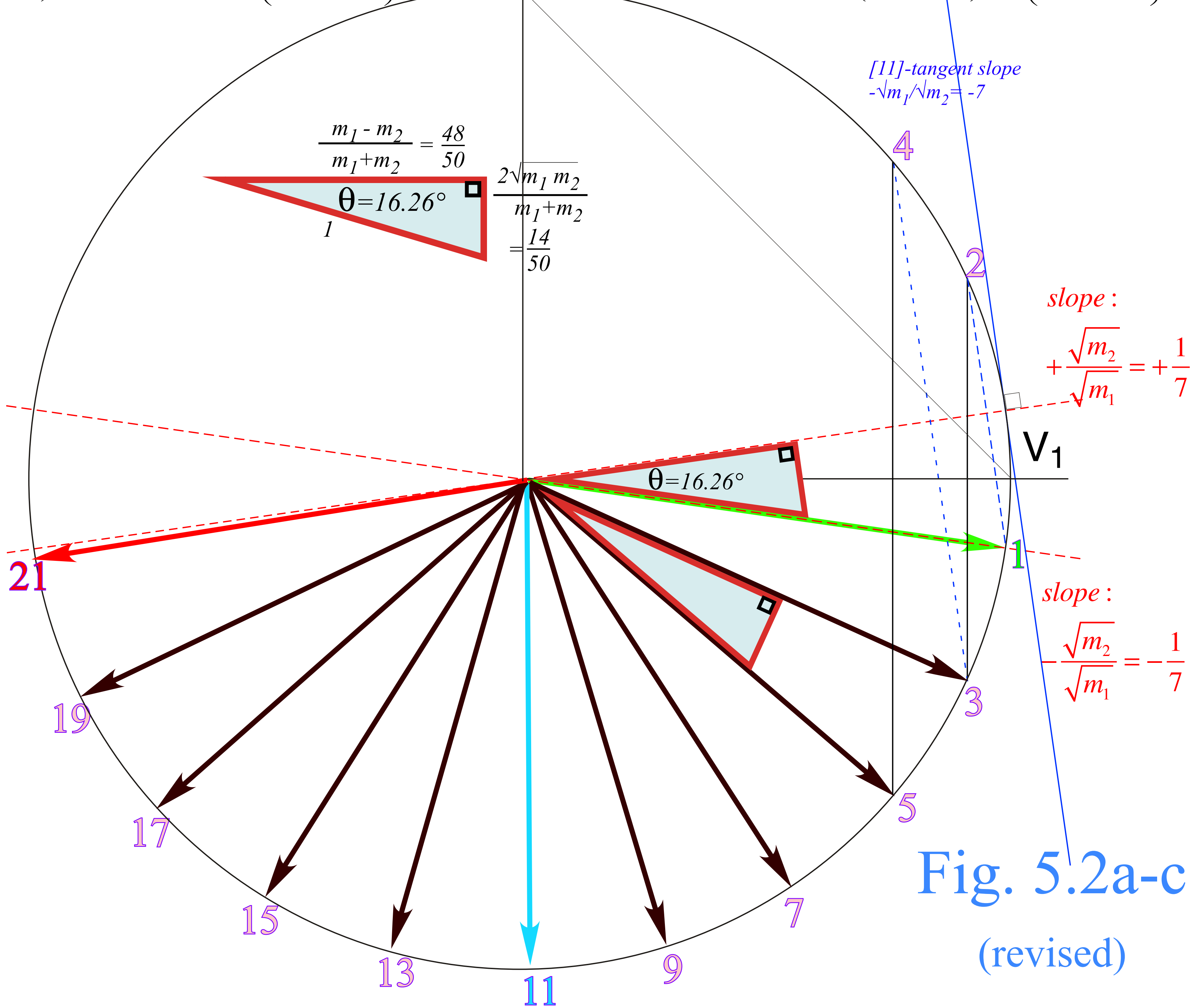


Fig. 5.2a-c

(revised)

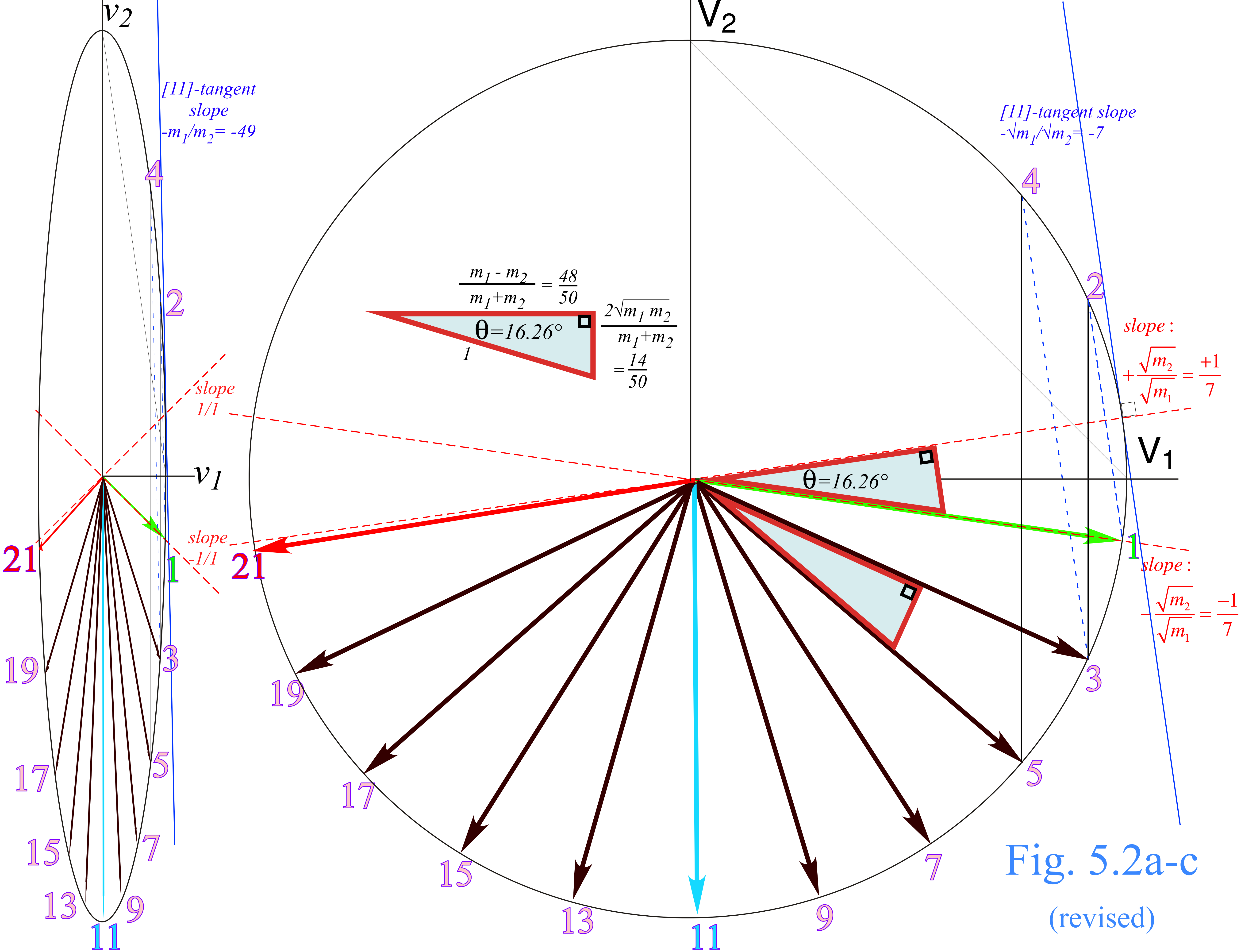


Fig. 5.2a-c

(revised)

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

What ellipse rescaling leads to...(in Ch. 9-12)

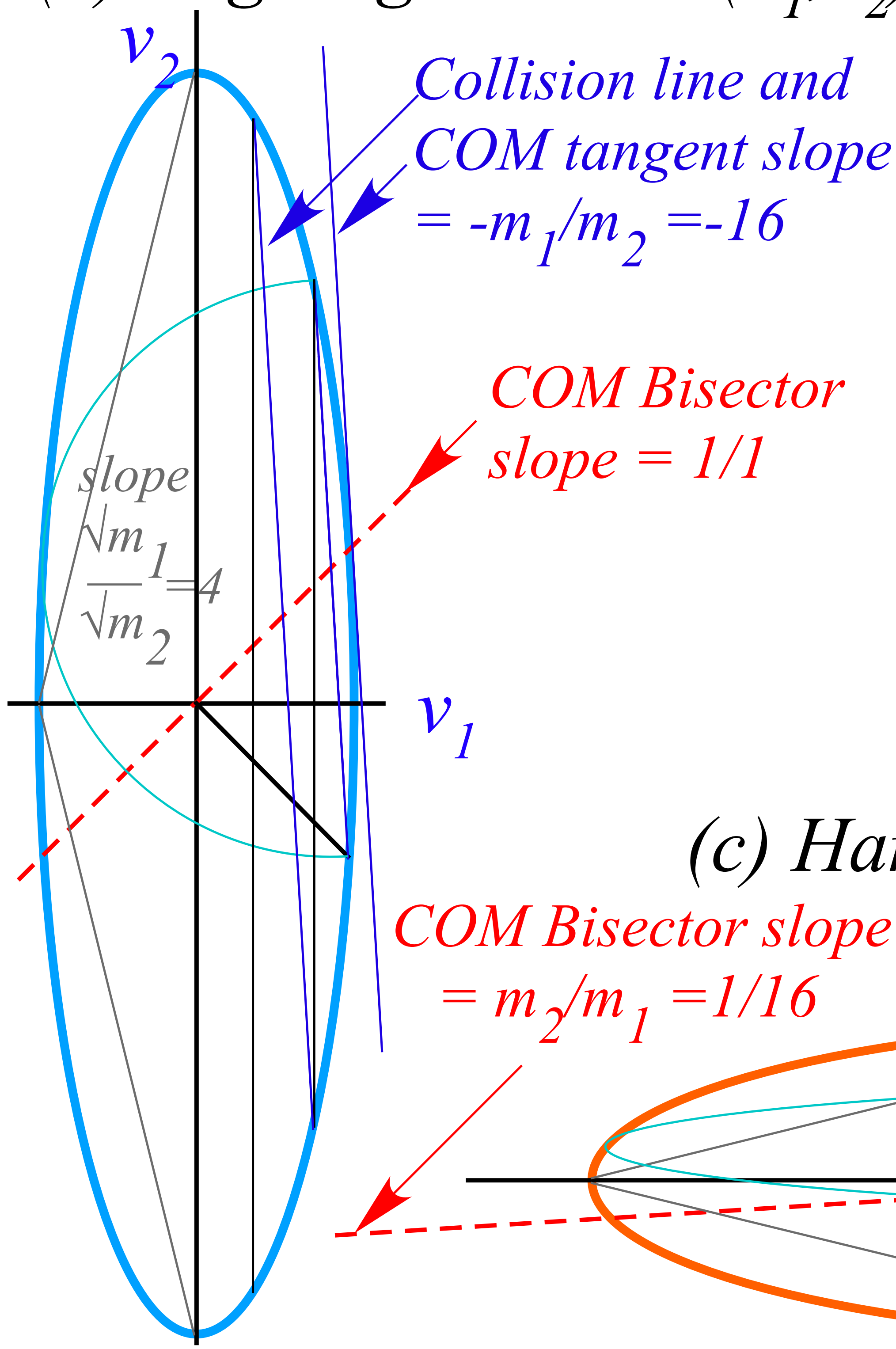
Fig. 12.1

How this relates to Lagrangian, and Hamiltonian mechanics in Ch. 12

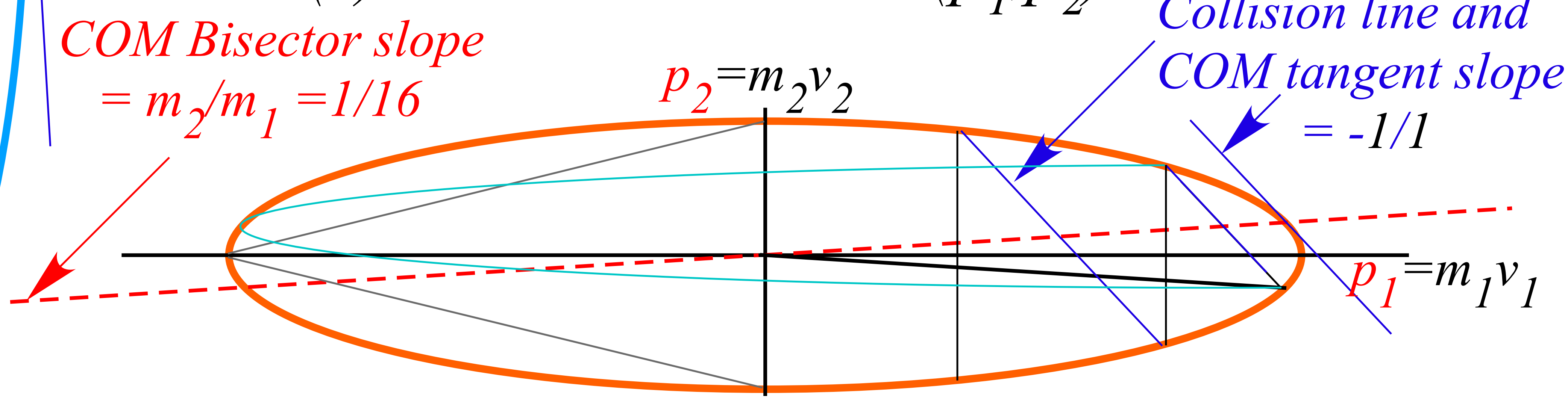
(a) Lagrangian $L = L(v_1, v_2)$

velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$

velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$



(c) Hamiltonian $H = H(p_1, p_2)$

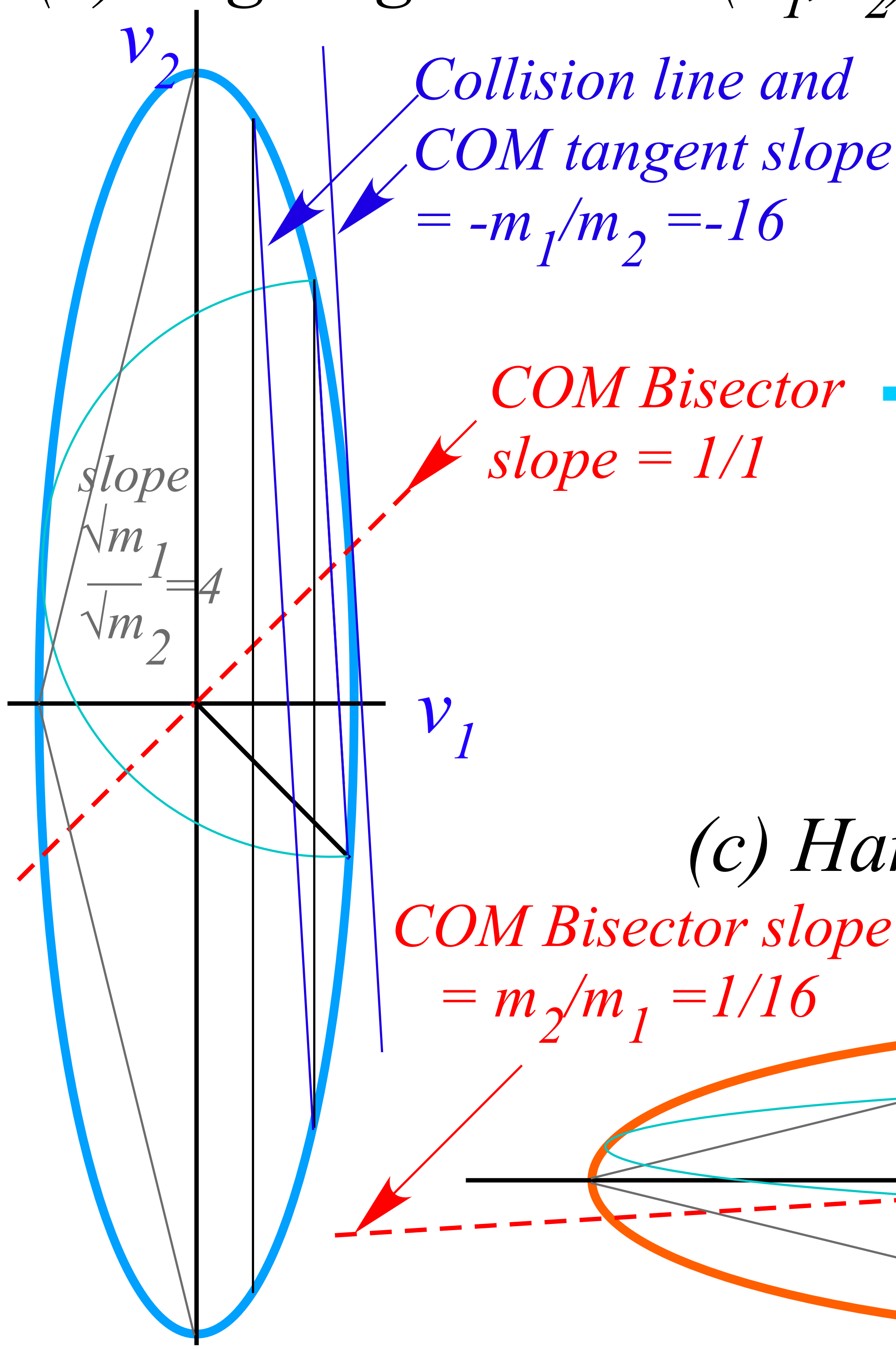


What ellipse rescaling leads to...(in Ch. 9-12)

Fig. 12.1

How this relates to Lagrangian, and Hamiltonian mechanics in Ch. 12

(a) Lagrangian $L = L(v_1,v_2)$



velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$

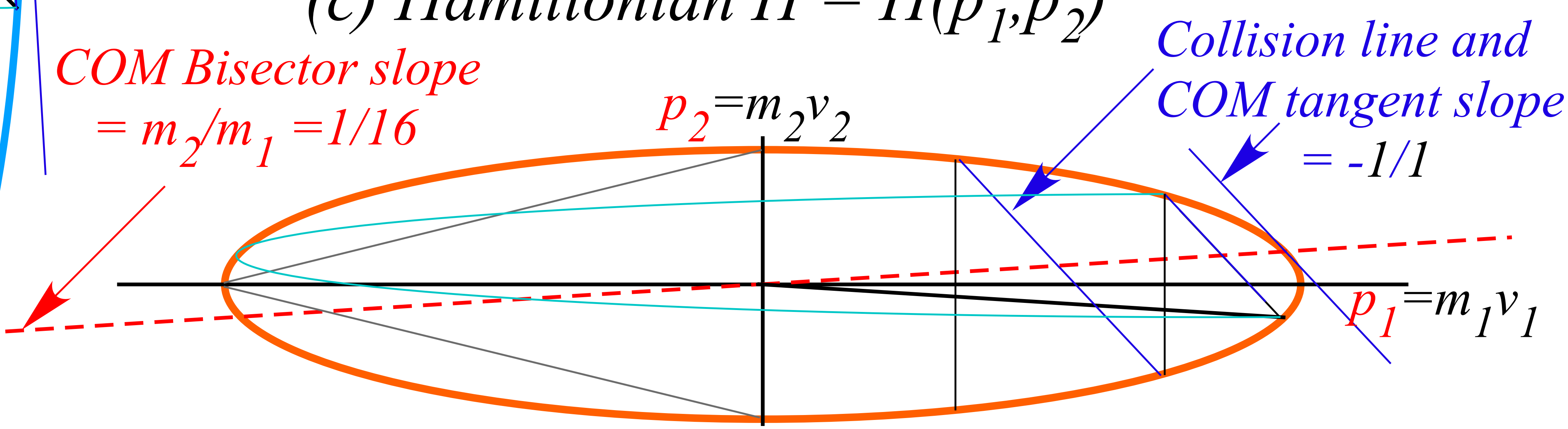
velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$

Lagrangian $L(v_1,v_2) = KE = \frac{1}{2}m_1 v_1^{\;2} + \frac{1}{2}m_2 v_2^{\;2}$

rescaled to

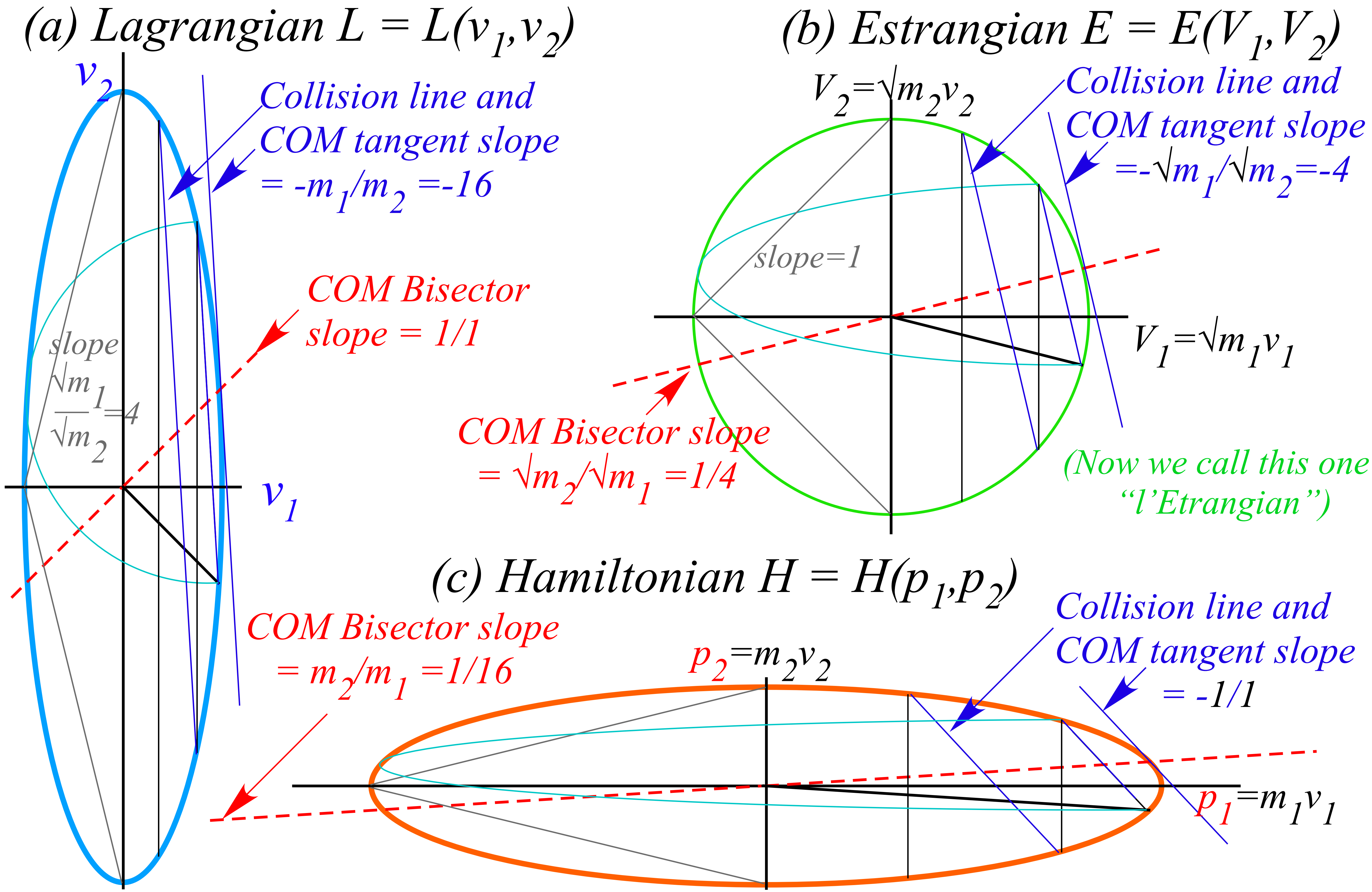
Hamiltonian $H(p_1,p_2) = KE = \frac{p_1^{\;2}}{2m_1} + \frac{p_2^{\;2}}{2m_2}$

(c) Hamiltonian $H = H(p_1,p_2)$



What ellipse rescaling leads to...(in Ch. 9-12)

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

→ *Reflections in the clothing store: "It's all done with mirrors!"*

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Reflections in clothing store mirrors

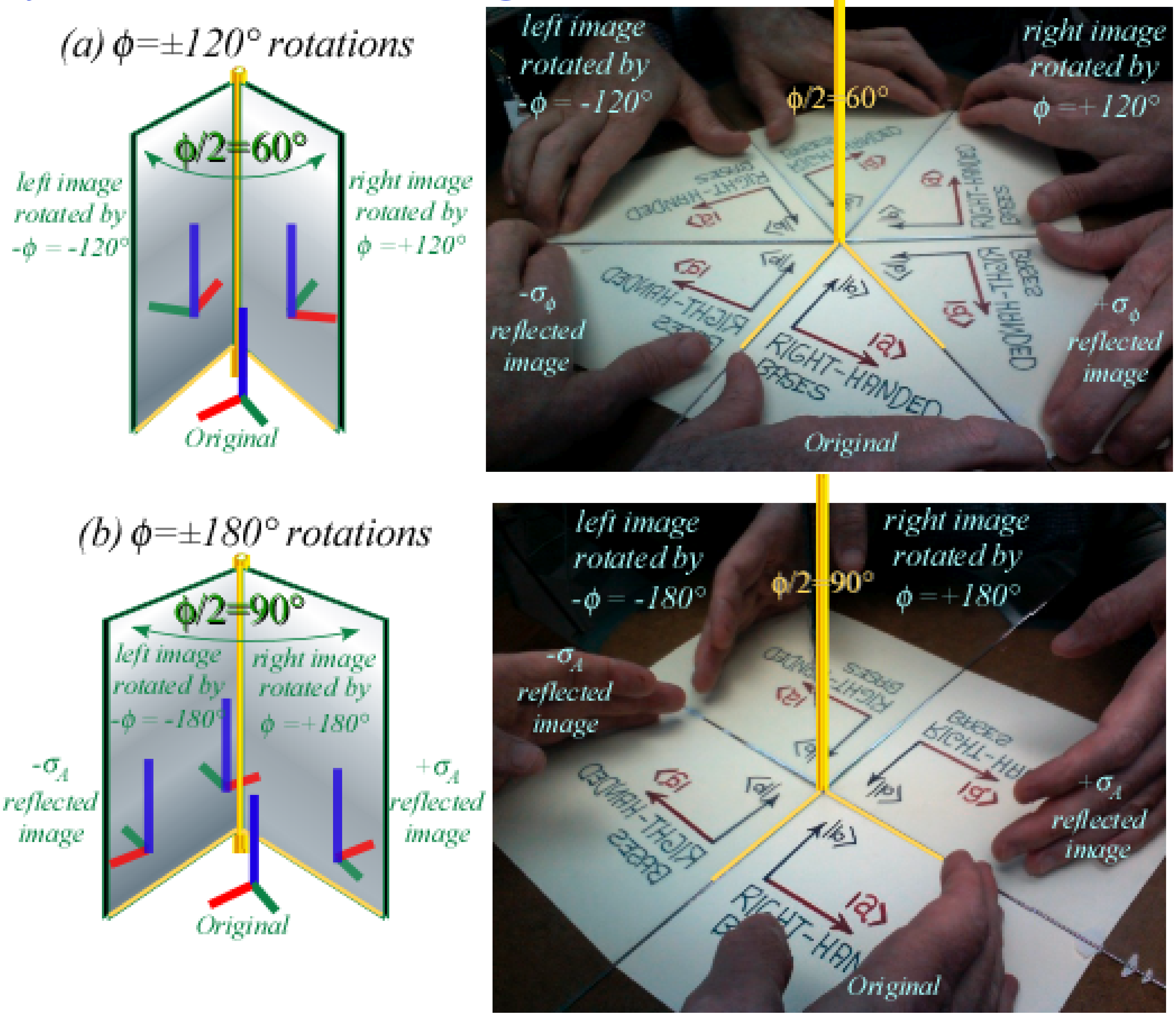


Fig. 5.4a-b

Symmetry: It's all done with mirrors!

(a)Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *(b)Reflections* $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

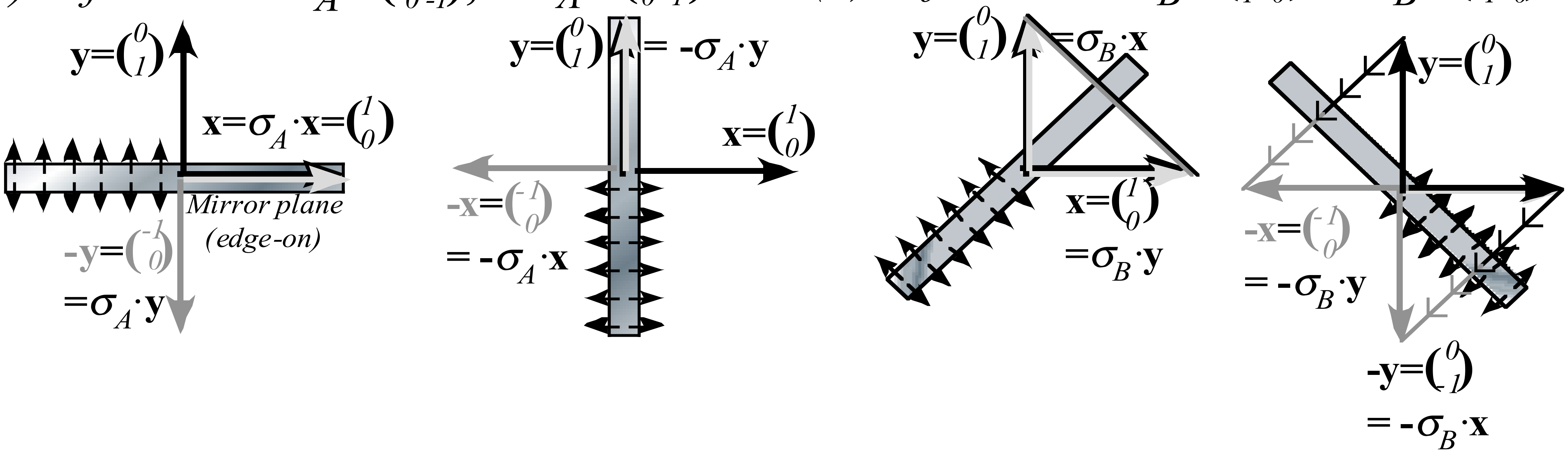
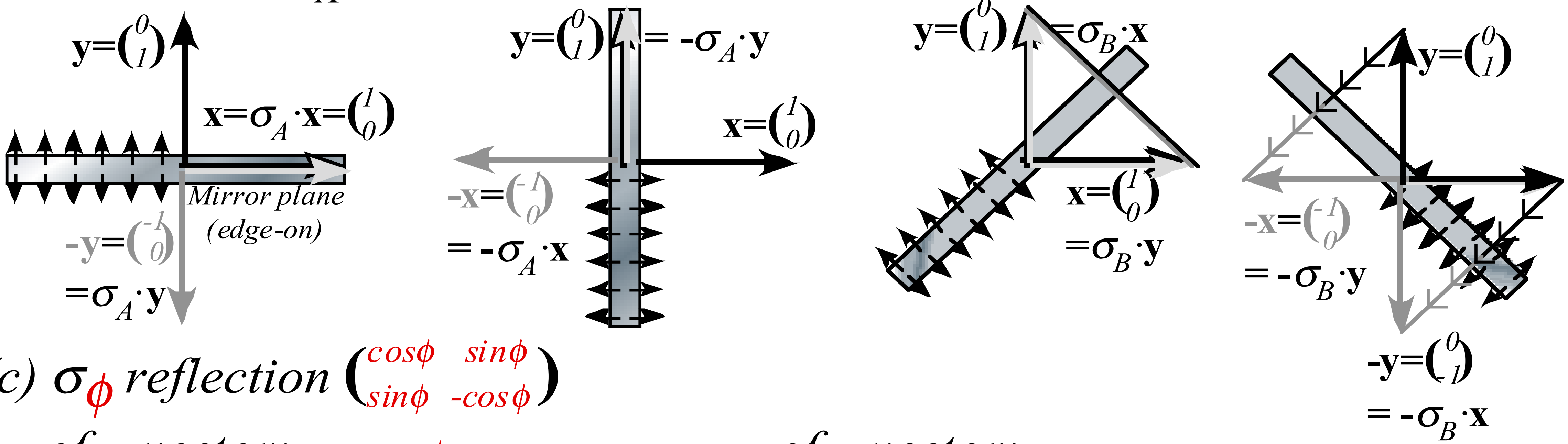


Fig. 5.3a-e

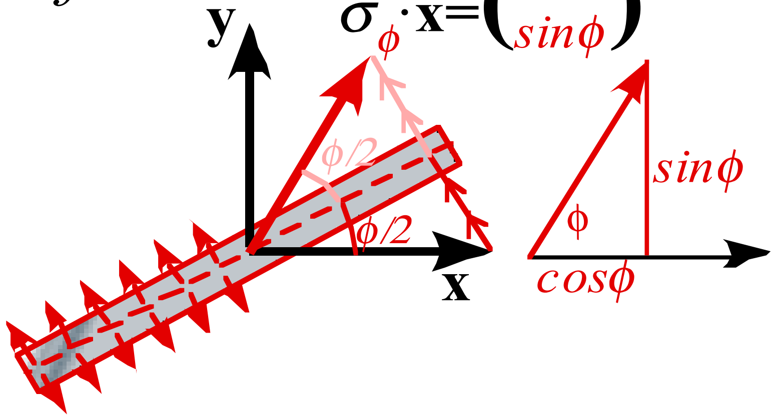
Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of **x**-vector: ...of **y**-vector:



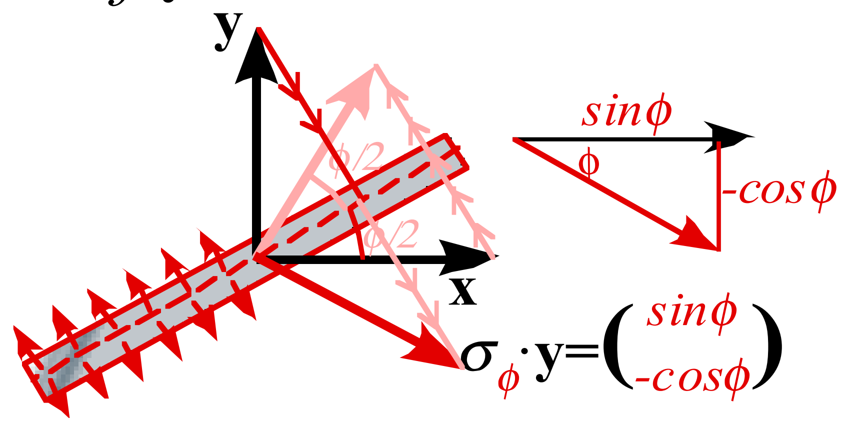


Fig. 5.3a-e

Symmetry: It's all done with mirrors!

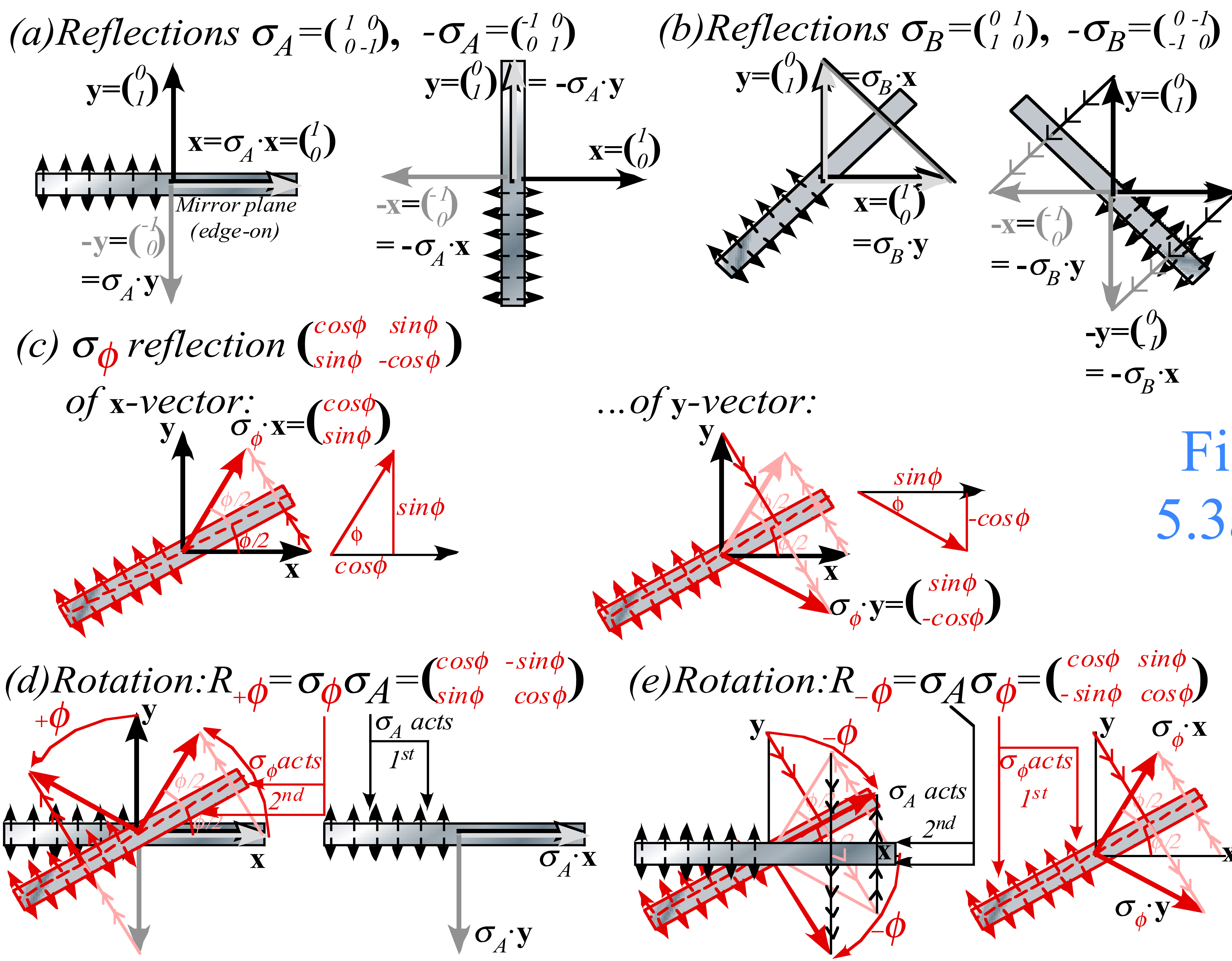


Fig. 5.3a-e

Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,.....,ND

Product of odd number of reflections is a reflection

... even number of reflections is a rotation (or unit-op **1***)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily
Waves patterns are non-rigid and reflect easily

Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,.....,ND

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... even number of reflections is a rotation (or unit-op **1***)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily
Waves patterns are non-rigid and reflect easily

∴ *...wave reflections underlie modern physics*

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

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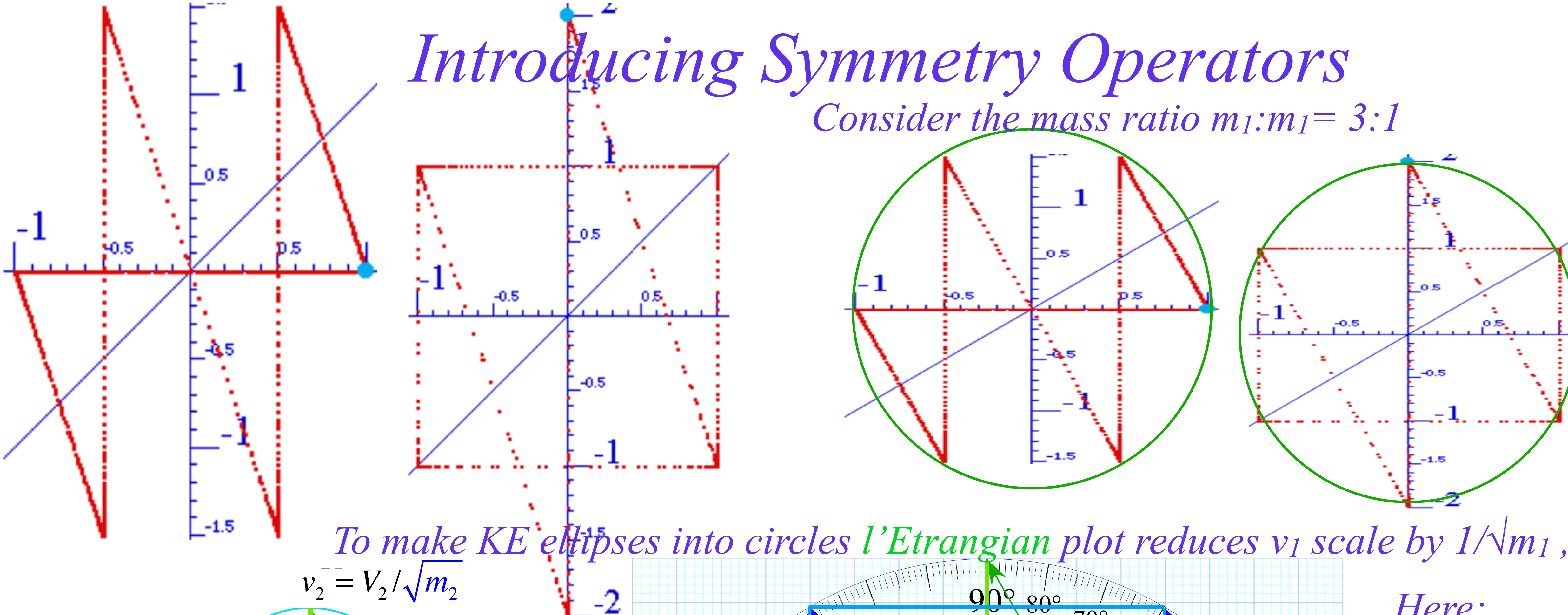
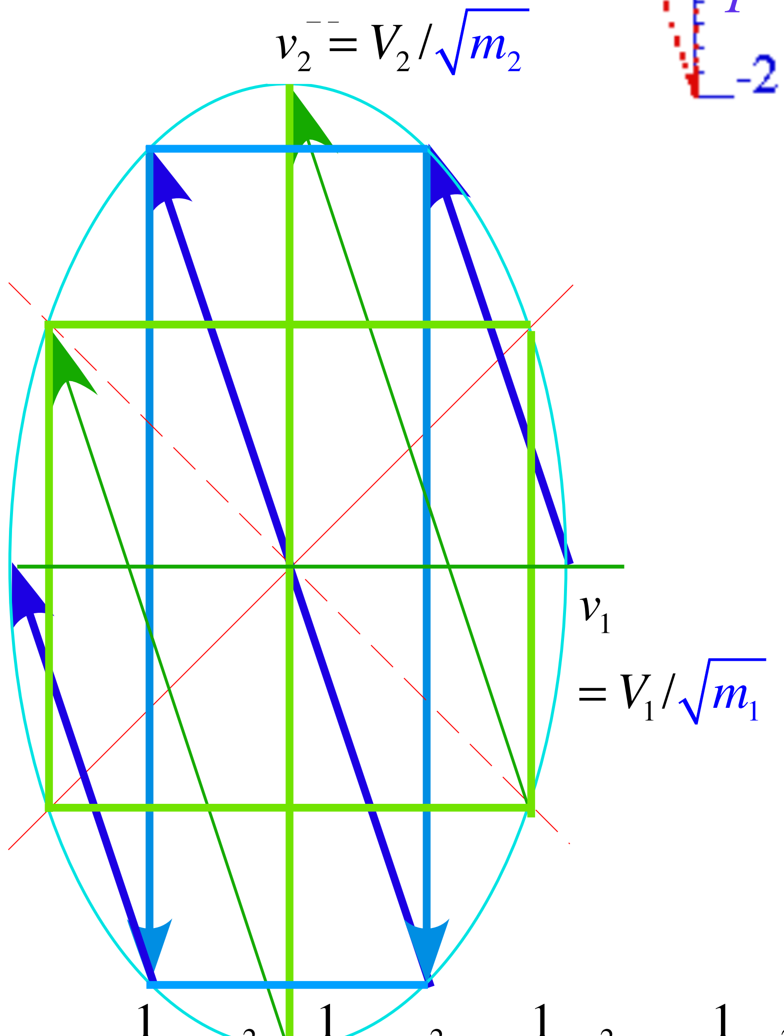
Introducing Symmetry Operators

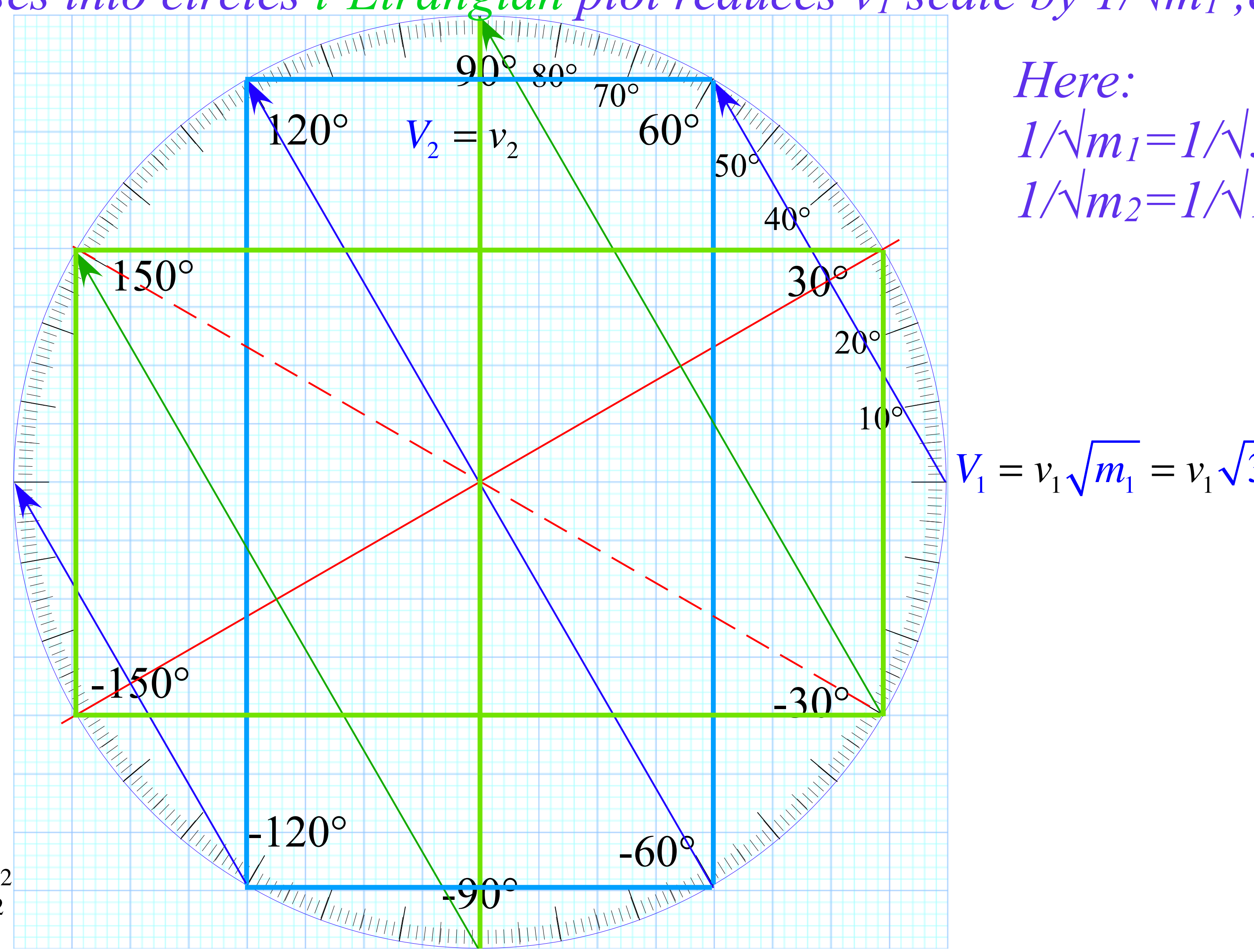
Consider the mass ratio m_1:m_1= 3:1



To make KE ellipses into circles l'Etrangian plot reduces v_1 scale by $1/\sqrt{m_1}$,etc.



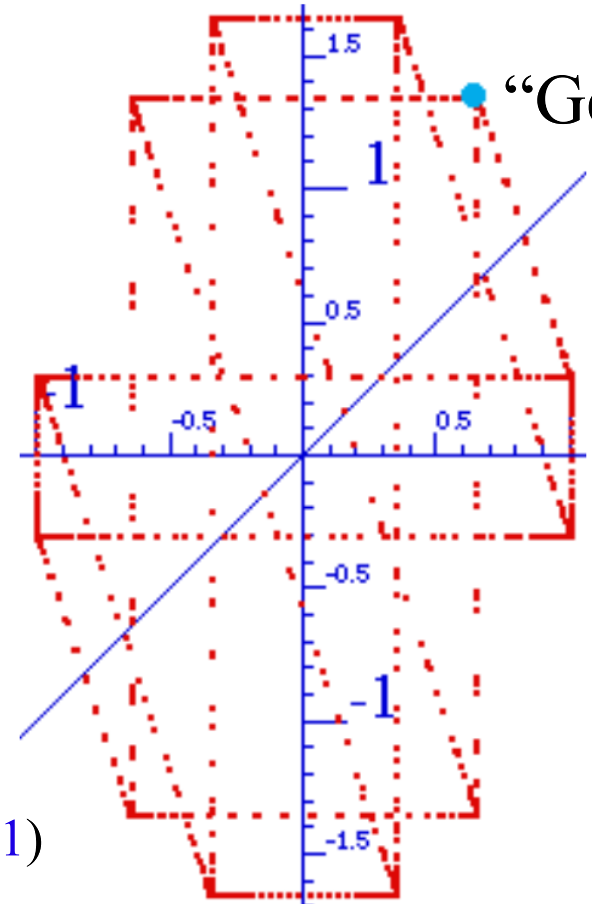
$$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad = \frac{1}{2}V_1^2 \quad + \frac{1}{2}V_2^2$$

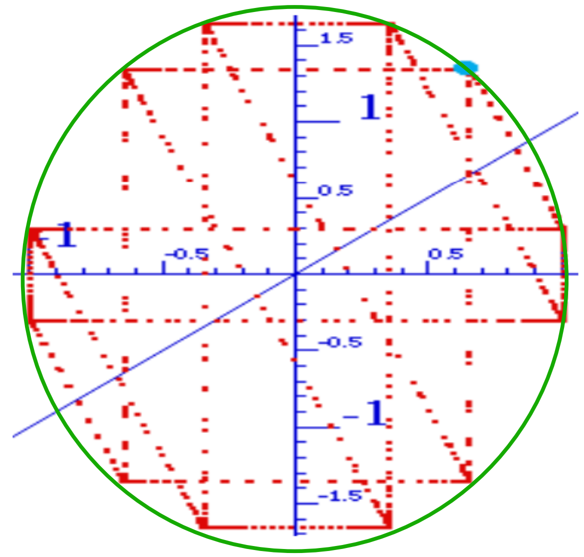


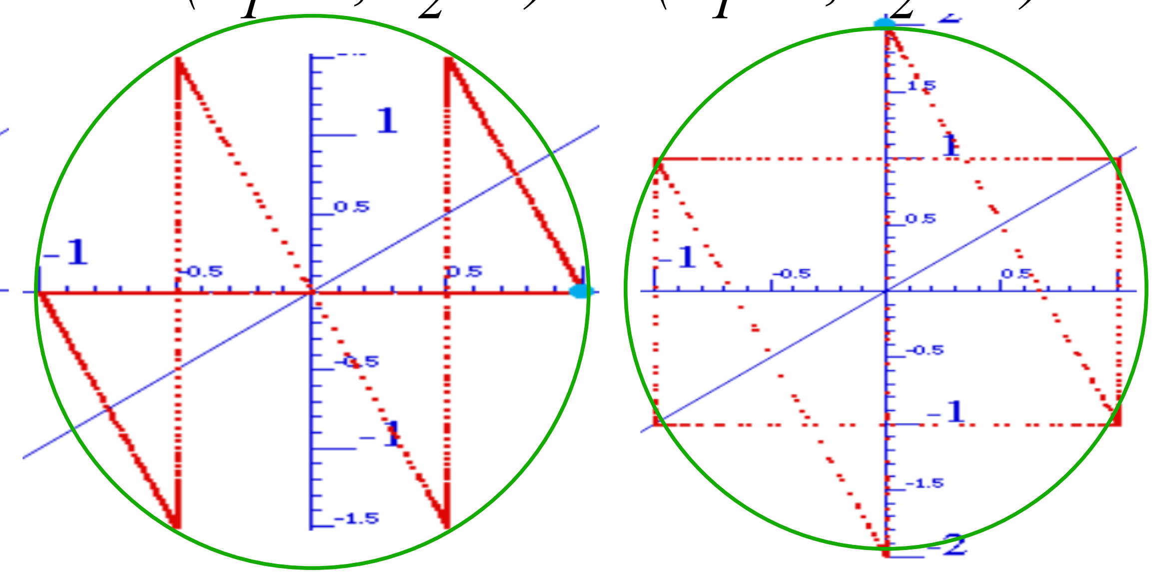
Here:
$1/\sqrt{m_1}=1/\sqrt{3}=0.577$
$1/\sqrt{m_2}=1/\sqrt{1}=1.0$

"Generic" initial velocity
(v_1=1.0, v_2=0.1)

"Symmetric" initial velocity
(v_1=1, v_2=0) or (v_1=1, v_2=-1)

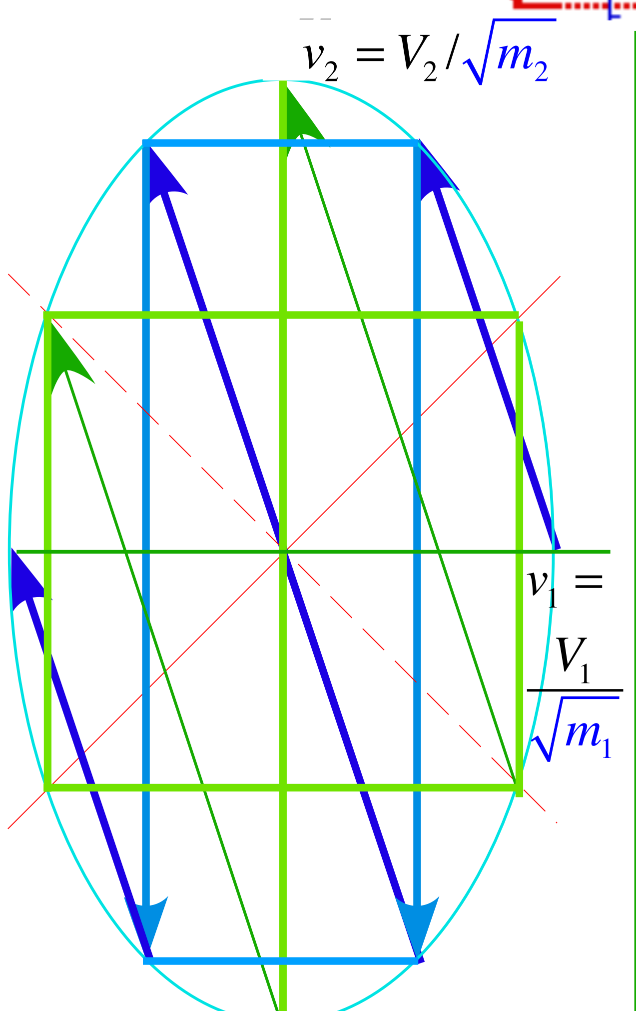


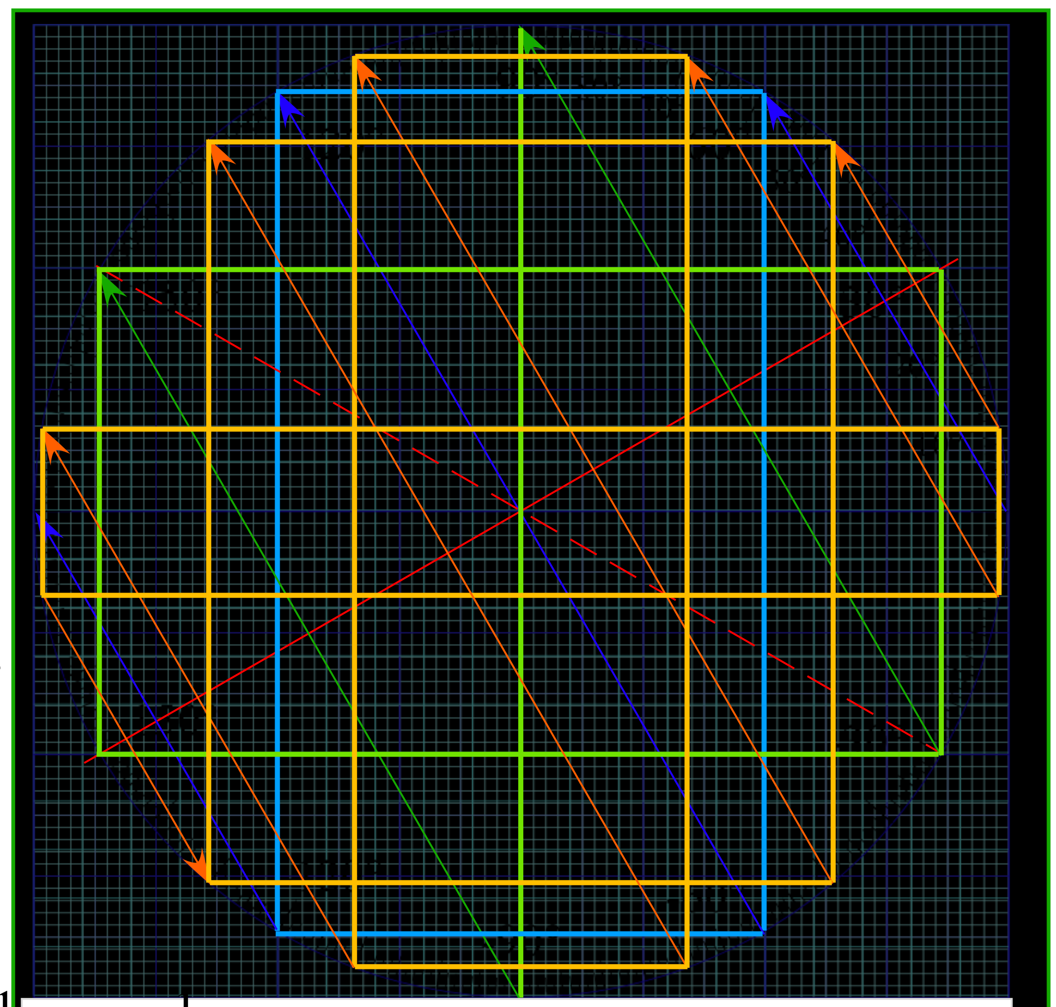


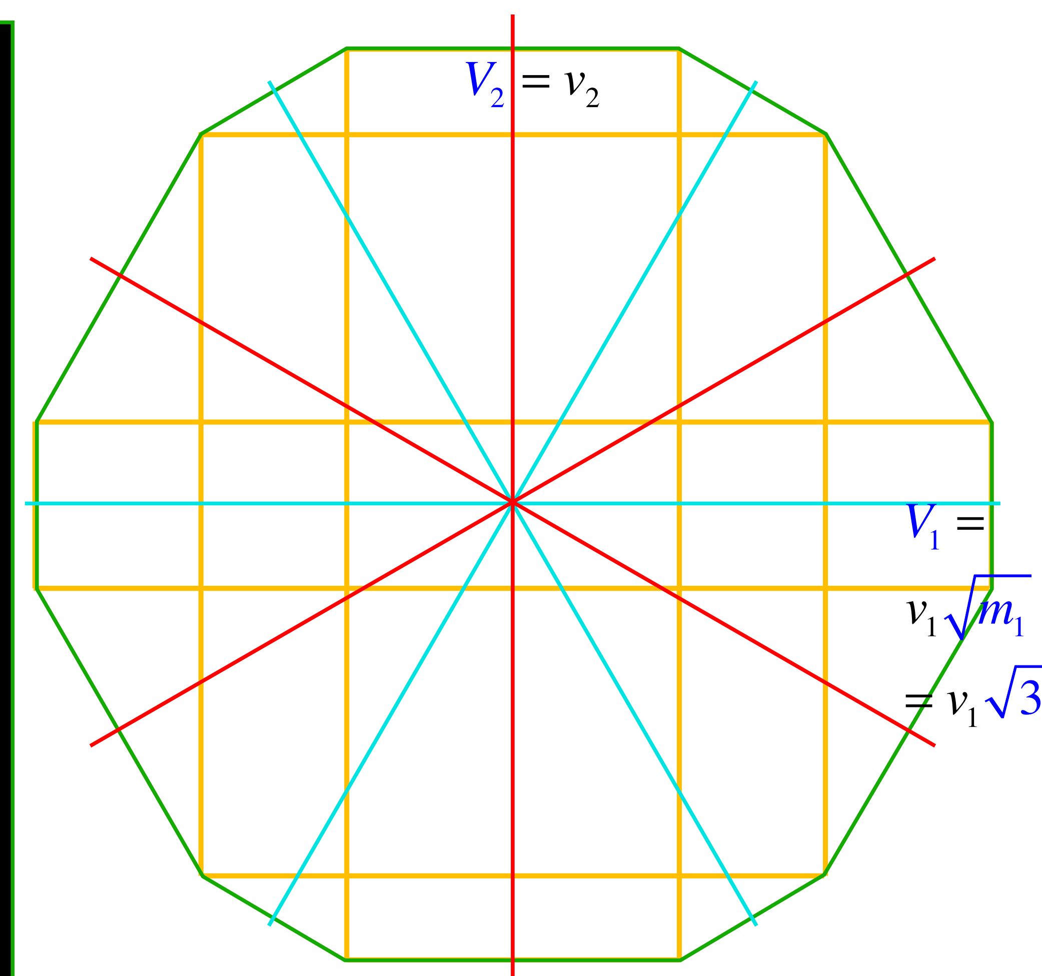


$m_1/m_2=(3)/(1)$

reduce v_1 scale by $1/\sqrt{m_1} = 1/\sqrt{3}=0.577$







$$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2$$

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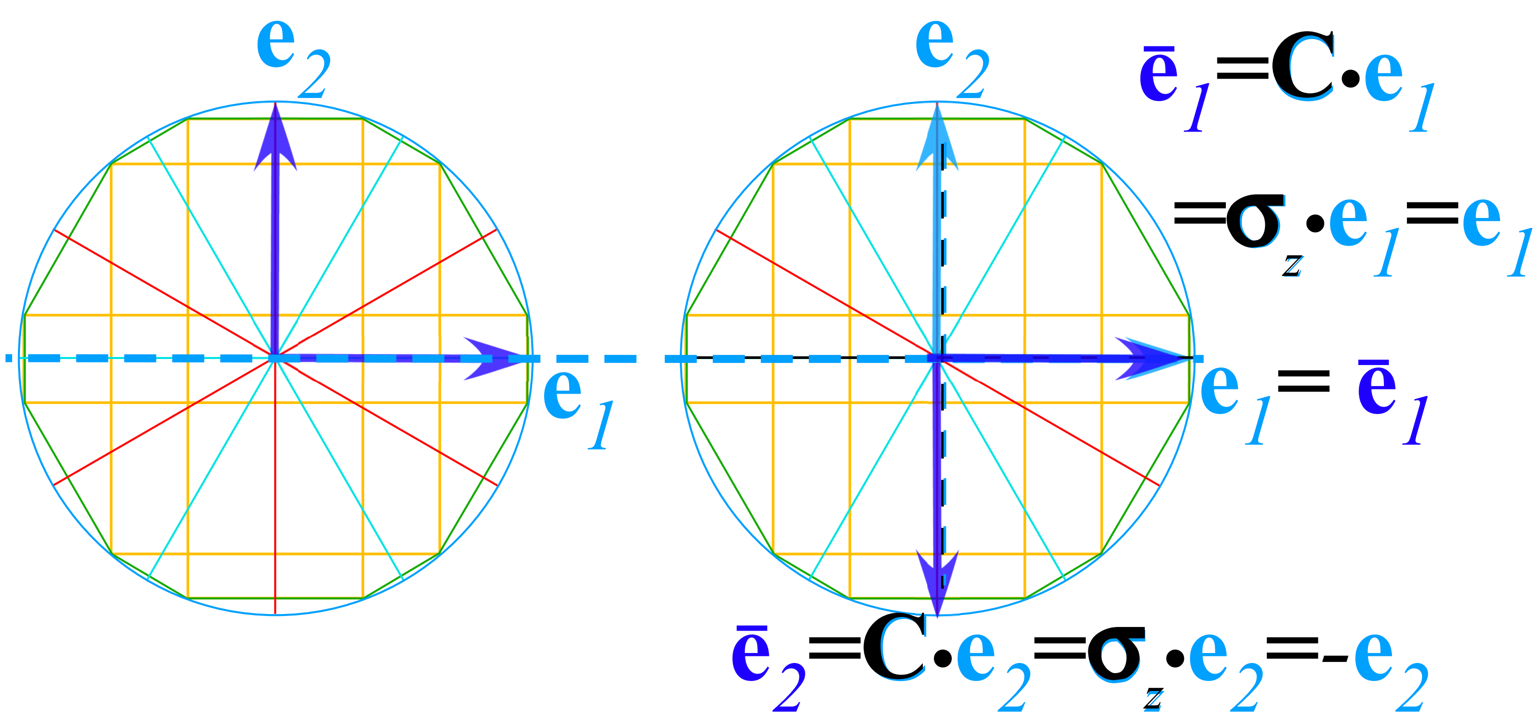
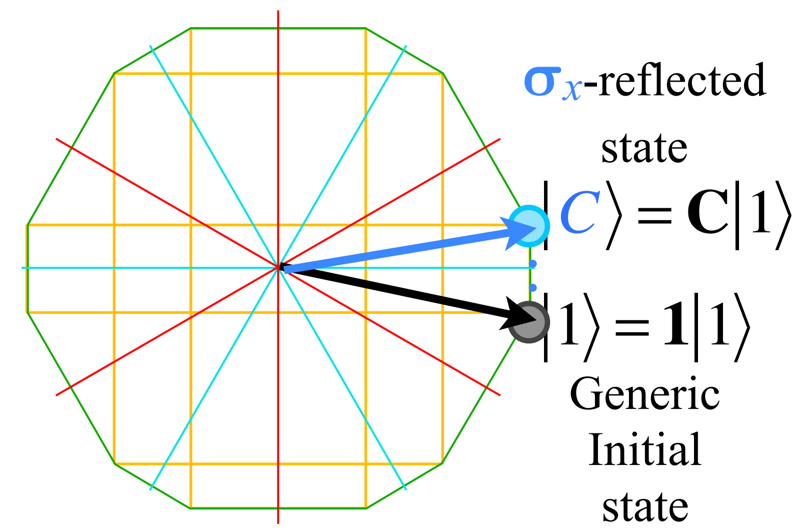
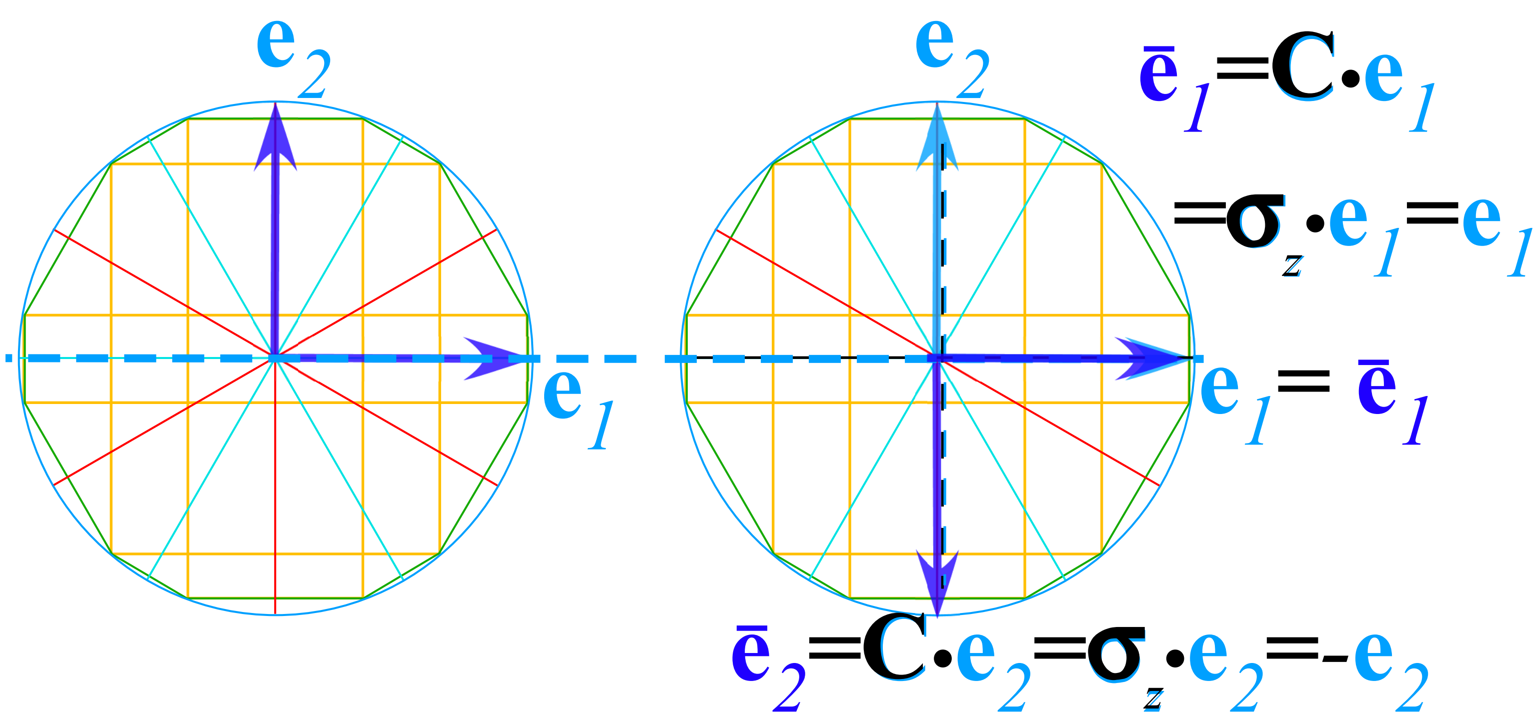
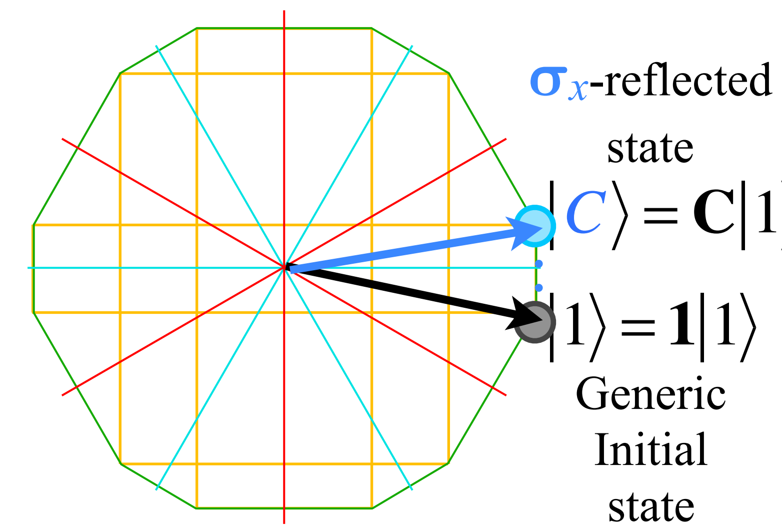
Effects of Ceiling Bang Matrix $\mathbf{C}=\boldsymbol{\sigma}_z=\begin{pmatrix} \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_2 \\ \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_2 \end{pmatrix}=\begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$





Effects of Ceiling Bang Matrix $\mathbf{C}=\boldsymbol{\sigma}_z = \begin{pmatrix} \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_2 \\ \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

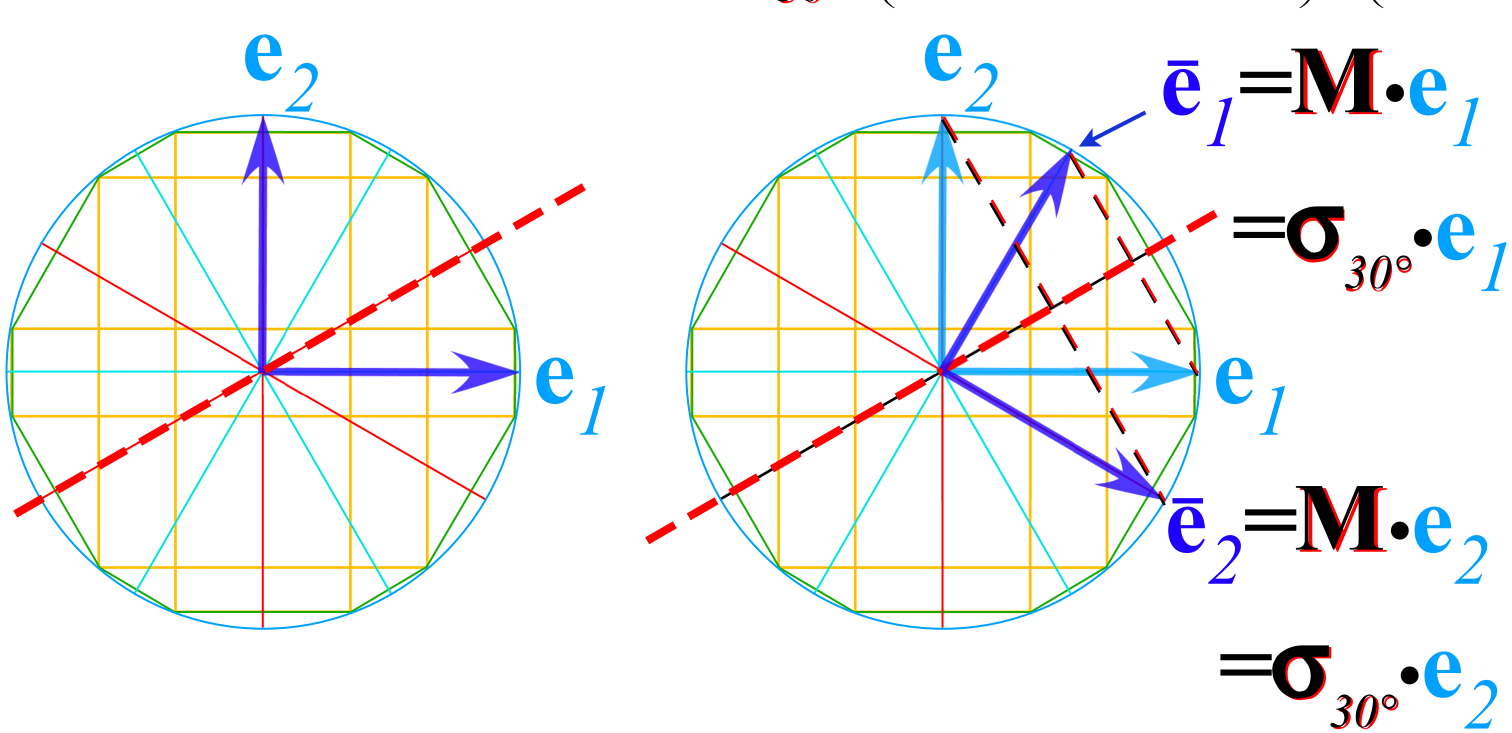


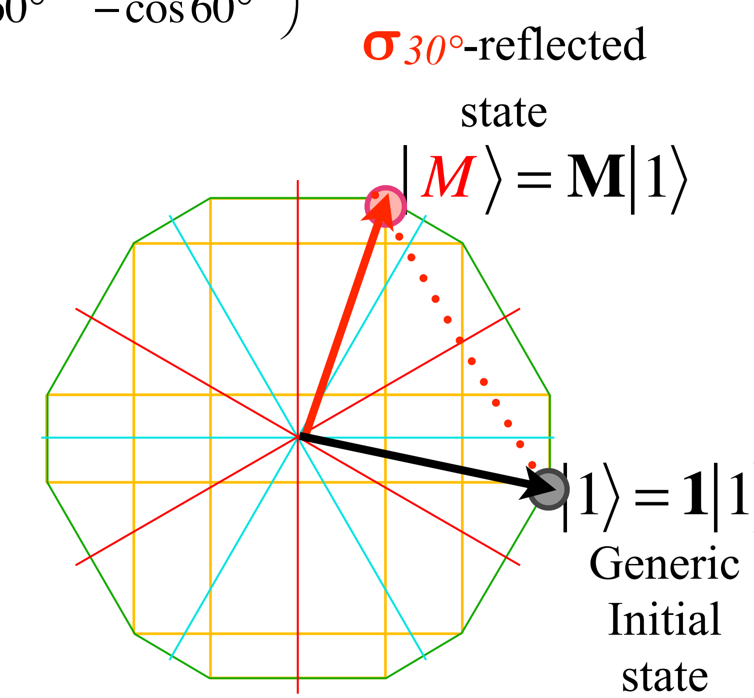


Known as *matrix elements* or *components*

Known as *relative direction cosines*

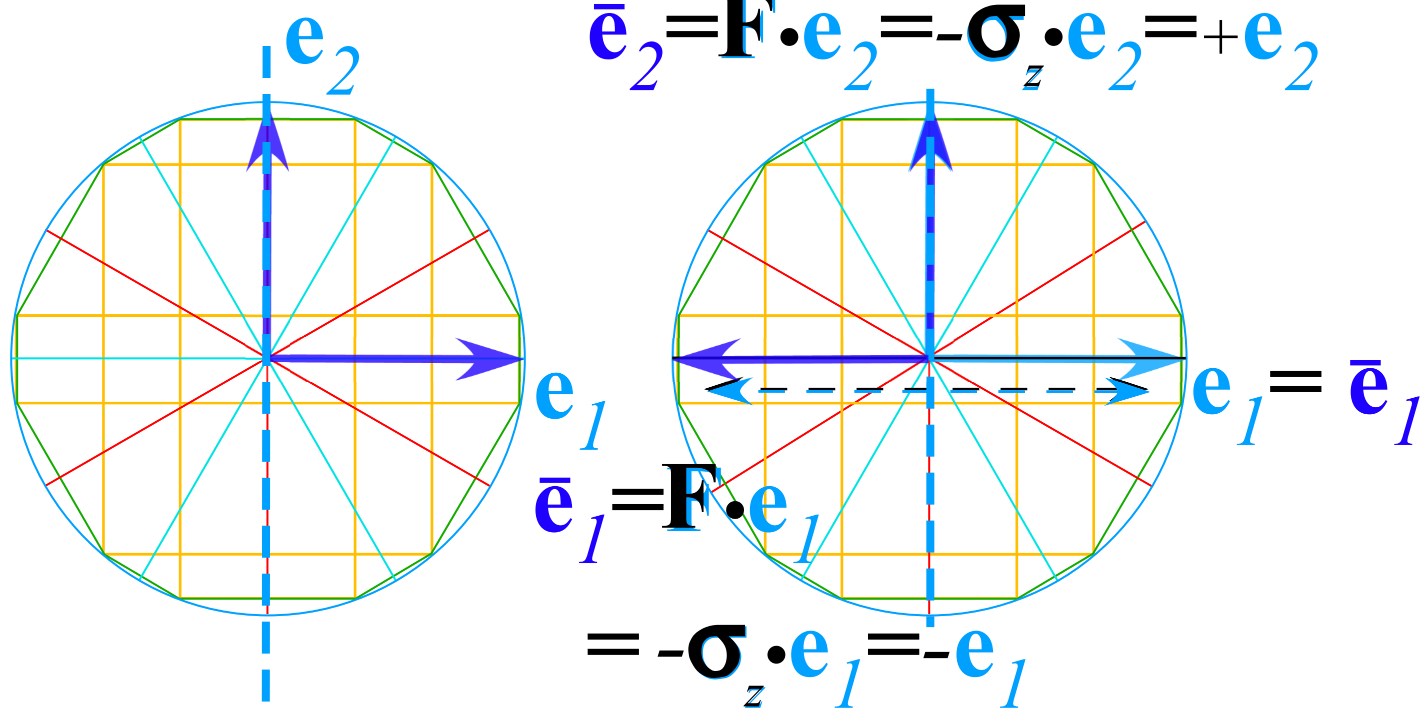
Effects of Mass Bang Matrix $\mathbf{M}=\boldsymbol{\sigma}_{30°} = \begin{pmatrix} \mathbf{e}_1\bullet\mathbf{M}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{M}\bullet\mathbf{e}_2 \\ \mathbf{e}_2\bullet\mathbf{M}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{M}\bullet\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{pmatrix}$





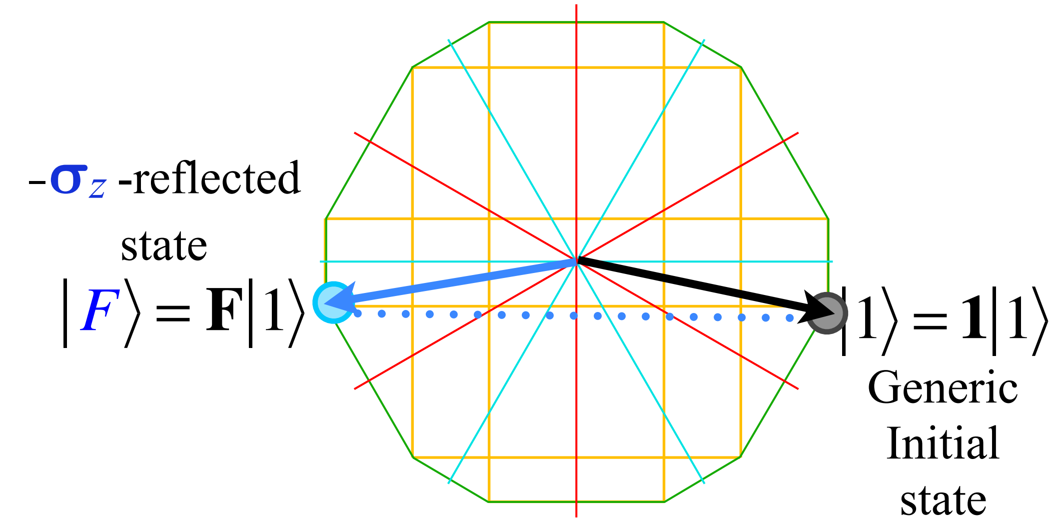
Effects of Floor Bang Matrix $\mathbf{F}=-\boldsymbol{\sigma}_z=\begin{pmatrix}\mathbf{e}_1\bullet\mathbf{F}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{F}\bullet\mathbf{e}_2\\ \mathbf{e}_2\bullet\mathbf{F}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{F}\bullet\mathbf{e}_2\end{pmatrix}=\begin{pmatrix}\mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2\\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$$\bar{\mathbf{e}}_2=\mathbf{F}\bullet\mathbf{e}_2=-\boldsymbol{\sigma}_z\bullet\mathbf{e}_2=+\mathbf{e}_2$$



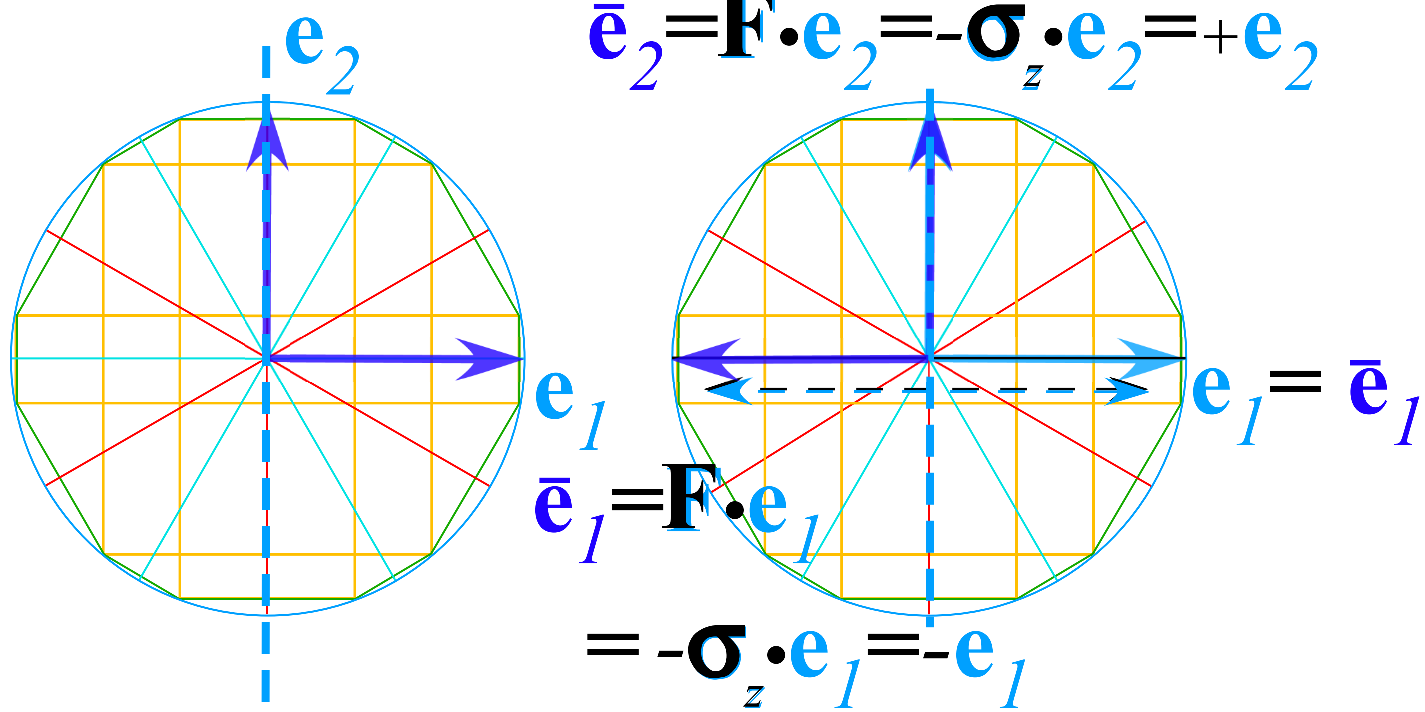
$$\bar{\mathbf{e}}_1=\mathbf{F}\bullet\mathbf{e}_1$$

$$=-\boldsymbol{\sigma}_z\bullet\mathbf{e}_1=-\mathbf{e}_1$$

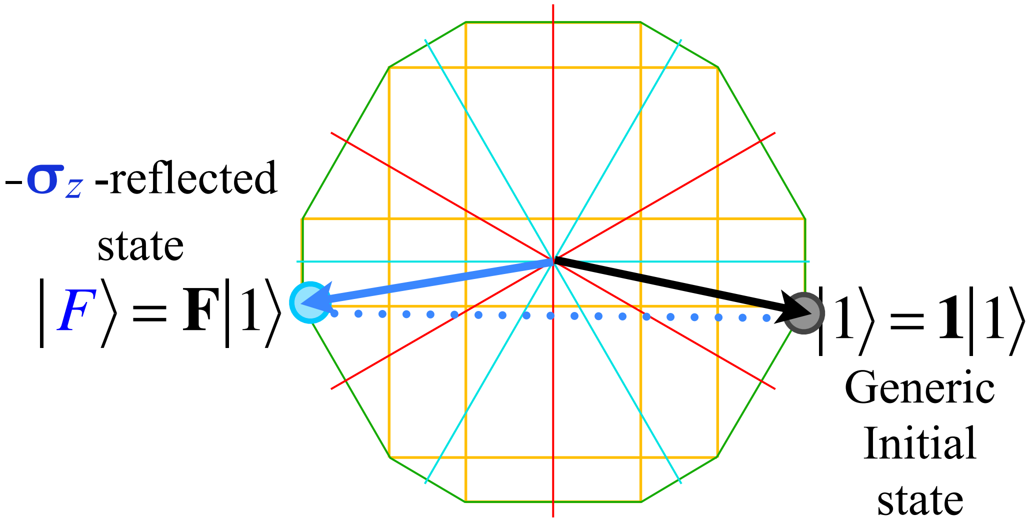


Effects of Floor Bang Matrix $\mathbf{F} = -\sigma_z = \begin{pmatrix} \mathbf{e}_1\bullet\mathbf{F}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{F}\bullet\mathbf{e}_2 \\ \mathbf{e}_2\bullet\mathbf{F}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{F}\bullet\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

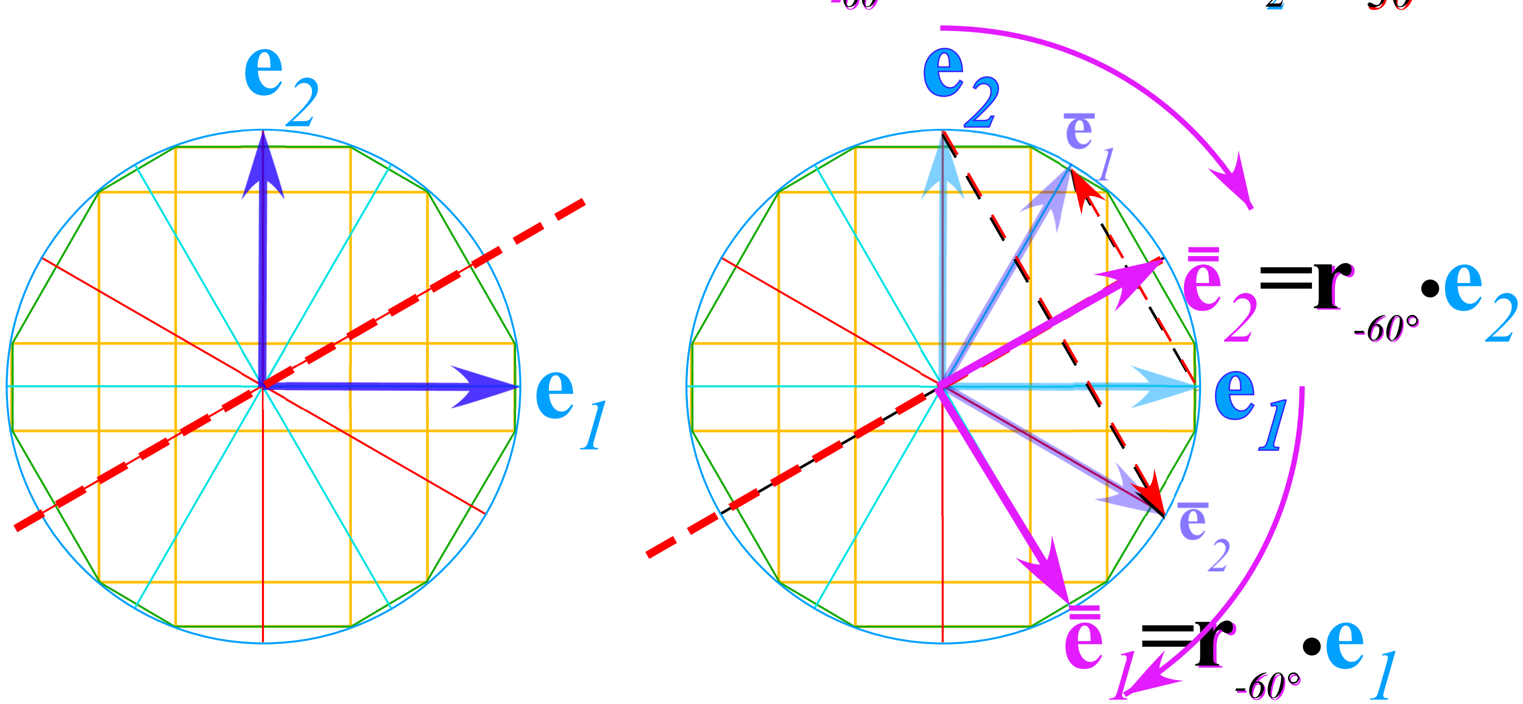
$\bar{\mathbf{e}}_2 = \mathbf{F}\bullet\mathbf{e}_2 = -\sigma_z\bullet\mathbf{e}_2 = +\mathbf{e}_2$

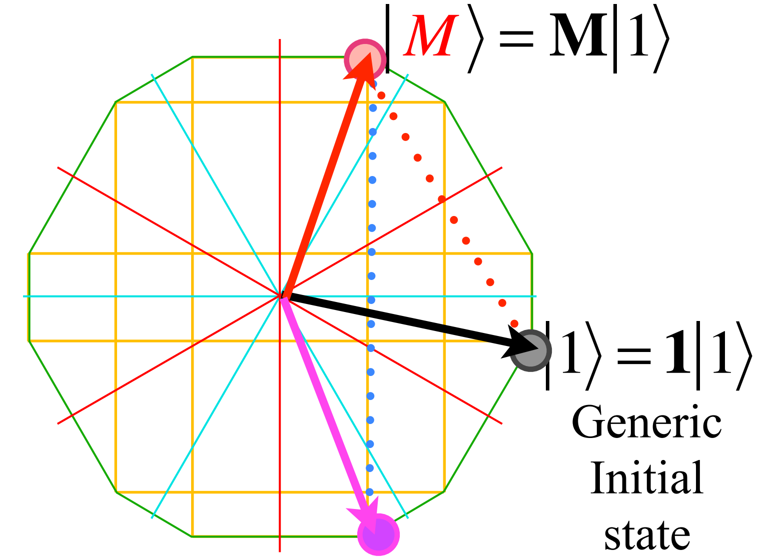


$\bar{\mathbf{e}}_1 = \mathbf{F}\bullet\mathbf{e}_1 = -\sigma_z\bullet\mathbf{e}_1 = -\mathbf{e}_1$



Effects of Ceiling **C** after Bang **M**: $\mathbf{r}_{-60^\circ} = \mathbf{C}\bullet\mathbf{M} = \sigma_z\bullet\sigma_{30^\circ}$





$|r_{-60^\circ}\rangle = \mathbf{C}\bullet\mathbf{M}|1\rangle = \mathbf{r}_{-60^\circ}|1\rangle$

$\sigma_{30^\circ}\,\sigma_{30^\circ}$-reflected state is a $\mathbf{r}_{-60^\circ}$-rotated state

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→ *Group multiplication and product table* ←

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D_6	1	r_{120}	$\bar{r}_{120}$	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	$\bar{r}_{60}$	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
1	1											
$\bar{r}_{120}$		1										
r_{120}			1									
σ_{60}				1								
$\bar{\sigma}_{60}$					1							
σ_z						1					$\bar{r}_{60}$	
I							1					
r_{60}								1				
$\bar{r}_{60}$									1			
$\bar{\sigma}_{30}$										1		
σ_{30}						r_{60}					1	
$\bar{\sigma}_z$												1

Note: $\bar{r}_{60} = Ir_{120} = r_{120}I = r_{-60}$ and: $I = r_{\pm 180}$

$\bar{r}_{120} = Ir_{60} = r_{60}I = r_{-120}$ and: $I^2 = 1$

$\sigma_{60} = I\bar{\sigma}_{30} = \bar{\sigma}_{30}I$

$\bar{\sigma}_{60} = I\sigma_{30} = \sigma_{30}I$

$\bar{\sigma}_z = I\sigma_z = \sigma_z I$

Easy to make hexagonal (D_6) symmetry group table:

Example 1: Find $\boldsymbol{\sigma}_{30°}\bullet\boldsymbol{\sigma}_{-60°}$ = _____?

Solution: Find $\boldsymbol{\sigma}_{30°}$*-plane and state-*$|\boldsymbol{\sigma}_{-60°}\rangle$

Operate former on latter to get: $\boldsymbol{\sigma}_{30°}|\boldsymbol{\sigma}_{-60°}\rangle = |\mathbf{I}\rangle$

That gives answer: $\boldsymbol{\sigma}_{30°}\bullet\boldsymbol{\sigma}_{-60°} = \mathbf{I}$.

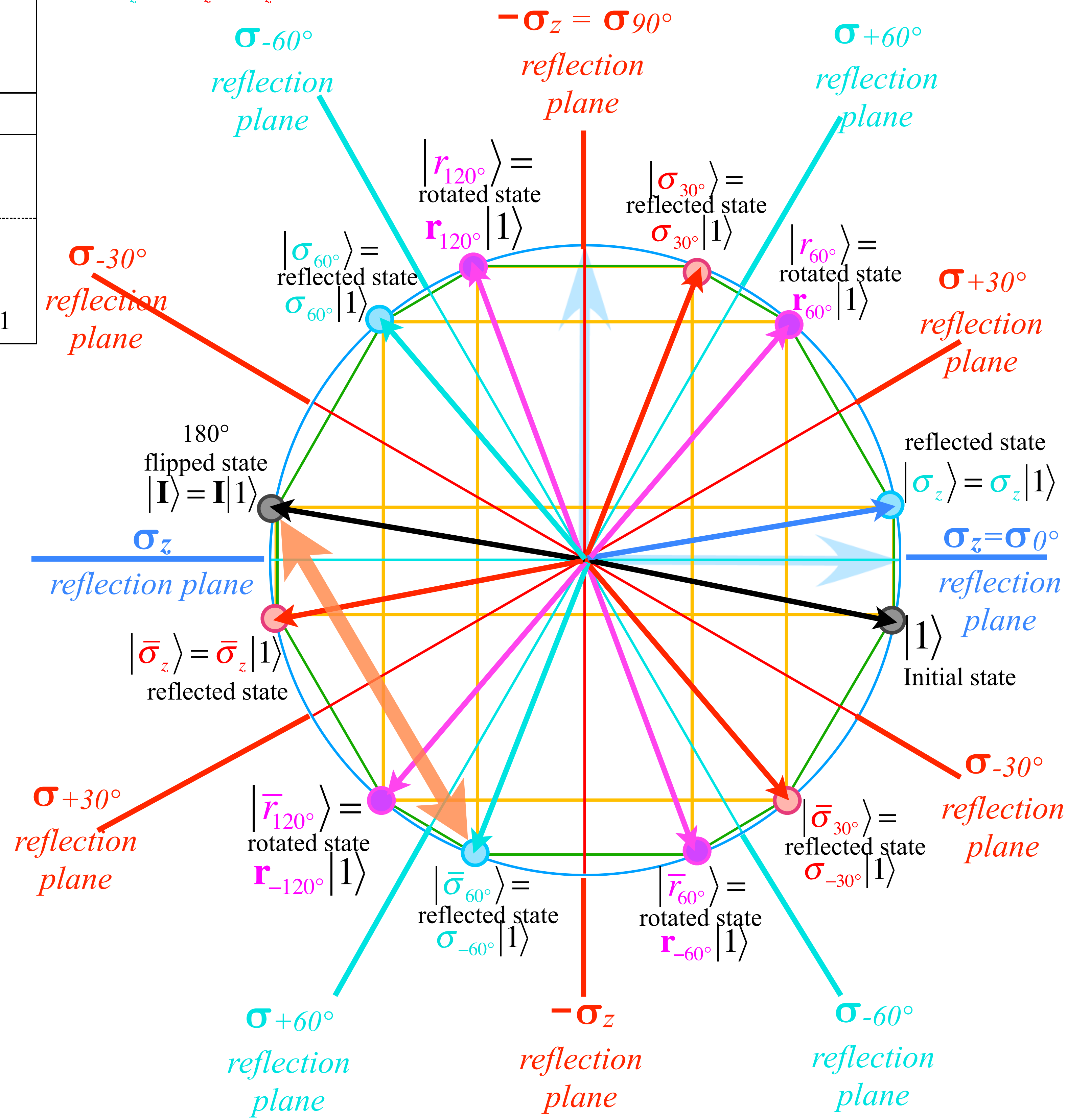
Rest of $\boldsymbol{\sigma}_{30°}$ *row follows:*

11^{th} row	1	r_{120}	$\bar{r}_{120}$	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	$\bar{r}_{60}$	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
σ_{30}	σ_{30}	$\bar{\sigma}_{30}$	$\bar{\sigma}_z$	$\bar{r}_{60}$	I	r_{60}	$\bar{\sigma}_{60}$	σ_{60}	σ_z	r_{120}	1	$\bar{r}_{120}$

Example 2: Find $\mathbf{r}_{60°}\bullet\boldsymbol{\sigma}_{-60°}$ = _____?

Solution: Do $\mathbf{r}_{60°}$*-rotation* $\mathbf{r}_{60°}|\boldsymbol{\sigma}_{-60°}\rangle = |\boldsymbol{\sigma}_{-30°}\rangle$

That gives answer: $\mathbf{r}_{60°}\bullet\boldsymbol{\sigma}_{-60°} = \boldsymbol{\sigma}_{-30°}$



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

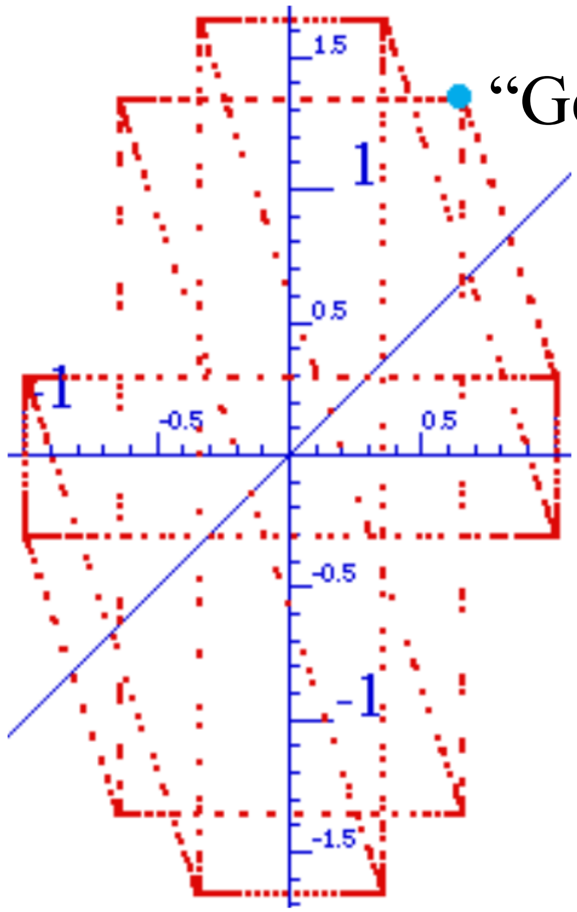
How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

→ *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*

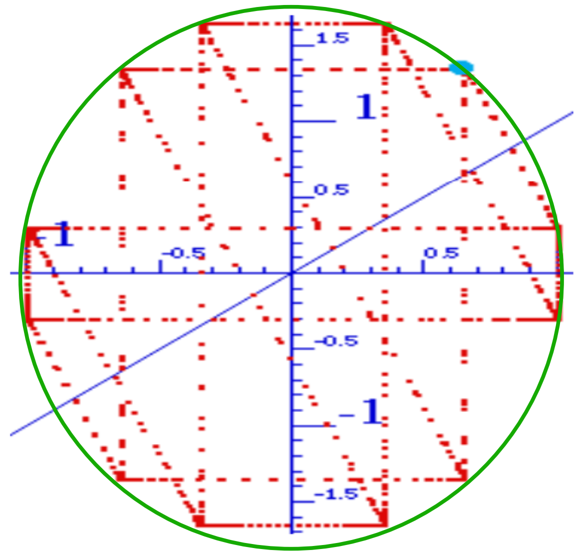


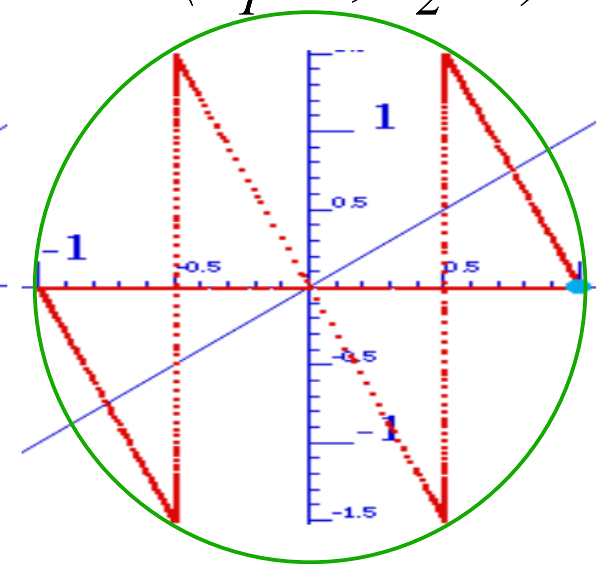
"Generic" initial velocity

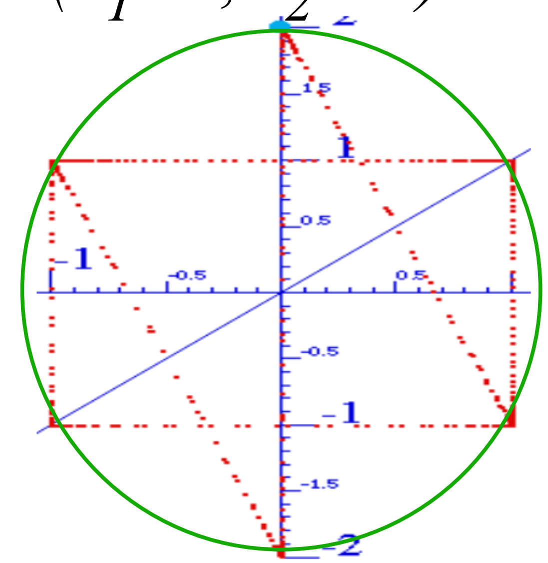
$(v_1=1.0, v_2=0.1)$

"Symmetric" initial velocity

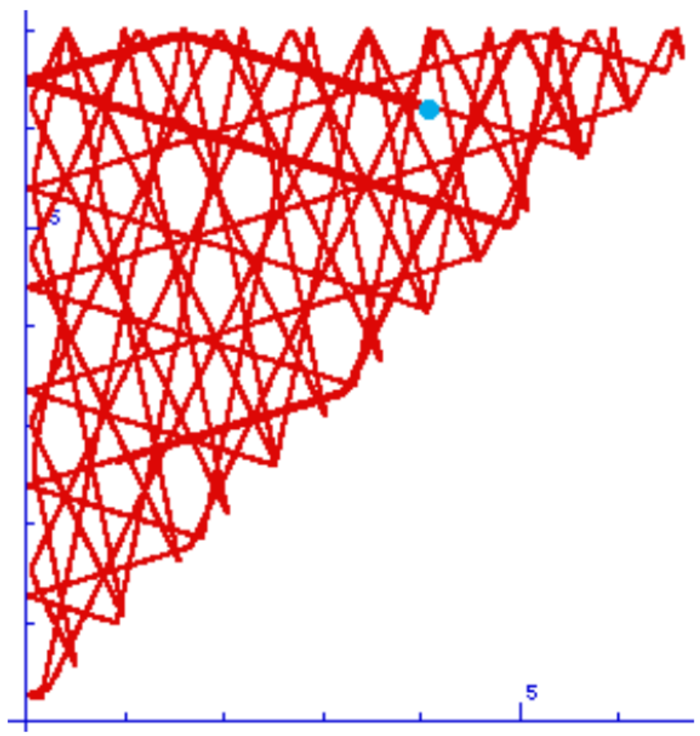
$(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$

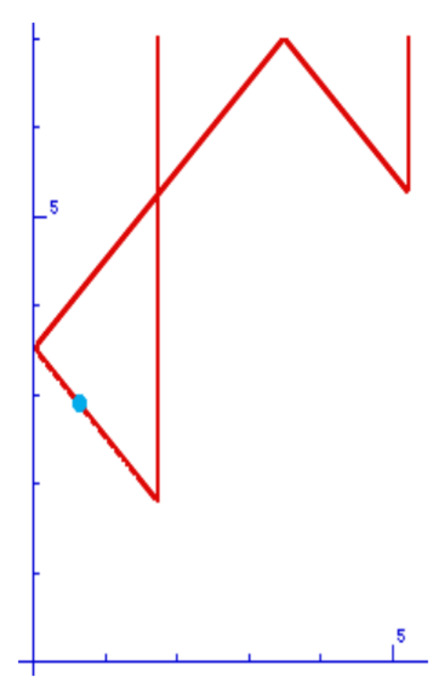


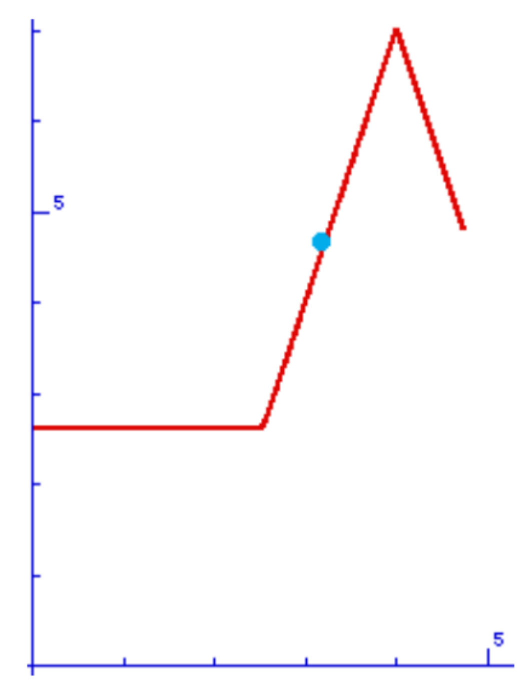


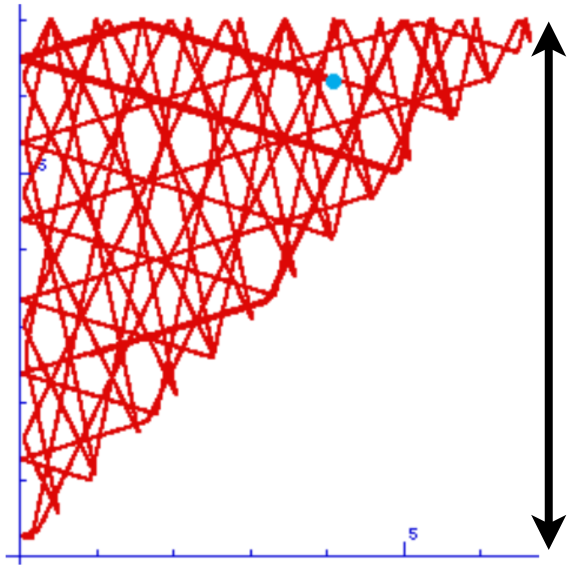


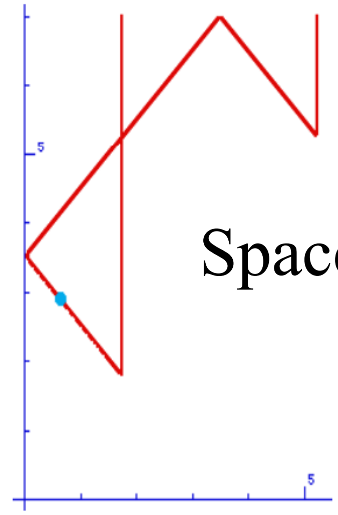
Corresponding space-space (y_1, y_2) paths



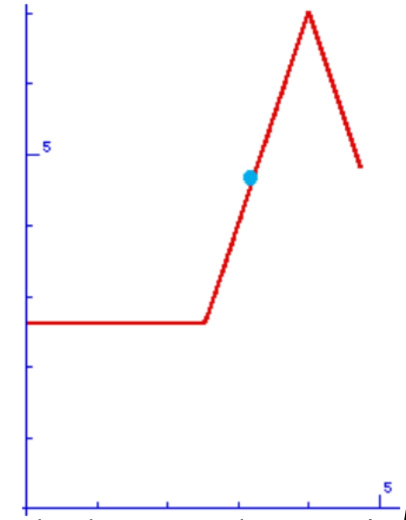






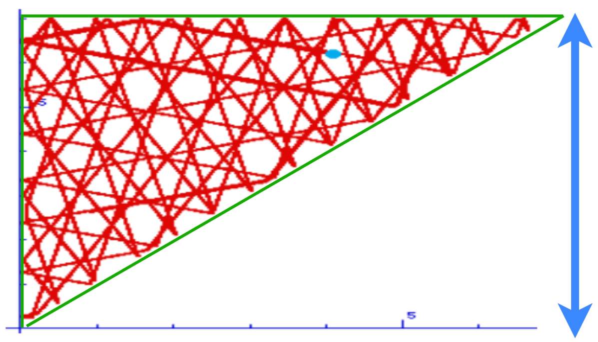


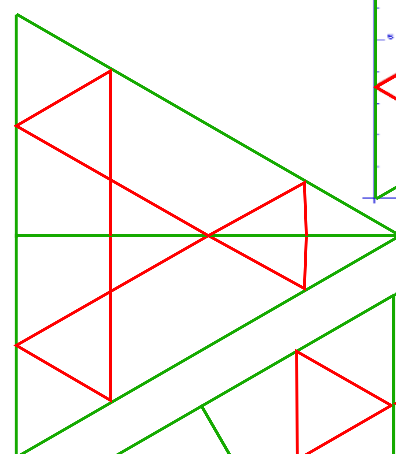
Space-space (y_1, y_2) paths

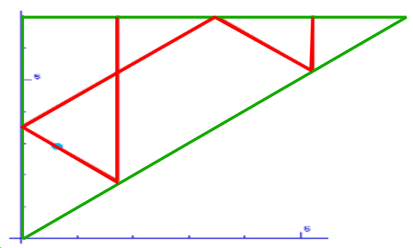


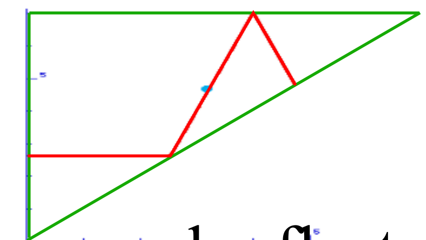
Space-space (y_1, y_2) paths scaled down by $1/\sqrt{3}$...

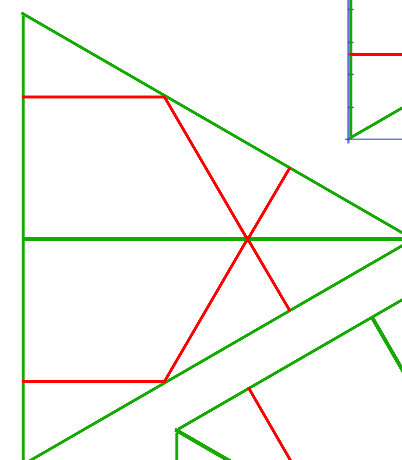
Scaled y down by $1/\sqrt{3}=0.577$





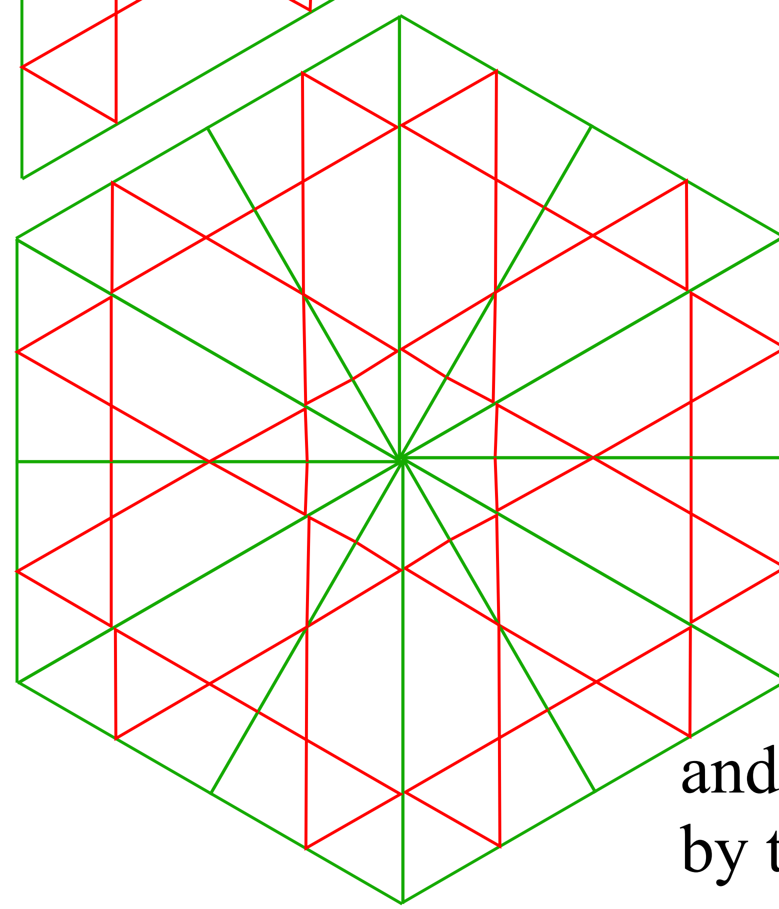


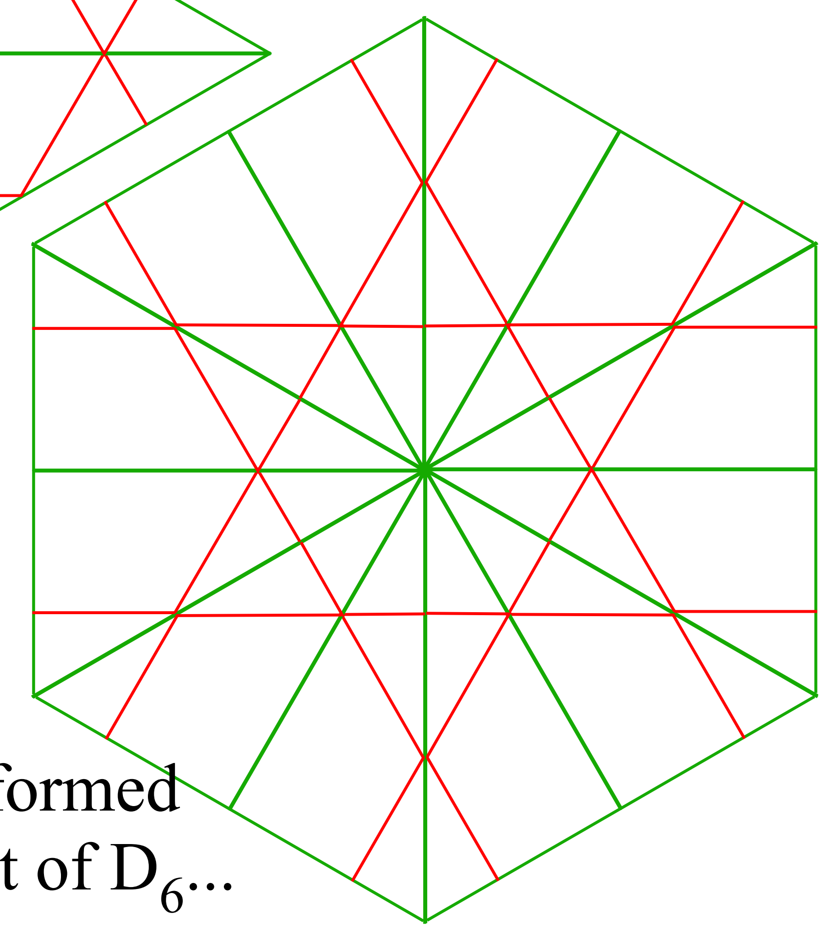




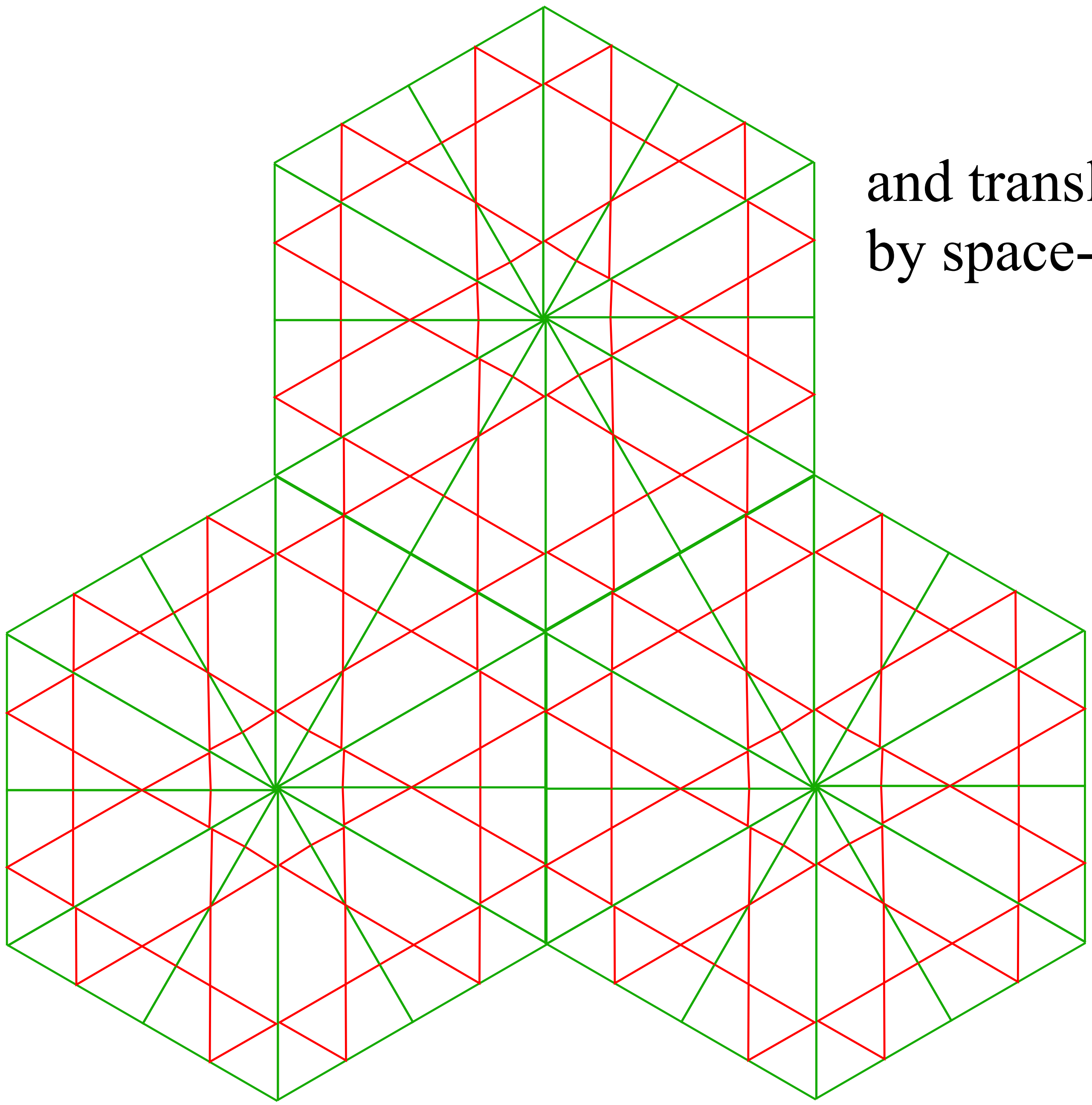
and reflected...

..or could have scaled x up by $\sqrt{3}=1.732$

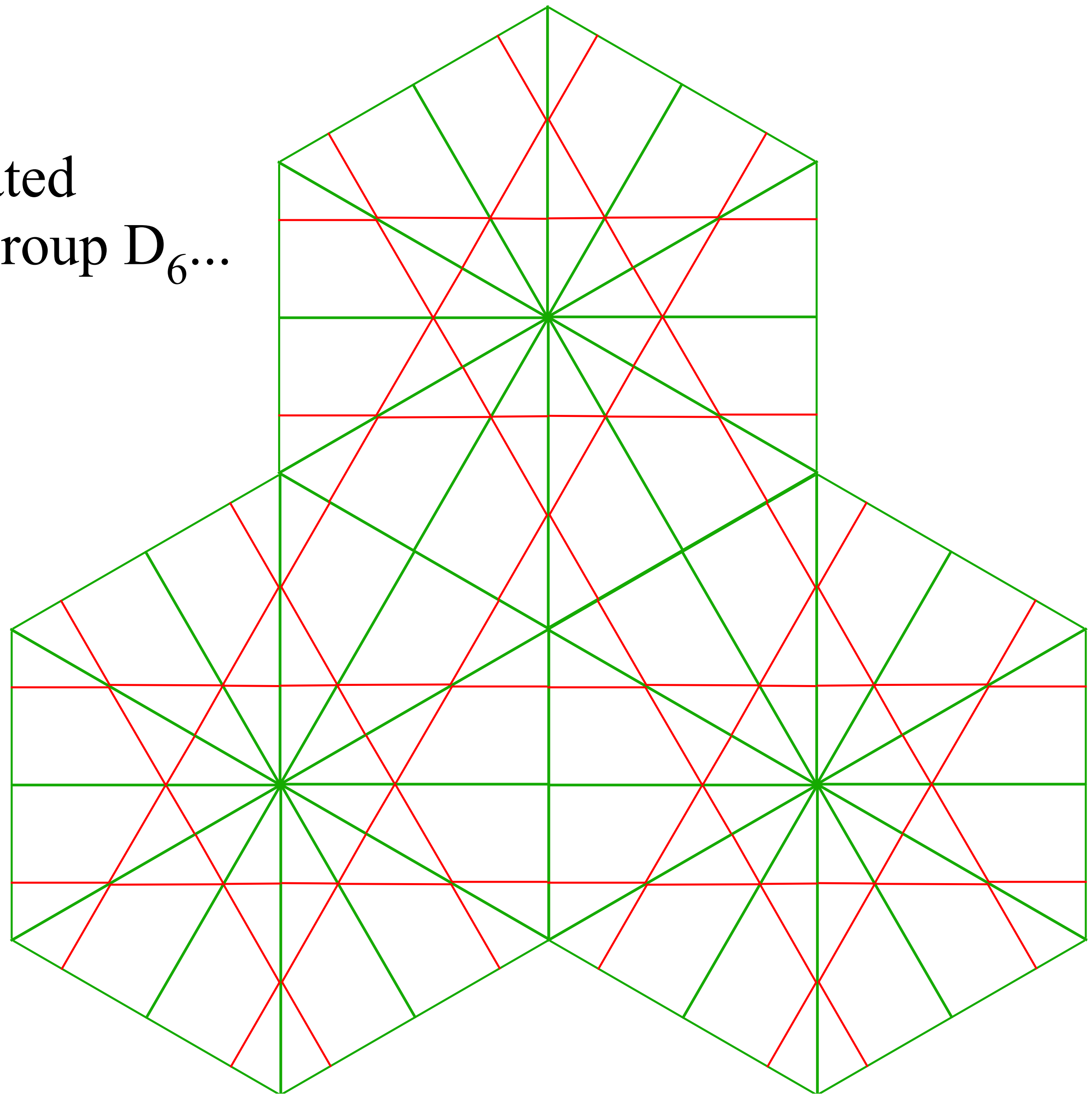




and transformed by the rest of D_6...

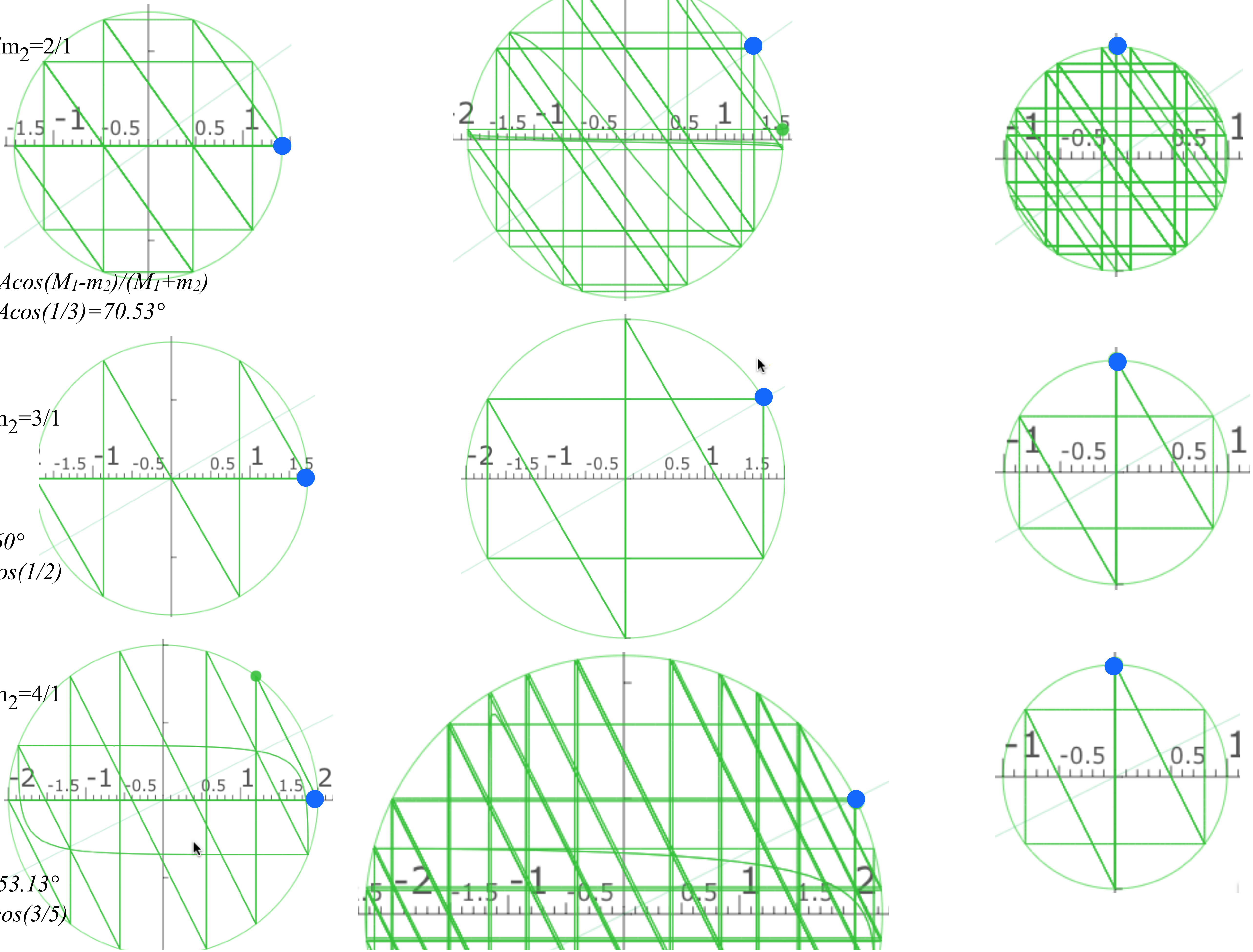


and translated
by space-group D_6...



...they're just straight lines going forever.

Initial velocity $v_1=1$, $v_2=0$ | *Initial velocity $v_1=1$, $v_2=1$* | *Initial velocity $v_1=0$, $v_2=1$*



$M_1/m_2=2/1$

$\phi = A\cos(M_1-m_2)/(M_1+m_2)$
$= A\cos(1/3) = 70.53°$

$M_1/m_2=3/1$

$\phi = 60°$
$= A\cos(1/2)$

$M_1/m_2=4/1$

$\phi = 53.13°$
$= A\cos(3/5)$

Geometric "Integration" (Converting Velocity data to Spacetime)

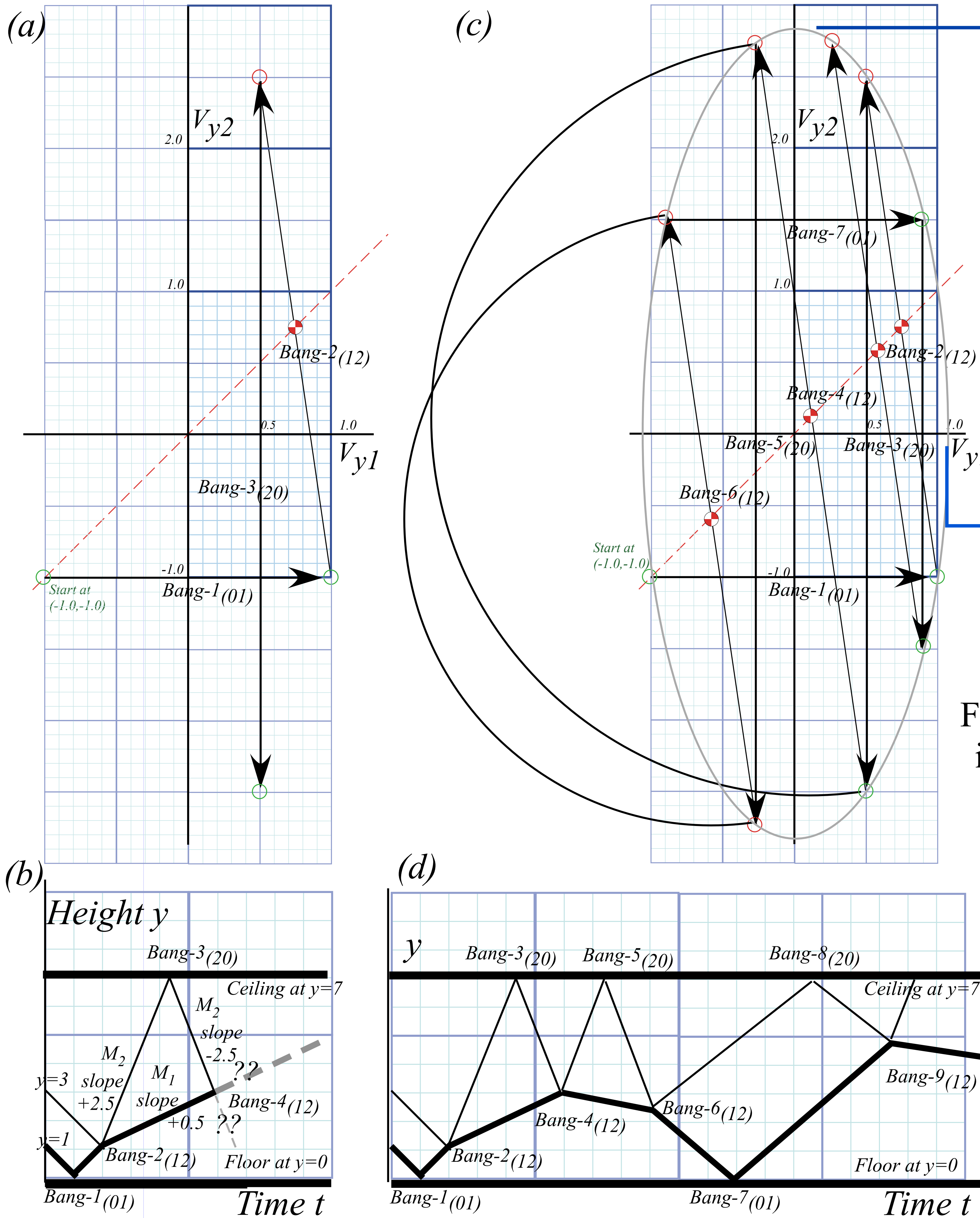


Fig. 4.7a-d in Unit 1

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE/M_1} = \sqrt{2KE/7} = \sqrt{8/7} = 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE/M_1} = \sqrt{2KE/1} = \sqrt{8/1} = 2.83$$

Geometric "Integration" (Converting Velocity data to Spacetime)

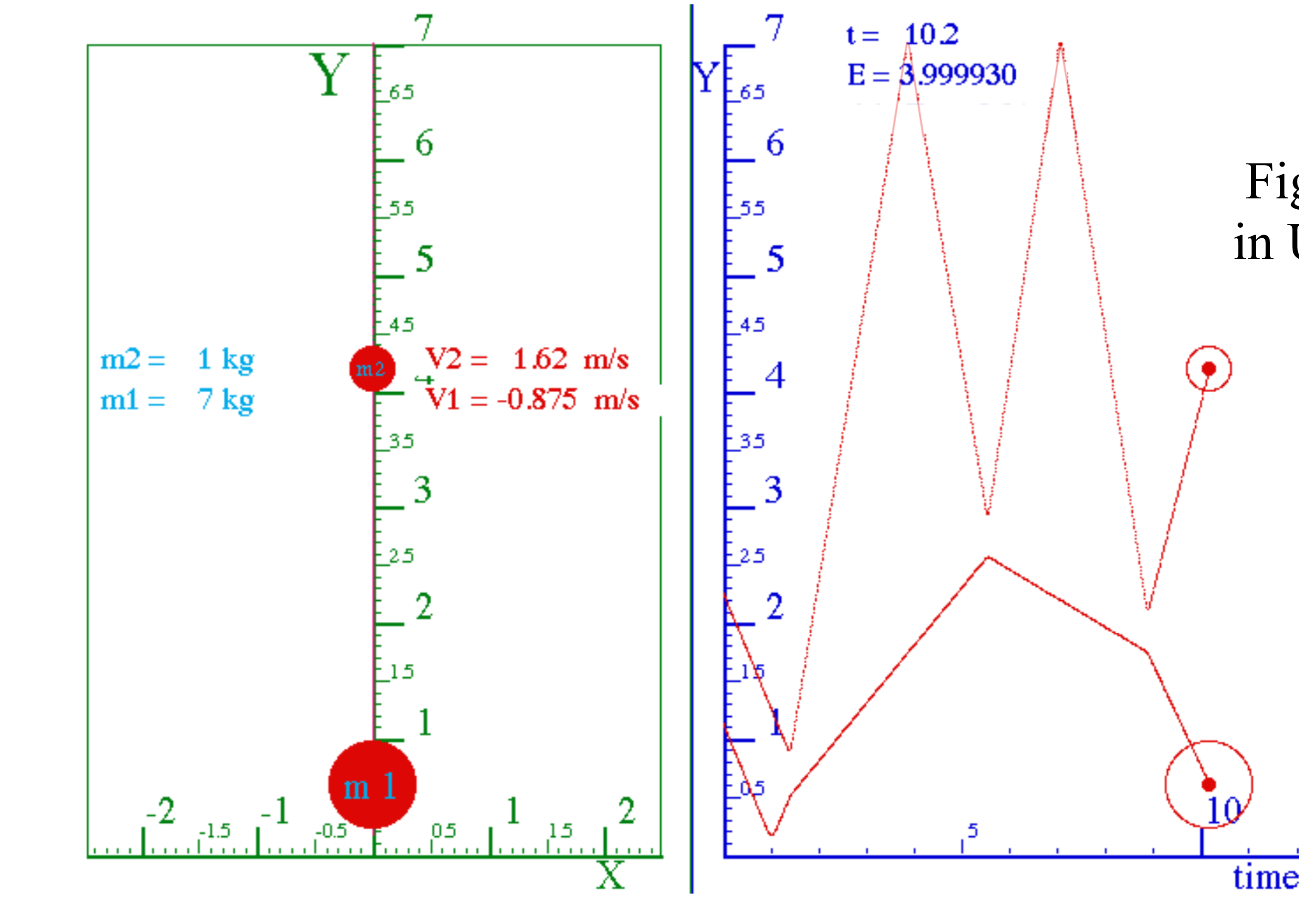


Fig. 4.8
in Unit 1

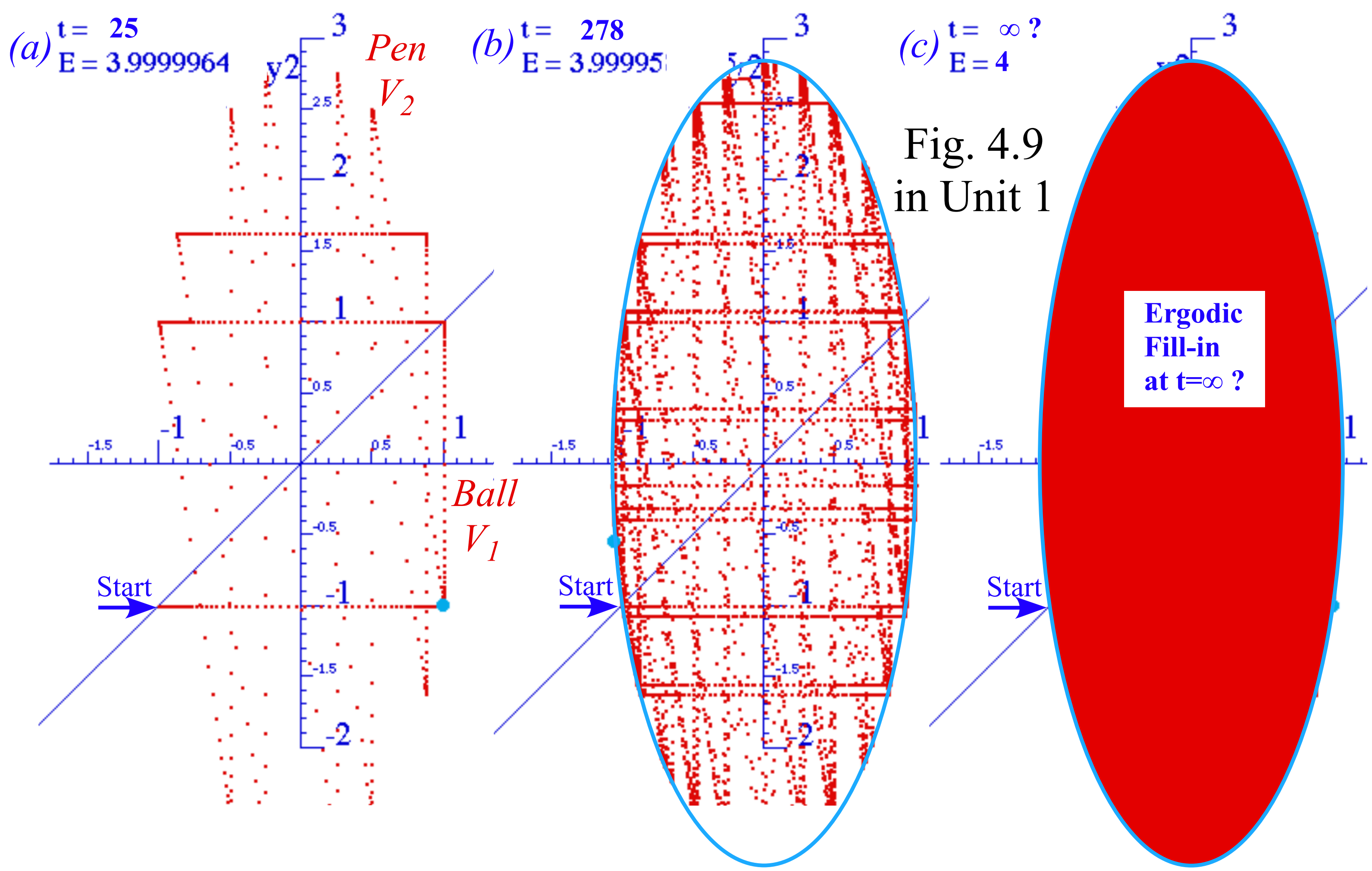


Fig. 4.9
in Unit 1

Exercise 1.4.1 and Exercise 1.4.2

Exercise 1.4.1: (a) Construct a bounce sequence plot of a mass ratio m_1: m_2 *=4:1* with the following initial values *($x_1(0)=1.5$, $x_2(0)=3.0$, $v_1(0)=-1$, $v_2(0)=-1$)* and ceiling height $y_{max}=7.0$. This *4:1* case is quasi-periodic. The collision sequence in the (v_1,v_2) plot path appears to repeat several steps then jumps to make new paths. Does the (x_1,x_2) plot also repeat those steps? Draw both plots for at least 16 collisions to analyze the sequences.

(b) Show that, with initial values *($x_1(0)=1.5$, $x_2(0)=3.0$, $v_1(0)=1$, $v_2(0)=0$),* the collision sequence is periodic after 12 steps in both the (v_1,v_2) plot and the (x_1,x_2) plot.

Exercise 1.4.2: Continue the (v_1,v_2) and (x_1,x_2) collision plots begun in class and shown in Fig. 4.7 and Fig. 4.11.
Continue until you reach the "gameover" point of last possible M_1-M_2 collision assuming the floor is open after *Bang-1* so both masses can fall thru indefinitely. Show where is this last last collision.

Exercise 1.4.2 solutions from Assignment 2 are given first followed by detailed solutions of Exercise 1.4.1 from Assignment 2.

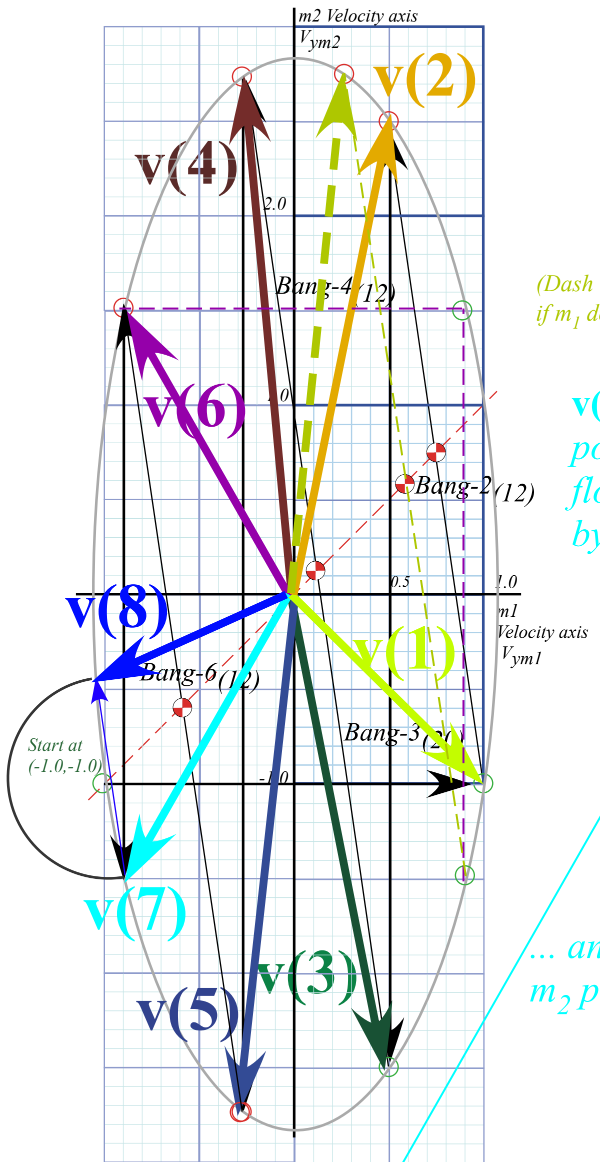
Solutions to Exercise 1.4.2 *(Fig. 1.4.12 completion)*

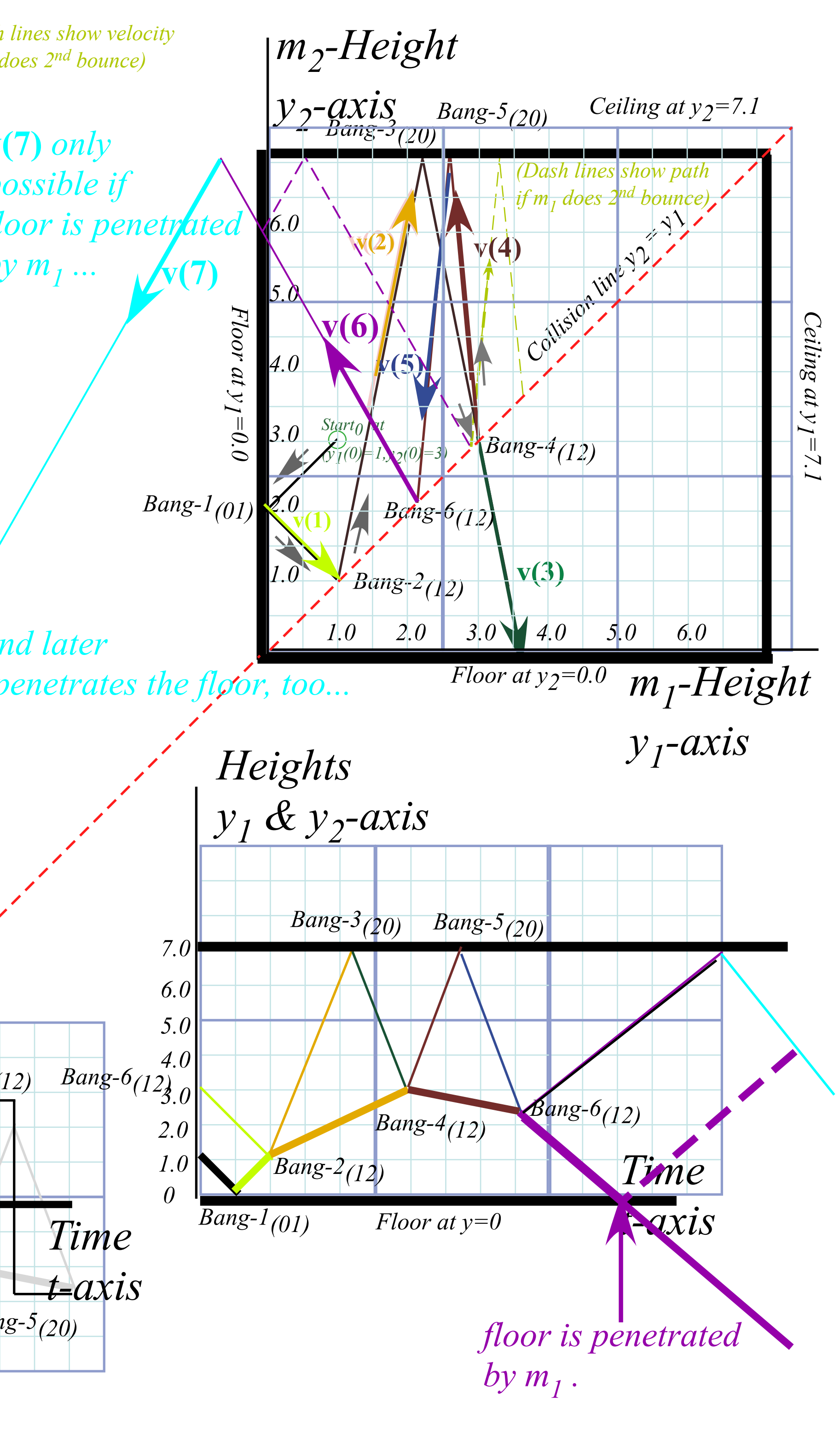
Step-2: Extend **v***(2) line to ceiling point* **y***(3)=(?,7.1) and draw Bang-3*$_{(20)}$ *velocity* **v***(3)=(1,-1) line. (Find* **v***(3) using V-V plot.)*

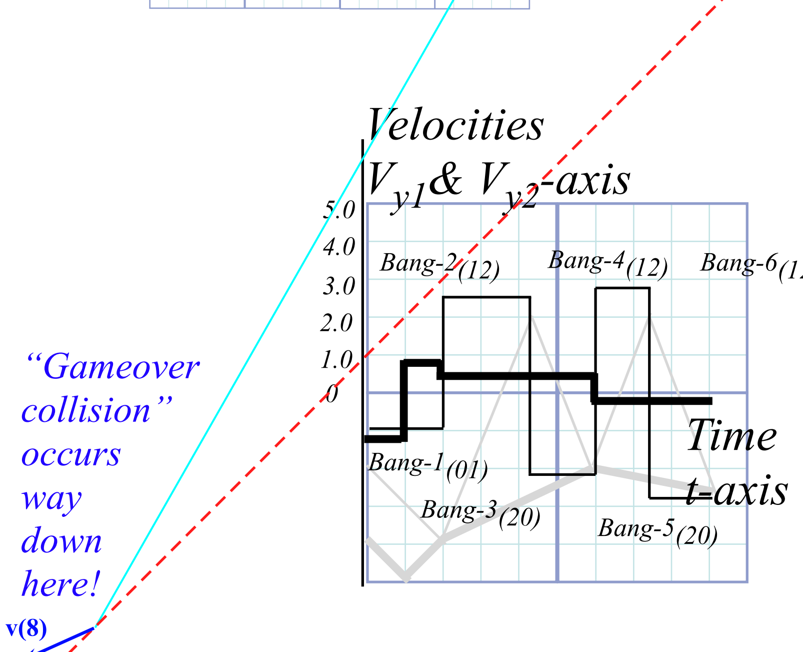
Step-3: Extend **v***(3) line to collision point* **y***(4)=(?,?) and draw Bang-4*$_{(12)}$ *velocity* **v***(4)=(0.5,2.5). (Find* **v***(4) using V-V plot.)*

Step-4: Extend **v***(4) line to ceiling point* **y***(4)=(?,7.1) and draw Bang-5*$_{(20)}$ *velocity* **v***(5)=(1,-1) line. (Find* **v***(5) using V-V plot.)*

Step-5: Extend **v***(5) line to collision point* **y***(6)=(?,?) and draw Bang-6*$_{(12)}$ *velocity* **v***(6)=(0.5,2.5). (Find* **v***(6) using V-V plot.)*

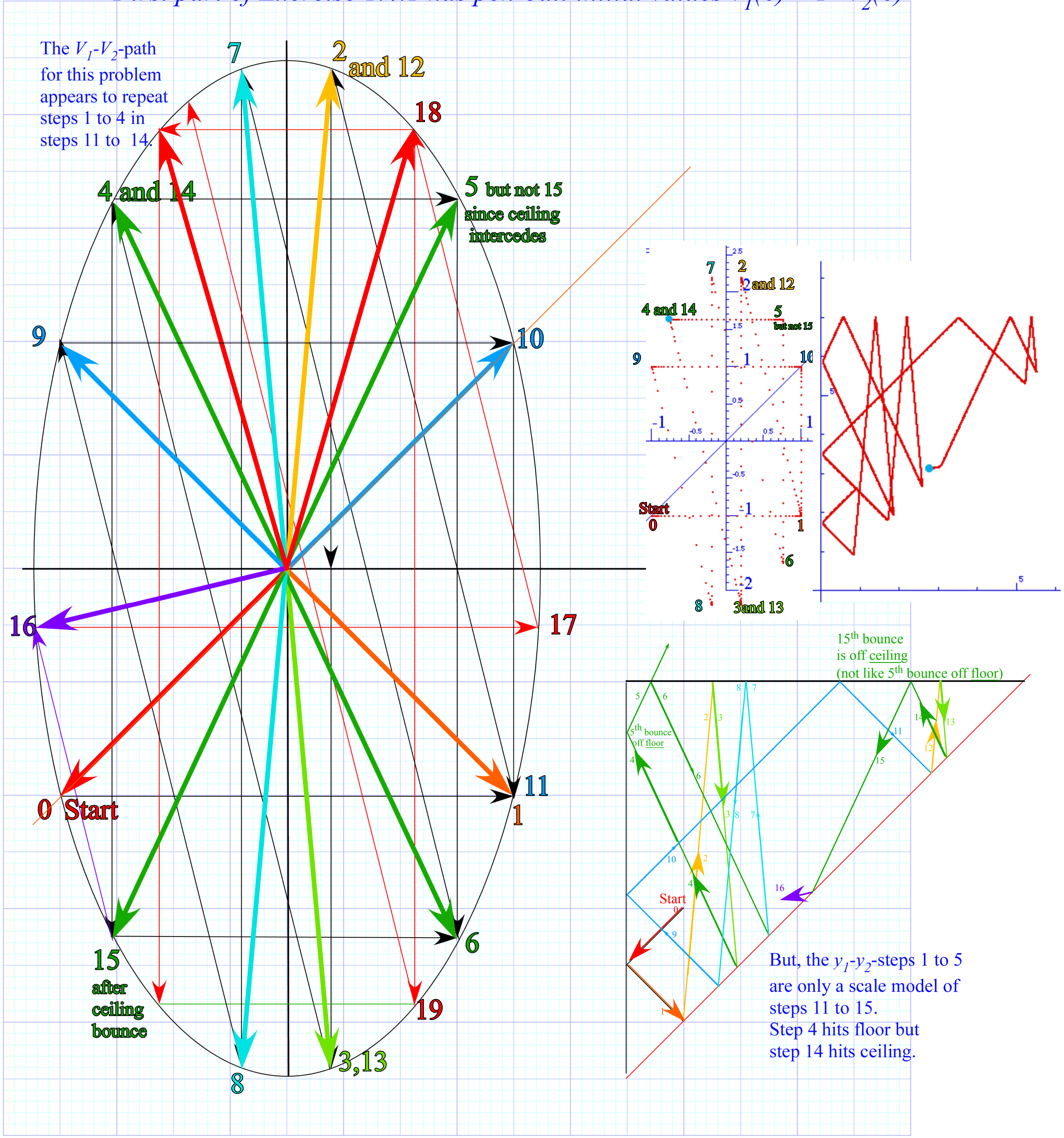




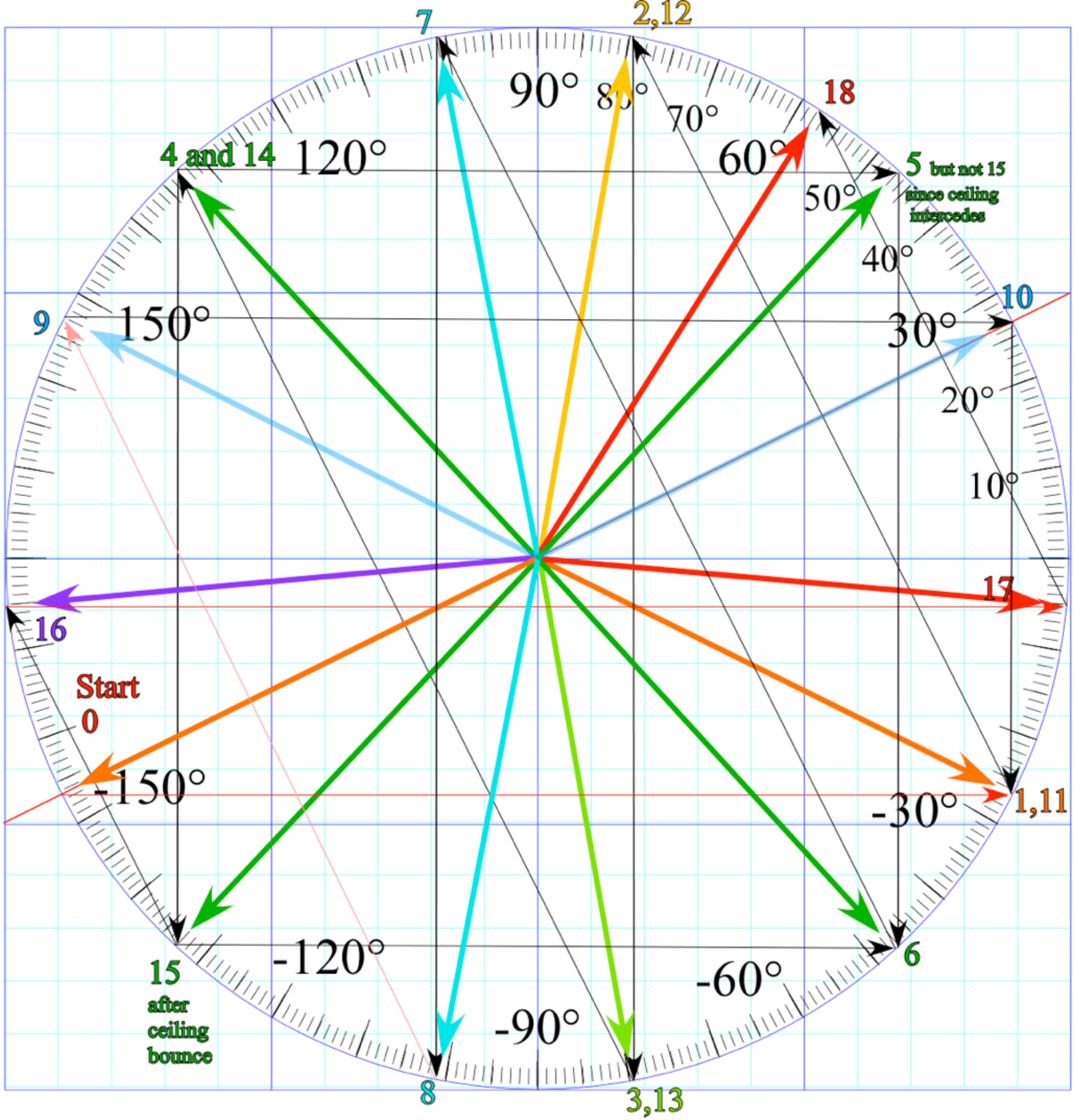


First part of Exercise 1.4.1 has pen-ball initial values $v_1(0)=-1=v_2(0)$

The V_1-V_2-path for this problem appears to repeat steps 1 to 4 in steps 11 to 14.

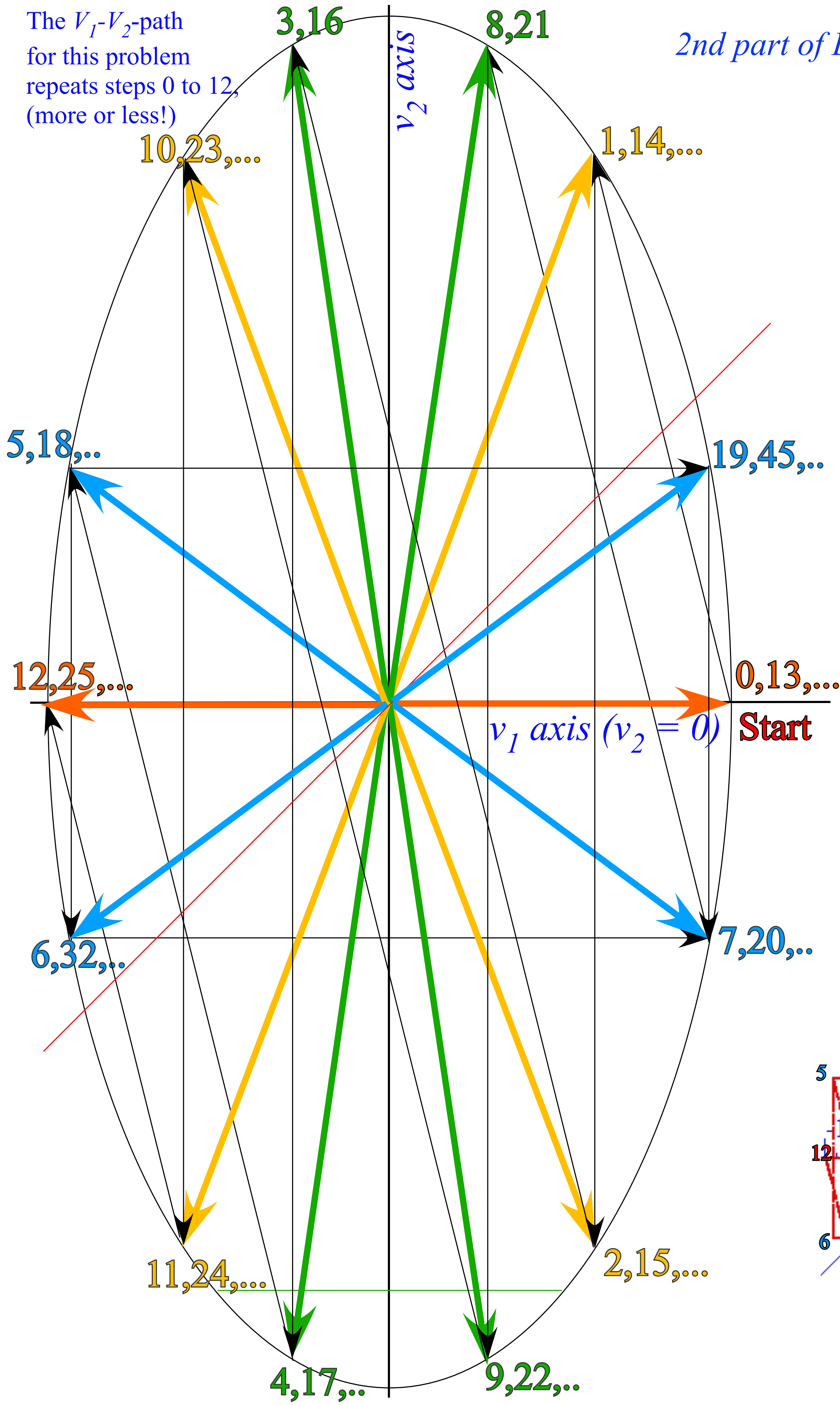


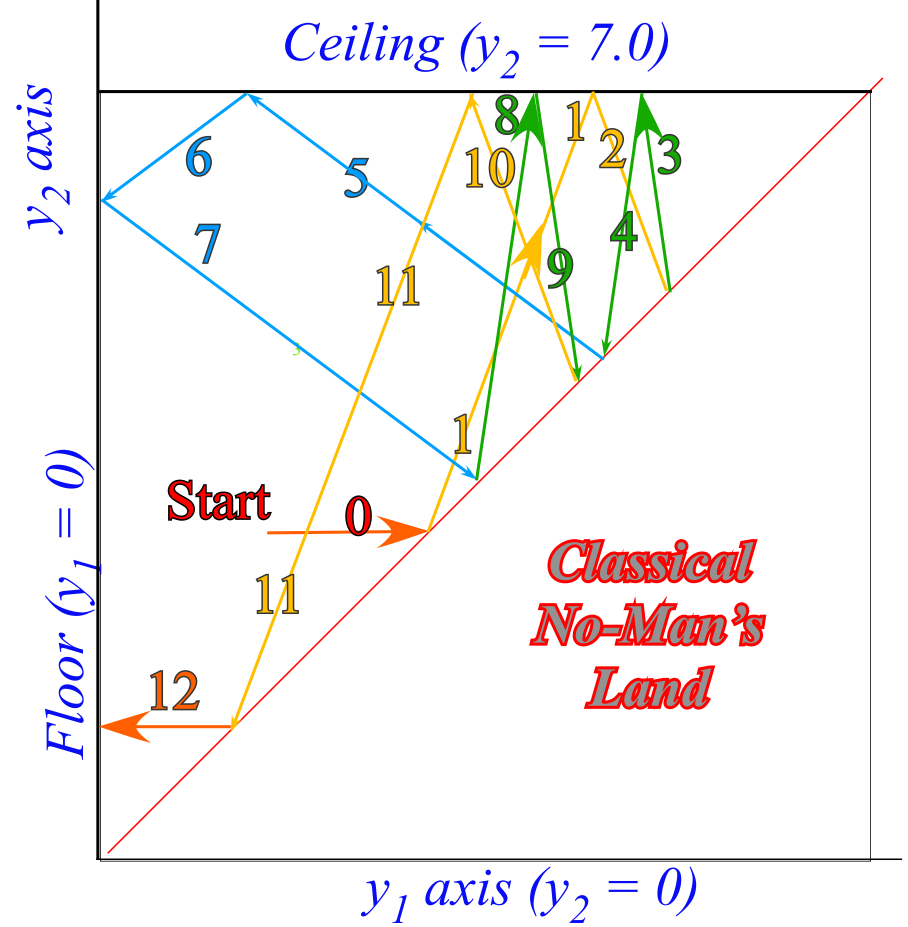
But, the y_1-y_2-steps 1 to 5 are only a scale model of steps 11 to 15. Step 4 hits floor but step 14 hits ceiling.


7
2,12
90°
80°
70°
18
60°
4 and 14
120°
5 but not 15 since ceiling intercedes
50°
40°
9
150°
10
30°
20°
10°
17
16
Start
0
-150°
1,11
-30°
-120°
15 after ceiling bounce
6
-60°
-90°
8
3,13

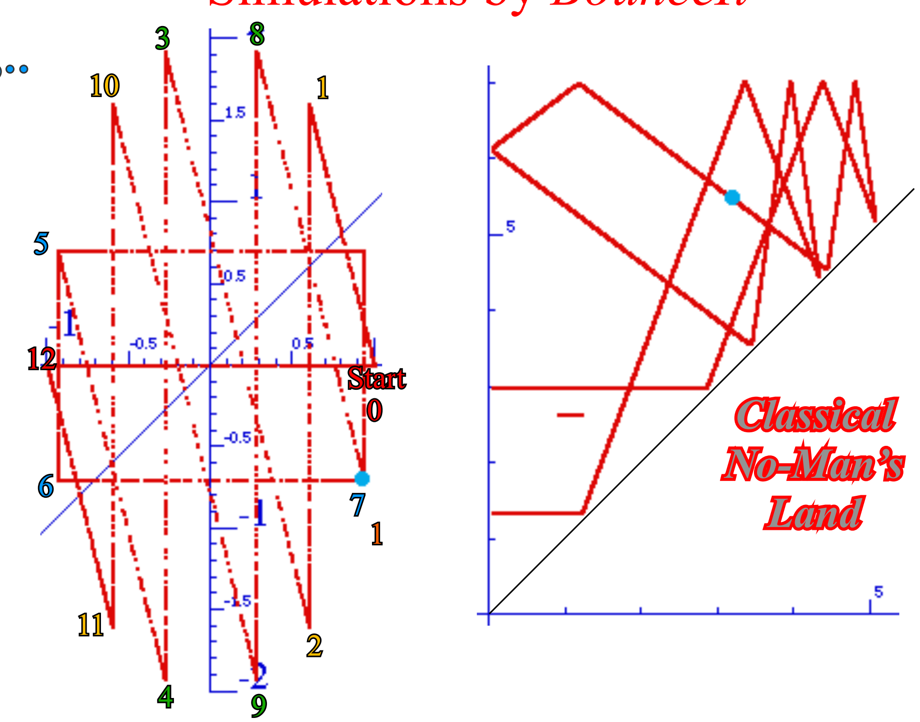
2nd part of Exercise 1.4.1 has pen-ball initial velocity values $v_1(0)=1$ and $v_2(0)=0$

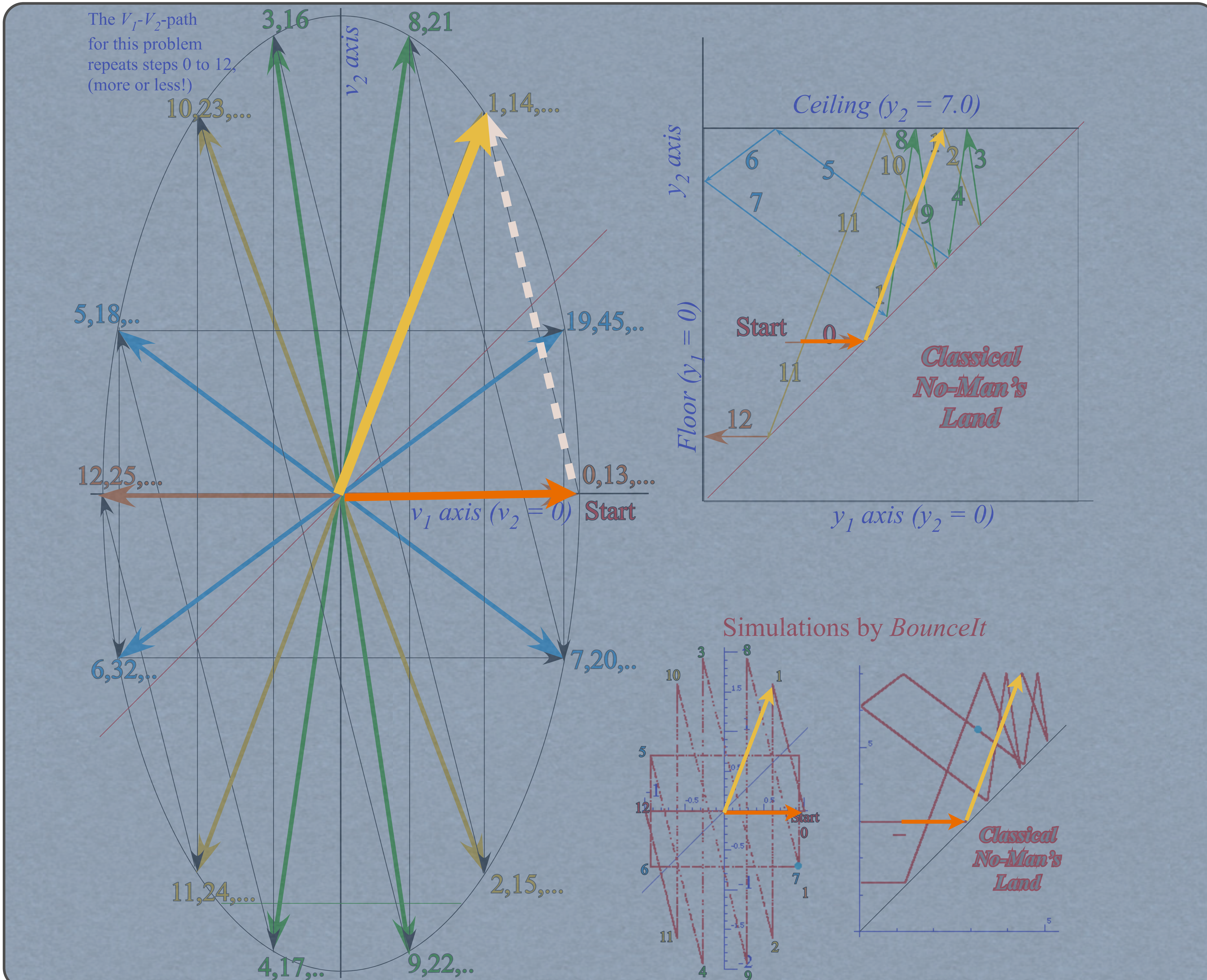
at: $x_1(0)=1.5$ and $x_2(0)=3.0$



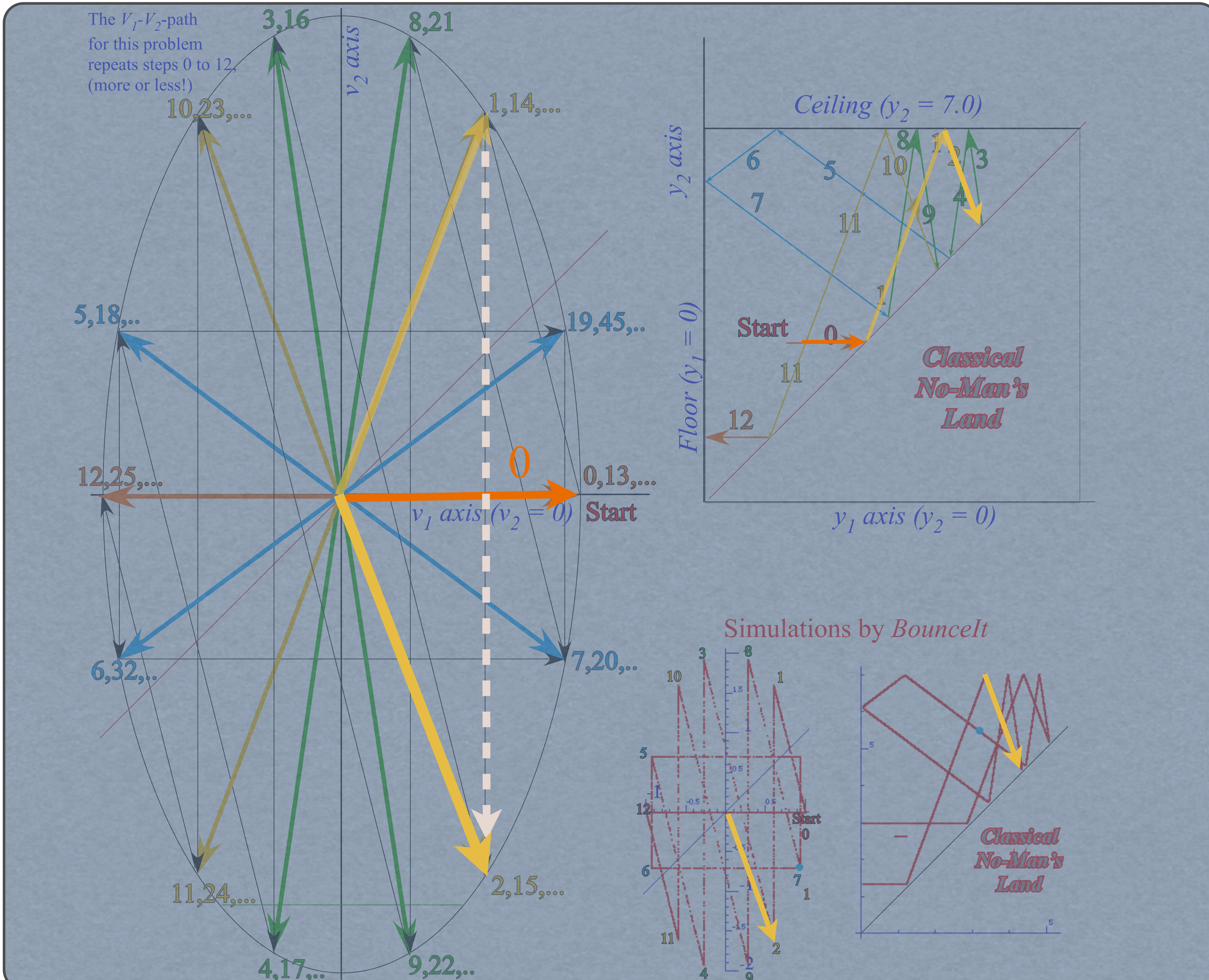


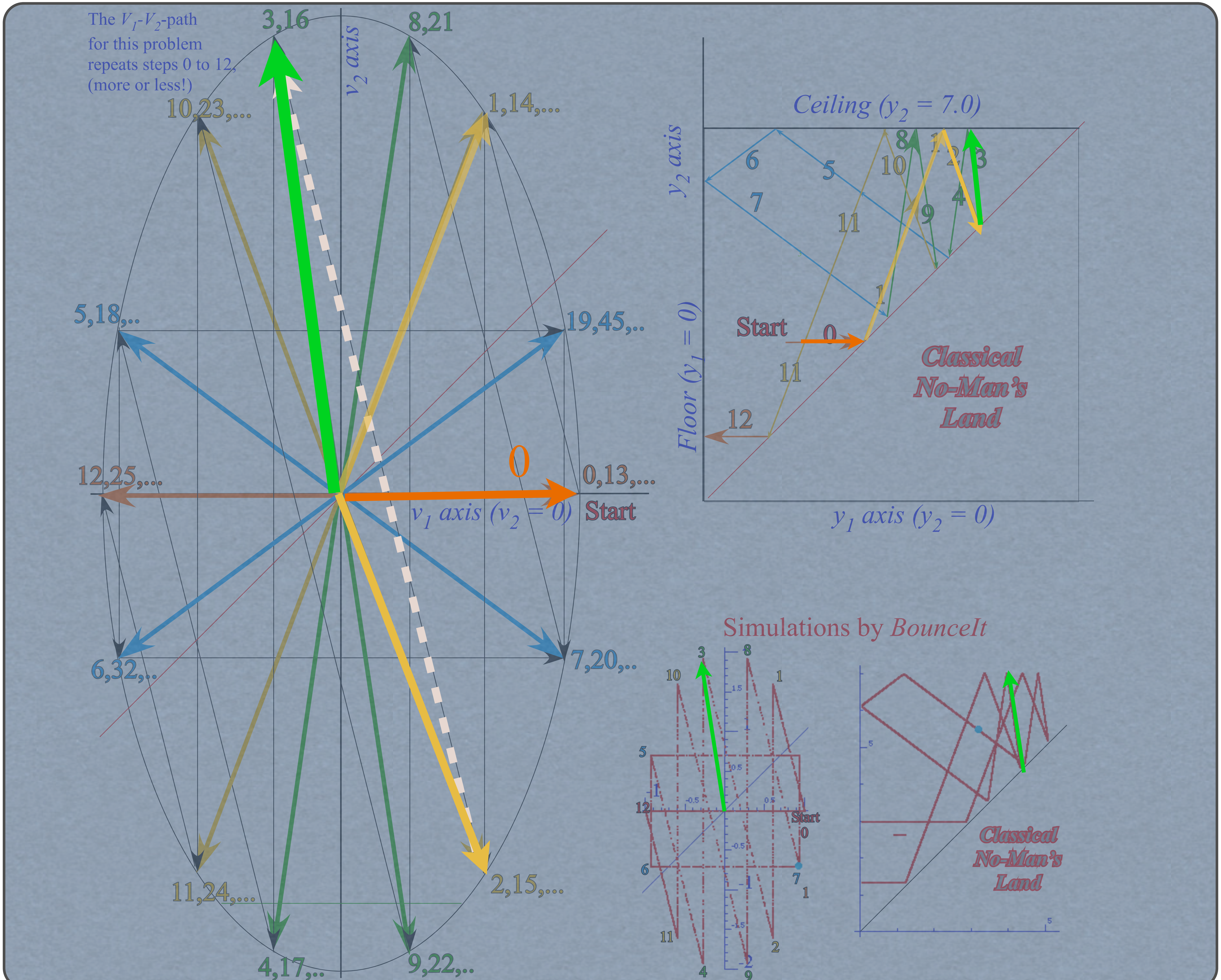
Simulations by *BounceIt*





The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)
3,16
8,21
v_2 axis
10,23,...
1,14,...
5,18,..
19,45,..
12,25,...
0,13,...
v_1 axis (v_2 = 0)
Start
6,32,..
7,20,..
11,24,...
2,15,...
4,17,..
9,22,..
Ceiling (y_2 = 7.0)
y_2 axis
Floor (y_1 = 0)
Start
Classical No-Man's Land
y_1 axis (y_2 = 0)
Simulations by *BounceIt*
Start
Classical No-Man's Land


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)
v_2 axis
3,16
8,21
1,14,...
10,23,...
5,18,..
19,45,..
12,25,...
0,13,...
0
Start
v_1 axis (v_2 = 0)
6,32,..
7,20,..
11,24,...
2,15,...
4,17,..
9,22,..
Ceiling (y_2 = 7.0)
y_2 axis
Floor (y_1 = 0)
y_1 axis (y_2 = 0)
Start
Classical
No-Man's
Land
Simulations by *BounceIt*
Classical
No-Man's
Land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(more or less!)
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Simulations by *BounceIt*
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No-Man's
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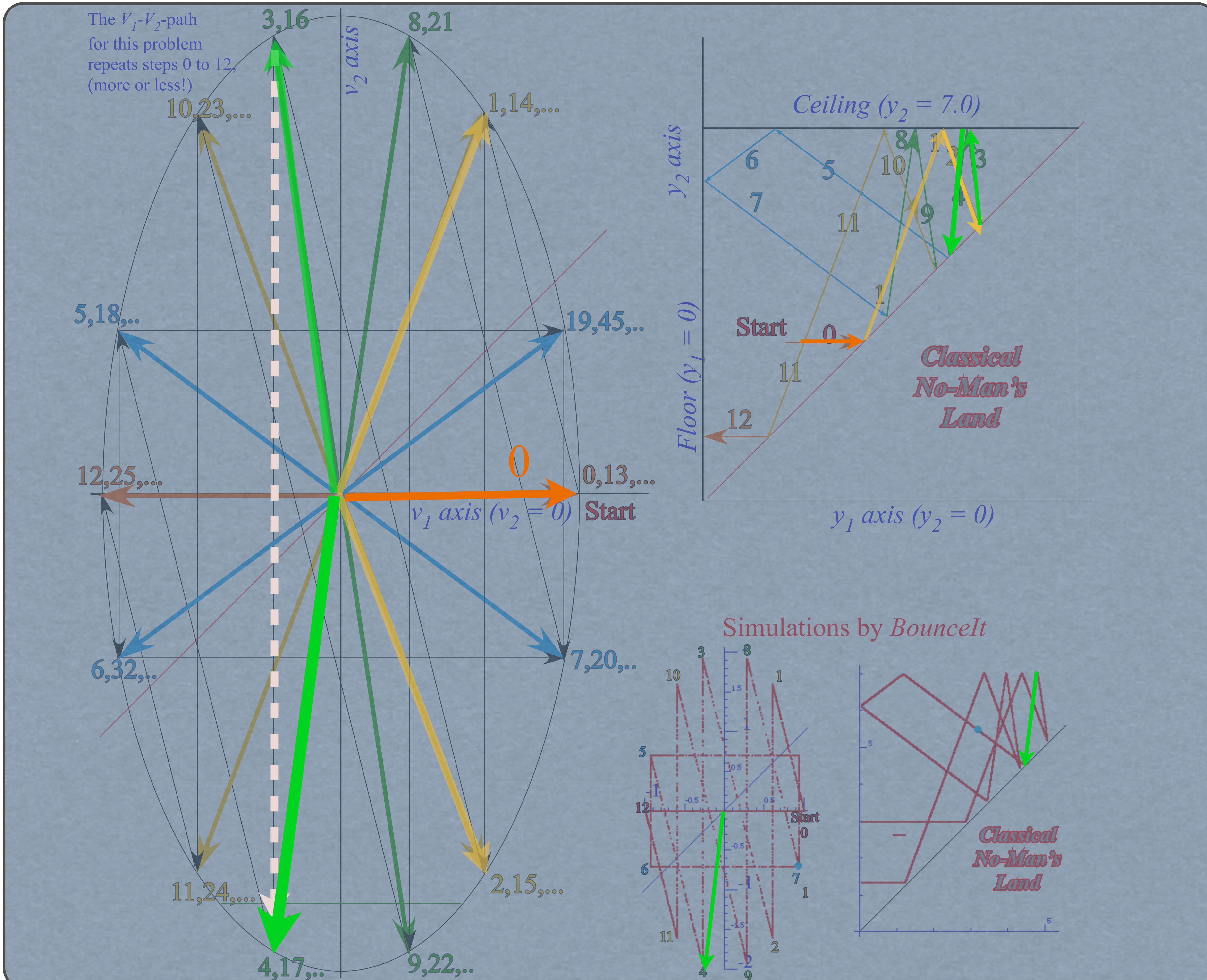
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8,21
v_2 axis
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10,23,...
5,18,..
19,45,..
0
12,25,...
0,13,...
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v_1 axis ($v_2 = 0$)
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11,24,...
2,15,...
4,17,..
9,22,..
Ceiling ($y_2 = 7.0$)
y_2 axis
Floor ($y_1 = 0$)
Start
Classical No-Man's Land
y_1 axis ($y_2 = 0$)
Simulations by *BounceIt*
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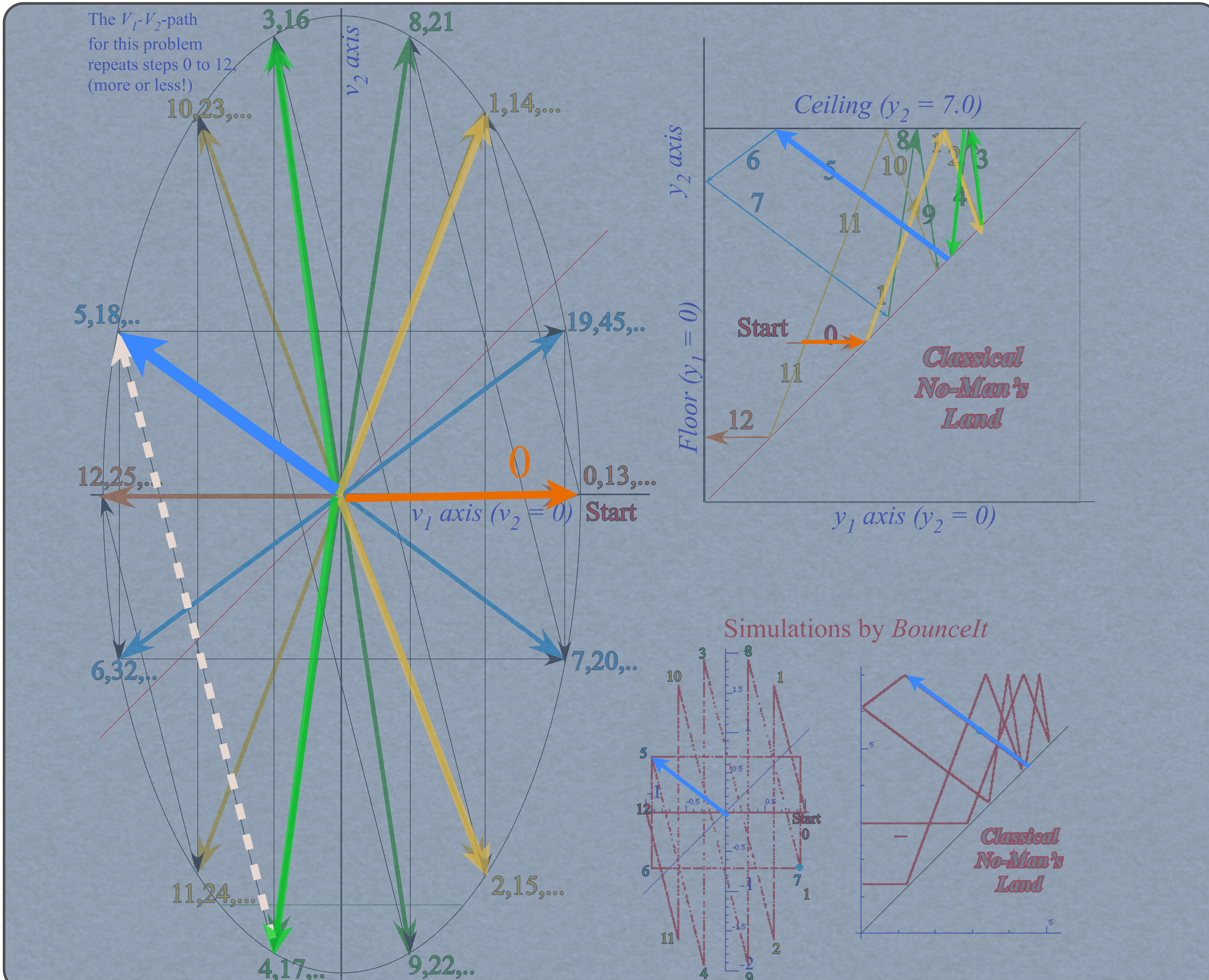
Classical No-Man's Land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repeats steps 0 to 12,
(more or less!)
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1,14,...
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19,45,..
12,25,..
0
0,13,...
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v_1 axis (v_2 = 0)
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Ceiling (y_2 = 7.0)
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Land
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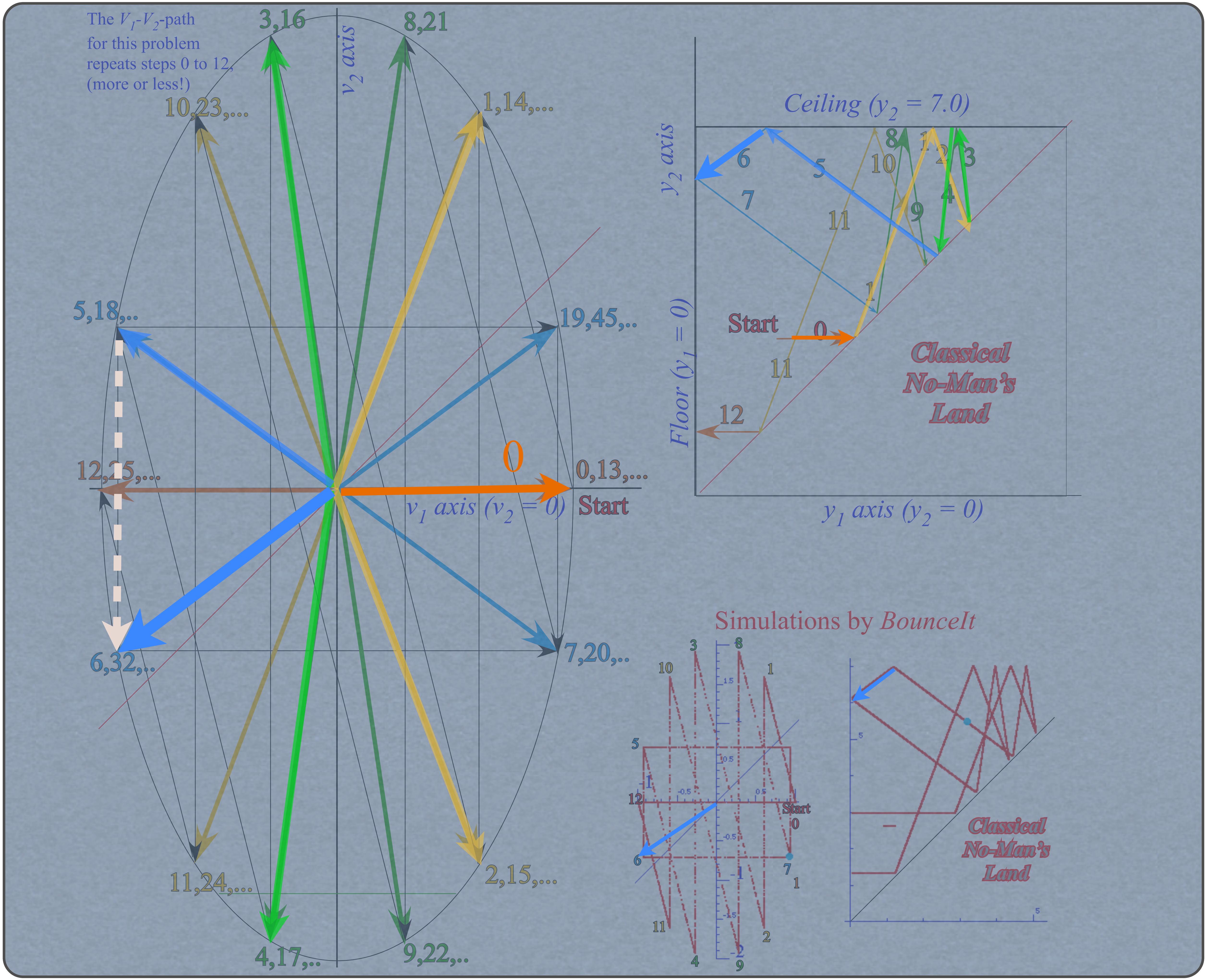
Land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10,23,...
1,14,...
5,18,..
19,45,..
0
12,25,...
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v_1 axis (v_2 = 0)
Start
6,32,..
7,20,..
11,24,...
2,15,...
4,17,..
9,22,..
Ceiling (y_2 = 7.0)
y_2 axis
Floor (y_1 = 0)
Start
Classical
No-Man's
Land
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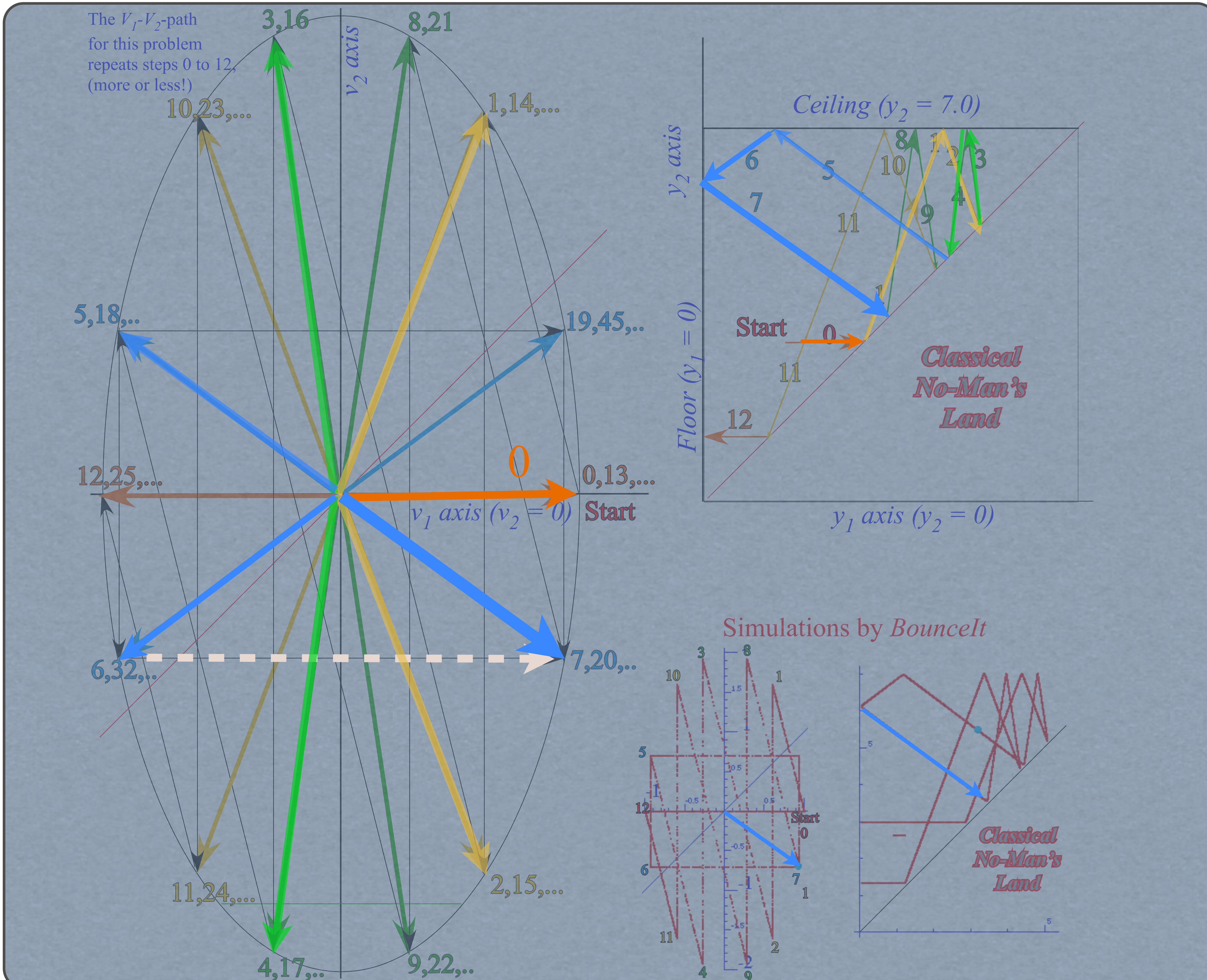
No-Man's
Land



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8,21
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1,14,...
5,18,..
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12,25,...
0,13,...
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v_1 axis (v_2 = 0)
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Start
Classical
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Land
Simulations by *BounceIt*
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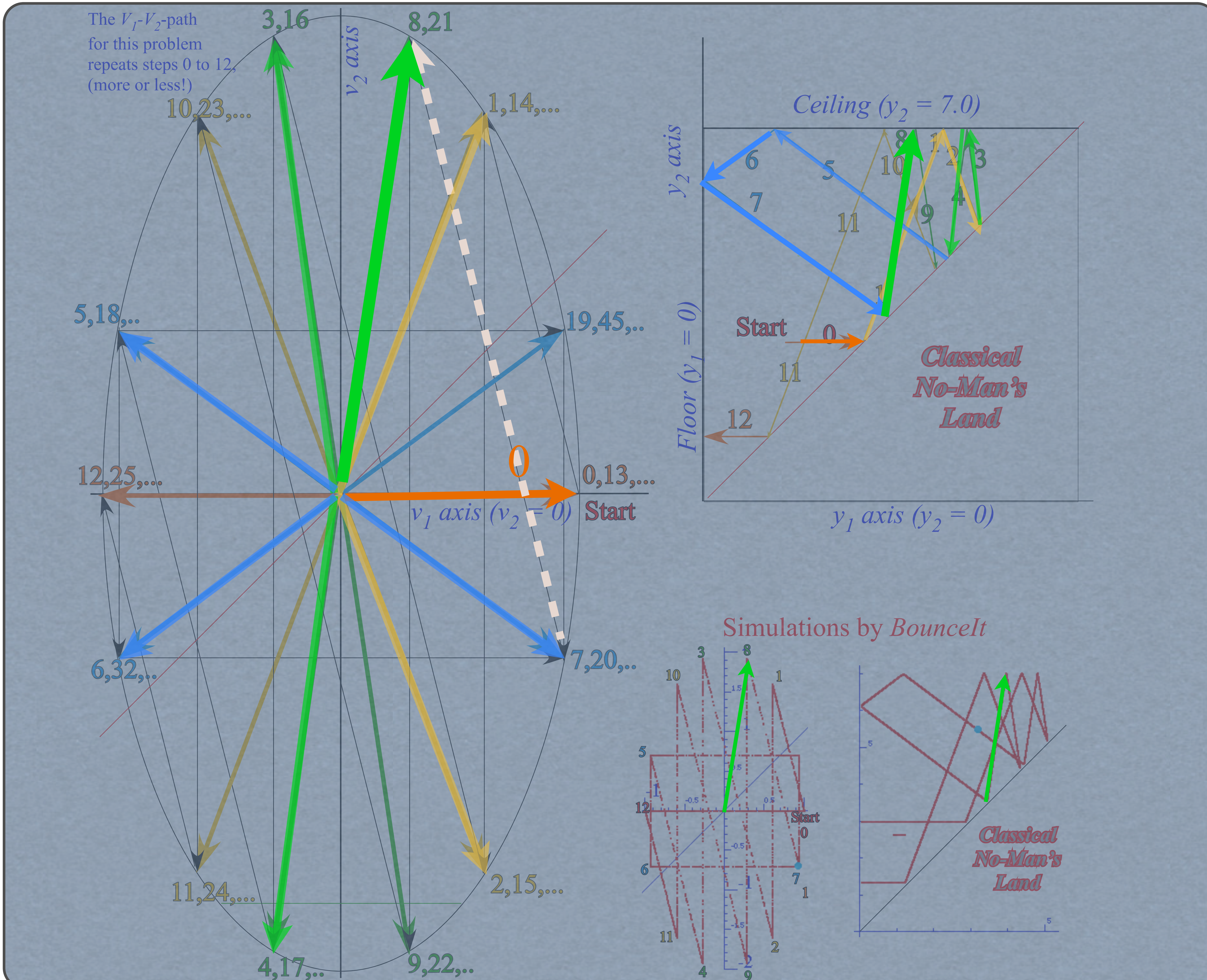

The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)
v_2 axis
v_1 axis (v_2 = 0)
Start
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1,14,...
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19,45,..
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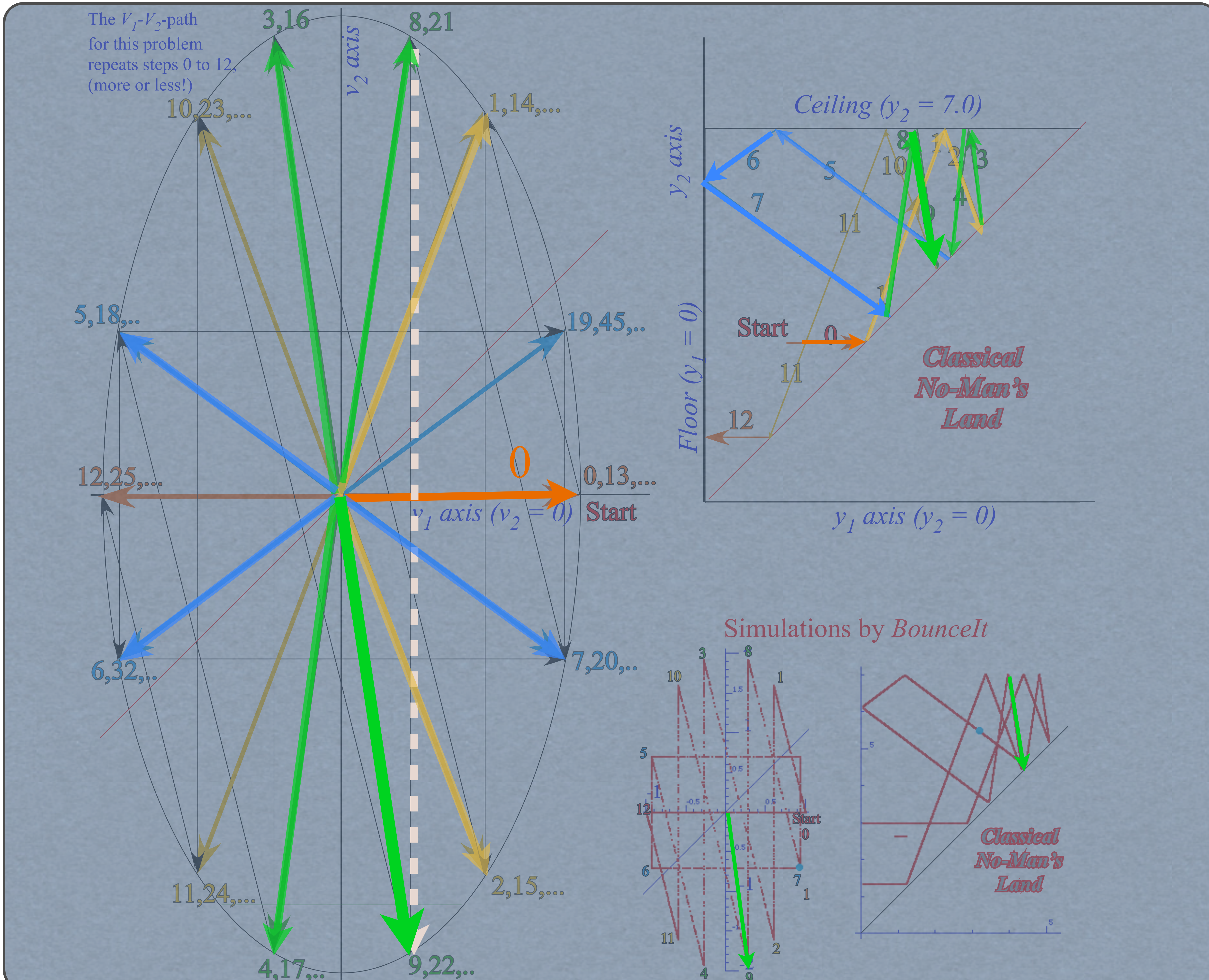
Floor (y_1 = 0)
y_1 axis (y_2 = 0)
Classical No-Man's Land
Simulations by *BounceIt*
Classical No-Man's Land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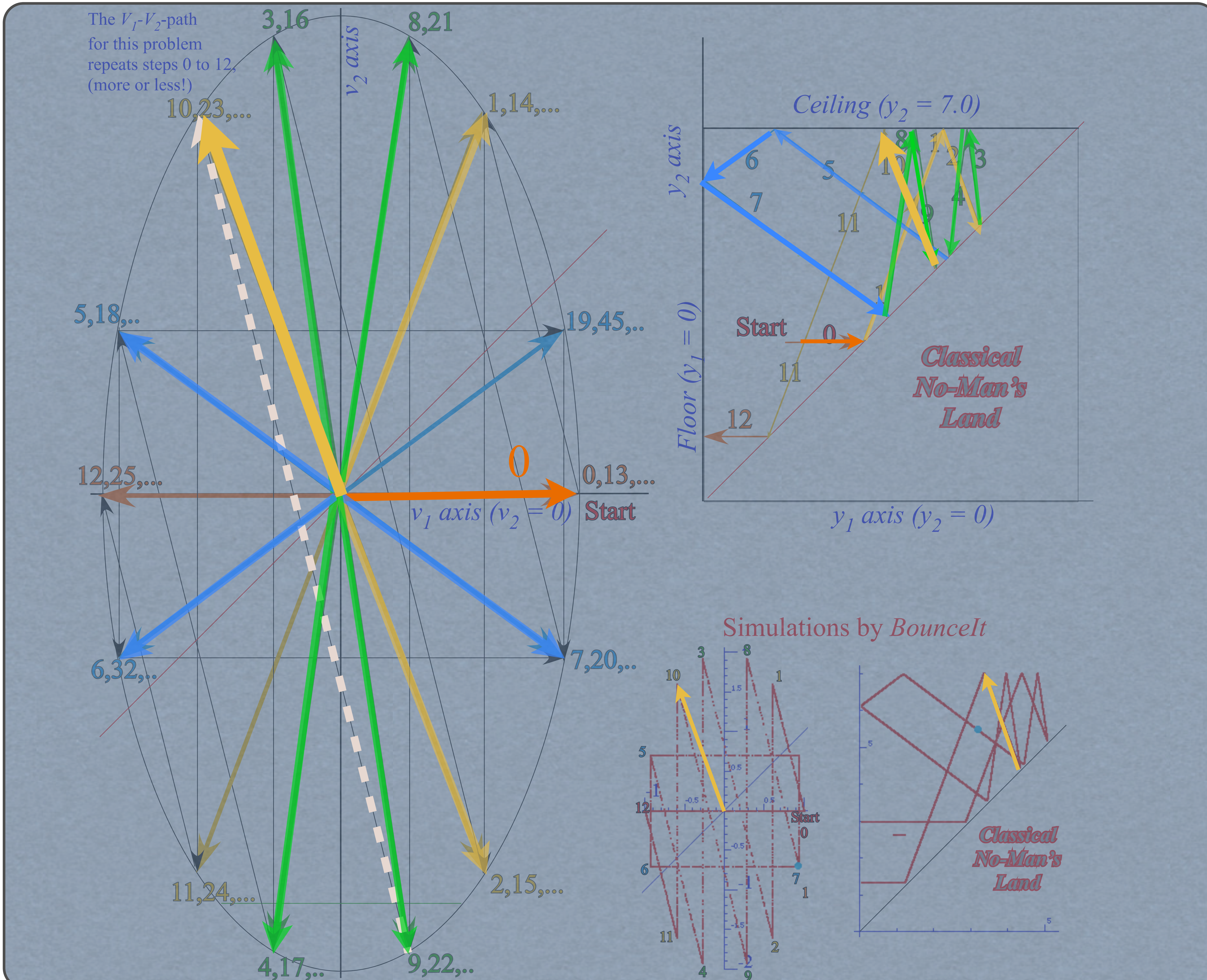
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Floor (y_1 = 0)
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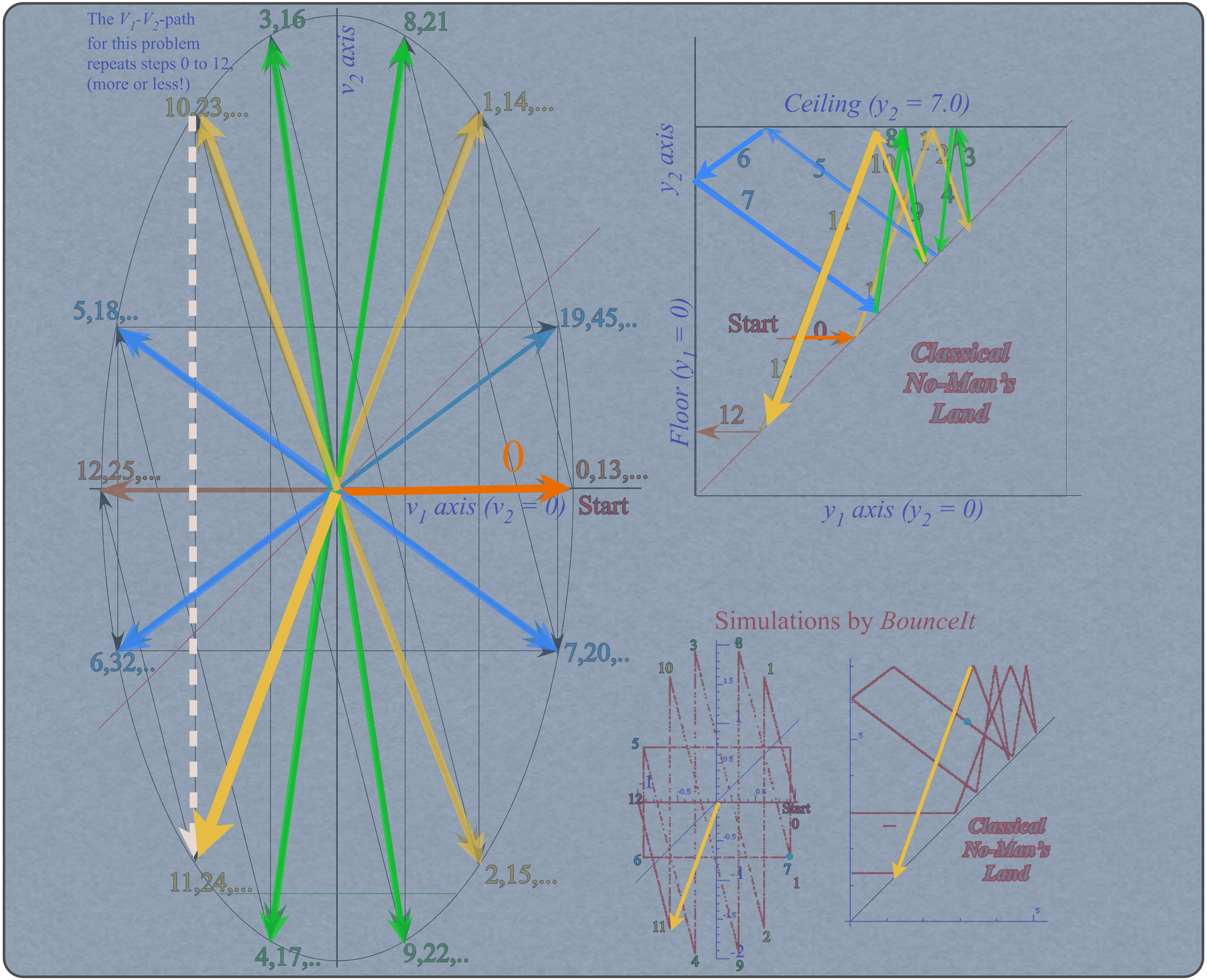
Start
Classical No-Man's Land
Simulations by *BounceIt*
Classical No-Man's Land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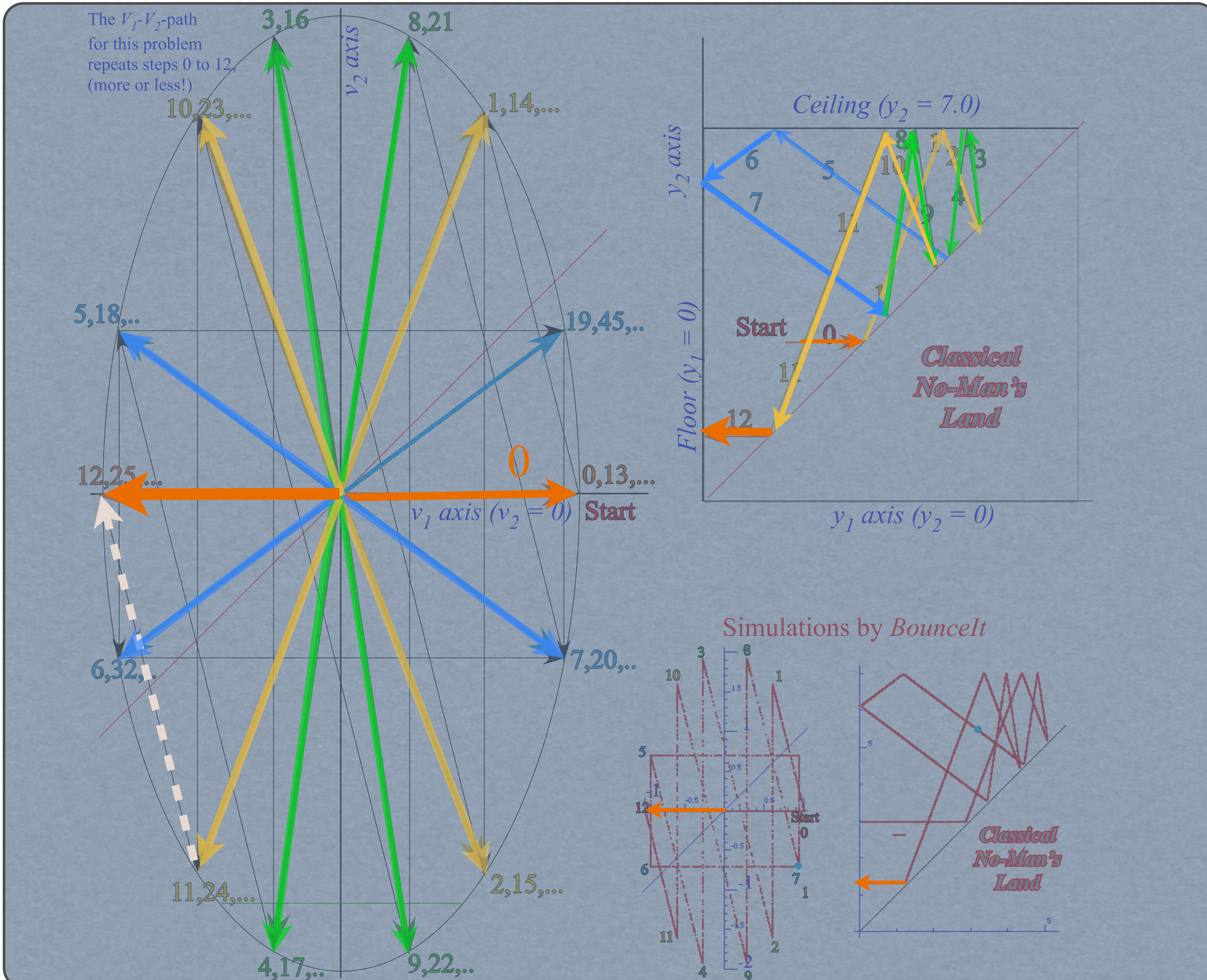
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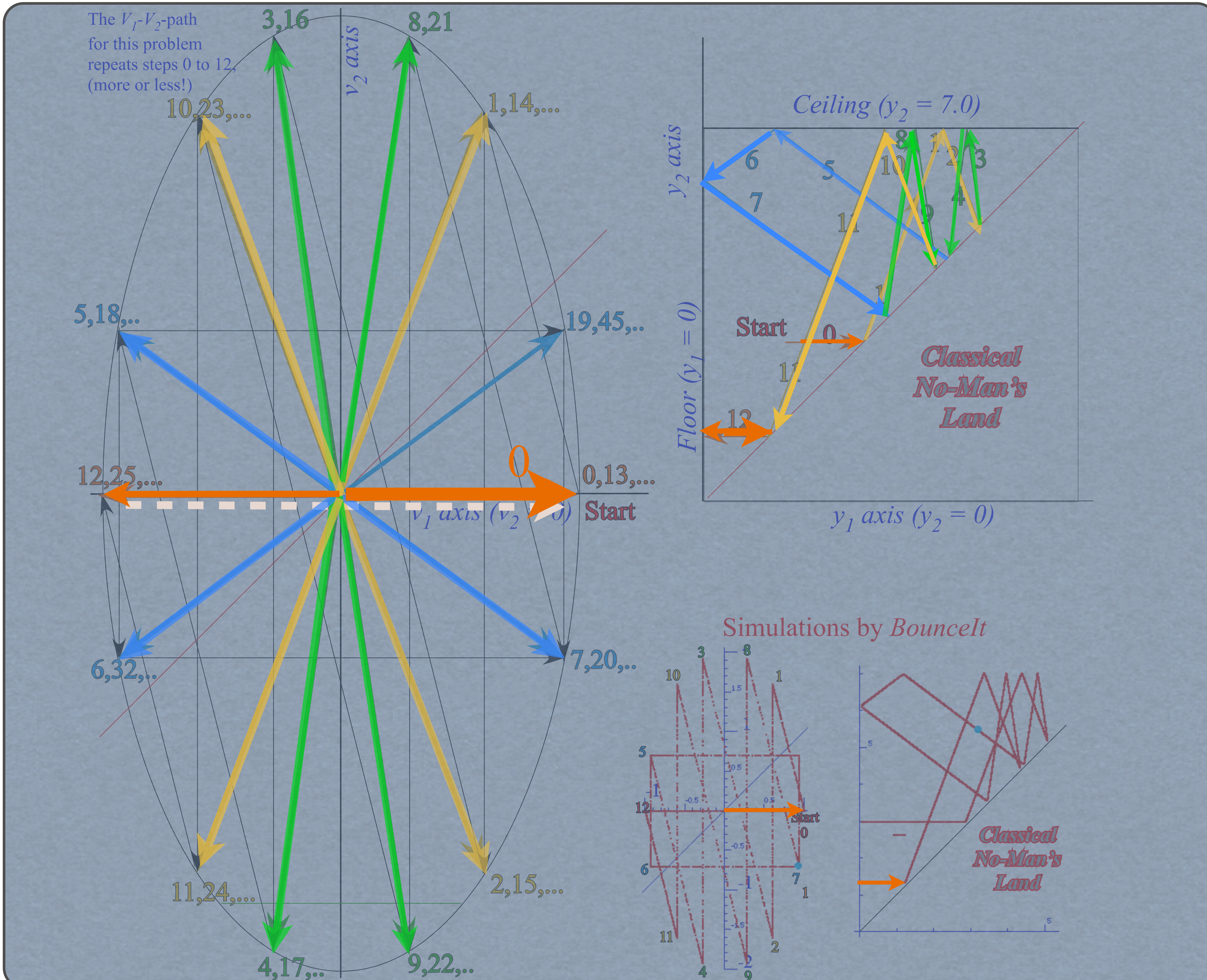
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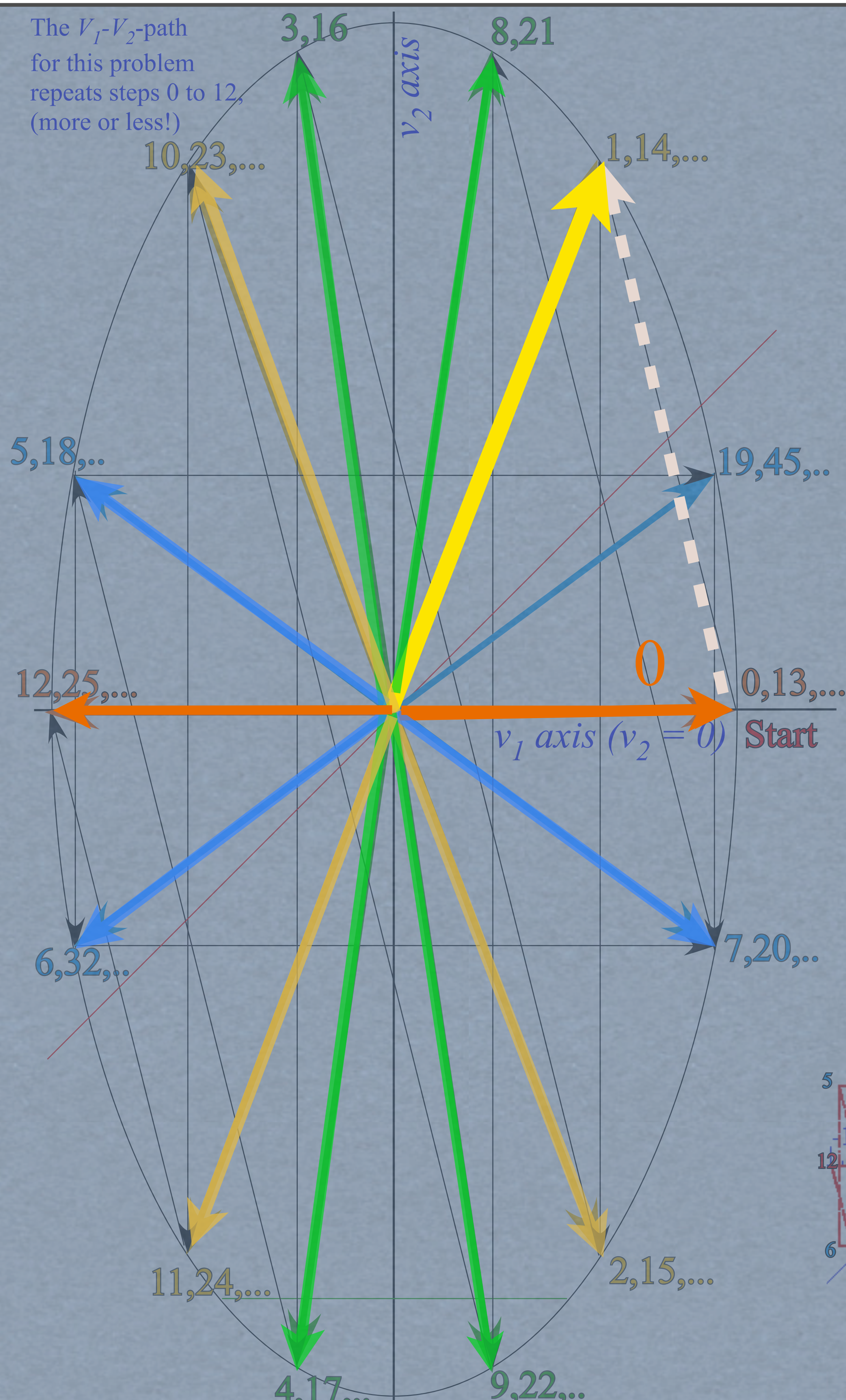
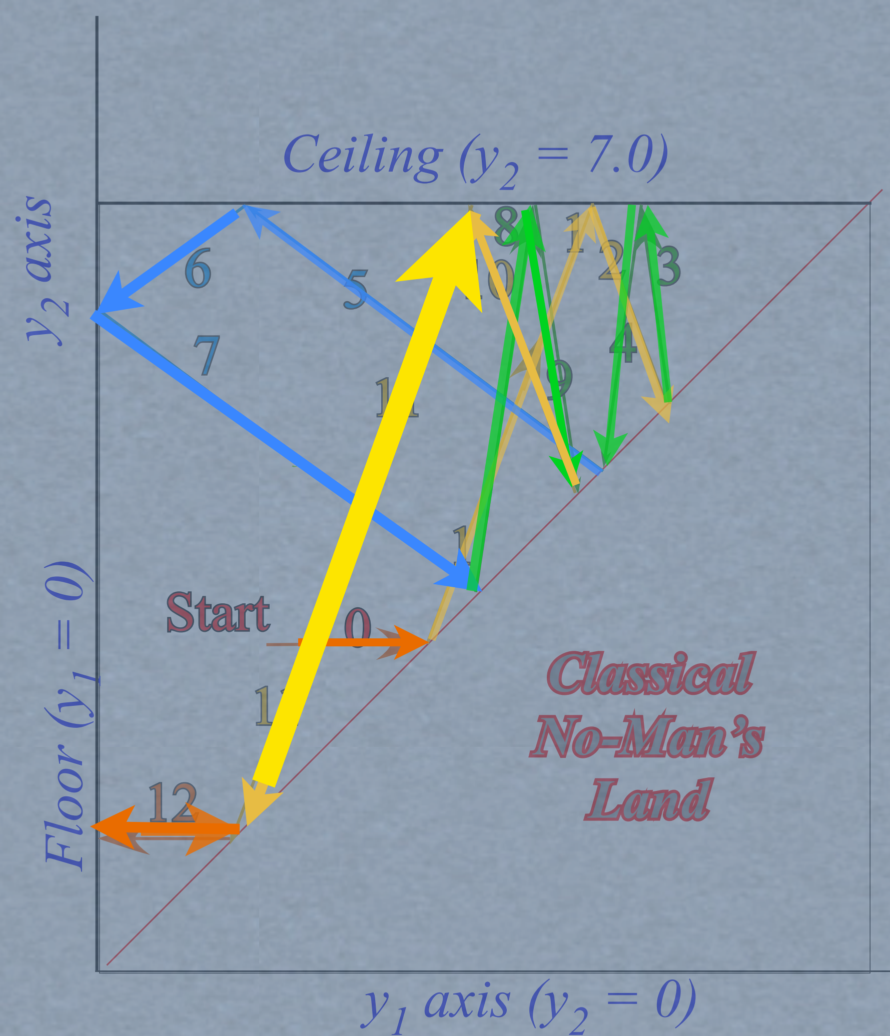
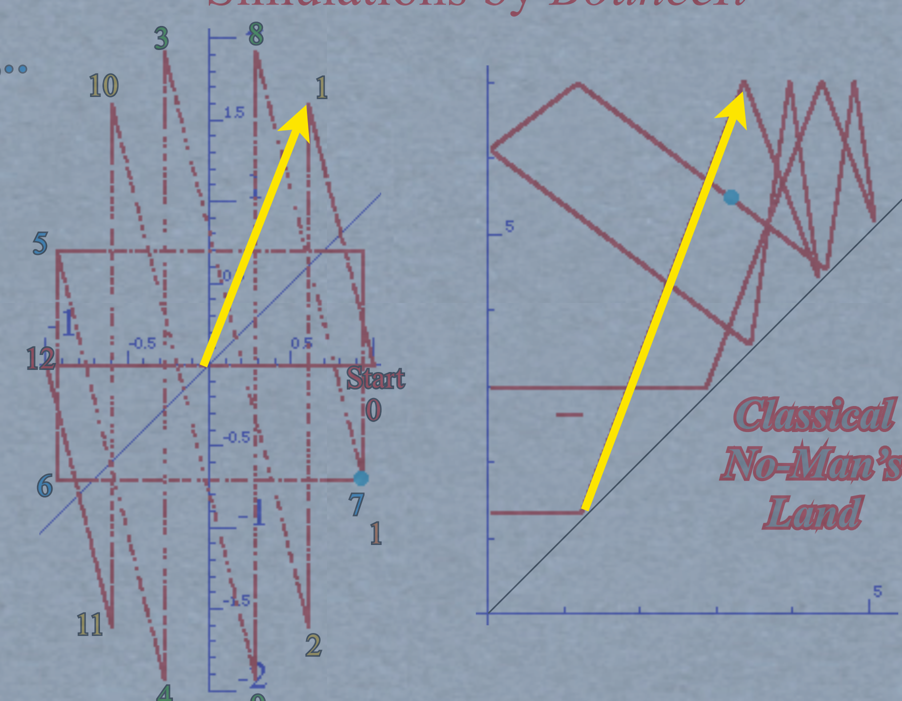
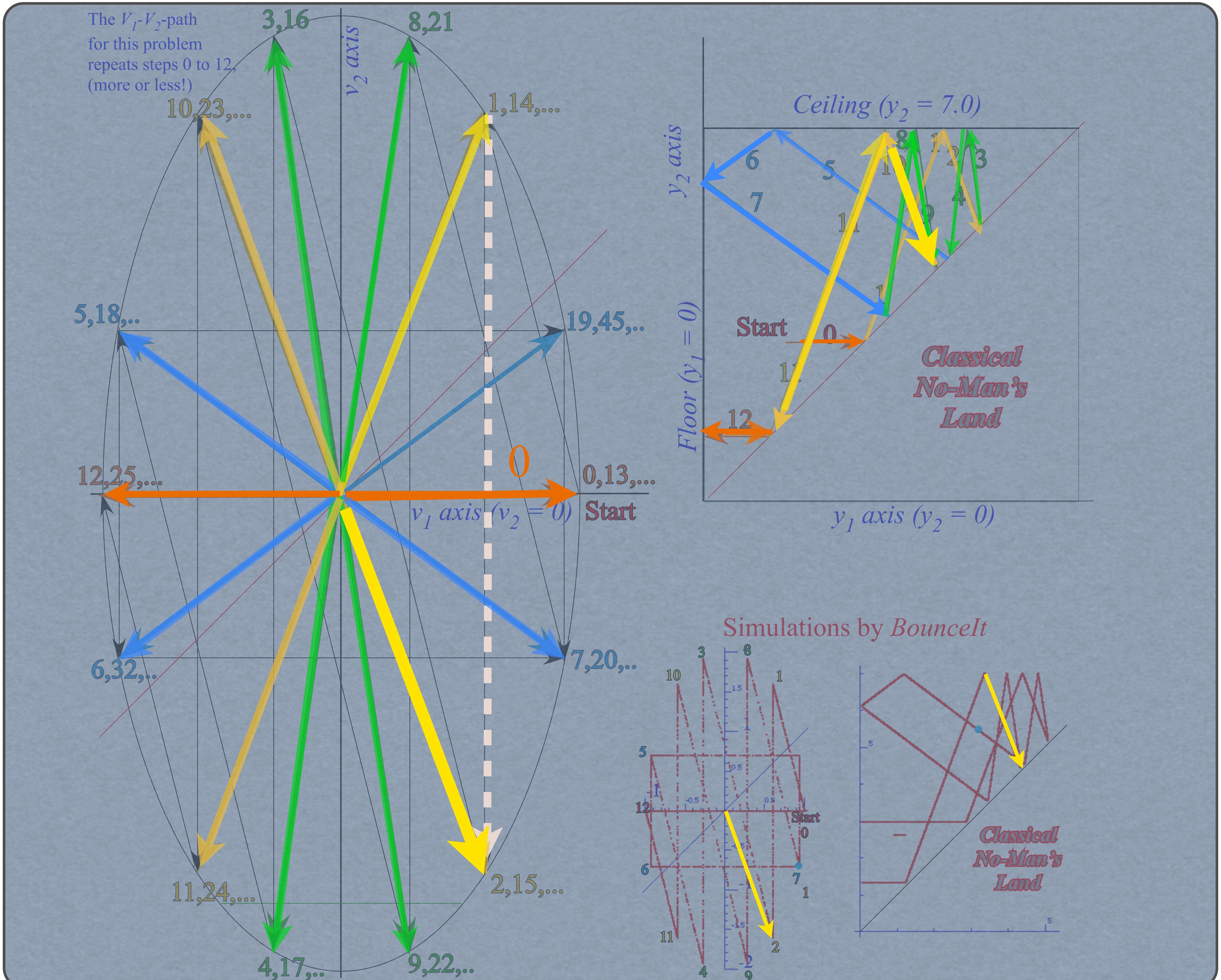




Simulations by *BounceIt*





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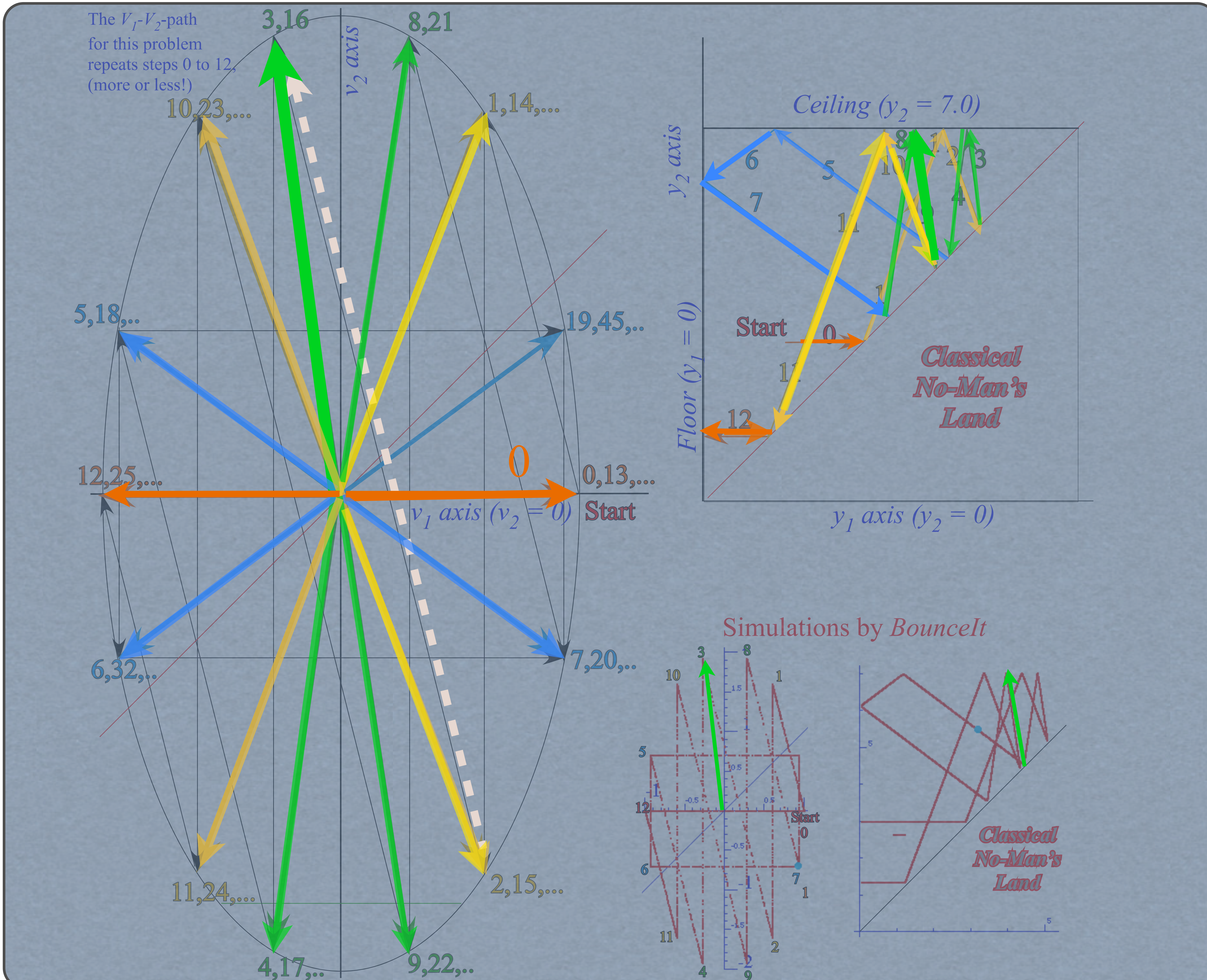

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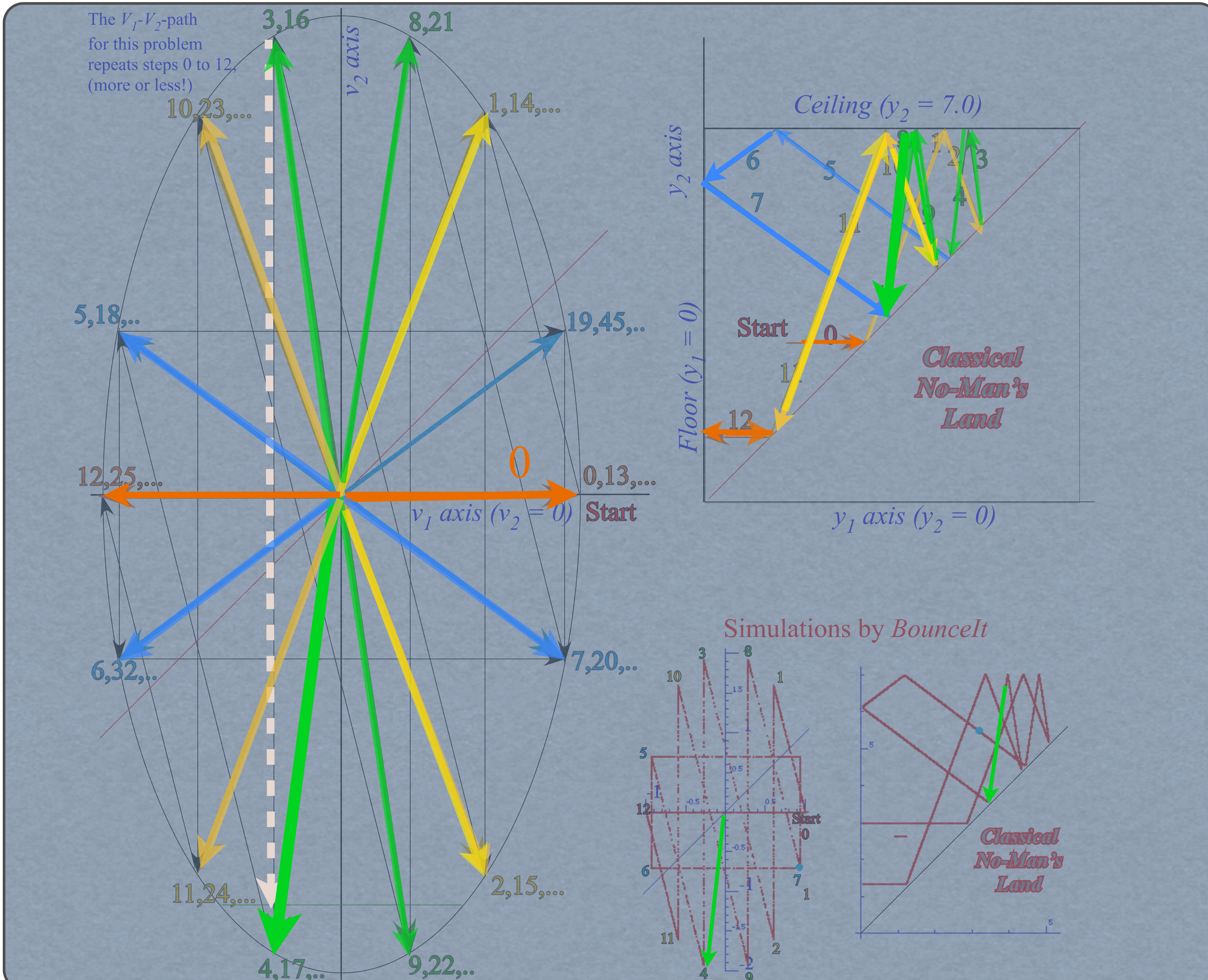
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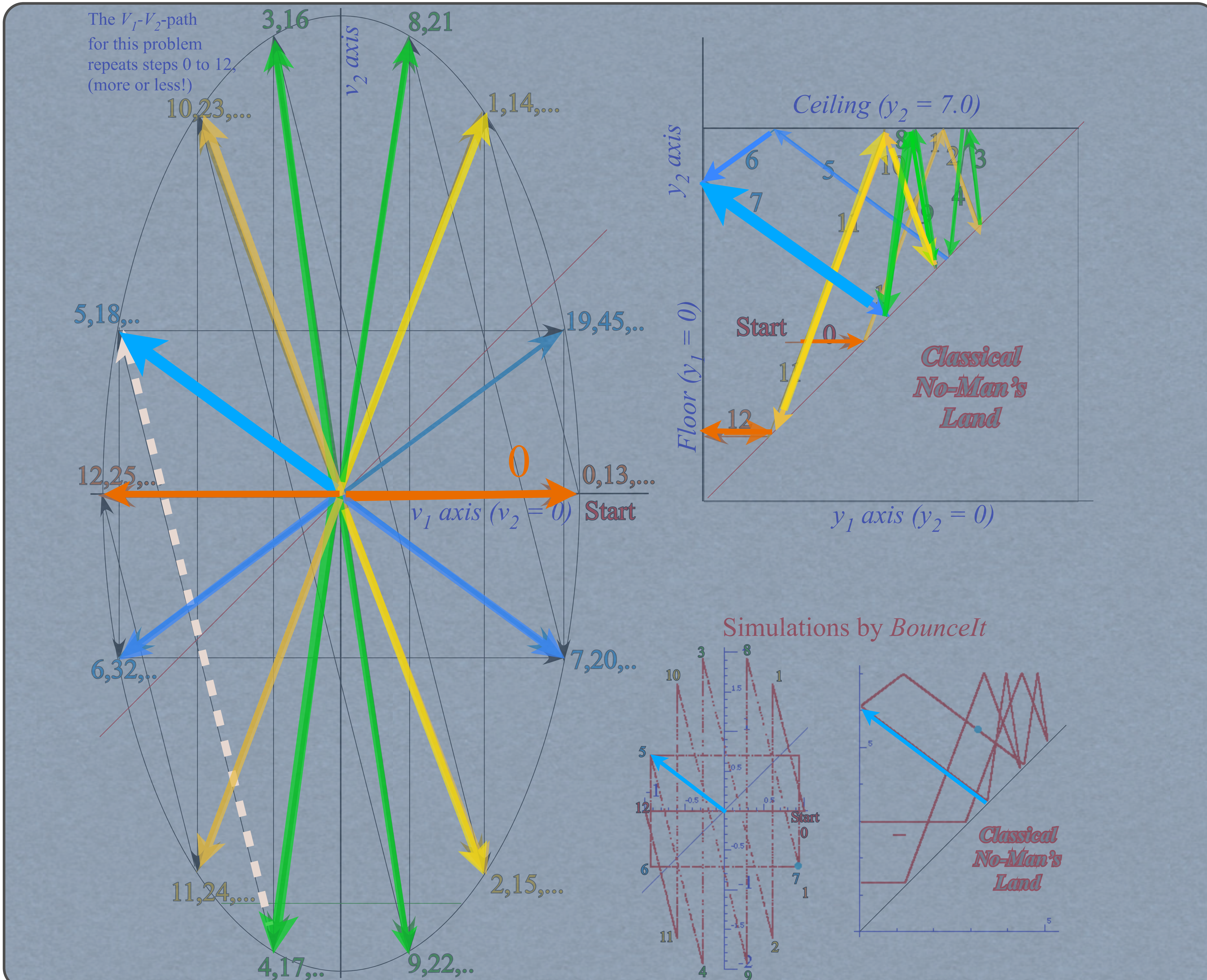
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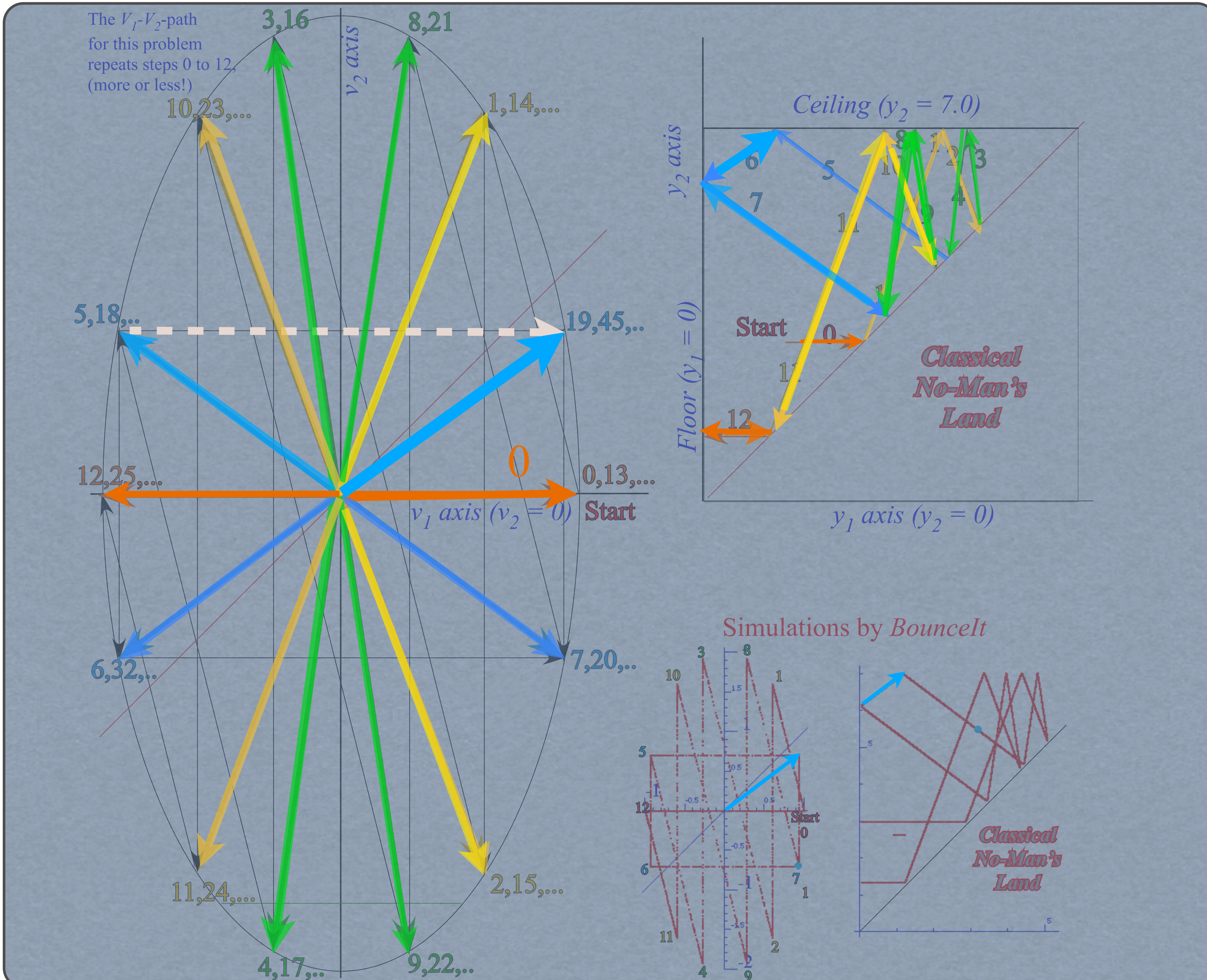

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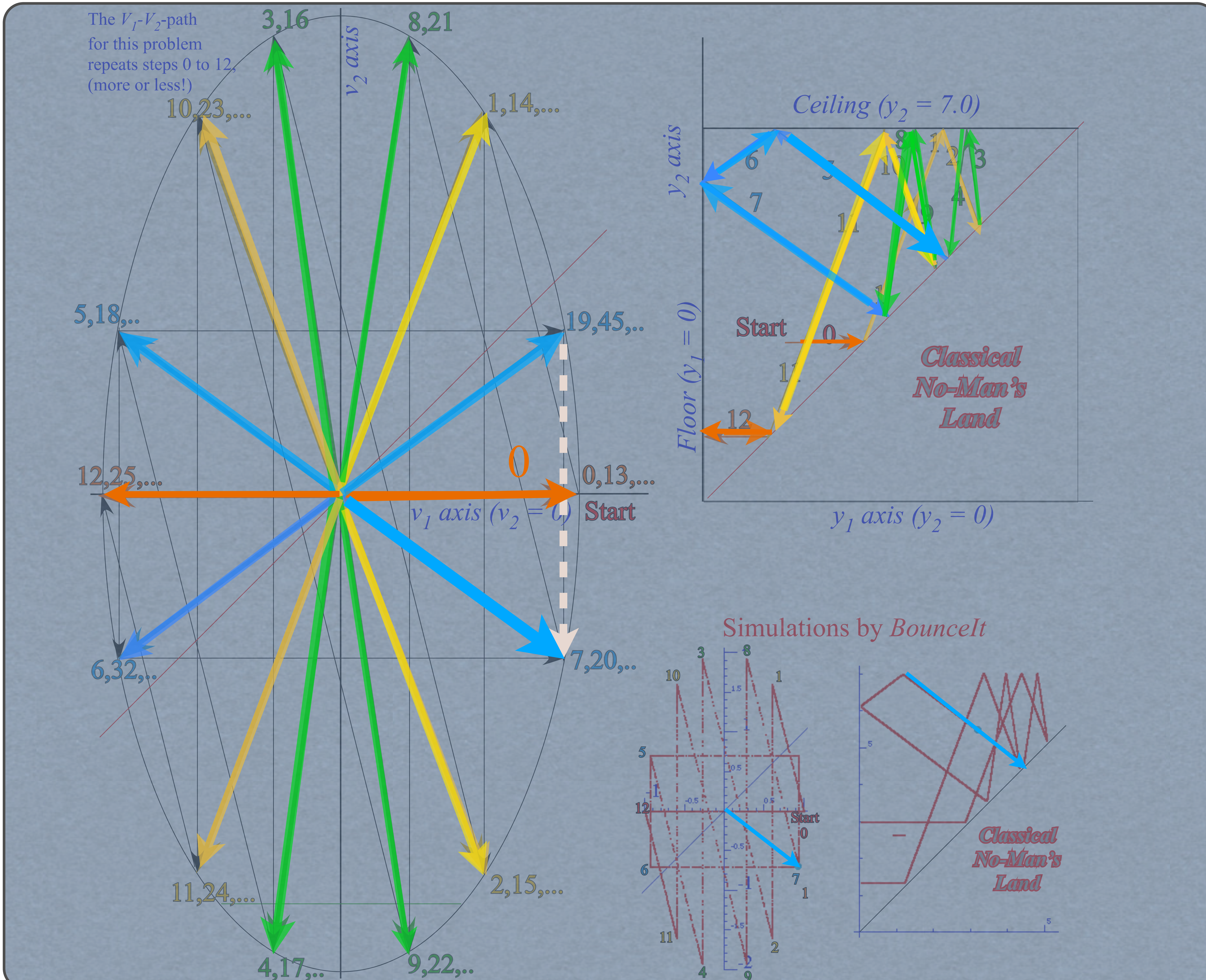
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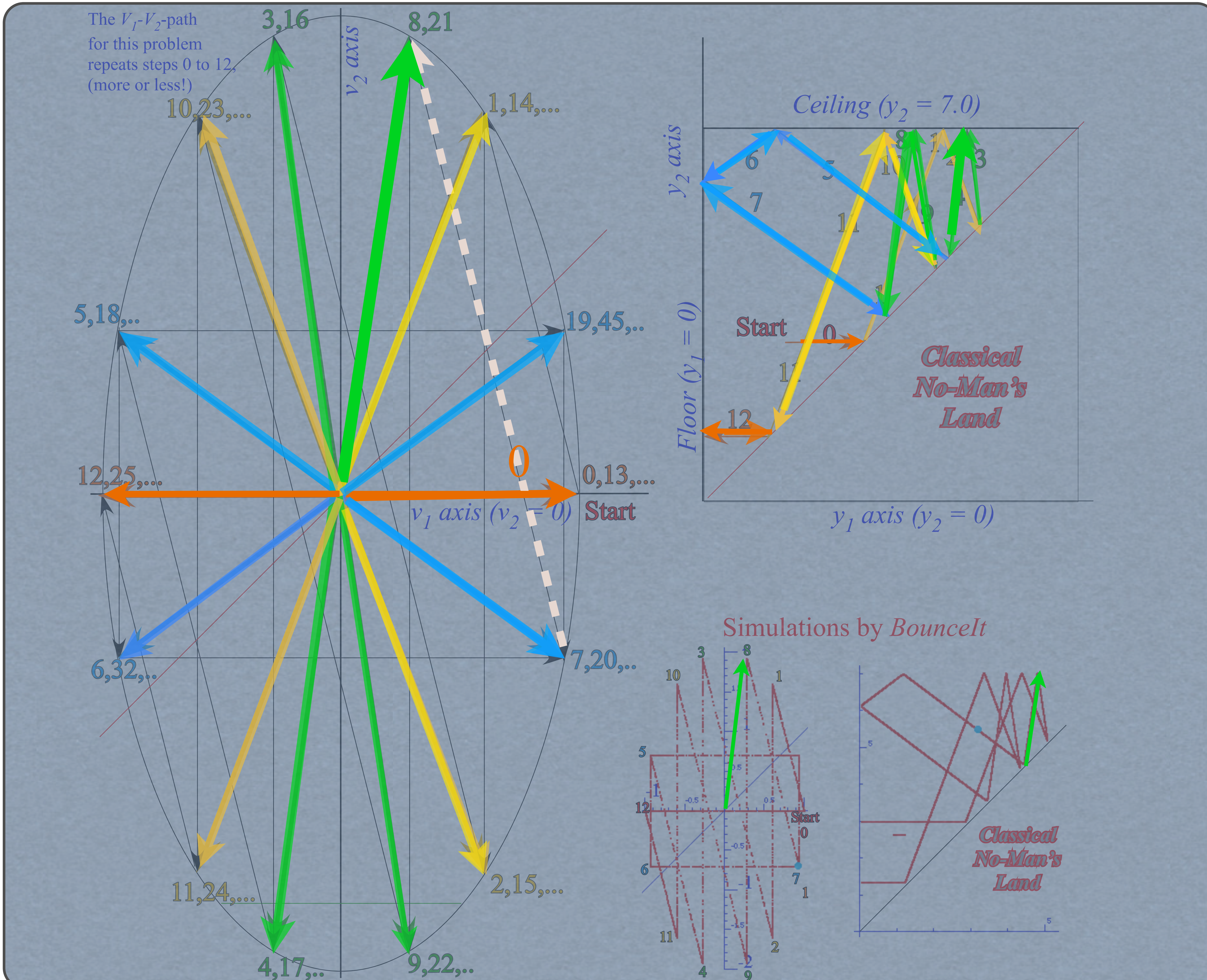
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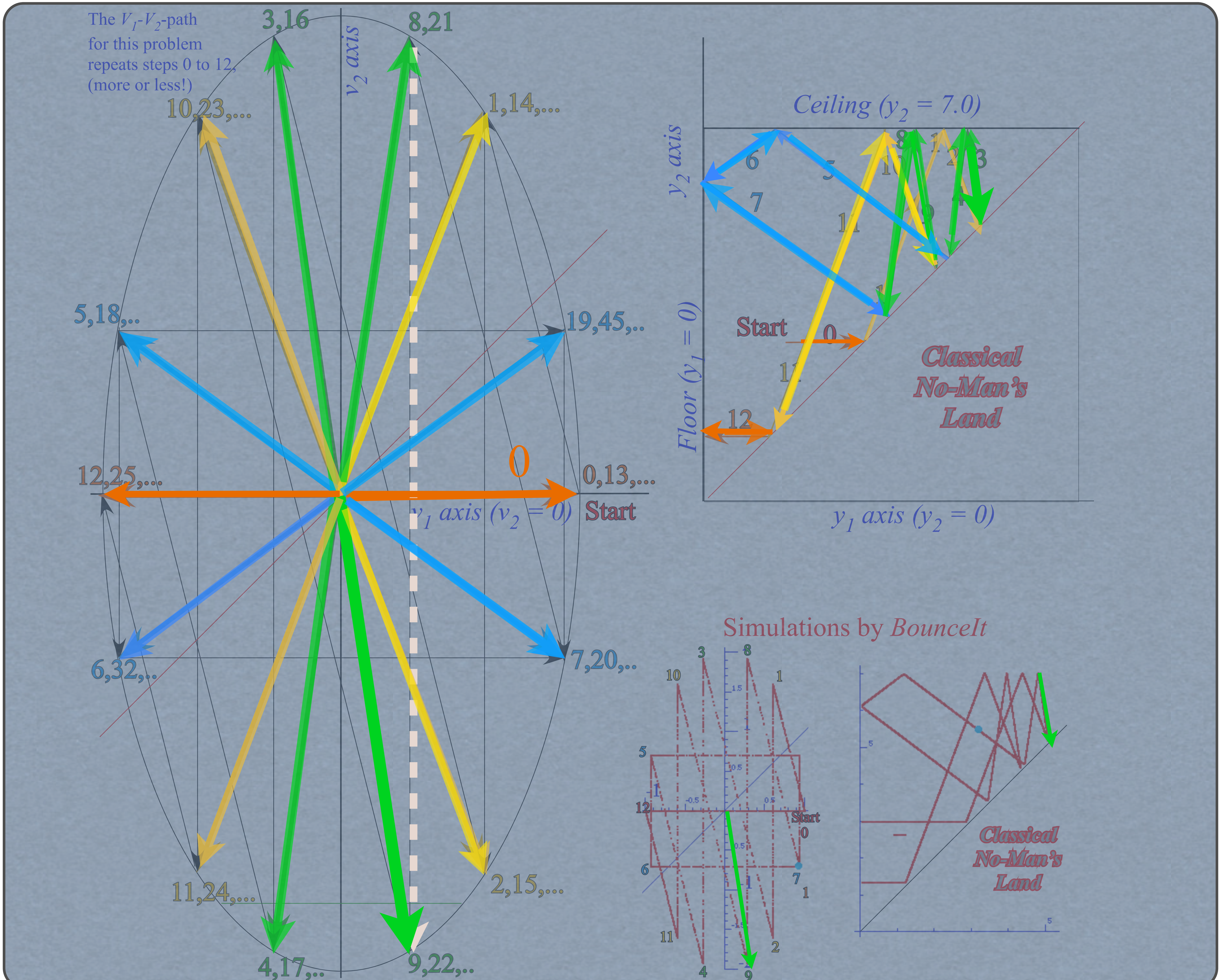
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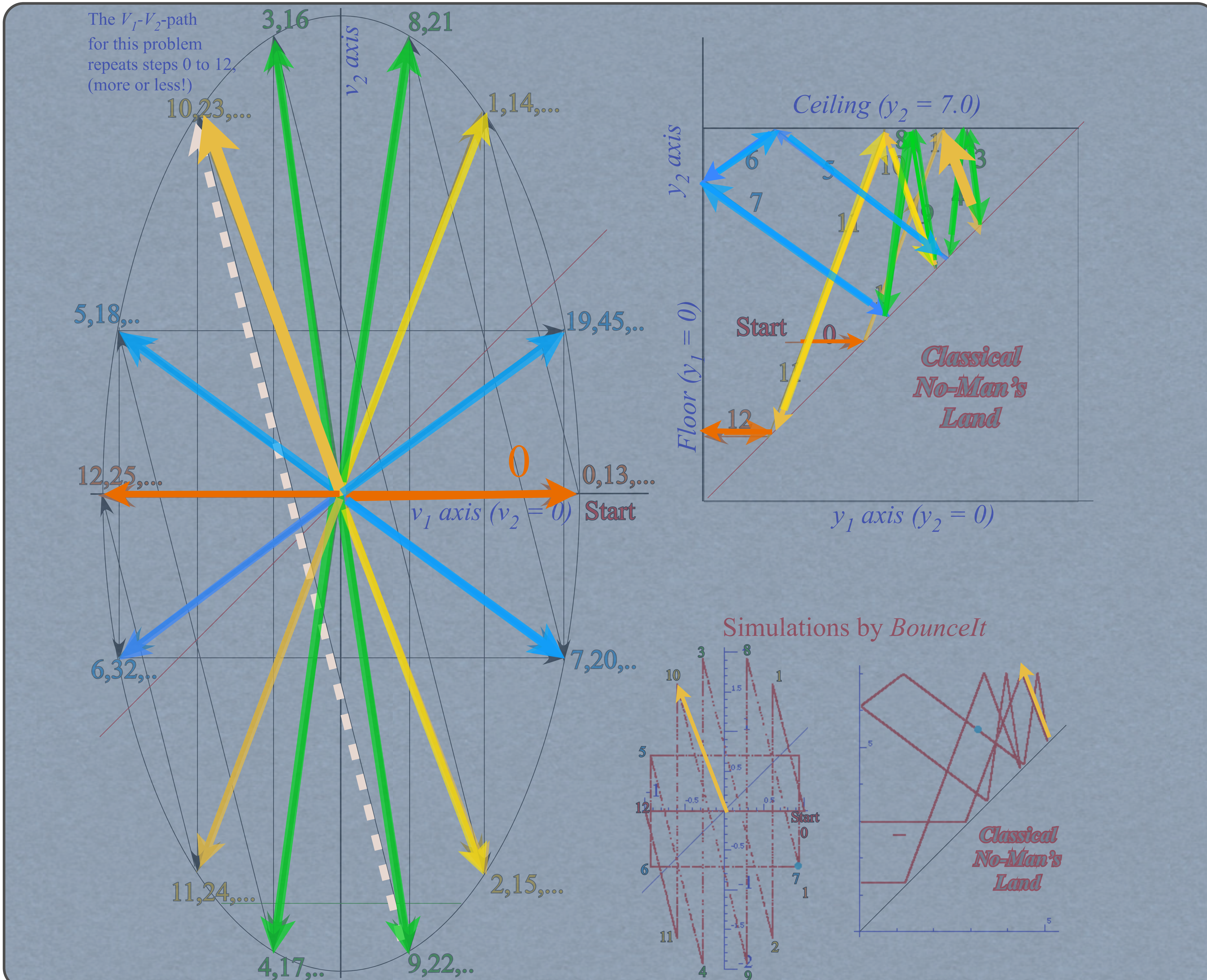
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12
Classical
No-Man's
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No-Man's
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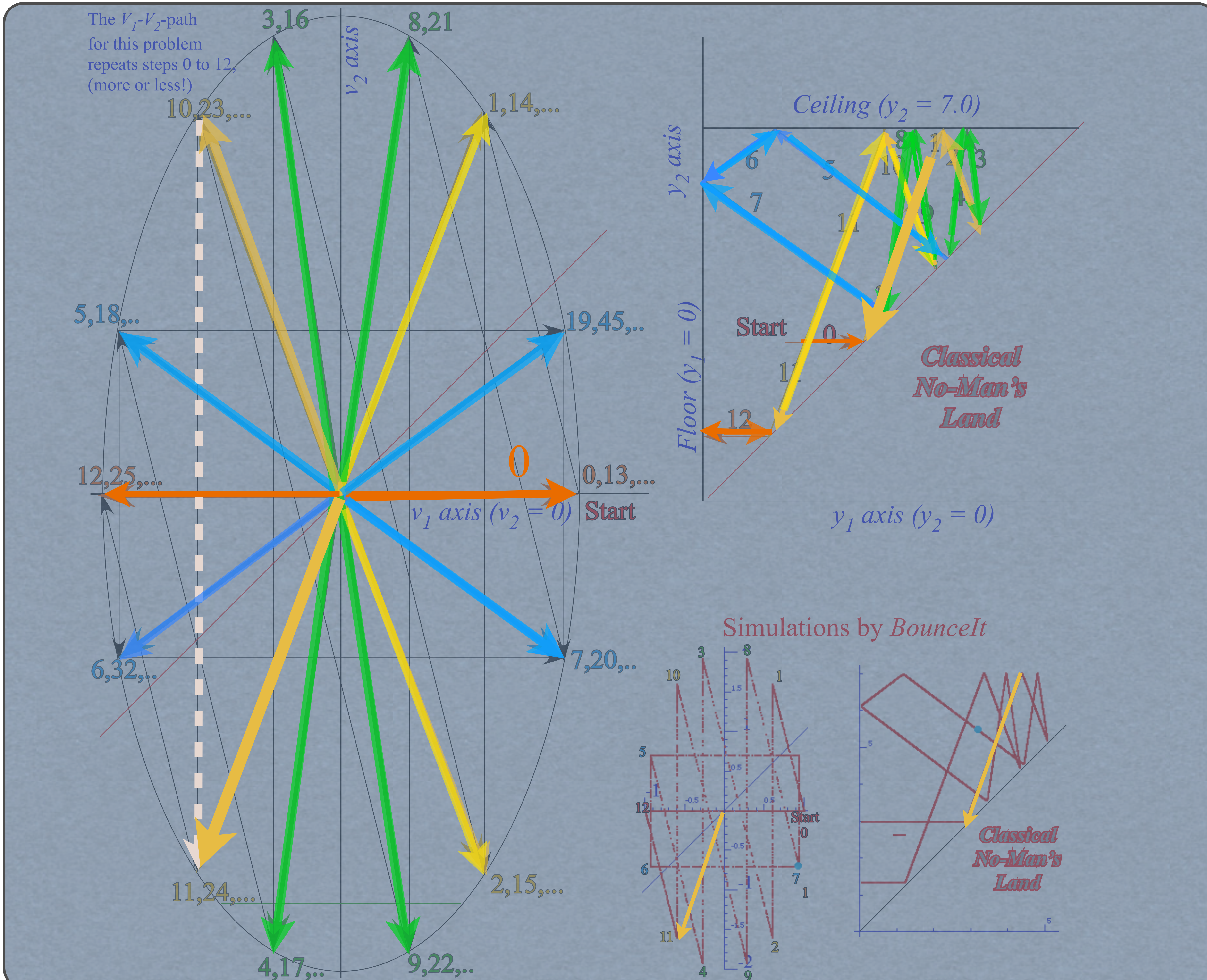
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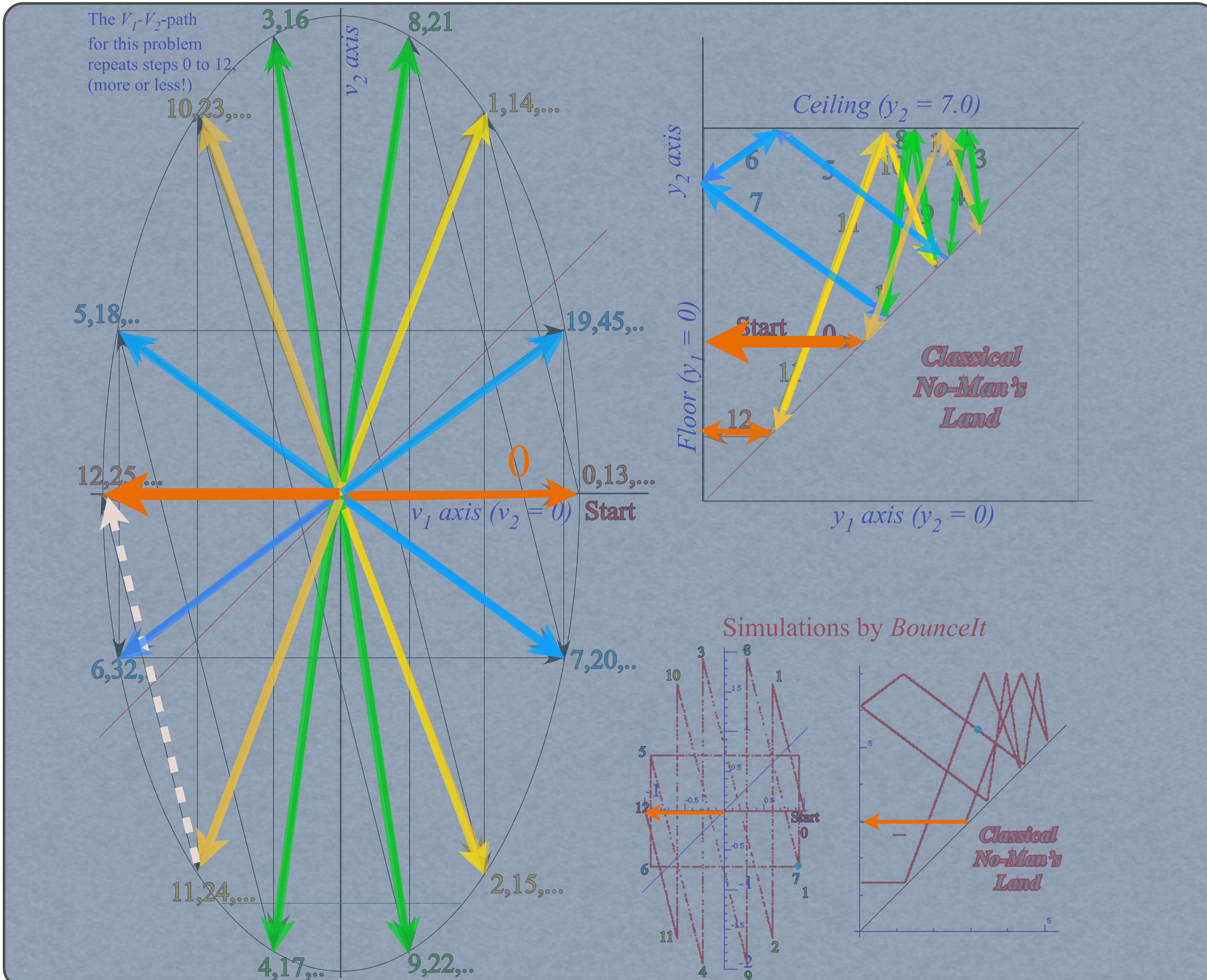

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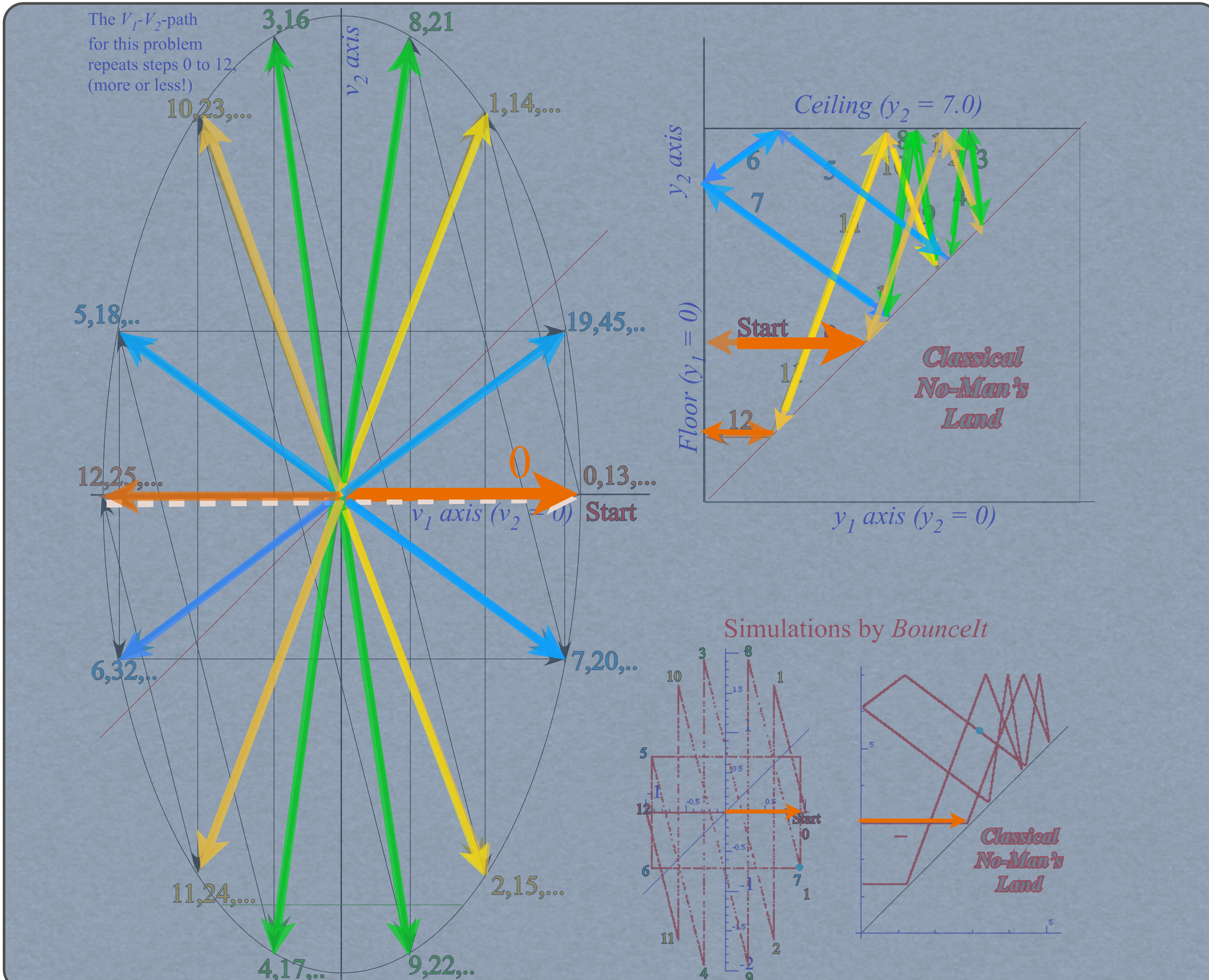

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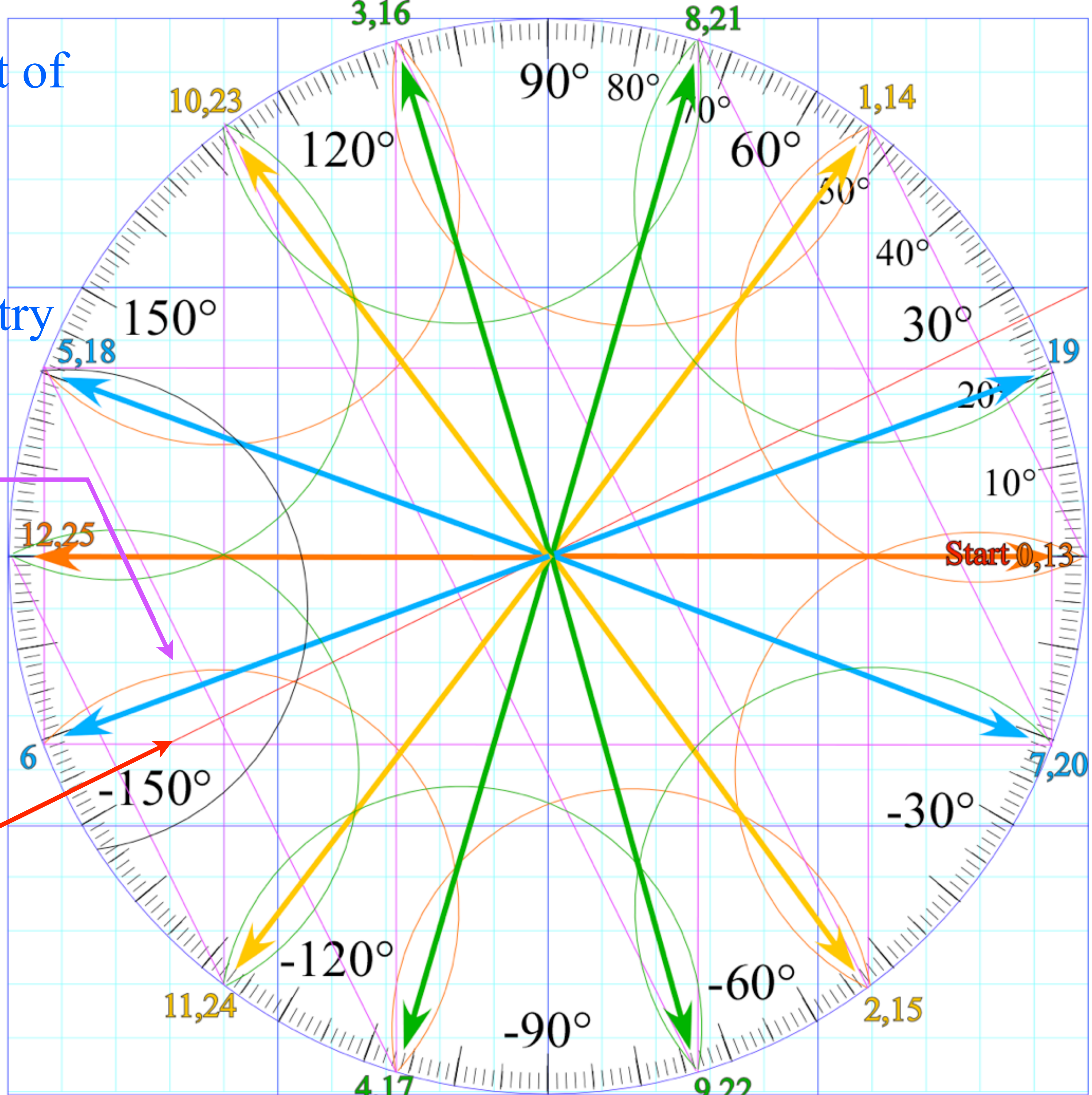
Estrangian plot of $m_1/m_2=4/1$ collision sequence shows symmetry

(sort of)



c.o.m. lines
(cons. of mom.)
have slope
$-\sqrt{m_2}/\sqrt{m_1}=-2/1$

COM line
has slope
$\sqrt{m_2}/\sqrt{m_1}=1/2$