Geometry and Symmetry of Coulomb Orbital Dynamics
(Ch. 2-4 of Unit 5  12.05.15)

Review of *Eccentricity vector* $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

Analytic geometry derivation of $\varepsilon$-construction

Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Detailed ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375$  elliptic orbit)

$(R=+0.5$  hyperbolic orbit)

Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
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Eccentricity vector $\mathbf{\varepsilon}$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V = V(r)$ guarantees conservation of angular momentum vector $\mathbf{L}$

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}
\]

(Review of Lect. 26)

Coulomb $V = -k/r$ also conserves eccentricity vector $\mathbf{\varepsilon}$

\[
\mathbf{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} - \mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}
\]

$A = km \cdot \mathbf{\varepsilon}$ is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of $\mathbf{\varepsilon}$ with a radial vector $\mathbf{r}$:

\[
\mathbf{\varepsilon} \cdot \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{r}}{r} - \frac{\mathbf{r} \cdot \mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} \times \mathbf{p} \cdot \mathbf{L}}{km} = \frac{\mathbf{r} \cdot \mathbf{L} \cdot \mathbf{L}}{km}
\]

Let angle $\phi$ be angle between $\mathbf{\varepsilon}$ and radial vector $\mathbf{r}$

\[
\varepsilon r \cos \phi = r - \frac{L^2}{km}
\]

For $\lambda = L^2 / km$ that matches:

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi}
\]

(a) Attractive ($k > 0$)
Elliptic ($E < 0$)

(b) Attractive ($k > 0$)
Hyperbolic ($E > 0$)

(c) Repulsive ($k < 0$)
Hyperbolic ($E > 0$)

(Rotational momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is normal to the orbit plane.)

IHO $V = (k/2)r^2$ also conserves Stokes vector $\mathbf{S}$

\[
S_A = \frac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)
\]

\[
S_B = x_1p_1 + x_2p_2
\]

\[
S_C = x_1p_2 - x_2p_1
\]

Generate symmetry groups: $U(2) \subset U(2)$
or: $R(3) \subset R(3) \times R(3) \subset O(4)$

...or of $\mathbf{\varepsilon}$ with momentum vector $\mathbf{p}$:

\[
\mathbf{\varepsilon} \cdot \mathbf{p} = \frac{\mathbf{p} \cdot \mathbf{r}}{km} - \frac{\mathbf{p} \cdot \mathbf{p} \times \mathbf{L}}{km} = \mathbf{p} \cdot \dot{\mathbf{r}} = p_r
\]

\[
\begin{cases}
\lambda & \text{if } \phi = 0 \text{ apogee} \\
\frac{\lambda}{1-\varepsilon} & \text{if } \phi = \frac{\pi}{2} \text{ zenith} \\
\frac{\lambda}{1+\varepsilon} & \text{if } \phi = \pi \text{ perigee}
\end{cases}
\]

Sunday, December 6, 2015
**Geometry of Coulomb orbits (Let: \( r = \rho \) here)**

\[
\frac{r}{\varepsilon} = \frac{\lambda}{\varepsilon} + r \cos \phi \\
\lambda \equiv 1 - \varepsilon \cos \phi
\]

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi}
\]

\[
\rho = \frac{\lambda}{(1 + \varepsilon)} \text{ perhelion} \\
\rho = \frac{\lambda}{(1 - \varepsilon)} \text{ aphelion}
\]

All conics defined by:

*Defining eccentricity* \( \varepsilon \)

**Distance to Focal-point = \( \varepsilon \cdot \text{Distance to Directrix-line} \)**

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\( (x, y) \) **physical parameters**

\( \varepsilon = \sqrt{\frac{k^2 m + 2 \frac{L^2}{k^2 m}}{\frac{k^2 m + 2 \frac{L^2}{k^2 m}}}} = \sqrt{1 \pm \frac{b^2}{a^2}} \)

\[
a = \frac{k}{2E} \\
E = \frac{k}{2a} \\
b = \frac{L}{\sqrt{2m E}} \\
L = \sqrt{km \lambda}
\]

**Minor radius**: \( b = \sqrt{(a^2 - a^2 \varepsilon^2)} = \sqrt{(a \lambda)} \) (ellipse: \( \varepsilon < 1 \))

**Minor radius**: \( b = \sqrt{(a^2 \varepsilon^2 - a^2)} = \sqrt{(\lambda a)} \) (hyperb: \( \varepsilon > 1 \))

\[
\varepsilon^2 = 1 - \frac{b^2}{a^2} \quad (\text{ellipse: } \varepsilon < 1) \\
\varepsilon^2 = 1 + \frac{b^2}{a^2} \quad (\text{hyperbola: } \varepsilon > 1)
\]

\[
\lambda = a(1 - \varepsilon^2) \quad (\text{ellipse: } \varepsilon < 1) \\
\lambda = a(\varepsilon^2 - 1) \quad (\text{hyperb: } \varepsilon > 1)
\]

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(Review of Lect. 25)

\[
\frac{1}{r} = \frac{1 - \varepsilon \cos \phi}{\lambda} = 1 - \frac{\varepsilon}{\lambda} \cos \phi
\]

\[
\rho = \frac{\mu^2}{m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m} \cos \phi
\]

**Major axis**: \( \rho_+ + \rho_- = 2a \)

\[
\rho_+ + \rho_- = \frac{[\lambda(1 + \varepsilon) + \lambda(1 - \varepsilon)]}{(1 - \varepsilon^2)} = 2\lambda/|1 - \varepsilon^2|
\]

**Focal axis**: \( \rho_+ - \rho_- = 2a \varepsilon \)

\[
\rho_+ - \rho_- = \frac{[\lambda(1 + \varepsilon) - \lambda(1 - \varepsilon)]}{(1 - \varepsilon^2)} = 2\lambda \varepsilon /|1 - \varepsilon^2|
\]

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Sunday, December 6, 2015
Review of Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

Analytic geometry derivation of $\varepsilon$-construction

Connection formulas for $(a, b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Detailed ruler & compass construction of $\varepsilon$-vector and orbits

$(R = -0.375 \text{ elliptic orbit})$

$(R = +0.5 \text{ hyperbolic orbit})$

Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
**ε-vector** and Coulomb orbit construction steps

- Pick launch point P (radius vector \( \mathbf{r} \))
- and elevation angle \( \gamma \) from radius (momentum initial \( \mathbf{p} \) direction)

- Copy F-center circle around launch point P
- Copy elevation angle \( \gamma (\angle \mathbf{FP}' \mathbf{P}) \) onto \( \angle \mathbf{P}' \mathbf{P} \mathbf{Q} \)
- Extend resulting line \( \mathbf{P} \mathbf{Q} \mathbf{P}' \) to make **focus locus**

**Reason for focus locus:**
- Line \( \mathbf{r} \) from 1st focus \( \mathbf{F} \) “reflects” off line \( \mathbf{p} \) (or \( \mathbf{P}' \mathbf{P} \)) toward 2nd focus \( \mathbf{F}' \) somewhere so incident-angle \( \gamma \) equals reflected-angle \( \gamma \)

**Draw ε-vector** from focus \( \mathbf{F} \) to \( \mathbf{R} \)-point and beyond to 2nd focus \( \mathbf{F}' \)

**Copy double angle** \( 2\gamma (\angle \mathbf{FP} \mathbf{Q}) \) onto \( \angle \mathbf{PFT} \)

**Extend \( \angle \mathbf{PFT} \) chord \( \mathbf{PT} \) to make \( \mathbf{R} \)-**ratio scale line**

**Label chord PT** with \( \mathbf{R}=0 \) at \( \mathbf{P} \) and \( \mathbf{R}=-1.0 \) at \( \mathbf{T} \).

**Mark R-line** fractions \( \mathbf{R}=0, +1/4, +1/2, ... \) above \( \mathbf{P} \) and \( \mathbf{R}=0, -1/8, -1/4, -1/2, ..., -3/4 \) below \( \mathbf{P} \) and \(-5/4, -3/2, ... \) below \( \mathbf{T} \).

**Focused Construction of Orbital Trajectory:**

**Focus \( \mathbf{F} \) and 2nd focus \( \mathbf{F}' \) allow final construction of orbital trajectory.**

Here it is an \( \mathbf{R}=-3/8 \) ellipse.

(Detailed Analytic geometry of ε-vector follows.)

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**From Lecture 26 p. 64**

\[
R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} - \frac{k}{r(0)}
\]

\[
= \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}
\]
**ε-vector** and Coulomb orbit construction steps

**Pick launch point** \( P \)

(radius vector \( \mathbf{r} \))
and elevation angle \( \gamma \) from radius
(momentum initial \( \mathbf{p} \) direction)

Copy F-center circle around launch point \( P \)

Copy elevation angle \( \gamma (\angle FPP') \) onto \( \angle P/\mathbf{P}Q \)

Extend resulting line \( \mathbf{Q}PQ' \) to make focus locus

**Copy double angle** \( 2\gamma (\angle \mathbf{FPQ}) \) onto \( \angle \mathbf{PFT} \)

Extend \( \angle \mathbf{PFT} \) chord \( PT \) to make \( R \)-ratio scale line

Label chord with \( R=0 \) at \( P \) and \( R=-1.0 \) at \( T \).

Mark \( R \)-line fractions \( R=0, +1/4, +1/2, \ldots \) above \( P \) and
\( R=0, -1/8, -1/4, -1/2, \ldots, -3/4 \) below \( P \) and \(-5/4, -3/2, \ldots \) below \( T \).

\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} \frac{-k}{r(0)} \]

\[ R = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]

(Ratio scale line)

(Extension of chord PT)

(Reason for focus locus: Line \( r \) from 1st focus \( F \) “reflects” off line \( \mathbf{P} \) (or \( \mathbf{P}' \)) toward 2nd focus \( F' \) somewhere so incident-angle \( \gamma \) equals reflected-angle \( \gamma \))

(Reason for focus locus: \( \text{Lines} \) \( r \) and chord \( \mathbf{P} \) “reflect” off line \( \mathbf{P}' \) toward 2nd focus \( F' \) somewhere so incident-angle \( \gamma \) equals reflected-angle \( \gamma \))

(From Lecture 26 p. 65)
Analytic geometry derivation of \( \varepsilon \)-constructions

\[
\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \frac{(mv_0)(mv_0r_0)\sin \gamma}{km}
\]

where: \( L_{px} \equiv p \times L \)

\[
\varepsilon = \hat{r} + 2\sin \gamma \frac{mv_0^2}{-k/r_0} \quad \hat{L}_{px} = \hat{r} + 2\sin \gamma \frac{KE}{PE} \hat{L}_{px}
\]

The eccentricity vector is:

\[
\varepsilon = \begin{pmatrix}
\cos \gamma \\
\sin \gamma
\end{pmatrix} + 2\sin \gamma \begin{pmatrix}
0 \\
1
\end{pmatrix} \quad R = \begin{pmatrix}
\cos \gamma \\
(2R+1)\sin \gamma
\end{pmatrix}
\]

For: \( \gamma = 45^\circ \) and: \( R = +\frac{1}{2} \)

\[
\varepsilon = \begin{pmatrix}
1/\sqrt{2} \\
1/\sqrt{2}(2R+1)
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{2} \\
2/\sqrt{2}
\end{pmatrix}
\]

Fig. 5.4.3 in Unit 5 of CMwBANG!

\[
\frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{-k/r(0)} = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}
\]
Analytic geometry derivation of $\varepsilon$-constructions

$\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \left( \frac{mv_0}{km} \right) r_0 \sin \gamma \hat{L}_{px}$

where: $L_{px} \equiv p \times L$

$\varepsilon = \hat{r} + 2 \sin \gamma \frac{mv_0^2}{-k/r_0}$

$\hat{L}_{px} = \hat{r} + 2 \sin \gamma \frac{KE}{PE} \hat{L}_{px}$

The *eccentricity* vector is:

$\varepsilon = \begin{pmatrix}
\cos \gamma \\
\sin \gamma
\end{pmatrix} + 2 \sin \gamma \begin{pmatrix}
0 \\
1
\end{pmatrix} R = \begin{pmatrix}
\cos \gamma \\
(2R+1) \sin \gamma
\end{pmatrix}$

For: $\gamma = 45^\circ$ and: $R = +\frac{1}{2}$

$\varepsilon = \begin{pmatrix}
1/\sqrt{2} \\
1/\sqrt{2}(2R+1)
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{2} \\
2/\sqrt{2}
\end{pmatrix}$

The *eccentricity* parameter defined by:

$e^2 = \cos^2 \gamma + (2R+1)^2 \sin^2 \gamma = 1 + a^2 = 1 + 4R(R+1)\sin^2 \gamma = \frac{a^2}{b^2}$

$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2/-k/r(0)}$

$= \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$

Fig. 5.4.3 in Unit 5 of CMwBANG!
Analytic geometry derivation of $\varepsilon$-constructions

$$\varepsilon = \hat{r} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \hat{r} - \left(\frac{mv_0}{km}\right)\sin \gamma \hat{L}_p$$

where: $\hat{L}_p = \hat{p} \times \hat{L}$

$\varepsilon = \hat{r} + 2 \sin \gamma \frac{mv_0^2}{-k/r_0} \hat{L}_p = \hat{r} + 2 \sin \gamma \frac{KE}{PE} \hat{L}_p$

The **eccentricity** vector is:

$$\varepsilon = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2 \sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos \gamma \\ (2R+1) \sin \gamma \end{pmatrix}$$

For: $\gamma = 45^\circ$ and: $R = +\frac{1}{2}$

$$\varepsilon = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} (2R+1) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 2/\sqrt{2} \end{pmatrix}$$

The **eccentricity** parameter defined by:

$$e^2 = \cos^2 \gamma + (2R+1)^2 \sin^2 \gamma \leq 1 + \frac{a^2}{b^2}$$

$$e^2 = 1 + 4R(2R+1)\sin^2 \gamma = \frac{a^2}{b^2}$$

$$R = \frac{\text{Initial } KE}{\text{Initial } PE} = \frac{mv^2(0)}{2(-k/r(0))}$$

$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$

Fig. 5.4.3 in Unit 5 of CMwBANG!
Analytic geometry derivation ofε-constructions

\[ \mathbf{r} = \hat{\mathbf{r}} - \mathbf{p} \times \mathbf{L} = \hat{\mathbf{r}} - \left( \frac{m v_0}{r_0} \right) \sin \gamma \hat{\mathbf{L}} \]

where: \( \hat{\mathbf{L}} = \mathbf{p} \times \mathbf{L} \)

\[ \mathbf{r} = \hat{\mathbf{r}} + 2 \sin \gamma \frac{m v_0^2}{-k/r_0} \hat{\mathbf{L}} = \hat{\mathbf{r}} + 2 \sin \gamma \frac{KE}{PE} \hat{\mathbf{L}} \]

The eccentricity vector is:

\[ \mathbf{e} = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2 \sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos \gamma \\ (2R+1) \sin \gamma \end{pmatrix} \]

For: \( \gamma = 45^\circ \) and: \( R = \frac{1}{2} \)

\[ \mathbf{e} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 2/\sqrt{2} \end{pmatrix} \]

The eccentricity parameter defined by:

\[ e^2 = \cos^2 \gamma + \left( \frac{2R+1}{2} \right)^2 \sin^2 \gamma = 1 + \frac{a^2}{b^2} \]

\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0)}{2} / \left(-k / r(0)\right) \]

\[ = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]
Analytic geometry derivation of ε-constructions

\[ \varepsilon = \hat{r} - \frac{\mathbf{p} \times \mathbf{L}}{\text{km}} = \hat{r} - \frac{(mv_0)(mv_0 r_0)\sin \gamma}{\text{km}} \hat{L}_{px} \]

where: \( \hat{L}_{px} \equiv \mathbf{p} \times \mathbf{L} \)

\[ \varepsilon = \hat{r} + 2\sin \gamma \frac{mv_0^2}{-k/r_0} \hat{L}_{px} = \hat{r} + 2\sin \gamma \frac{KE}{PE} \hat{L}_{px} \]

The **eccentricity** vector is:

\[ \varepsilon = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2\sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos \gamma \\ (2R+1)\sin \gamma \end{pmatrix} \]

For: \( \gamma = 45^\circ \) and: \( R = \frac{1}{2} \)

\[ \varepsilon = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 2/\sqrt{2} \end{pmatrix} \]

The **eccentricity** parameter defined by:

\[ \varepsilon^2 = \cos^2 \gamma + (2R+1)^2 \sin^2 \gamma \approx 1 + \frac{a^2}{b^2} = 1 + 4R(R+1)\sin^2 \gamma = \frac{a^2}{b^2} \]

\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{-k/r(0)} \]

\[ = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]
Analytic geometry derivation of $\varepsilon$-constructions

$$R = \frac{KE}{PE}$$

where: $L_{px} = p \times L$

The eccentricity vector is:

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \left( \frac{mv_0}{km} \right) \left( \frac{mv_0 r_0}{km} \right) \sin \gamma \hat{L}_{px}$$

For: $\gamma = 45^\circ$ and: $R = +\frac{1}{2}$

$$\varepsilon = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2\sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos \gamma \\ (2R+1)\sin \gamma \end{pmatrix}$$

The eccentricity parameter defined by:

$$e^2 = \cos^2 \gamma + \frac{[2R+1]^2}{2}\sin^2 \gamma = 1 + \frac{a^2}{b^2}$$

$$R = \frac{Initial KE}{Initial PE} = \frac{mv^2(0)/2}{-k/r(0)}$$

$$= \pm \left( \frac{Initial velocity}{Escape velocity} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$

Fig. 5.4.3 in Unit 5 of CMvBANG!
Review of Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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Connection formulas for $(a, b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Detailed ruler & compass construction of $\varepsilon$-vector and orbits

$(R = -0.375 \text{ elliptic orbit})$

$(R = +0.5 \text{ hyperbolic orbit})$

Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
Algebra of $\varepsilon$-construction geometry

The eccentricity parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$.

Three pairs of parameters for Coulomb orbits:
1. Cartesian (a, b), 2. Physics (E, L), 3. Polar ($\varepsilon, \lambda$)

Now we relate a 4th pair: 4. Initial ($\gamma, R$)

\[\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma \]

- for ellipse ($\varepsilon < 1$) where: $4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) < 0$ (or $-R^2 > R$)
- for hyperbola ($\varepsilon > 1$) where: $4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) > 0$ (or $-R^2 < R$)

Total $\frac{-k}{2a} = E = energy = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

Total $\frac{-k}{2a} = E = KE + PE = R\ PE + PE = (R + 1)\ PE = (R + 1)\ \frac{-k}{r}$ or: $\frac{1}{2a} = (R + 1)\ \frac{1}{r}$

\[a = \frac{r}{2(R + 1)} = \left(\frac{1}{2(R + 1)}\right)\text{ assuming unit initial radius (}r \equiv 1).\]

\[4R(R+1)\sin^2\gamma = \mp\frac{b^2}{a^2} \text{ implies: } 2\sqrt{\mp R(R+1)}\sin\gamma = \frac{b}{a} \text{ or: } b = 2a\sqrt{\mp R(R+1)}\sin\gamma\]

\[b = r\sqrt{\frac{\mp R}{R+1}}\sin\gamma = \sqrt{\frac{\mp R}{R+1}}\sin\gamma \text{ assuming unit initial radius (}r \equiv 1)\]

Latus radius is similarly related:

\[\lambda = \frac{b^2}{a} = \mp 2r\ R \sin^2\gamma\]

(Review of Lect. 26 p.107-108)
Algebra of $\epsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\epsilon^2 = 1 + 4R(R+1)\sin^2 \gamma$$

$$= 1 - \frac{b^2}{a^2} \text{ ellipse } (\epsilon < 1) \quad 4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2}$$

$$= 1 + \frac{b^2}{a^2} \text{ hyperbola } (\epsilon > 1) \quad 4R(R+1)\sin^2 \gamma = +\frac{b^2}{a^2}$$

$$a = \frac{r}{2(R+1)} = \left( \frac{1}{2(R+1)} \right) \text{ assuming unit initial radius } (r \equiv 1).$$

$$b = r \sqrt{\frac{\mp R}{R+1}} \sin \gamma = \sqrt{\frac{\mp R}{R+1}} \sin \gamma \text{ assuming unit initial radius } (r \equiv 1).$$

*Latus radius* is similarly related:

$$\lambda = \frac{b^2}{a} = \mp 2R \sin^2 \gamma$$

From $\epsilon^2$ result (at top):

$$\frac{b}{a} = 2\sqrt{\mp R(R+1)\sin \gamma} = \sqrt{\pm(1-\epsilon^2)}$$

(Review of Lect. 26 p.107-108)

Three pairs of parameters for Coulomb orbits:
1. Cartesian $(a,b)$, 2. Physics $(E,L)$, 3. Polar $(\epsilon, \lambda)$

Now we relate a 4th pair: 4. Initial $(\gamma, R)$.
Review of Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Detailed ruler & compass construction of $\varepsilon$-vector and orbits

- $(R=-0.375 \text{ elliptic orbit})$
- $(R=+0.5 \text{ hyperbolic orbit})$

Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

- Launch angle fixed-Varied launch energy
- Launch energy fixed-Varied launch angle
- Launch optimization and orbit family envelopes
Extend FP to make major axis sum FPP' : (r + r' = 2a) at intersect of r'-arc F'P' F'P' R = -3/8 elliptic orbit construction

γ = 45°
Strike radius-\( r \) arc about point \( P' \) to intersect original radius-\( r \) circle about focus \( F \) at ends of bisection line \( BB' \). Draw radius-\( a \) circle at \( F \) tangent to bisection line \( BB' \).
Strike radius-\( r \) arc about point \( P' \) to intersect original radius-\( r \) circle about focus \( F \) at ends of bisection line \( BB' \). Draw radius-\( a \) circle at tangent to bisection line \( BB' \).

Draw radius-\( a \) circle at \( F' \). Draw radius-\( a \) and radius-\( b \) circles at \( O \) (Center of bisection line \( \pm b \)).

Extend FP to make major axis sum \( FPP': (r + r' = 2a) \) at \( P' \) intersect of \( r' \)-arc of \( r' \)-arc.

\( R = \frac{-3}{8} \) elliptic orbit construction

\( R = \frac{-3}{8} \)

\( \gamma = 45^\circ \)
Strike radius-r arc about point P' to intersect original radius-r circle about focus at ends of bisection line BB'. Draw radius-a circle at tangent to bisection line BB'.

\[ R = -3/8 \] elliptic orbit construction

\[ \gamma = 45^\circ \]

Extend FP to make major axis sum FPP':(r+r'=2a) at P' intersect of r'-arc of \( r' \)-arc

Draw radius-a circle at \( F' \)

Draw radius-a and radius-b circles at O (Center of bisection line \((\pm b)\)).
\[ \epsilon = \sqrt{1 + 4R(R+1)\sin^2\gamma} = \frac{\sqrt{34}}{8} = .73 \]

\[ a = \frac{1}{2(R+1)} = \frac{4}{5} \]

\[ b = \sqrt{\frac{R}{R+1}} \sin \gamma = \sqrt{\frac{3}{10}} = .54 \]

\[ \lambda = \frac{b^2}{a} = 2R\sin^2\gamma = \frac{3}{8} = .375 \]

\[ \frac{b}{a} = 2\sqrt{R(R+1)\sin \gamma} = \tan 34^\circ \]

---

**Draw radius-\(a\) circle at \(F'\)**

**Draw radius-\(a\) and radius-\(b\) circles at \(O\)**

*(Center of bisection line \((\pm b)\).)*

**Do \((a,b)\)-ellipse construction.**
Review of *Eccentricity vector* $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

Analytic geometry derivation of $\varepsilon$-construction

Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Detailed ruler & compass construction of $\varepsilon$-vector and orbits

$(R = -0.375 \text{  elliptic orbit})$

$(R = +0.5 \text{  hyperbolic orbit})$

Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
Major diameter \(2a\) is difference \((r-r')=2a\).
Major radius \(a\) is half of difference \((r-r')/2=a\)
Major diameter \(2a\) needs to be centered on \(F-F'\) focal axis

\[R=+1/2\] hyperbolic orbit construction

\[R=+1/2\]
\[\gamma=45^\circ\]
Major diameter $2a$ is difference $v - r' = 2a$. 

1. Bisect $F-P$ radius raising $F-P$ circle intersections to define $r^2$ sections.

$R = +1/2$ hyperbolic orbit construction.

$\gamma = 45^\circ$

$R = +1/2$ hyperbolic orbit construction.
Major diameter $2a$ is difference $(r-r'=2a)$.
Major radius $a$ is half of difference $(r-r')/2=a$.
Major diameter $2a$ needs to be centered on F-F‘ focal axis.
1. Bisect F-P radius $r$ using F-P circle intersections to define $r/2$ sections.
2. Bisect F-F‘ focal axis using F-F‘ circle intersections to locate orbit center C.

$R=+1/2$ hyperbolic orbit construction

$\gamma=45^\circ$
Major diameter 2a is difference (r-r' = 2a).
Major radius a is half of difference (r-r')/2 = a.
Major diameter 2a needs to be centered on F-F' focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F' focal axis using F-F' circle intersections to locate orbit center C.

\[ R = +1/2 \ \text{hyperbolic orbit construction} \]

\[ \gamma = 45^\circ \]
Major diameter \(2a\) is difference \((r-r' = 2a)\).

Major radius \(a\) is half of difference \((r-r')/2 = a\).

Major diameter \(2a\) needs to be centered on \(F-F'\) focal axis.

1. Bisect \(F-P\) radius \(r\) using \(F-P\) circle intersections to define \(r/2\) sections.
2. Bisect \(F-F'\) focal axis using \(F-F'\) circle intersections to locate orbit center \(C\).
3. Bisect \(F'-P\) radius \(r'\) using \(F'-P\) circle intersections.
4. Swing radius \(r'/2\) onto \(r/2\) section to make major radius \(a=(r-r'/2)\).
Major diameter $2a$ is difference $(r-r')=2a$.
Major radius $a$ is half of difference $(r-r')/2=a$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$.
5. Copy circle of major radius $a=(r-r')/2$ about orbit center $C$.

$R=+1/2$ hyperbolic orbit construction

$\gamma=45^\circ$
Major diameter 2a is difference (r-r' = 2a).
Major radius a is half of difference (r-r')/2 = a
Major diameter 2a needs to be centered on F-F' focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F' focal axis using F-F' circle intersections to locate orbit center C.
4. Swing radius r'/2 onto r/2 section to make major radius a=(r-r')/2.
5. Copy circle of major radius a=(r-r')/2 about orbit center C.
6. Draw focal circle of diameter 2ae about orbit center C.
Major diameter 2a is difference (r-r’=2a).
Major radius a is half of difference (r-r’)/2=a
Major diameter 2a needs to be centered on F-F’ focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F’ focal axis using F-F’ circle intersections to locate orbit center C.
4. Swing radius r’/2 onto r/2 section to make major radius a=(r-r’)/2.
5. Copy circle of major radius a=(r-r’)/2 about orbit center C.
6. Draw focal circle of diameter 2aε about orbit center C.
7. Erect minor radius b tangent to a-circle from point a on Cε-axis to point b on focal circle.
Major diameter $2a$ is difference $(r-r' = 2a)$.
Major radius $a$ is half of difference $(r-r')/2 = a$.

Major diameter $2a$ needs to be centered on F-F' focal axis.

1. Bisect F-P radius $r$ using F-P circle intersections to define $r/2$ sections.
2. Bisect F-F' focal axis using F-F' circle intersections to locate orbit center C.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a = (r-r')/2$.
5. Copy circle of major radius $a = (r-r')/2$ about orbit center C.
6. Draw focal circle of diameter $2ae$ about orbit center C.
7. Erect minor radius $b$ tangent to a-circle from point $a$ on CE-axis to point $b$ on focal circle.
8. Complete orbit $a-x-b$ box between focal circle and a-circle and its diagonal asymptotes.

$R = +1/2$ hyperbolic orbit construction

$R = +1/2$

$\gamma = 45^\circ$
9. Draw section of hyperbolic orbit.

\[ R = +\frac{1}{2} \text{ hyperbolic orbit construction} \]

\[ \gamma = 45^\circ \]
9. Draw section of hyperbolic orbit.

Construction based on: $r - r' = 2a$ or $r' = r - 2a$

$T$ draw an $r$-arc about focus $F$. 

$r = r'$

$R = +1/2$ hyperbolic orbit construction

$R = +1/2$

$\gamma = 45^\circ$
9. Draw section of hyperbolic orbit.

Construction based on: \( r-r' = 2a \) or: \( r' = r-2a \)

1st draw an \( r \)-arc about focus \( F \).
2nd set compass to \( (r-2a) \) using \( r \)-arc-minus-\( 2a \) on \( CE \)-line.
9. Draw section of hyperbolic orbit.

$R = +1/2$ hyperbolic orbit construction

$\gamma = 45^\circ$

Construction based on: $r - r' = 2a$ or: $r' = r - 2a$

1st draw an $r$-arc about focus $F$.

2nd set compass to $(r-2a)$ using $r$-arc-minus-2a on $C\varepsilon$-line.

3rd draw $(r-2a)$-arc about focus $F'$.

Sunday, December 6, 2015
9. Draw section of hyperbolic orbit.

Construction based on: $r-r'=2a$ or $r'=r-2a$

1\(^{st}\) draw an $r$-arc about focus $F$.
2\(^{st}\) set compass to $(r-2a)$ using $r$-arc-minus-2a on $C\varepsilon$-line.
3\(^{rd}\) draw $(r-2a)$-arc about focus $F'$.

Orbit points at intersections.

$R=+1/2$ hyperbolic orbit construction

$R=+1/2$

$\gamma=45^\circ$
9. Draw section of hyperbolic orbit.

$R = +1/2$ hyperbolic orbit construction

$\gamma = 45^\circ$
9. Draw section of hyperbolic orbit.
9. Draw section of hyperbolic orbit.

$R = +\frac{1}{2}$ hyperbolic orbit construction

$\gamma = 45^\circ$

Sunday, December 6, 2015
9. Draw section of hyperbolic orbit.

\[ \varepsilon = \sqrt{1 + 4R(R+1)\sin^2 \gamma} = \sqrt{\frac{3}{2}} = 1.58 \]

\[ a = \frac{1}{2(R+1)} = \frac{1}{3} = 0.33 \]

\[ b = \sqrt{\frac{R}{R+1} \sin \gamma} = \frac{1}{\sqrt{6}} = 0.408 \]

\[ \lambda = \frac{b^2}{a} = 2R\sin^2 \gamma = \frac{1}{2} = 0.5 \]

\[ \frac{b}{a} = 2\sqrt{R(R+1)\sin \gamma} = \tan 50.7^\circ \]
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Detailed ruler & compass construction of $\varepsilon$-vector and orbits

- $(R=-0.375)$ elliptic orbit
- $(R=+0.5)$ hyperbolic orbit

Properties of Coulomb trajectory families and envelopes

- Graphical $\varepsilon$-development of orbits
- Launch angle fixed-Varied launch energy
- Launch energy fixed-Varied launch angle
- Launch optimization and orbit family envelopes
Graphs and protractors make Coulomb trajectory analysis easier.
Start with initial angle
\[ \alpha = 20^\circ \]
(horiz. elev.)
or
\[ \gamma = 70^\circ \]
(rad. elev.)

for velocity \( v(0) \) or \(-v(0)\)

or \(-v(0)\)

\[ 2(\alpha - \gamma) = 100^\circ \]

Label Main Focus \( F \) and construct focus locus for 2\textsuperscript{nd} foci \( F' \).
Start with initial angle

\[ \alpha = 20^\circ \] (horiz. elev.)

or

\[ \gamma = 70^\circ \] (rad. elev.)

for velocity\n\[ v(0) \text{ or } -v(0) \]

Construct focus locus for 2nd foci \( F' \)

Construct R-scale line to initial velocity \( \pm v(0) \) line

\[ 2(\alpha - \gamma) = 100^\circ \]
Start with initial angle
\( \alpha = 20^\circ \)
(horiz. elev.)
or
\( \gamma = 70^\circ \)
(rad. elev.)
for velocity
\( v(0) \) or \(-v(0)\)

Construct \textit{R-scale line} to initial velocity \( v(0) \) line

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)\)-sect \textit{R-scale line} to mark \( R = K E / P E = 0, \pm 1/8, \pm 2/8, \pm 3/8 \)
for eccentricity \( \epsilon \)-vector scale

Extend eccentricity \( \epsilon \)-vectors from the main Focus \( F \) to each \textit{R-line}-point

\( R \)-scale line is normal to initial \( v(0) \)-line
Start with initial angle
\[ \alpha = 20^\circ \]
(horiz. elev.)
or
\[ \gamma = 70^\circ \]
(rad. elev.)
for velocity
\[ v(0) \] or \[ -v(0) \]

Label Main Focus \( F \)

Construct R-scale line to initial velocity \( v(0) \) line

Construct focus locus for prime foci \( F' \)

\((N=8)-\text{sect R-scale line}
\text{to mark } R=KEPE=0, \pm 1/8, \pm 2/8, \pm 3/8
\text{for eccentricity } \varepsilon\text{-vector scale}

Extend eccentricity \( \varepsilon \)-vectors
from the main Focus \( F \)
to each \( R\)-line-point and
beyond to prime foci \( F' \)

R-scale line is normal to initial \( v(0) \)-line
Start with initial angle
\[ \alpha = 20^\circ \] (horiz. elev.)

or
\[ \gamma = 70^\circ \] (rad. elev.)

for velocity \( v(0) \) or \( -v(0) \)
Start with initial angle
\( \alpha = 20^\circ \) (horiz. elev.)
or
\( \gamma = 70^\circ \) (rad. elev.)
for velocity \( v(0) \) or \(-v(0)\)

\((R=\pm \infty) \ e\text{-line is parallel to } R\text{-scale line.}\)
This \((R=-1)\) case \(\Rightarrow\) R-scale line is parallel to \(\varepsilon-line\)

Start with initial angle \(\alpha=20^\circ\) or \(\gamma=70^\circ\) for velocity \(\pm v(0)\)

\((\text{horiz. elev.)}\) (rad. elev.)

Construct: \(\pm \infty\) point and \(\pm \infty\) line (attractive loci for \(R<1\))

Construct: \(\varepsilon\)-line hits focus-locus \((R=\pm \infty)\) parallel to \(R\)-scale line

Focus-locus \(F\) mark \(R \neq \pm \infty\) from left

Extend eccentricity \(e\) vectors from the main Focus \(F\)

to each \(R\)-line-point and beyond to prime foci \(F'\)

Range Longitude

\((N=8)\) scale line is normal to initial \(v(0)\)-line
This (R=-1) case
Start with initial angle
α = 20° (horiz. elev.)
or γ = 70° (rad. elev.)
for velocity v(0) or -v(0)

Construct focus locus for R<1
or R-scale line is normal to initial v(0)-line

Extend eccentricity e vectors from the main Focus F to each R-line-point and beyond to prime foci F′

(R=±∞) ε-line is parallel to R-scale line.

(R=1/8) ε-line hits focus locus

Focus locus for R<1 intersects ε-line

Range Longitude
This (R=-1) \( \epsilon \)-line is parallel to \( R \)-scale line.

Start with initial angle \( \alpha = 20^\circ \)

\( \gamma = 70^\circ \) (rad. elev.)

(0)

\( (R=-9/8) \) - line intersects focus-locus for \( -1 < R < 0 \)

\( (R=\infty) \) - line is normal to initial \( v(0) \)-line

\( v(0) \) or \(-v(0)\)

Focus-locus for \( -\infty < R < -1 \)

Focus-locus for \( R > 1 \)

(30\( \infty \) - line is normal to initial \( v(0) \)-line

Extend eccentricity \( e \)-vectors from the main Focus \( F \) to each \( R \)-line-point and beyond to prime foci \( F' \).
Review of **Eccentricity vector** $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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$(R=+0.5)$ hyperbolic orbit

Properties of Coulomb trajectory families and envelopes

- Graphical $\varepsilon$-development of orbits
  - Launch angle fixed-Varied launch energy
  - Launch energy fixed-Varied launch angle
  - Launch optimization and orbit family envelopes
Start with initial velocity \( \mathbf{v}(0) \) or \(-\mathbf{v}(0)\)

Label Main Focus \( F \)

Construct \textit{R-line normal} to initial velocity \( \mathbf{v}(0) \) line

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)\)-sect \textit{R-line normal} to mark \( R=\frac{KE}{PE}=0,\pm 1/8,\pm 2/8,\pm 3/8,\ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \)
to each \textit{R-line}-point and beyond to prime foci \( F' \)

Range bisection circles (these are not orbits) indicate reentry ranges
Start with initial velocity $v(0)$ or $-v(0)$

Label Main Focus $F$

Construct $R$-line normal to initial velocity $v(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$
to each $R$-line-point and beyond to prime foci $F'$

$2a = r' + r = r' + 1$

Range bisection circles (these are not orbits) indicate rentry ranges

$\Rightarrow$ Same arc centered on unit circle measures “string length”
Start with initial velocity $\mathbf{v}(0)$ or $-\mathbf{v}(0)$

- Label Main Focus $F$
- Construct $R$-line normal to initial velocity $\mathbf{v}(0)$
- Construct focus locus for prime foci $F'$
- $(N=8)$-sect $R$-line normal to mark $R=KE, PE=0, \pm 1/8, \pm 2/8, \pm 3/8...$
- For eccentricity $\varepsilon$-vector scale
- Extend eccentricity $\varepsilon$-vectors from the main Focus $F$
- To each $R$-line point and beyond to prime foci $F'$

$2a = r' + r = r' + 1$

Range bisection circles (these are not orbits) indicate reentry ranges

Construct ellipse point by point

Same arc centered on unit circle measures "string length"
Label Main Focus $F$

Construct $R$-line normal to initial velocity $\mathbf{v}(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8,$...

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$
to each $R$-line-point and beyond to prime foci $F'$

Range bisection indicates re-entry ranges

Maximum range limit for this elevation angle $\alpha=20^\circ$ has $R=-1$
and $\varepsilon=1$ (parabola)

This puts 2nd focus at $\infty$. 
Label Main Focus \( F \)

Construct \( R \)-line normal to initial velocity \( \mathbf{v}(0) \) line

Construct focus locus for prime foci \( F' \)

\((N=8)\)-sect \( R \)-line normal to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8\),...

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \)
to each \( R \)-line-point and beyond to prime foci \( F' \)

Maximum range limit for this elevation angle \( \alpha=20^\circ \) is range \( \phi=280^\circ \)

\( (R=-1) \varepsilon = 1 \)-line parallel to focus-locus

This puts 2nd focus at \( \infty \).
Label Main Focus $F$

Construct $R$-line normal to initial velocity $\mathbf{v}(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8$.

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$
to each $R$-line-point and
beyond to prime foci $F'$

Maximum range limit for this elevation angle $\alpha=20^\circ$ is range $\phi=280^\circ$.

Maximum range limit for this elevation angle $\alpha=20^\circ$ has $R=-1$
and $\varepsilon=1$ (parabola)

$\{ (R=-1) \ \varepsilon = 1 \text{-line parallel to focus-locus} \}$

This puts 2nd focus at $\infty$. 

Range bisection indicates re-entry ranges
Revu: geometry of parabola "kites"

Parabola

\[ 4p \cdot y = x^2 = 2\lambda y \]

This puts 2nd focus at \( \infty \).

Maximum range limit for this elevation angle \( \alpha = 20^\circ \) is range \( \Phi = 280^\circ \).

Maximum range limit for this elevation angle \( \alpha = 20^\circ \) has \( R = -1 \) and \( \varepsilon = 1 \) (parabola).

\[ (R=-1) \; \varepsilon = 1 \text{-line parallel to focus-locus} \]

Range Longitude

This puts 2nd focus at \( \infty \).
Review of Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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$(R=-0.375$ elliptic orbit$)$
$(R=+0.5$ hyperbolic orbit$)$

Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

- Launch angle fixed-Varied launch energy
- Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
Orbit with $\gamma = 47^\circ$ and $R = -3/8$
Orbit with $\gamma = 47^\circ$ and $R = -3/8$

Do one with $\gamma = 60^\circ$

(...and same R)

**Elevation angle $\gamma$**

**Eccentricity vector**

$P$, $P'$, $P_{60^\circ}$, $P_{47^\circ}$

$Q$, $R = KE/PE$

$F$, $R = -3/8$

$T$
Orbits with the same $R$ have the same energy $E$ and the same major radii $a$. Hence their foci lie on a circle of radius $2a-r$ around launch point $P$. 

Orbit with $\gamma=47^\circ$ and $R=-3/8$ 
Do one with $\gamma=60^\circ$ (...and same $R$)
Orbits with the same $R$ have the same energy $E$ and the same major radii $a$.

Hence their foci lie on a circle of radius $2a-r$ around launch point $P$.
focus locus for KE/PE
= $R = -\frac{3}{8}$

Envelope for KE/PE
= $R = -\frac{3}{8}$

Contact Pt. for KE/PE
= $R = -\frac{3}{8}$
Review of *Eccentricity vector* $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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$(R = +0.5$ hyperbolic orbit$)$

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Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
Coulomb envelope geometry

 Ideal comet “heads” or “tails” in solar wind

Fig. 5.4.4 in Unit 5 of CMwBANG!

Fig. 5.4.5 in Unit 5 of CMwBANG!

(a) Focus locus for KE/PE
   \( R = -\frac{3}{8} \)

(b) Caustic for KE/PE
   \( R = -\frac{3}{8} \)

(c) Diving orbit

(d) \( R \in (0, \infty) \)
CoulIt Web Simulation
Attractive Coulomb Burst

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CoulIt Web Simulation

Repulsive Coulomb Burst
Review of Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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- $(R=+0.5$ hyperbolic orbit$)$

Properties of Coulomb trajectory families and envelopes

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  - Launch energy fixed-Varied launch angle
  - Launch optimization and orbit family envelopes
Start with initial velocity \( \mathbf{v}(0) \) or \(-\mathbf{v}(0)\). Construct \textit{R-line normal} to initial velocity \( \mathbf{v}(0) \) line. Construct \textit{focus locus} for prime foci \( F' \). 

\((N=8)\)-sect \textit{R-line normal} to mark \( R = KE/PE = 0, \pm 1/8, \pm 2/8, \pm 3/8 \). 

for eccentricity \( \varepsilon \)-vector scale. Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \textit{R-line}-point and beyond to prime foci \( F' \).

Range Longitude
Label Main Focus \( F \)

Construct \( R\)-line normal to initial velocity \( v(0) \) line

Construct focus locus for prime foci \( F' \)

\((N=8)\)-sect \( R\)-line normal to mark \( R = KE/PE = 0, \pm 1/8, \pm 2/8, \pm 3/8 \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \( R\)-line-point and beyond to prime foci \( F' \)

\( v(0) \)

focus locus for fixed Energy or fixed \( R = KE/PE = -5/8 \)

Range Longitude
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

**Problem:**

Find trajectory angle of minimum energy to fly $90^\circ$ of arc (1/4 around planet)
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

**Problem:**

Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

**Solution:** Prime focus $F'$ lies on radial line that bisects longitude angle.
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

**Problem:**
Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

**Solution:** Prime focus $F'$ lies on radial line that bisects longitude angle

Optimal prime focus $F'$ lies on line connecting $\text{START}$ and $\text{FINISH}$ at tangent point of minimal energy circle $SF'$. 

Range Longitude
Problem: Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus $F'$ lies on radial line that bisects longitude angle

Optimal prime focus $F'$ lies on line connecting START and FINISH at tangent point of minimal energy circle $SF'$.

$R$-line normal must bisect angle $FSF'$ connecting foci $F$ and $F'$ and is normal to initial launch vector $v_0$. 

Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

Problem:
Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus F' lies on radial line that bisects longitude angle

Optimal prime focus F' lies on line connecting START and FINISH
at tangent point of minimal energy circle SF'.

R-line normal must bisect angle FSF' connecting foci F and F' and is normal to initial launch vector v₀ with launch angle α = 22.5°

The ε-vector and R-value: slightly below R = -3/8...

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Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

**Problem:**

With launch angle $\alpha = 22.5^\circ$ find trajectory to fly $207^\circ$ of longitude.

**Solution:** Prime focus $F'$ lies on radial line at $103.5^\circ$ that bisects longitude angle $207^\circ$.

The $\varepsilon$-vector and $R$-value: slightly below $R = -5/8$...
Problem: With launch angle $\alpha = 22.5^\circ$ find maximum range of trajectory.

Solution: Prime focus $F'$ lies at infinity and gives parabola ($\varepsilon_\infty = 1, R = -1$) trajectory.

Trajectory axis is at $135^\circ$.

Trajectory would hit Earth at $270^\circ$ …if it actually returned ...

...but a parabola cannot!

But at slightly less energy a very long ellipse would return after a very long time but at slightly less range than $270^\circ$.

Maximum range $269.999^\circ$: (with launch angle $\alpha = 22.5^\circ$)

Parabola escapes'... ...does not return...
Launch optimization

For low range $\rho$ the optimum angle $\theta$ approaches $\theta = \pi/4$

(The well-known sophomore physics result.)