Lecture 19
Tue. 11.05.2015

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential
Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential
Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential
Canonical momentum in ($\boldsymbol{A}$,Φ) potential
Hamiltonian formulation
Hamilton's equations

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations
Vector theory vs. complex variable theory
Mechanical analog of cyclotron and FBI rule
Cycloidal ruler&compass geometry
Cycloidal geometry of flying levers
Practical poolhall application

This mechanical analog of (E_x,B_z) field mimics **A**-*field with tabletop* **v**-*field*

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So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
$e = -1.602176 \cdot 10^{-19}$*Coulombs*

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

scalar potential field $\Phi=\Phi(\mathbf{r},t)$

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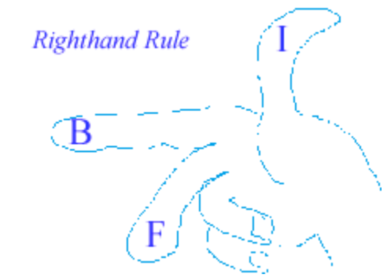
$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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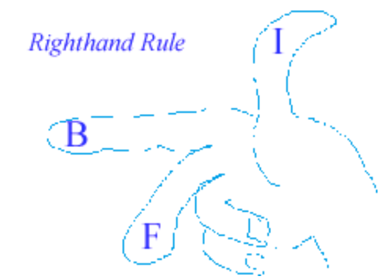

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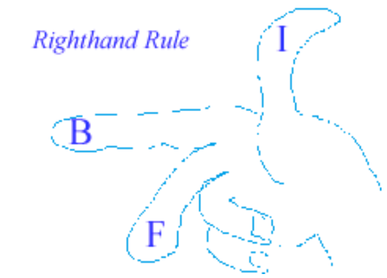

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Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i(\nabla\times\mathbf{A})_j$$

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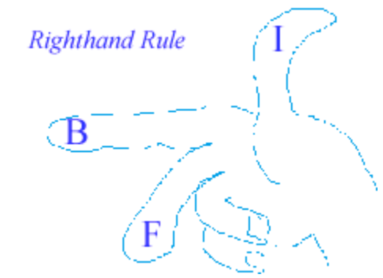

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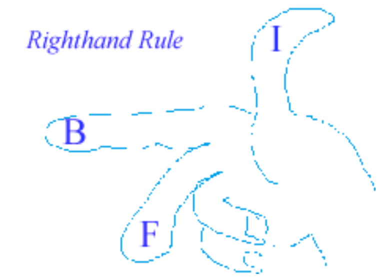

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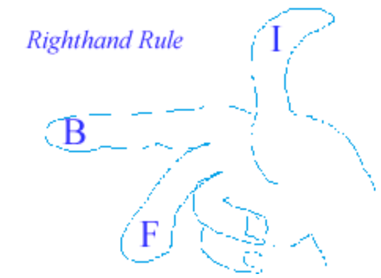

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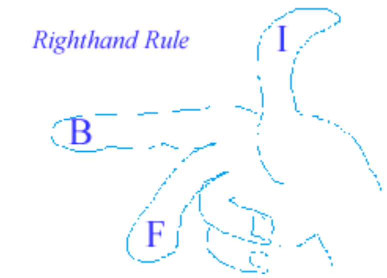

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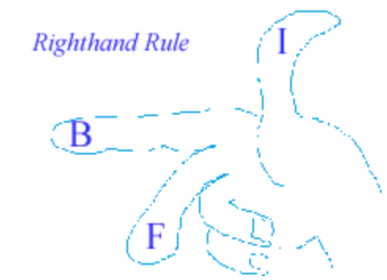

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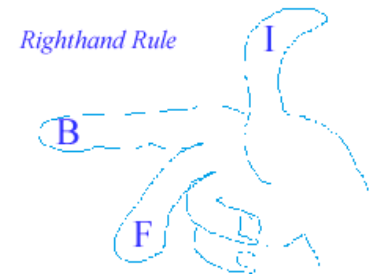

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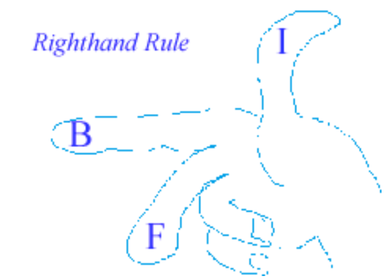

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&= \left(\partial_k A_b\right) v_b \quad - v_a\left(\partial_a A_k\right) = (\nabla\mathbf{A})\bullet\mathbf{v} - \mathbf{v}\bullet\nabla\mathbf{A} \\
&= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right) A_b - v_a\left(\partial_a A_k\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - (\nabla\mathbf{v})\bullet\mathbf{A} - \mathbf{v}\bullet\nabla\mathbf{A}
\end{aligned}$$

Applying Levi-Civita ε-identity:

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Converting back to Gibbs's **bold** notation involves *tensors* like $\nabla\mathbf{A}$ and $\nabla\mathbf{v}$.

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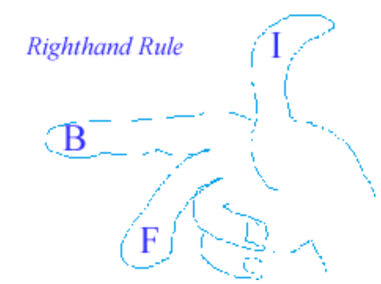

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$$= \left(\partial_k A_b\right)v_b \quad - v_a\left(\partial_a A_k\right) = (\nabla\mathbf{A})\bullet\mathbf{v} - \mathbf{v}\bullet\nabla\mathbf{A}$$
$$= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right)A_b - v_a\left(\partial_a A_k\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - (\nabla\mathbf{v})\bullet\mathbf{A} - \mathbf{v}\bullet\nabla\mathbf{A}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj} = \delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}$$

Converting back to Gibbs's **bold** notation involves *tensors* like ∇**A** and ∇**v**.

Newtonian mechanics has *no explicit dependence* of position **r** and velocity **v**.

r-partial derivative of **v** (or vice-versa) is identically zero. $\partial_k v^j \equiv 0$ *iff* : $\nabla\mathbf{v} = \frac{\partial\mathbf{v}}{\partial\mathbf{r}} = \mathbf{0}$

Vector analysis for particle-in-(A,Φ)-potential

So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

scalar potential field $\Phi=\Phi(\mathbf{r},t)$

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$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

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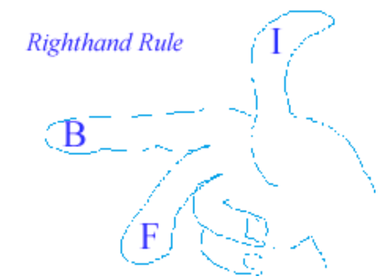

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

$$\varepsilon_{ijk} = \varepsilon_{ikj} = \varepsilon_{kij} = 1$$
$$= -\varepsilon_{jik} = -\varepsilon_{jki} = -\varepsilon_{kji}$$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij} v_i \left(\nabla\times\mathbf{A}\right)_j$$
$$= \varepsilon_{kij} v_i \left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$
$$= \varepsilon_{kij}\varepsilon_{abj} \quad v_i\left(\partial_a A_b\right)$$
$$= (\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}) v_i\left(\partial_a A_b\right)$$
$$= \delta_{ka}\delta_{ib} v_i\left(\partial_a A_b\right) - \delta_{kb}\delta_{ia} v_i\left(\partial_a A_b\right)$$
$$= v_b\left(\partial_k A_b\right) \quad - v_a\left(\partial_a A_k\right)$$
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$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - \quad 0 \quad - \mathbf{v}\bullet\nabla\mathbf{A} \qquad \textit{for particle mechanics}$$

Summary of Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Tensor index notation helps to distinguish $(\nabla\mathbf{A})\bullet\mathbf{v}$, $\mathbf{v}\bullet(\nabla\mathbf{A})$, and $\nabla(\mathbf{A}\bullet\mathbf{v}) = (\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k = \frac{\partial A_j}{\partial x_k} v_j = (\partial_k A_j) v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k = v_j \frac{\partial A_k}{\partial x_j} = (v_j \partial_j A_k)$$

$$\left[\nabla(\mathbf{A}\bullet\mathbf{v})\right]_k = \left[(\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}\right]_k$$

$$\partial_k\left(A_b v_b\right) = \left(\partial_k v_b\right) A_b - \left(\partial_k v_a\right) A_a$$

$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - \quad 0 \quad - \mathbf{v}\bullet\nabla\mathbf{A}$$ *for particle mechanics*

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

→ *Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential*

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- *Hamiltonian formulation*
- *Hamilton's equations*

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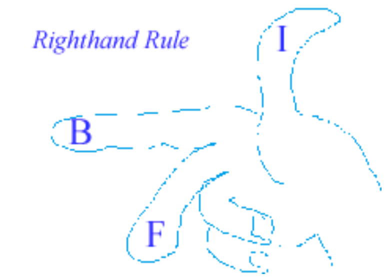

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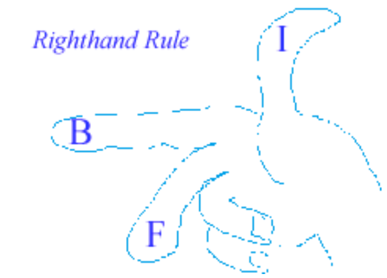

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Chain rule expansion of vector potential total t-derivative: $\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial x}\dot{x} + \frac{\partial\mathbf{A}}{\partial y}\dot{y} + \frac{\partial\mathbf{A}}{\partial z}\dot{z} + \frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{A}$

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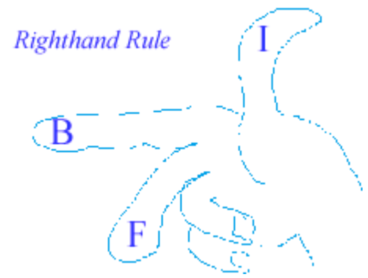

F=*q***v**×**B**=**I**×**B**

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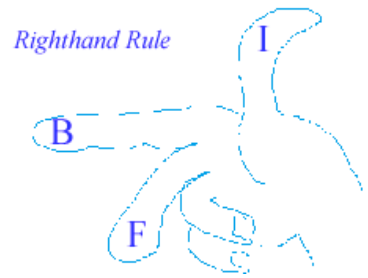

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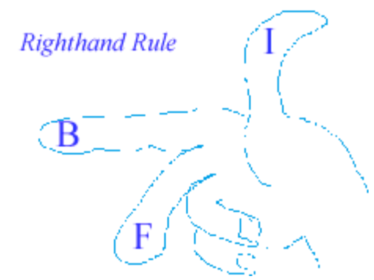

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$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\left(-e\mathbf{A}\right)\quad-\nabla(e\Phi-\mathbf{v}\bullet e\mathbf{A})\quad-e\frac{d\mathbf{A}}{dt}$$

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scalar potential field **Φ=Φ(r**,*t*)

vector potential field **A=A(r**,*t*)

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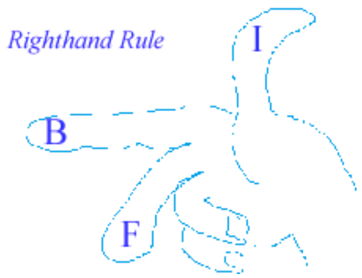
$$\mathbf{B}=\nabla\times\mathbf{A}$$



F=*q*v×B=I×B

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

Inserting Φ-term that $\partial_{\mathbf{v}}$ zeros :

$\left(\textit{This step requires that}: \frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right)=0\right)\left(\textit{and}: \frac{\partial}{\partial\mathbf{v}}\left(\mathbf{v}\bullet e\mathbf{A}\right)=e\mathbf{A}\right)$

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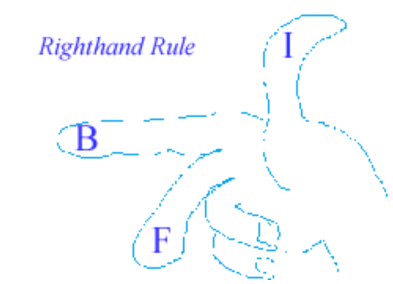

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right) = \frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right)$$

Inserting **v•v**-term that $\partial_{\mathbf{r}}$ zeros :

This step requires that :

$$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) \equiv \frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) = 0$$

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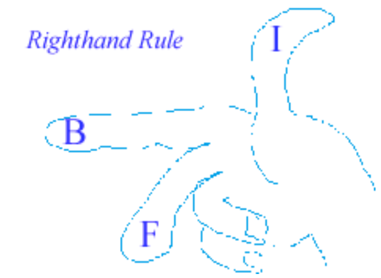

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\underbrace{\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})}\right)=\frac{\partial}{\partial\mathbf{r}}\left(\underbrace{\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})}\right)$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}=\frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian for particle-in-$(\mathbf{A},\Phi)$-potential

So-called *pondermotive* form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

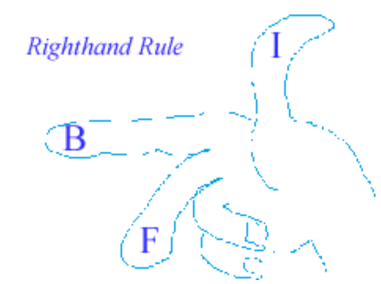
Electric field **E** and magnetic field **B**

scalar potential field $\boldsymbol{\Phi}=\boldsymbol{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\nabla(\mathbf{v}\bullet\mathbf{A})-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

Chain rule expansion of vector potential total t-derivative: $\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{A}$

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}=\frac{\partial}{\partial\mathbf{r}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}=\frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L=L(\mathbf{r},\mathbf{v},t)=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

→ *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*

Hamiltonian formulation

Hamilton's equations

Hamiltonian for particle-in-(A,Φ)-potential

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

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Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t)$$

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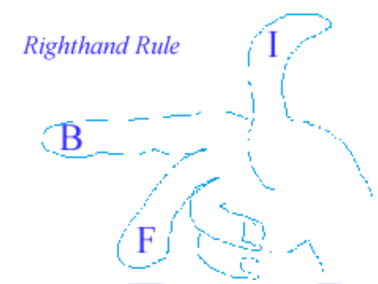

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

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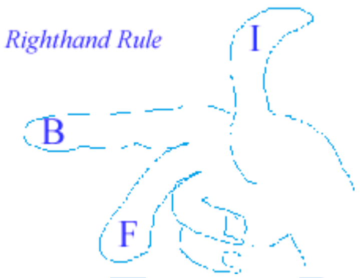

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Lagrangian is usual form $L= T - V$ with electric (scalar) potential $V=\mathbf{\Phi}(\mathbf{r},t)$

<u>if</u> magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

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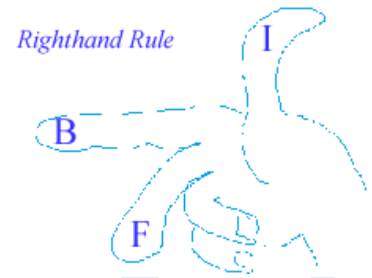

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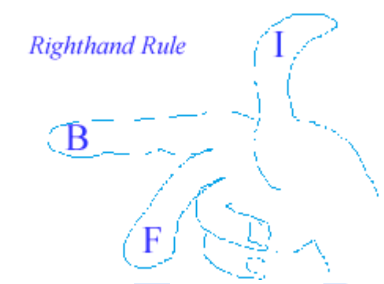

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Lagrangian is usual form $L= T - V$ with electric (scalar) potential $V=\Phi(\mathbf{r},t)$
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Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\qquad (For\ \mathbf{A}(\mathbf{r},t)=0)$

Otherwise vector potential term $-\mathbf{v}\cdot e\mathbf{A}$ leads to an extraordinary *canonical momentum:* $\mathbf{p} = m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)$.
Particle momentum $m\mathbf{v}$ is not canonical, but related to *canonical* $\mathbf{p}$ as follows: $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*

*Lagrangian for particle-in-(**A**,Φ)-potential*

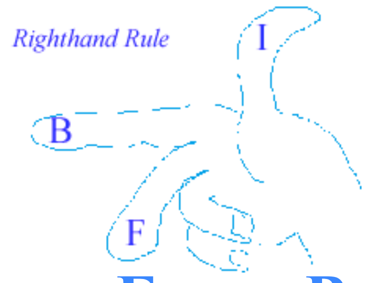
*Hamiltonian for particle-in-(**A**,Φ)-potential*

- *Canonical momentum in (**A**,Φ) potential*
- → *Hamiltonian formulation*
- *Hamilton's equations*

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t) \right) \right)$$



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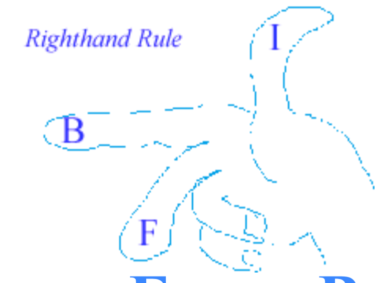


$$\mathbf{F}{=}q\mathbf{v}\times\mathbf{B}{=}\mathbf{I}\times\mathbf{B}$$

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + \cancel{e\mathbf{A}(\mathbf{r},t)} \right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet \cancel{e\mathbf{A}(\mathbf{r},t)} \right) \right)$$

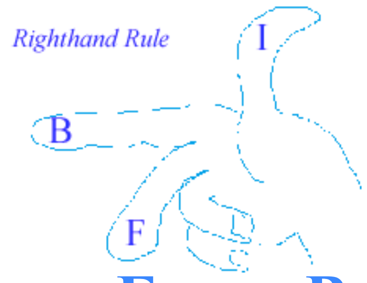
$$H = \frac{1}{2} m\mathbf{v} \bullet \mathbf{v} + e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

Righthand Rule



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+\cancel{e\mathbf{A}(\mathbf{r},t)}\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet\cancel{e\mathbf{A}(\mathbf{r},t)}\right)\right)$$

$$H=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}+e\Phi(\mathbf{r},t)$$

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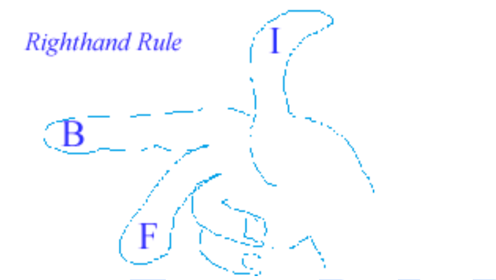
Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

Hamiltonian for charged particle in fields



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Hamiltonian for charged particle in fields

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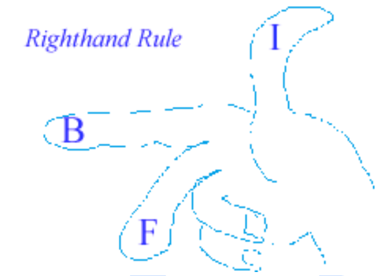

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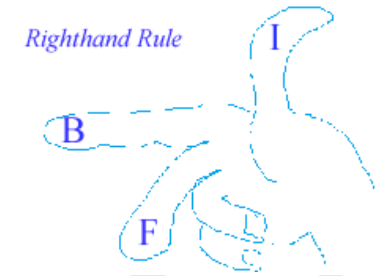
(Correct formally and numerically)

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.



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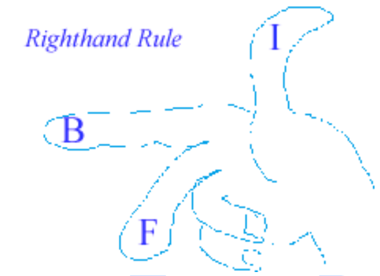
$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$



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Charge mechanics in electromagnetic fields

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Canonical momentum in ($\boldsymbol{A}$,Φ) potential

Hamiltonian formulation

Hamilton's equations

Hamilton's equations for charged particle in fields

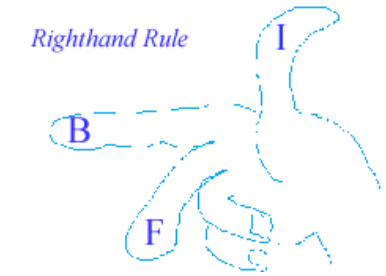
Hamiltonian is explicit function of ***momentum*** $\mathbf{p}$. Must replace velocity $\mathbf{v}$ using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) \bullet \left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H = \frac{\mathbf{p} \bullet \mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p} \bullet \mathbf{A} + \mathbf{A} \bullet \mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A} \bullet \mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Hamiltonian for charged particle in fields

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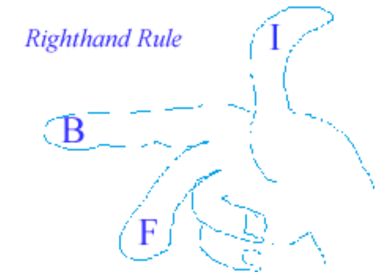

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(Expanded)

Hamilton's **v** equation:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) \bullet \left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H = \frac{\mathbf{p} \bullet \mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p} \bullet \mathbf{A} + \mathbf{A} \bullet \mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A} \bullet \mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)

Hamilton's **v** equation: $\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of ***momentum*** $\mathbf{p}$. Must replace velocity $\mathbf{v}$ using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

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(Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A} + \mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A} + e\Phi(\mathbf{r},t)$$

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Hamilton's equations for charged particle in fields

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$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$ ---------------- $$\dot{p}_a = m\dot{v}_a + e\dot{A}_a = +\frac{\left(p_\mu - eA_\mu\right)}{m}e\frac{\partial A_\mu}{\partial x_a} - e\frac{\partial \Phi}{\partial x_a}$$

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$$m\dot{\mathbf{v}}=e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\mathbf{E}\right)=e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

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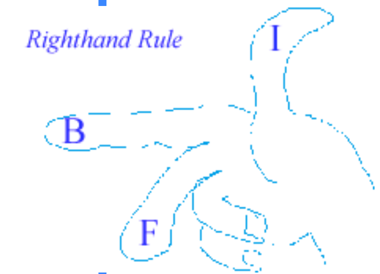
(Expanded)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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$$\frac{\partial A_a}{\partial t} = \boxed{\dot{A}_a - \sum_\mu v_\mu \frac{\partial A_a}{\partial x_\mu}}$$

...and now

we come back

full circle...

$$m\dot{v}_a = e\left(\underbrace{v_\mu \frac{\partial A_\mu}{\partial x_a} - v_\mu \frac{\partial A_a}{\partial x_\mu}} + E_a \right)$$

$$m\dot{\mathbf{v}} = e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right) + \mathbf{E} \right) = e(\mathbf{v}\times\mathbf{B} + \mathbf{E})$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

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for particle mechanics

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

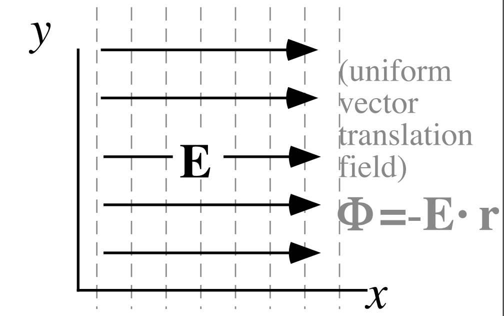
Practical poolhall application

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = \nabla\left(-\mathbf{E} \bullet \mathbf{r}\right) = \mathbf{E} = const.$$

Fig. 2.4.1.



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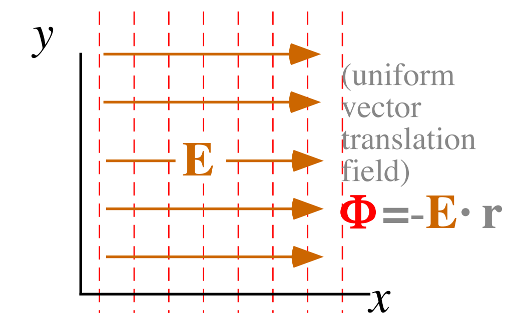

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A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

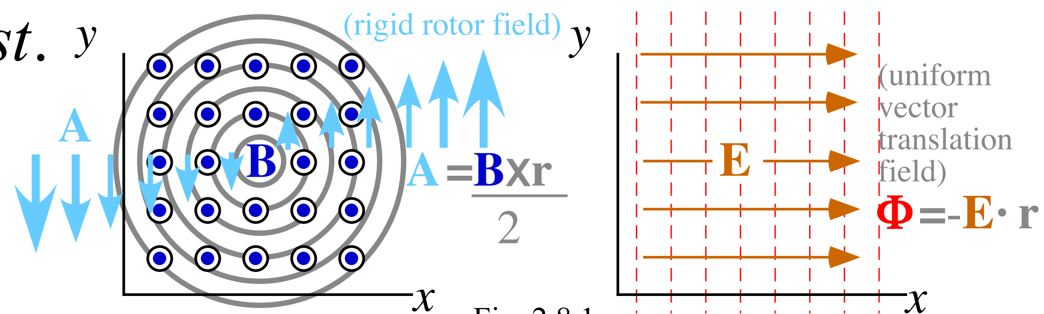

Fig. 2.8.1.

*This mechanical analog of (E_x,B_z) field mimics **A**-field with tabletop **v**-field*

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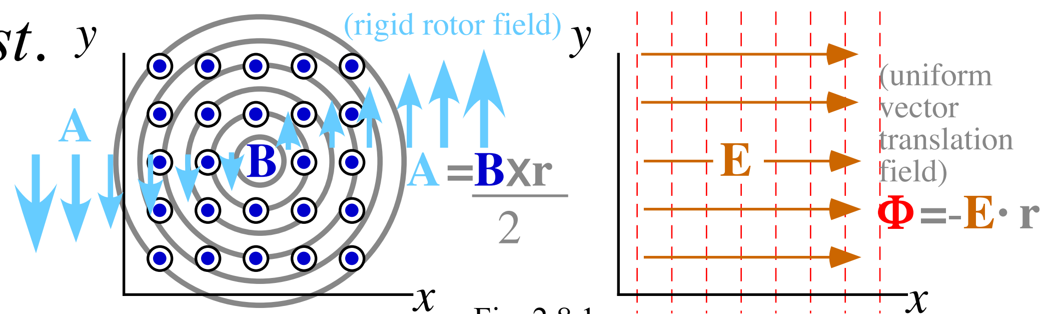

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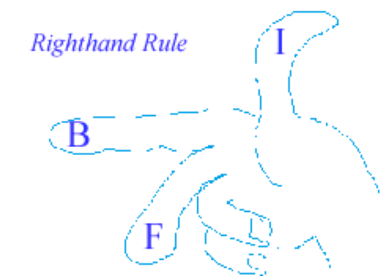

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v} \times \frac{e}{m}\mathbf{B}.$$

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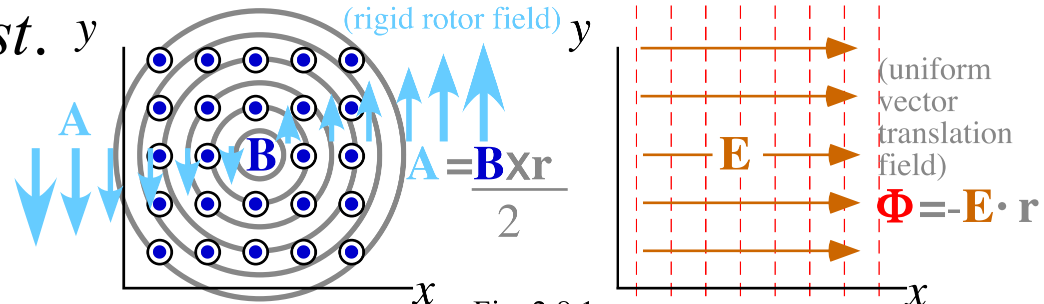

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Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

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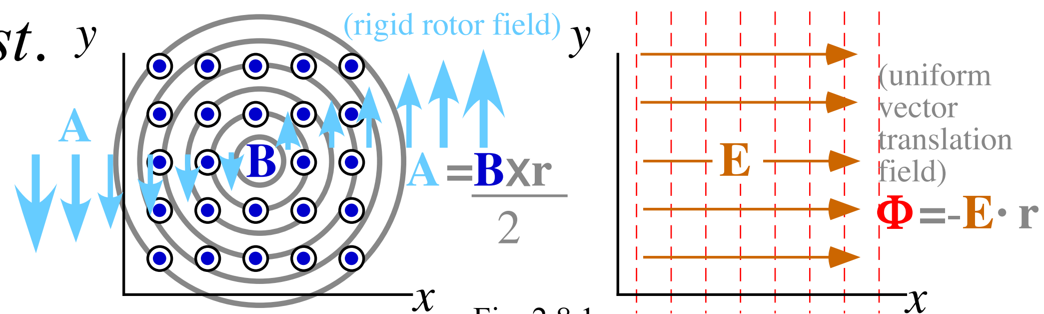

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$$\boxed{\begin{gathered}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{gathered}}$$

Gibb's notation:

$$\dot{\mathbf{v}} = \boldsymbol{\varepsilon} + \mathbf{v} \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit equations
Vector theory vs. complex variable theory
Mechanical analog of cyclotron and FBI rule
Cycloid geometry and flying sticks
Practical poolhall application

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\mathbf{\Phi}(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\mathbf{\Phi}(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

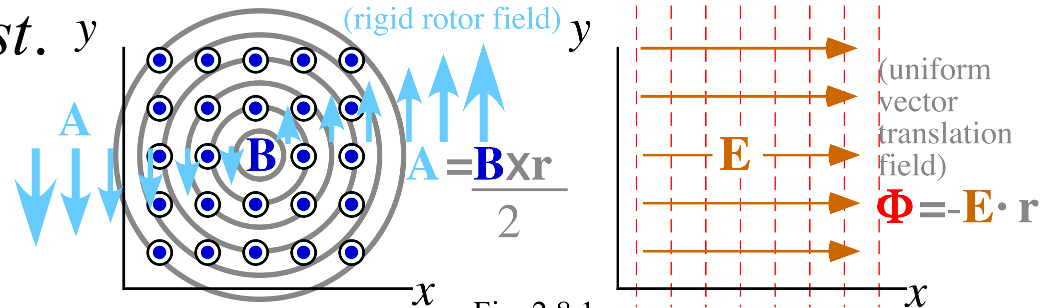

Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Gibb's notation:

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Complex variable ***velocity:*** $v=v_x+iv_y$ *and* ***electric field:*** $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

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with replacements : $\hat{\mathbf{e}}_\mathbf{x}\to 1$ and : $\hat{\mathbf{e}}_\mathbf{y}\to i = \sqrt{-1}$

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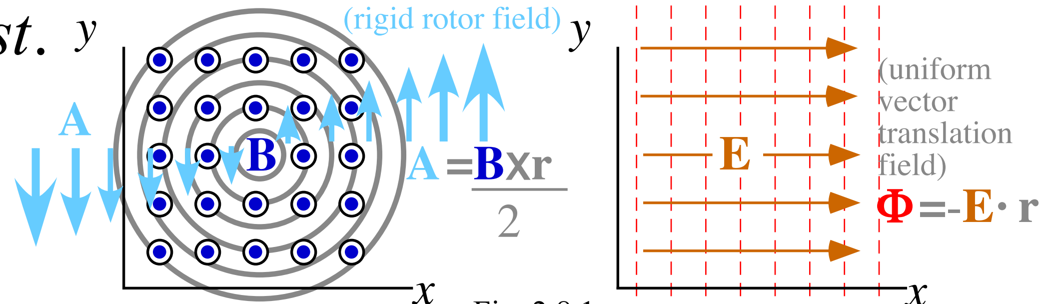

Fig. 2.8.1.

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$$\boxed{\begin{gathered}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{gathered}}$$

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Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

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$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

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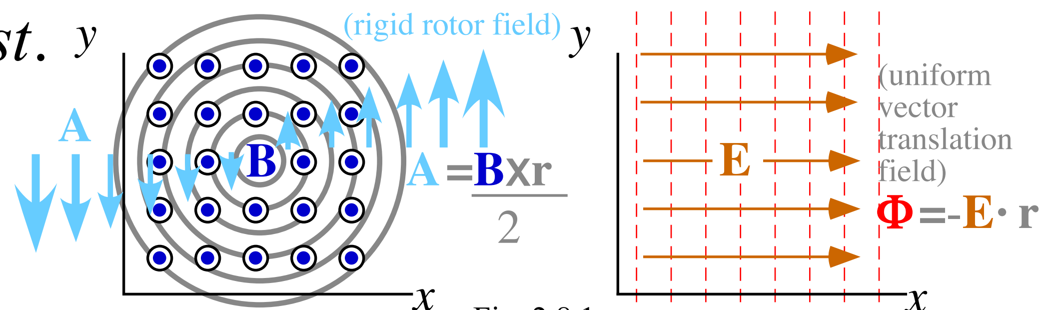

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where : $\beta = -\dfrac{\varepsilon}{iB} = i\dfrac{\varepsilon}{B}$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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Shorthand Labeling

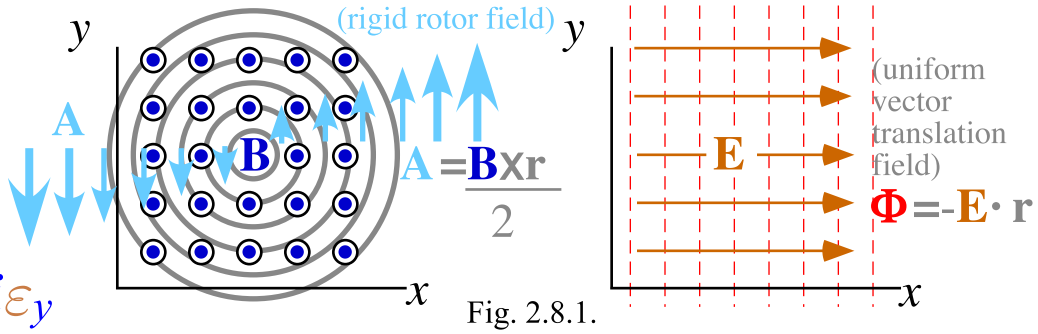

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

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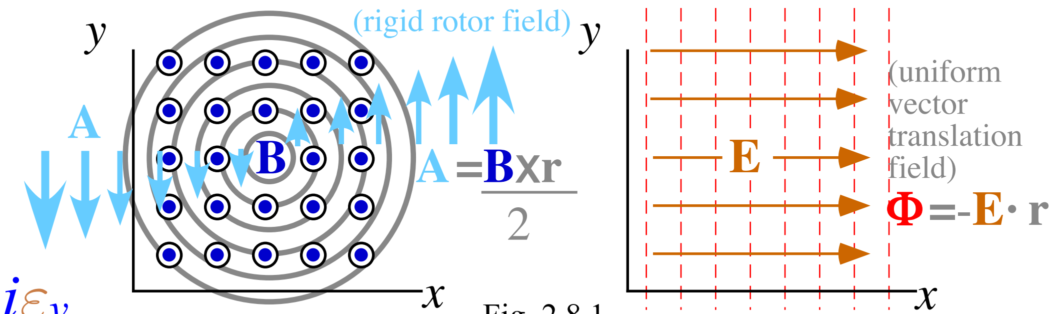
Shorthand Labeling



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An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

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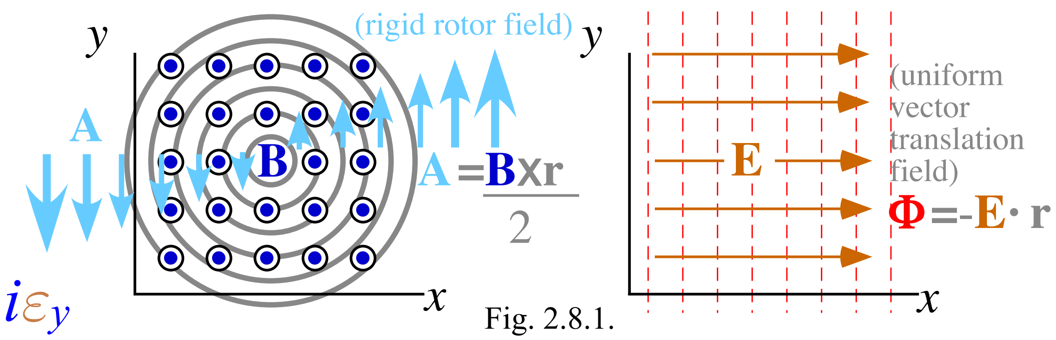

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$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta)$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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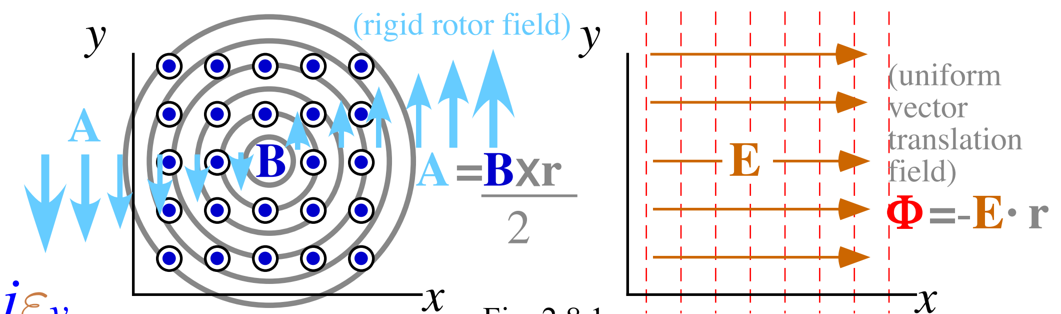
Shorthand Labeling



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complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}\left(v(0)+i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}$$

Crossed E and B field mechanics (Solution by complex variables)

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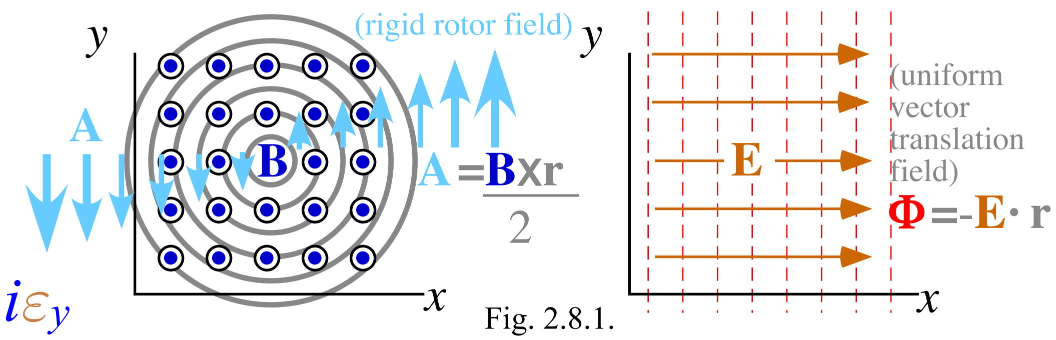
Shorthand Labeling



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A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

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complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}\left(v(0) + i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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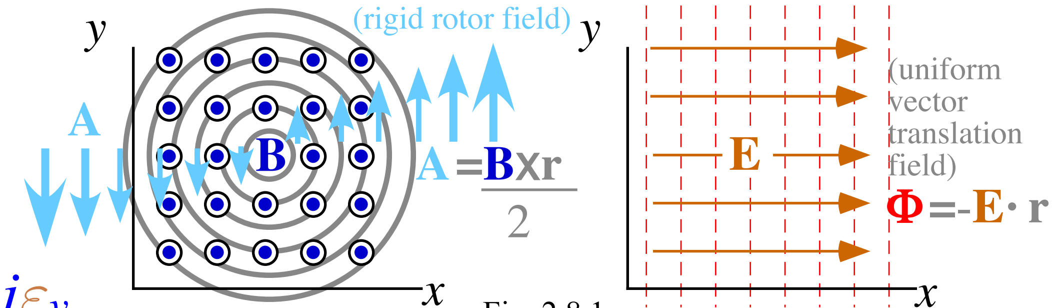
Shorthand Labeling



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Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

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A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

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complex form

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}\left(v(0) + i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}$$

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Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

vector form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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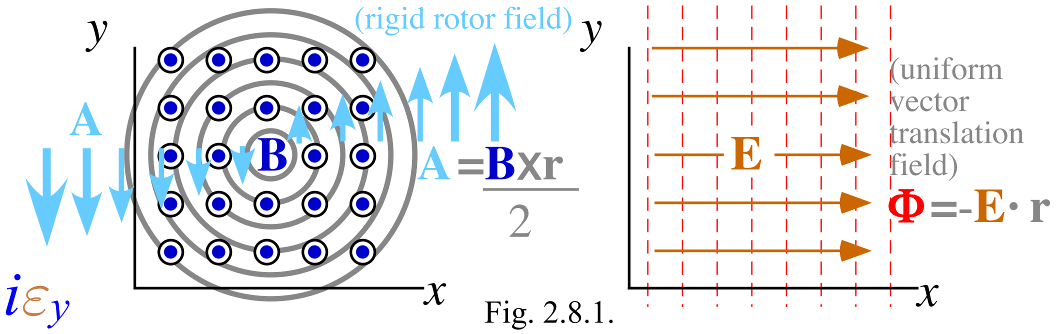
Shorthand Labeling



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$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \rightarrow 1 \qquad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

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An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)= e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}\left(v(0)+i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}\left(v(0)+i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - \left(\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2}\right)$$

complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

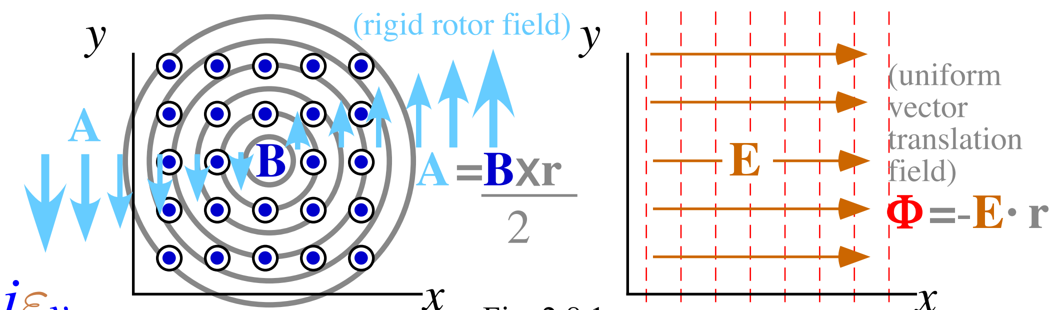

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \to 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

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complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

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complex form

$$x(t) + iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

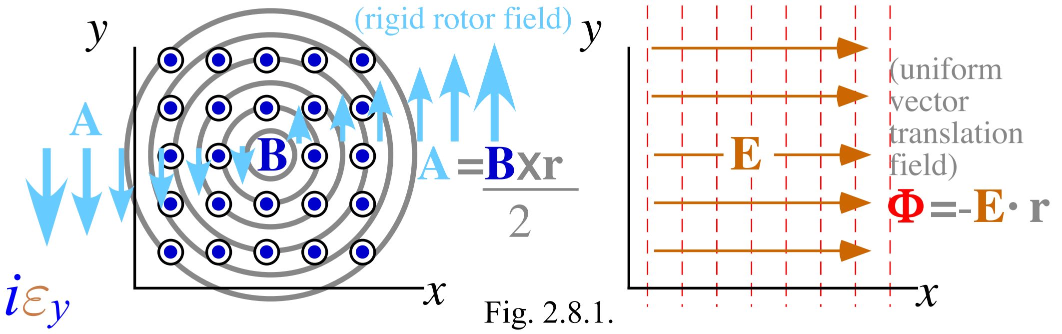

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

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complex form

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Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

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complex form

$$x(t) + iy(t) = e^{-iB\cdot t}\left(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}\right) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where :} \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

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complex form

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$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$ *complex form*

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$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where : } \beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}$$

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vector form

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$ *complex form*

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$ *complex form*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$ *vector form*

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where : } \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

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complex form

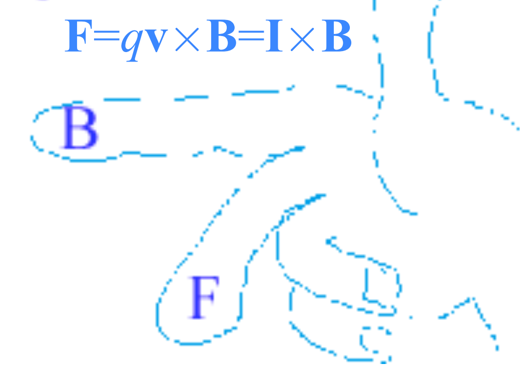
$$x(t)+iy(t)= \quad e^{-iB\cdot t} \quad (i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}) \quad -i\frac{\varepsilon}{B}\cdot t \quad +x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

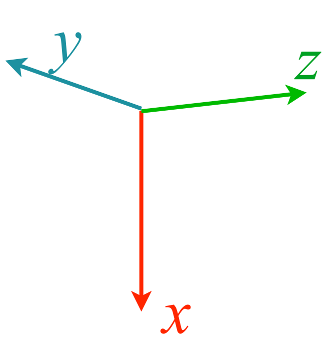
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form

Righthand Rule

F=q**v**×**B**=**I**×**B**





Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where}: \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

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complex form

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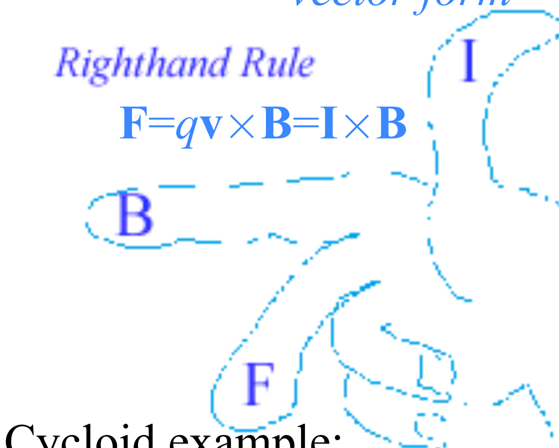
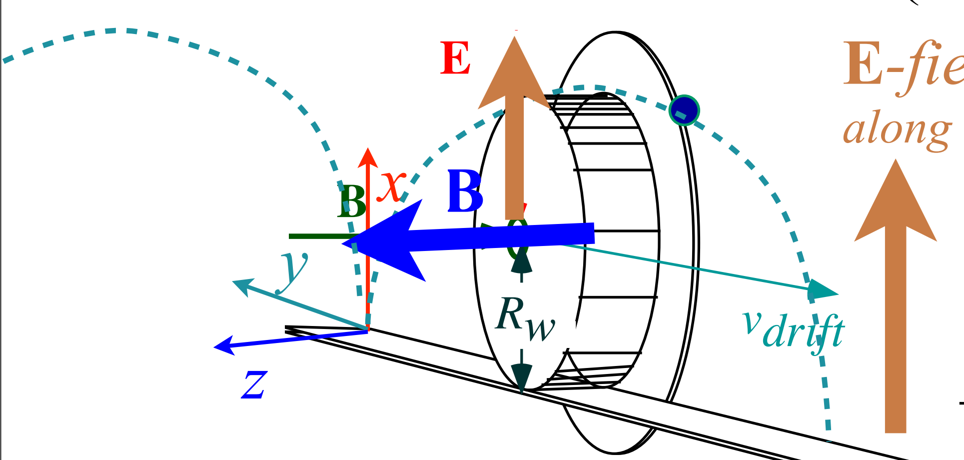
vector form



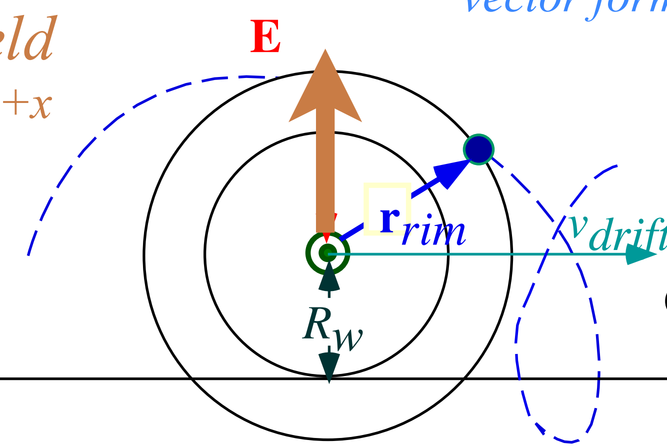
Cycloid example:
initial $(x(0), y(0)) = (0,0)$
and $(v_x(0), v_y(0)) = (0,0)$

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} =$ is on rim of a wheel of radius $R_W = E/B^2$

$$\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix} + \begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$



E-field
along +x



Constant v_{drift} in -y-direction
$v_{drift} = -E/B$

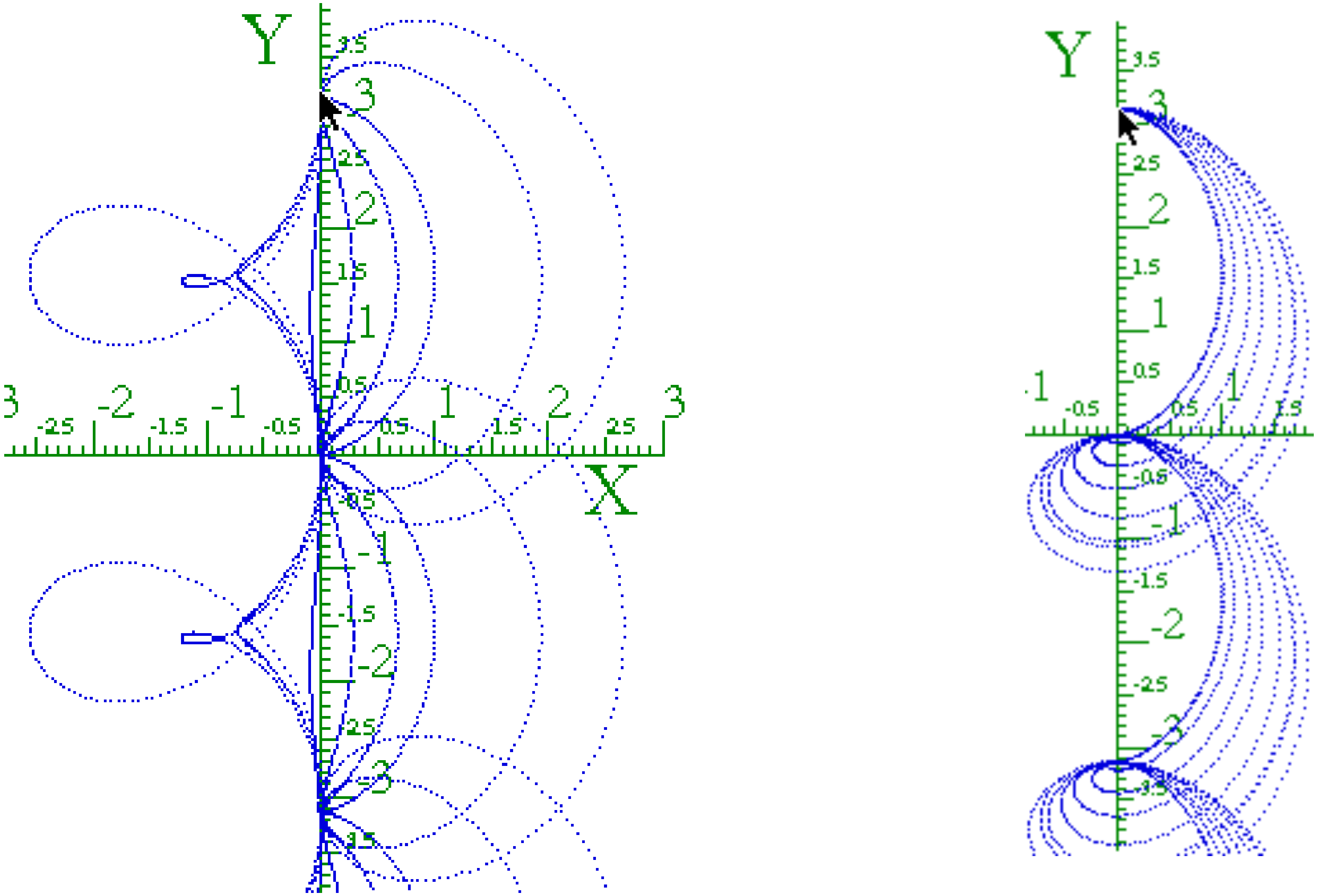
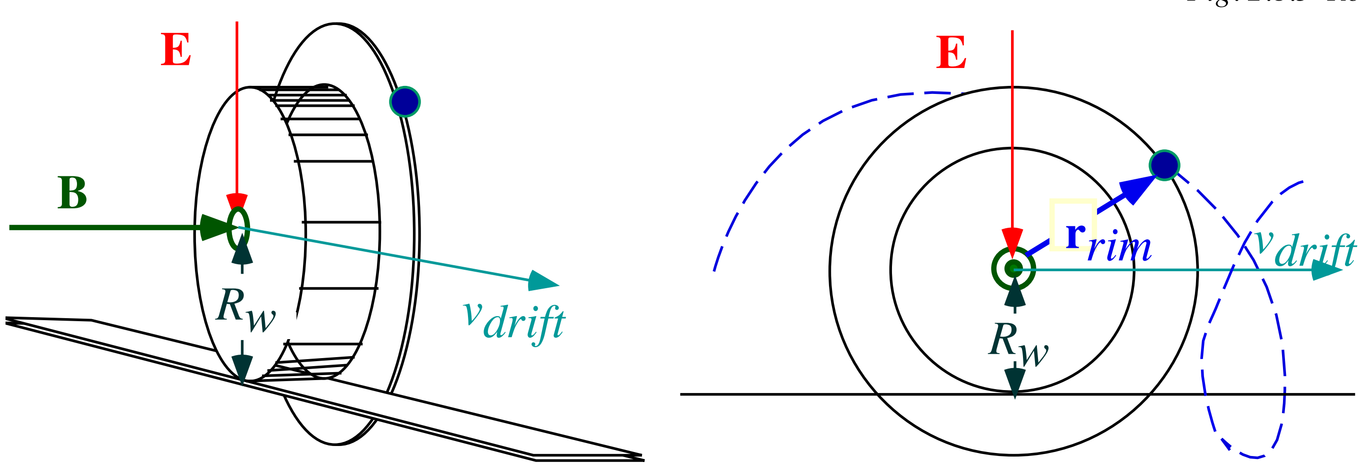


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (*E=1/2, B=1*)

Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits



Initial position x(0) = 1.38263

Initial position y(0) = 1.49839

Initial momentum px(0) = 0

Initial momentum py(0) = 0

Terminal time t(off) = 6.28318

Maximum step size dt = 0.08

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = 0

Core thickness r = 0.00000

x-Stark field Ex = 0

y-Stark field Ey = -0.1

Zeeman field Bz = 1

Diamagnetic strength k = 0

Plank constant h-bar = 1.57079

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), x = 8

Particle Size = 8

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐

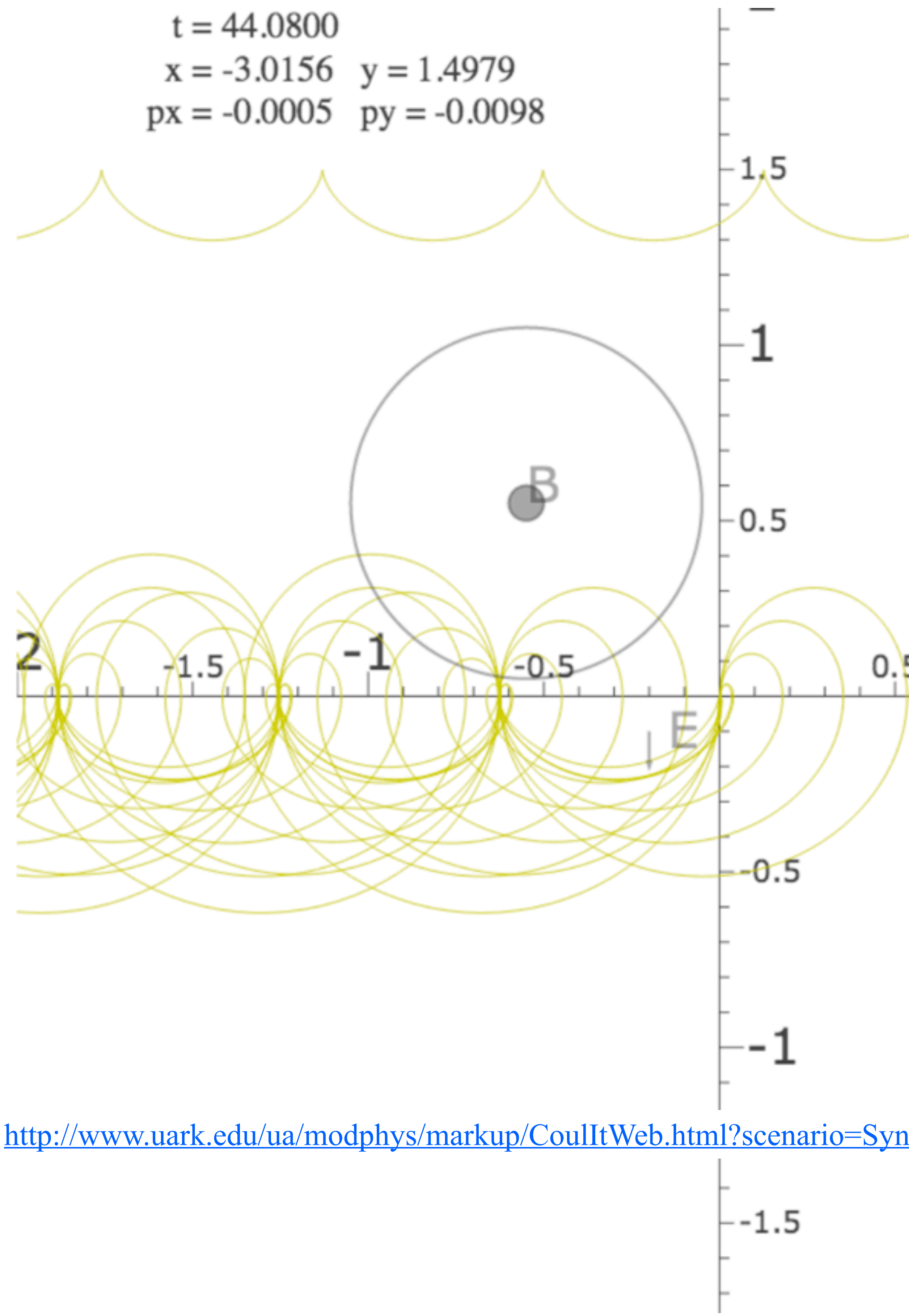
Plot r(t) ☑ Plot p(t) ☐

Color action ☐ No stops ☑ Field vectors ☑ Info ☑

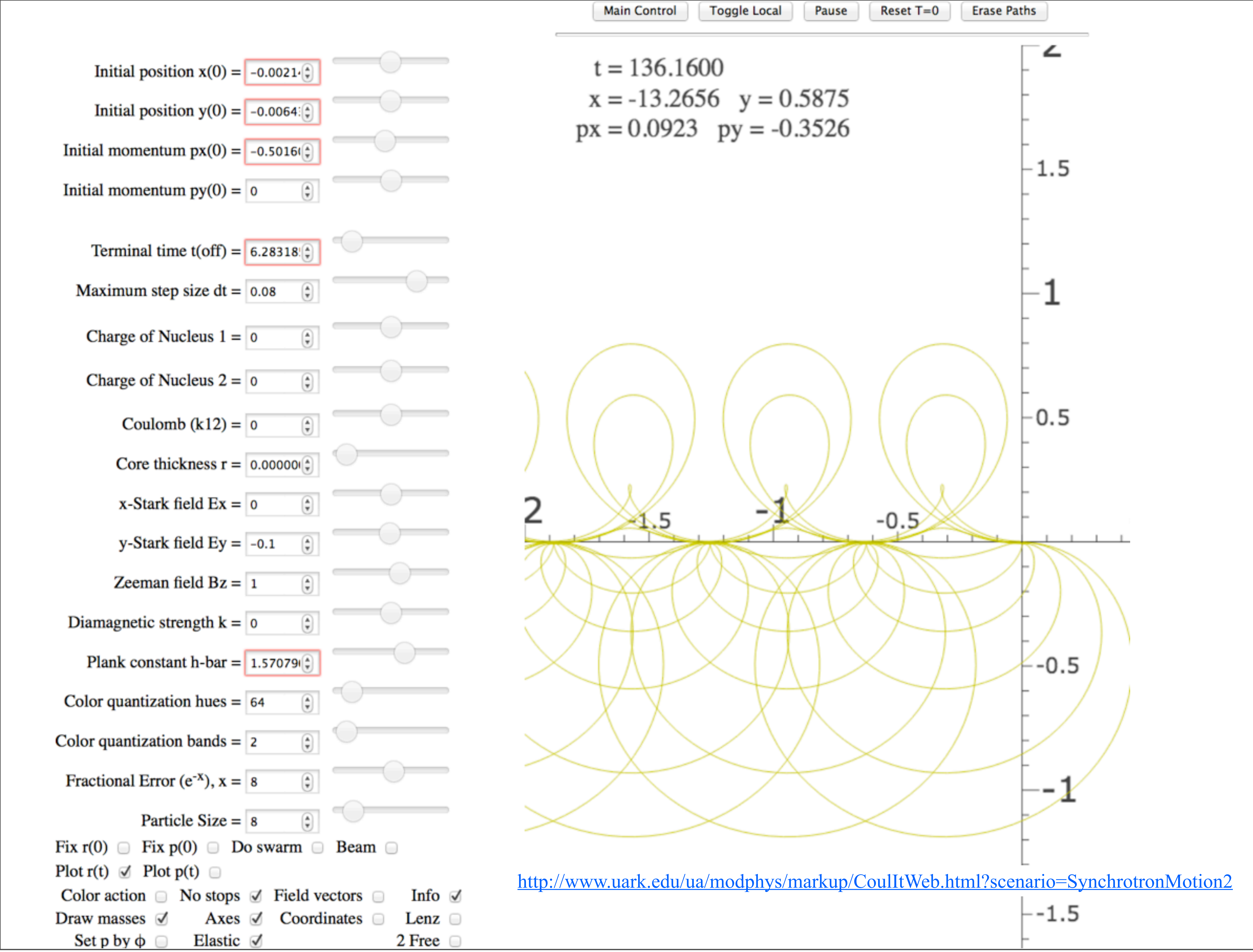
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☐

Set p by ϕ ☐ Elastic ☑ 2 Free ☐

Save to GIF ☐



http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=SynchrotronMotion



http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=SynchrotronMotion2

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit equations

Vector theory vs. complex variable theory

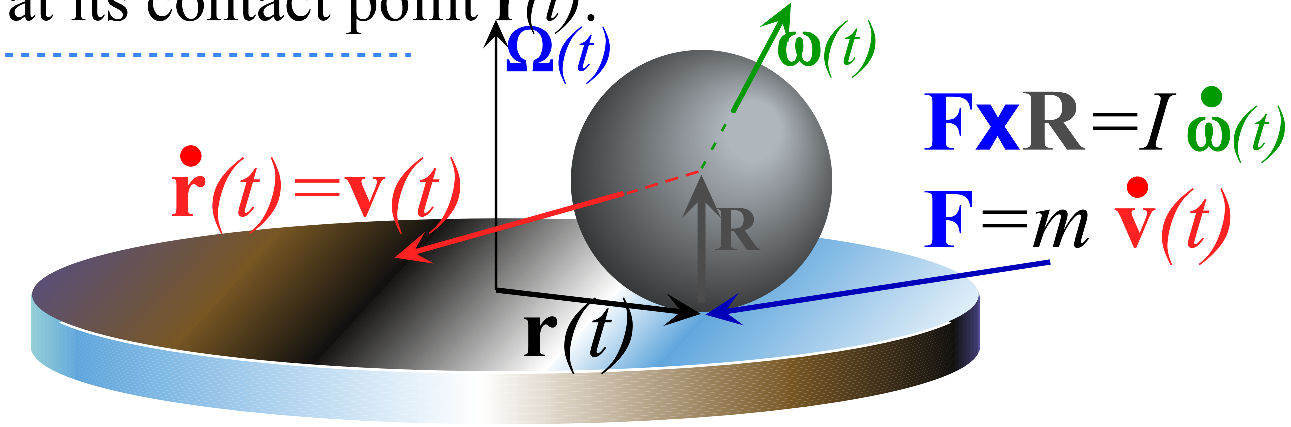
Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

Practical poolhall application

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals
table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

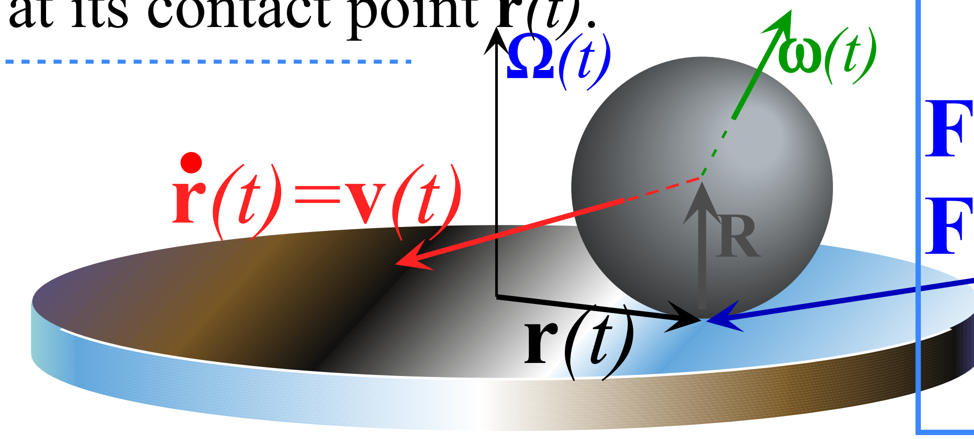


Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

[YouTube Video of Analog to Syncrotron Motion]

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

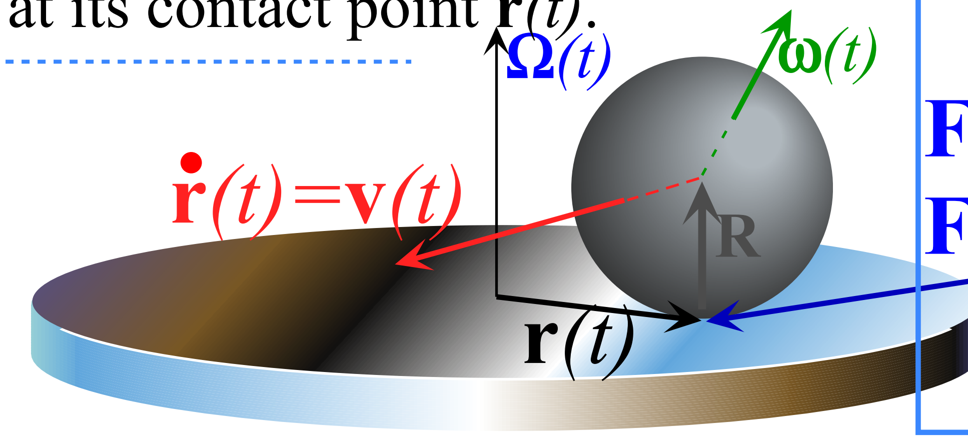
Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t) = \mathbf{F}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

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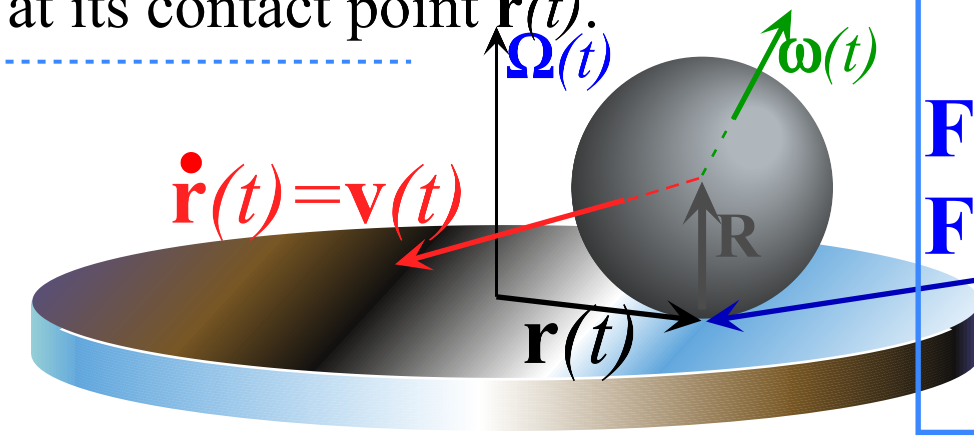

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$$\begin{aligned} I\dot{\boldsymbol{\omega}}(t) &= \mathbf{F}(t)\times\mathbf{R} \\ &= m\dot{\mathbf{v}}(t)\times\mathbf{R} \\ &= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R \end{aligned}$$

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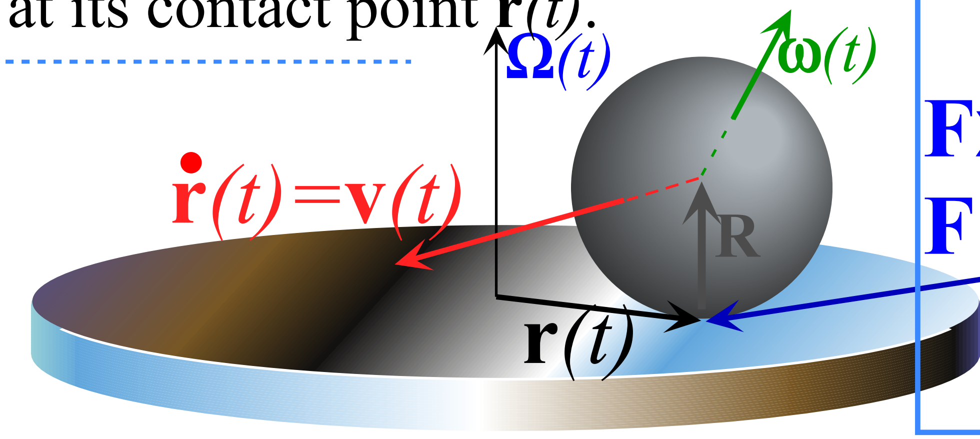

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Do time-derivative

$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R = \boldsymbol{\Omega}\times\mathbf{v}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

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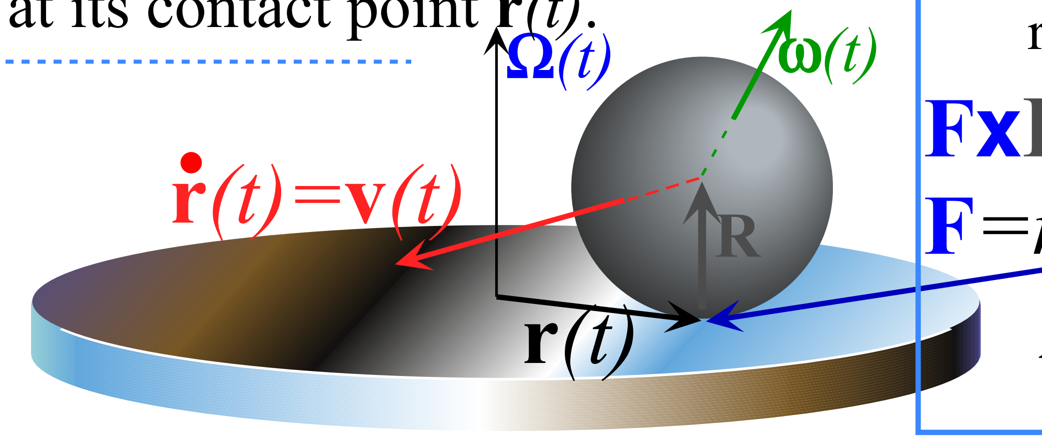
Rolling Constraint



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use: $\dot{\boldsymbol{\omega}}(t) = \dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

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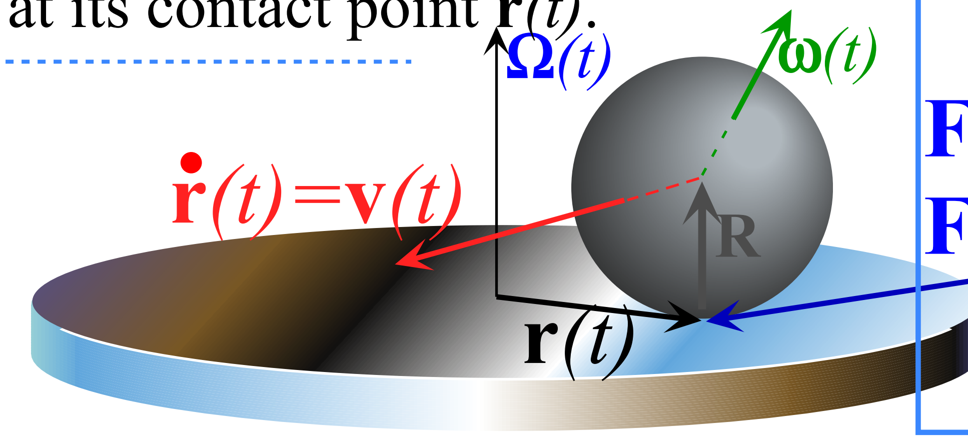
Rolling Constraint



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$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

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$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A} = (\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B} = \dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A} = \hat{\mathbf{z}}R = \mathbf{C}$

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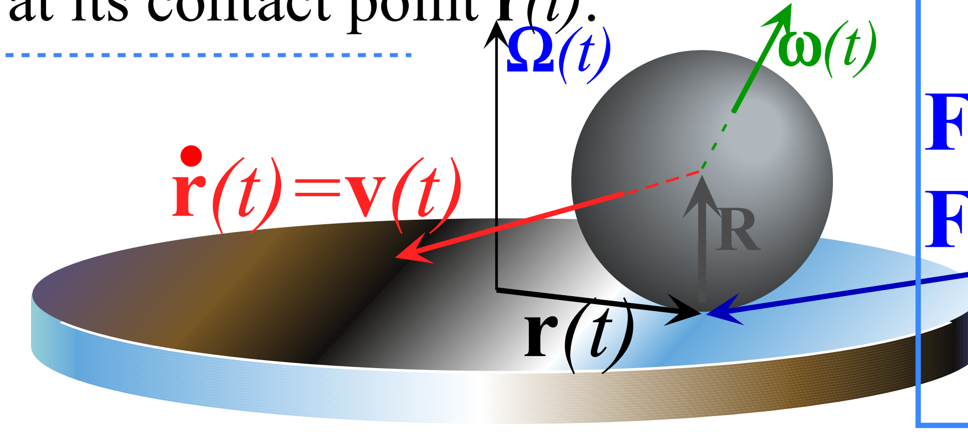

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$$\dot{\mathbf{v}}(t) = \mathbf{\Omega}\times\dot{\mathbf{r}}(t) \;+\; \dot{\omega}(t) \;\times\hat{\mathbf{z}}R$$

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($\mathbf{v}(t)$ *always normal to* $\hat{\mathbf{z}}$)

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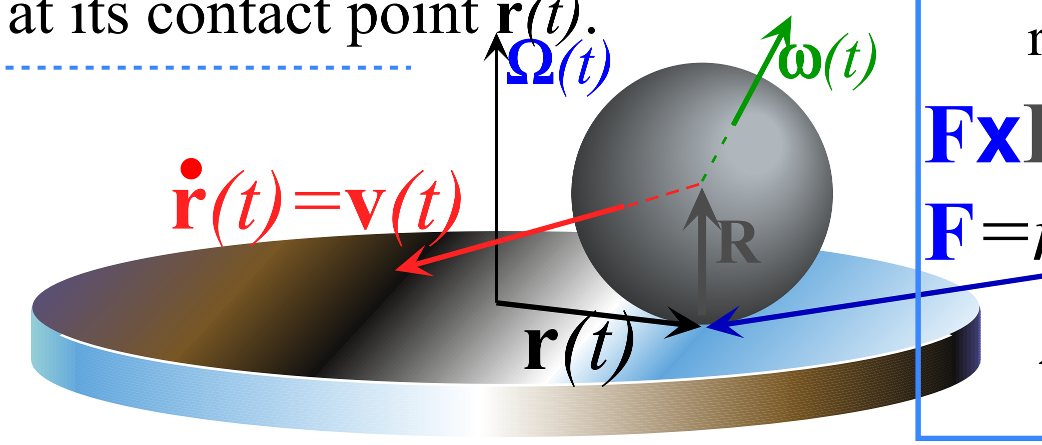

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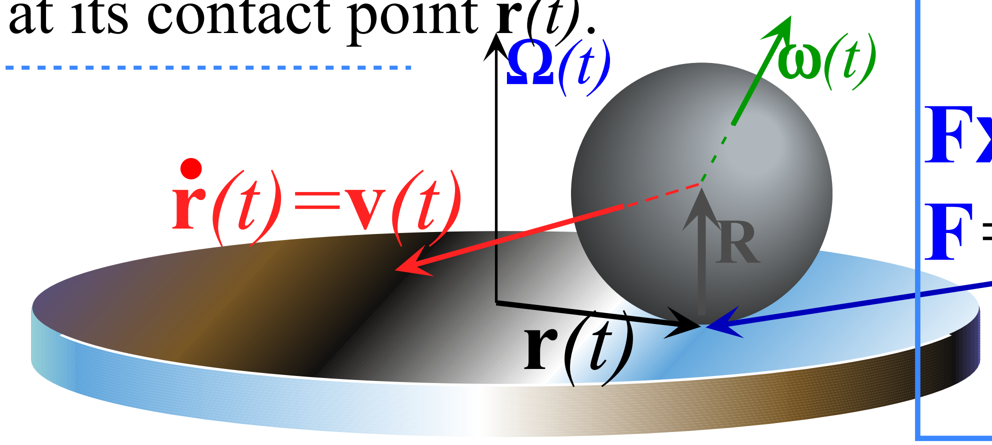

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$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)$$

or :

F=B×v *mechanical analog:*

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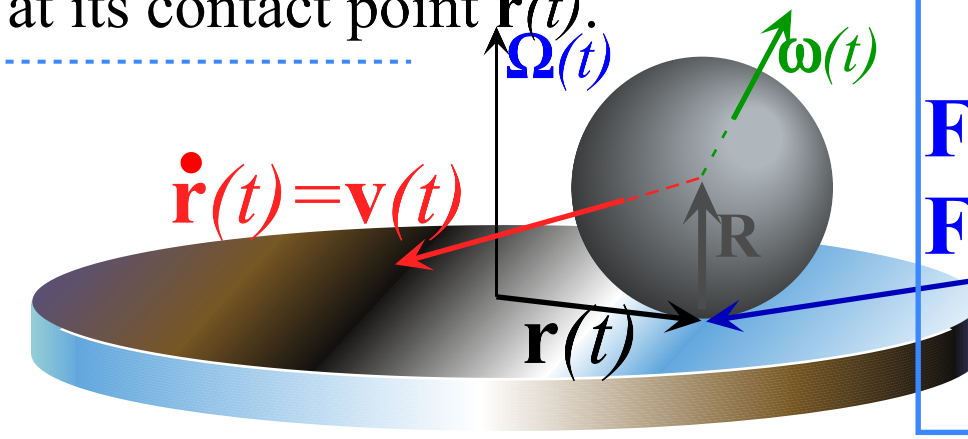

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Mechanical analog cyclotron frequency

$$\omega=\frac{e}{m}B=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}$$

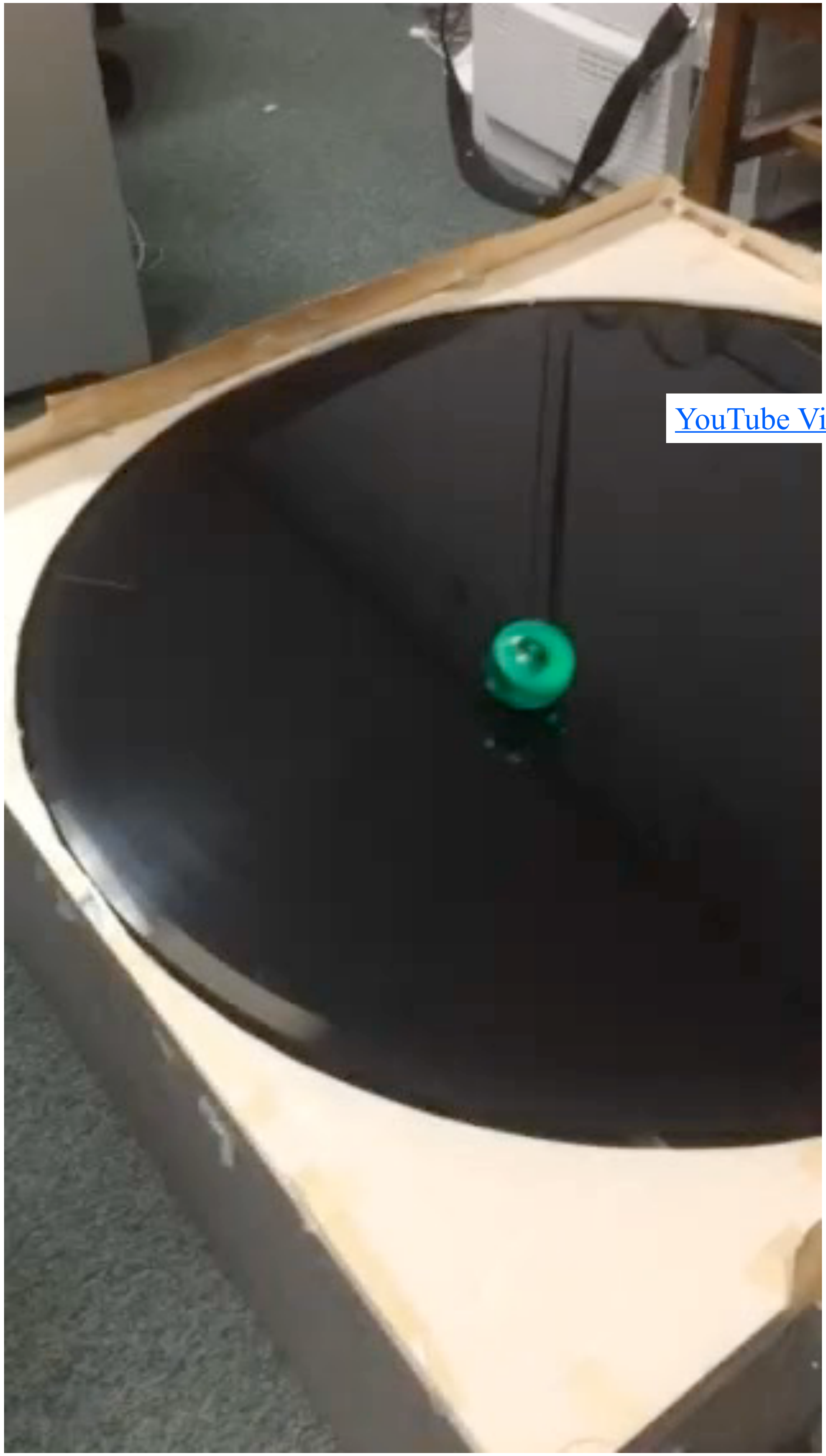
$\omega=\frac{2}{7}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{5}$

$=\frac{2}{5}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{3}$

$m\mathbf{a}=e\mathbf{B}\times\mathbf{v}$ mechanical analog:

or : $$\dot{\mathbf{v}}(t)=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}\times\mathbf{v}(t)$$

[YouTube Video of Analog to Syncrotron Motion]



[YouTube Video of Analog to Syncrotron Motion]

Solid ball has 2 orbits
as table turns 7 rotations

Mechanical analog
cyclotron frequency

$$\omega = \frac{e}{m} B = \frac{\boldsymbol{\Omega}}{1 + \frac{mR^2}{I}}$$

$\omega = \frac{2}{7}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2} = \frac{2}{5}$

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Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

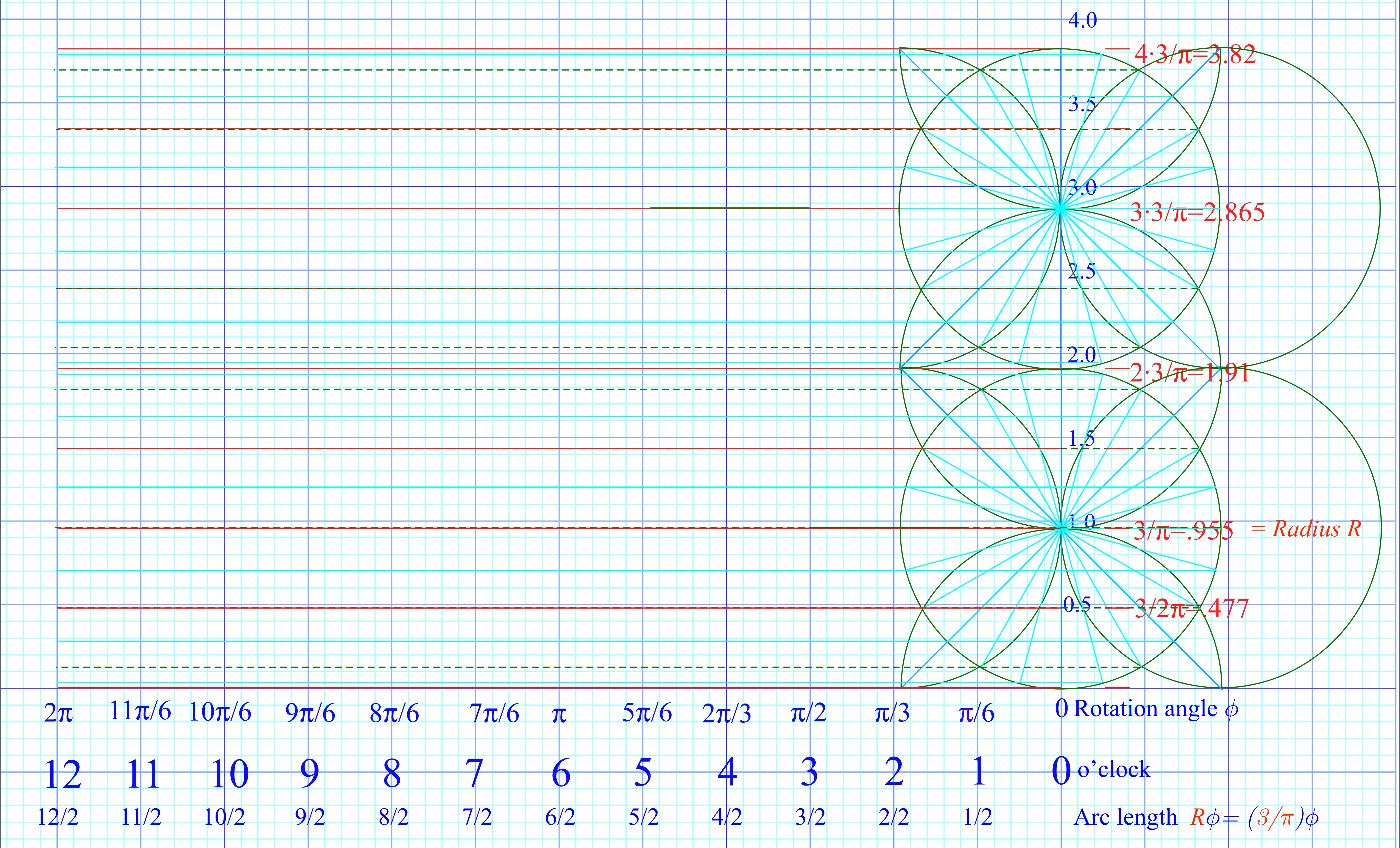
Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

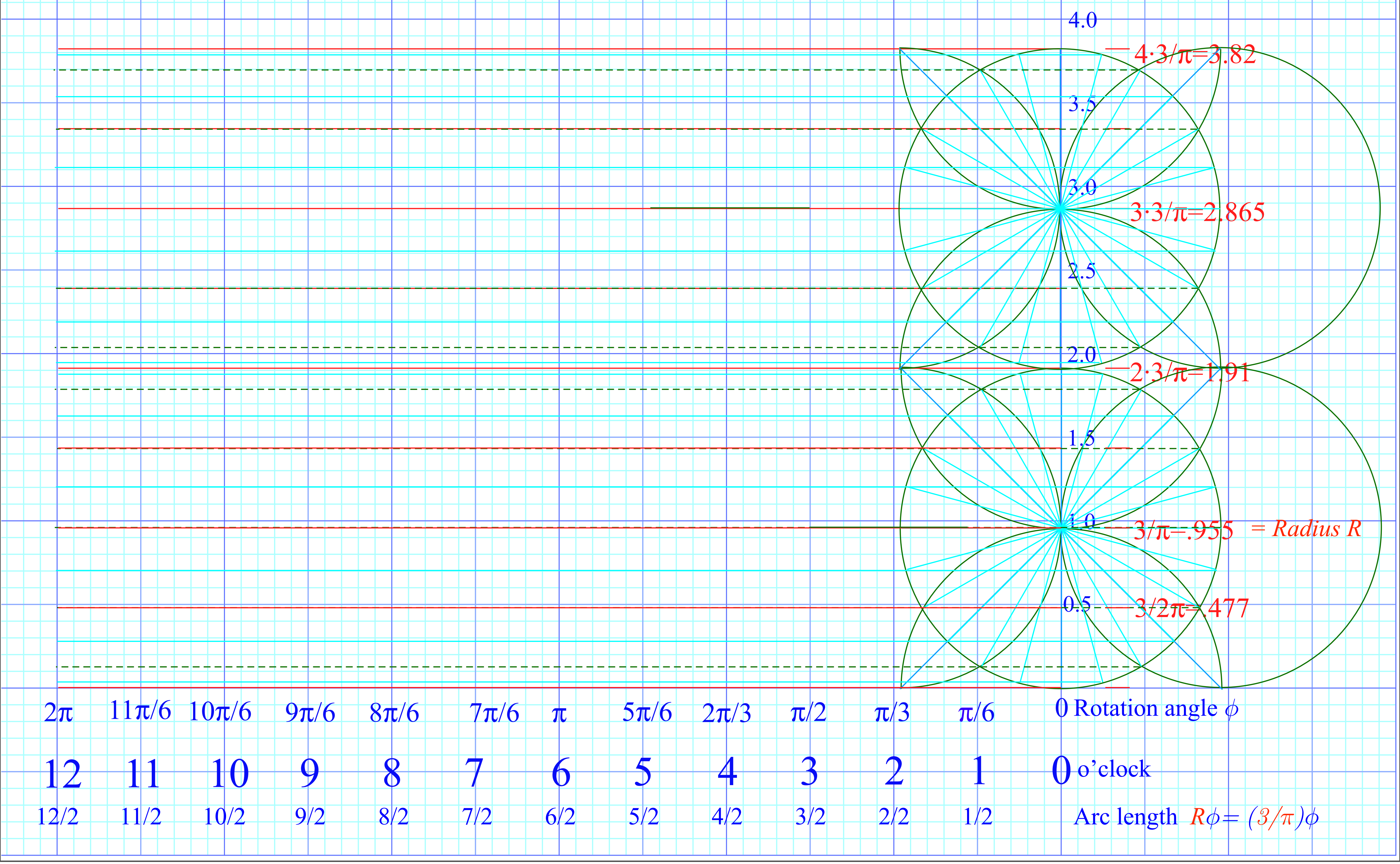
→ *Cycloid geometry and flying sticks*

Practical poolhall application

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$
is a rational length of rolled -out circumference $R\phi = (3/\pi)m\pi/n=3m/n$.



Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled-out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$



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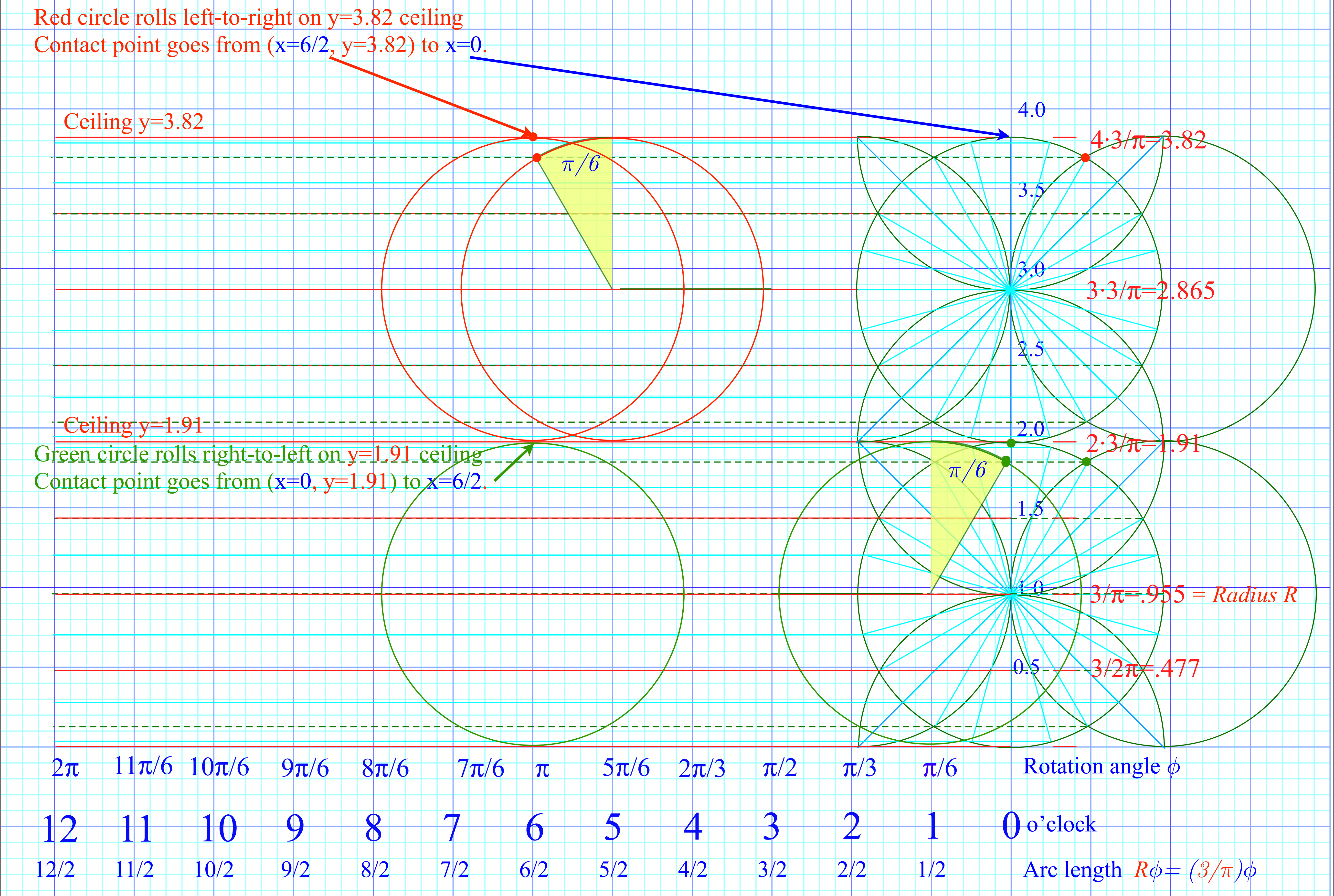

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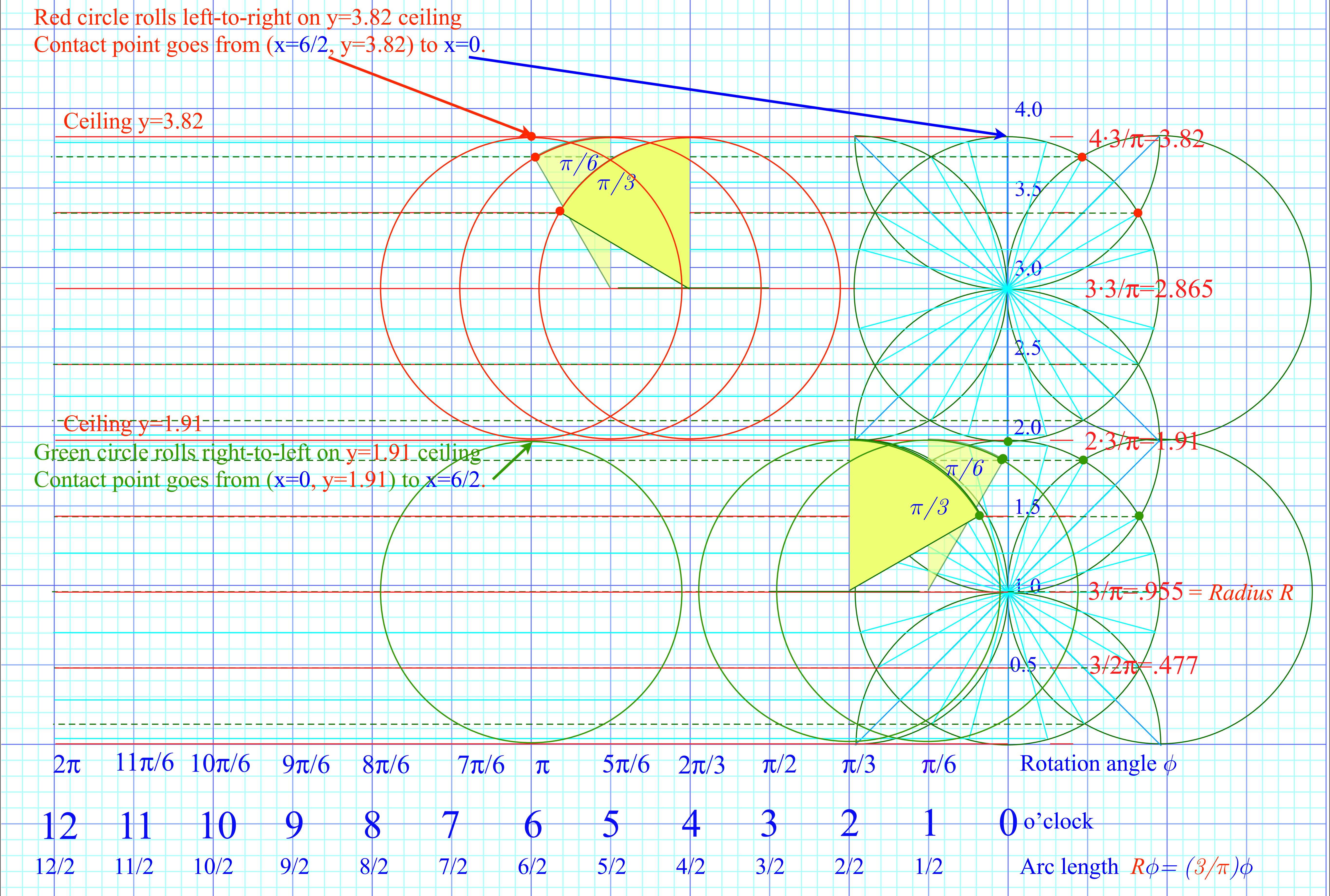

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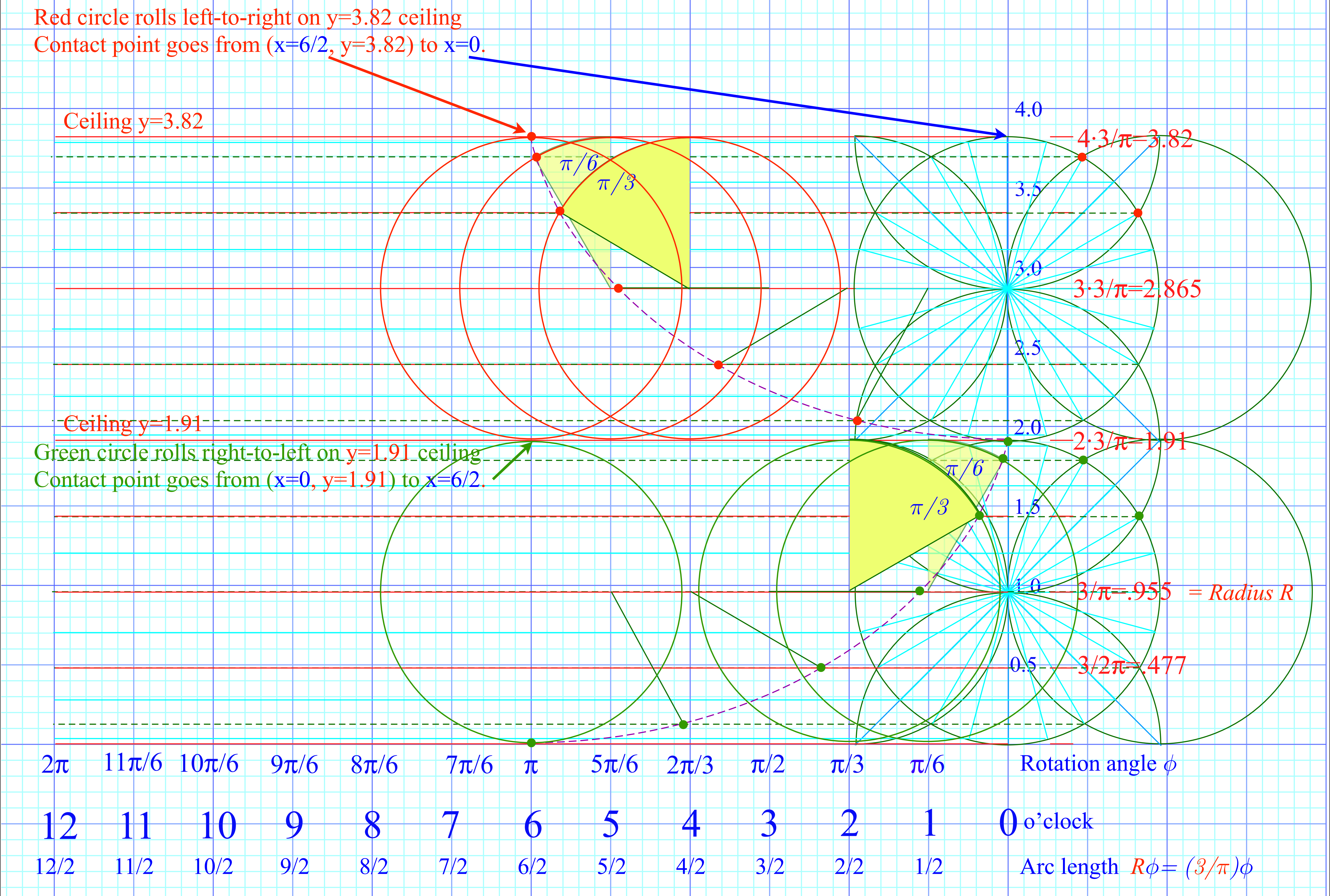

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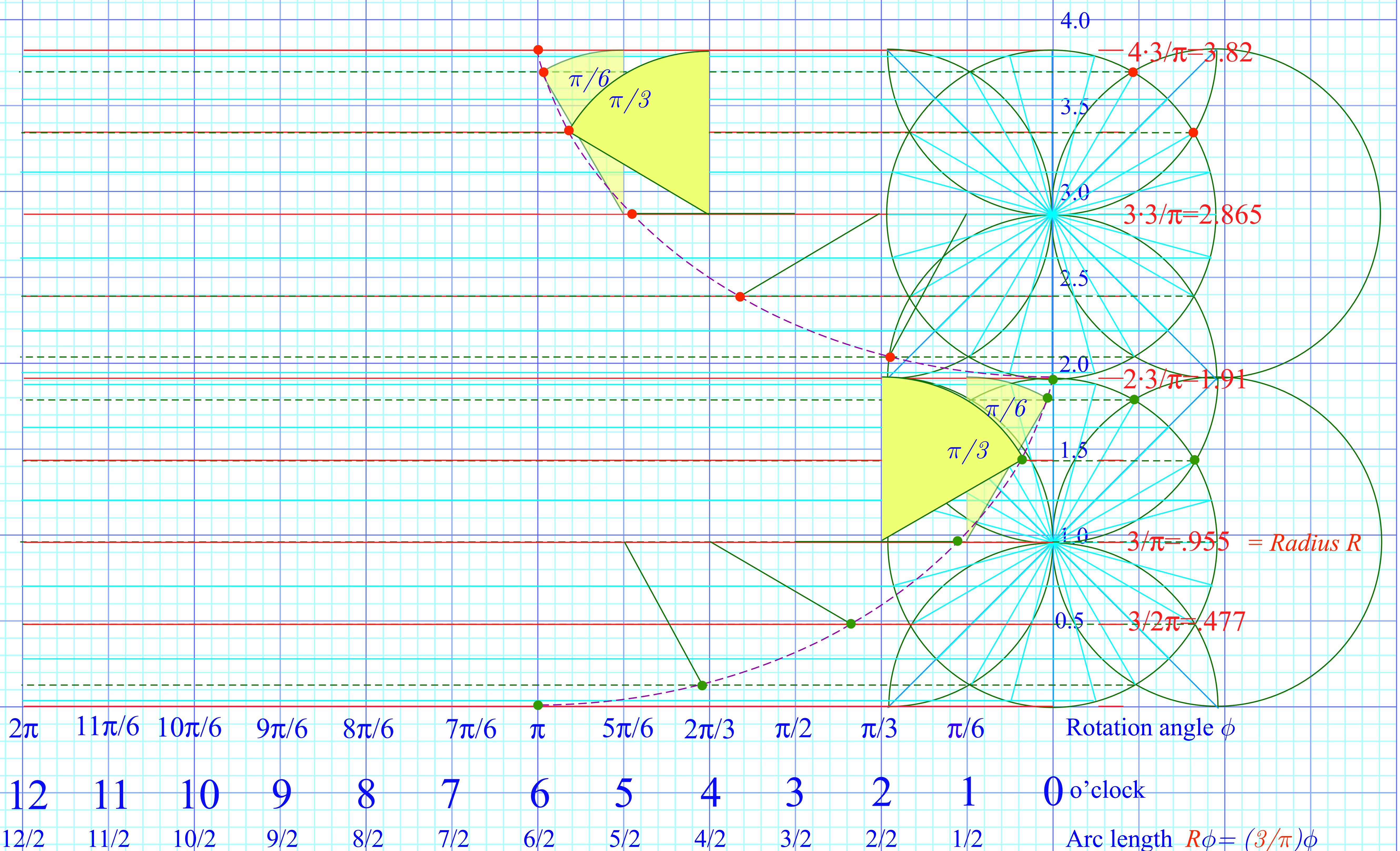

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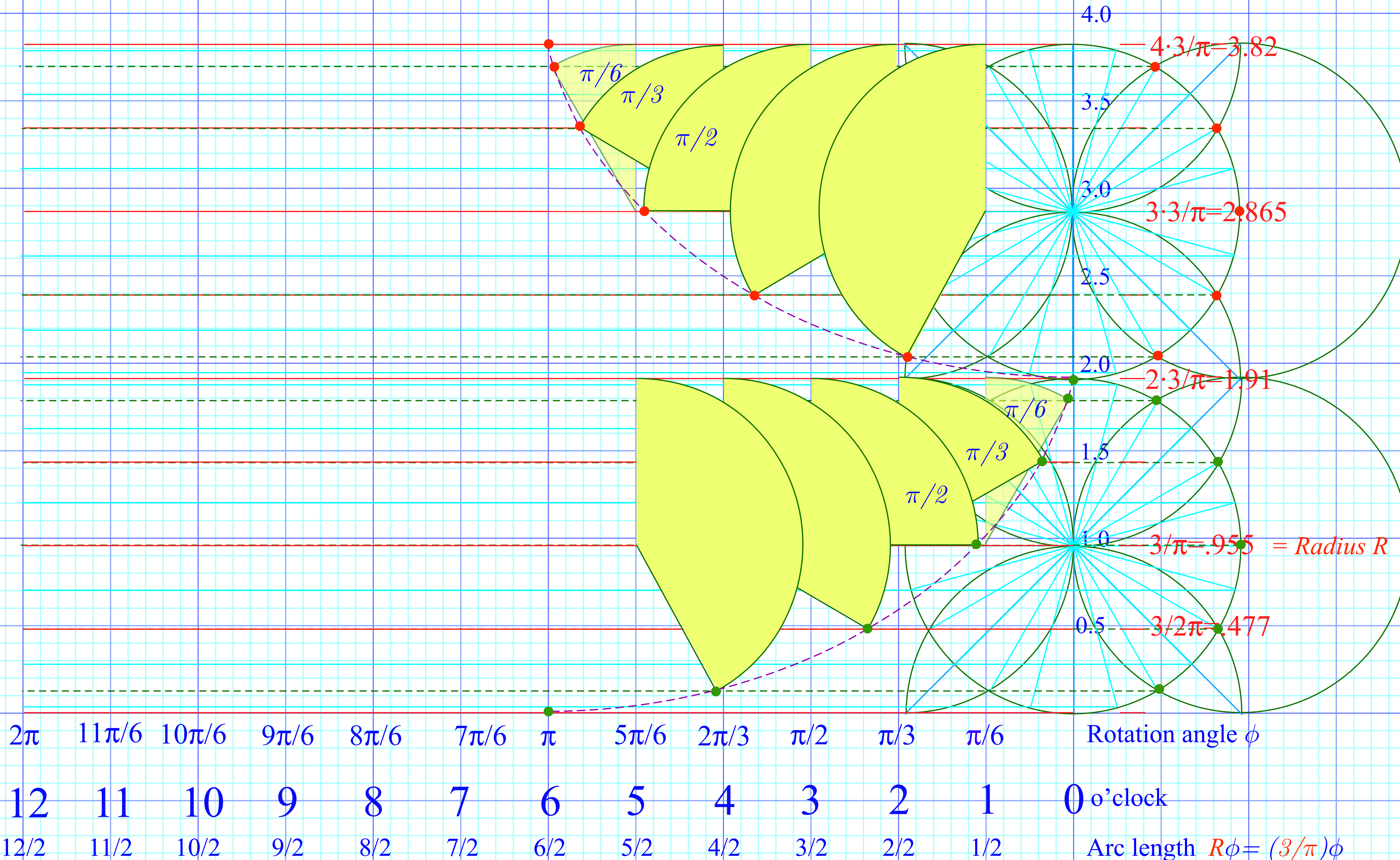

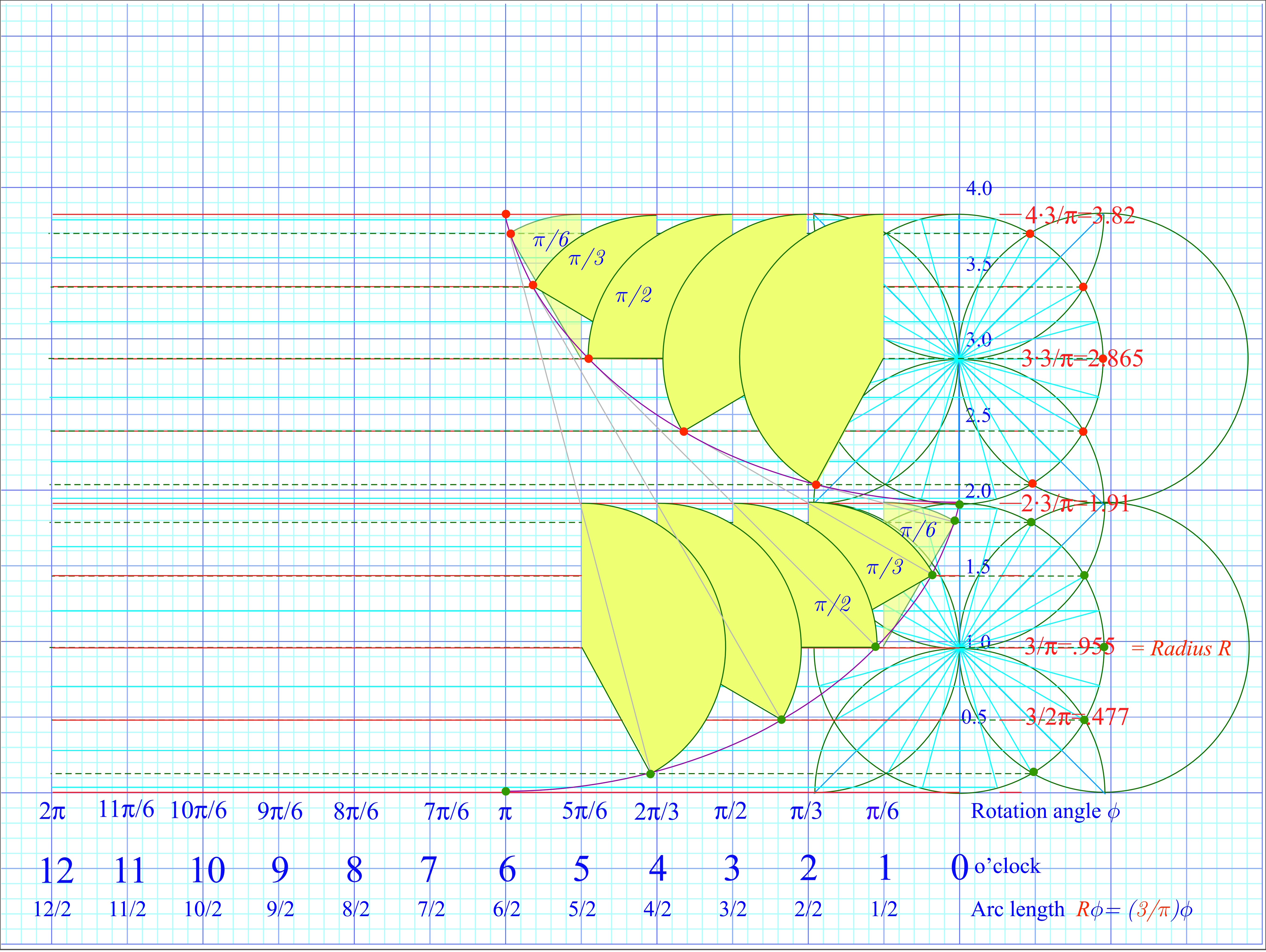
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/2
π/6
π/3
π/2
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle φ
12
11
10
9
8
7
6
5
4
3
2
1
0 o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rφ= (3/π)φ

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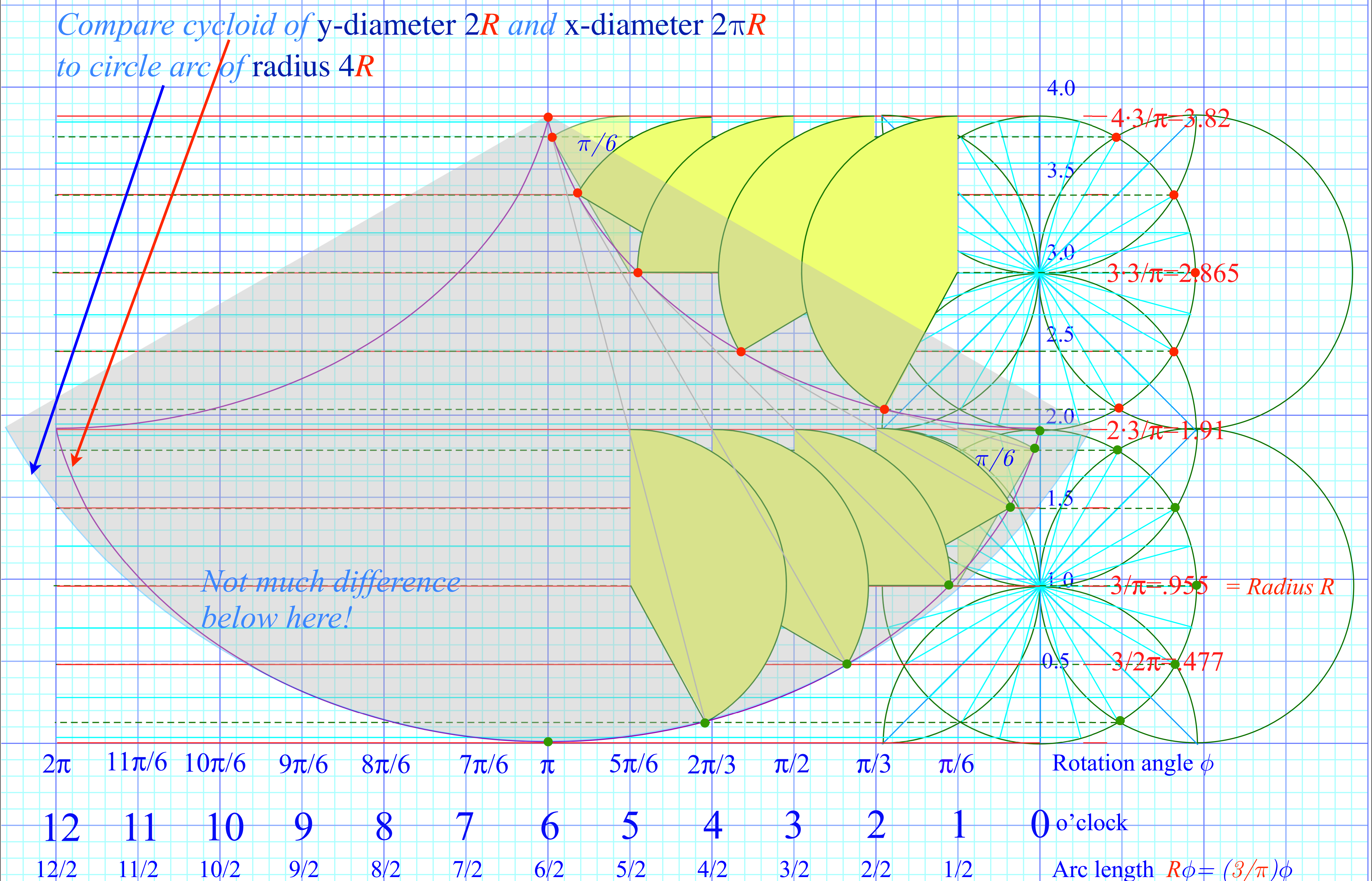

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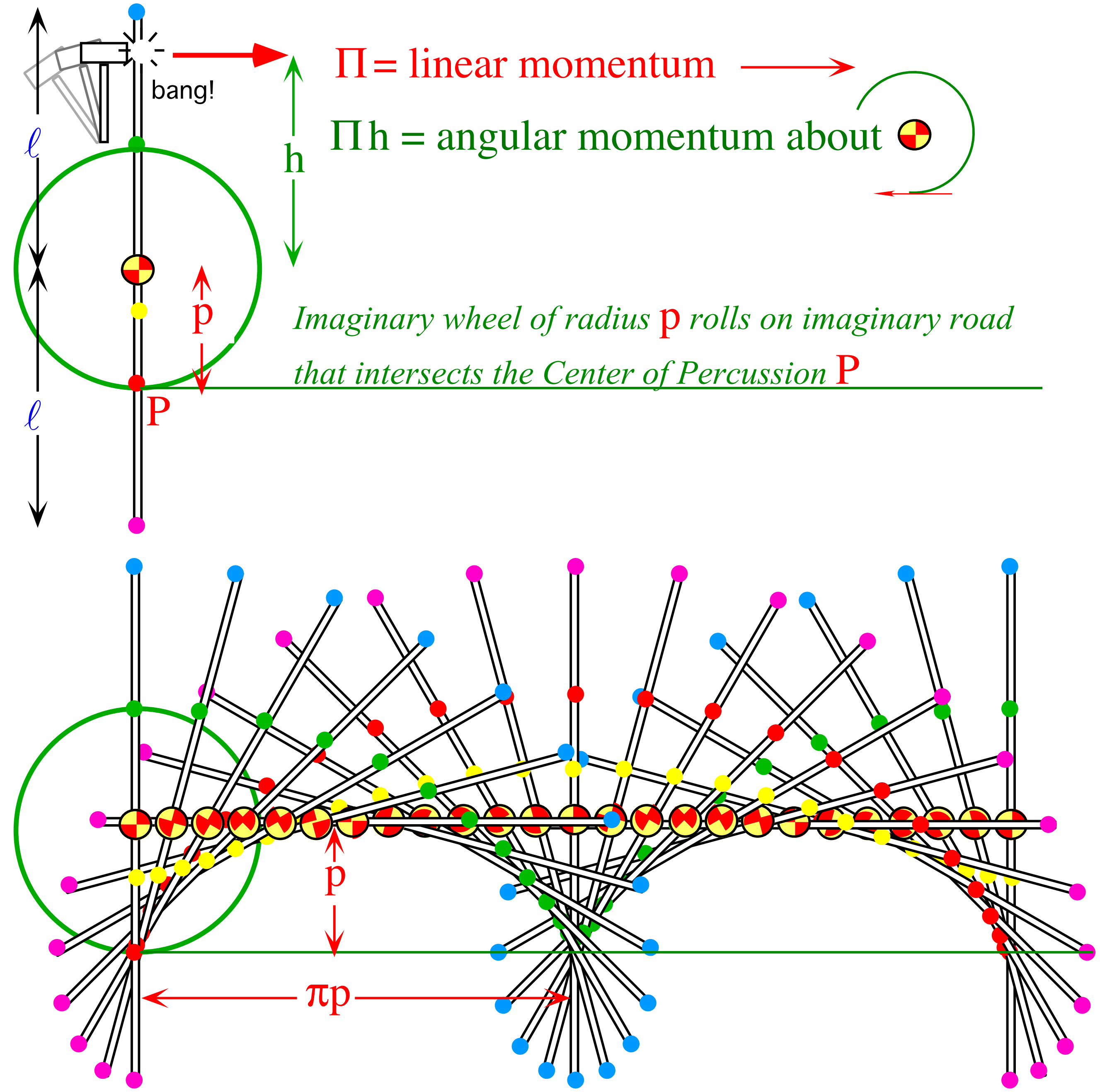

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point *h* meters from its center
you give it some linear momentum Π
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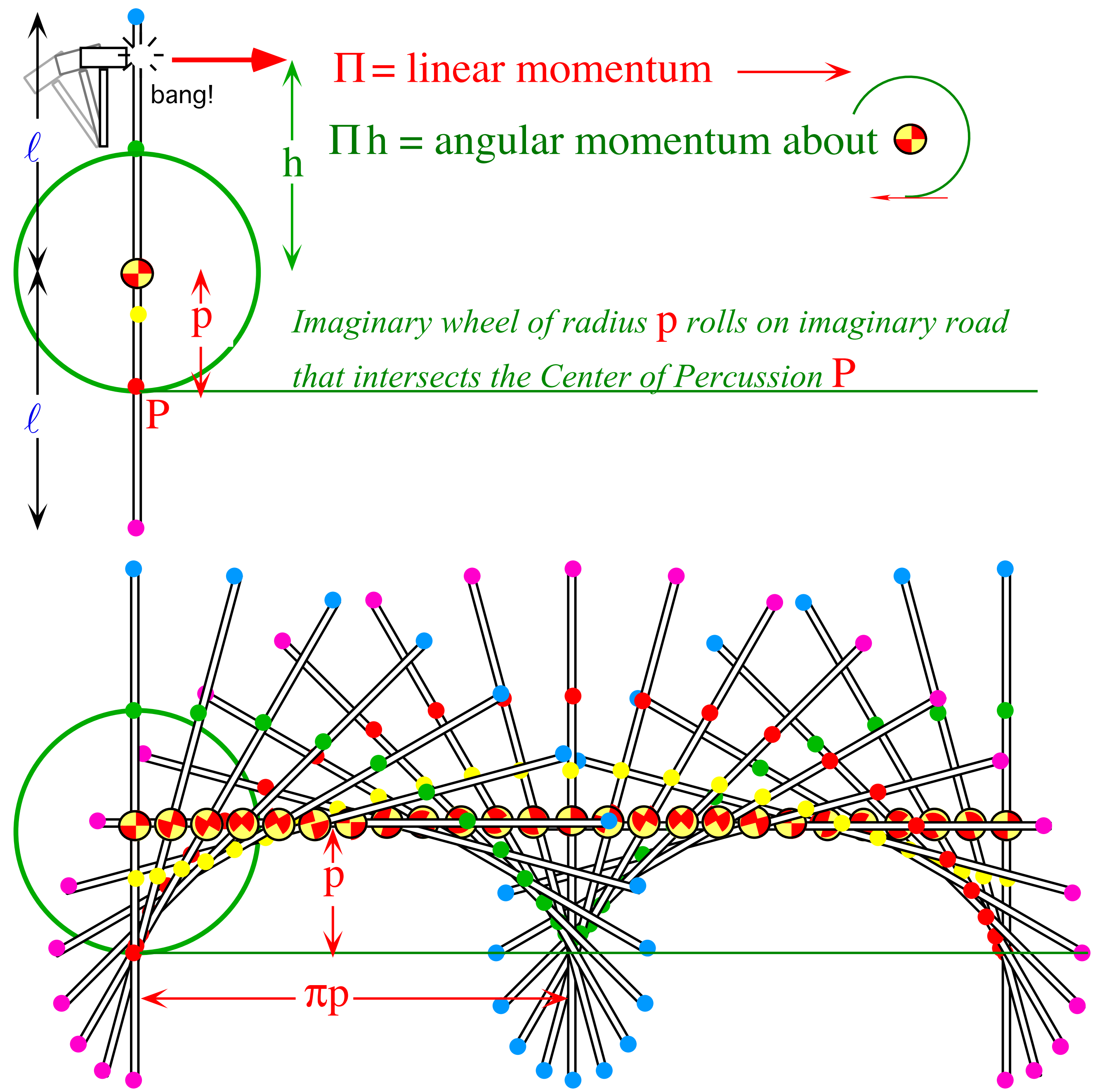
Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.



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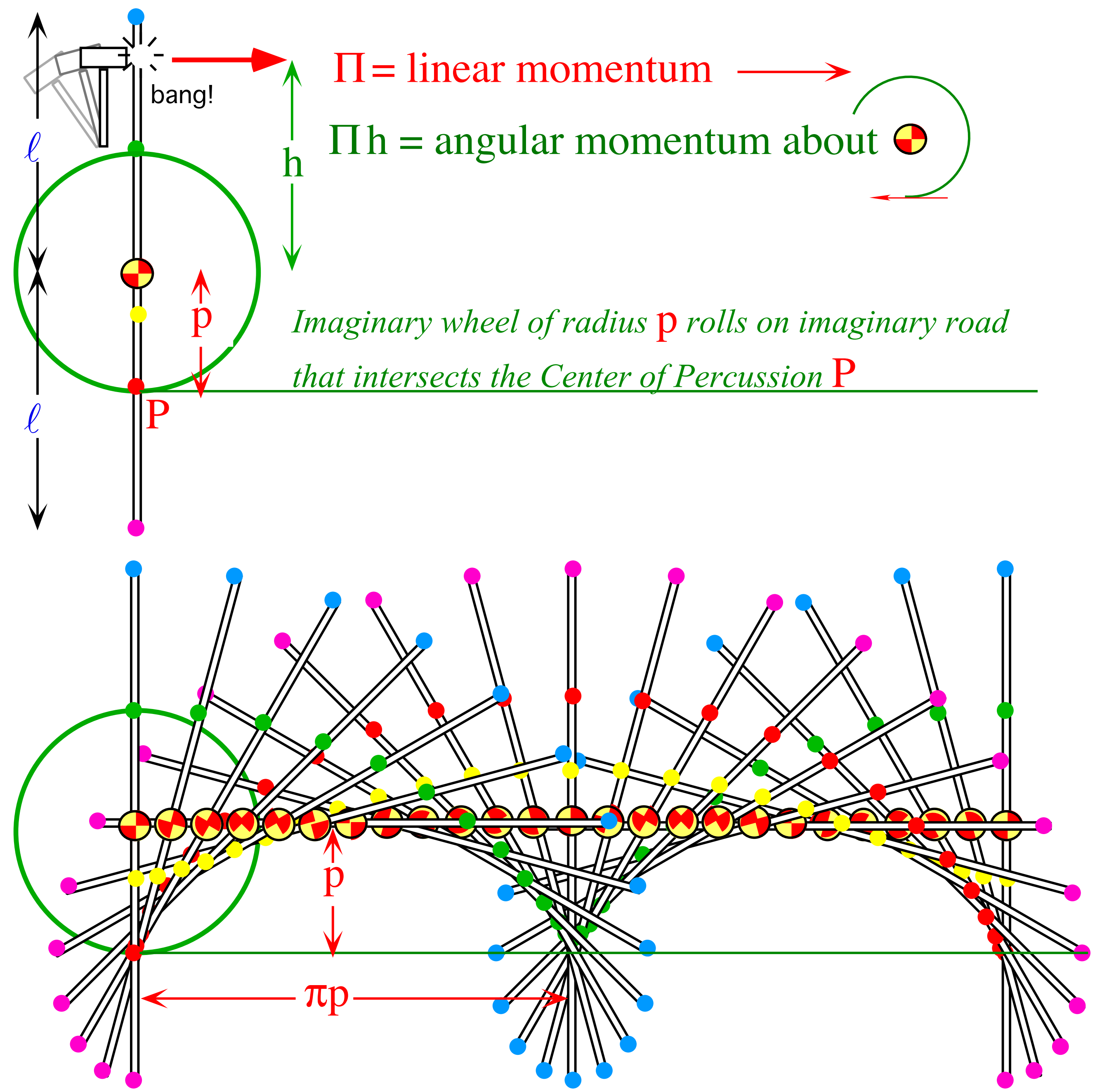

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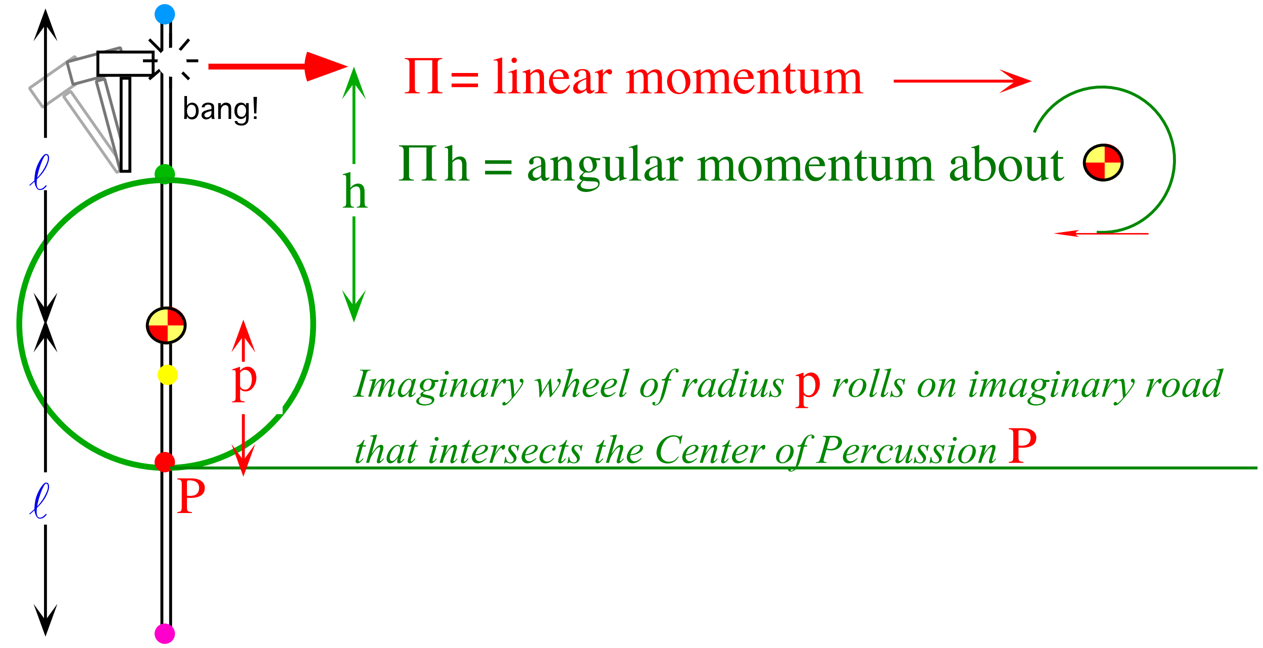

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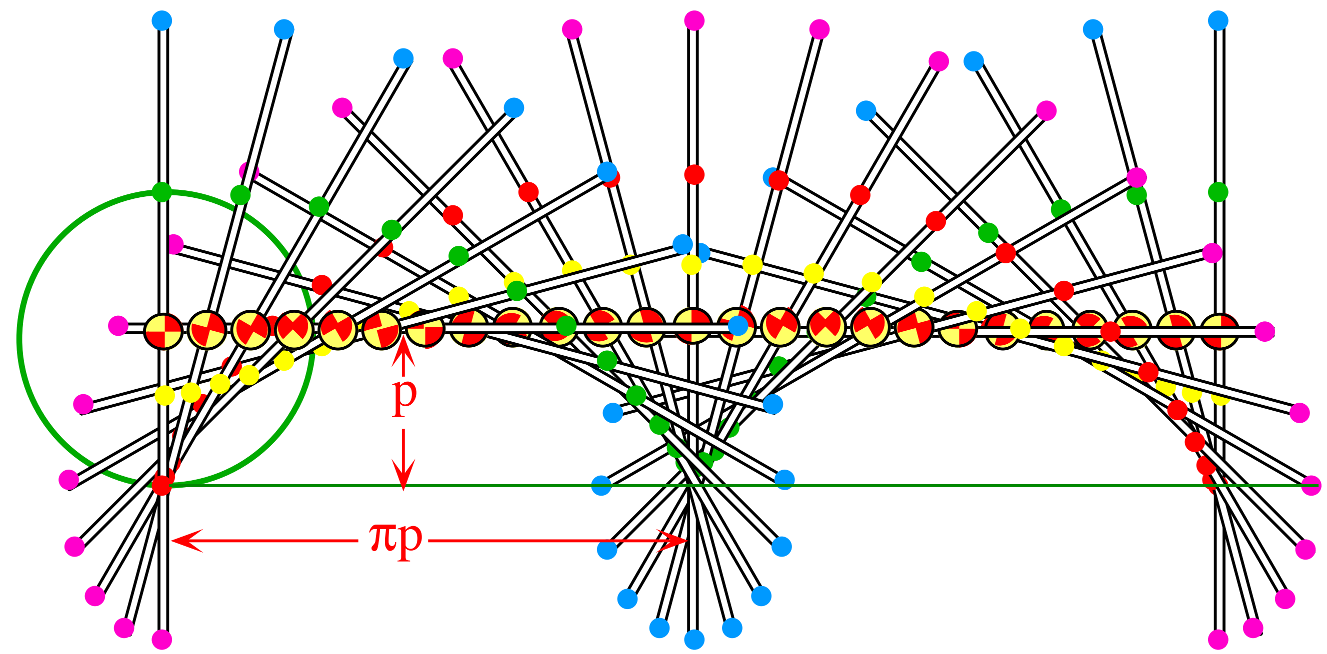

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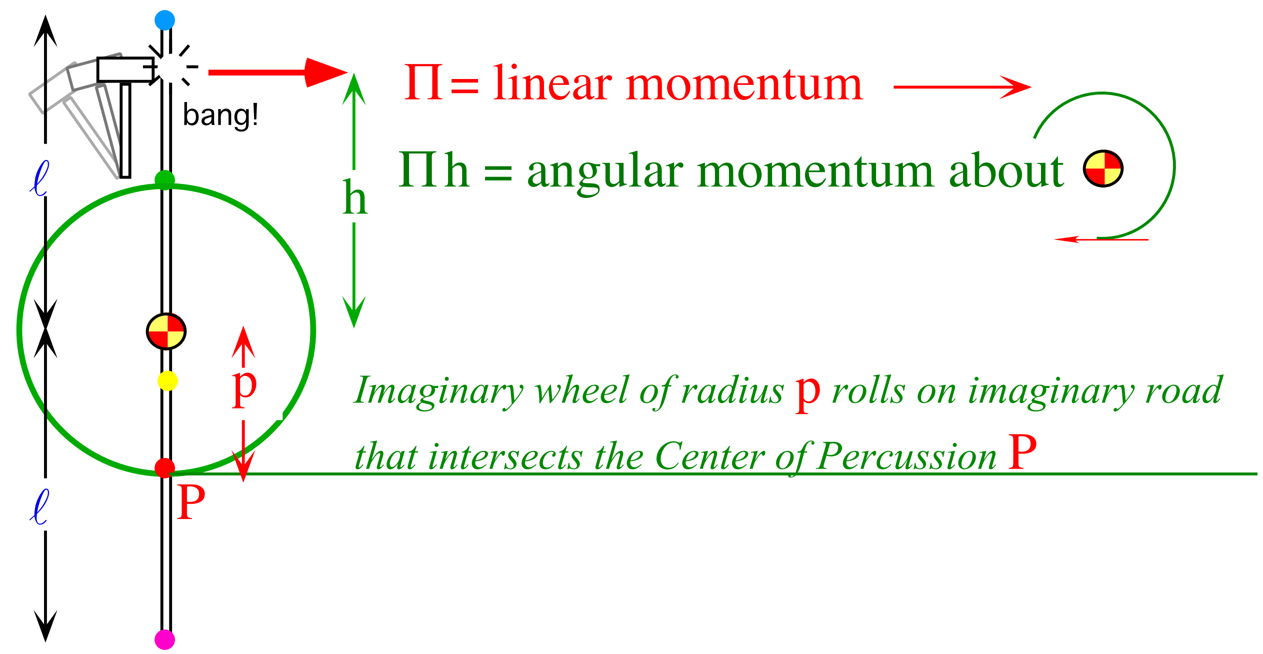

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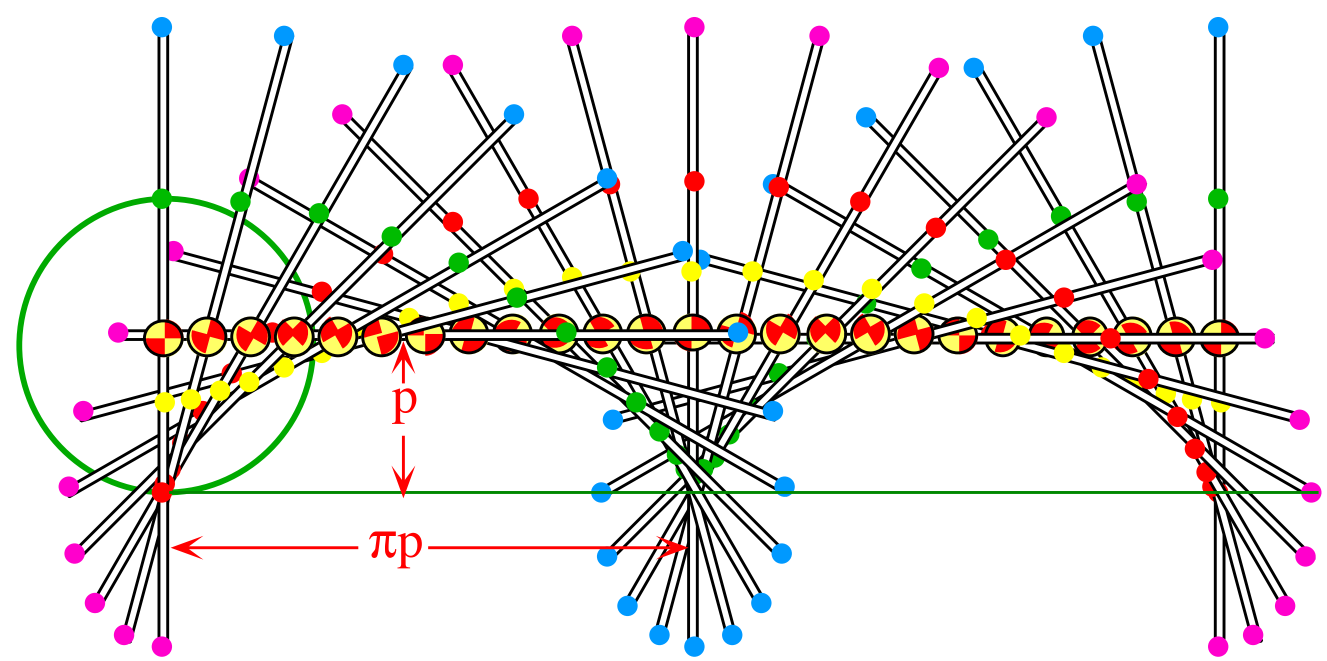

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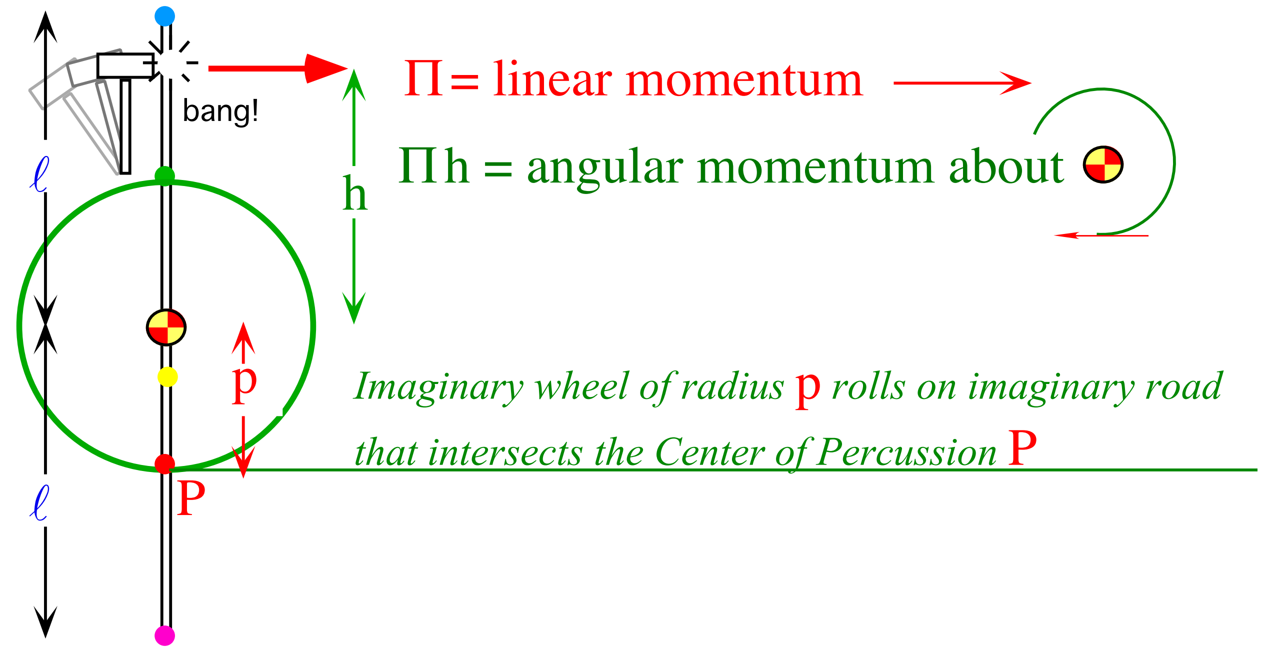

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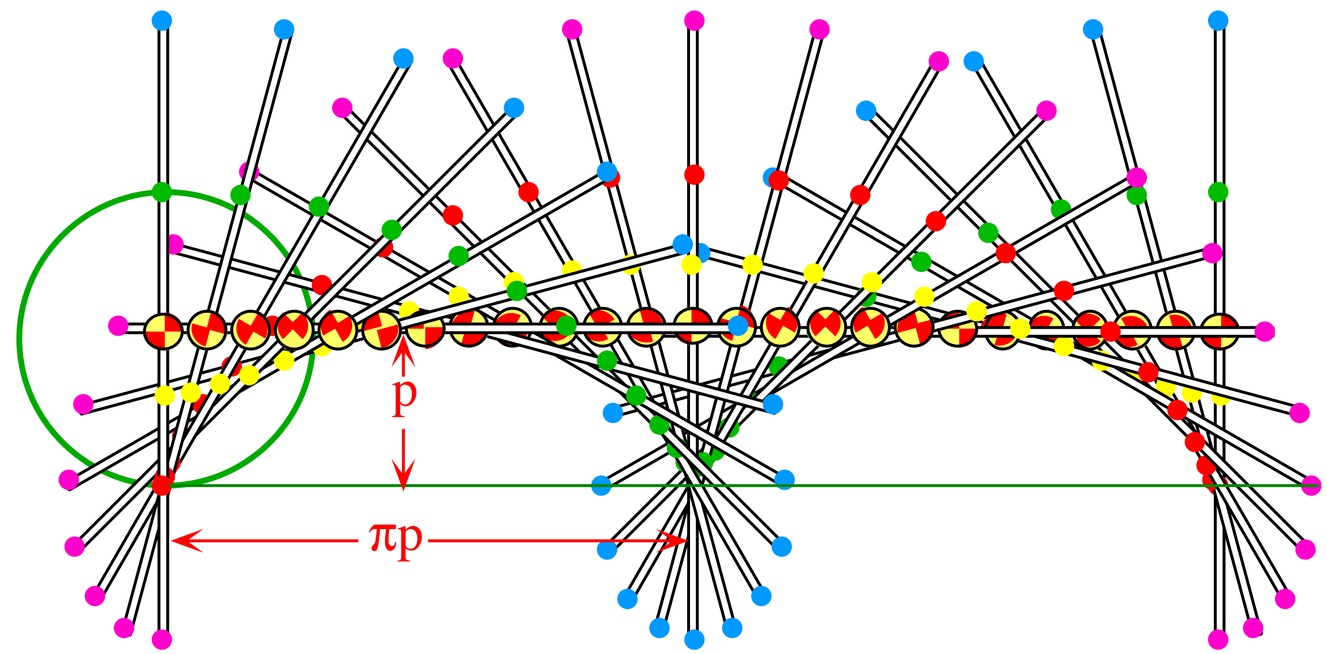

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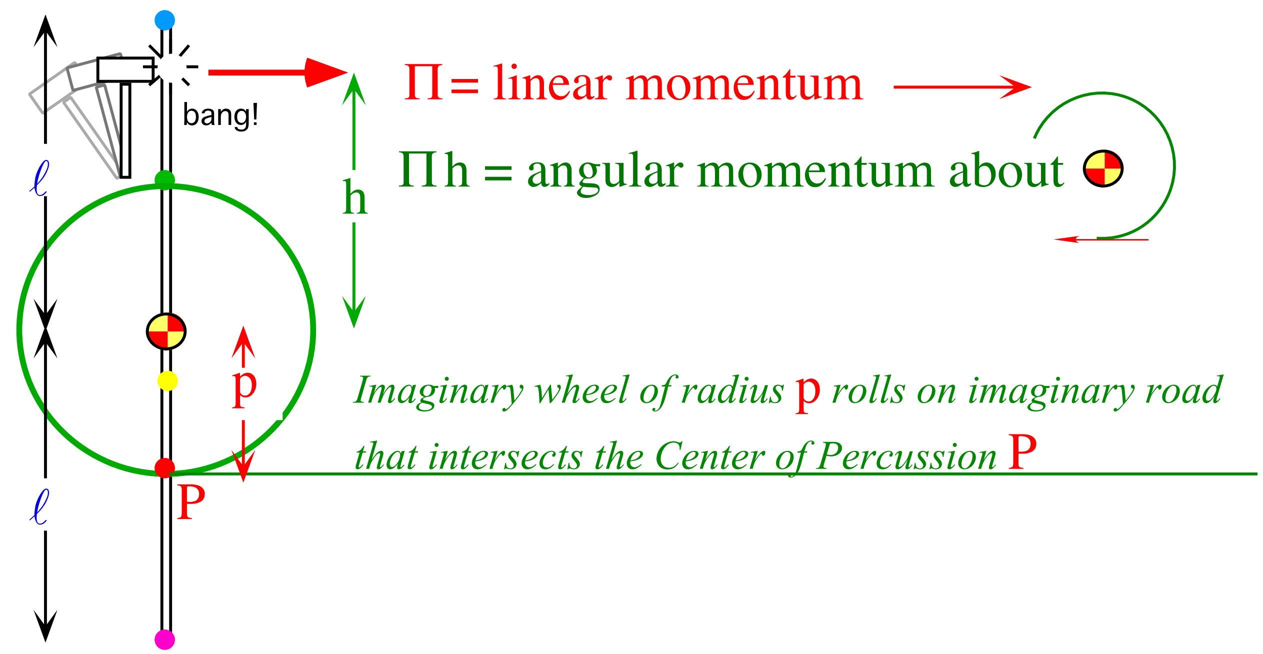

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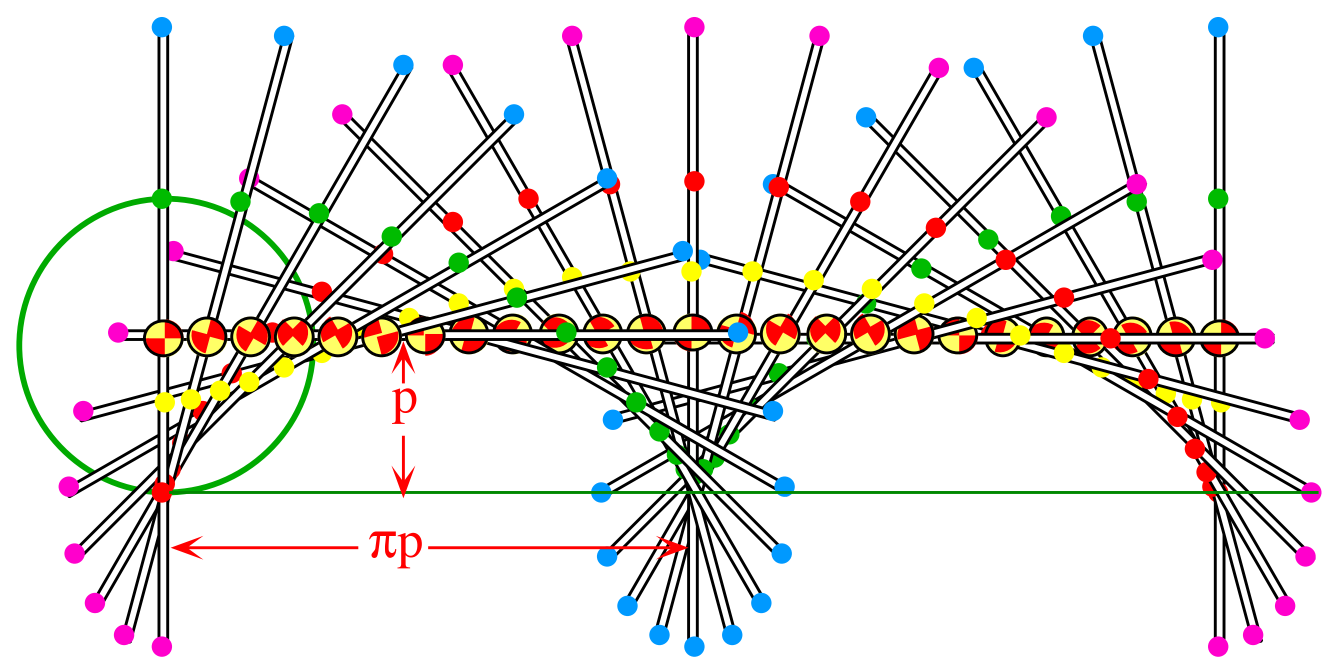

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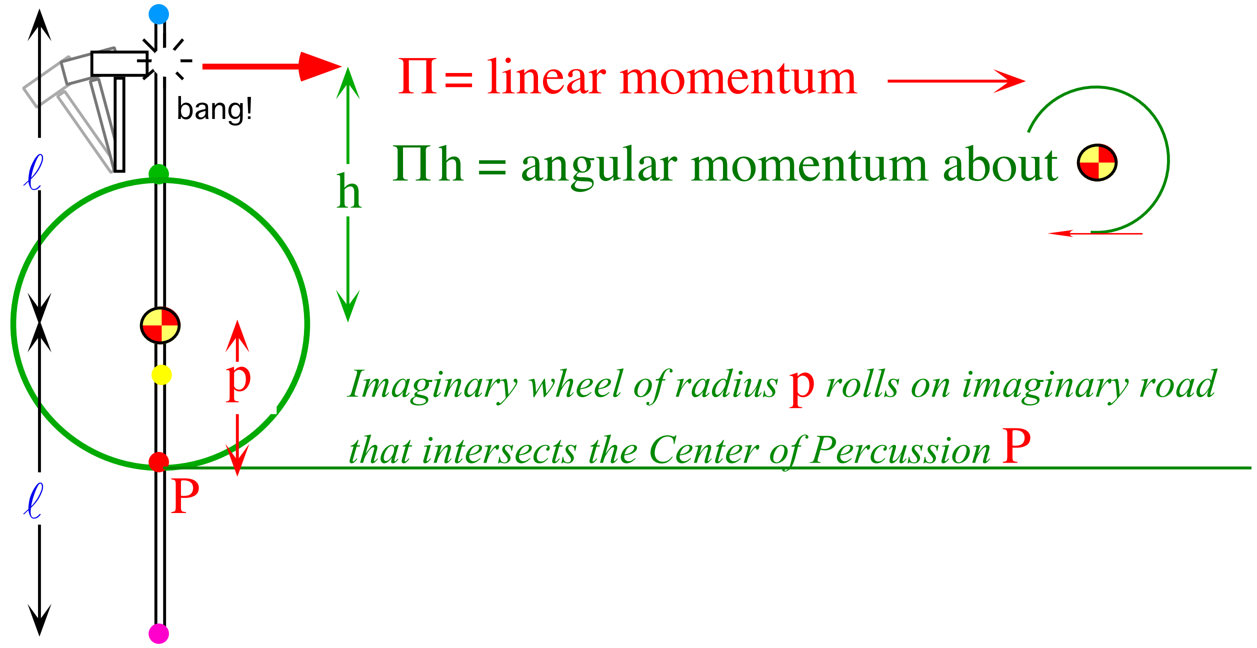

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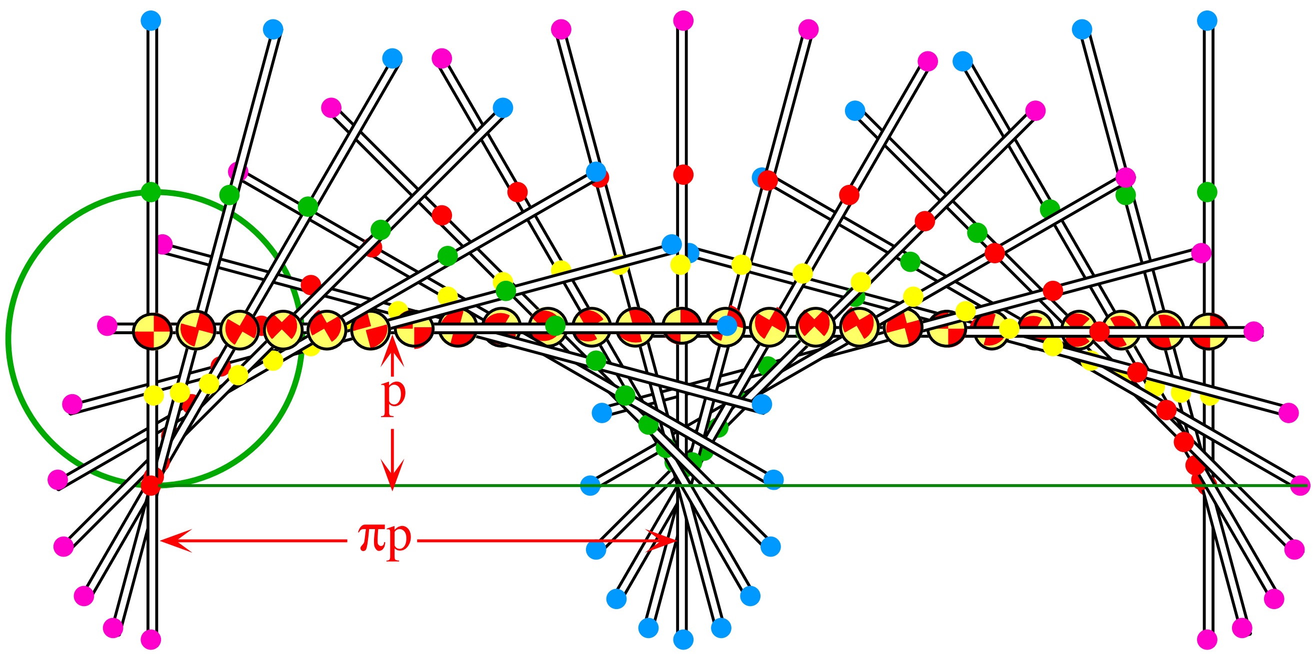
$$\Pi\,/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$
$$I\,/M = \qquad = \qquad = p\cdot h \qquad \textit{or: } p = I/(Mh)$$

P follows a normal cycloid made by a circle of radius $p=I/(Mh)$ rolling on an imaginary road thru point P in direction of Π.



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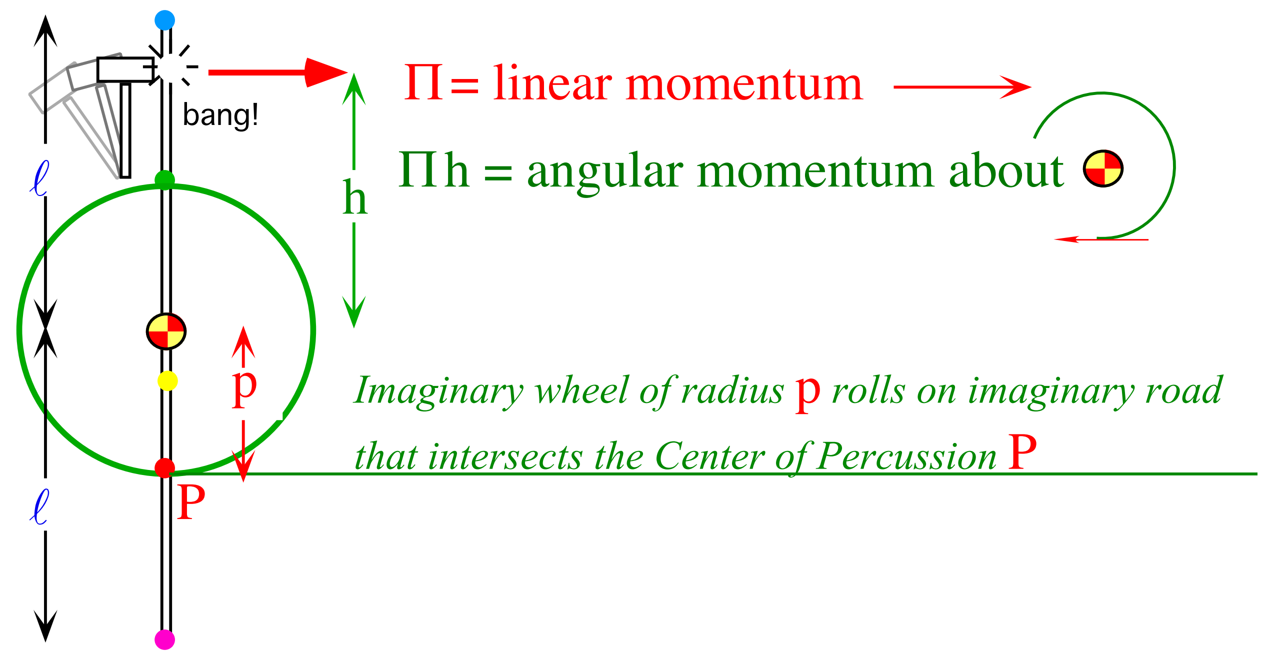

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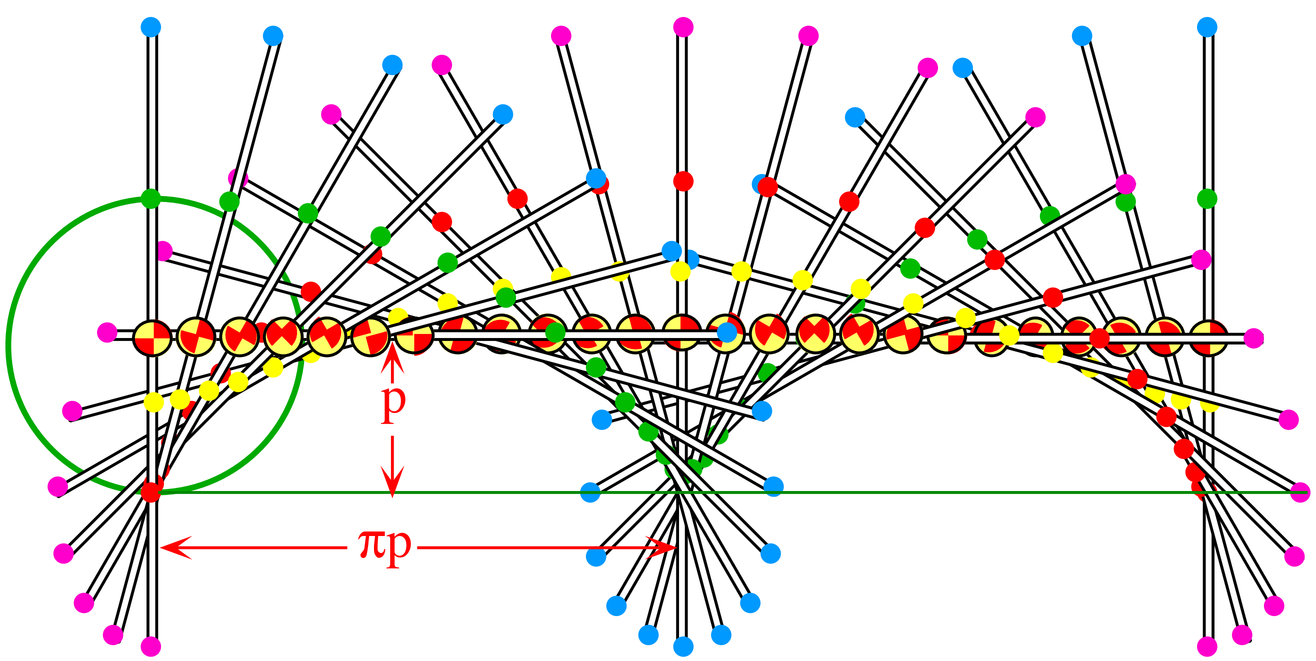

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at h.

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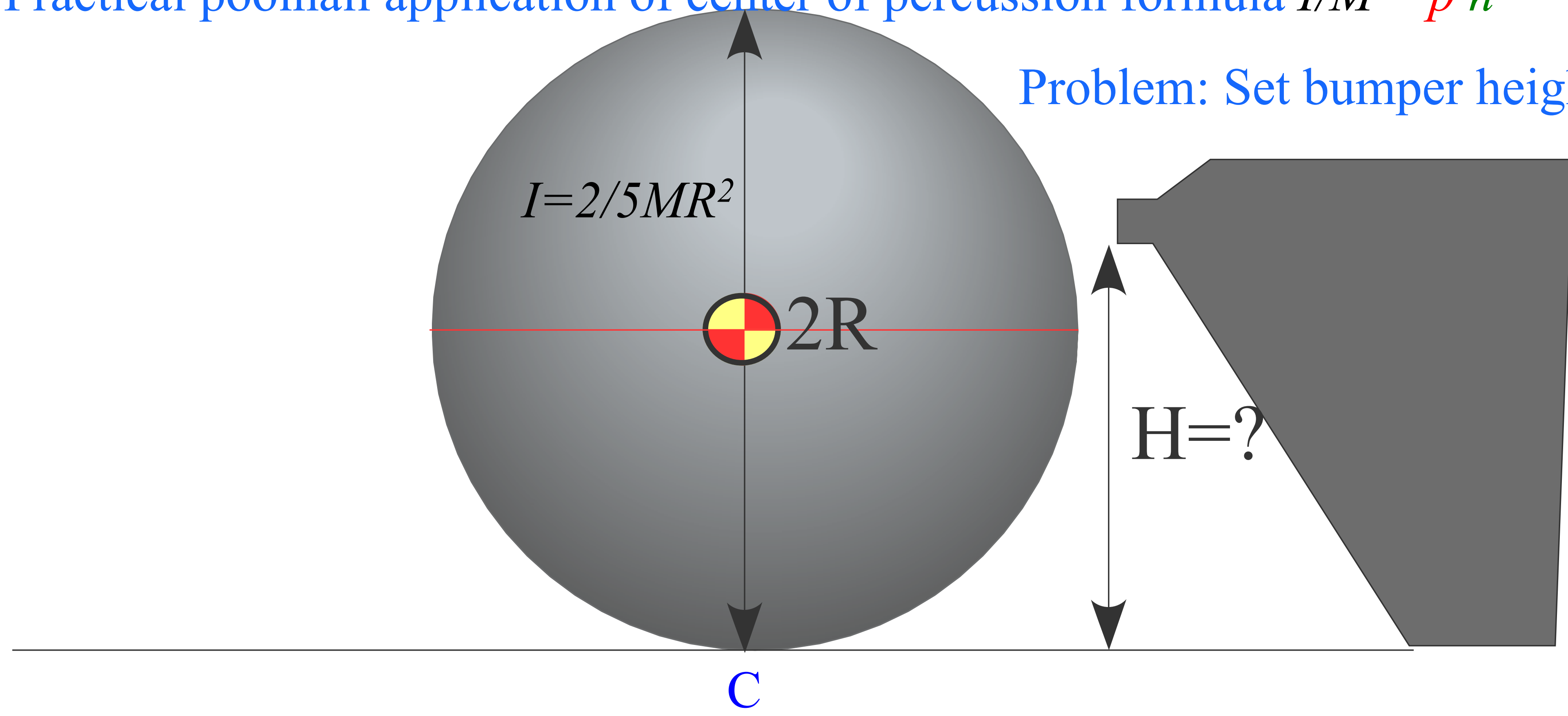
Practical poolhall application

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

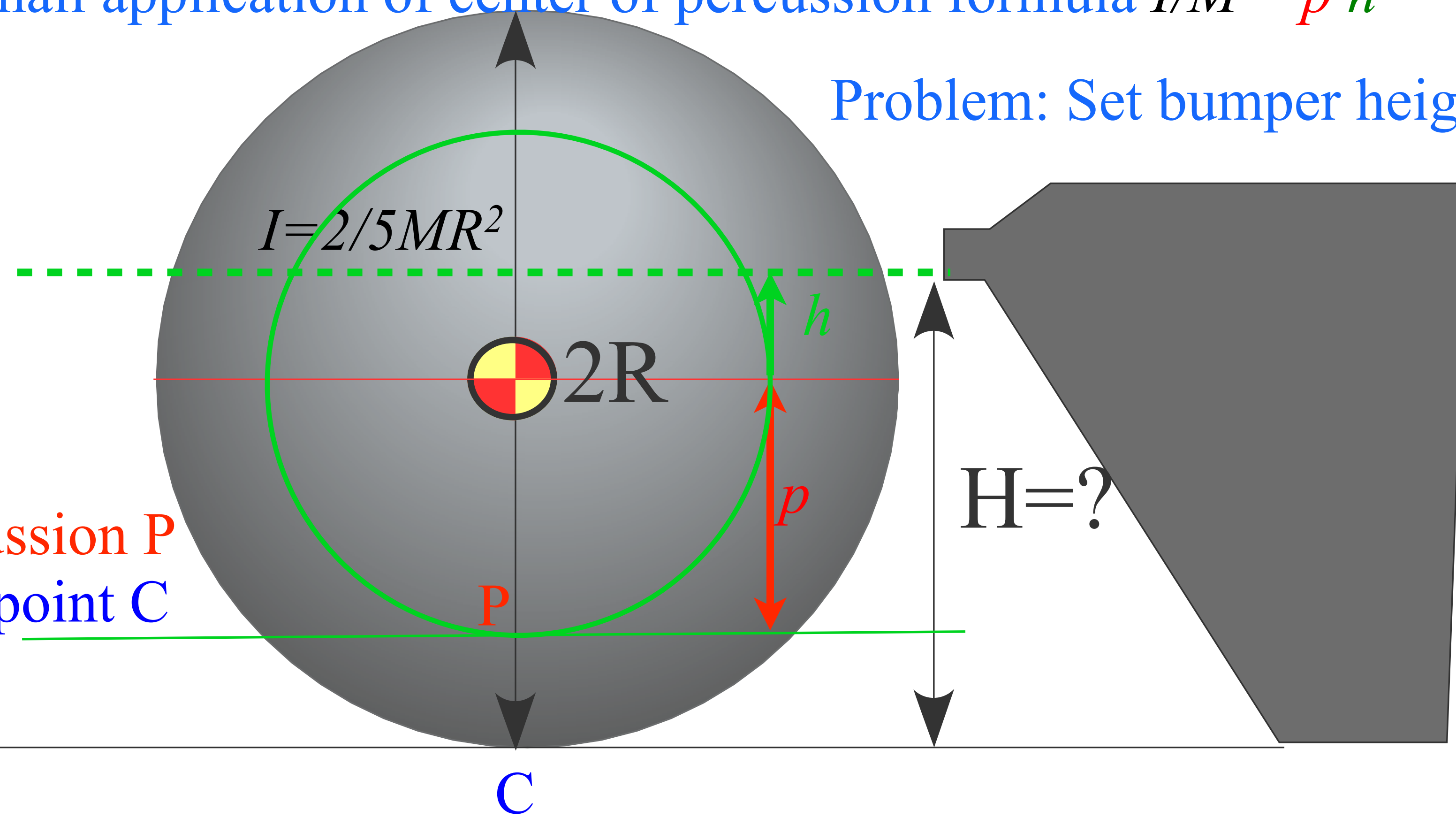


Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
above contact point C



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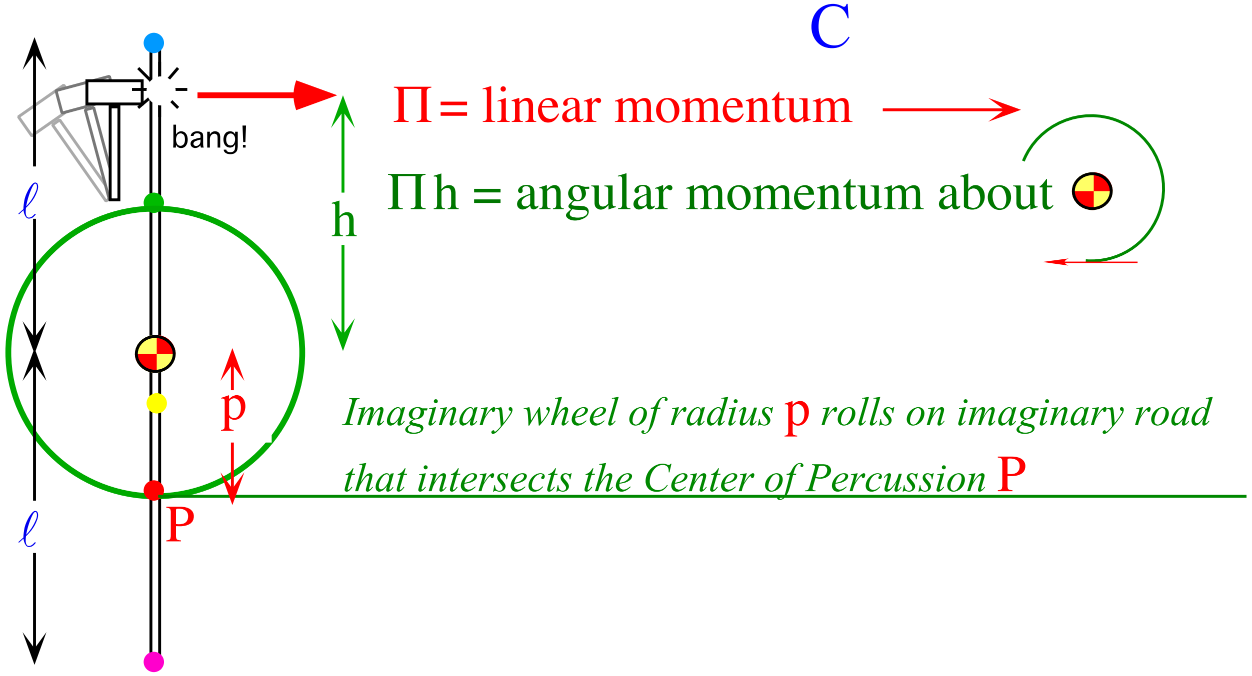

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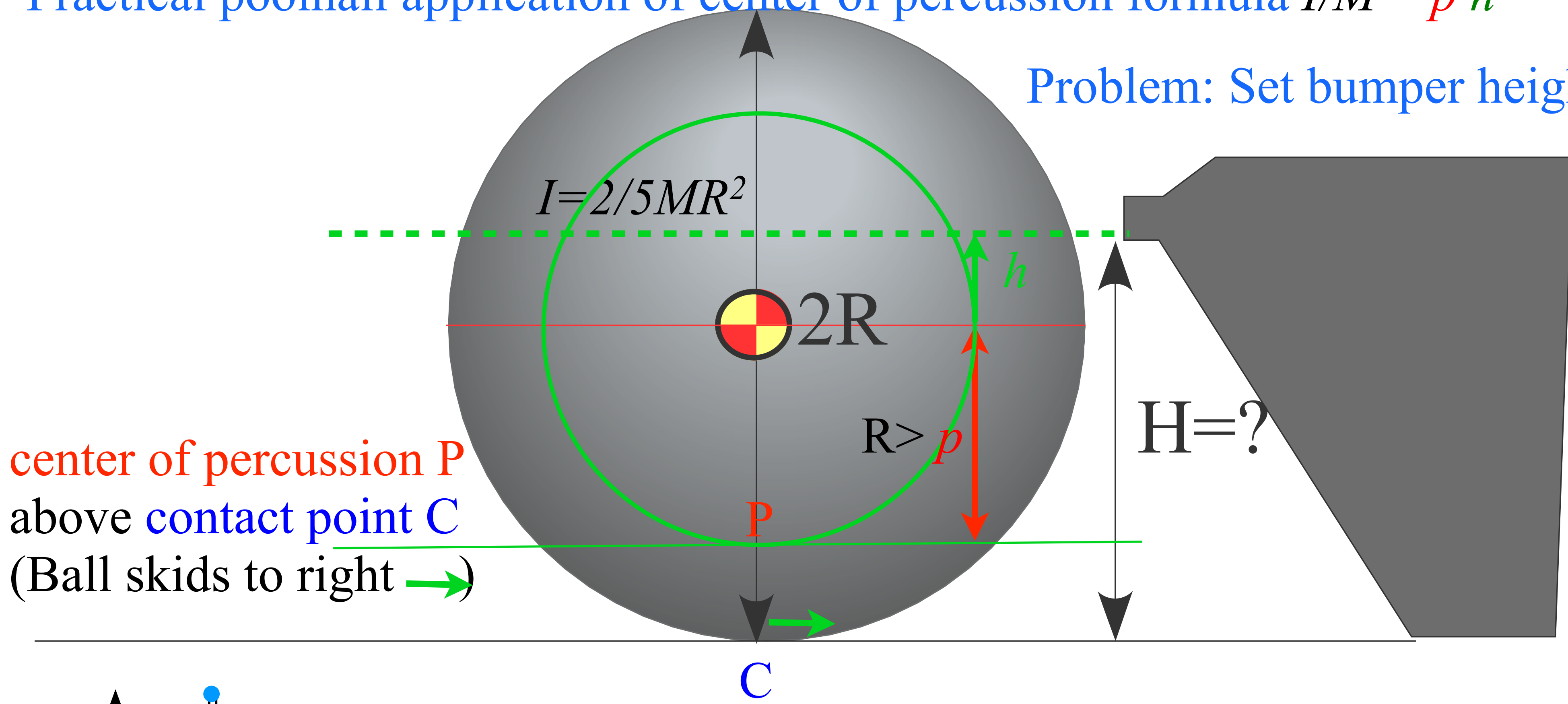
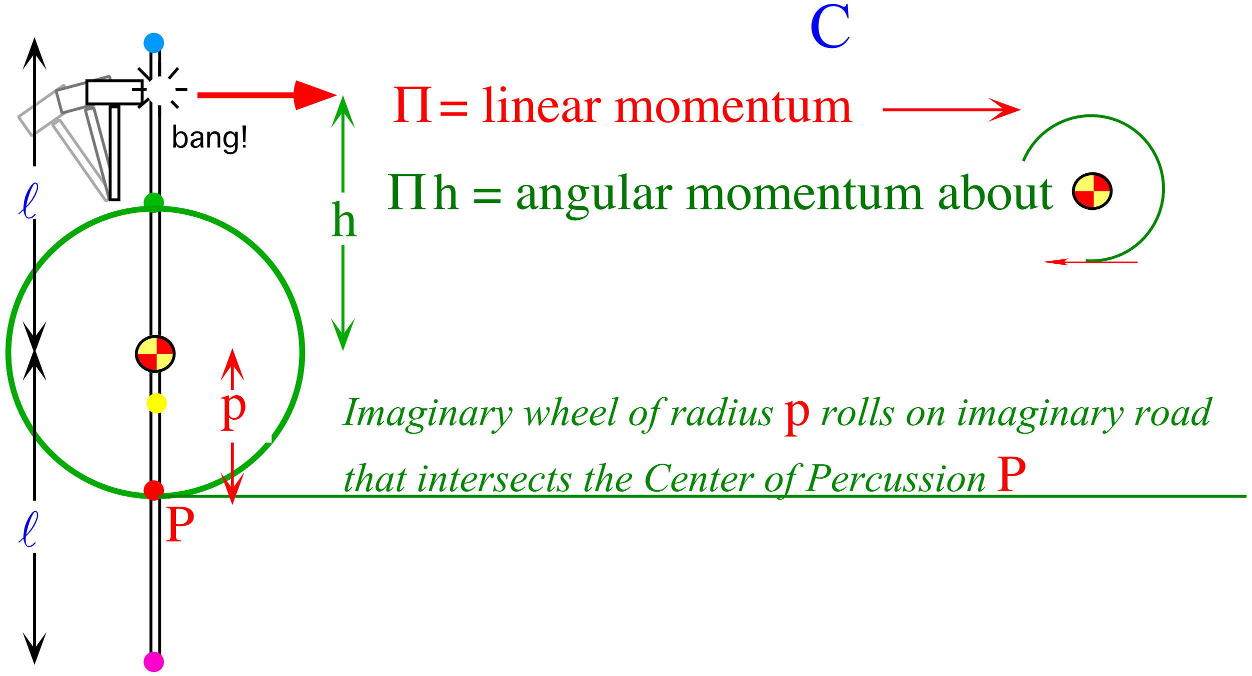


Where should bumper height H be set to make ball contact point C at the center of percussion P?



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Practical poolhall application of center of percussion formula $I/M = p \cdot h$

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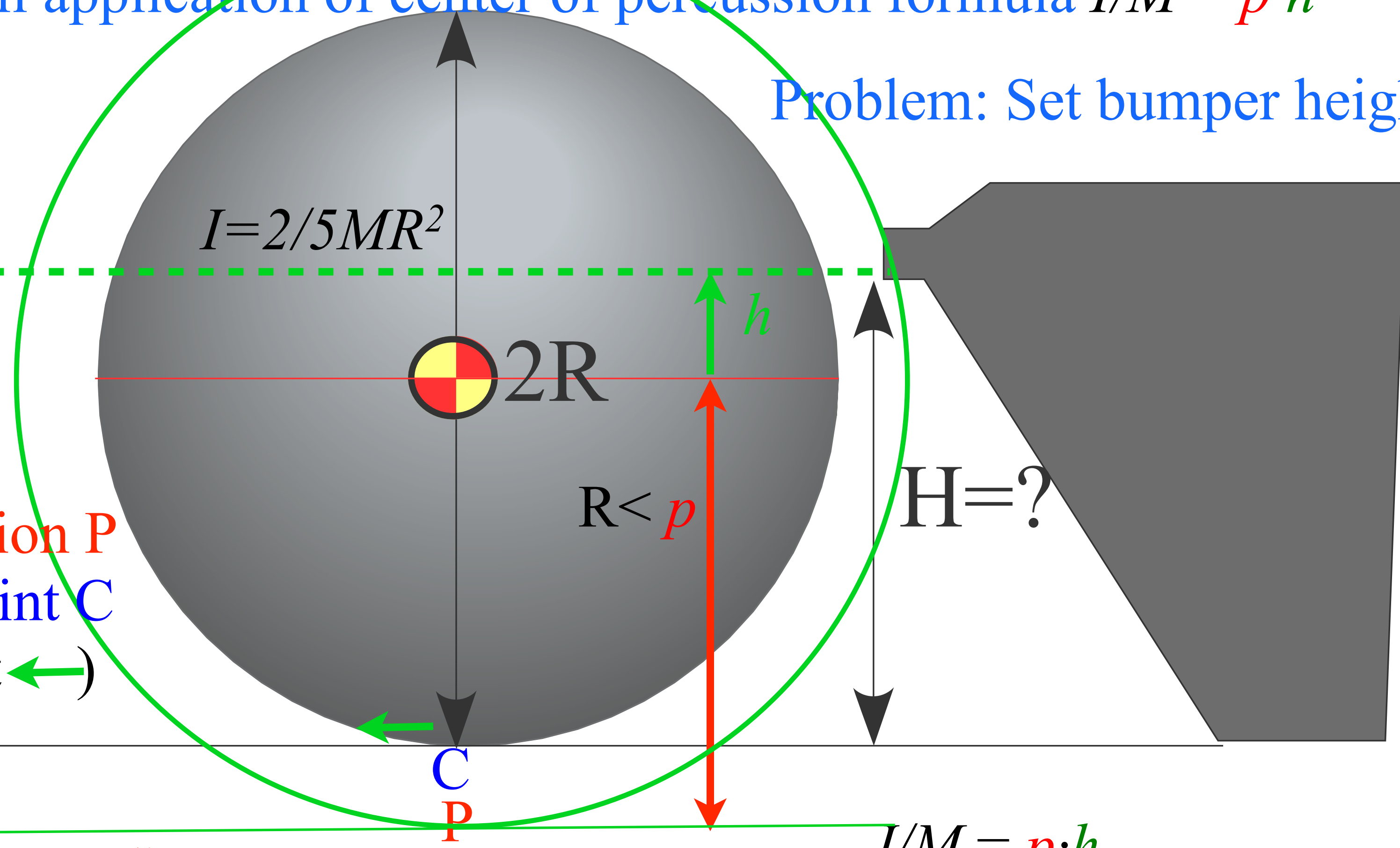
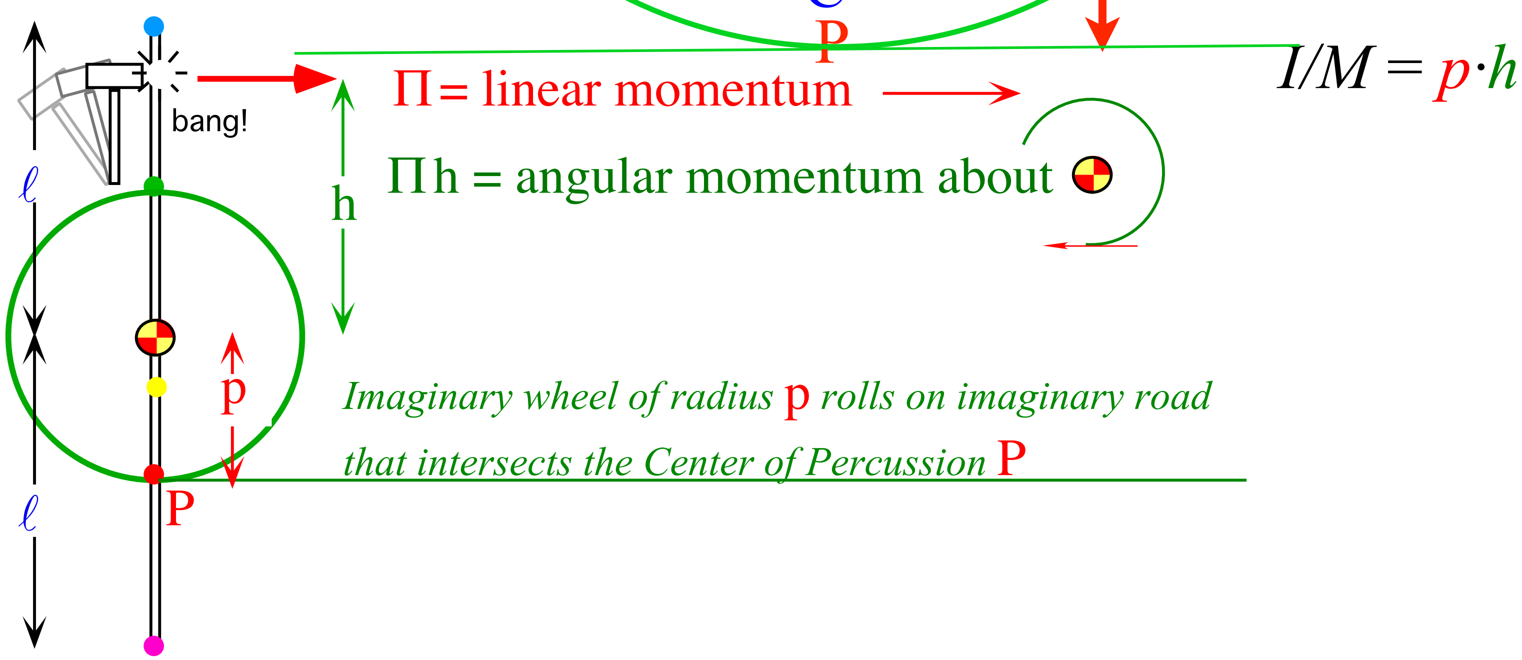
center of percussion P below contact point C (Ball skids to left ←)

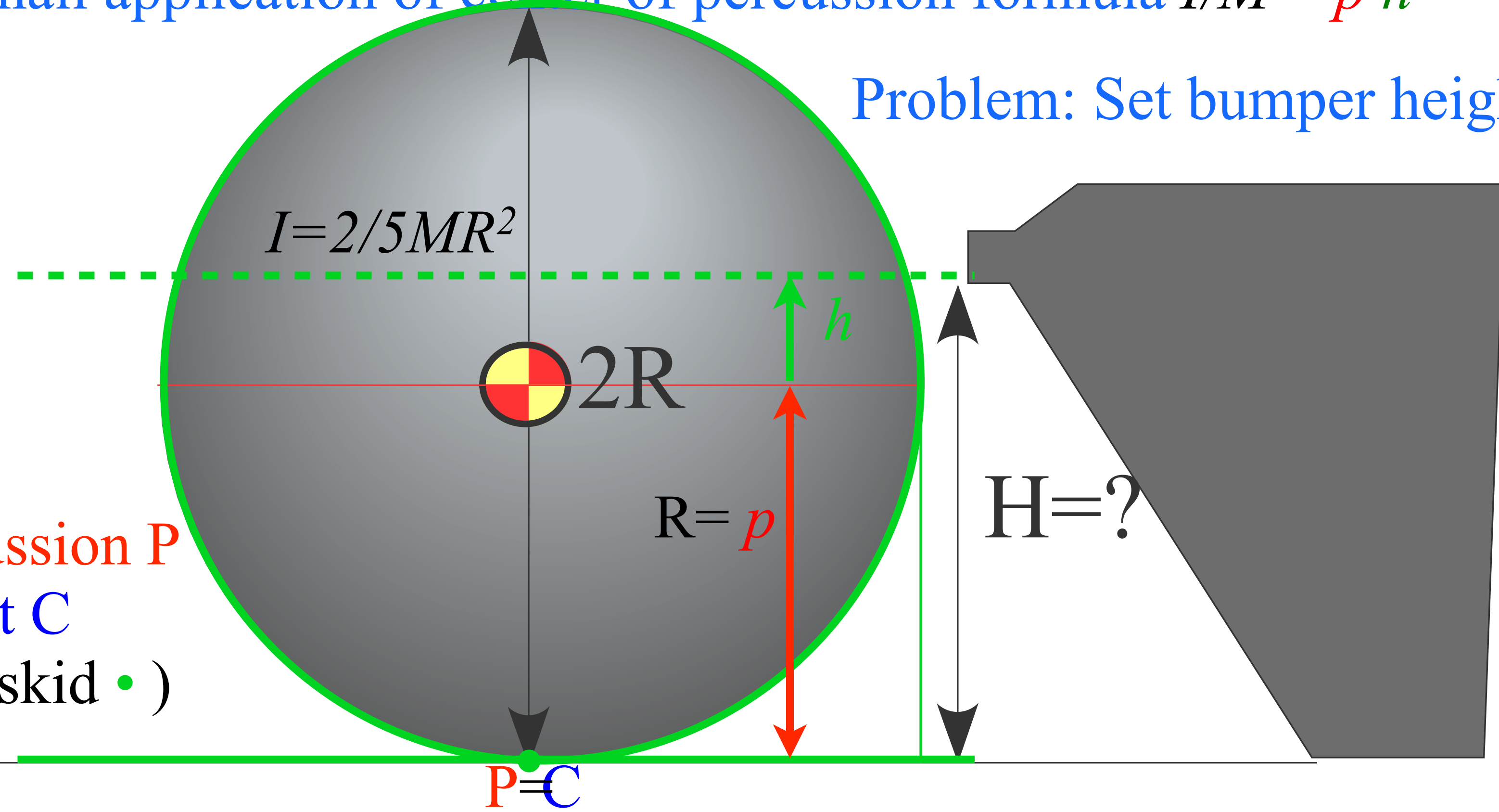




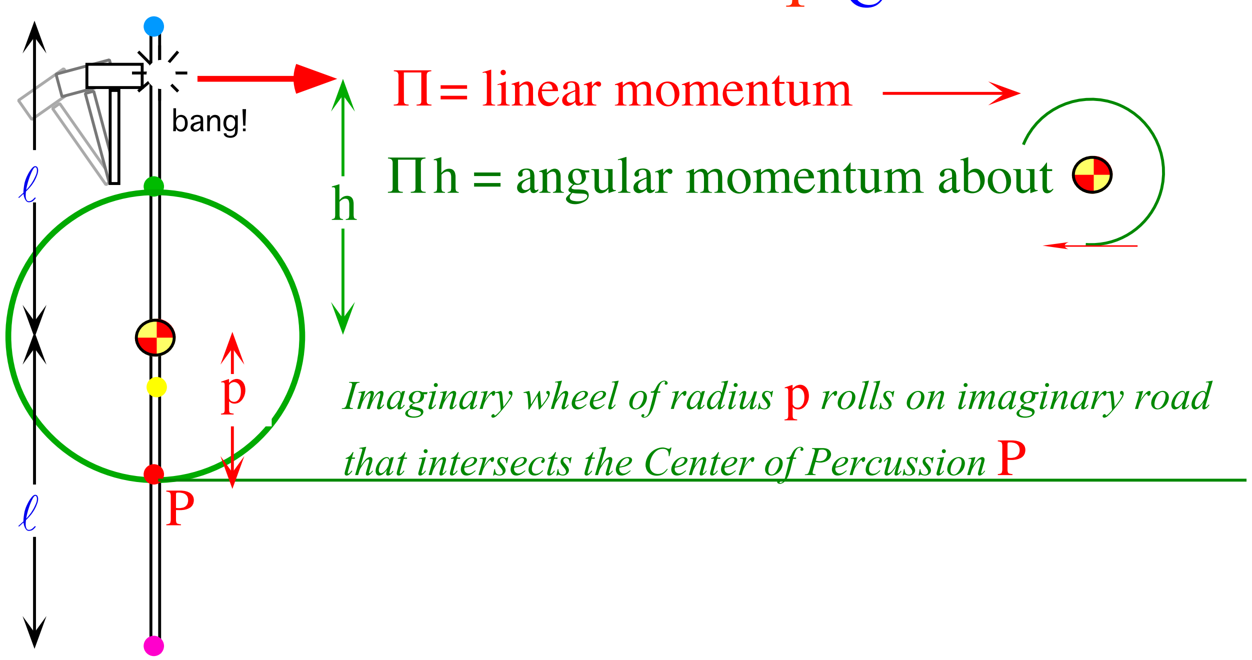
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P?



center of percussion P
at contact point C
(Ball does not skid •)



$I/M = p \cdot h$

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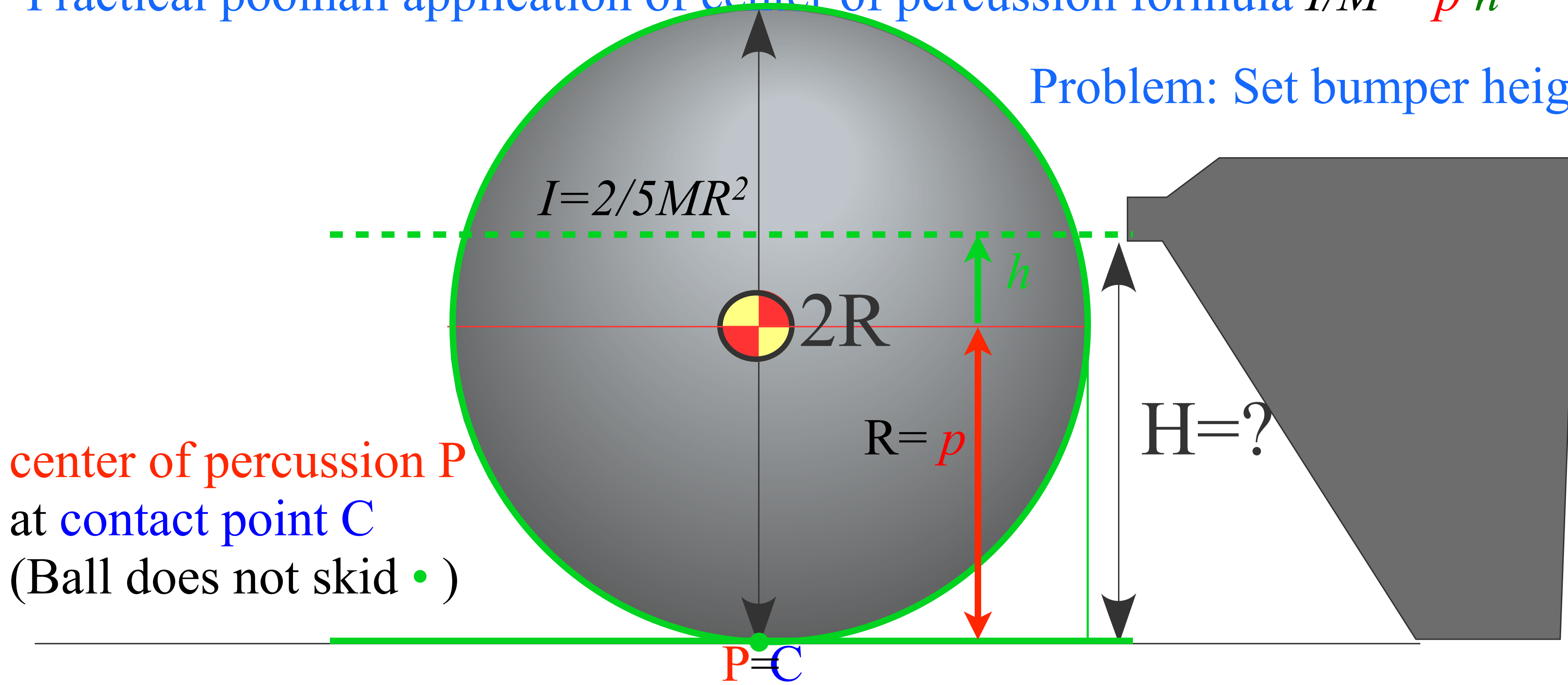

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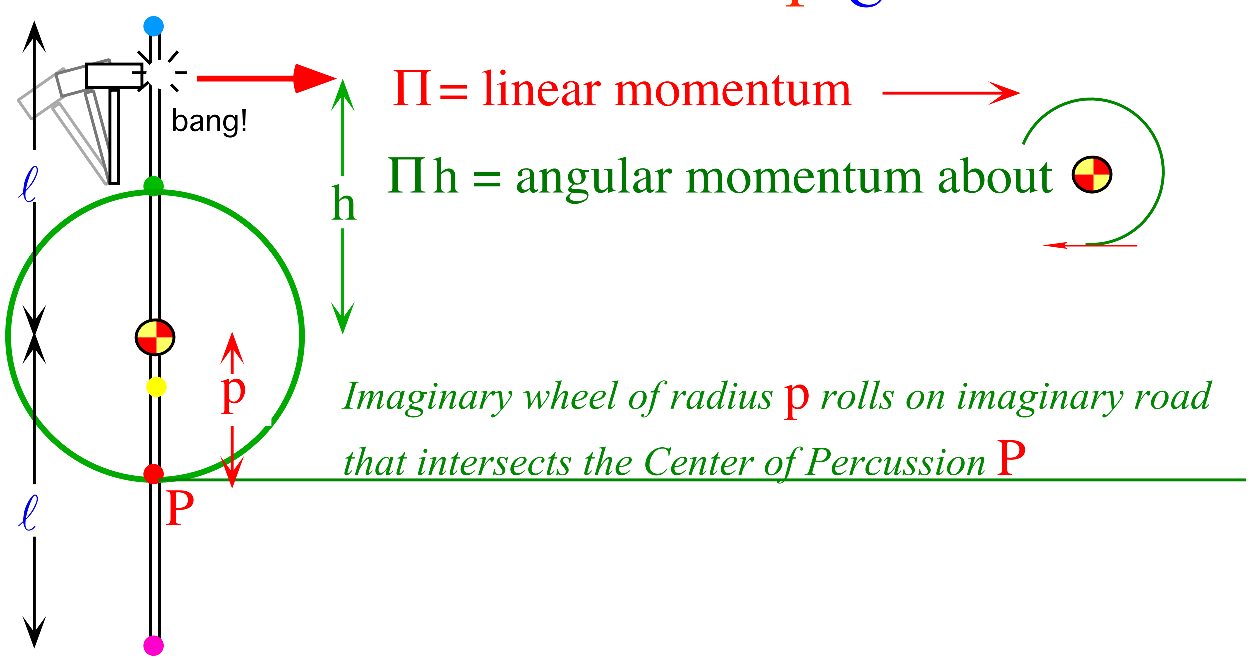

$I/M = p \cdot h$

$h = I/Mp = I/MR$ (For R= p)

$= 2/5MR^2/MR$

$= 2/5R$

For: $H = R+h = 7/10(2R)$ ball does not skid.

Thats all folks!