Lecture 15
Thur. 10.22.2015

*treb-yew-shay

Introducing GCC Lagrangian `a la Trebuchet Dynamics*

Ch. 1-3 of Unit 2 and Unit 3 (Mostly Unit 2.)

*The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. **273**, 66 (July 1995))*
- *The medieval ingenium (9th to 14th century) and modern re-enactments*
- *Human kinesthetics and sports kinesiology*

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)
- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Summary of Lagrange equations and force analysis (Mostly Unit 2.)
- *Forces: total, genuine, potential, and/or fictitious*

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
- *Multivalued functionality and connections*
- *Covariant and contravariant relations*
    - *Tangent space vs. Normal space*
- *Metric* g_{mn} *tensor geometric relations to length, area, and volume*

Chapter 1. The Trebuchet: A dream problem for Galileo?

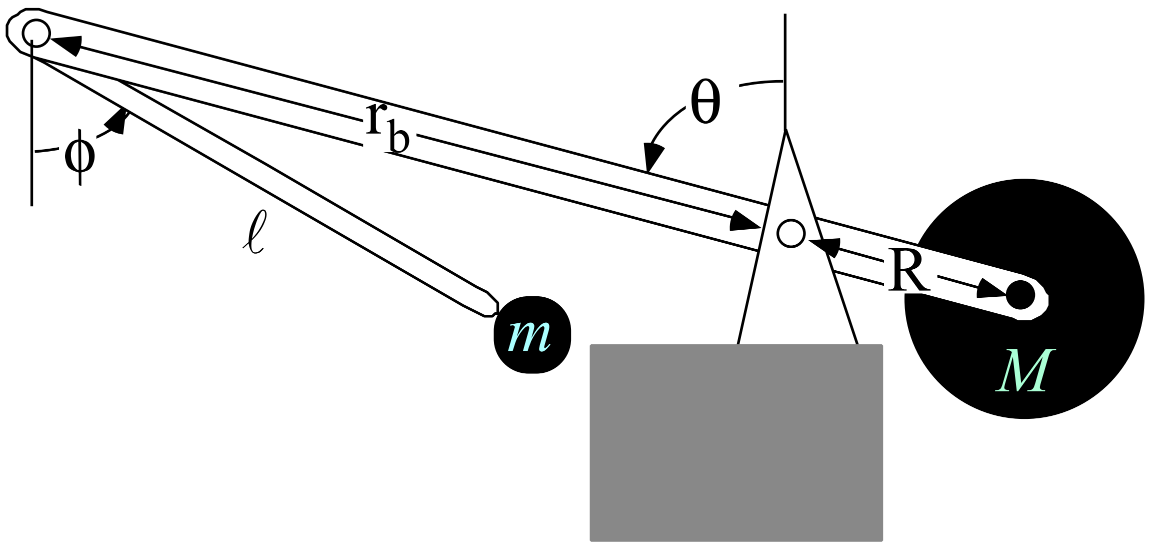
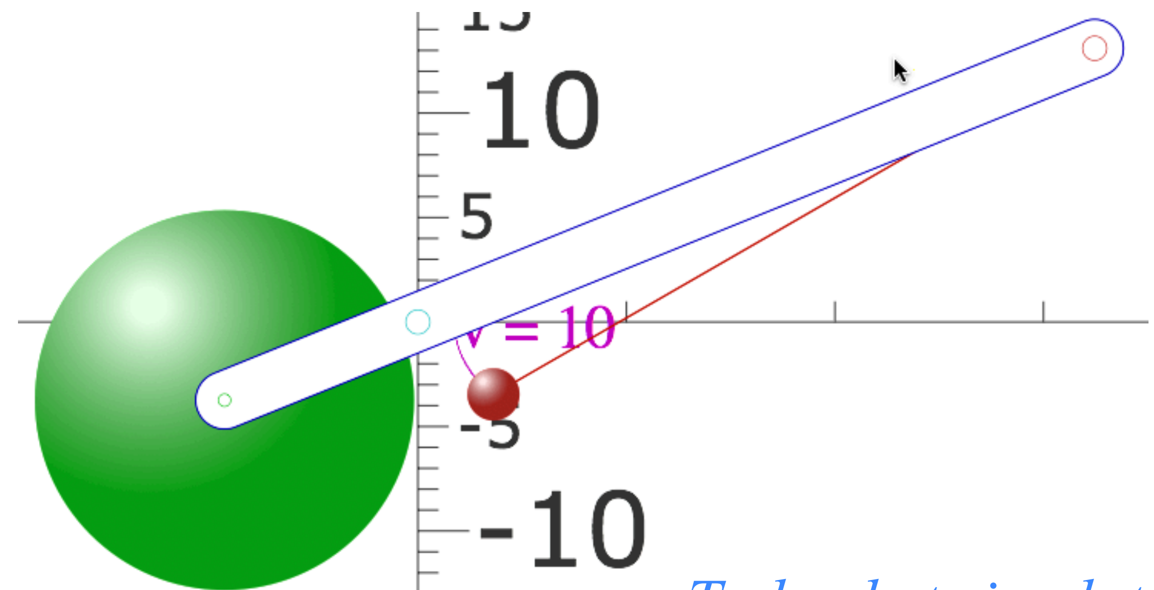


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

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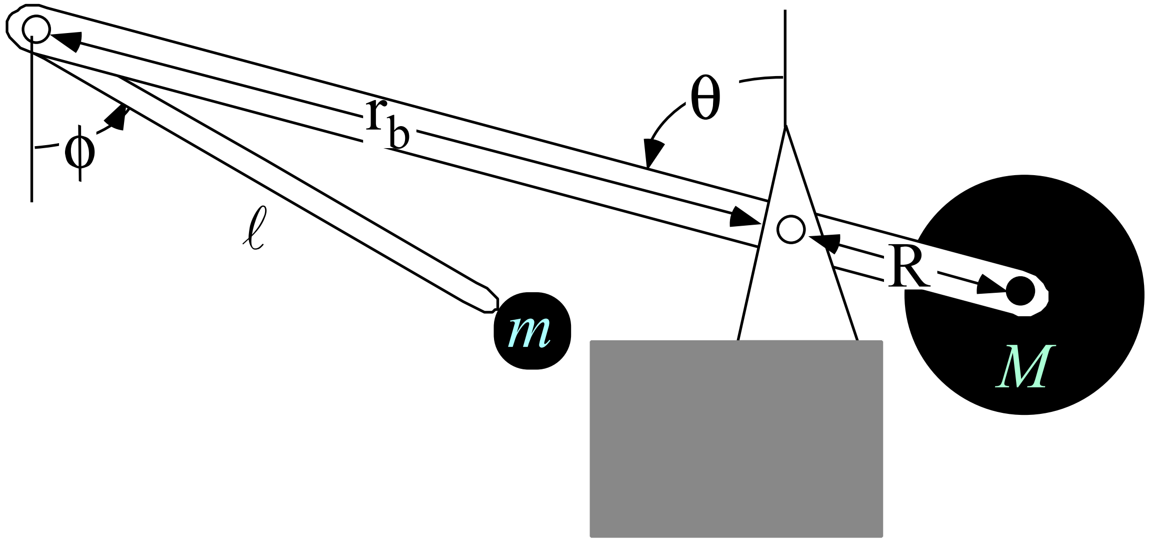
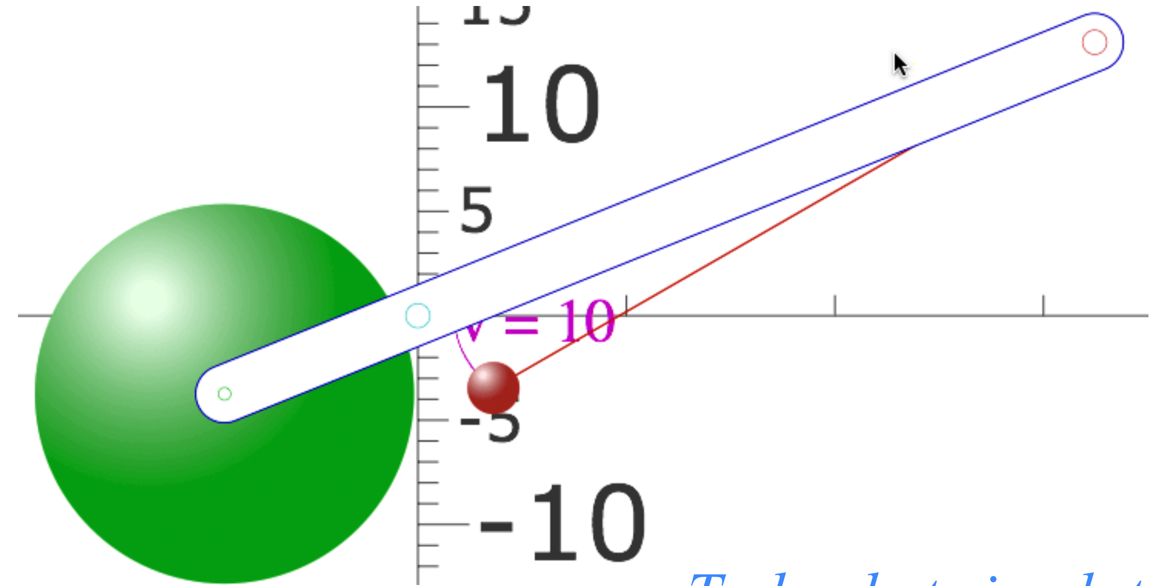


Fig. 2.1.1 An elementary ground-fixed trebuchet



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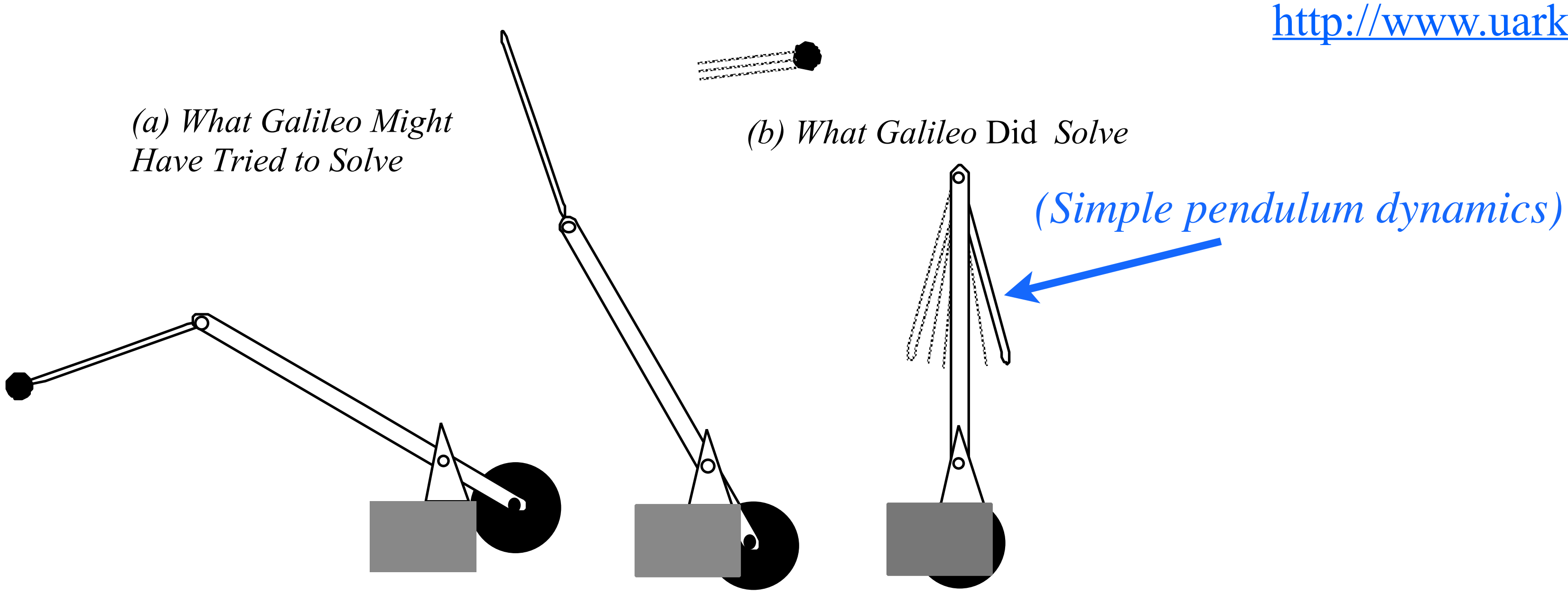
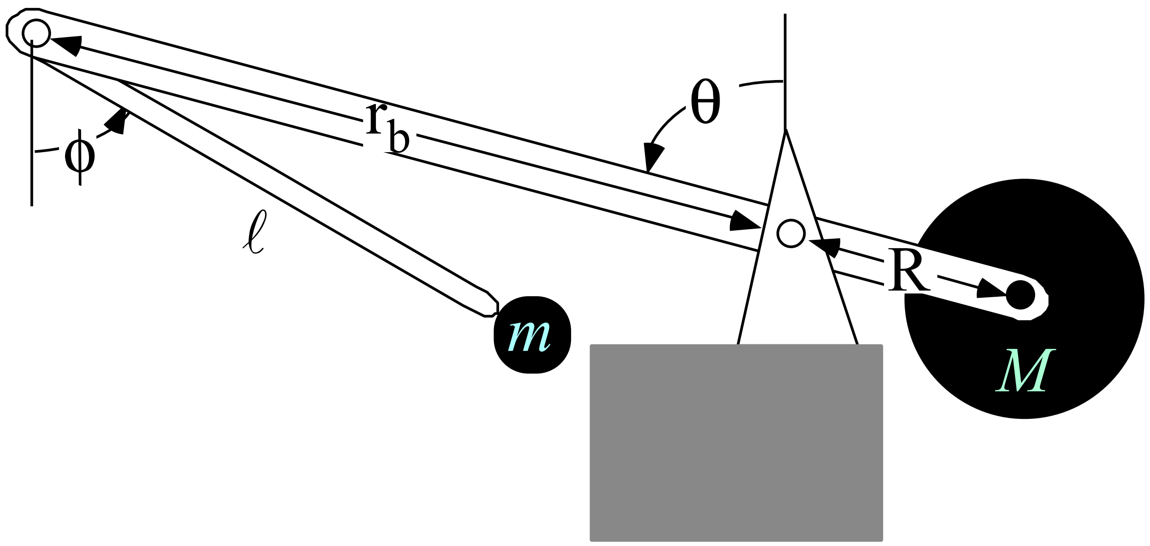


Fig. 2.1.2 Galileo's (supposed fictitious) problem

Chapter 1. The Trebuchet: A dream problem for Galileo?



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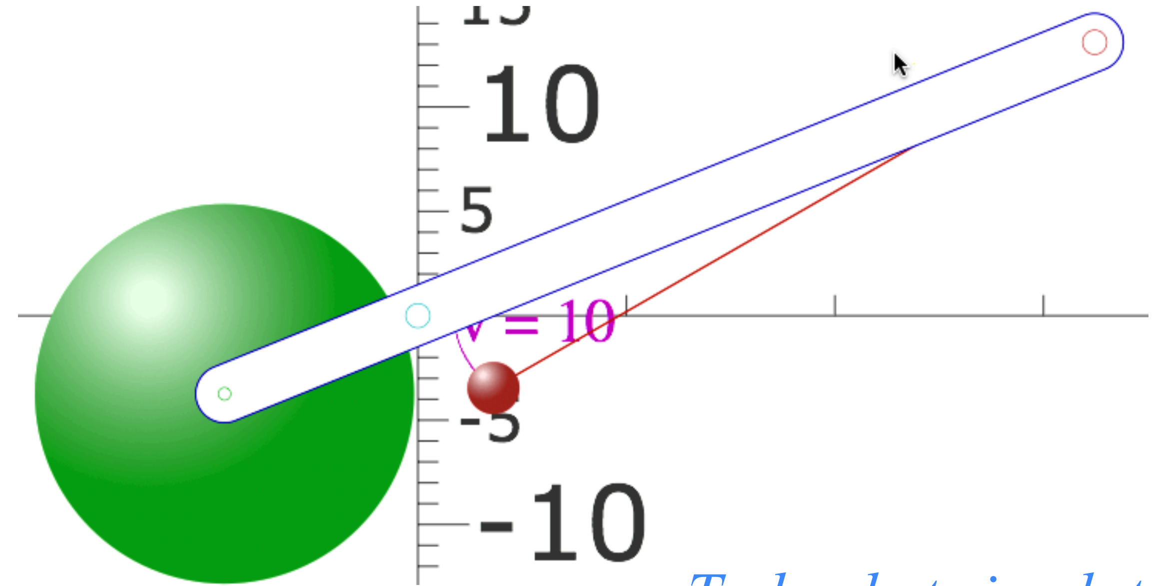

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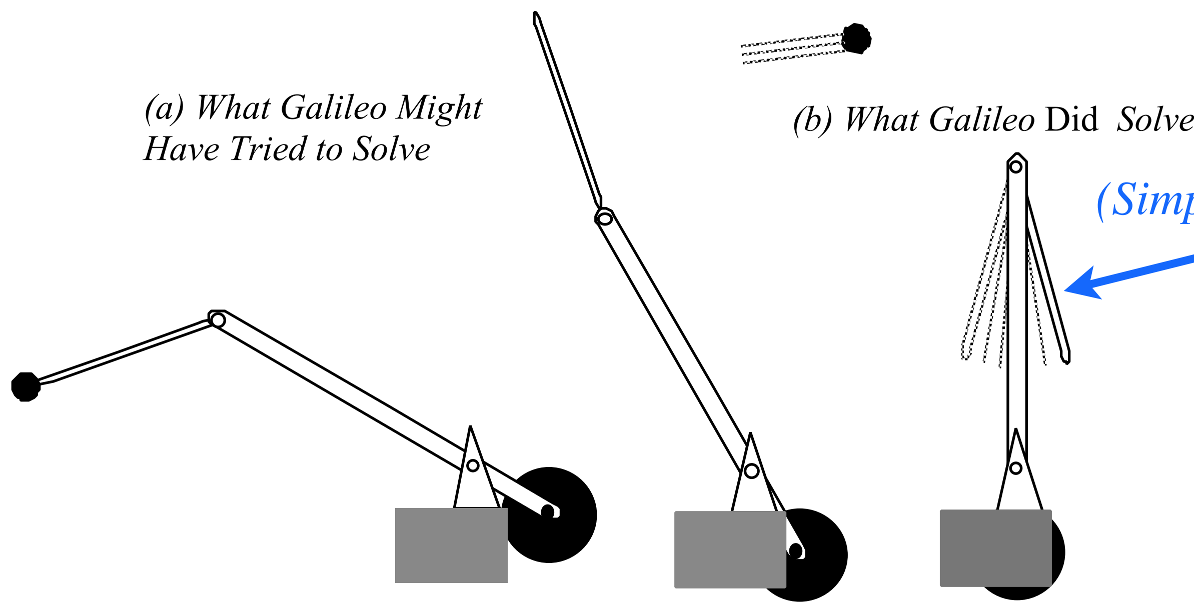

(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed fictitious) problem

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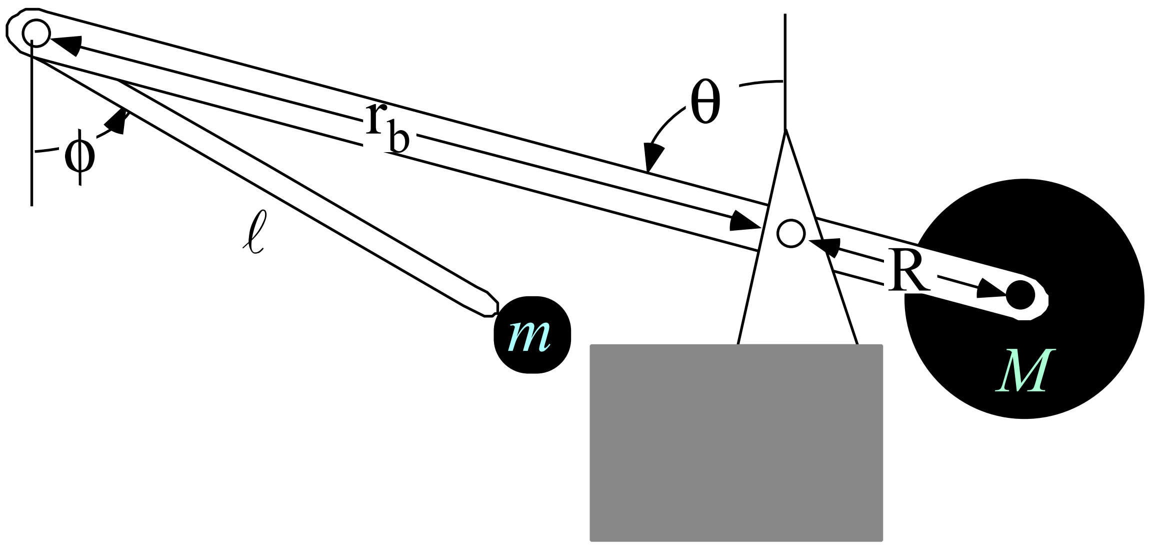

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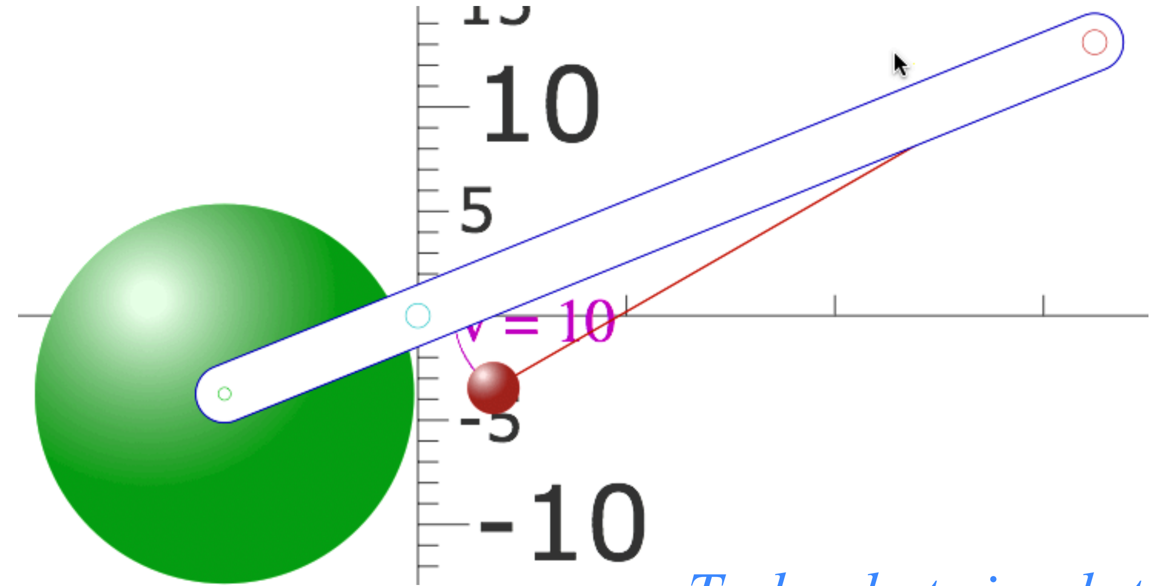

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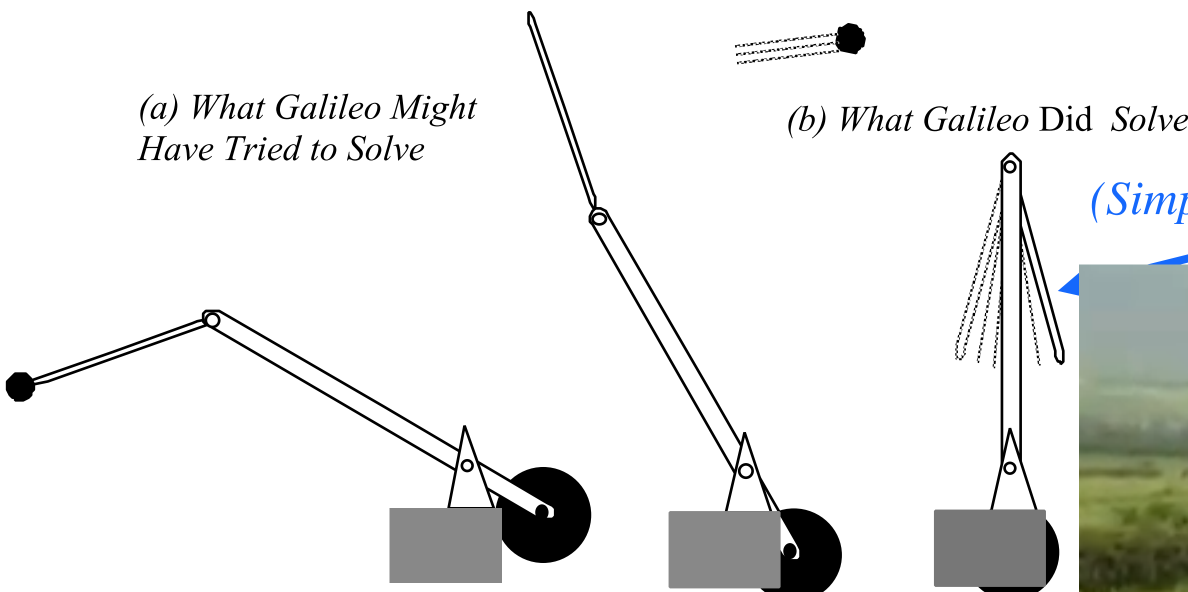


Fig. 2.1.2 Galileo's (supposed) problem

(Simple pendulum dynamics)

It’s Halloween!...and time for Punkin’ Chunkin’ Trebuchets

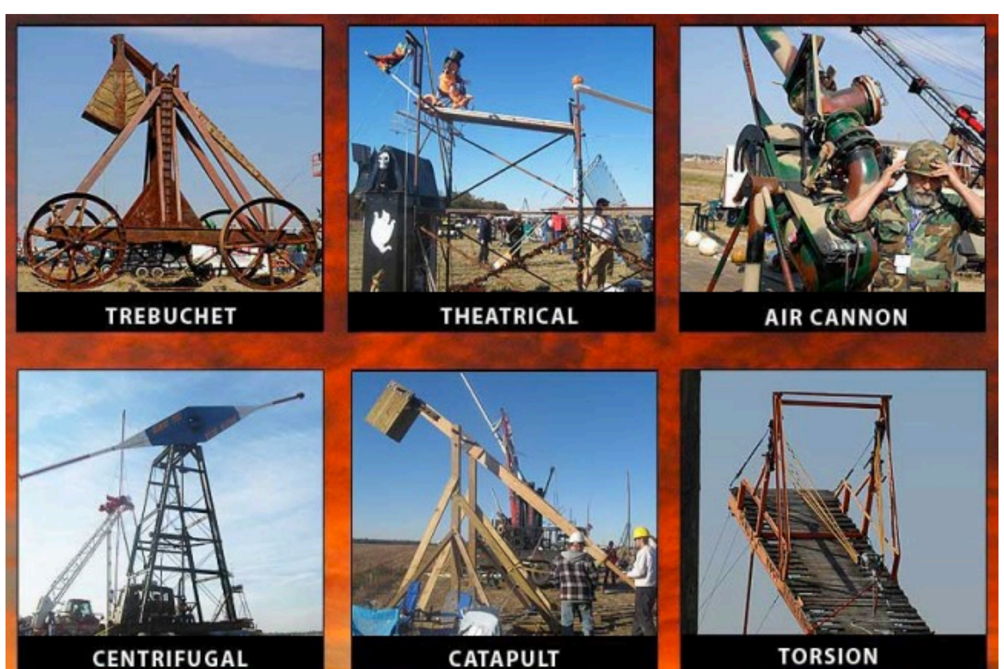
http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html

As happened in history...Trebuchet is replaced by higher-tech (or lower tech)

Giant cannons can chunk-a-punkin over 4,000 ft. *Trebuchet range max ~1,200ft.*

http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows

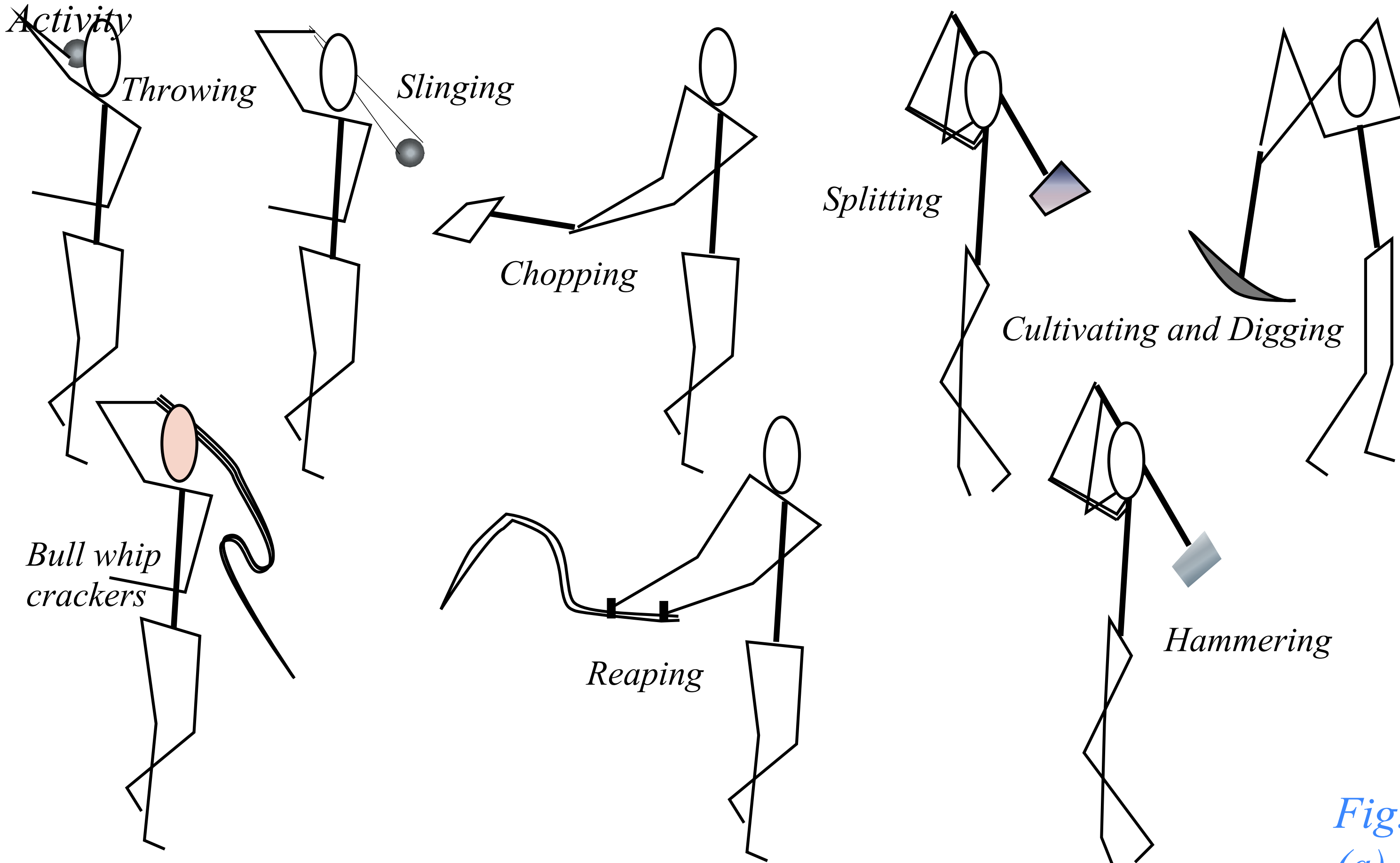
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html



*The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. **273**, 66 (July 1995))*

The medieval ingenium (9th to 14th century) and modern re-enactments

→ *Human kinesthetics and sports kinesiology* *(a) Early Human Agriculture and Infrastructure Building Activity*

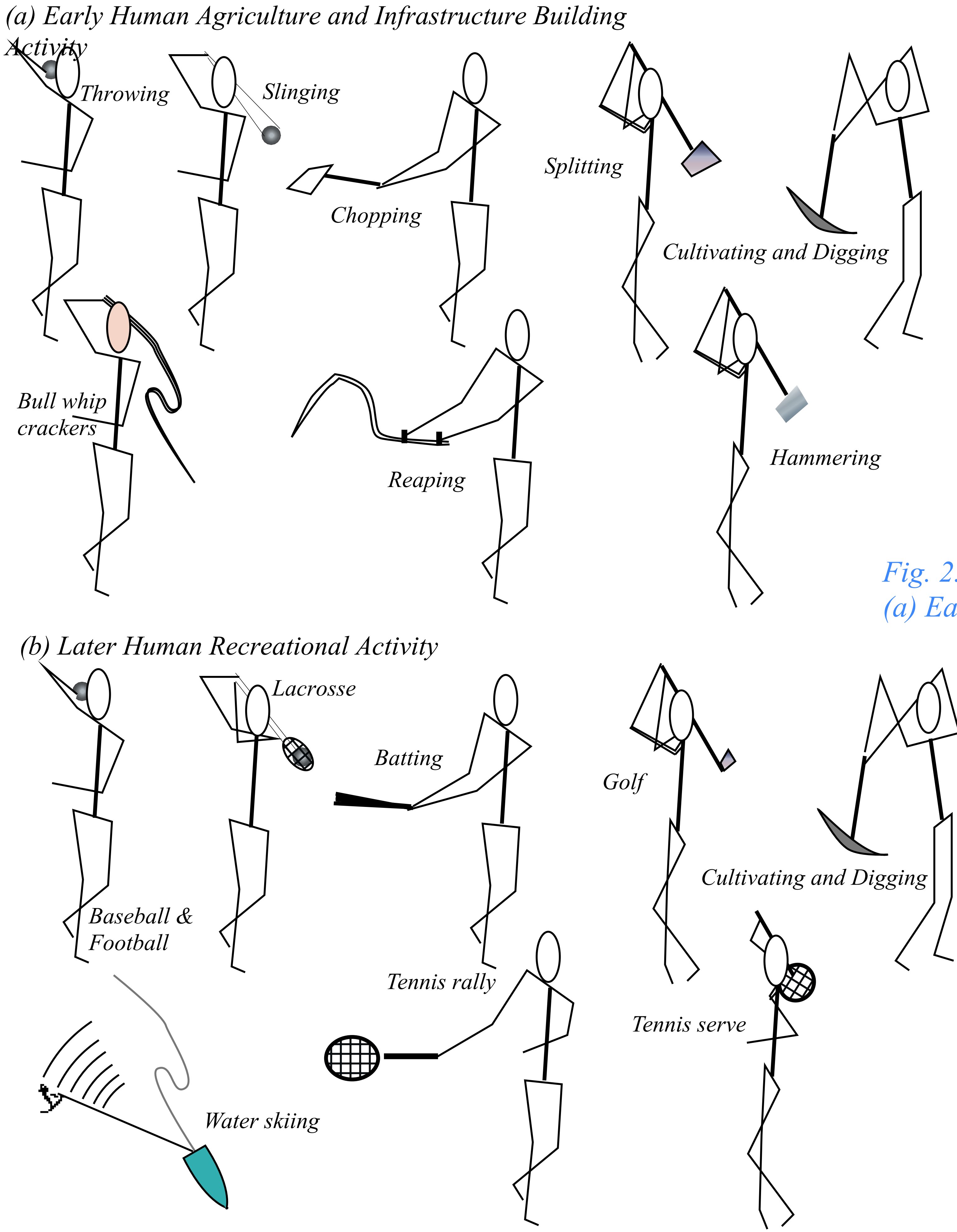


Some technique required!

KE achieved by non-linear whip action

Must avoid injury

Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work.



Some technique required!
KE achieved by non-linear whip action
Must avoid injury

Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work. (b) Later recreational kinesthetics.

Some technique required!
KE achieved by non-linear whip action
Must avoid injury

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

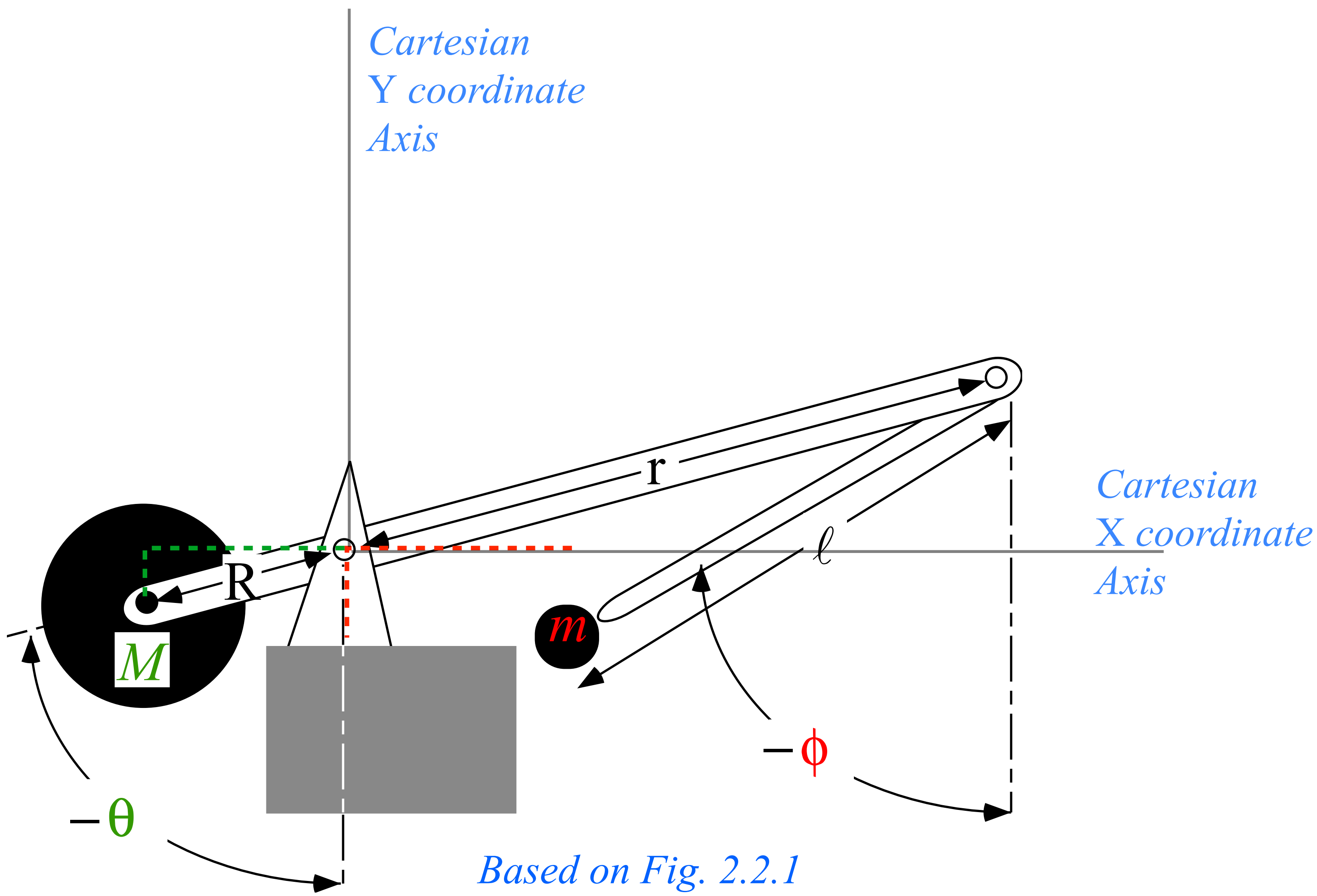
Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

geometry of trebuchet

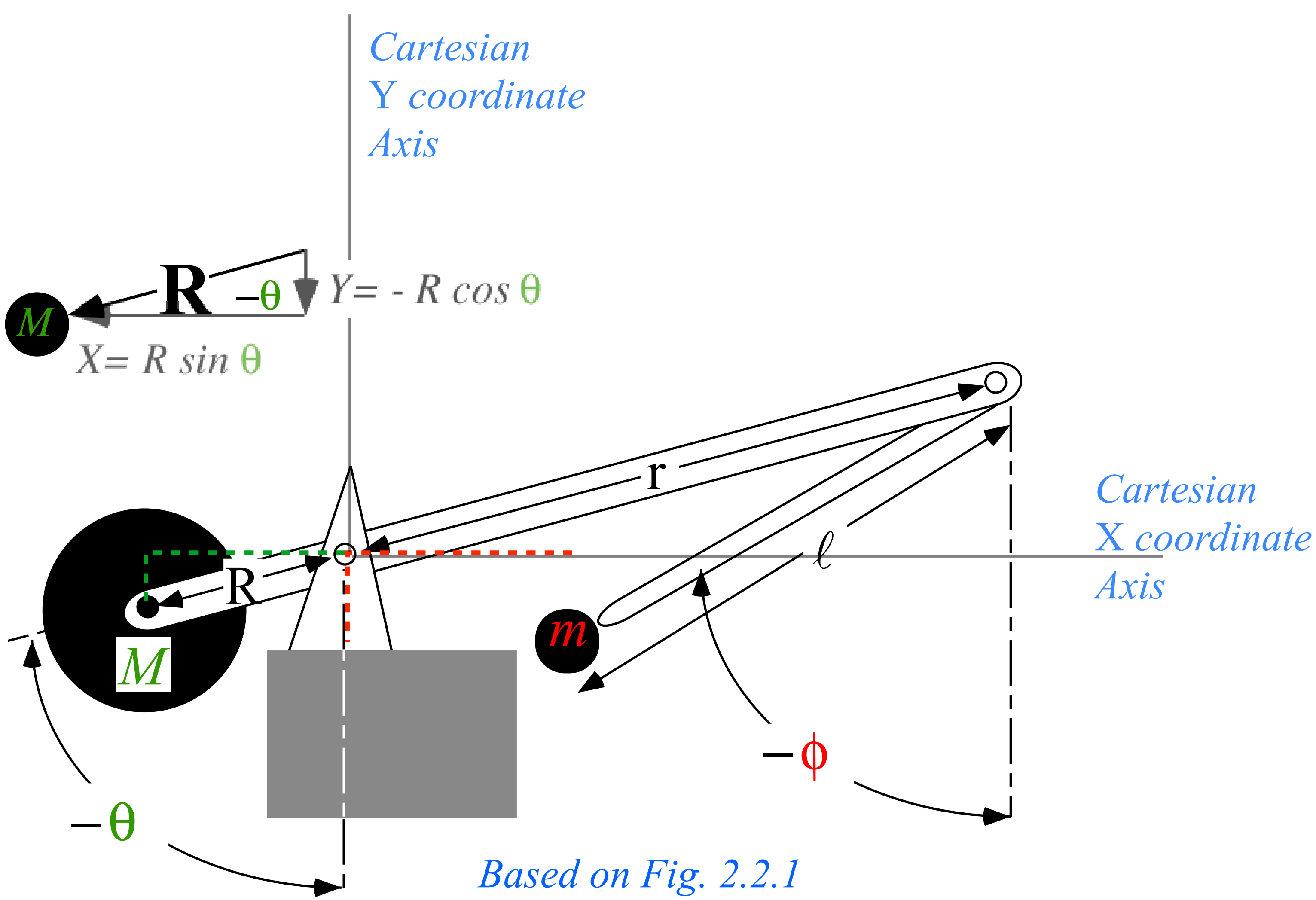


Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):
$X = R \sin \theta$
$Y = -R \cos \theta$

geometry of trebuchet



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Coordinates of mass m
(Payload or projectile):
$x = x_r + x_\ell = -r \sin\theta + \ell \sin\phi$
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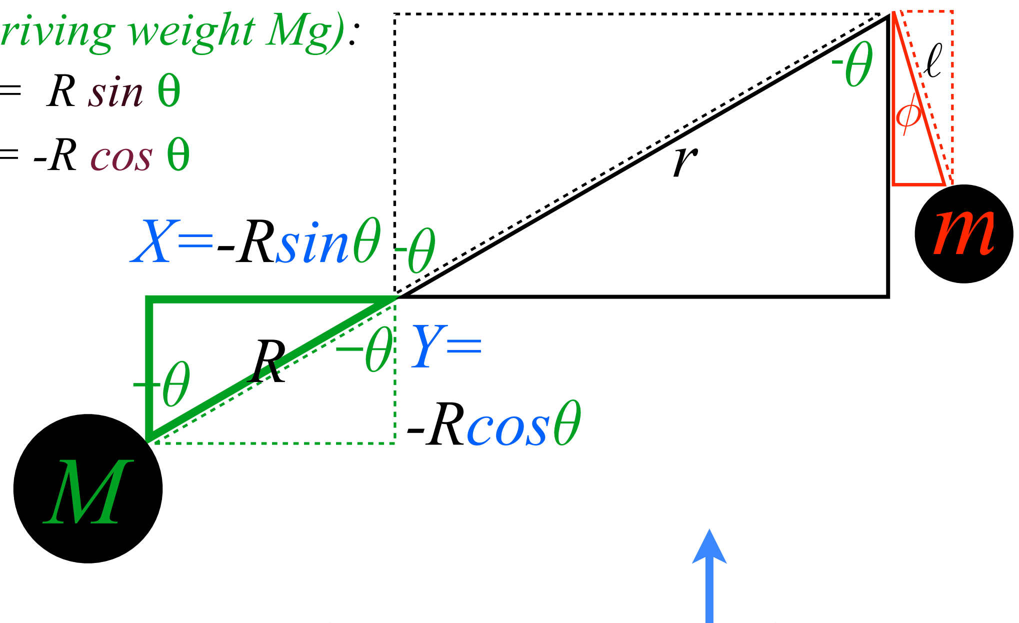
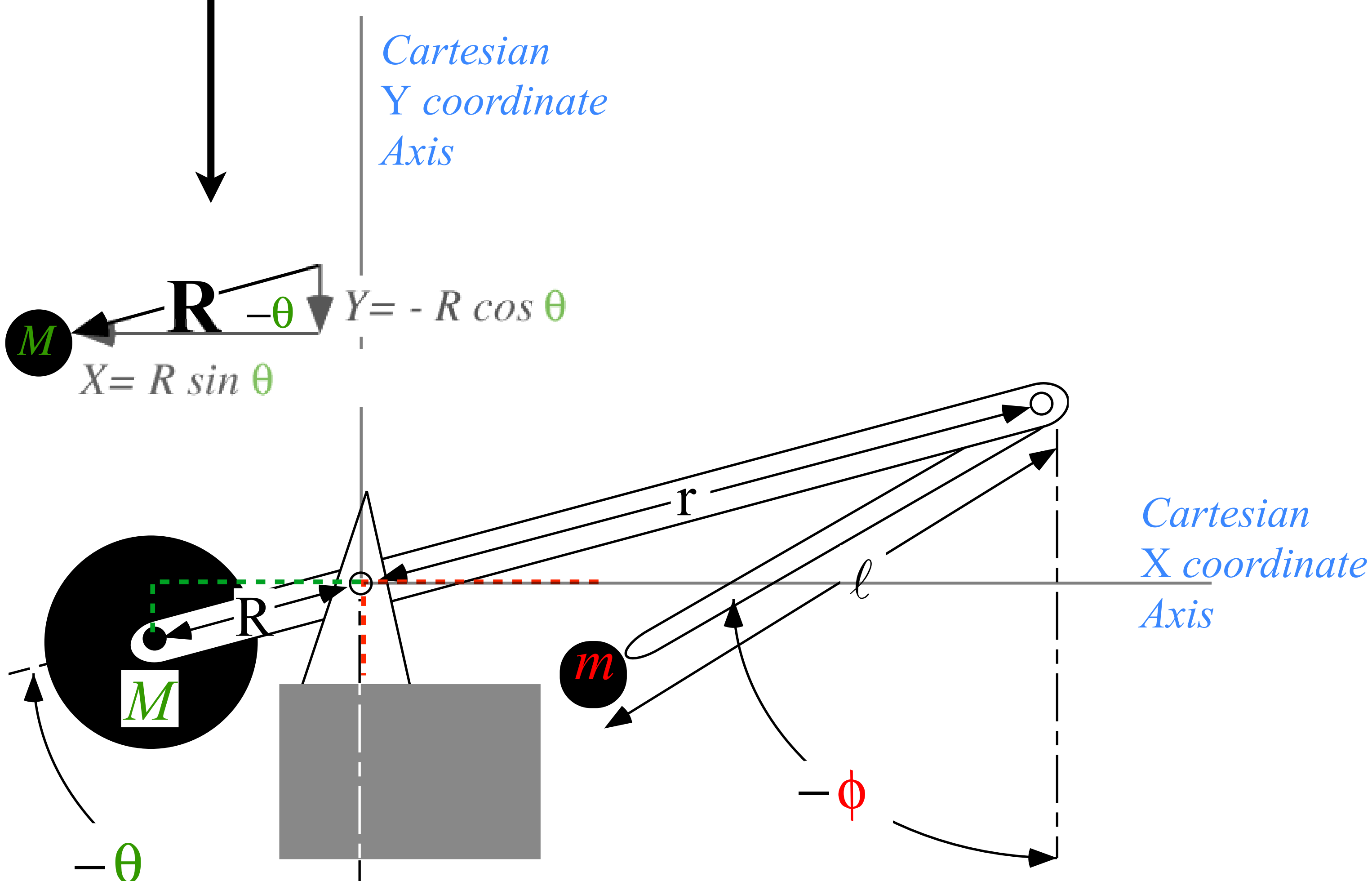


geometry of trebuchet simplified somewhat...



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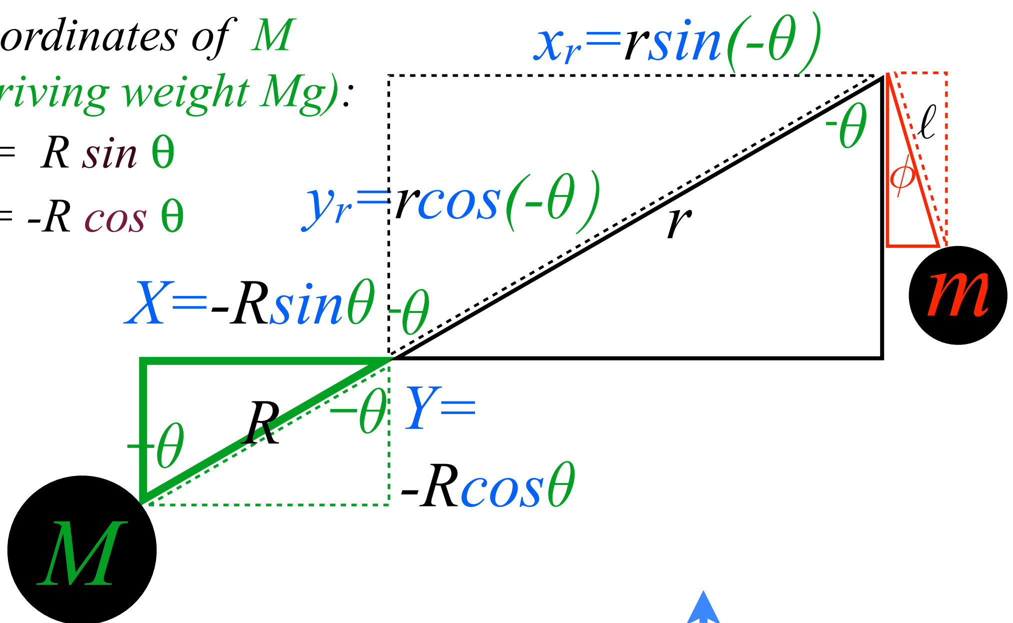

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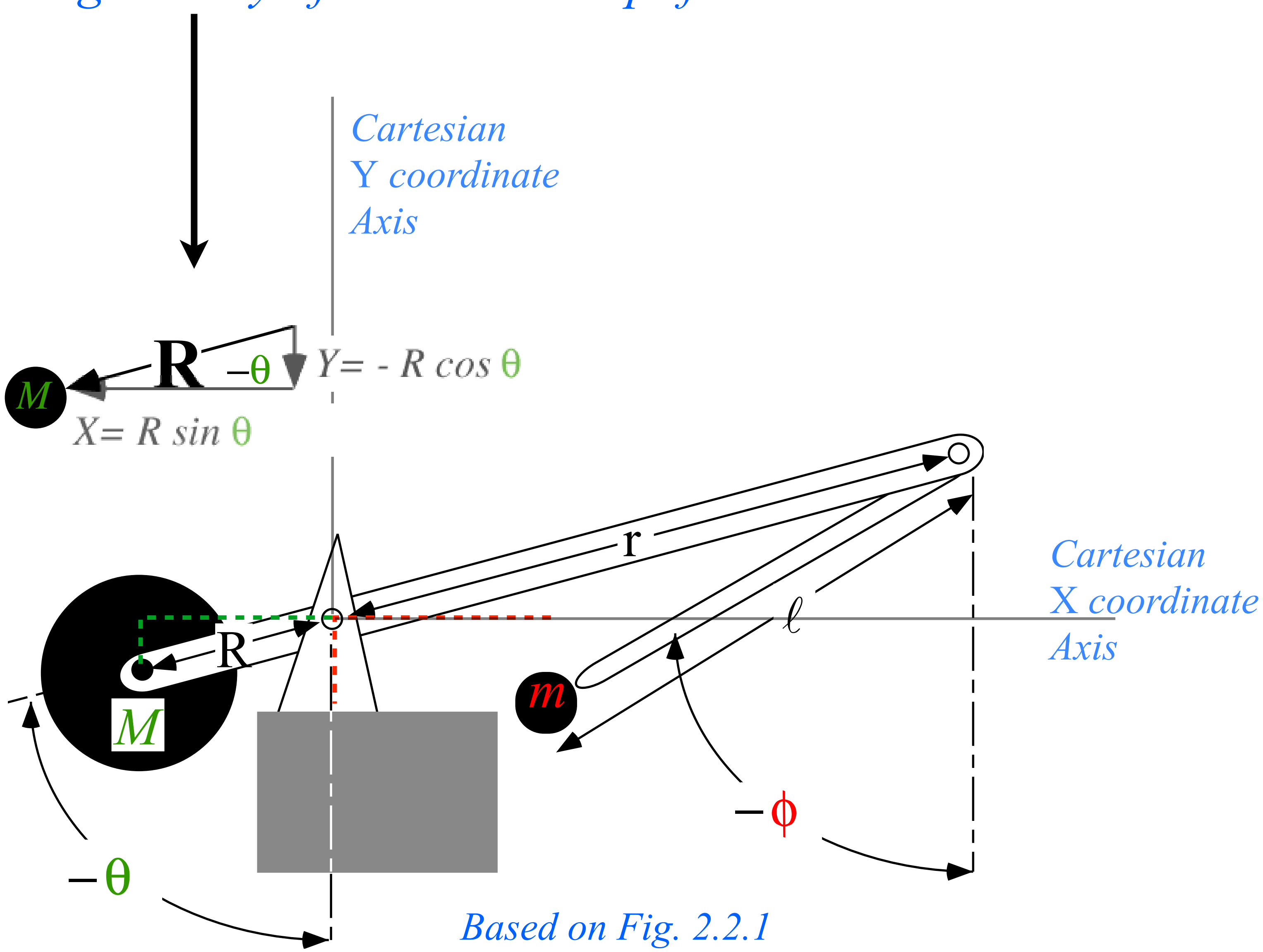

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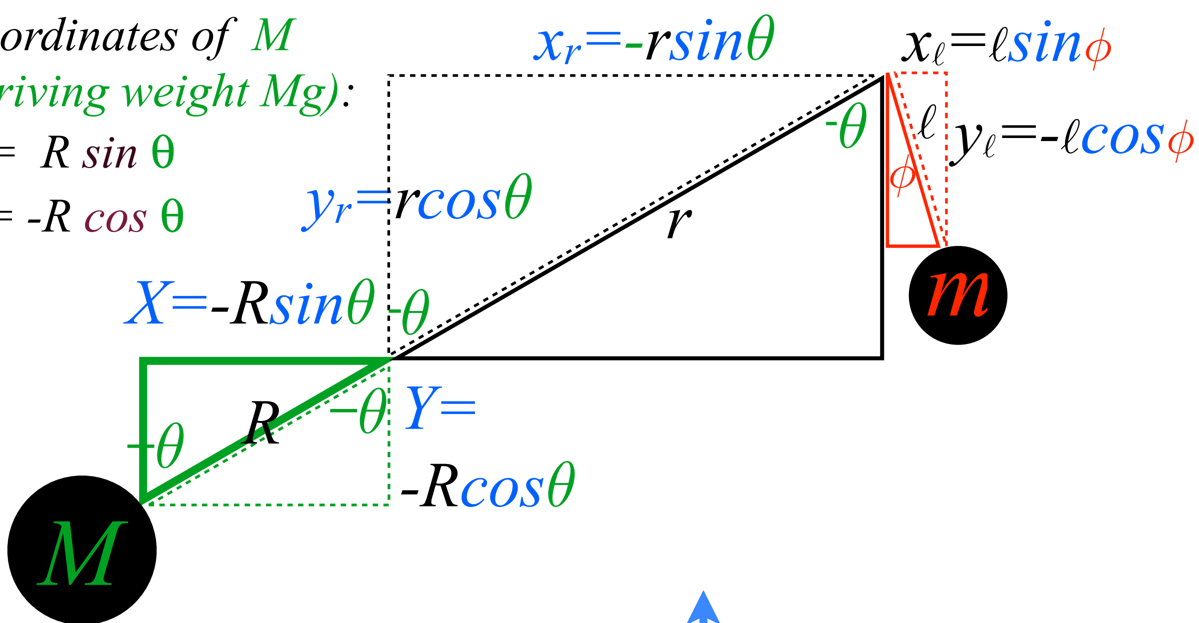

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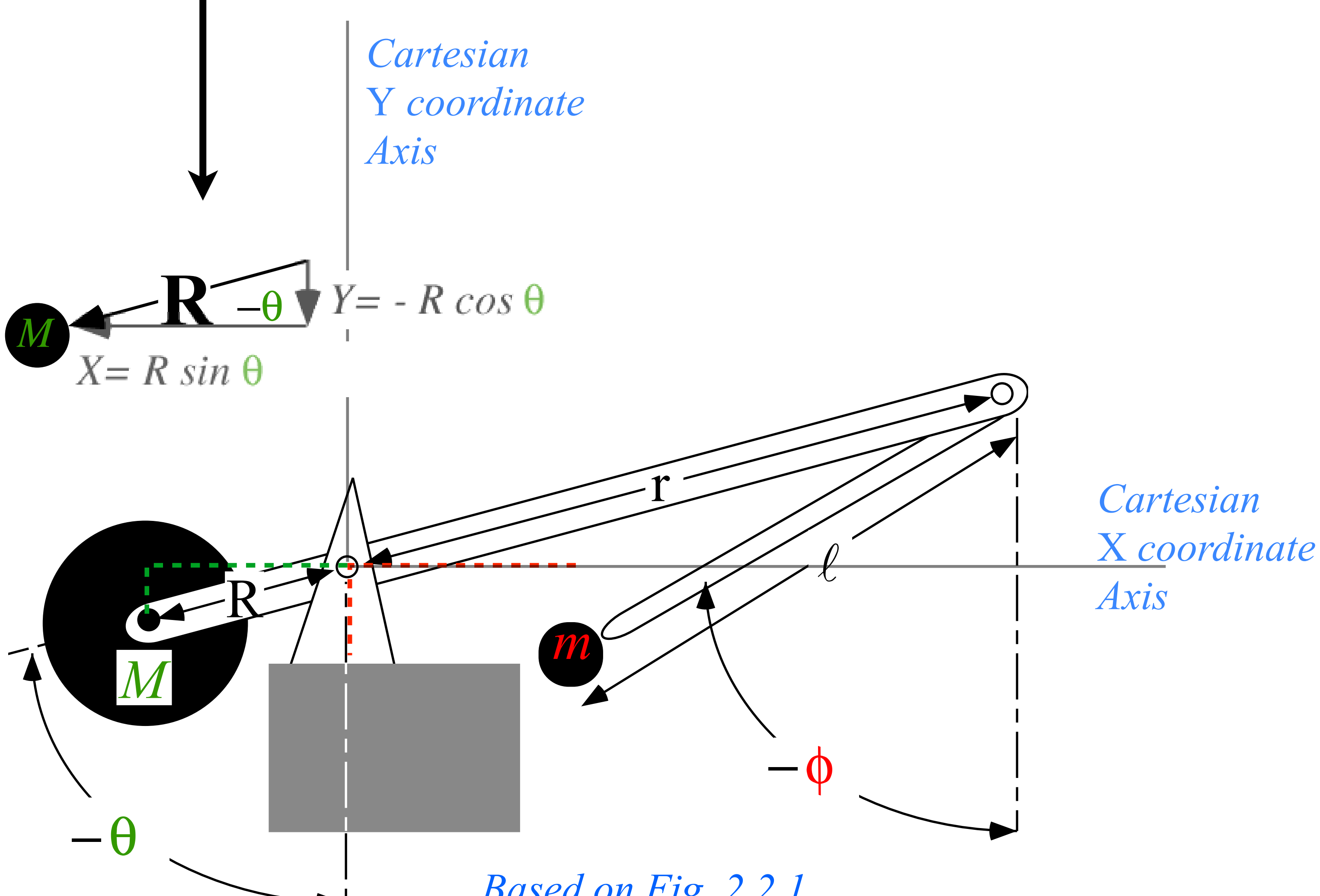

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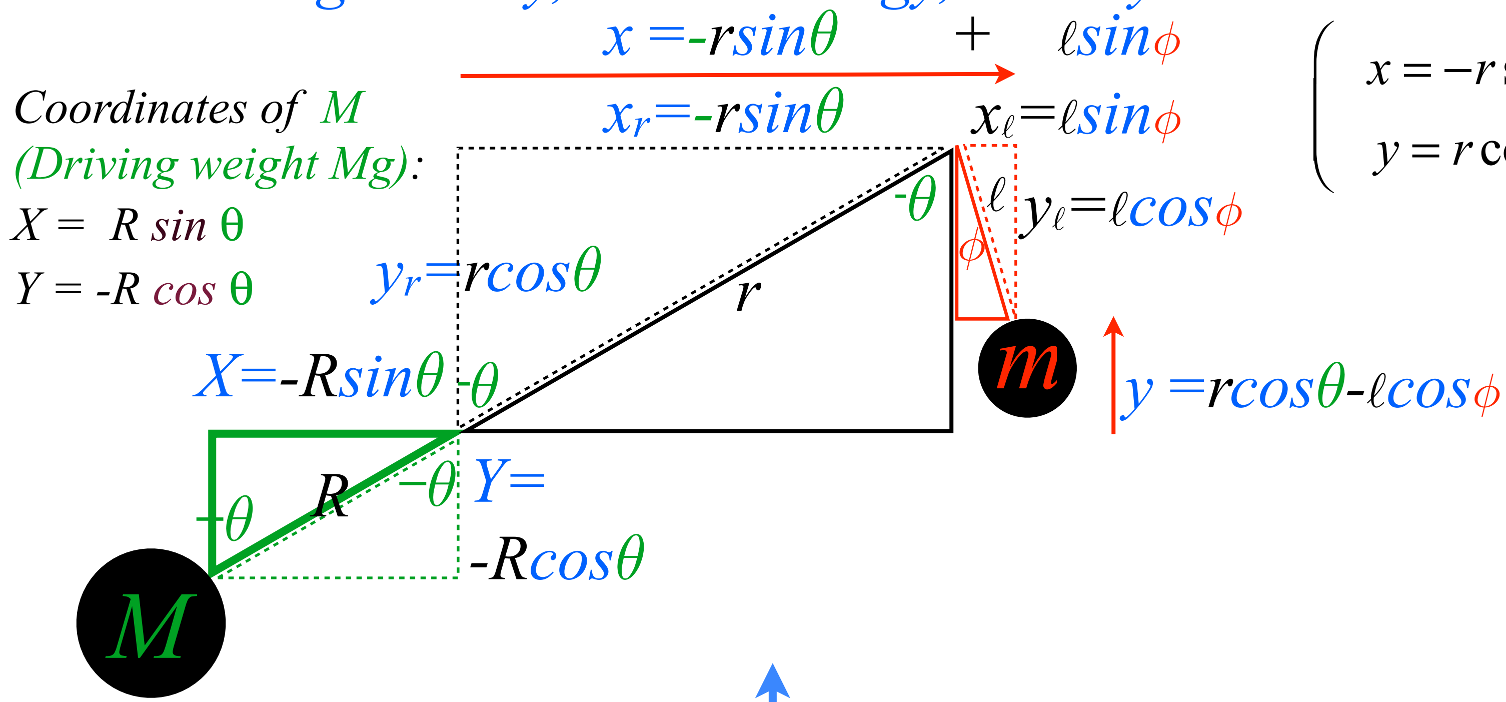

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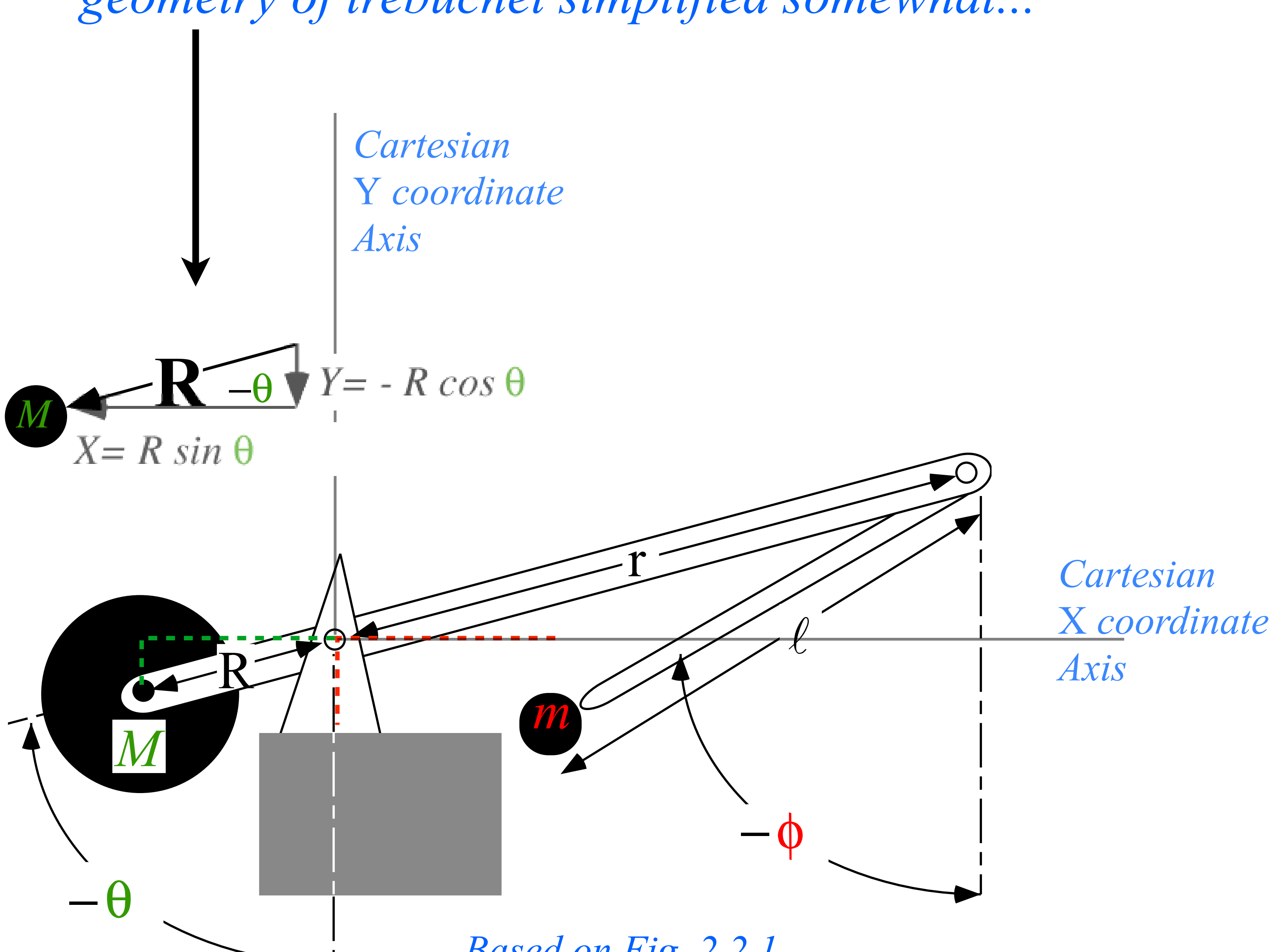

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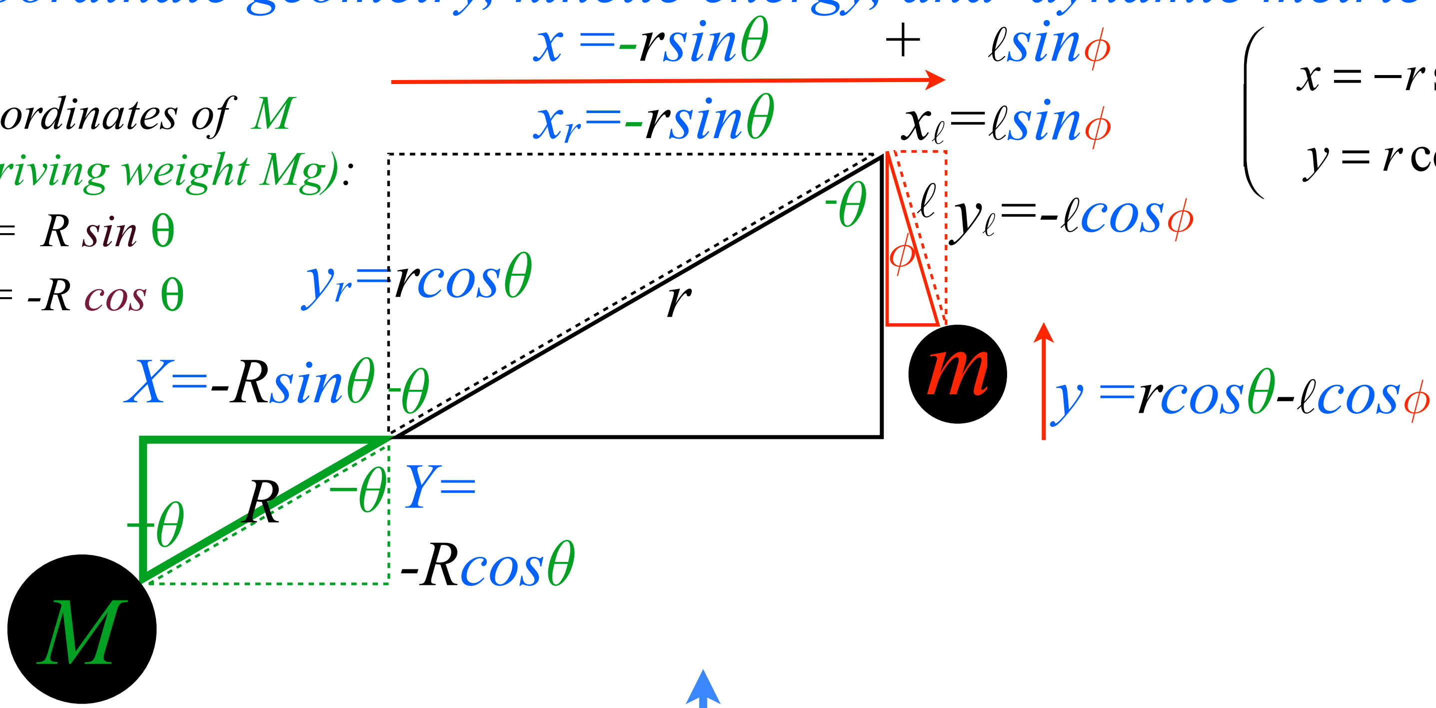

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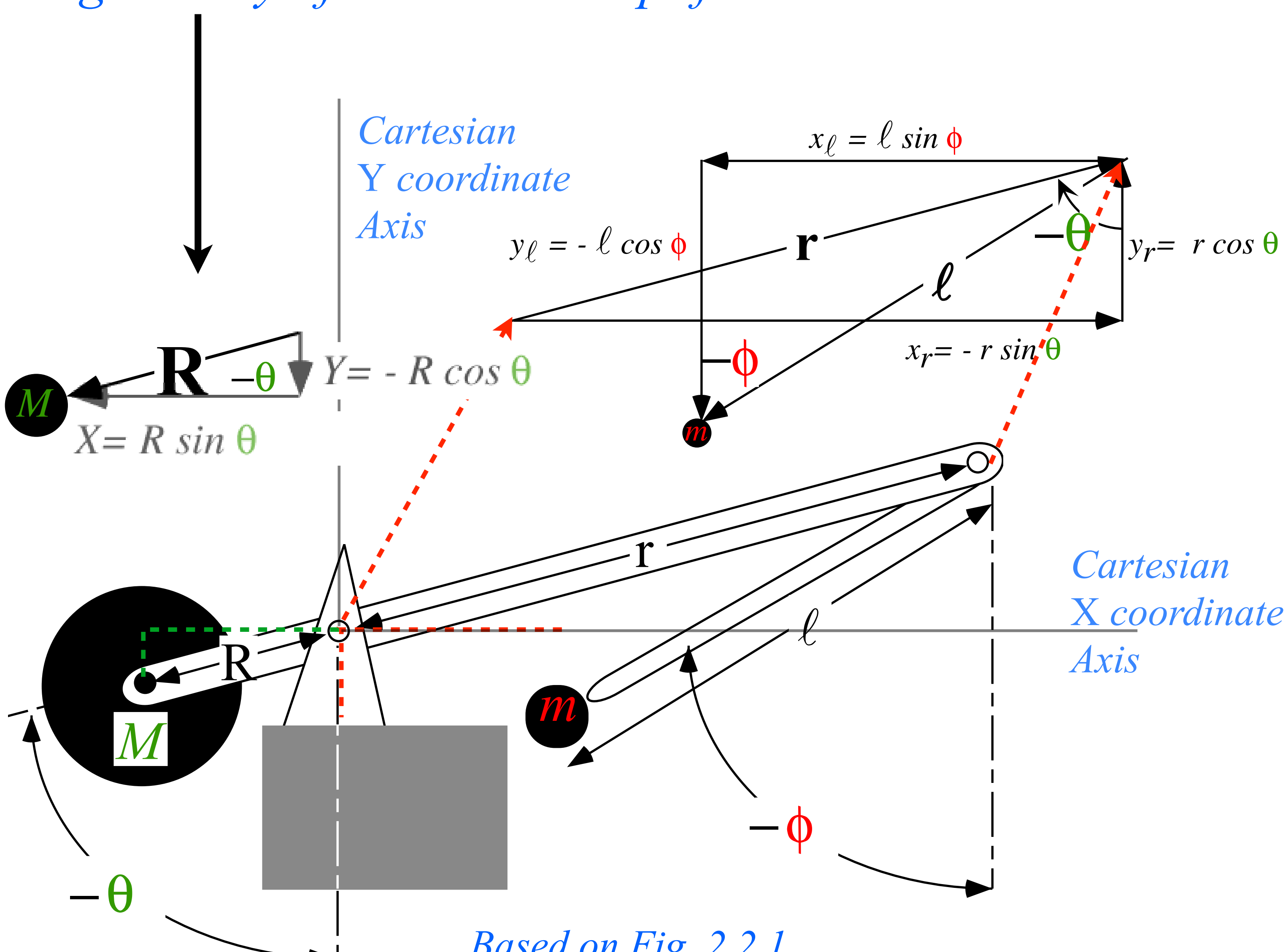

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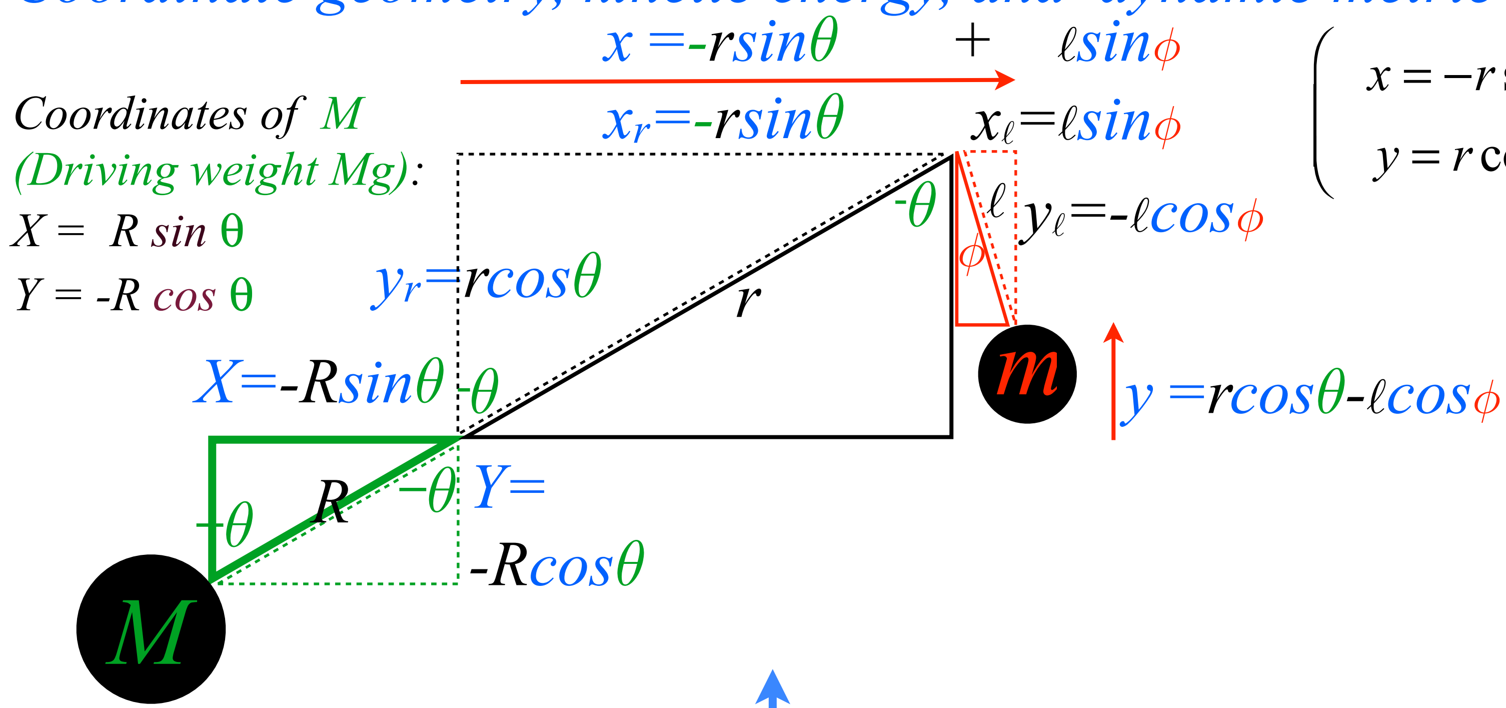
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



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1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta} d\theta + \frac{\partial X}{\partial\phi} d\phi, \qquad dx = \frac{\partial x}{\partial\theta} d\theta + \frac{\partial x}{\partial\phi} d\phi,$$

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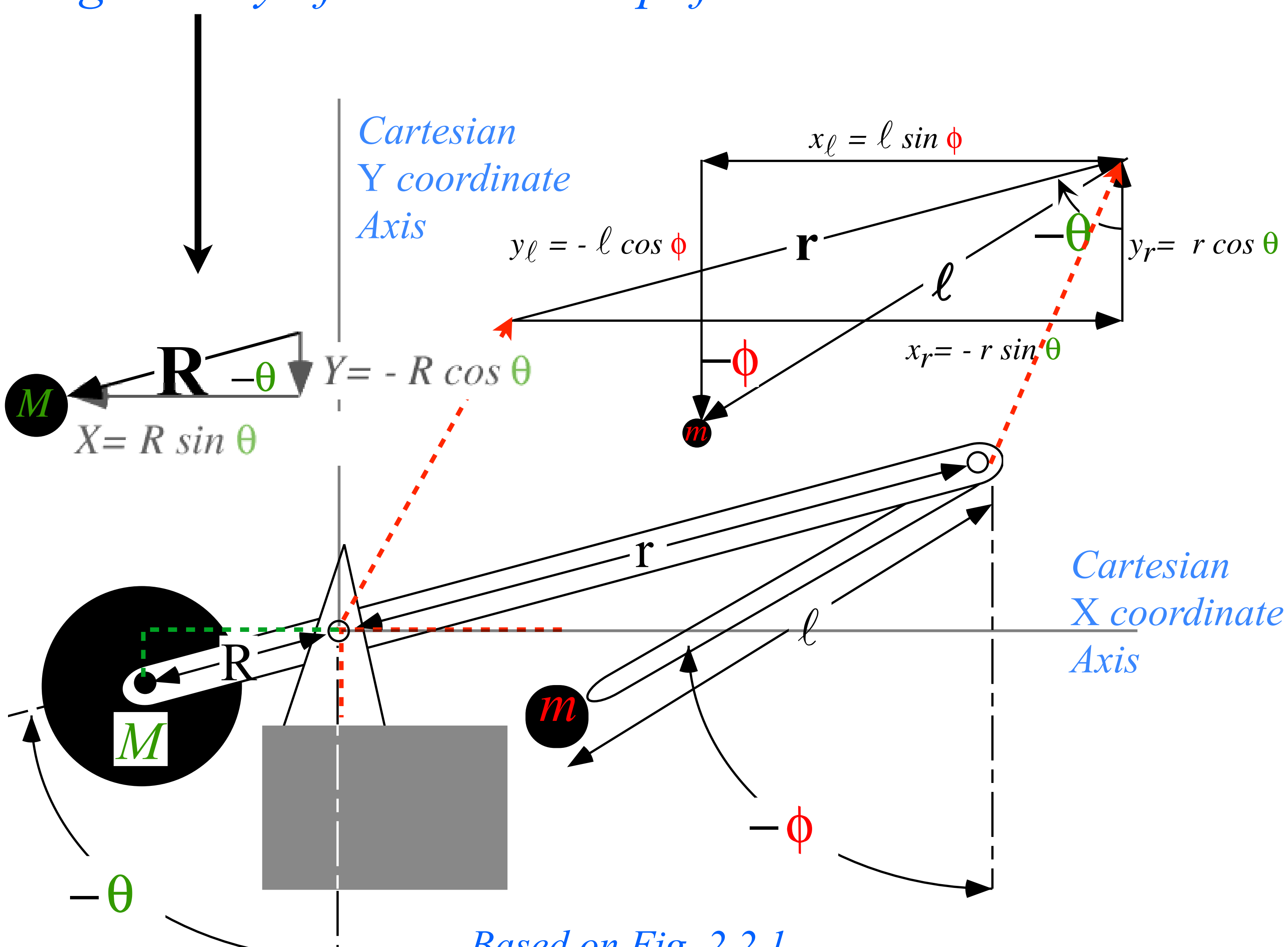
geometry of trebuchet simplified somewhat...



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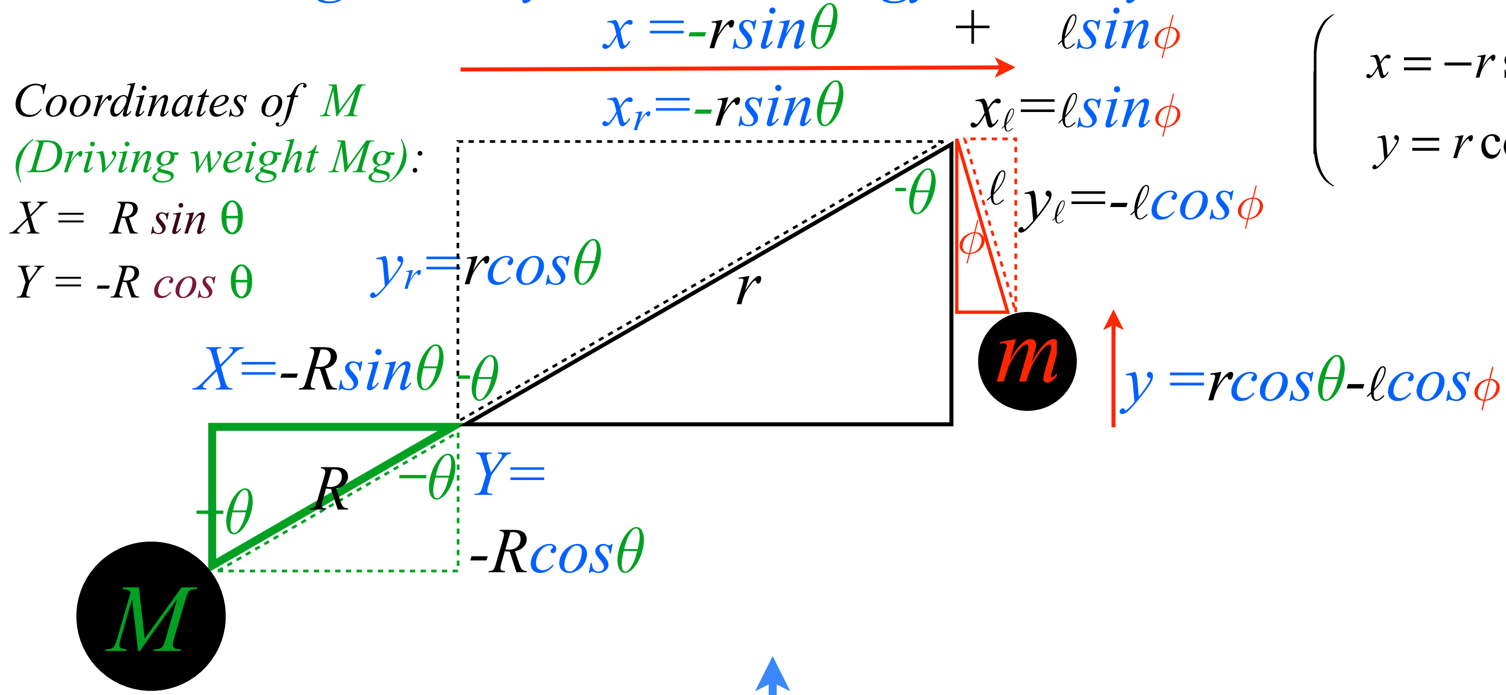

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$$dX = R\cos\theta \ d\theta + 0, \qquad dx = -r\cos\theta \ d\theta + \ell\cos\phi \ d\phi$$

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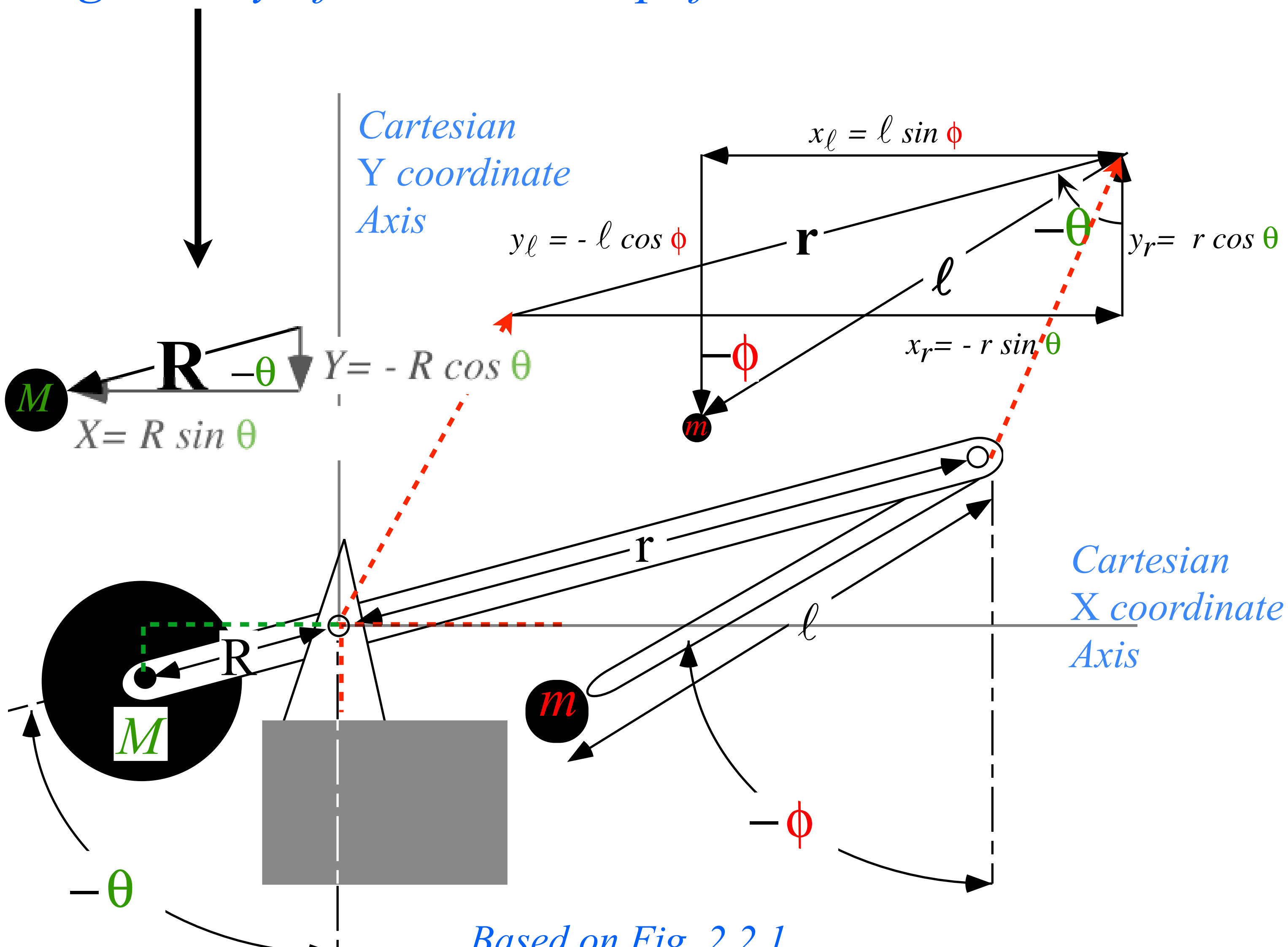
geometry of trebuchet simplified somewhat...



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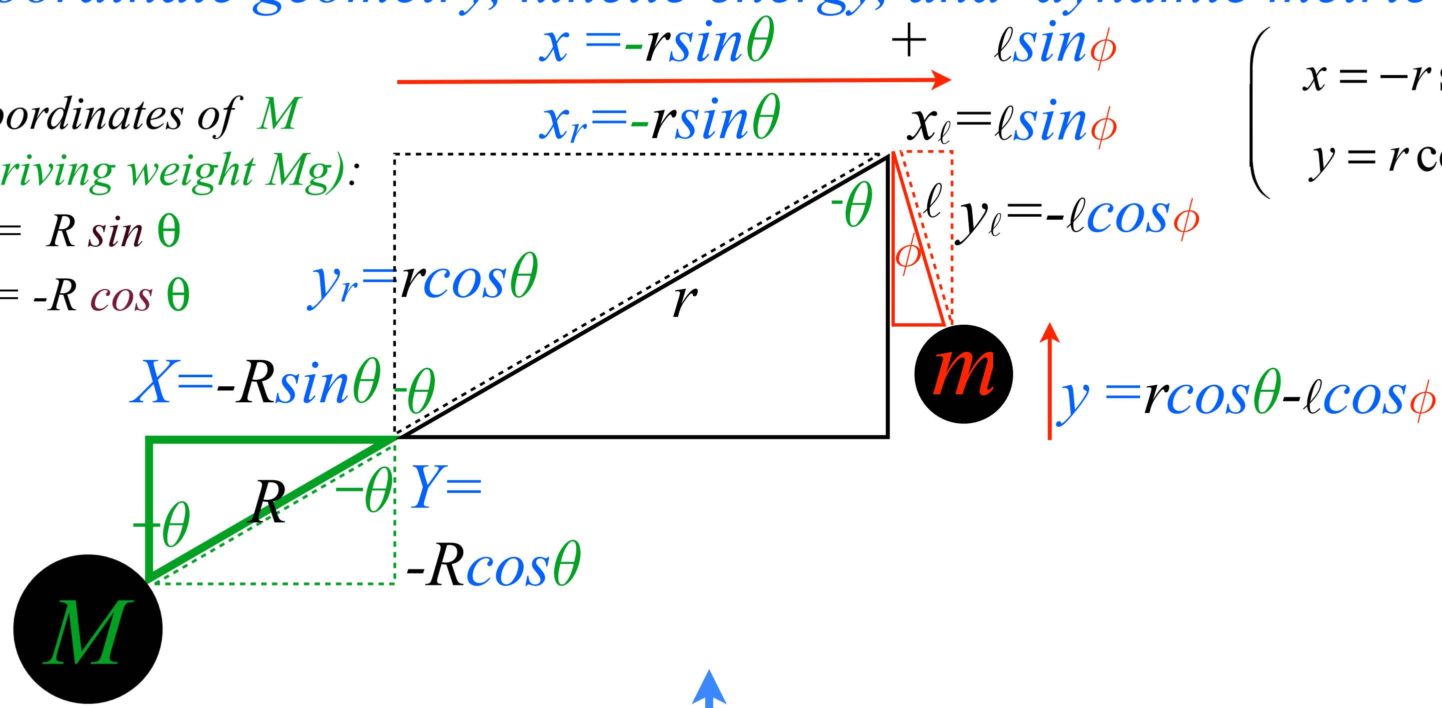

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$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

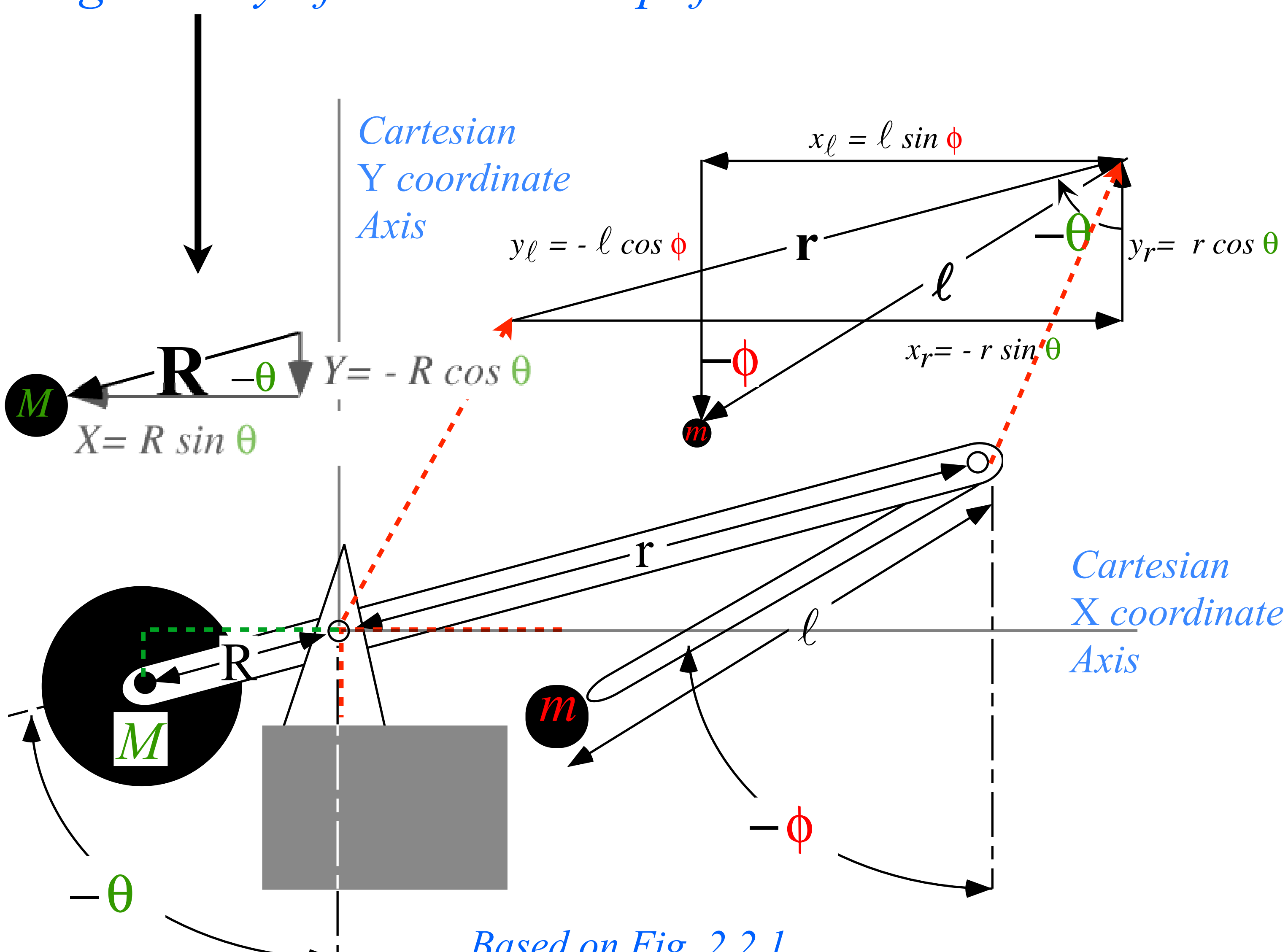
$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

Constraint relations:
$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$
$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$
$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$

geometry of trebuchet simplified somewhat...

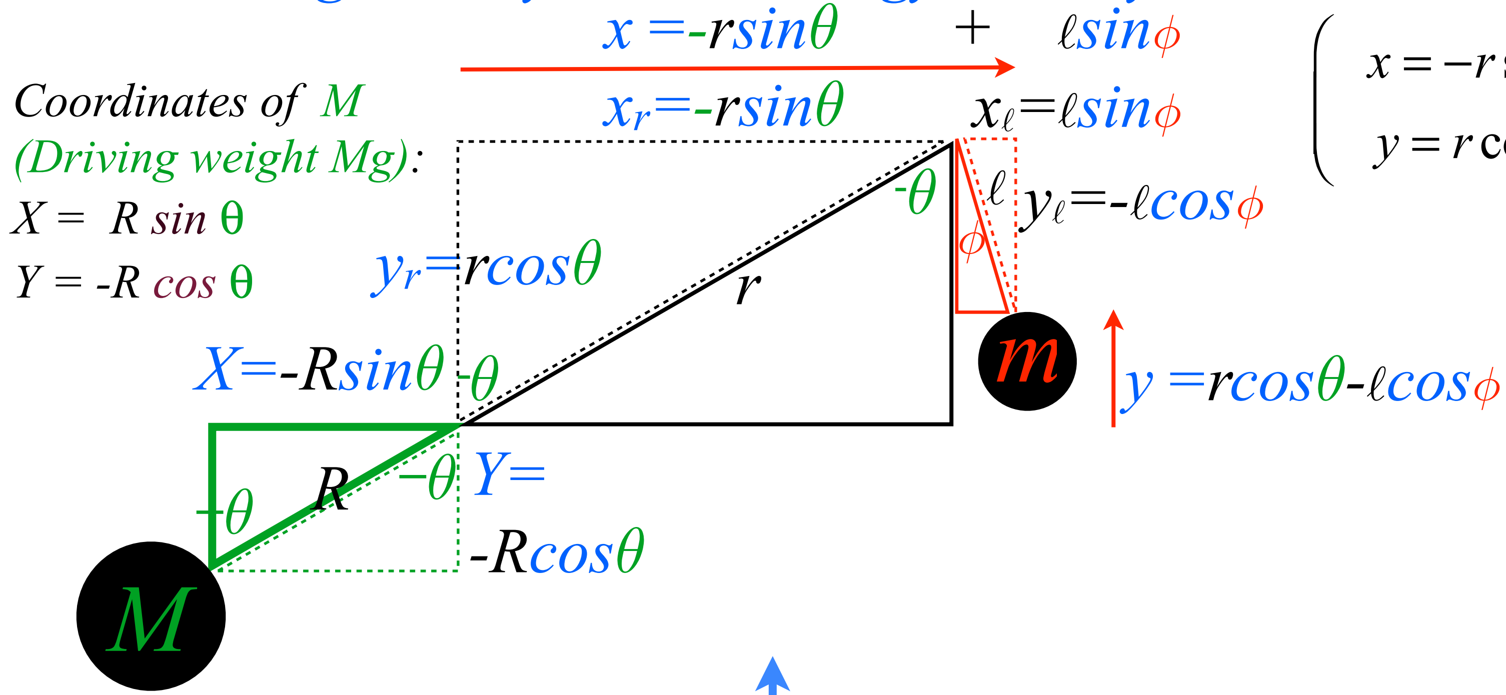
'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Based on Fig. 2.2.1

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
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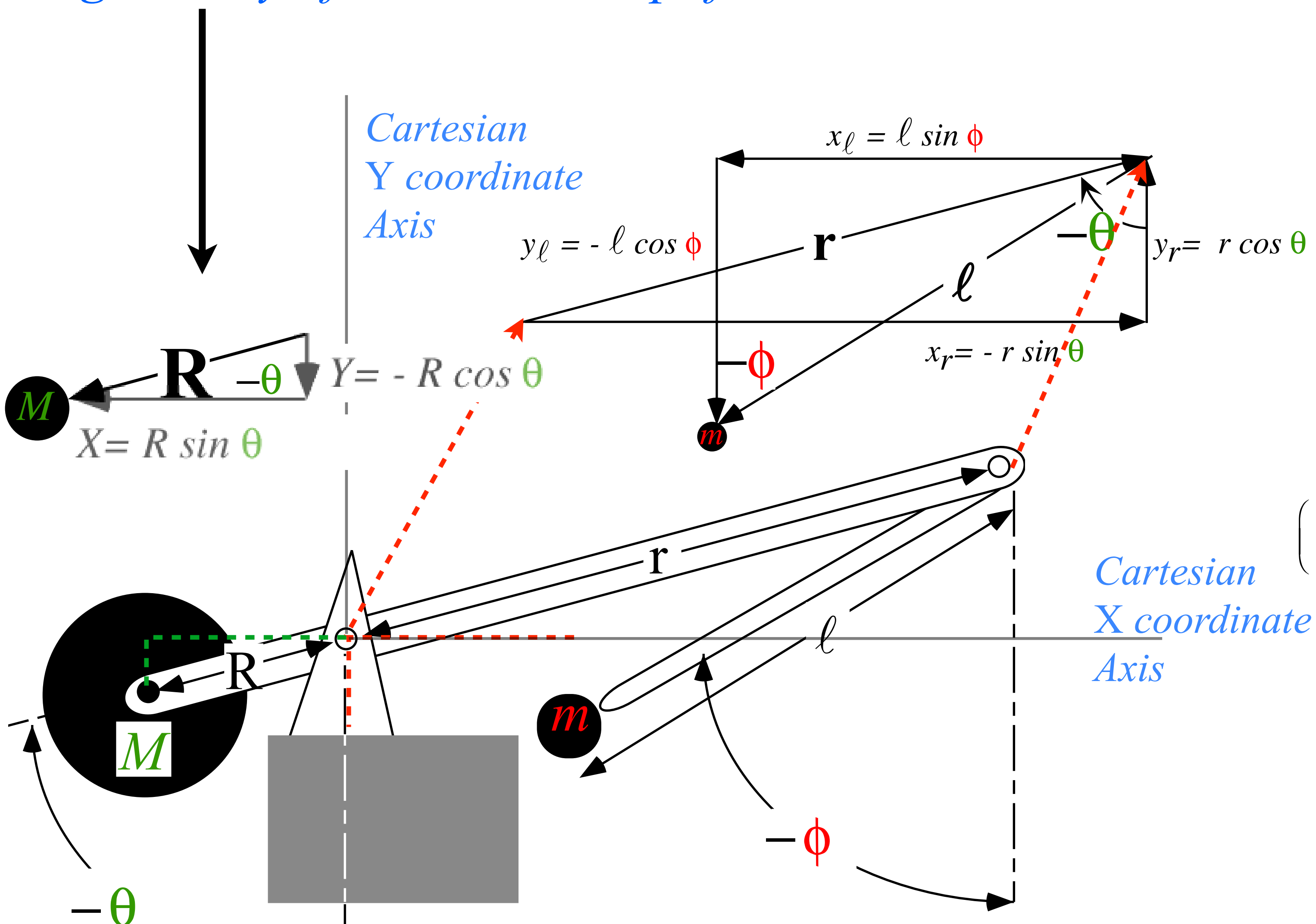
geometry of trebuchet simplified somewhat...



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$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

Constraint relations: $c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

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$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$

'Raw' Jacobian form

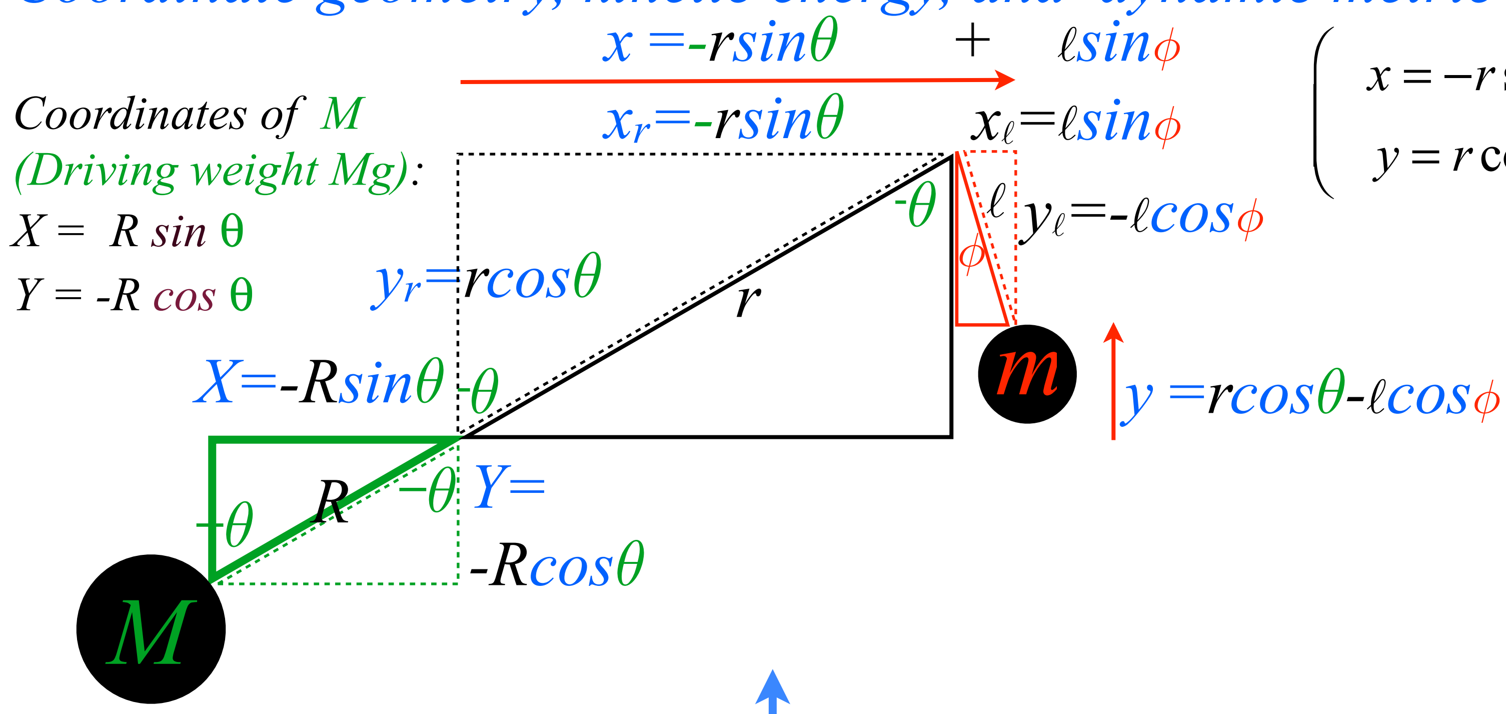
$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: det} \begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

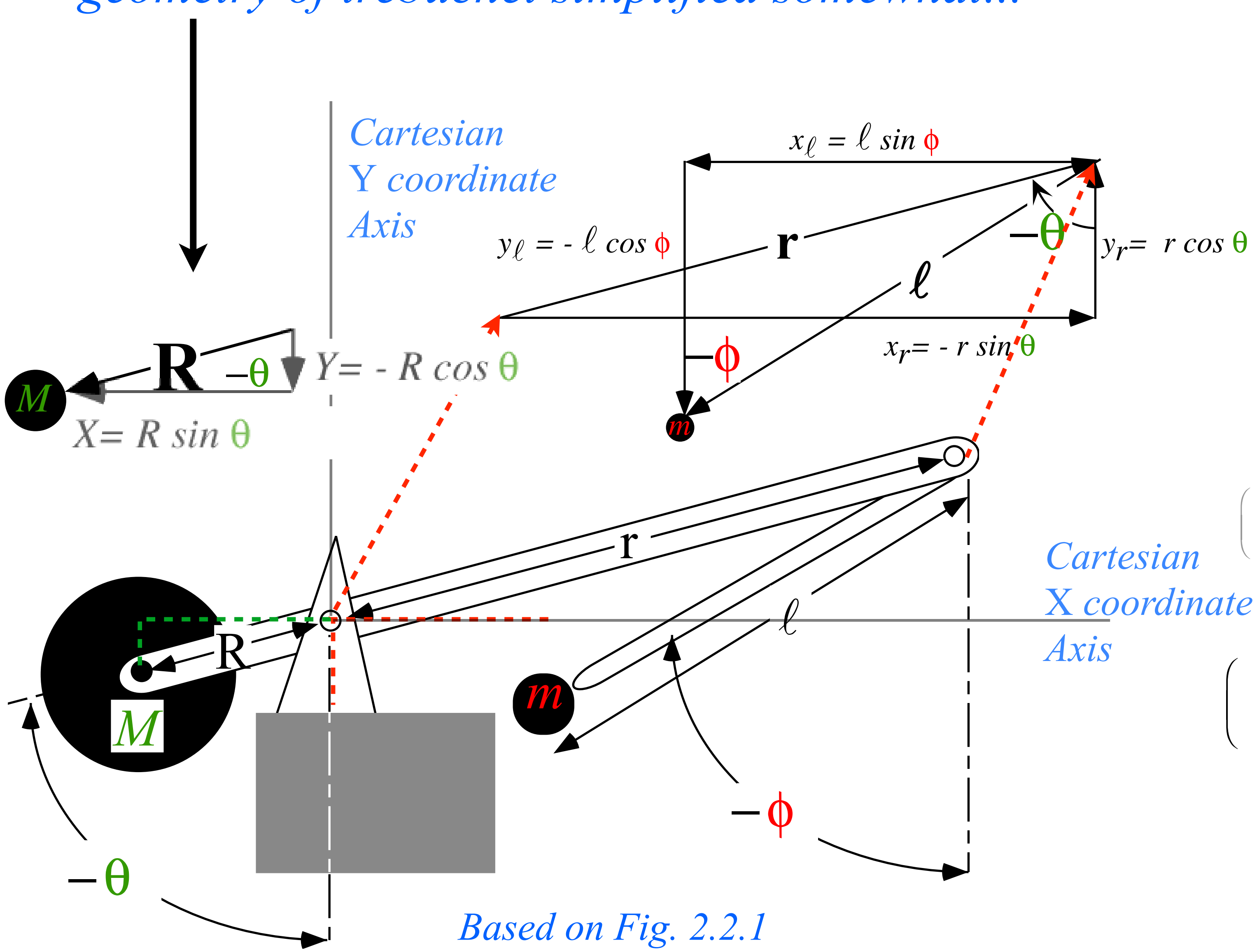
FAILS! (Always singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\left(\begin{array}{l} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{array}\right)$$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Coordinates of mass m
(Payload or projectile):

$x = x_r + x_\ell = -\,r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = \ \ r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX=\frac{\partial X}{\partial\theta}d\theta+\frac{\partial X}{\partial\phi}d\phi,\qquad dx=\frac{\partial x}{\partial\theta}d\theta+\frac{\partial x}{\partial\phi}d\phi,$$

$$dY=\frac{\partial Y}{\partial\theta}d\theta+\frac{\partial Y}{\partial\phi}d\phi,\qquad dy=\frac{\partial y}{\partial\theta}d\theta+\frac{\partial y}{\partial\phi}d\phi.$$

$$dX=R\cos\theta\ d\theta+0,\qquad dx=-r\cos\theta\ d\theta+\ell\cos\phi\ d\phi$$

$$dY=R\sin\theta\ d\theta+0,\qquad dy=-r\sin\theta\ d\theta+\ell\sin\phi\ d\phi$$

Constraint relations:

$c_R(X,Y)=X^2+Y^2=R^2=const.$

$c_\ell(x_\ell,y_\ell)=x_\ell^2+y_\ell^2=\ell^2=const.$

$c_r(x_r,y_r)=x_r^2+y_r^2=r^2=const.$

'Raw' Jacobian form

$$\begin{pmatrix} dX\\ dY\\ dx\\ dy\end{pmatrix}=\begin{pmatrix}\frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi}\\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi}\\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix} d\theta\\ d\phi\end{pmatrix}=\begin{pmatrix} R\cos\theta & 0\\ R\sin\theta & 0\\ -r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix} d\theta\\ d\phi\end{pmatrix}$$

Finding a reduced Jacobian form

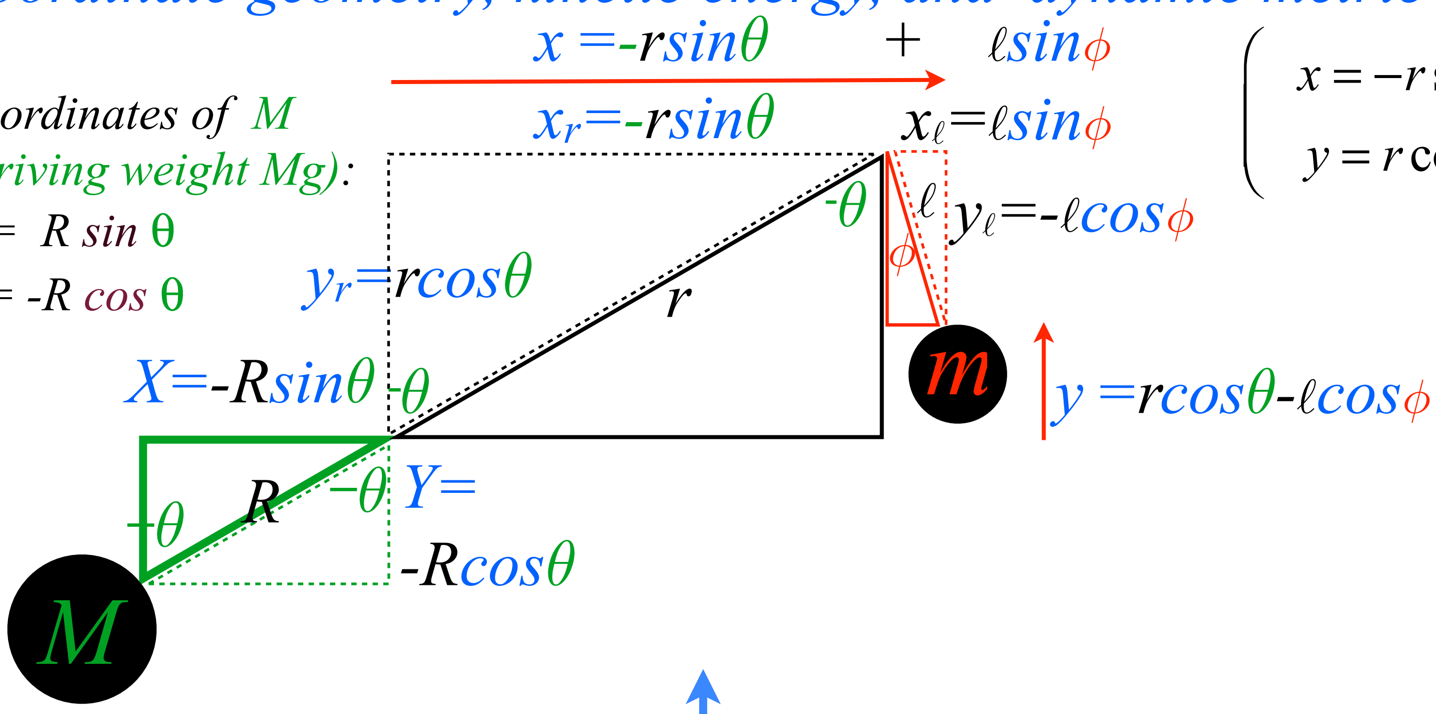
$$\begin{pmatrix} dX\\ dY\end{pmatrix}=\begin{pmatrix}\frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi}\\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi}\end{pmatrix}\begin{pmatrix} d\theta\\ d\phi\end{pmatrix}=\begin{pmatrix} R\cos\theta & 0\\ R\sin\theta & 0\end{pmatrix}\begin{pmatrix} d\theta\\ d\phi\end{pmatrix}\ \text{FAILS since: det}\begin{vmatrix} R\cos\theta & 0\\ R\sin\theta & 0\end{vmatrix}=0$$

FAILS! (Always singular)

$$\begin{pmatrix} dx\\ dy\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix} d\theta\\ d\phi\end{pmatrix}=\begin{pmatrix} -r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix} d\theta\\ d\phi\end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):
$X = R \sin\theta$
$Y = -R \cos\theta$



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
(Payload or projectile):
$x = x_r + x_\ell = - r \sin\theta + \ell \sin\phi$
$y = y_r + y_\ell = r \cos\theta - \ell \cos\phi$

geometry of trebuchet simplified somewhat...

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta} d\theta + \frac{\partial X}{\partial\phi} d\phi, \qquad dx = \frac{\partial x}{\partial\theta} d\theta + \frac{\partial x}{\partial\phi} d\phi,$$

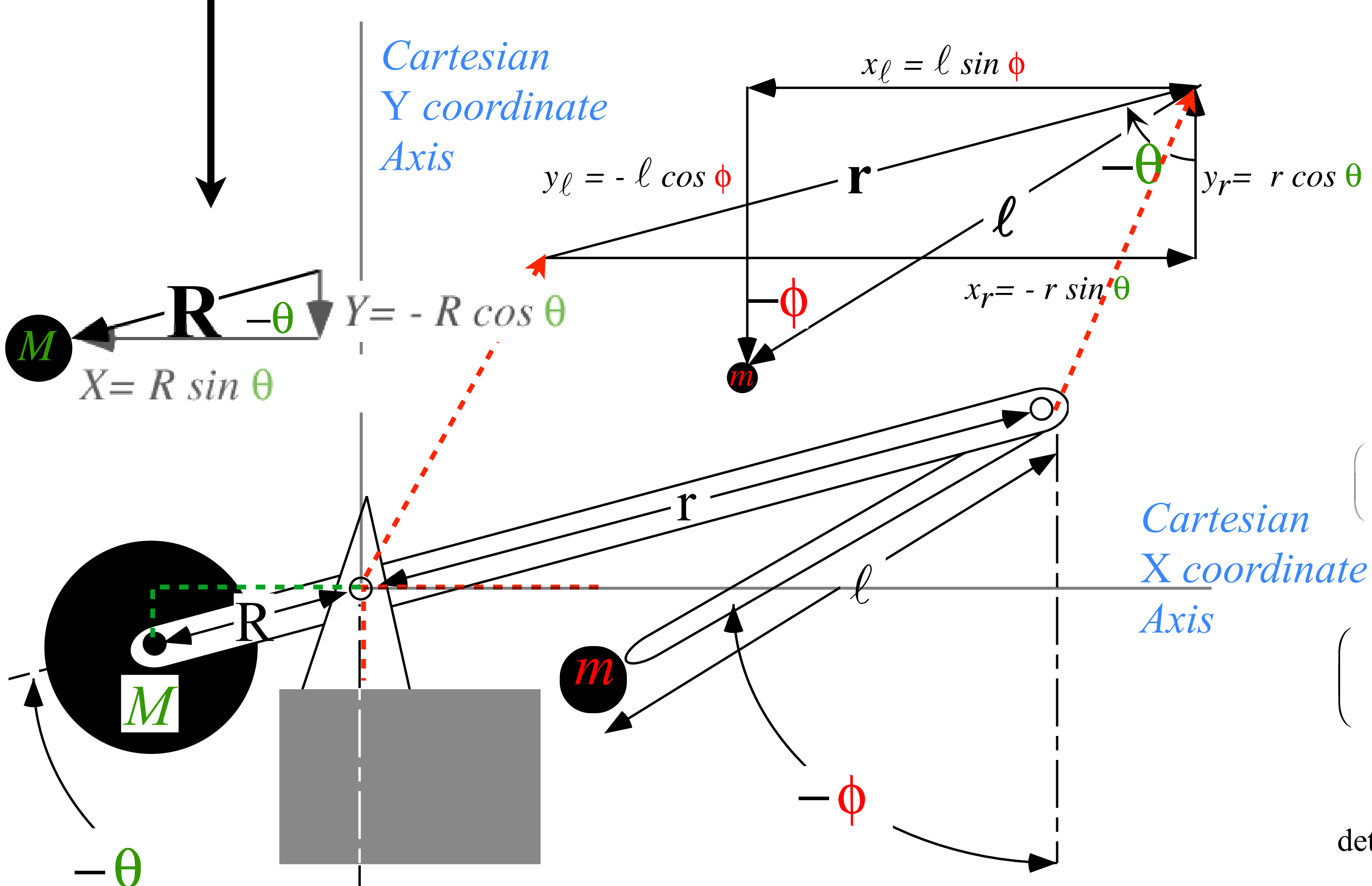
$$dY = \frac{\partial Y}{\partial\theta} d\theta + \frac{\partial Y}{\partial\phi} d\phi, \qquad dy = \frac{\partial y}{\partial\theta} d\theta + \frac{\partial y}{\partial\phi} d\phi.$$

$$dX = R\cos\theta \; d\theta + 0, \qquad dx = -r\cos\theta \; d\theta + \ell\cos\phi \; d\phi$$

$$dY = R\sin\theta \; d\theta + 0, \qquad dy = -r\sin\theta \; d\theta + \ell\sin\phi \; d\phi$$

Constraint relations:

$$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$$
$$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$$
$$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$$



Based on Fig. 2.2.1

'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \quad \text{FAILS since: det} \begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

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$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta - \phi)$$

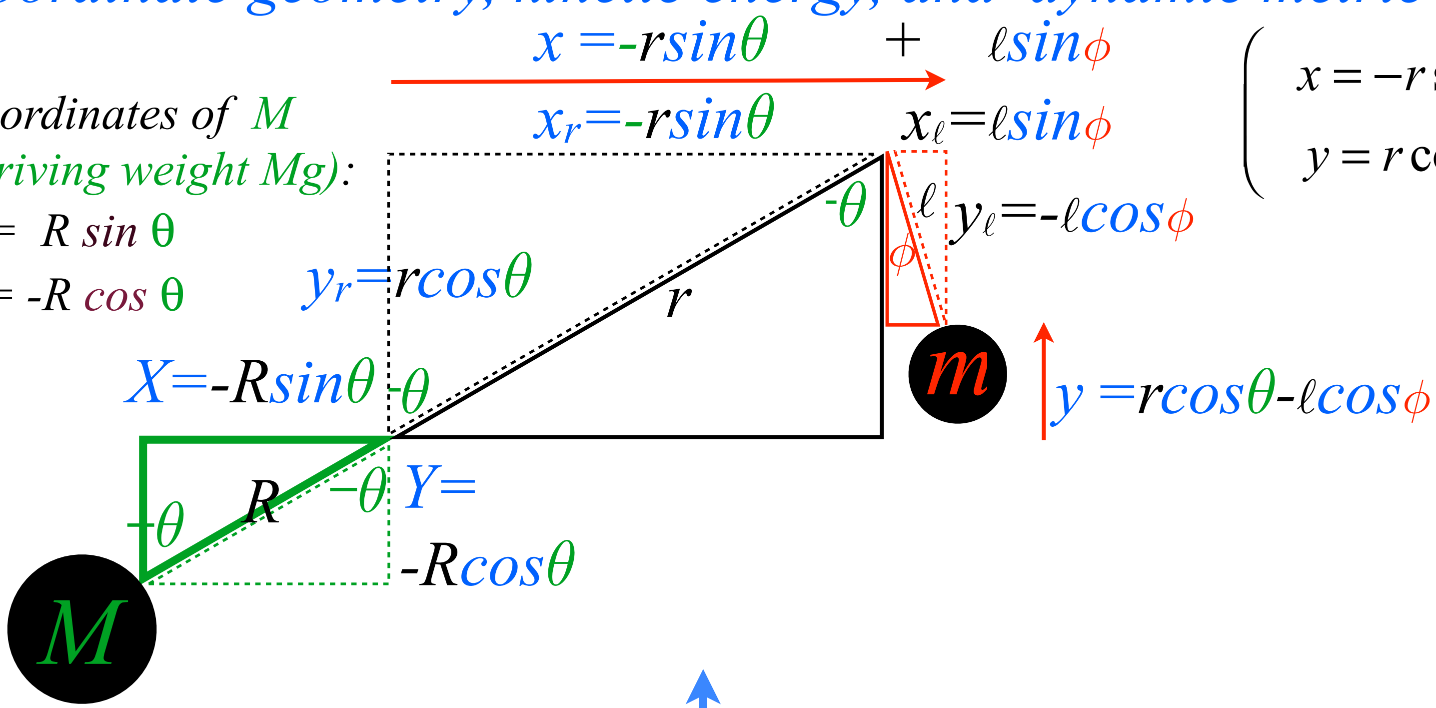
SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M (Driving weight Mg):

$X = R\sin\theta$

$Y = -R\cos\theta$



geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\; d\theta + 0, \qquad dx = -r\cos\theta\; d\theta + \ell\cos\phi\; d\phi$$

$$dY = R\sin\theta\; d\theta + 0, \qquad dy = -r\sin\theta\; d\theta + \ell\sin\phi\; d\phi$$

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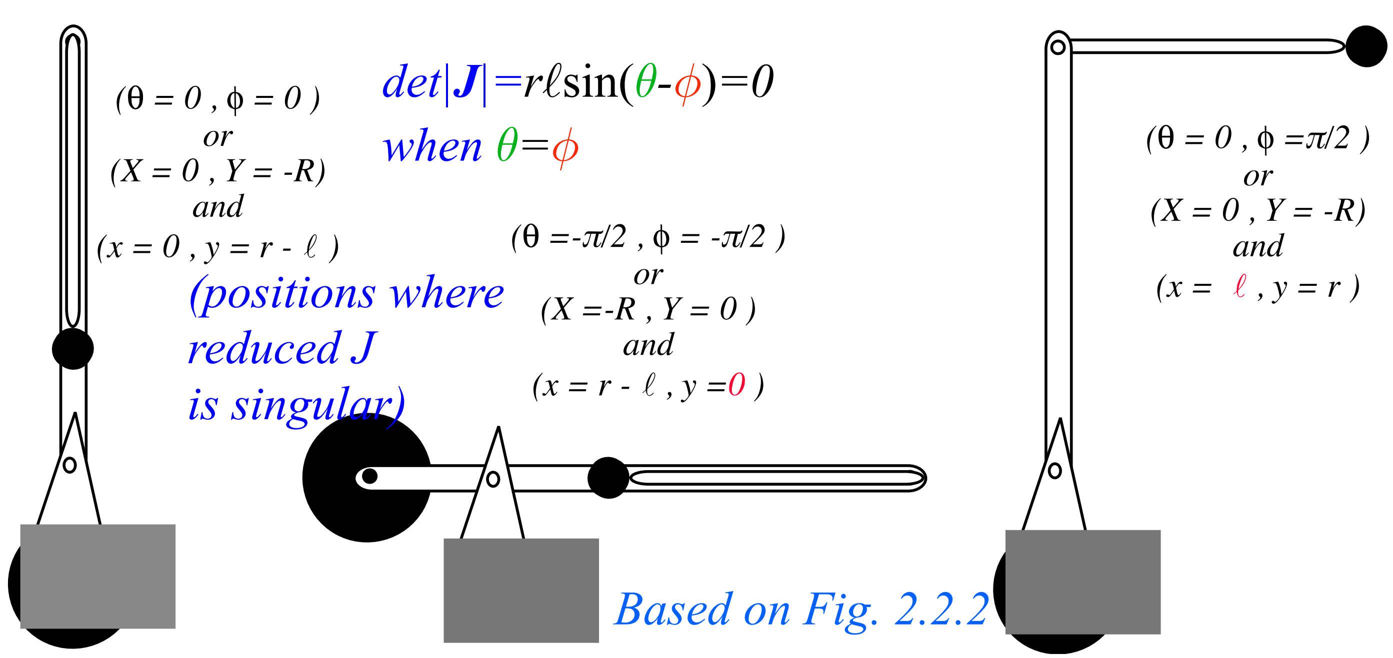
'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Fig. 2.2.2 Singular positions of the trebuchet



Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: det} \begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

Jacobian **J**-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

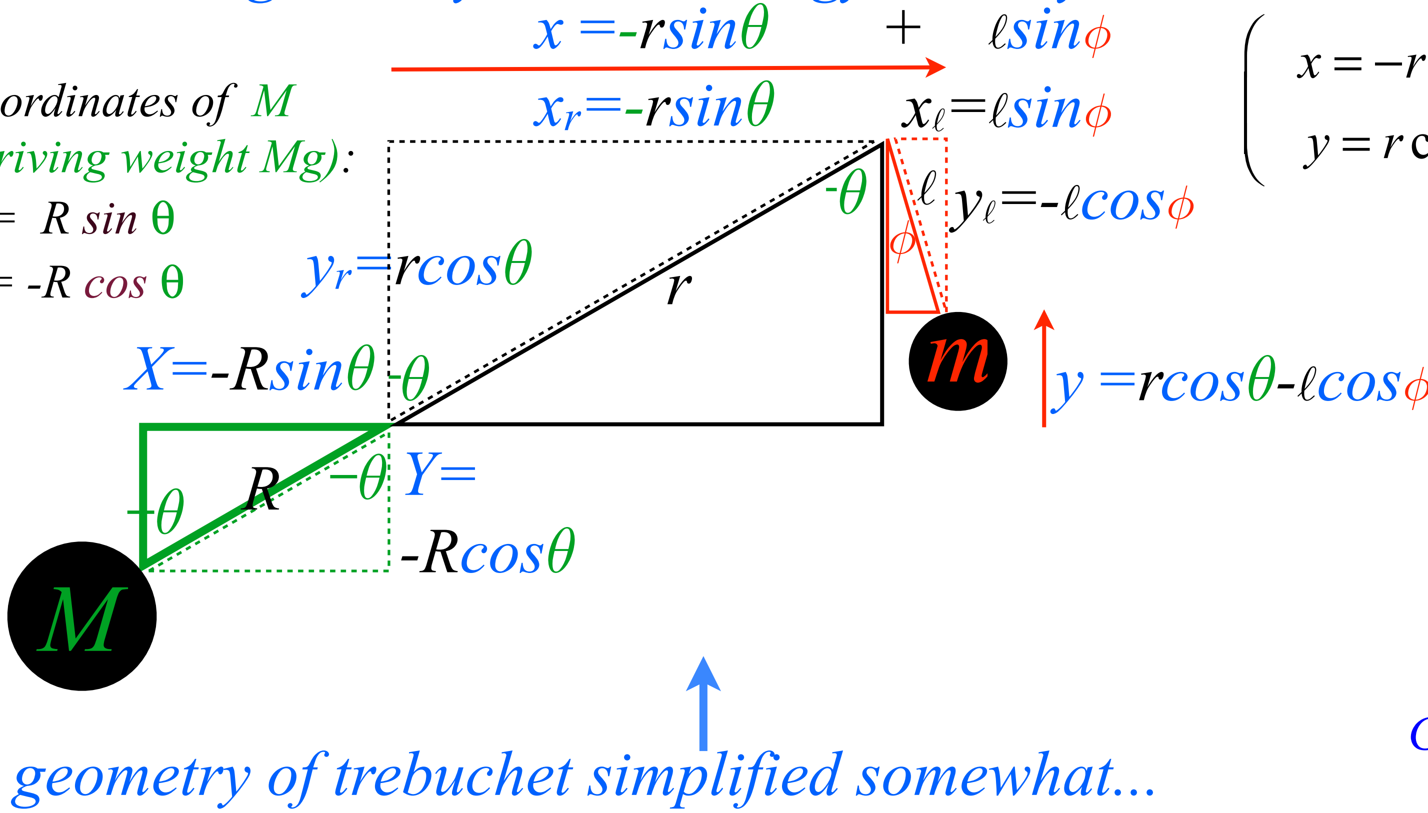
SUCCESS! (Usually non-singular)

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):
$X = R \sin\theta$
$Y = -R\cos\theta$



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
(Payload or projectile):
$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$
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1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \qquad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

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$$dX = R\cos\theta \; d\theta + 0, \qquad dx = -r\cos\theta \; d\theta + \ell\cos\phi \; d\phi$$

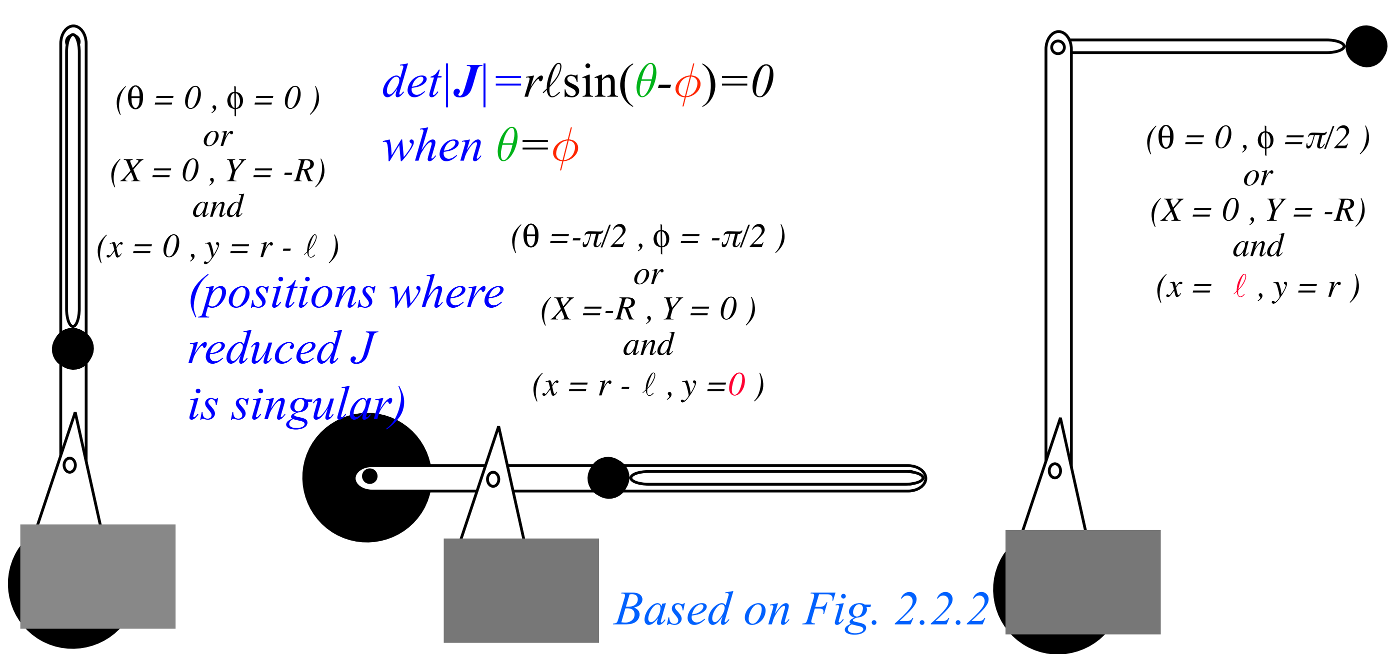
$$dY = R\sin\theta \; d\theta + 0, \qquad dy = -r\sin\theta \; d\theta + \ell\sin\phi \; d\phi$$

GCC Velocity relations:

$$\dot{X} = R\cos\theta \;\; \dot{\theta} + 0, \qquad \dot{x} = -r\cos\theta \;\; \dot{\theta} + \ell\cos\phi \;\; \dot{\phi}$$

$$\dot{Y} = R\sin\theta \;\; \dot{\theta} + 0, \qquad \dot{y} = -r\sin\theta \;\; \dot{\theta} + \ell\sin\phi \;\; \dot{\phi}$$

Fig. 2.2.2 Singular positions of the trebuchet



Jacobian
J-matrix

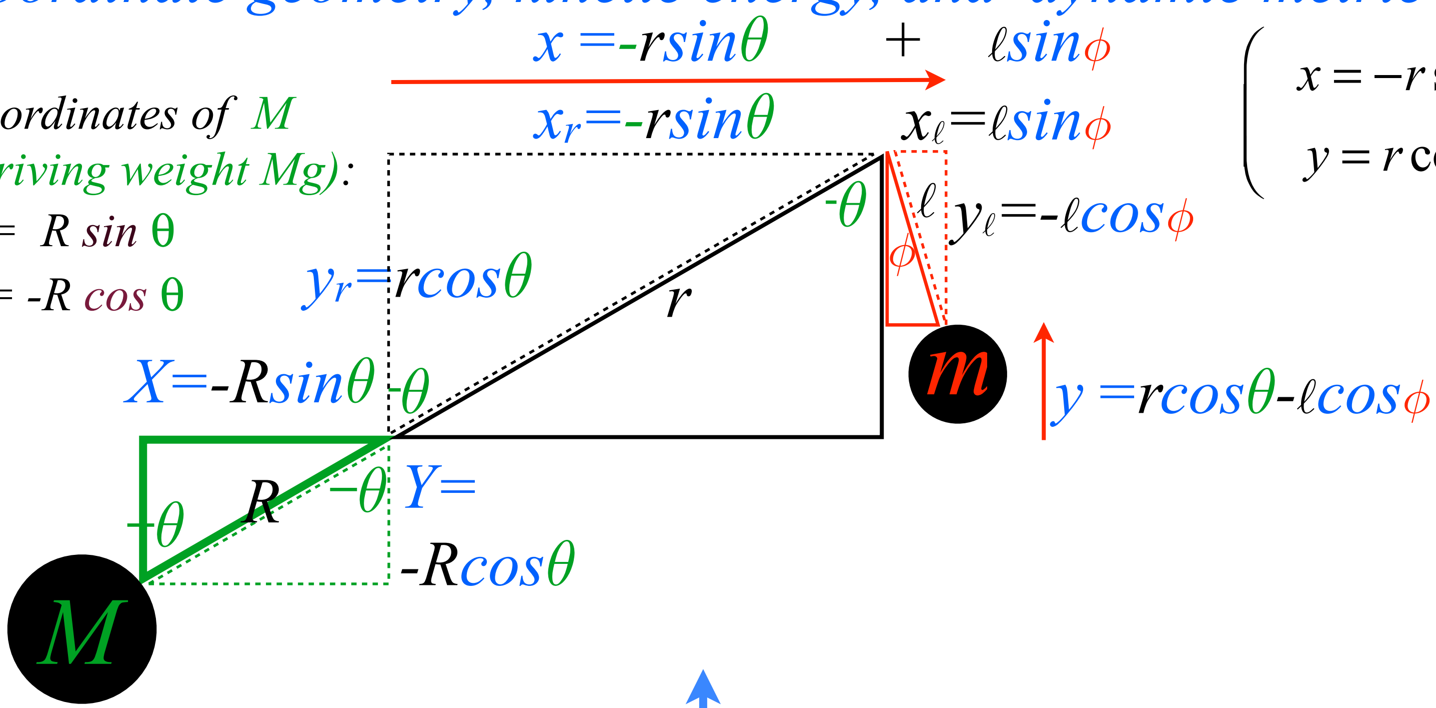
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{matrix}} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det \begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta - \phi)$$

SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M
(Driving weight Mg):
$X = R \sin\theta$
$Y = -R\cos\theta$



geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
(Payload or projectile):
$x = x_r + x_\ell = -\, r\sin\theta + \ell\sin\phi$
$y = y_r + y_\ell = \;\; r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

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$$dX = R\cos\theta\; d\theta + 0, \qquad dx = -r\cos\theta\; d\theta + \ell\cos\phi\; d\phi$$

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GCC Velocity relations:

$$\dot{X} = R\cos\theta\;\; \dot\theta + 0, \qquad \dot{x} = -r\cos\theta\;\; \dot\theta + \ell\cos\phi\;\; \dot\phi$$

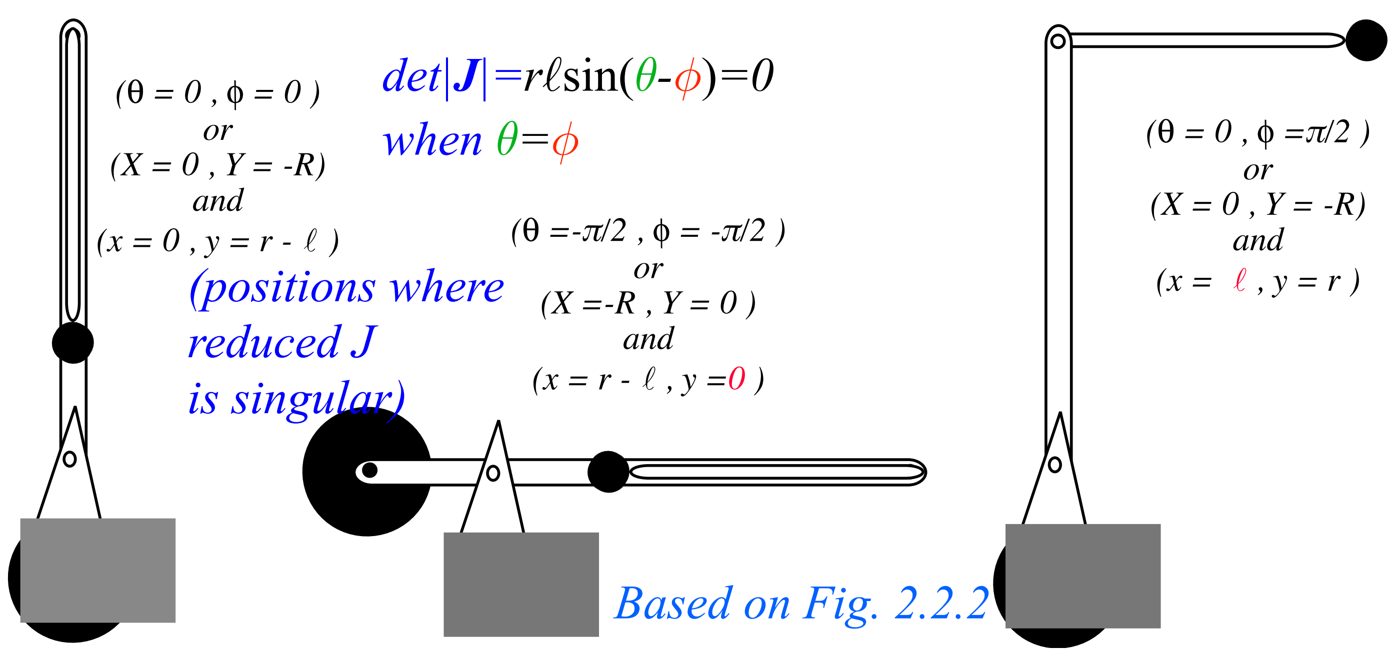
$$\dot{Y} = R\sin\theta\;\; \dot\theta + 0, \qquad \dot{y} = -r\sin\theta\;\; \dot\theta + \ell\sin\phi\;\; \dot\phi$$

Jacobian **J**-matrix velocity relations:

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{\textbf{J}\text{-matrix}} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Fig. 2.2.2 Singular positions of the trebuchet

$\det|\textbf{J}| = r\ell\sin(\theta-\phi) = 0$
when $\theta = \phi$



Based on Fig. 2.2.2

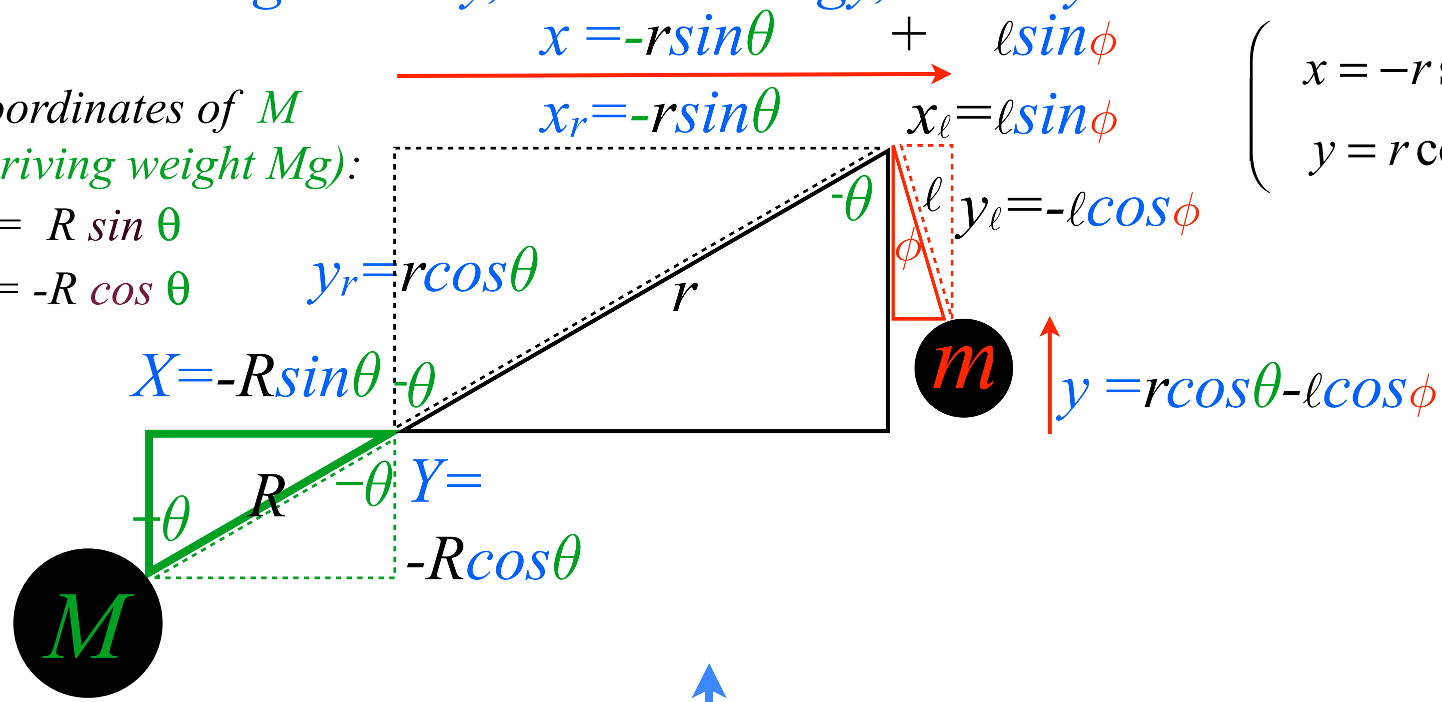
Jacobian **J**-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{matrix}} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass m
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$$dX = R\cos\theta \; d\theta + 0, \qquad dx = -r\cos\theta \; d\theta + \ell\cos\phi \; d\phi$$

$$dY = R\sin\theta \; d\theta + 0, \qquad dy = -r\sin\theta \; d\theta + \ell\sin\phi \; d\phi$$

GCC Velocity relations:

$$\dot{X} = R\cos\theta \;\; \dot{\theta} + 0, \qquad \dot{x} = -r\cos\theta \;\; \dot{\theta} + \ell\cos\phi \;\; \dot{\phi}$$

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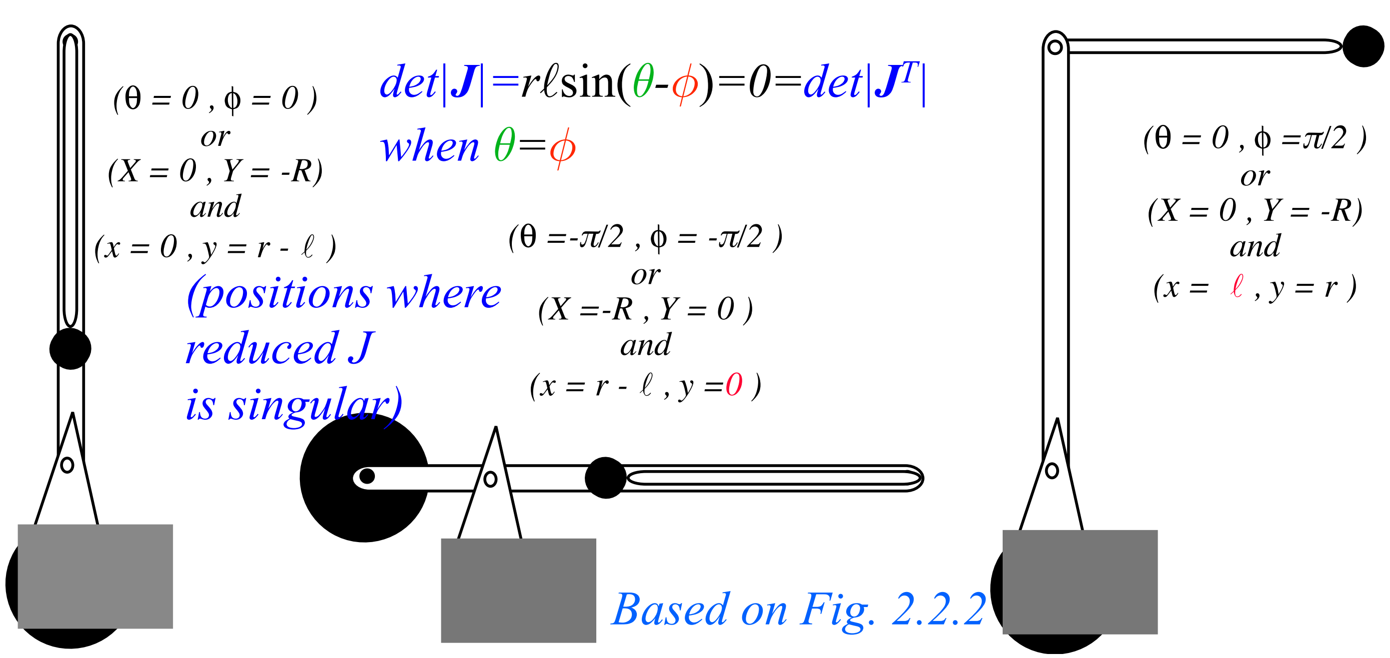
*Jacobian **J**-matrix velocity relations:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{\boldsymbol{J}\text{-matrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

*Transpose Jacobian **J**^{T}-matrix velocity relations:*

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \overbrace{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}^{\boldsymbol{J}^T\text{-matrix}}$$

Fig. 2.2.2 Singular positions of the trebuchet



*Jacobian **J**-matrix*

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{matrix}} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta - \phi)$$

SUCCESS! (Usually non-singular)

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
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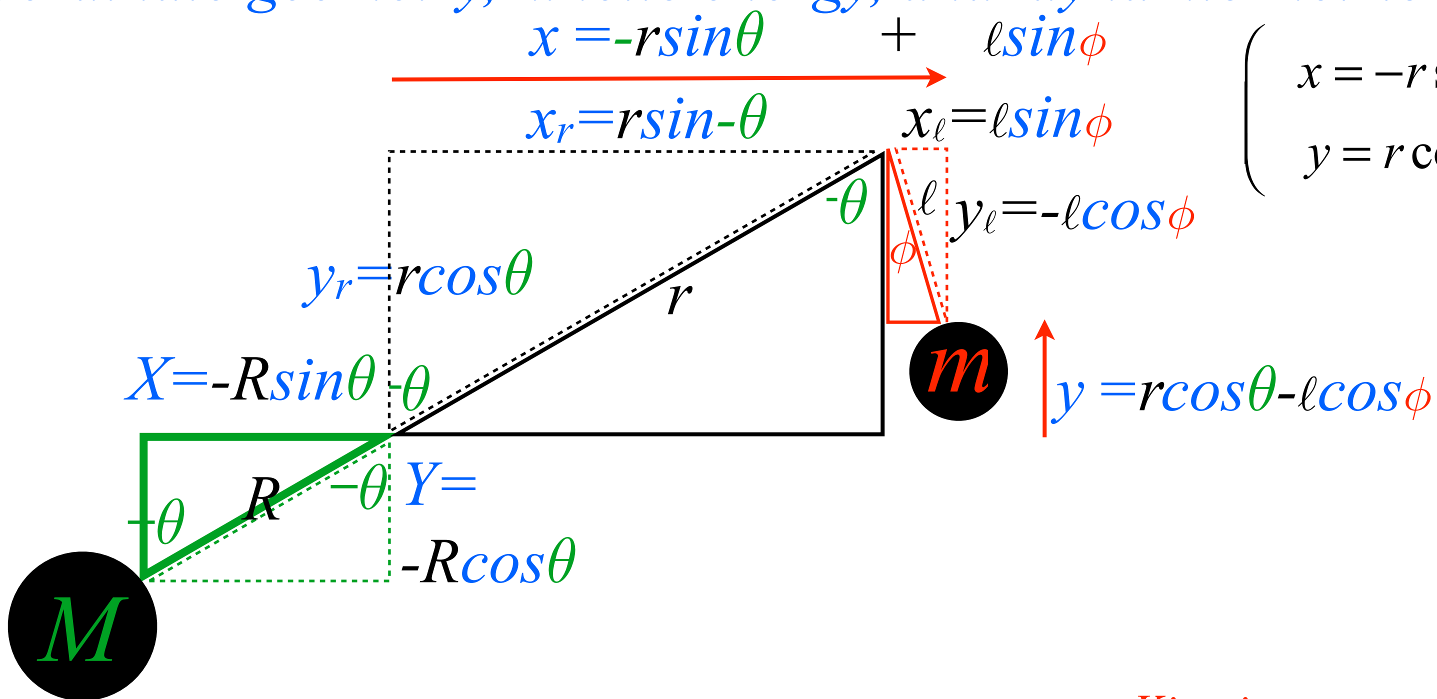
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} Velocity, Jacobian, and KE



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

J-matrix

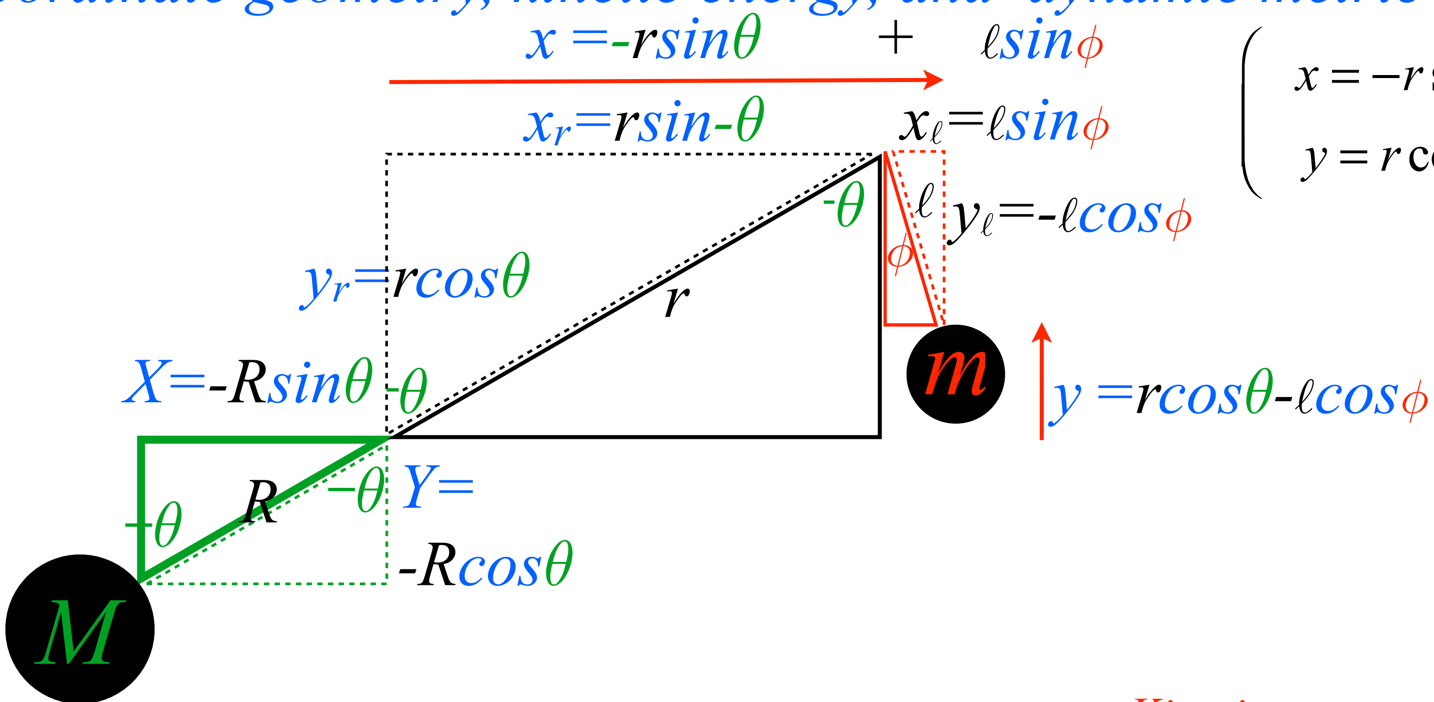
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

Velocity, Jacobian, and KE

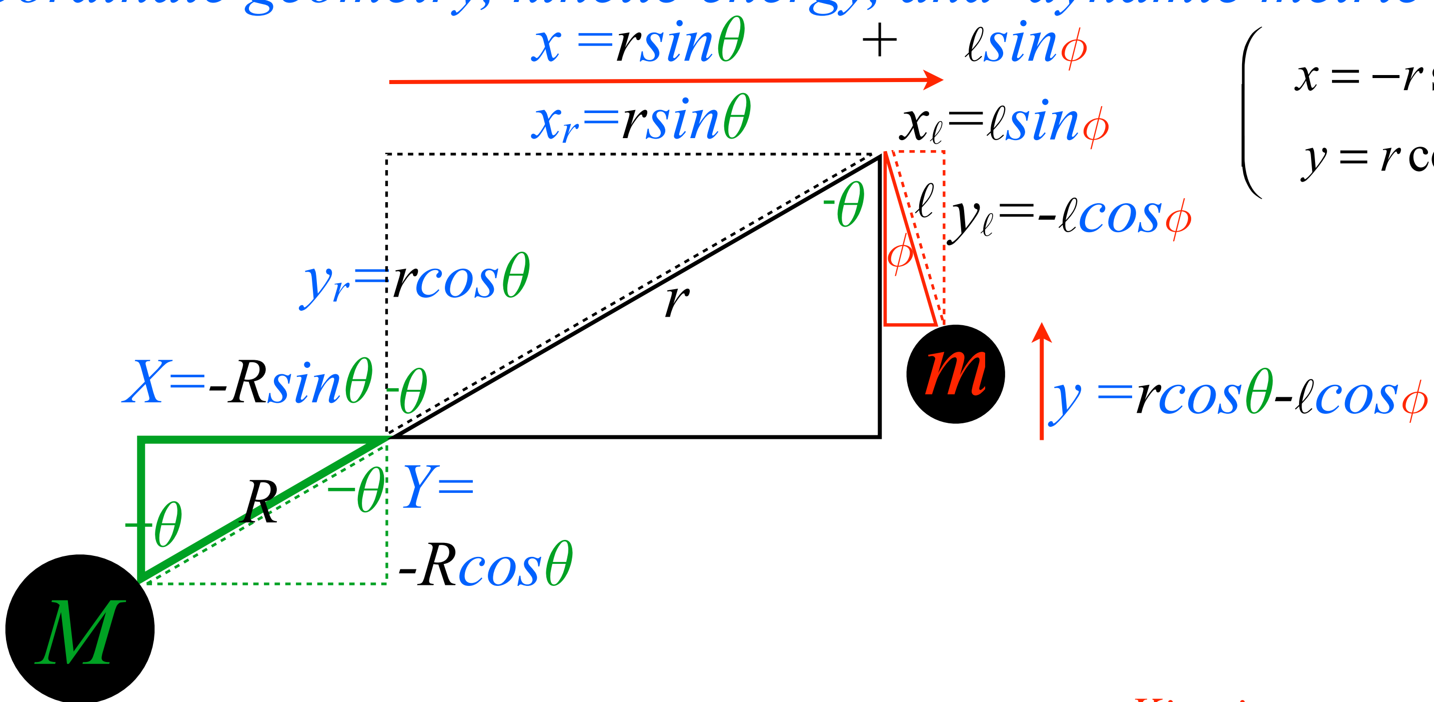
$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\overset{J\text{-matrix}}{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}=\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\overset{J^T\text{-matrix}}{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}$$

Kinetic energy of projectile m $T(m)=\frac{1}{2}\left[m\dot{x}^2+m\dot{y}^2\right]=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \overset{\textbf{J-matrix}}{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \overset{\textbf{J}^T\textbf{-matrix}}{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}$$

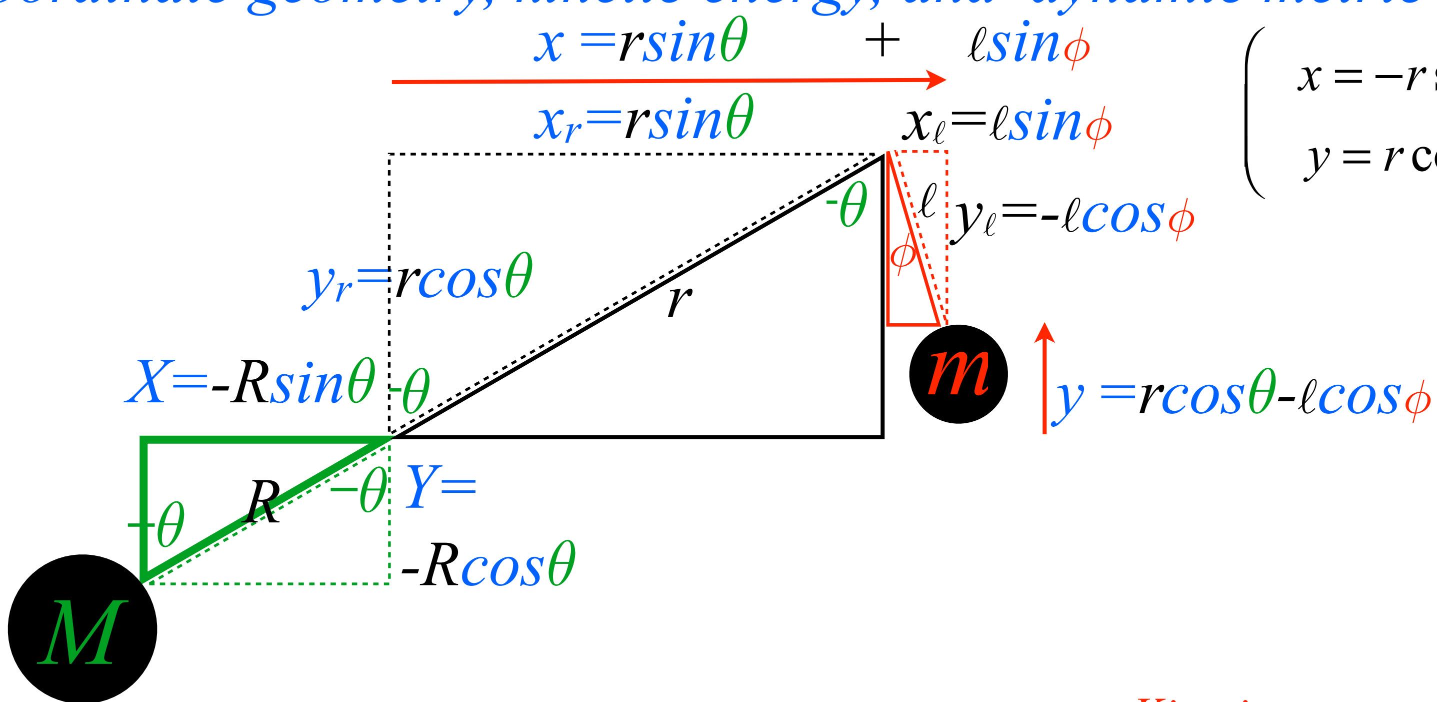
Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \overset{\textbf{J}^T\textbf{-matrix}}{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}} \overset{\textbf{J-matrix}}{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn} — Velocity, Jacobian, and KE



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Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot\theta^2$$

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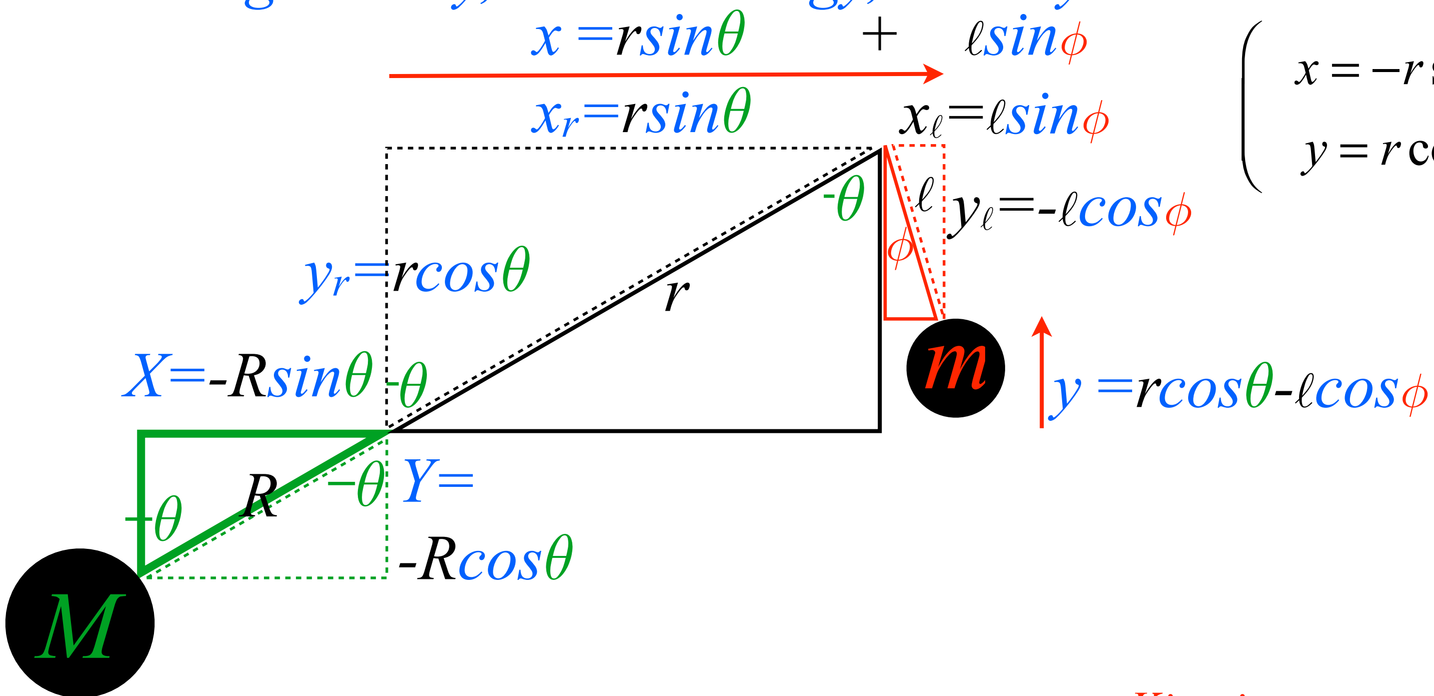
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

'Raw' Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

$$= \frac{1}{2} MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}_{J^T\text{-matrix}} \underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{J\text{-matrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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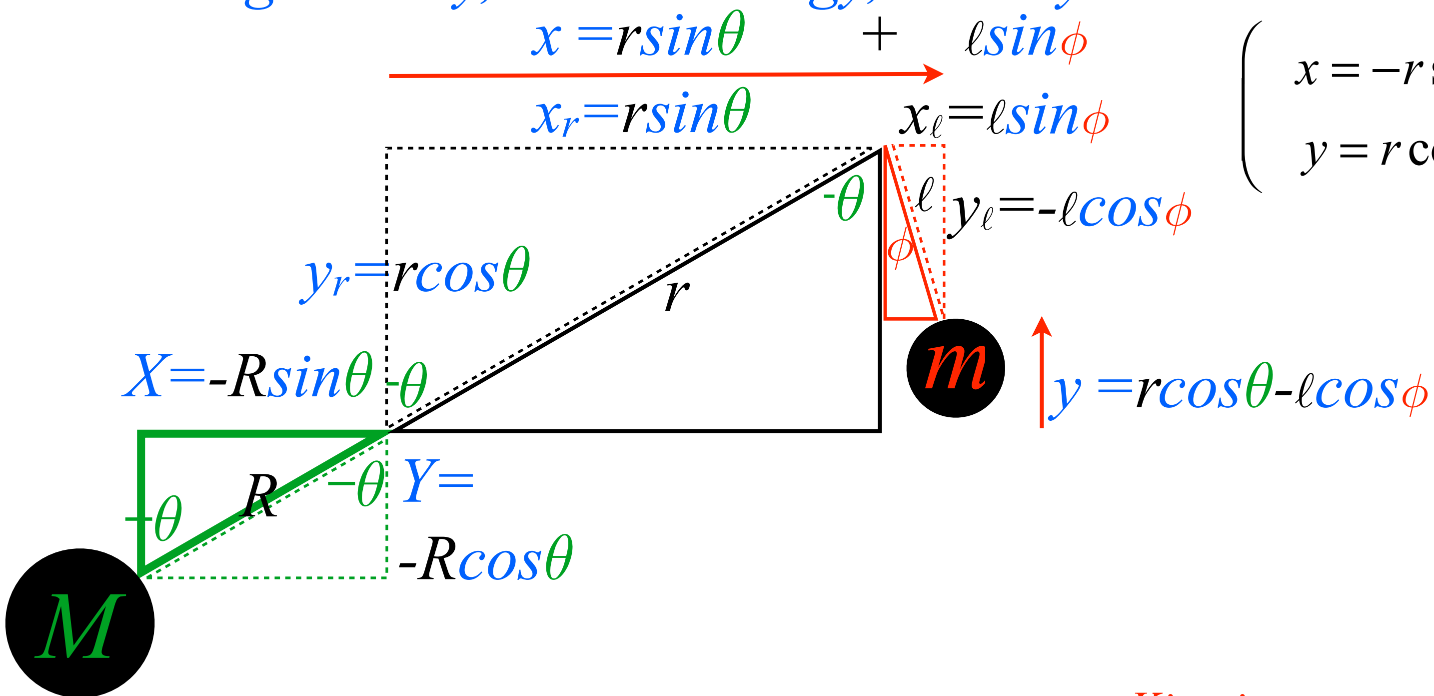
$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

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Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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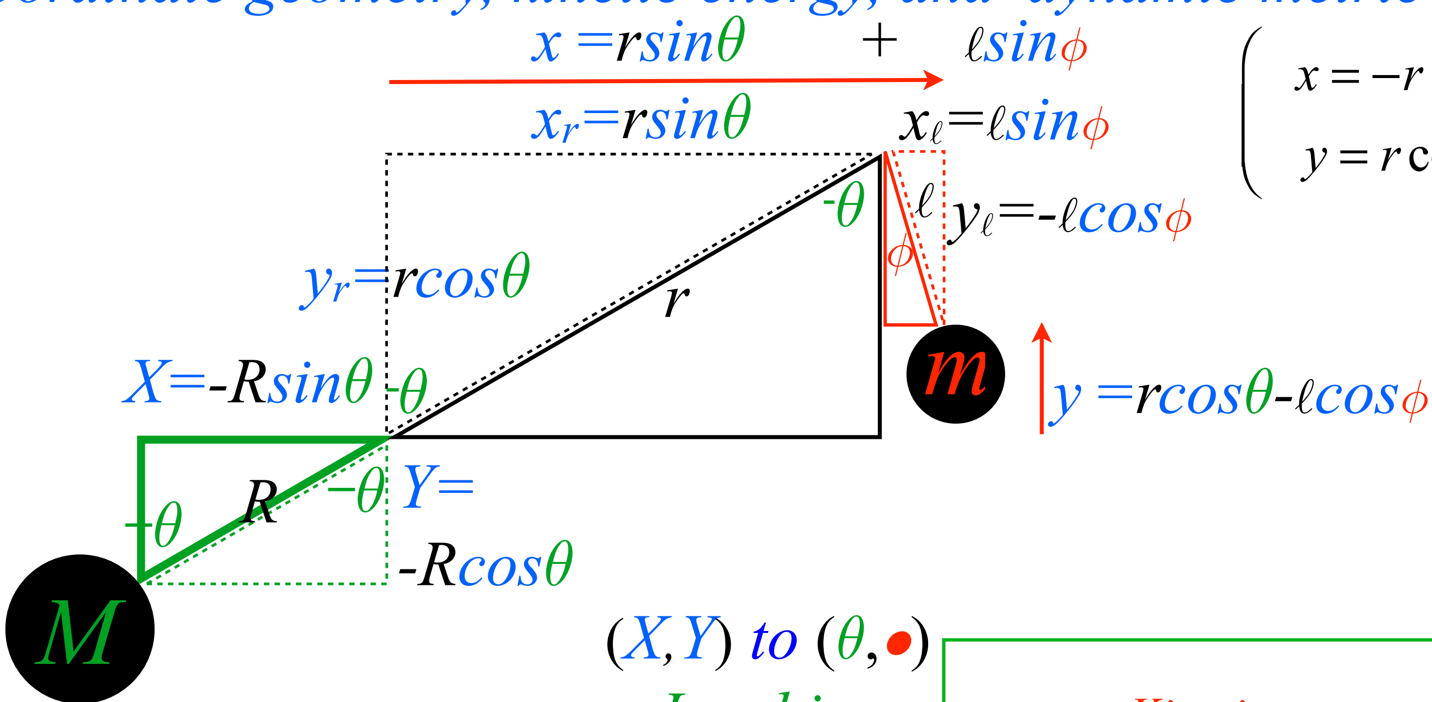
$$\textit{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix} \quad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX\\ dY\\ dx\\ dy\end{pmatrix}=\begin{pmatrix}\frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi}\\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi}\\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}d\theta\\ d\phi\end{pmatrix}=\begin{pmatrix}R\cos\theta & 0\\ R\sin\theta & 0\\ -r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}d\theta\\ d\phi\end{pmatrix}$$

(X,Y) to $(\theta,\bullet)$ Jacobian

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot\theta^2$$

Kinetic energy of projectile m $T(m)=\frac{1}{2}\left[m\dot{x}^2+m\dot{y}^2\right]$

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

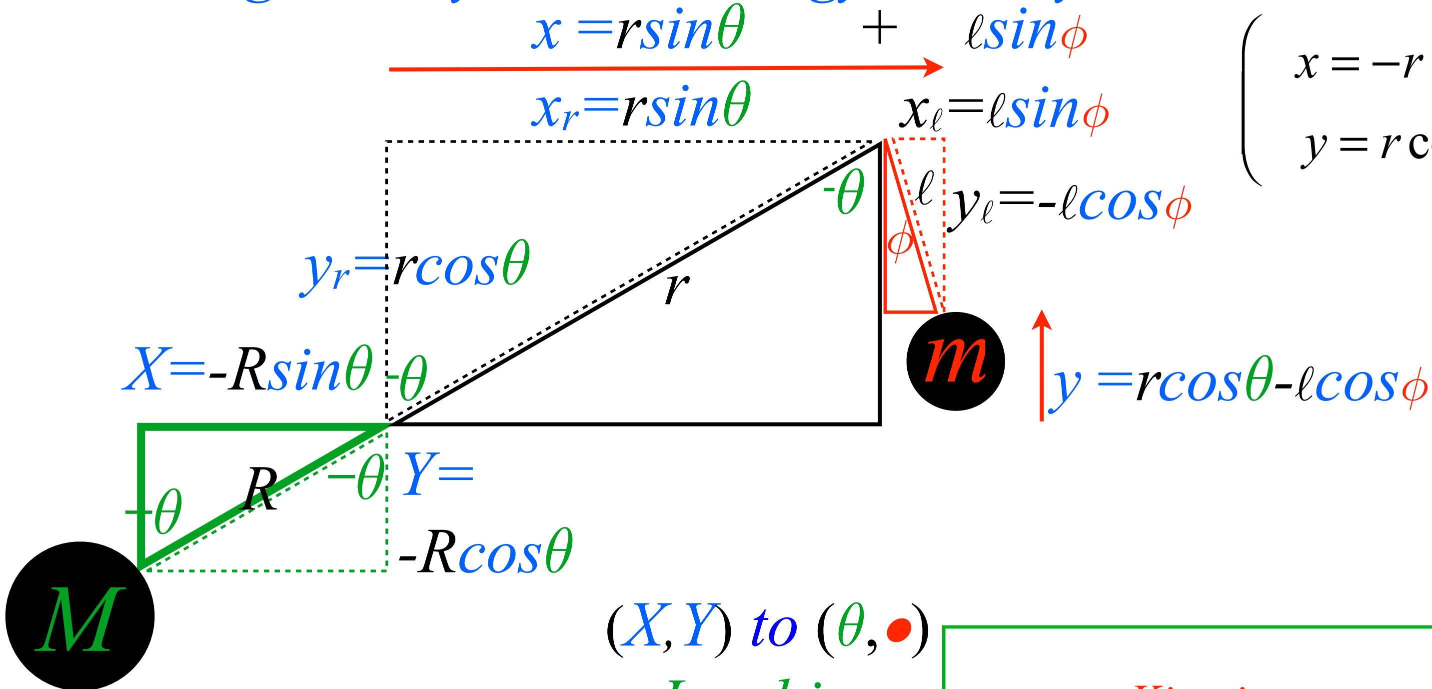
$$\text{Total KE}=T=\frac{1}{2}\left[(MR^2+mr^2)\dot\theta^2-2mr\ell\cos(\theta-\phi)\,\dot\theta\dot\phi+m\ell^2\dot\phi^2\right]$$

$$T=\frac{1}{2}\left[M\dot{X}^2+M\dot{Y}^2+m\dot{x}^2+m\dot{y}^2\right]$$

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}*
- *Structure of dynamic metric tensor γ_{mn}*
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and γ_{mn} tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(X,Y) to $(\theta,\bullet)$ Jacobian

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $\quad T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

J^T-matrix $\qquad$ J-matrix

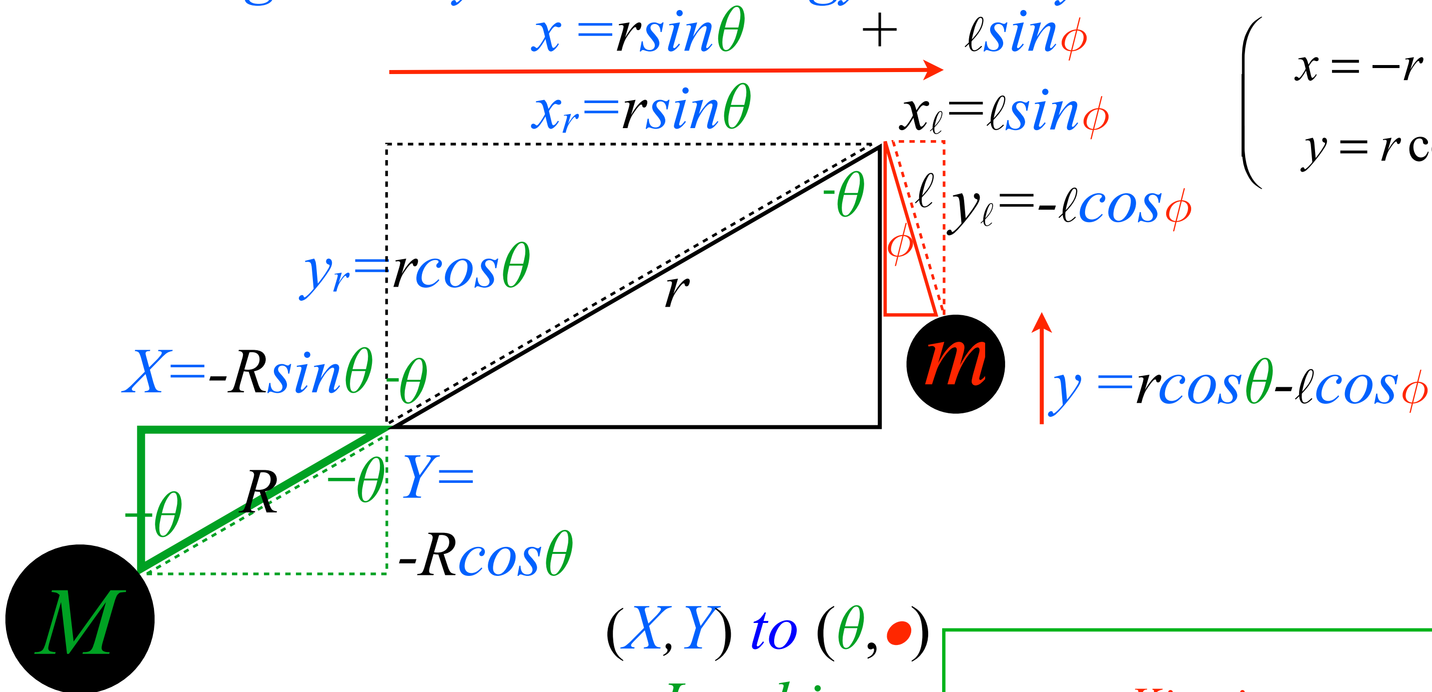
$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix} \qquad \text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix}$$

Velocity, Jacobian, and KE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

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(X,Y) to $(\theta,\bullet)$ Jacobian

(x,y) to (θ,ϕ) Jacobian

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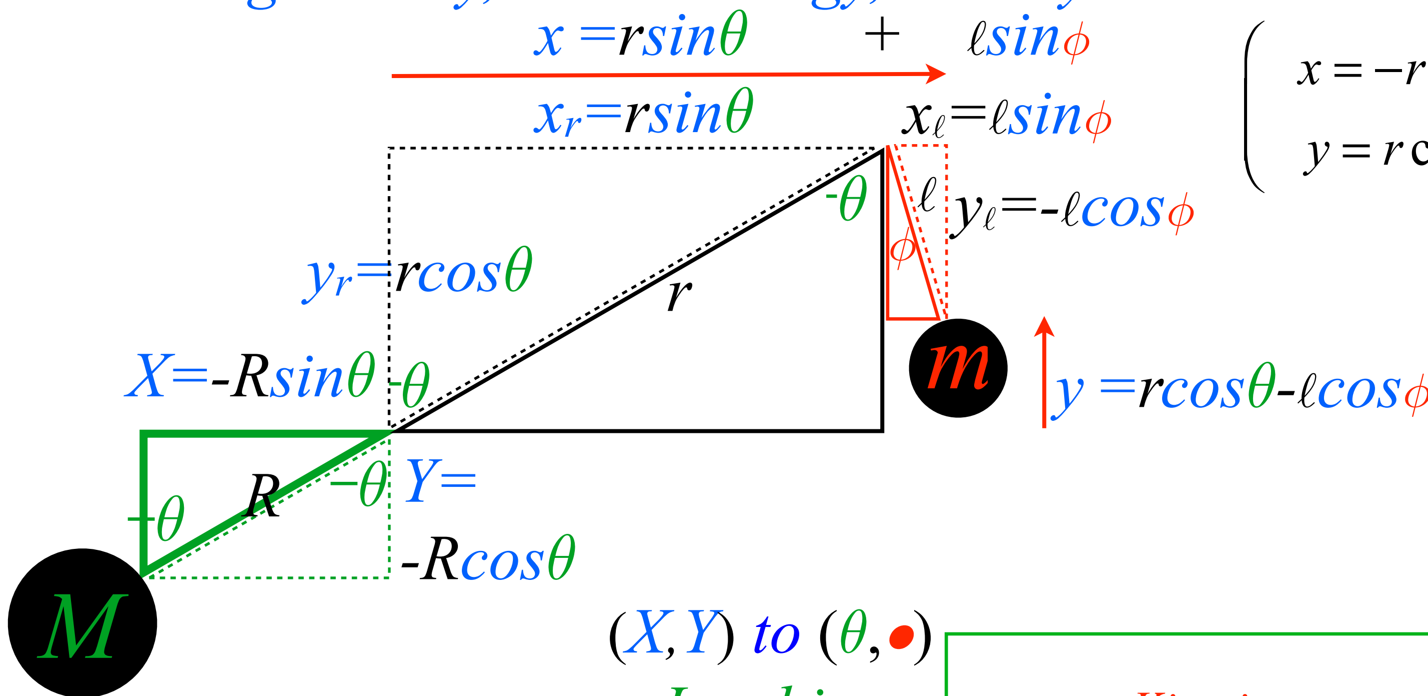
$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix}=\begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



(X,Y) to $(\theta,\bullet)$ Jacobian

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Raw Jacobian

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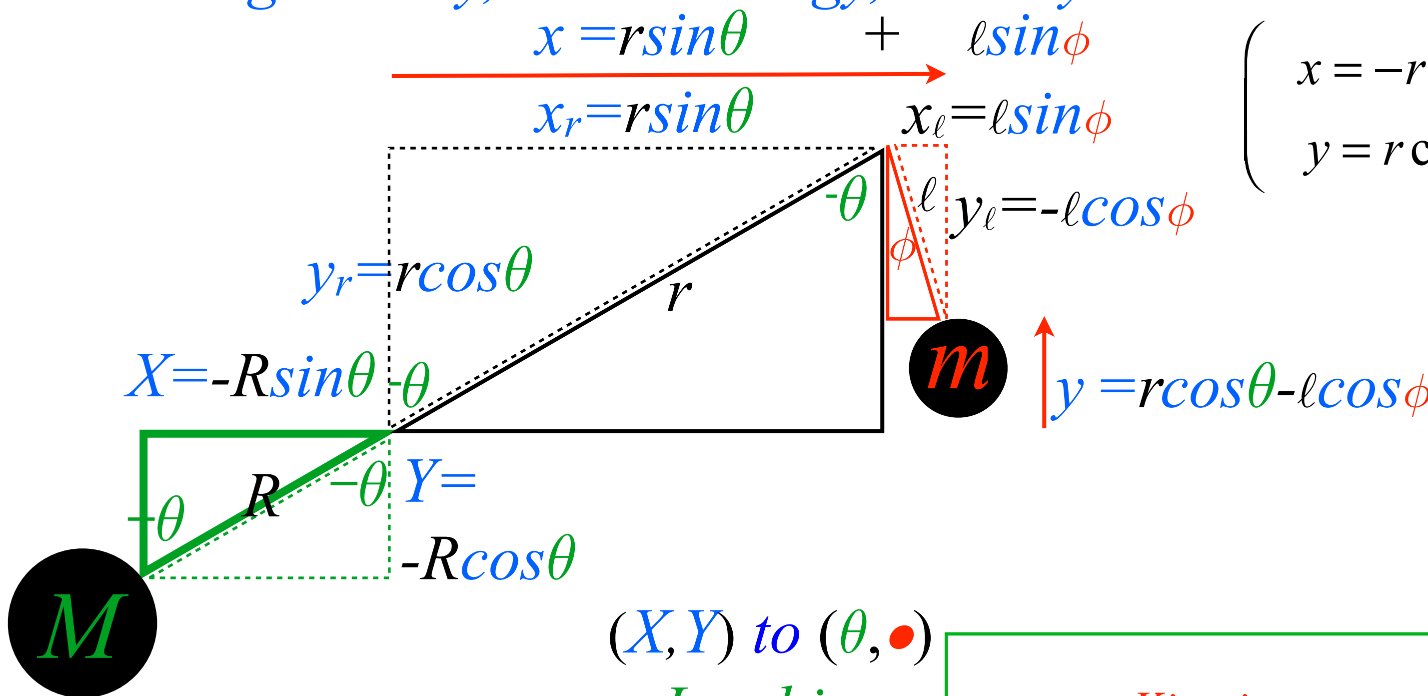
$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

$$T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right] = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



Velocity, Jacobian, and KE

$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(X,Y) to $(\theta,\bullet)$ Jacobian

(x,y) to (θ,ϕ) Jacobian

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot\theta^2$$

Kinetic energy of projectile m $\quad T(m)=\frac{1}{2}\left[m\dot{x}^2+m\dot{y}^2\right]$

$$T(m)=\frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\frac{1}{2}m\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\underbrace{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}_{J^T\text{-matrix}}\underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{J\text{-matrix}}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\text{Total KE}=T=\frac{1}{2}\left[M\dot{X}^2+M\dot{Y}^2+m\dot{x}^2+m\dot{y}^2\right]$$

$$T=\frac{1}{2}\left[(MR^2+mr^2)\dot\theta^2-2mr\ell\cos(\theta-\phi)\,\dot\theta\dot\phi+m\ell^2\dot\phi^2\right]=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor γ_{mn}
in raw Cartesian X,Y and x,y

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta\text{-}\phi) \\ -mr\ell\cos(\theta\text{-}\phi) & m\ell^2 \end{pmatrix} = \begin{pmatrix} MR^2 & 0 \\ 0 & 0 \end{pmatrix} + m\underbrace{\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}}_{J^T\text{-matrix}}\underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{J\text{-matrix}}$$

Dynamic metric tensor γ_{mn} in GCC θ and ϕ

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* γ_{mn}
- *Structure of dynamic metric tensor* γ_{mn}
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* γ_{mn} *tensor*

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

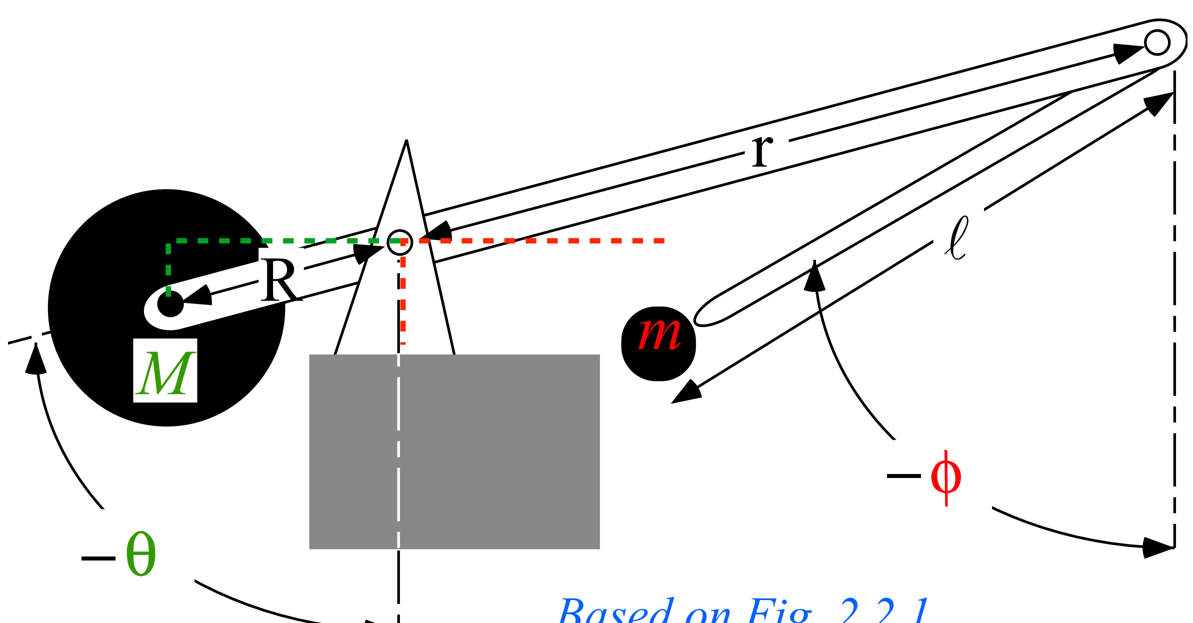
$$= \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} r^2\cos^2\theta + r^2\sin^2\theta & -r\ell\cos\theta\cos\phi - r\ell\sin\theta\sin\phi \\ -\ell r\cos\phi\cos\theta - r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi + \ell^2\sin^2\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total } KE = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}$$



Based on Fig. 2.2.1

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total } KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

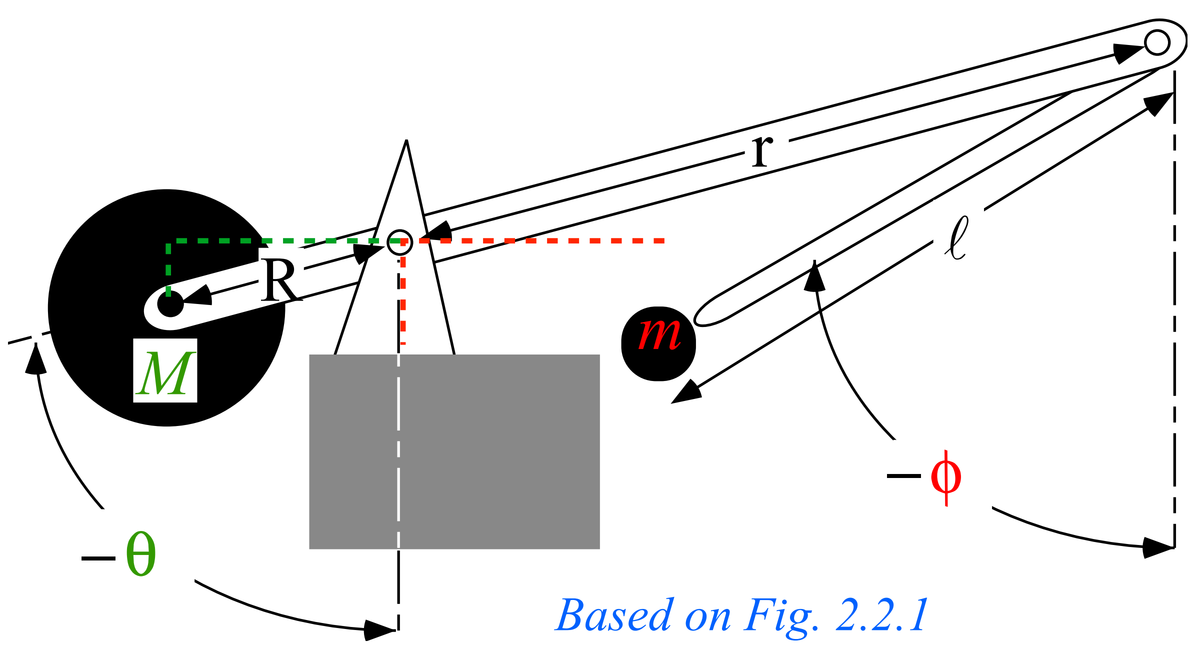
$$\gamma_{mn}=\sum_{mass\ \mu}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\frac{\partial\mathbf{r}(\mu)}{\partial q^m}\bullet\frac{\partial\mathbf{r}(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\mathbf{E_m}(\mu)\bullet\mathbf{E_n}(\mu)$$

$$KE=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\dot{x}^j(\mu)\dot{x}^j(\mu)=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}\dot{q}^m\dot{q}^n$$

$$=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$



Based on Fig. 2.2.1

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\textit{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

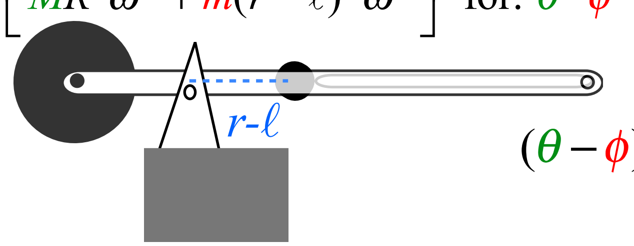
Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

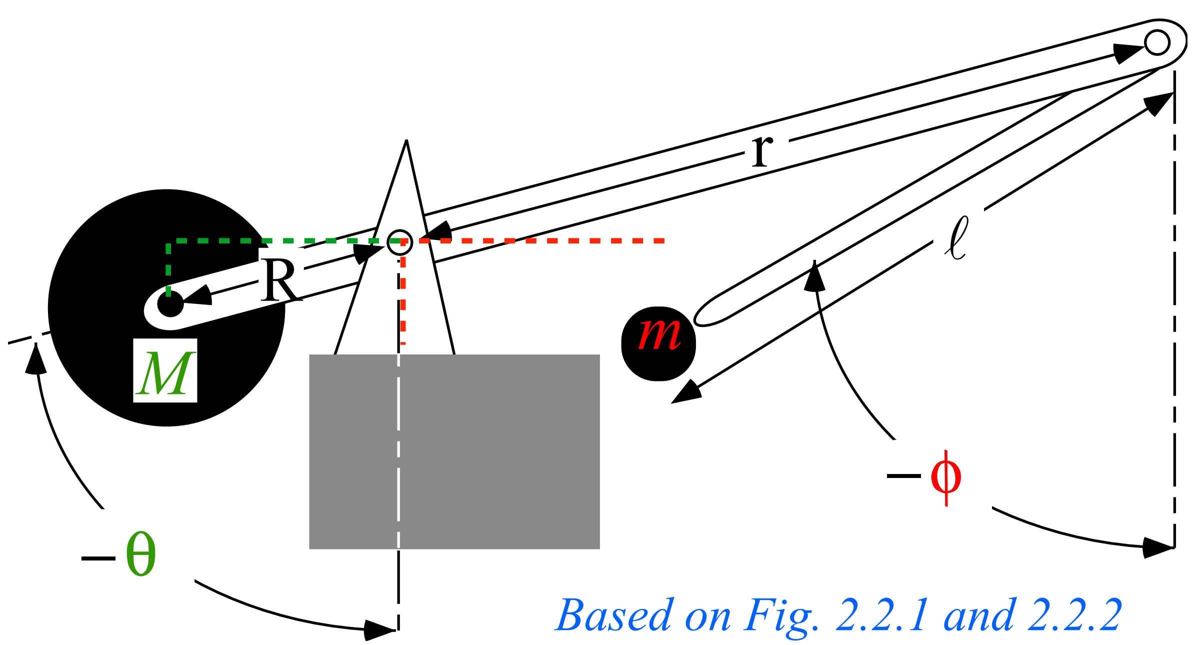
Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$

(J is Singular)



$(\theta-\phi)=0$



Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

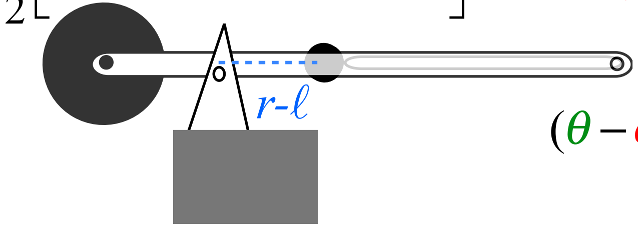
$$\textit{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

Special cases (rigid rotation)

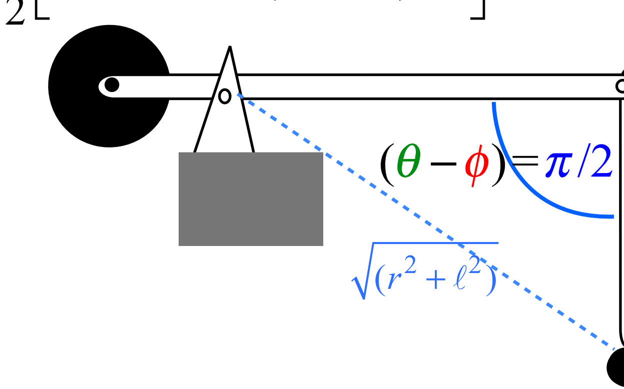
$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$



(J is Singular)

$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=\pi/2$$



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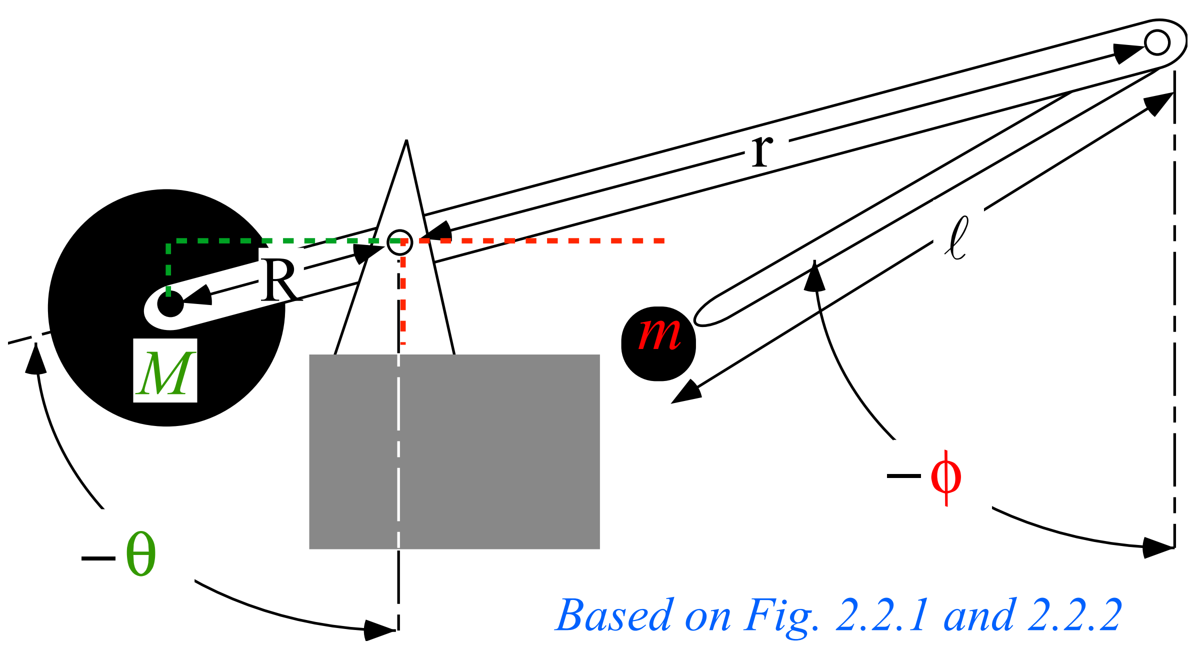

Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total }KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

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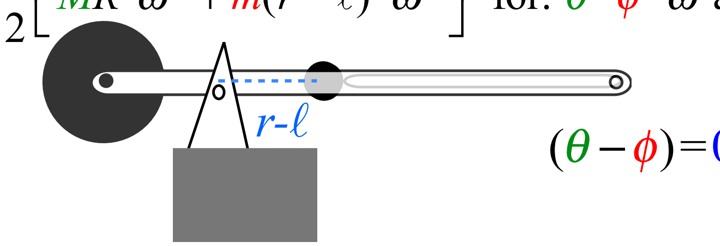
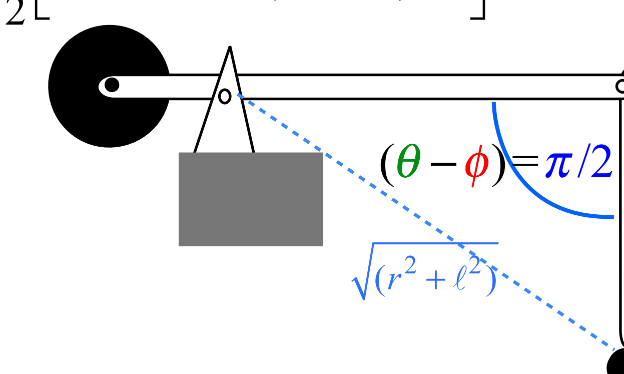


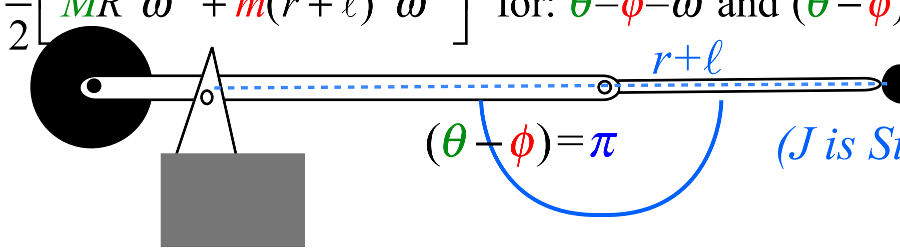
(J is Singular)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$



(J is Orthogonal)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$



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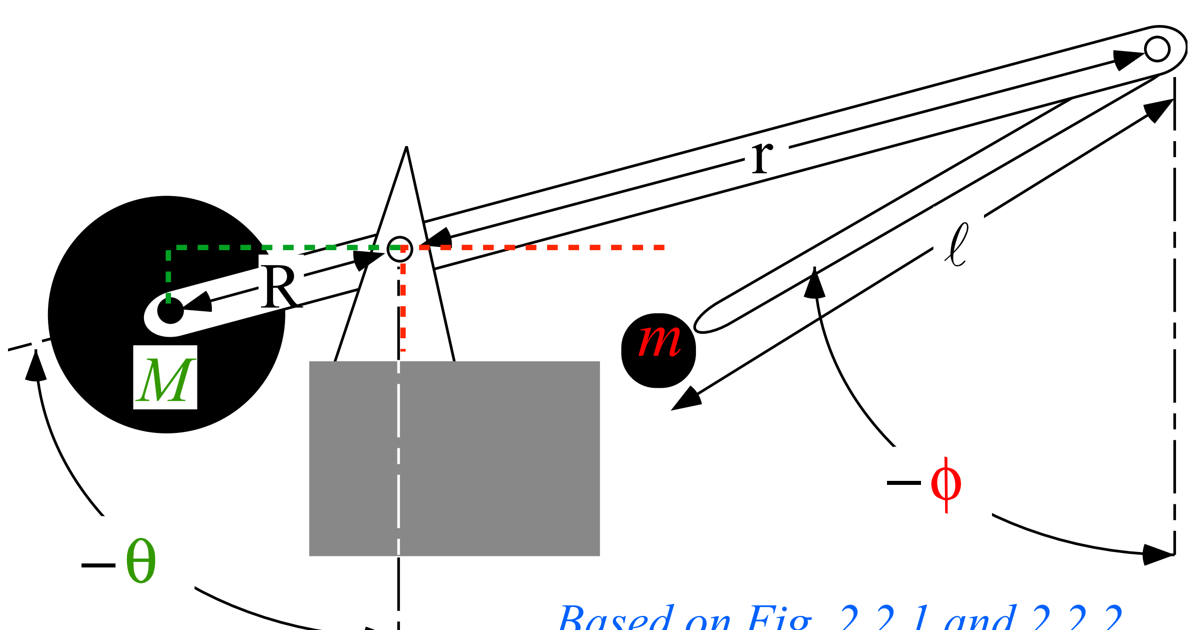

Based on Fig. 2.2.1 and 2.2.2

Kinetic energy of driver M

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$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

SPECIAL CASE

(a) When (θ,ϕ) coordinates are orthogonal

USUAL CASE

(b) When (θ,ϕ) coordinates are not orthogonal

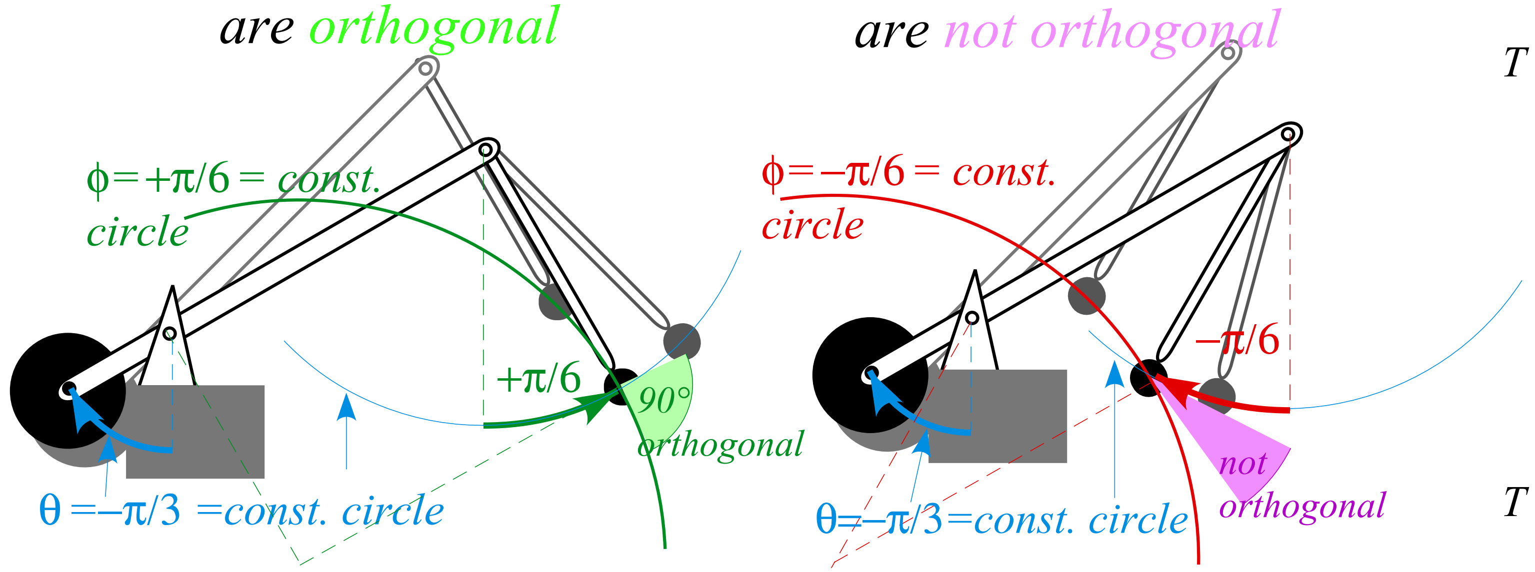
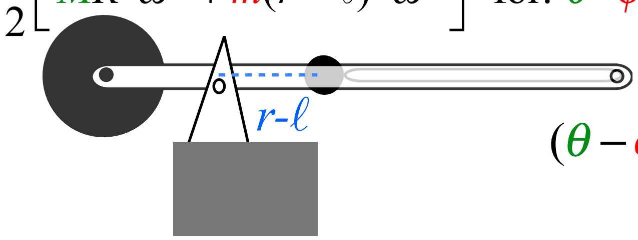


Fig. 2.3.1 Examples of (θ,ϕ) intersections (a) othogonal (special case), (b) non-orthogonal (typical).

Based on Fig. 2.3.1 and 2.2.2

Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$

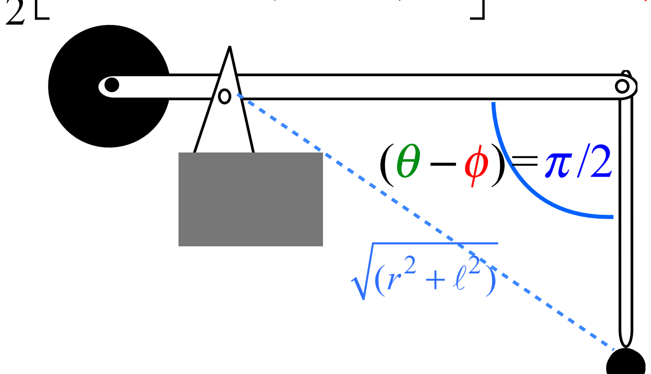


(J is Singular)

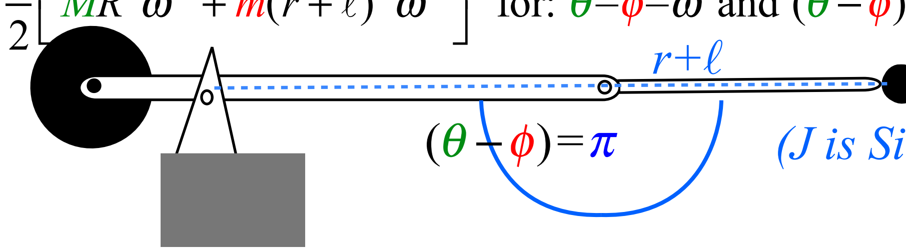
$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=\pi/2$$

(J is Orthogonal)



$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=\pi$$



(J is Singular)

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

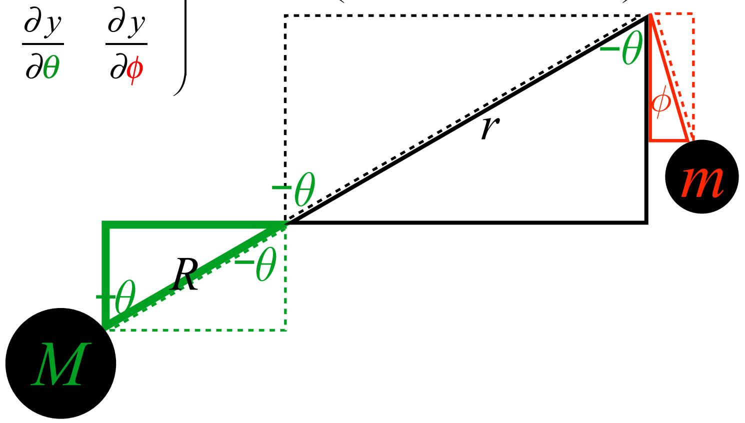
Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:

$$dW = \quad F_X\ dX \quad = \quad M\ddot{X}\,dX$$

$$+F_Y\ dY \qquad +M\ddot{Y}\,dY$$

$$+F_x\ dx \qquad +m\ddot{x}\,dx$$

$$+F_y\ dy \qquad +m\ddot{y}\,dy$$

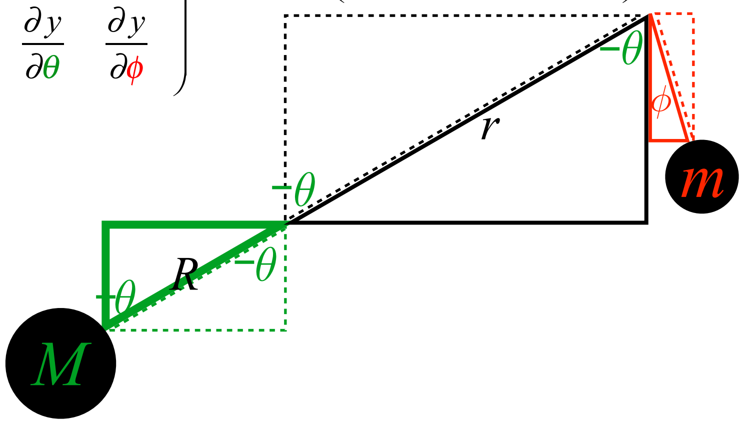
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \; & F_X\, dX &&= M\ddot{X}\,dX &&= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &&= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
& + F_Y\, dY && + M\ddot{Y}\,dY && + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
& + F_x\, dx && + m\ddot{x}\,dx && + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
& + F_y\, dy && + m\ddot{y}\,dy && + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

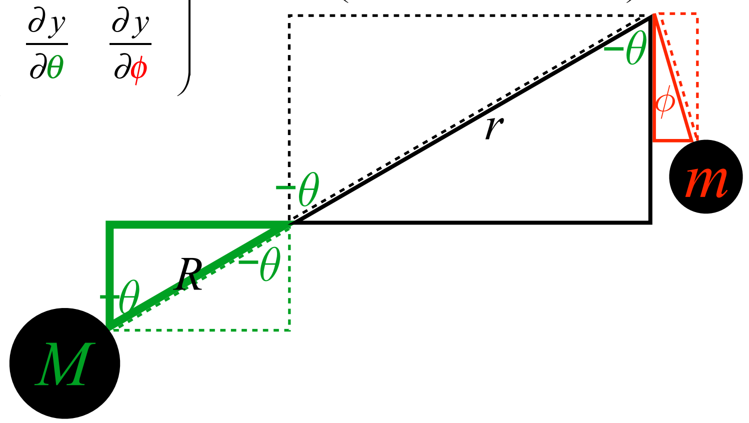
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \; & F_X\, dX && = M\ddot{X}\,dX && = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi && = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
& + F_Y\, dY && + M\ddot{Y}\,dY && + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
& + F_x\, dx && + m\ddot{x}\,dx && + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
& + F_y\, dy && + m\ddot{y}\,dy && + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Assuming variables θ and ϕ are independent...

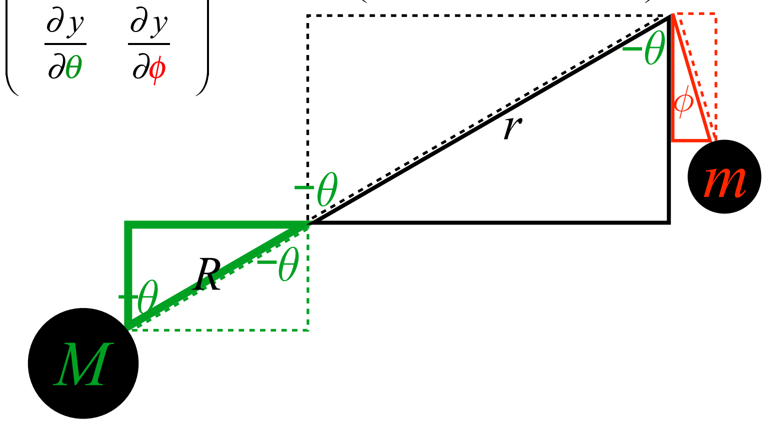
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{array}{lllll}
dW = F_X\, dX & = M\ddot{X}\, dX & = F_X \frac{\partial X}{\partial\theta}d\theta + F_X \frac{\partial X}{\partial\phi}d\phi & = & M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
\quad + F_Y\, dY & + M\ddot{Y}\, dY & + F_Y \frac{\partial Y}{\partial\theta}d\theta + F_Y \frac{\partial Y}{\partial\phi}d\phi & & + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
\quad + F_x\, dx & + m\ddot{x}\, dx & + F_x \frac{\partial x}{\partial\theta}d\theta + F_x \frac{\partial x}{\partial\phi}d\phi & & + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
\quad + F_y\, dy & + m\ddot{y}\, dy & + F_y \frac{\partial y}{\partial\theta}d\theta + F_y \frac{\partial y}{\partial\phi}d\phi & & + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{array}$$

Assuming variables θ and ϕ are independent...

Set: $d\theta=1 \quad d\phi=0$

$$\begin{array}{lcl}
F_X \frac{\partial X}{\partial\theta} & = & M\ddot{X}\frac{\partial X}{\partial\theta} \\
+ F_Y \frac{\partial Y}{\partial\theta} & & + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+ F_x \frac{\partial x}{\partial\theta} & & + m\ddot{x}\frac{\partial x}{\partial\theta} \\
+ F_y \frac{\partial y}{\partial\theta} & & + m\ddot{y}\frac{\partial y}{\partial\theta}
\end{array}$$

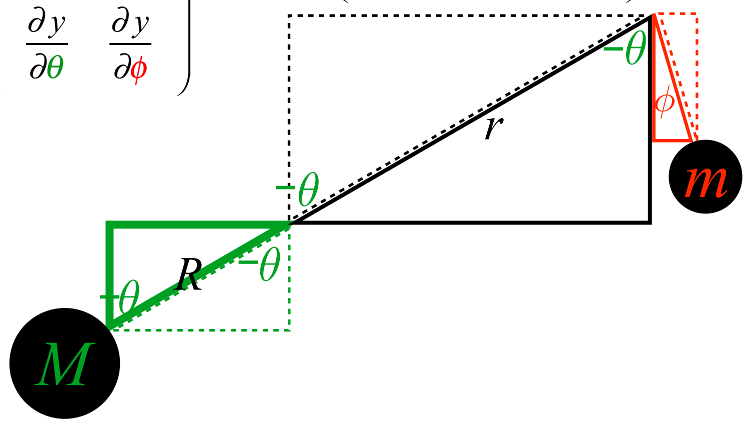
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned} dW = \; & F_X\,dX &=\; & M\ddot{X}\,dX &=\; & F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &=\; & M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\ & + F_Y\,dY && + M\ddot{Y}\,dY && + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\ & + F_x\,dx && + m\ddot{x}\,dx && + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\ & + F_y\,dy && + m\ddot{y}\,dy && + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi \end{aligned}$$

Assuming variables θ and ϕ are independent...

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned} & F_X\frac{\partial X}{\partial\theta} &=\; & M\ddot{X}\frac{\partial X}{\partial\theta} \\ +\; & F_Y\frac{\partial Y}{\partial\theta} && + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\ +\; & F_x\frac{\partial x}{\partial\theta} && + m\ddot{x}\frac{\partial x}{\partial\theta} \\ +\; & F_y\frac{\partial y}{\partial\theta} && + m\ddot{y}\frac{\partial y}{\partial\theta} \end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned} & F_X\frac{\partial X}{\partial\phi} &=\; & M\ddot{X}\frac{\partial X}{\partial\phi} \\ +\; & F_Y\frac{\partial Y}{\partial\phi} && + M\ddot{Y}\frac{\partial Y}{\partial\phi} \\ +\; & F_x\frac{\partial x}{\partial\phi} && + m\ddot{x}\frac{\partial x}{\partial\phi} \\ +\; & F_y\frac{\partial y}{\partial\phi} && + m\ddot{y}\frac{\partial y}{\partial\phi} \end{aligned}$$

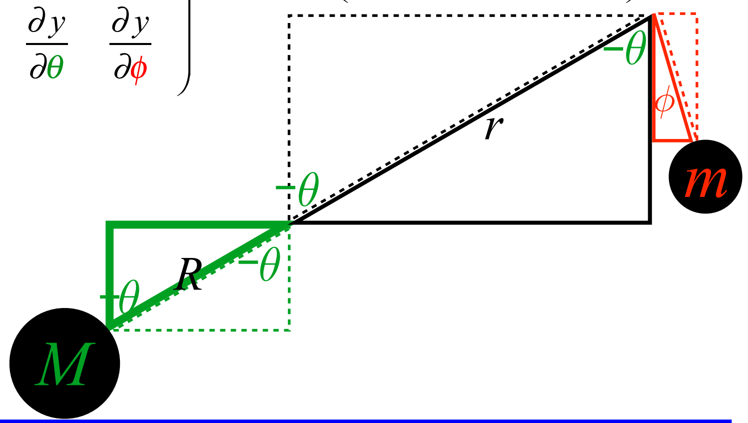
Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\ dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\ dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\ dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\ dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

$$\text{(using } \frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}\text{)}$$

STEP
A

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad +m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad +m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad +m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

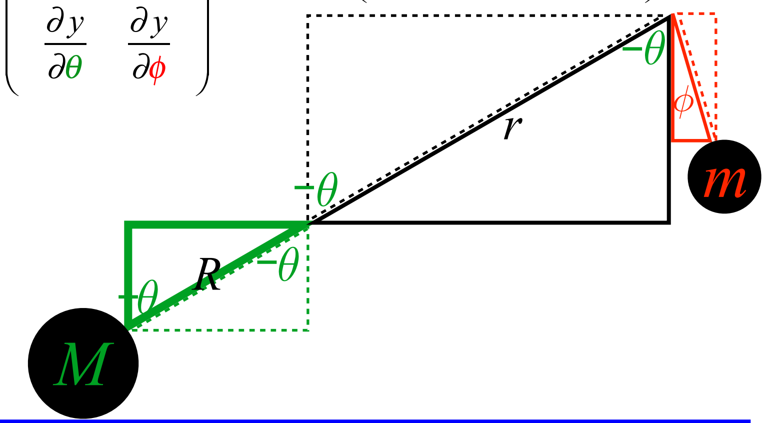
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &\quad + M\ddot{Y}\,dY &\quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &\quad + m\ddot{x}\,dx &\quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &\quad + m\ddot{y}\,dy &\quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial\dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial\dot{X}}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP
A

STEP
B

by lemma 1: $\frac{\partial\dot{X}}{\partial\dot{q}} = \frac{\partial X}{\partial q}$

and lemma 2: $\frac{\partial\dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

Lemmas from Lect.10

lemma 1: p.10.13

lemma 2: p.10.24

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad + m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad + m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad + m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad + m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

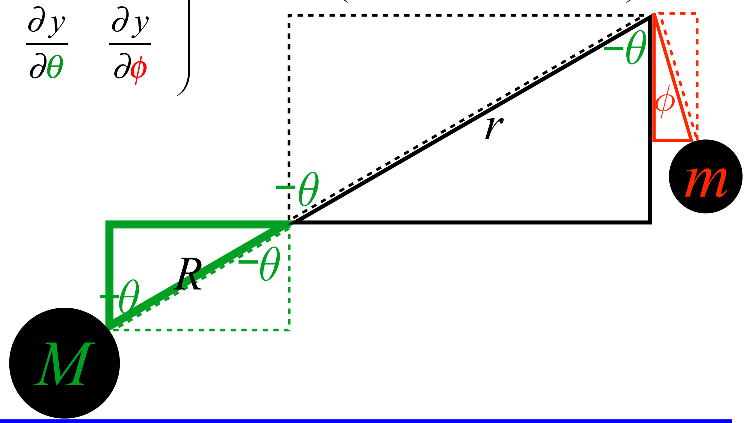
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\, dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\, dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\, dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

STEP **A**

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **B**

$$= \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial\dot{X}}{\partial\theta}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial\dot{X}}{\partial\dot{q}}$

and lemma 2: $\frac{\partial\dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP **C**

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial\dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial\theta}$$

(using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad +m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

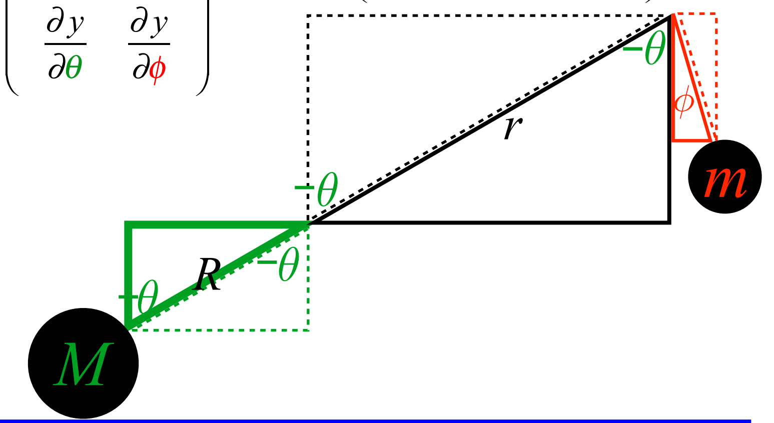
$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad +m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad +m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$
$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned} dW = F_X\, dX &= M\ddot{X}\, dX = F_X \frac{\partial X}{\partial\theta} d\theta + F_X \frac{\partial X}{\partial\phi} d\phi = M\ddot{X} \frac{\partial X}{\partial\theta} d\theta + M\ddot{X} \frac{\partial X}{\partial\phi} d\phi \\ + F_Y\, dY &\quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial\theta} d\theta + F_Y \frac{\partial Y}{\partial\phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial\theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial\phi} d\phi \\ + F_x\, dx &\quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial\theta} d\theta + F_x \frac{\partial x}{\partial\phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial\theta} d\theta + m\ddot{x} \frac{\partial x}{\partial\phi} d\phi \\ + F_y\, dy &\quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial\theta} d\theta + F_y \frac{\partial y}{\partial\phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial\theta} d\theta + m\ddot{y} \frac{\partial y}{\partial\phi} d\phi \end{aligned}$$

Lagrange trickery:

STEP **A**

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **B**

$$= \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial \dot{X}}{\partial\theta}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial \dot{X}}{\partial \dot{q}}$

and lemma 2: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP **C**

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial\dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial\theta}$$

(using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned} F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta} \\ + F_Y\frac{\partial Y}{\partial\theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta} \\ + F_x\frac{\partial x}{\partial\theta} &\quad + m\ddot{x}\frac{\partial x}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta} \\ + F_y\frac{\partial y}{\partial\theta} &\quad + m\ddot{y}\frac{\partial y}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta} \end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned} F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi} \\ + F_Y\frac{\partial Y}{\partial\phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi} \\ + F_x\frac{\partial x}{\partial\phi} &\quad + m\ddot{x}\frac{\partial x}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi} \\ + F_y\frac{\partial y}{\partial\phi} &\quad + m\ddot{y}\frac{\partial y}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi} \end{aligned}$$

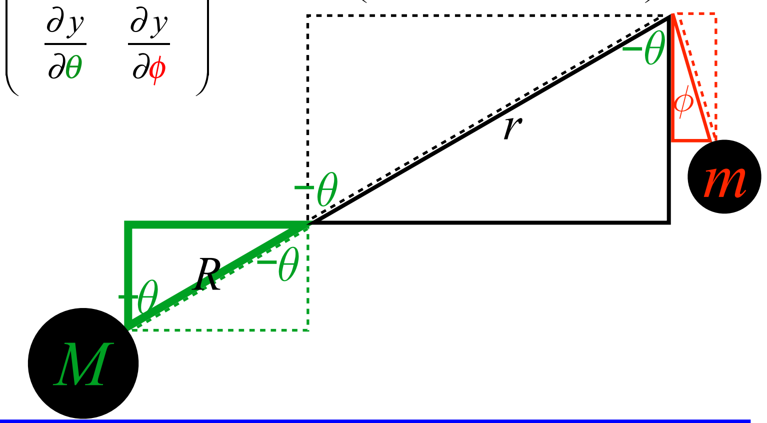
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X}\frac{\partial X}{\partial \theta} d\theta + M\ddot{X}\frac{\partial X}{\partial \phi} d\phi$$

$$+F_Y\, dY \quad + M\ddot{Y}\,dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi} d\phi$$

$$+F_x\, dx \quad + m\ddot{x}\,dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x}\frac{\partial x}{\partial \theta} d\theta + m\ddot{x}\frac{\partial x}{\partial \phi} d\phi$$

$$+F_y\, dy \quad + m\ddot{y}\,dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y}\frac{\partial y}{\partial \theta} d\theta + m\ddot{y}\frac{\partial y}{\partial \phi} d\phi$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ*

Set: $d\theta=1 \quad d\phi=0$

$$F_X\frac{\partial X}{\partial \theta} = M\ddot{X}\frac{\partial X}{\partial \theta} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+F_Y\frac{\partial Y}{\partial \theta} \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+F_x\frac{\partial x}{\partial \theta} \quad + m\ddot{x}\frac{\partial x}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+F_y\frac{\partial y}{\partial \theta} \quad + m\ddot{y}\frac{\partial y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: $d\theta=0 \quad d\phi=1$

$$F_X\frac{\partial X}{\partial \phi} = M\ddot{X}\frac{\partial X}{\partial \phi} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+F_Y\frac{\partial Y}{\partial \phi} \quad + M\ddot{Y}\frac{\partial Y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+F_x\frac{\partial x}{\partial \phi} \quad + m\ddot{x}\frac{\partial x}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+F_y\frac{\partial y}{\partial \phi} \quad + m\ddot{y}\frac{\partial y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

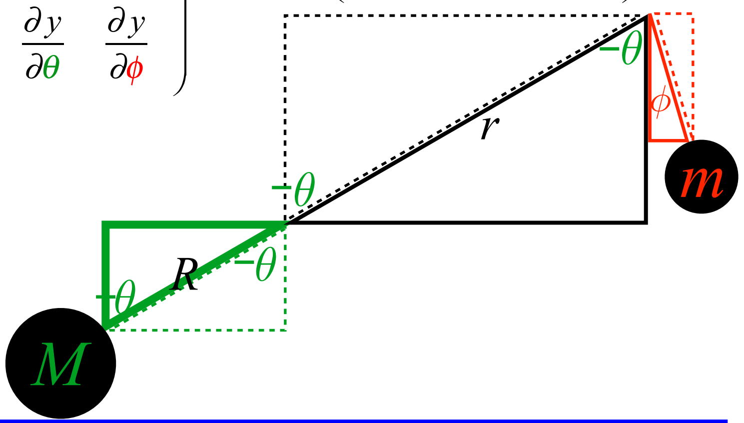
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
+ F_Y\, dY &\quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
+ F_x\, dx &\quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
+ F_y\, dy &\quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$

STEP **D** *Add up first and last columns for each variable θ and ϕ for:*

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

Lagrange trickery:

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$$

Defines F_θ

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

Set: $d\theta$=1 $d\phi$=0

$$\begin{aligned}
F_X \frac{\partial X}{\partial \theta} &= M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta} \\
+ F_Y \frac{\partial Y}{\partial \theta} &\quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta} \\
+ F_x \frac{\partial x}{\partial \theta} &\quad + m\ddot{x} \frac{\partial x}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta} \\
+ F_y \frac{\partial y}{\partial \theta} &\quad + m\ddot{y} \frac{\partial y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}
\end{aligned}$$

Set: $d\theta$=0 $d\phi$=1

$$\begin{aligned}
F_X \frac{\partial X}{\partial \phi} &= M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi} \\
+ F_Y \frac{\partial Y}{\partial \phi} &\quad + M\ddot{Y} \frac{\partial Y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi} \\
+ F_x \frac{\partial x}{\partial \phi} &\quad + m\ddot{x} \frac{\partial x}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi} \\
+ F_y \frac{\partial y}{\partial \phi} &\quad + m\ddot{y} \frac{\partial y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}
\end{aligned}$$

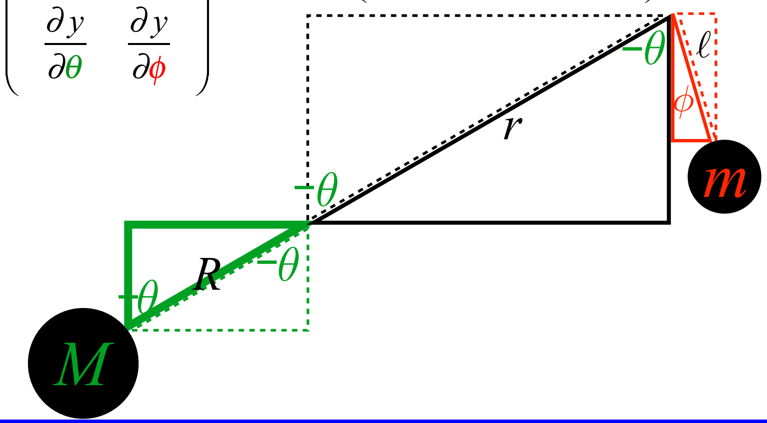
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} &= \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad +m\ddot{x}\frac{\partial x}{\partial\theta} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad +m\ddot{y}\frac{\partial y}{\partial\theta} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} &= \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad +m\ddot{x}\frac{\partial x}{\partial\phi} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad +m\ddot{y}\frac{\partial y}{\partial\phi} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi}
\end{aligned}$$

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

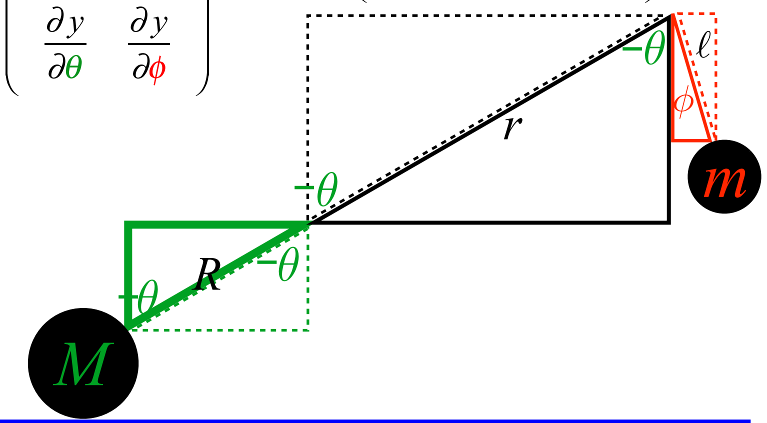
$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \; & F_X\, dX &= \; & M\ddot{X}\, dX &= \; & F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi &= \; & M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
& + F_Y\, dY & & + M\ddot{Y}\, dY & & + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi & & + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
& + F_x\, dx & & + m\ddot{x}\, dx & & + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi & & + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
& + F_y\, dy & & + m\ddot{y}\, dy & & + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi & & + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$



Lagrange trickery:

STEP **D**

Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

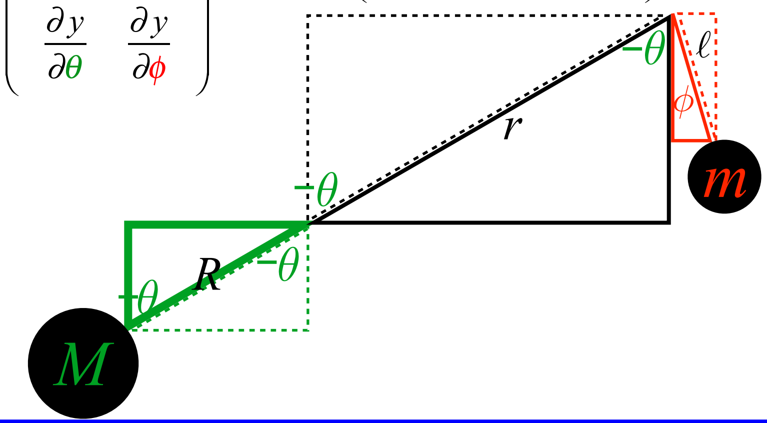
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Raw Jacobian



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\, dX &= F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi &= M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
+ F_Y\, dY &\quad + M\ddot{Y}\, dY &\quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi &\quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
+ F_x\, dx &\quad + m\ddot{x}\, dx &\quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi &\quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
+ F_y\, dy &\quad + m\ddot{y}\, dy &\quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi &\quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:*

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta \quad \text{Defines } F_\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi \quad \text{Defines } F_\phi$$

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Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

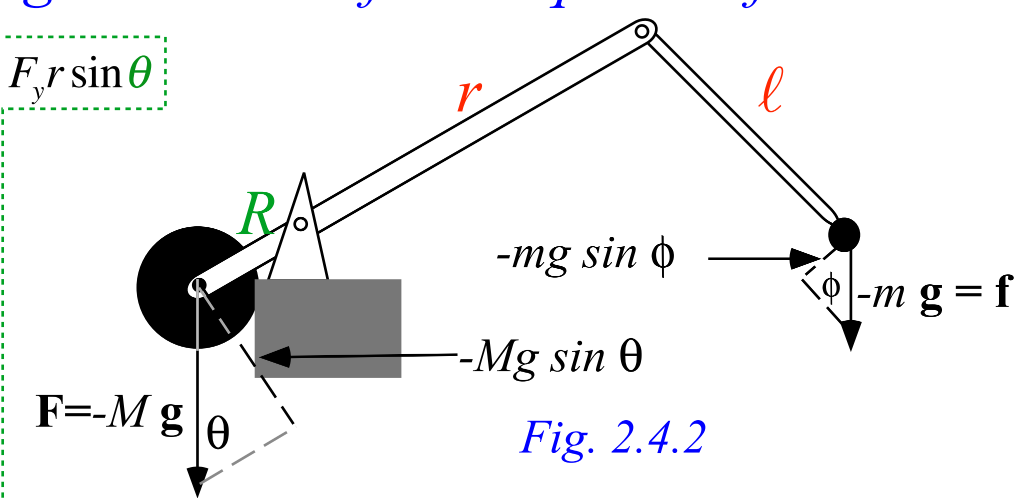


Fig. 2.4.2

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Add F_θ gravity given

$(F_X = 0,\ F_Y = -Mg)$

$(F_x = 0,\ F_y = -mg)$

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR\sin\theta + mgr\sin\theta$$

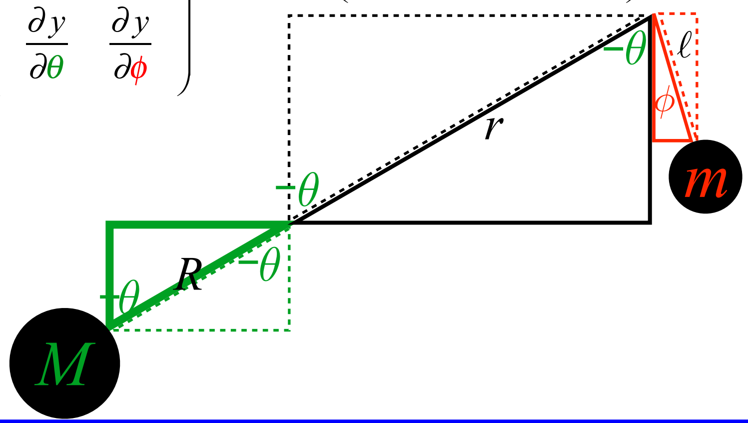
These are competing torques on main beam R

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\, dY &\quad +M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\, dx &\quad +m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\, dy &\quad +m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

Lagrange trickery:

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$
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$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given
$(F_X = 0\ ,\ F_Y = -Mg)$
$(F_x = 0\ ,\ F_y = -mg)$

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

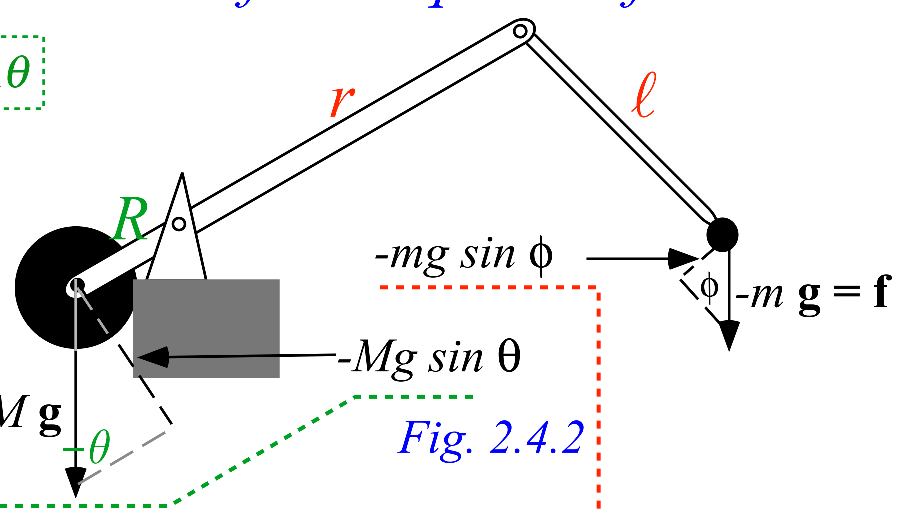

Fig. 2.4.2

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given
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$(F_x = 0\ ,\ F_y = -mg)$

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R...   *... and a torque on throwing lever ℓ*

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$
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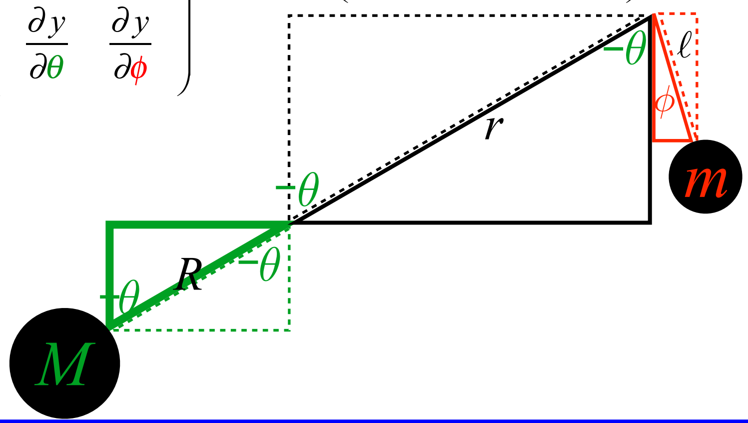
Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X}\frac{\partial X}{\partial \theta} d\theta + M\ddot{X}\frac{\partial X}{\partial \phi} d\phi$$

$$+F_Y\, dY \quad + M\ddot{Y}\,dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi} d\phi$$

$$+F_x\, dx \quad + m\ddot{x}\,dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x}\frac{\partial x}{\partial \theta} d\theta + m\ddot{x}\frac{\partial x}{\partial \phi} d\phi$$

$$+F_y\, dy \quad + m\ddot{y}\,dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y}\frac{\partial y}{\partial \theta} d\theta + m\ddot{y}\frac{\partial y}{\partial \phi} d\phi$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

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Defines F_θ

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta}$$

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Add F_θ gravity given

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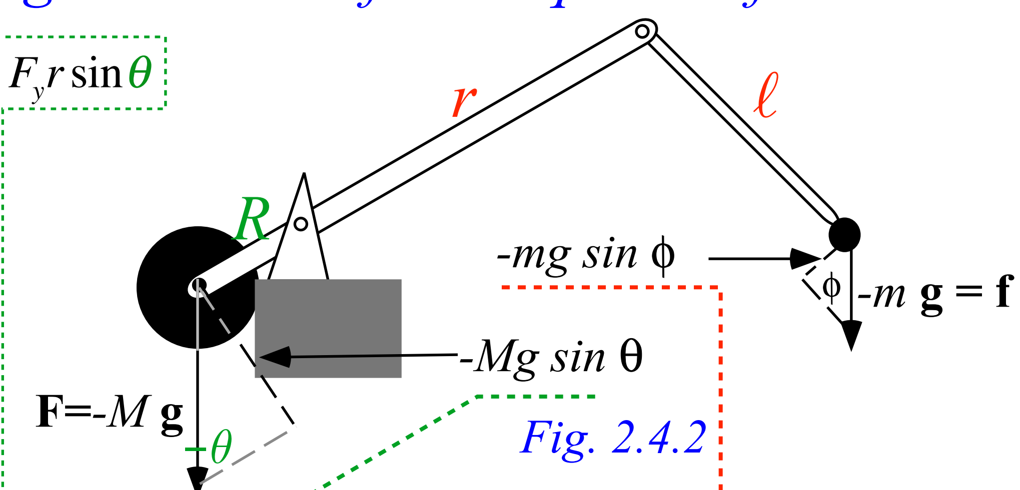

Fig. 2.4.2

Q: Are there
± sign errors here?

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$
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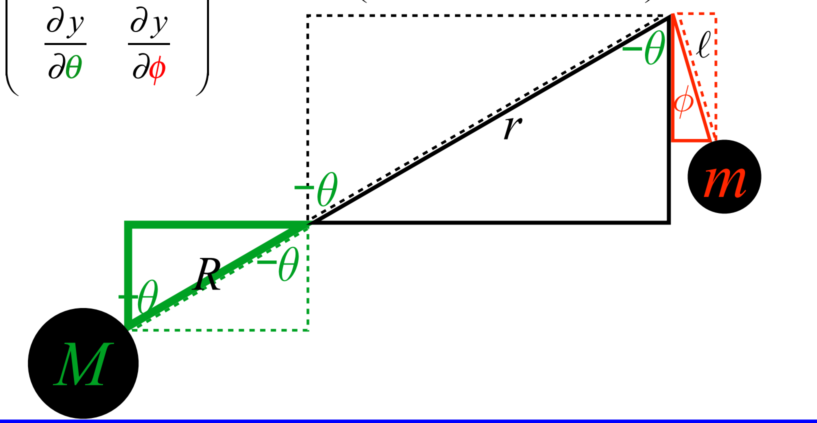

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

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$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR\sin\theta + mgr\sin\theta$$

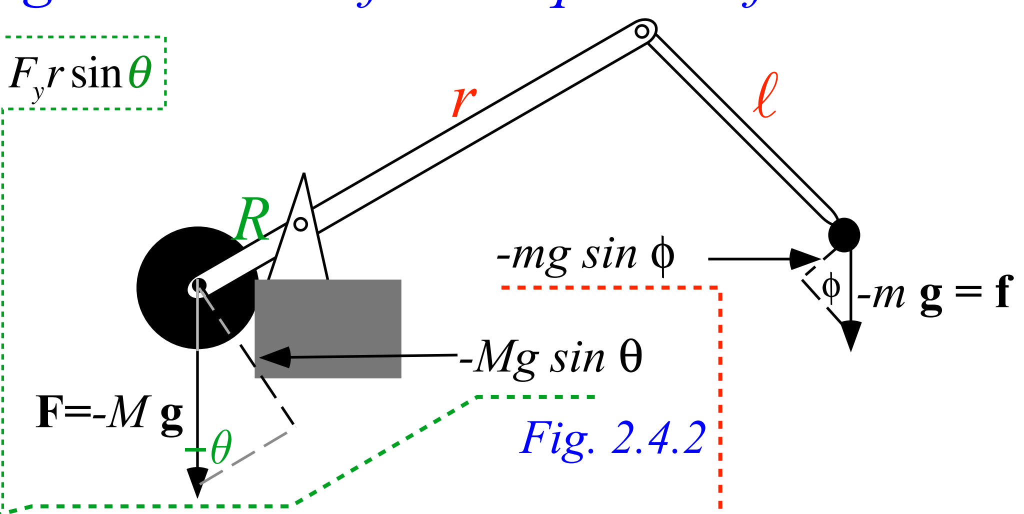


Fig. 2.4.2

Q: Are there ± sign errors here?
A: No. Beam in $-\theta$ position.

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Add F_ϕ gravity given
($F_X = 0$, $F_Y = -Mg$)
($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... *... and a torque on throwing lever ℓ*

Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)

Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor γ_{mn}

Structure of dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

Canonical momentum and γ_{mn} *tensor*

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

The γ_{mn} tensor/matrix formulation

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where:
γ_{mn} tensor is
$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is
$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: *γ_{mn} tensor is* $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

Momentum γ_{mn}-matrix theorem: *(matrix-proof on page 78)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2+mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2+mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2+mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is
$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

Total KE $= T = T(M) + T(m)$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)$$

The γ_{mn} tensor/matrix formulation

Total KE $= T = T(M) + T(m)$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where:
γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: *(matrix-proof on page 43)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: *(proof here)*

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof: *Given*: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ *where*: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

Then: $p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm}$$

QED

Momentum γ_{mn}-matrix theorem: *(matrix-proof here on page 43)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot\theta} \\ \frac{\partial T}{\partial \dot\phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial \dot\theta}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial \dot\theta}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \\ \frac{\partial}{\partial \dot\phi}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial \dot\phi}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\phi,\theta} \\ \gamma_{\theta,\phi} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

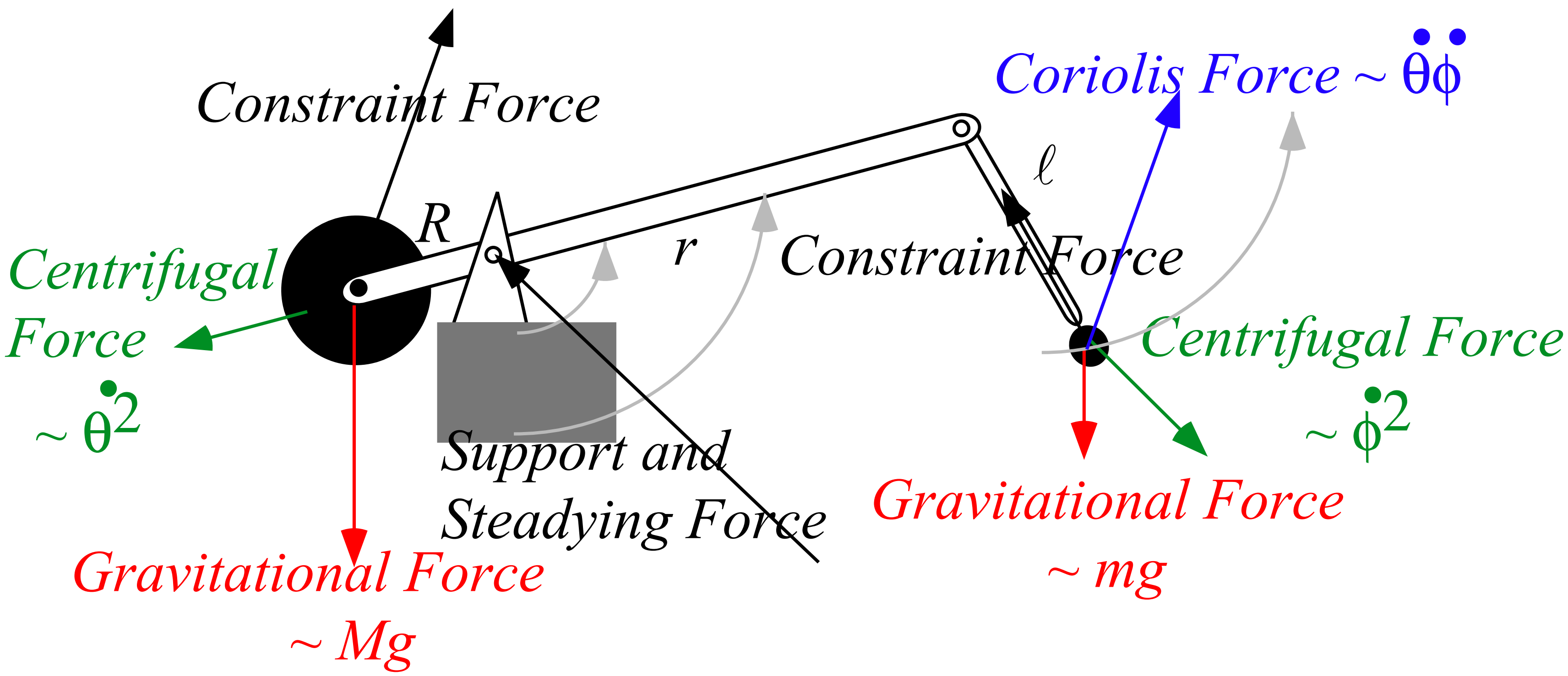
$$= \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \qquad QED$$

Summary of Lagrange equations and force analysis (Mostly Unit 2.)

→ *Forces: total, genuine, potential, and/or fictitious*

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

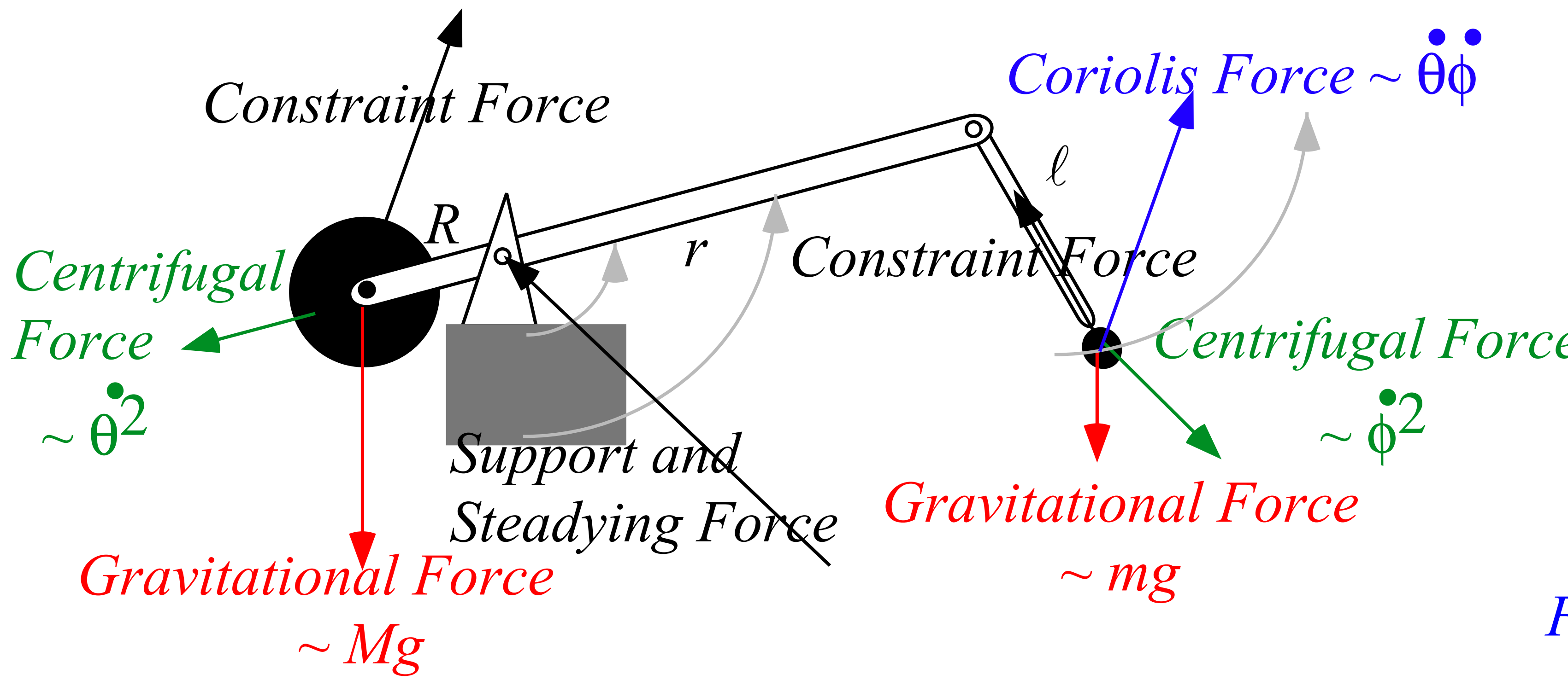
Lagrange Force equations

(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\dfrac{\partial V}{\partial \theta}$ and: $\dfrac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\dfrac{\partial V}{\partial \phi}$ and: $\dfrac{\partial V}{\partial \dot{\phi}} = 0$

Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
$L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

→ *Multivalued functionality and connections*

Covariant and contravariant relations

Tangent space vs. Normal space

Metric g_{mn} tensor geometric relations to length, area, and volume

Trebuchet Cartesian projectile coordinates are double-valued

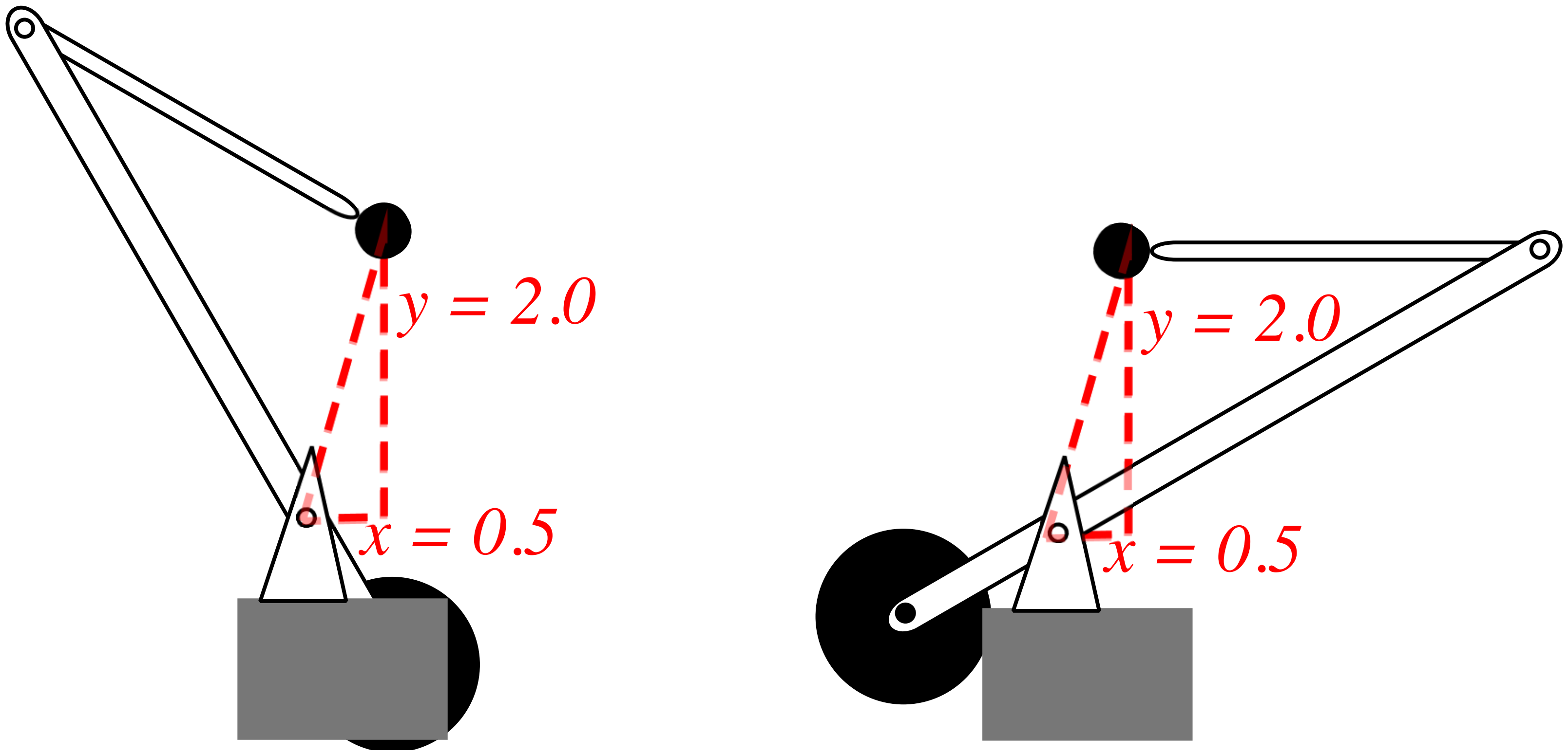


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)

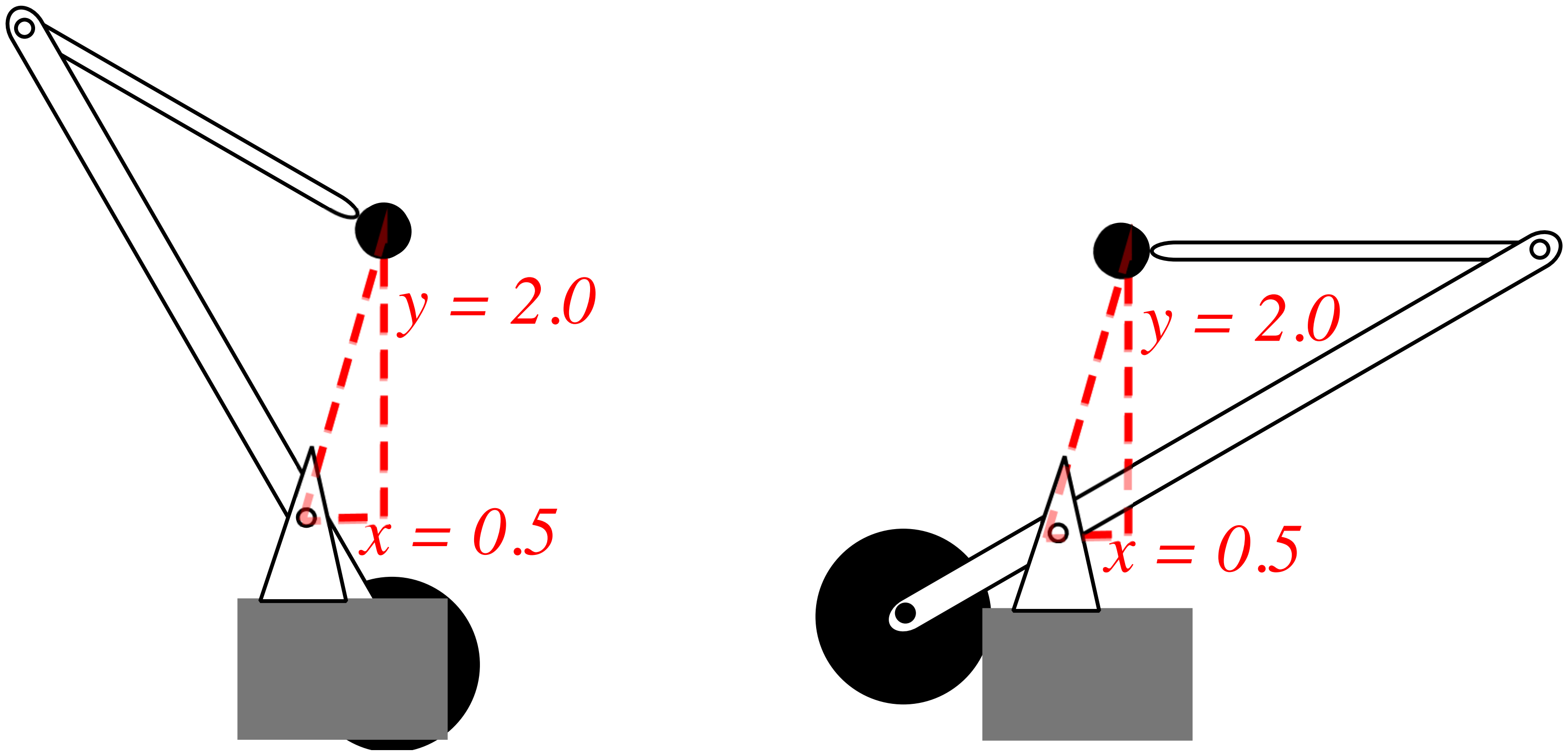


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile m.

So, for example, are polar coordinates … (for each angle there are two r-values)

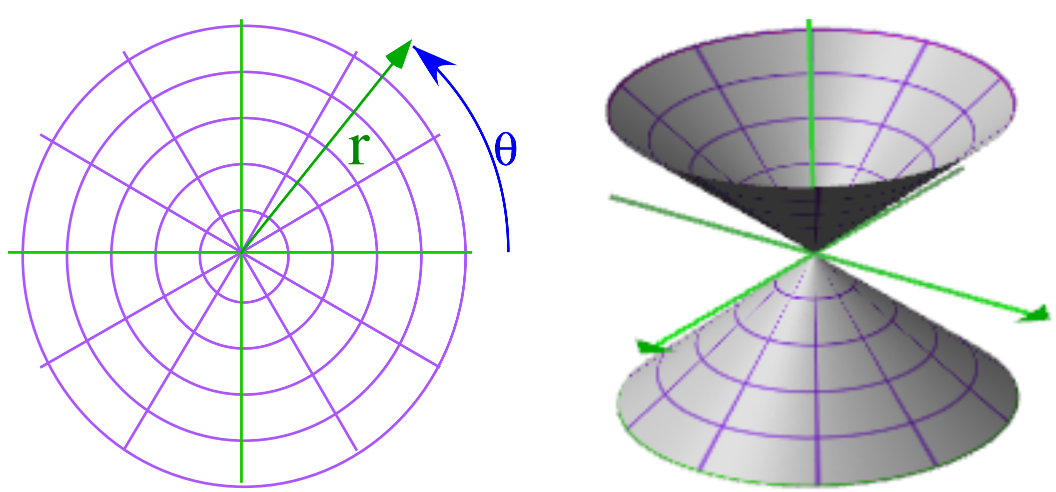
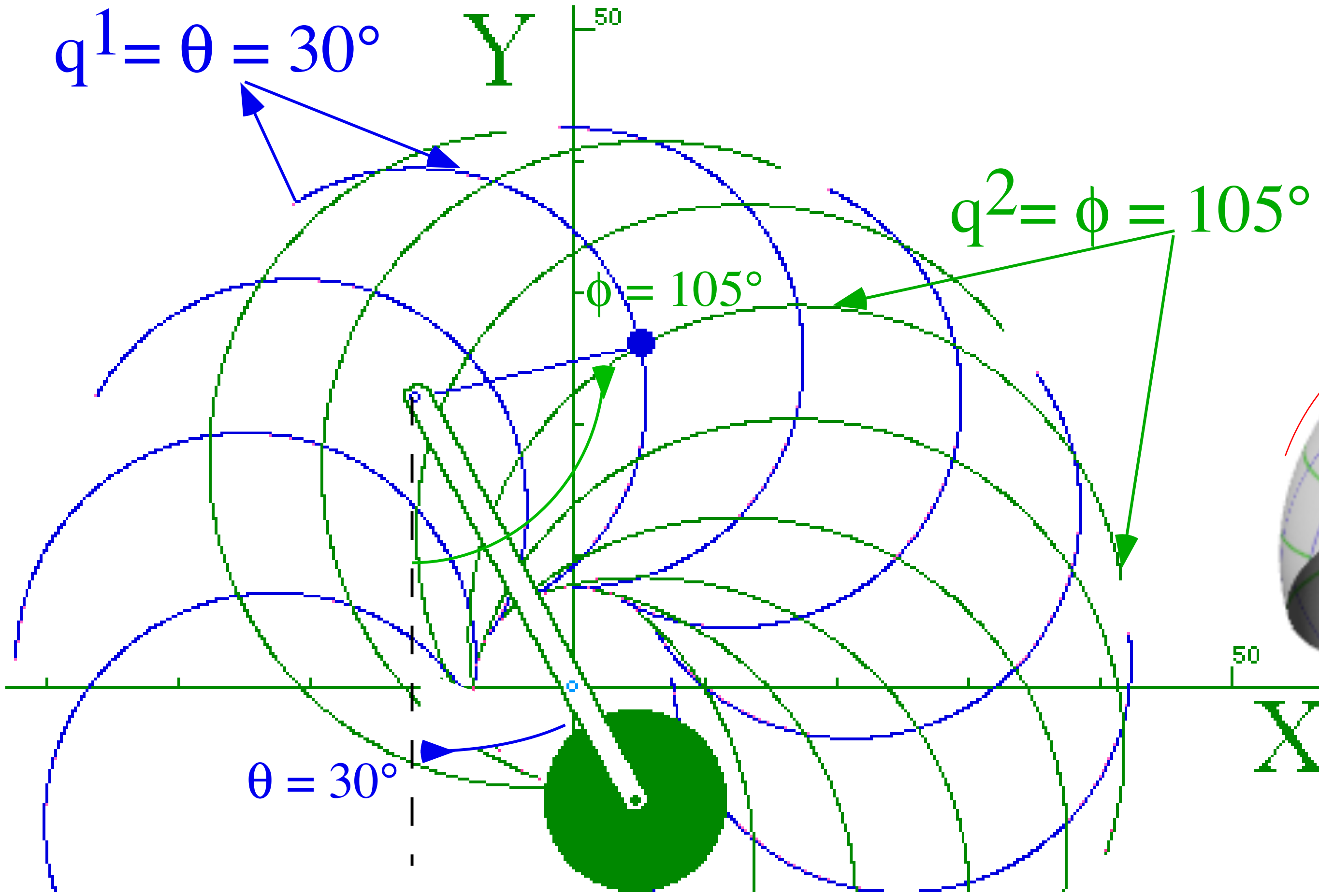


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.



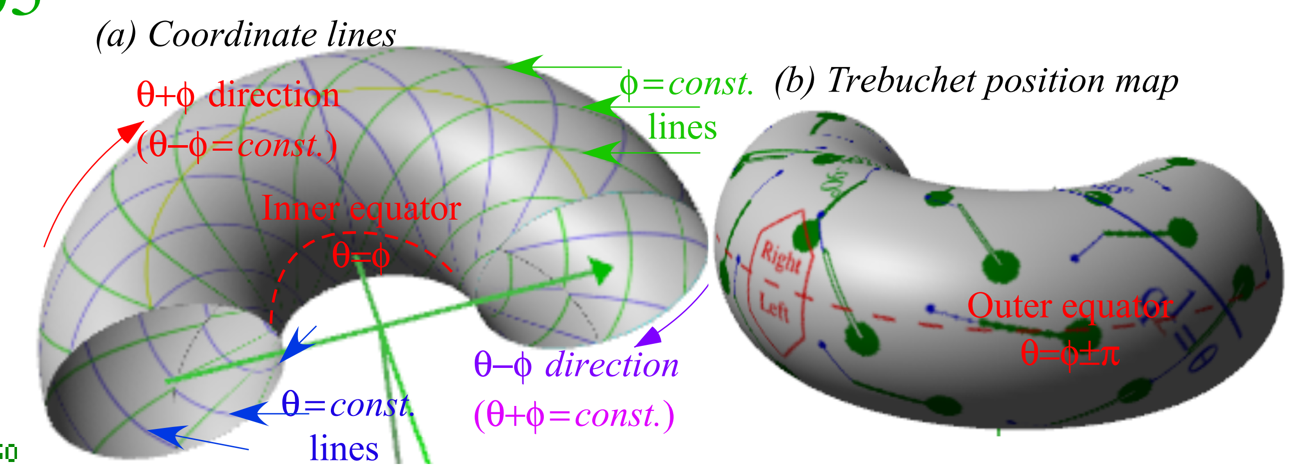


Fig. 3.1.2 Trebuchet torus.

(a) ($q^1=\theta$, $q^2=\phi$) coordinate lines. (b)Trebuchet position map and equators.

Fig. 3.1.1a ($q^1=\theta$, $q^2=\phi$)Coordinate manifold for trebuchet (Left handed sheet.)

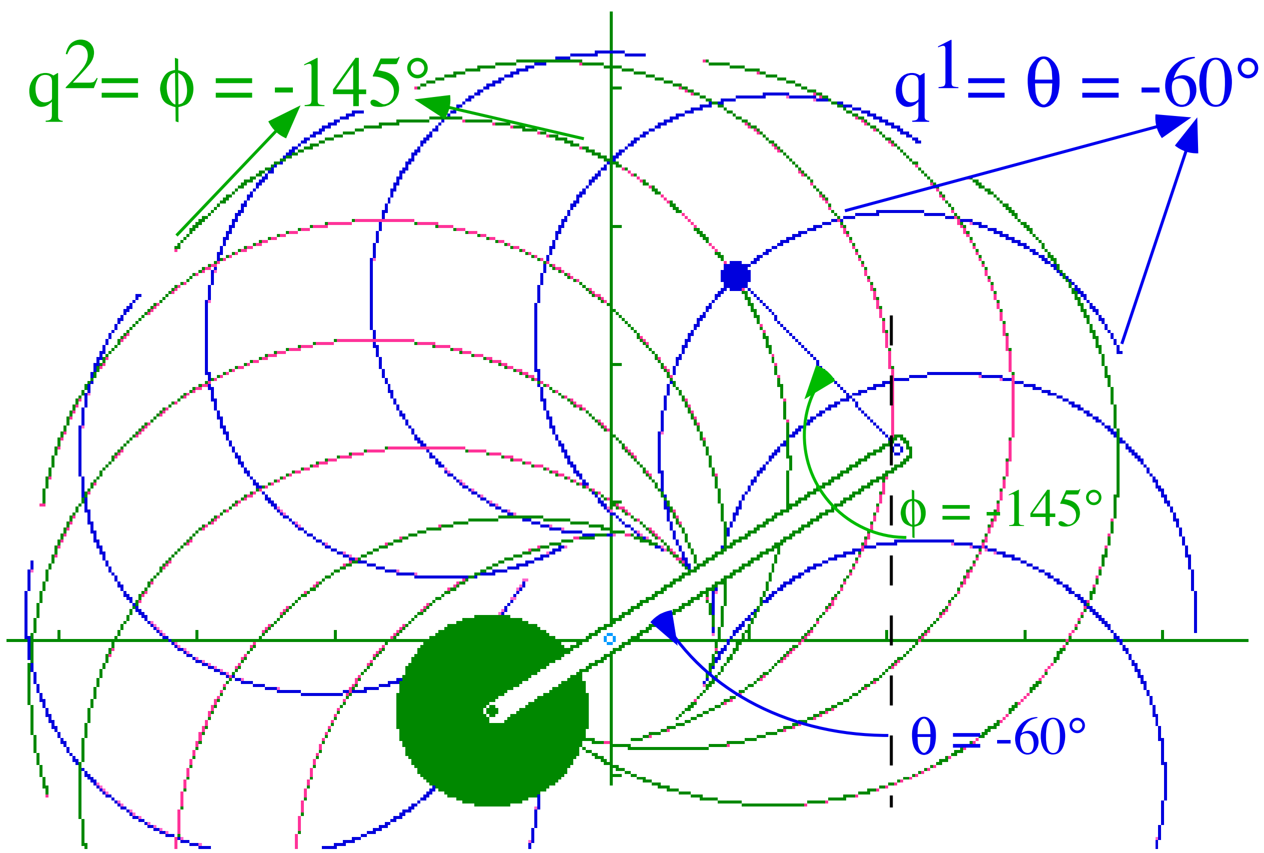


Fig. 3.1.1b ($q^1=\theta$, $q^2=\phi$)Coordinate manifold for trebuchet (Right handed sheet.)

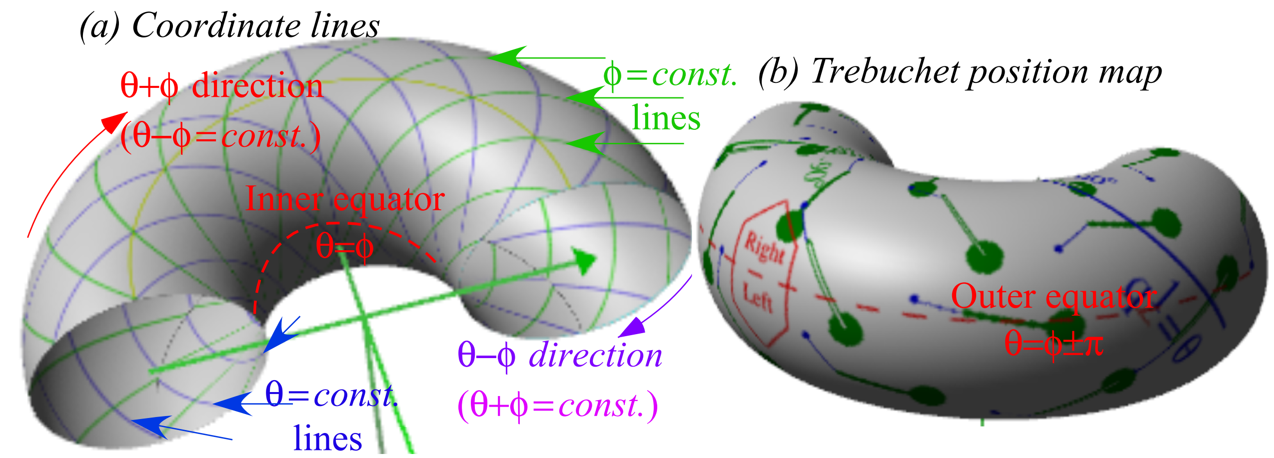
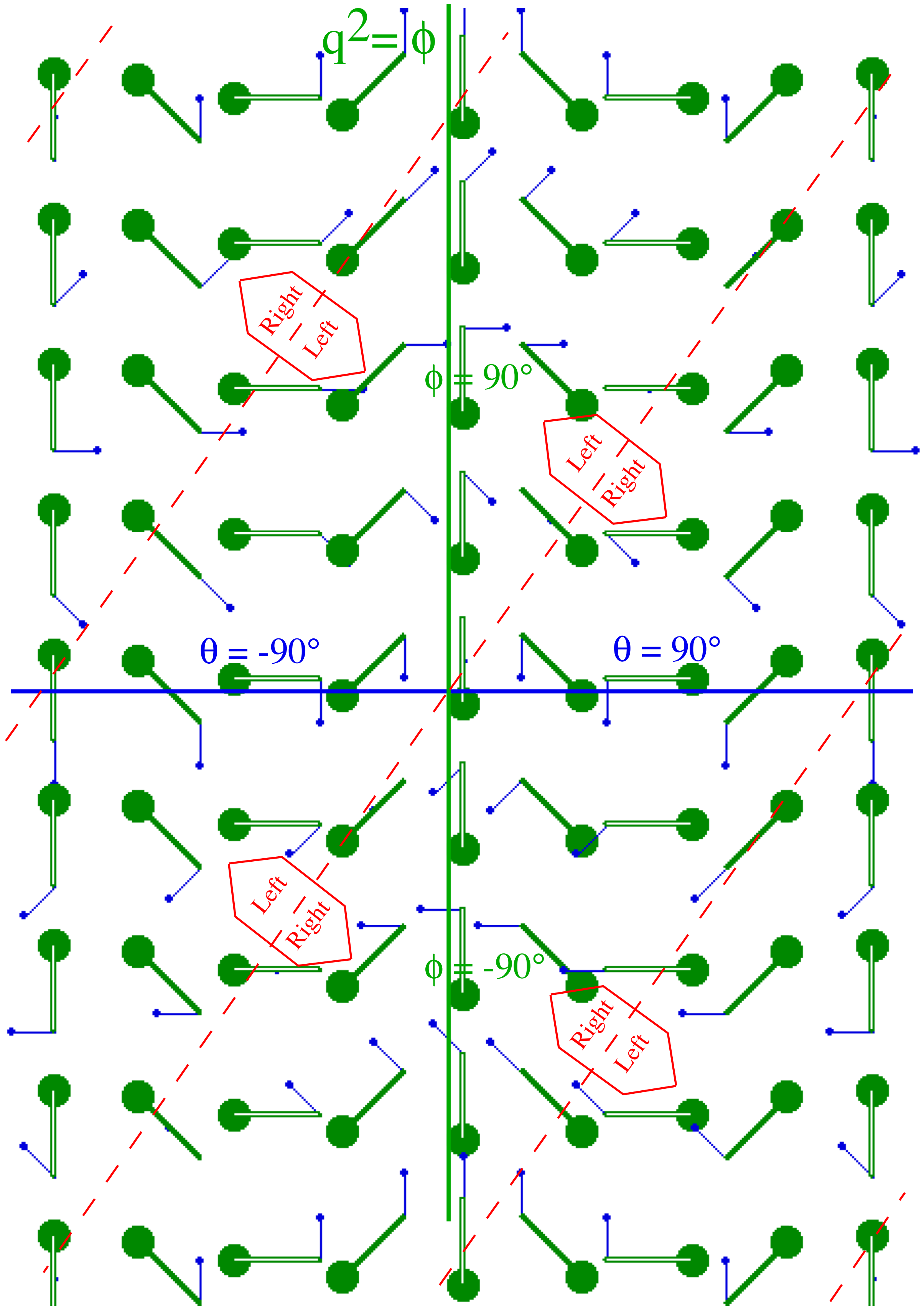


Fig. 3.1.2 Trebuchet torus.
(a) (q^1=θ, q^2=ϕ) coordinate lines.(b)Trebuchet position map and equators.



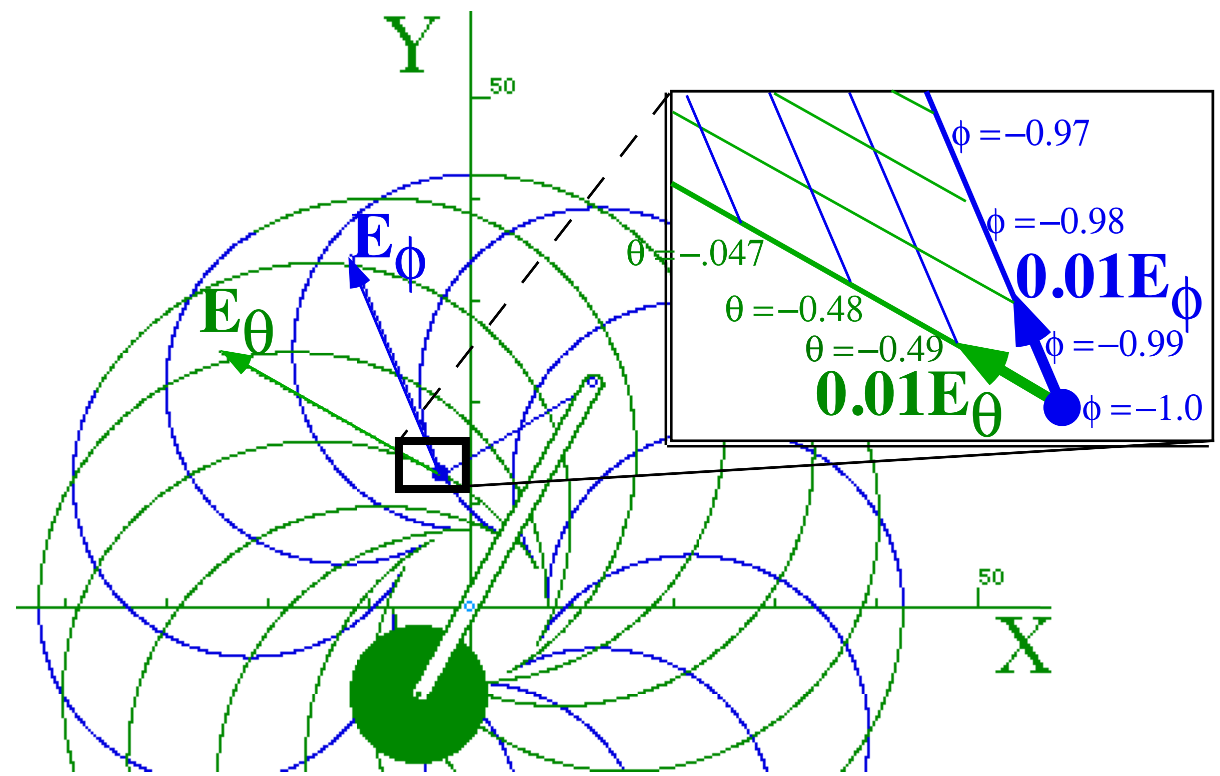


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Fig. 3.1.3 "Flattened" (q^1=θ, q^2=ϕ) coordinate manifold for trebuchet

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

→ *Covariant and contravariant relations*

Tangent space vs. Normal space

Metric g_{mn} tensor geometric relations to length, area, and volume

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

$$\begin{array}{c} \left| \begin{array}{ccc} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| \end{array} \begin{array}{c} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{array} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{array}{cc} \ell \sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{array} \right| \begin{array}{c} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

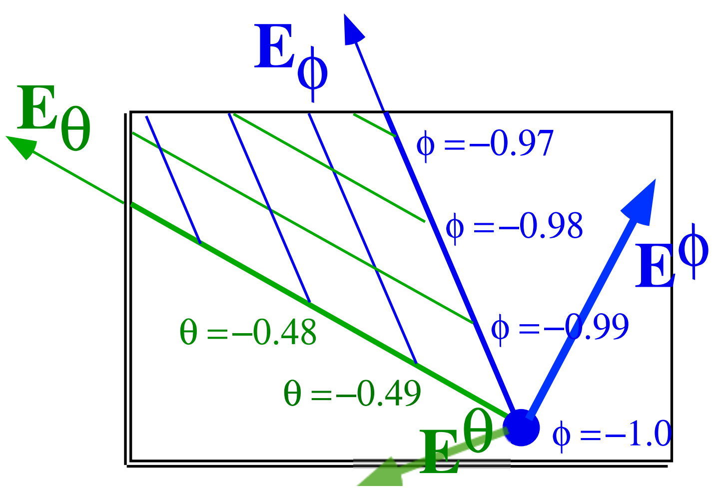


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

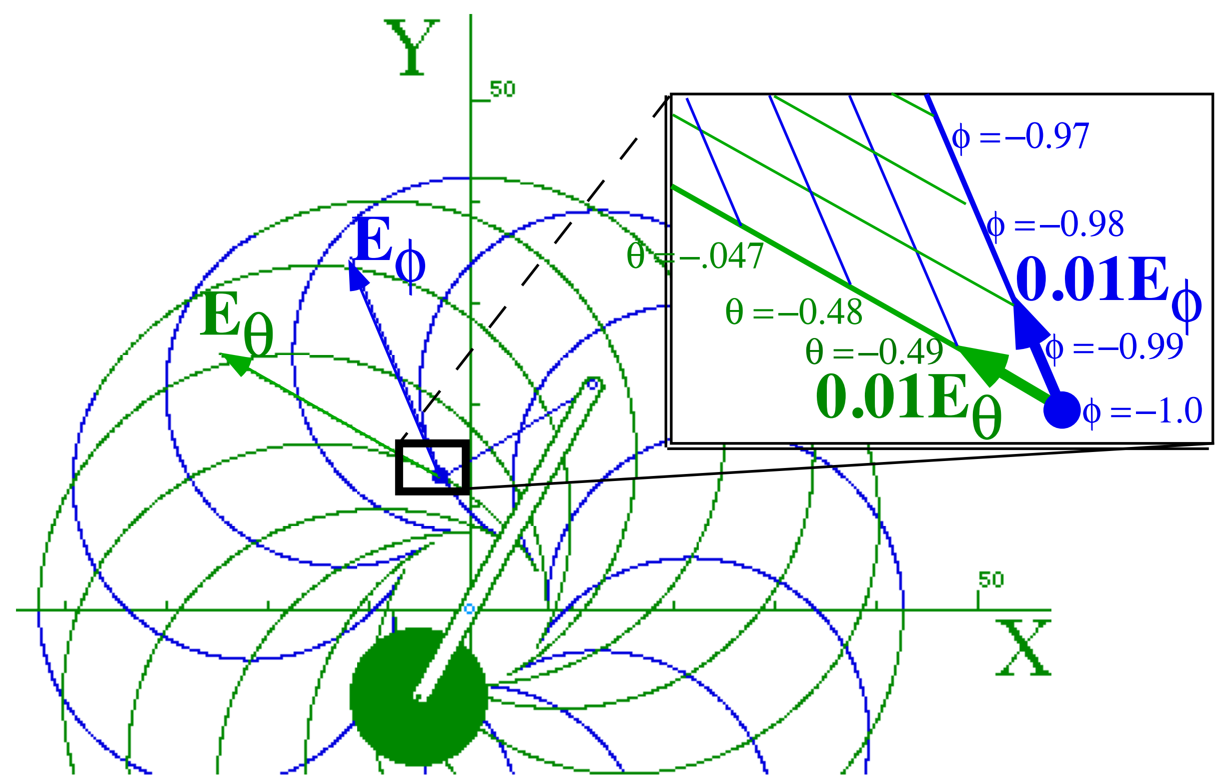


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

→ *Tangent space vs. Normal space*

Metric g_{mn} tensor geometric relations to length, area, and volume

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\quad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

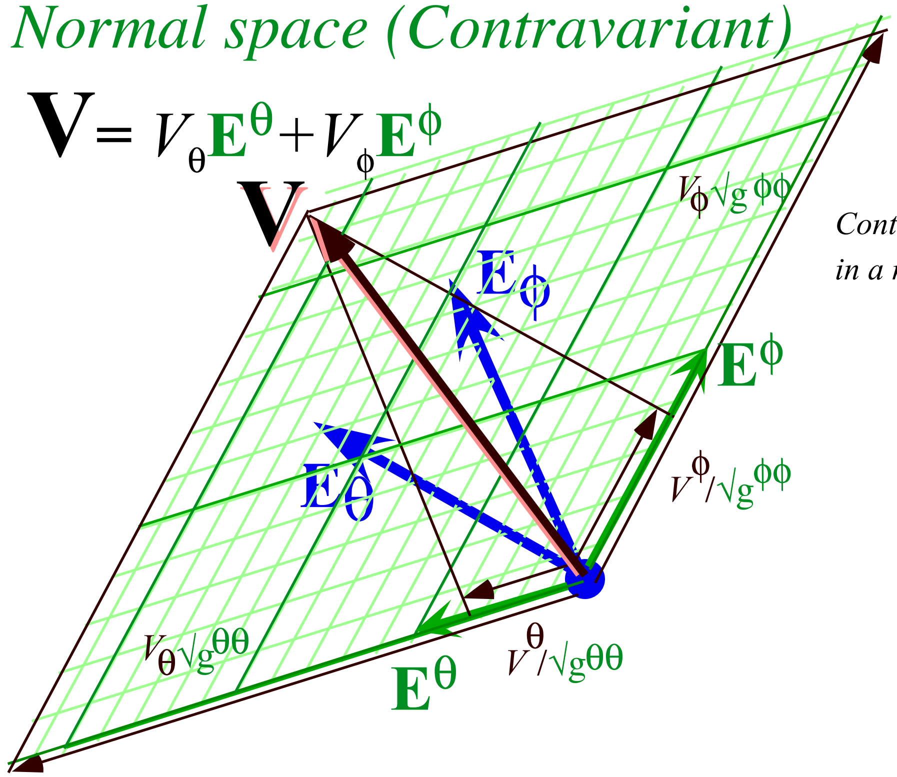


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta, \mathbf{E}^\phi$).

Tangent space (Covariant)

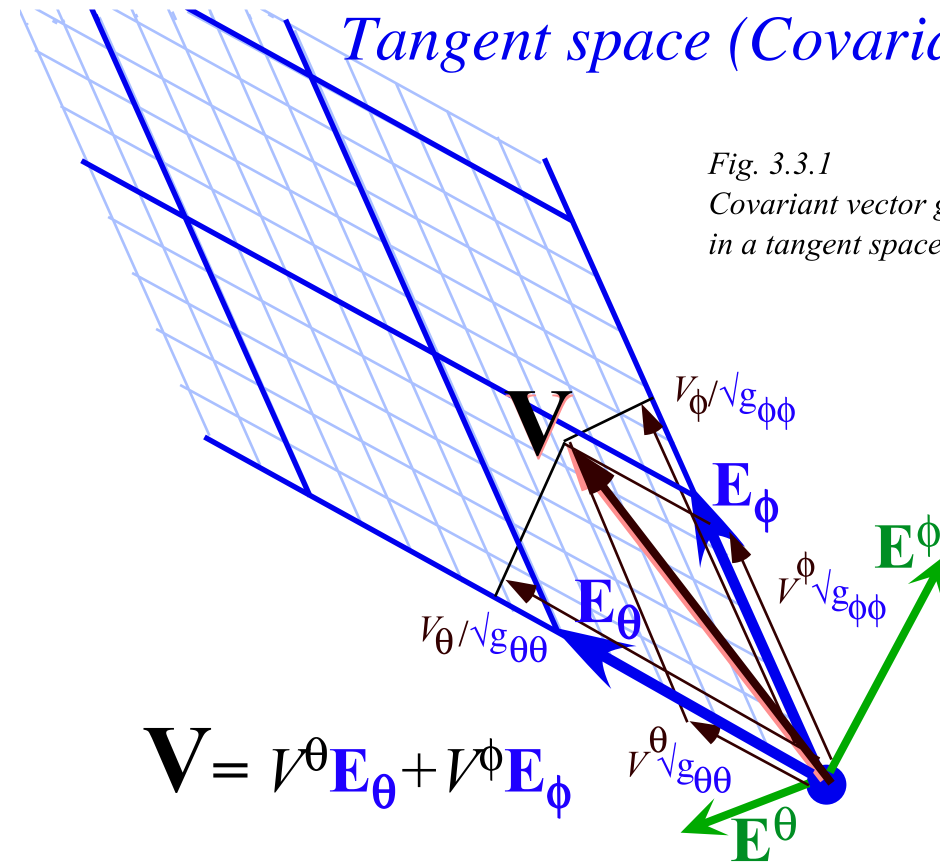


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta, \mathbf{E}_\phi$).

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U\bullet E^m}$, $V^m = \mathbf{V\bullet E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\quad U_n = \mathbf{U\bullet E_n}$, $V_n = \mathbf{V\bullet E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

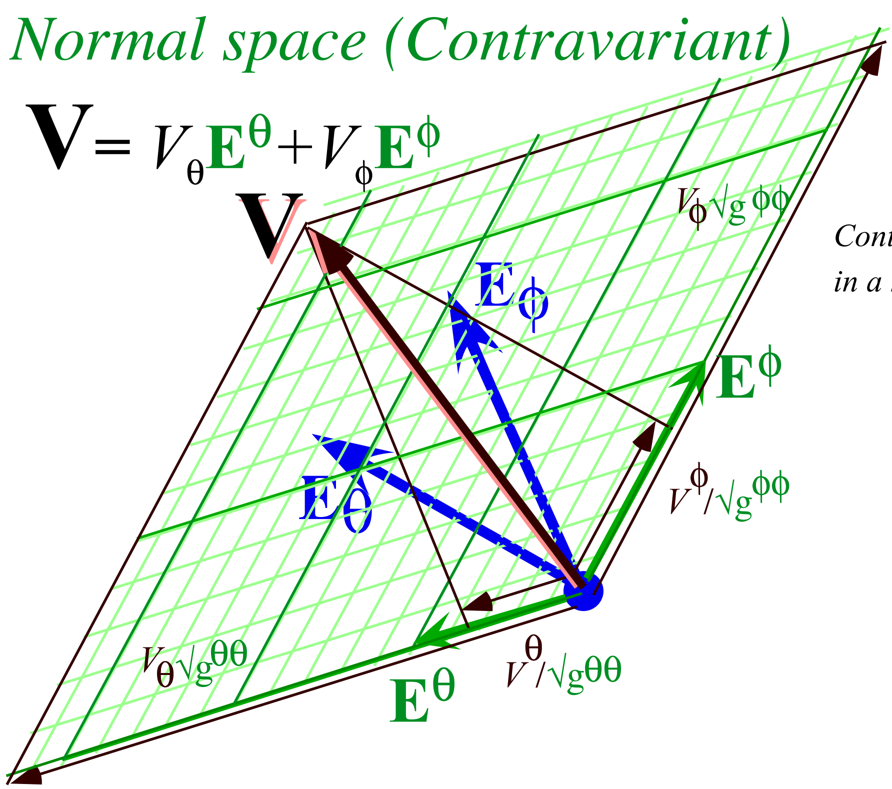


Fig. 3.3.2
Contravariant vector geometry in a normal space $(\mathbf{E}^\theta,\mathbf{E}^\phi)$.

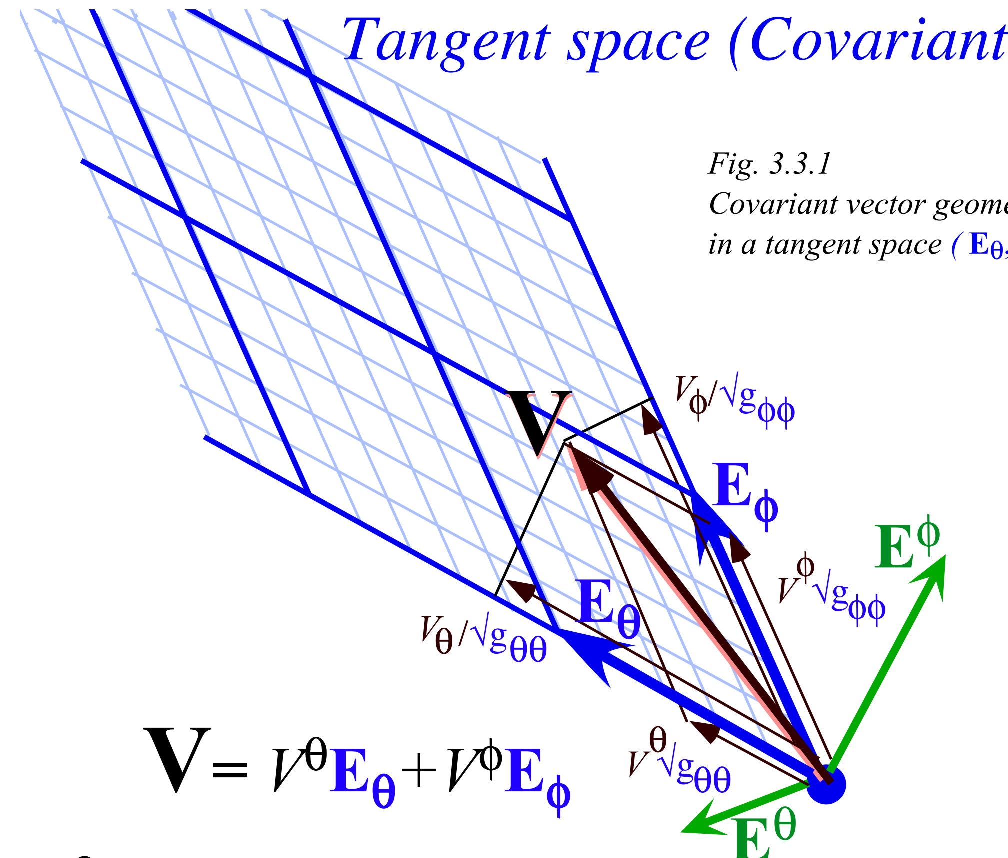


Fig. 3.3.1
Covariant vector geometry in a tangent space $(\mathbf{E}_\theta,\mathbf{E}_\phi)$.

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1,q^2,\cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}},\bar{q}^{\bar{2}},\cdots\}$ using a *"chain-saw-sum rule"*….

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\phantom{\partial \bar{q}^{\bar{m}}}}\frac{\phantom{\partial \bar{q}^{\bar{m}}}}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ , \quad \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\qquad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

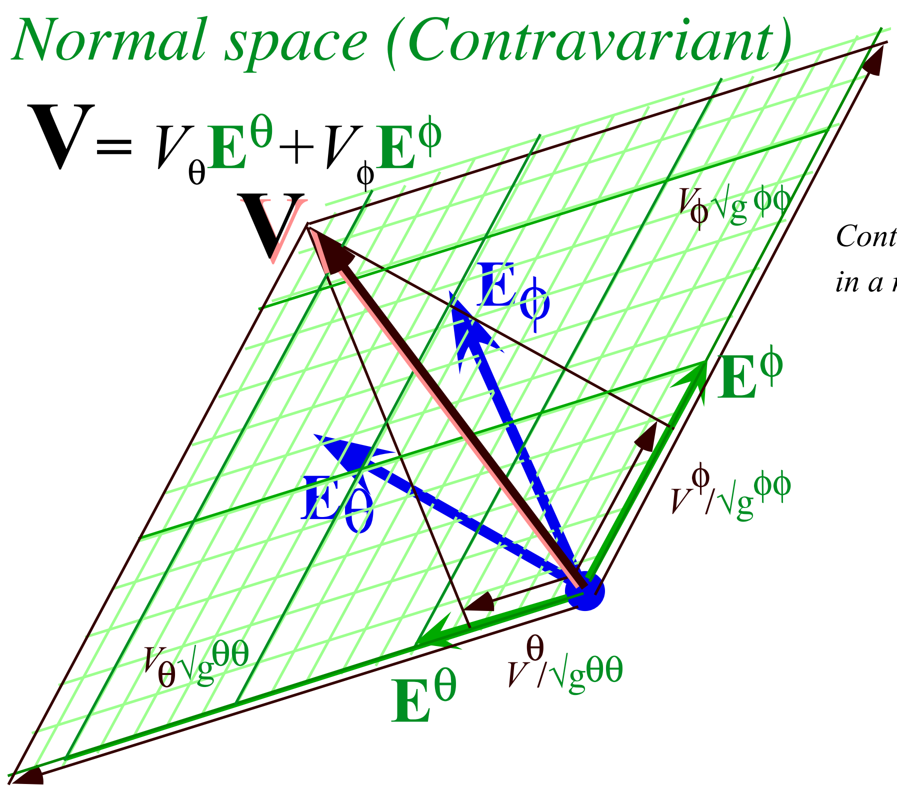

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

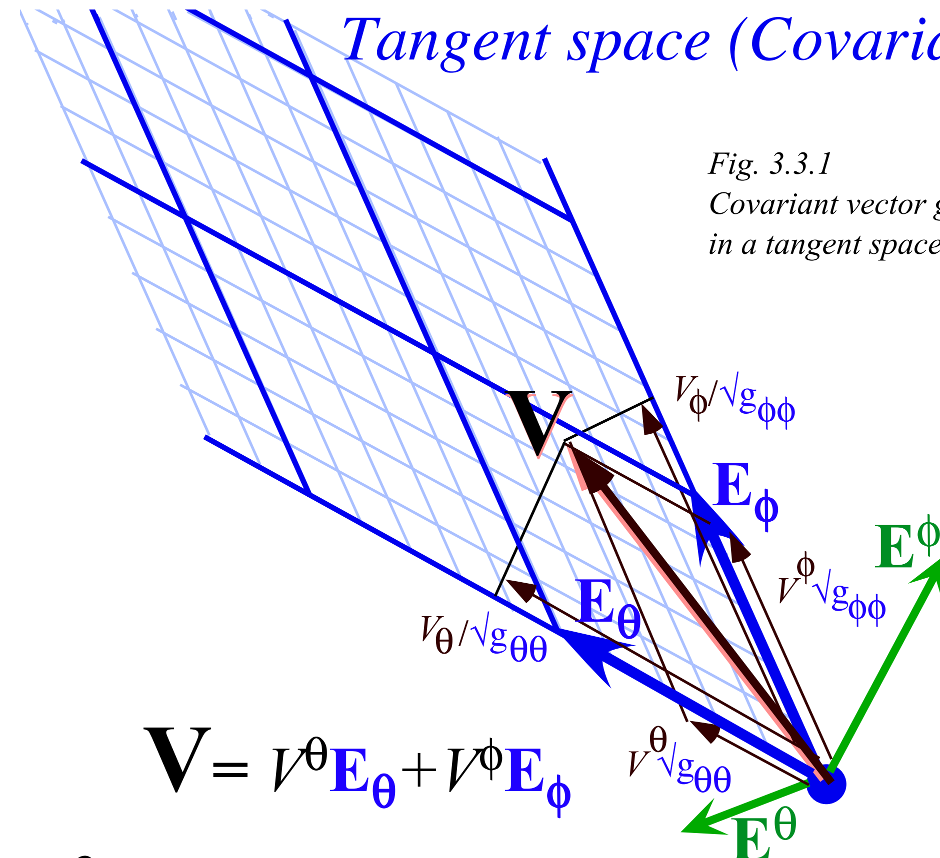

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\phantom{\partial \bar{q}^{\bar{m}}}}\frac{\phantom{\partial \bar{q}^{\bar{m}}}}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\phantom{\partial \bar{q}^{\bar{m}}}}{\partial q^m}\frac{\partial \mathbf{r}}{\phantom{\partial \bar{q}^{\bar{m}}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ \boxed{V^m = \mathbf{V}\bullet\mathbf{E^m}},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}},\ , \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}},\ etc.$$

Normal space (Contravariant)

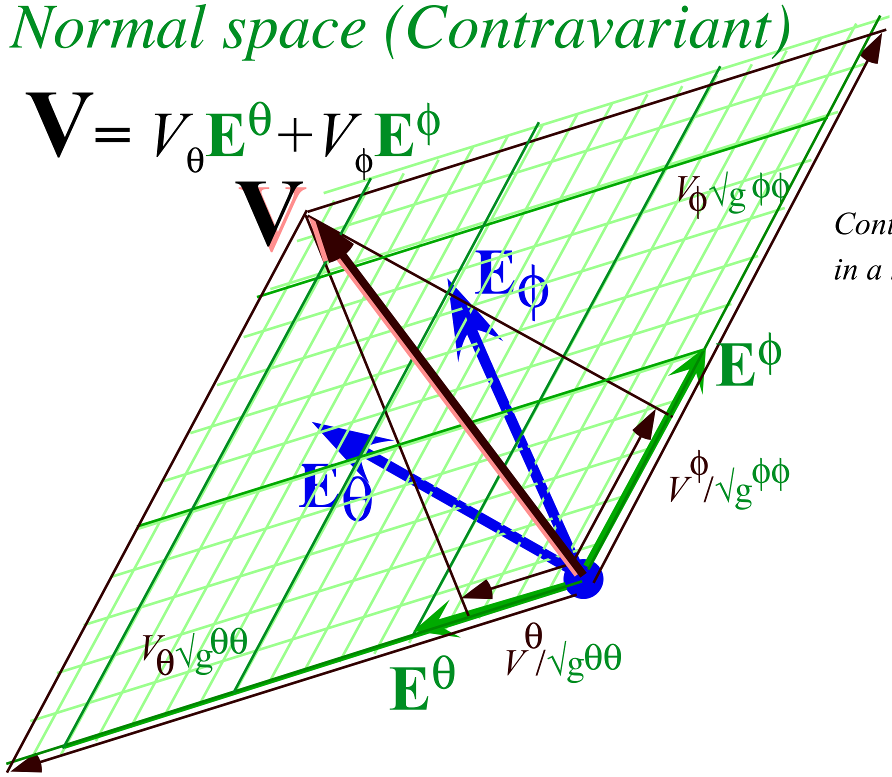

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

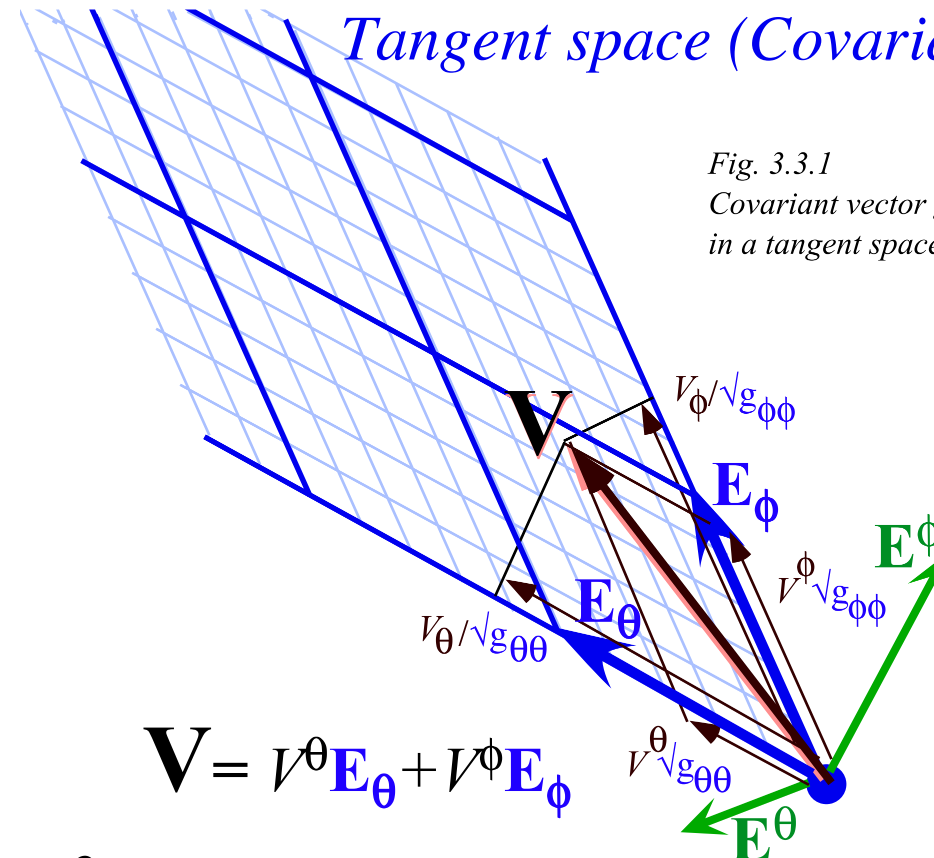

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*….

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or:} \quad \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or:} \quad \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

implies: $V_m = \dfrac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{V}_{\bar{m}}$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

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where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\qquad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

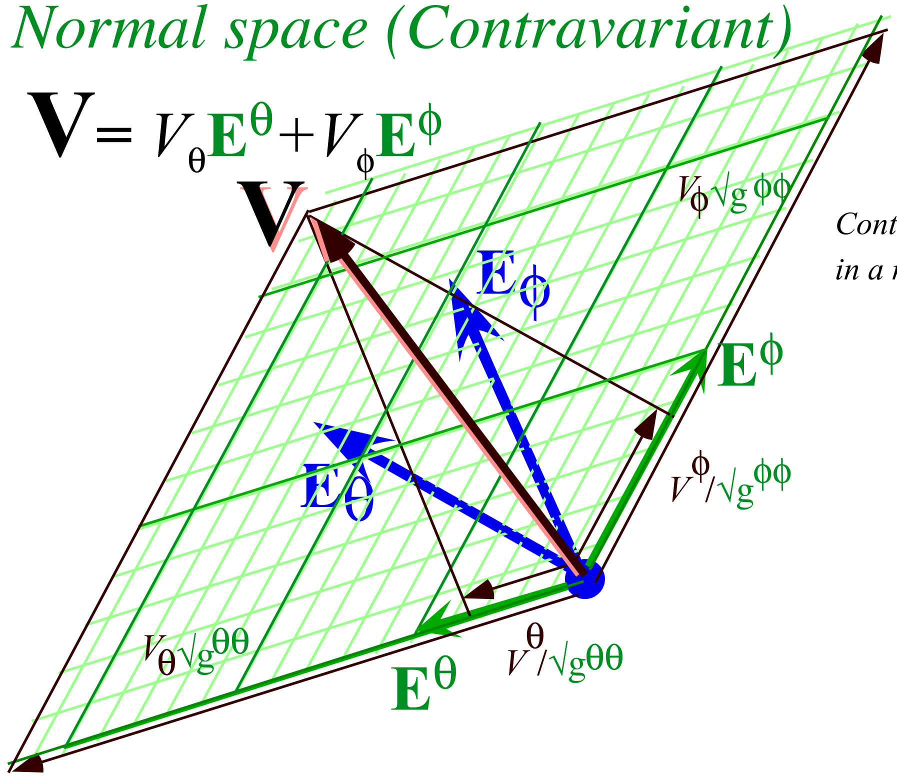

Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

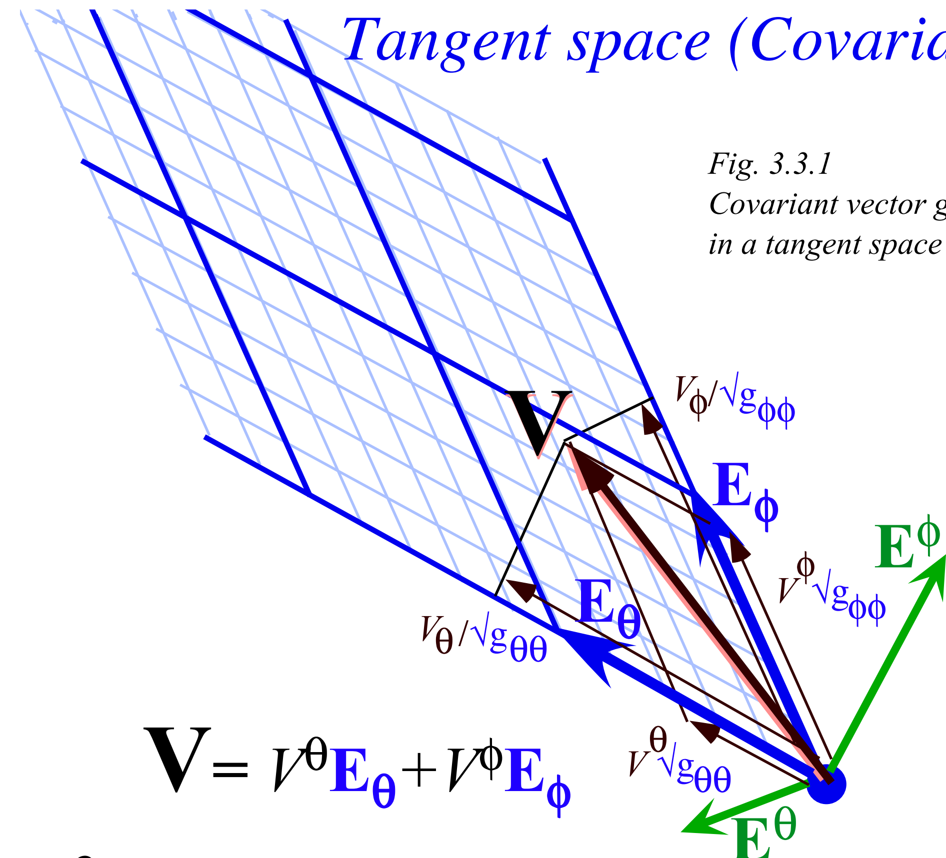

Fig. 3.3.1
Covariant vector geometry
in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}} \quad \text{implies: } V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}| \quad \text{implies: } \langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

Tangent space vs. Normal space

→ *Metric g_{mn} tensor geometric relations to length, area, and volume*

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \; , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \; .$$

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"Mixed" covariant-contravariant metric components

$$g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = \delta_m^n = \begin{cases} 0 \;\text{ if: } m \neq n \\ 1 \;\text{ if: } m = n \end{cases}$$

Caution: δ_{mn} is g_{mn} and <u>not</u> δ_n^m in GCC.

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Metric coefficients express covariant unitary vectors in terms of contras and vice-versa

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Co-and-Contra vector and tensor components are related by g-transformation. (So are g's themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \ , \quad T^{mm'} = g^{mn}g^{m'n'}T_{nn'} \ , \ etc.$$

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$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \;, \quad T^{mm'} = g^{mn}g^{m'n'}V_{nn'} \;, \; etc.$$

Diagonal square roots $\sqrt{g_{mm}}$ are the lengths of the covariant unitary vectors. $\left|\mathbf{E_m}\right| = \sqrt{\mathbf{E_m} \bullet \mathbf{E_m}} = \sqrt{g_{mm}}$

$$\left|\mathbf{E^m}\right| = \sqrt{\mathbf{E^m} \bullet \mathbf{E^m}} = \sqrt{g^{mm}}$$

tangent space area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1E_1,V^2E_2\right)=V^1V^2\left|\mathbf{E_1}\times\mathbf{E_2}\right|=V^1V^2\sqrt{\left(\mathbf{E_1}\times\mathbf{E_2}\right)\bullet\left(\mathbf{E_1}\times\mathbf{E_2}\right)}$$

$$Area\left(V^1E_1,V^2E_2\right)=V^1V^2\sqrt{\left(\mathbf{E_1}\bullet\mathbf{E_1}\right)\left(\mathbf{E_2}\bullet\mathbf{E_2}\right)-\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)}$$

$$=V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{12}}=V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}}$$

3D Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2,V^3\mathbf{E}_3\right)=V^1V^2V^3\left|\mathbf{E_1}\times\mathbf{E_2}\bullet\mathbf{E_3}\right|=V^1V^2V^3\det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3}\\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3}\\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3}\end{vmatrix}$$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}=\begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1}\\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2}\\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3}\end{pmatrix}\bullet\begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3}\\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3}\\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3}\end{pmatrix}=J^T\bullet J$$

Determinant product ($det|A|\ det|B| = det|A\bullet B|$) and symmetry ($det|A^T| = det|A|$) gives

$$Volume\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2,V^3\mathbf{E}_3\right)=V^1V^2V^3\det\left|J\right|=V^1V^2V^3\sqrt{\det\left|g\right|}$$