

Lecture 13

Tue. 10.13.2015

Complex Variables, Series, and Field Coordinates I.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great \$Interest\$)

How good are those power series?

Taylor-Maclaurin series, imaginary interest, and complex exponentials

Lecture 14 Tue. 10.15
starts here

2. What good are complex exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

Easy rotation and “dot” or “cross” products

3. Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

4. Riemann-Cauchy relations (What's analytic? What's not?)

Easy 2D curvilinear coordinate discovery

Easy 2D circulation and flux integrals

Easy 2D monopole, dipole, and 2^n -pole analysis

Easy 2^n -multipole field and potential expansion

Easy stereo-projection visualization

Cauchy integrals, Laurent-Maclaurin series

5. Mapping and Non-analytic 2D source field analysis

1. Complex numbers provide "automatic trigonometry"

2. Complex numbers add like vectors.

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

4. Complex products provide 2D rotation operations.

5. Complex products provide 2D “dot”(•) and “cross”(x) products.

6. Complex derivative contains “divergence”(∇•F) and “curl”(∇x F) of 2D vector field

7. Invent source-free 2D vector fields [∇•F=0 and ∇x F=0]

8. Complex potential ϕ contains “scalar”(F=∇Φ) and “vector”(F=∇xA) potentials

The **half-n'-half** results: (Riemann-Cauchy Derivative Relations)

9. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

10. Complex integrals $\int f(z)dz$ count 2D “circulation”(∫F•dr) and “flux”(∫Fxdx)

11. Complex integrals define 2D **monopole** fields and potentials

12. Complex derivatives give 2D dipole fields

13. More derivatives give 2D 2^N -pole fields...

14. ...and 2^N -pole multipole expansions of fields and potentials...

15. ...and Laurent Series...

16. ...and non-analytic source analysis.

Lecture 15 Thur. 10.17
starts here

The Story of e (A Tale of Great \$Interest\$)

Simple *interest* at some rate r based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time t later they would pay you $p(t)=(1+r\cdot t)p(0)$.

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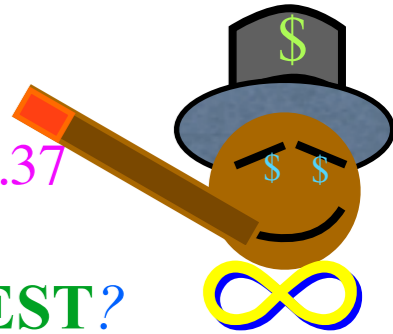
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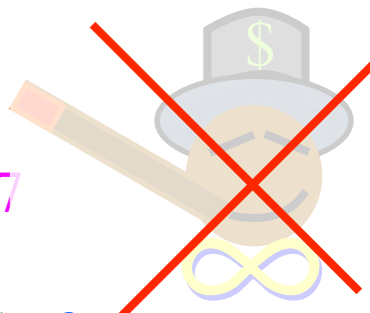
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$$p^{\frac{1}{2}}(t) = (1 + r \cdot \frac{t}{2})^2 p(0) = \left(\frac{3}{2}\right)^2 \cdot 1 = \frac{9}{4} = 2.25$$

+25¢

$$p^{\frac{1}{3}}(t) = (1 + r \cdot \frac{t}{3})^3 p(0) = \left(\frac{4}{3}\right)^3 \cdot 1 = \frac{64}{27} = 2.37$$

+12¢

$$p^{\frac{1}{4}}(t) = (1 + r \cdot \frac{t}{4})^4 p(0) = \left(\frac{5}{4}\right)^4 \cdot 1 = \frac{625}{256} = 2.44$$

+7¢

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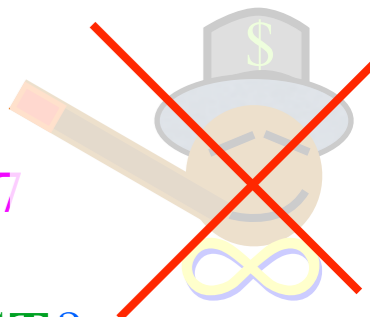
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$$\text{Monthly: } p^{\frac{1}{12}}(t) = (1 + r \cdot \frac{t}{12})^{12} p(0) = \left(\frac{13}{12}\right)^{12} \cdot 1 = 2.613$$

$$\text{Weekly: } p^{\frac{1}{52}}(t) = (1 + r \cdot \frac{t}{52})^{52} p(0) = \left(\frac{53}{52}\right)^{52} \cdot 1 = 2.693$$

$$\text{Daily: } p^{\frac{1}{365}}(t) = (1 + r \cdot \frac{t}{365})^{365} p(0) = \left(\frac{366}{365}\right)^{365} \cdot 1 = 2.7145$$

$$\text{Hrly: } p^{\frac{1}{8760}}(t) = (1 + r \cdot \frac{t}{8760})^{8760} p(0) = \left(\frac{8761}{8760}\right)^{8760} \cdot 1 = 2.7181$$

NOT!!



Interest product formula is really inefficient: 10^6 products for 6-figures! .. 10^9 products for 9 ...

$$p^{1/m}(1) = \left(1 + \frac{1}{m}\right)^m \xrightarrow{m \rightarrow \infty} \mathbf{2.718281828459..} = e$$

Let: $m \cdot r \cdot t = n$

$$\left(1 + \frac{1}{m}\right)^{m \cdot r \cdot t} \xrightarrow{m \rightarrow \infty} e^{r \cdot t}$$

or: $1/m = r \cdot t / n$

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Can improve computational efficiency using binomial theorem:

$$(x + y)^n = x^n + n \cdot x^{n-1}y + \frac{n(n-1)}{2!} x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3}y^3 + \dots + n \cdot xy^{n-1} + y^n$$

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Define: *Factorials(!)*:
 $0! = 1 = 1!$, $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3, \dots$

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Precision order:	(o=1)-e-series	= 2.00000	= 1+1
	(o=2)-e-series	= 2.50000	= 1+1+1/2
	(o=3)-e-series	= 2.66667	= 1+1+1/2+1/6
	(o=4)-e-series	= 2.70833	= 1+1+1/2+1/6+1/24
	(o=5)-e-series	= 2.71667	= 1+1+1/2+1/6+1/24+1/120
	(o=6)-e-series	= 2.71805	= 1+1+1/2+1/6+1/24+1/120+1/720
	(o=7)-e-series	= 2.71825	
	(o=8)-e-series	= 2.71828	

About 12 summed quotients for 6-figure precision (A lot better!)

Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{ etc.}$ Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + \dots + c_nt^n +$$

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Rate of change of position $x(t)$ is *velocity* $v(t)$.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt}x(t) = 0 + c_1 + 2c_2t + 3c_3t^2 + 4c_4t^3 + 5c_5t^4 + \dots + nc_nt^{n-1} +$$

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Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + \dots + c_nt^n +$$

Rate of change of position $x(t)$ is *velocity* $v(t)$.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt}x(t) = 0 + c_1 + 2c_2t + 3c_3t^2 + 4c_4t^3 + 5c_5t^4 + \dots + nc_nt^{n-1} +$$

Change of velocity $v(t)$ is *acceleration* $a(t)$.

Set $t=0$ to get $c_2 = \frac{1}{2}a(0)$.

$$a(t) = \frac{d}{dt}v(t) = 0 + 2c_2 + 2 \cdot 3c_3t + 3 \cdot 4c_4t^2 + 4 \cdot 5c_5t^3 + \dots + n(n-1)c_nt^{n-2} +$$

Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{ etc.}$

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + \dots + c_nt^n +$$

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Change of acceleration $a(t)$ is *jerk* $j(t)$. (*Jerk* is NASA term.)

Set $t=0$ to get $c_3 = \frac{1}{3!}j(0)$.

$$j(t) = \frac{d}{dt}a(t) = 0 + 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4t + 3 \cdot 4 \cdot 5c_5t^2 + \dots + n(n-1)(n-2)c_nt^{n-3} +$$

Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{ etc.}$

Set $t=0$ to get $c_0 = x(0)$.

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Change of jerk $j(t)$ is *inauguration* $i(t)$. (Be silly like NASA!)

Set $t=0$ to get $c_4 = \frac{1}{4!}i(0)$.

$$i(t) = \frac{d}{dt}j(t) = 0 + 2 \cdot 3 \cdot 4c_4 + 2 \cdot 3 \cdot 4 \cdot 5c_5t + \dots + n(n-1)(n-2)(n-3)c_nt^{n-4} +$$

Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{ etc.}$

Set $t=0$ to get $c_0 = x(0)$.

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Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!}a(0)t^2 + \frac{1}{3!}j(0)t^3 + \frac{1}{4!}i(0)t^4 + \frac{1}{5!}r(0)t^5 + \dots + \frac{1}{n!}x^{(n)}t^n +$$

Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{ etc.}$

Set $t=0$ to get $c_0 = x(0)$.

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$$x(t) = x(0) + v(0)t + \frac{1}{2!}a(0)t^2 + \frac{1}{3!}j(0)t^3 + \frac{1}{4!}i(0)t^4 + \frac{1}{5!}r(0)t^5 + \dots + \frac{1}{n!}x^{(n)}t^n +$$

Good old UP I formula!

Power Series Good! Need general power series development

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Set $t=0$ to get $c_0 = x(0)$.

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Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \dots + n c_n t^{n-1} +$$

Change of velocity $v(t)$ is *acceleration* $a(t)$.

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$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \dots + n(n-1)c_n t^{n-2} +$$

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Change of jerk $j(t)$ is *inauguration* $i(t)$. (Be silly like NASA!)

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Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \dots + \frac{1}{n!} x^{(n)} t^n +$$

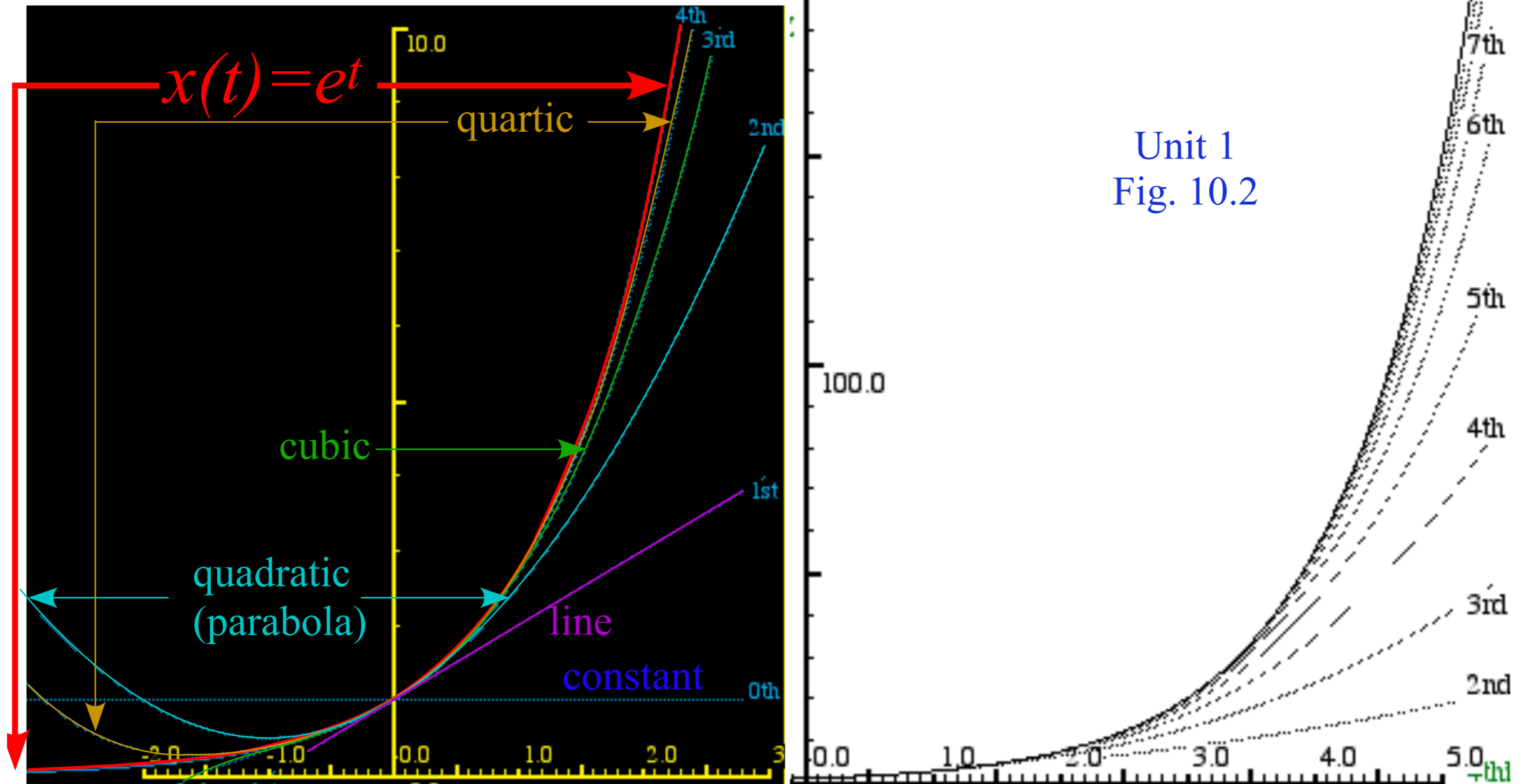
Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \dots$

Good old UP I formula!

gives *exponential*:

$$e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} t^5 + \dots + \frac{1}{n!} t^n +$$

But, how good are power series?



Unit 1
Fig. 10.2

Gives Maclaurin (or Taylor) power series

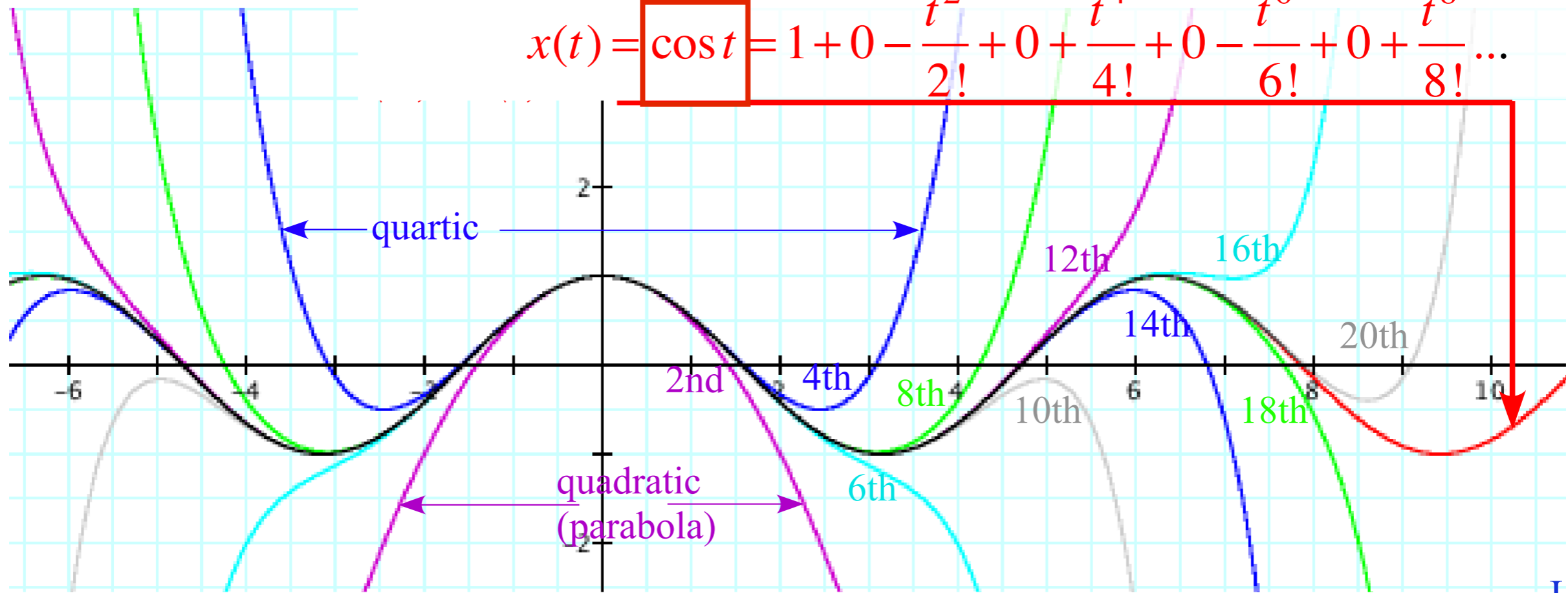
$$x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \dots + \frac{1}{n!} x^{(n)} t^n +$$

Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \dots$

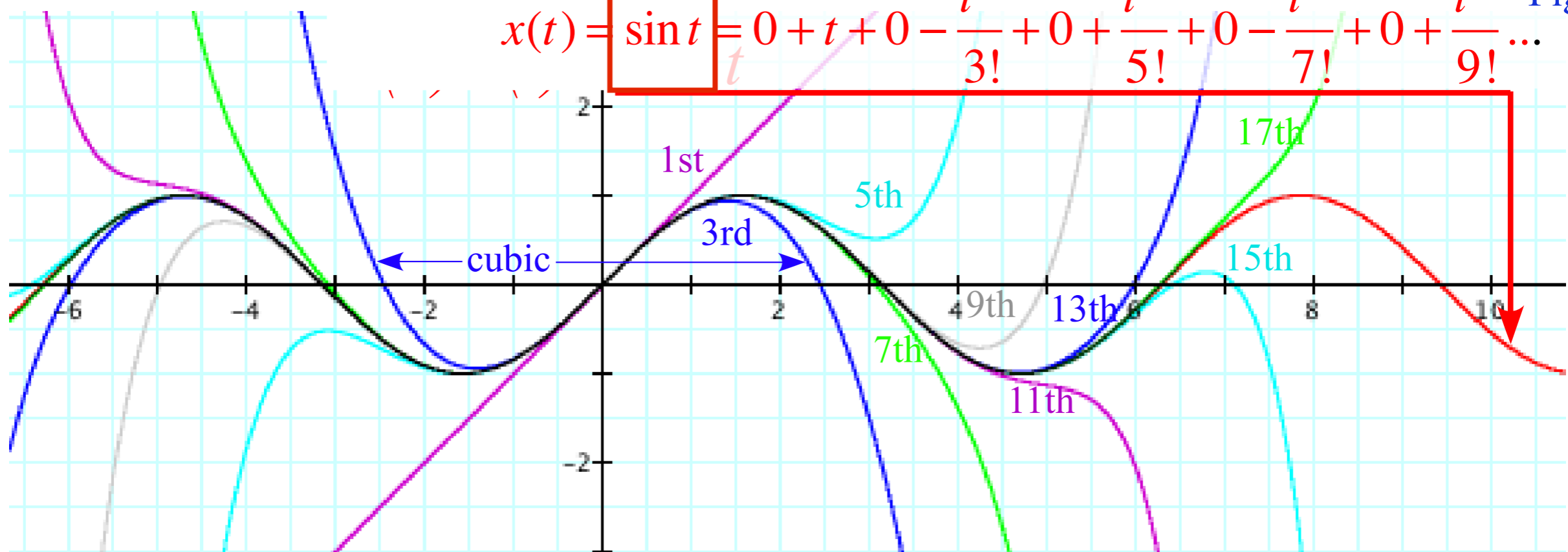
gives exponential:
$$e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} t^5 + \dots + \frac{1}{n!} t^n +$$

How good are power series? Depends...

$$x(t) = \boxed{\cos t} = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \dots$$



$$x(t) = \boxed{\sin t} = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \dots$$



Unit 1
Fig. 10.3

1. The Story of e (A Tale of Great \$Interest\$)

How good are those power series?

Taylor-Maclaurin series,



imaginary interest, and complex exponentials

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern*: $i^0=1, i^1=i, i^2=-1, i^3=-i, i^4=1, etc...$

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots && \text{(From exponential series)} \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots && (i = \sqrt{-1} \text{ implies: } i^1=i, i^2=-1, i^3=-i, i^4=+1, i^5=i, \dots) \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \dots \right) \end{aligned}$$

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

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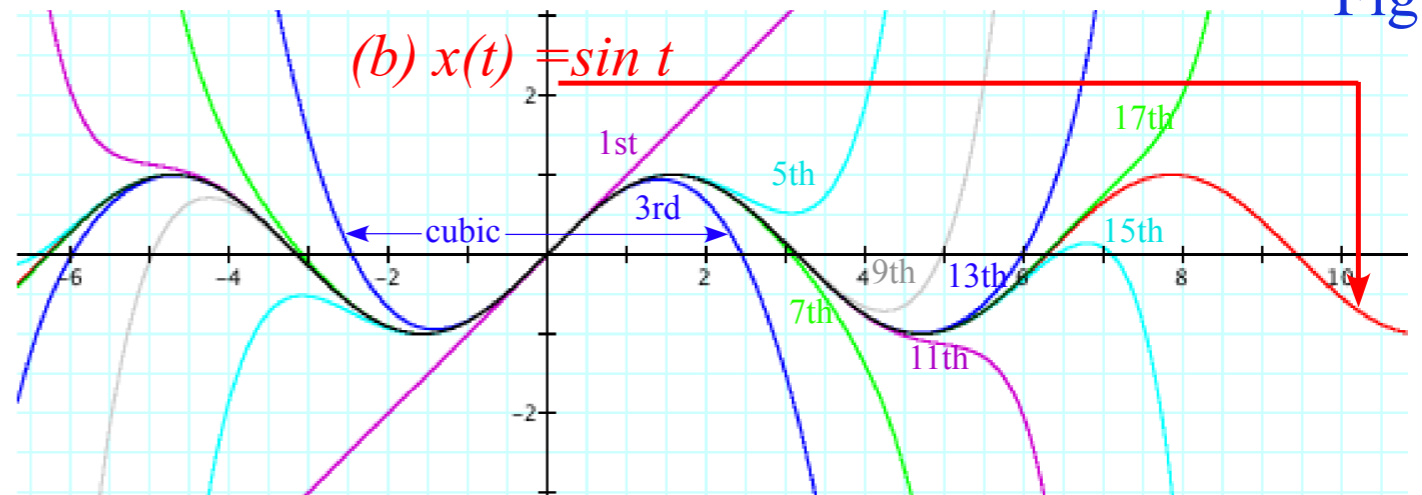
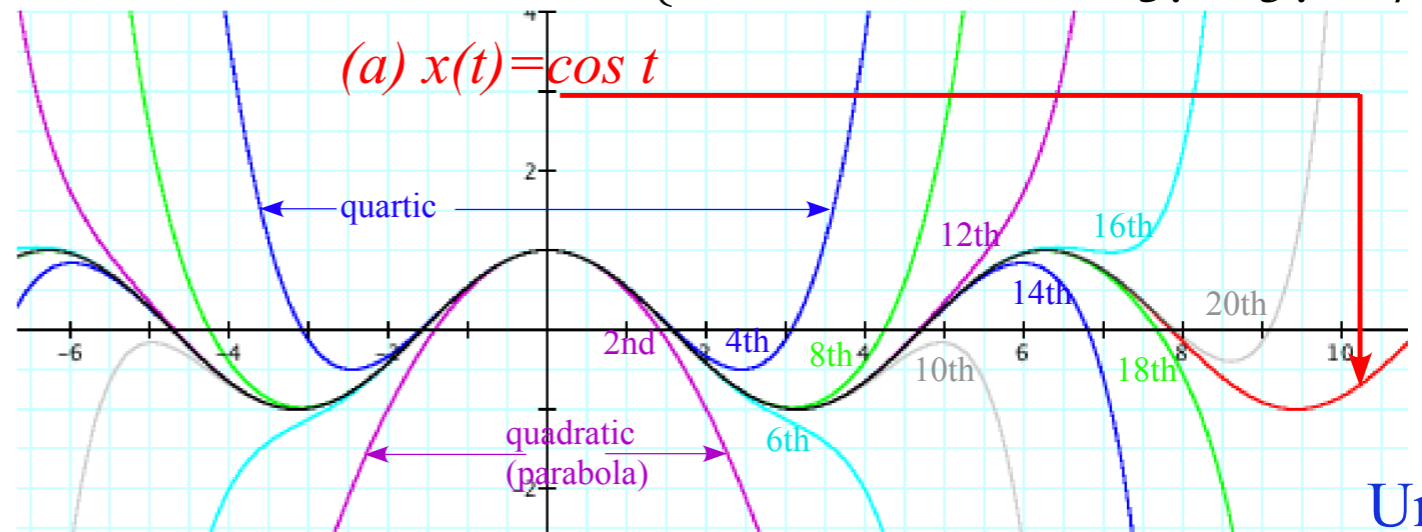
$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \quad (\text{From exponential series})$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots \quad (i = \sqrt{-1} \text{ implies: } i^1=i, i^2=-1, i^3=-i, i^4=+1, i^5=i, \dots)$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \dots \right) \quad \text{To match series for } \begin{cases} \text{cosine : } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \text{sine : } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{cases}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Euler-DeMoivre Theorem



Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern*: $i^0=1, i^1=i, i^2=-1, i^3=-i, i^4=1, etc...$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \quad (\text{From exponential series})$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots \quad (i = \sqrt{-1} \text{ implies: } i^1=i, i^2=-1, i^3=-i, i^4=+1, i^5=i, \dots)$$

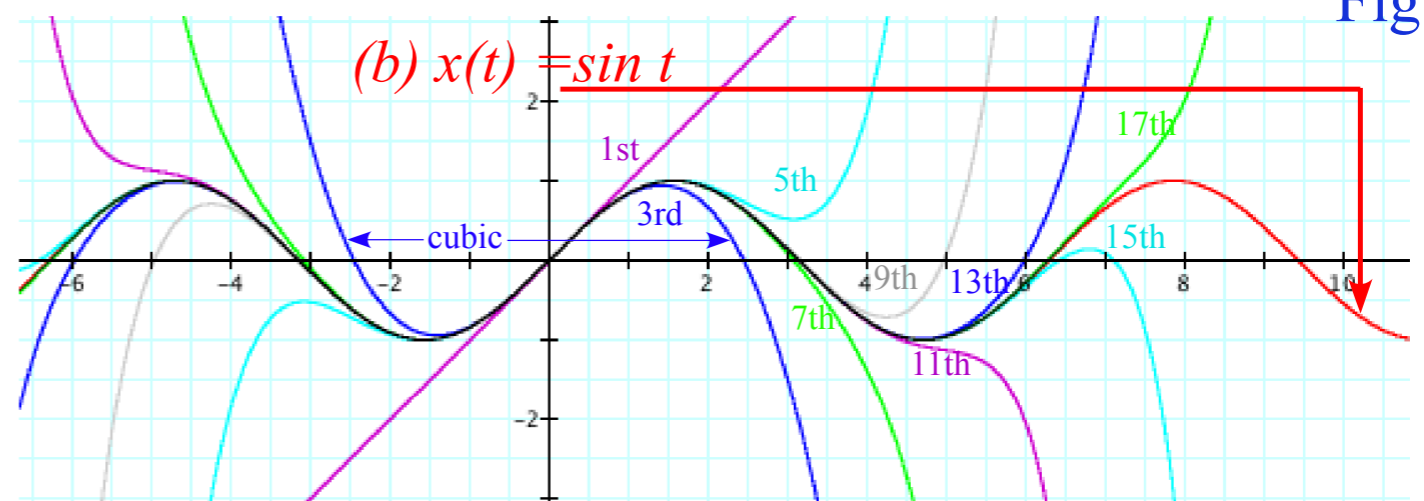
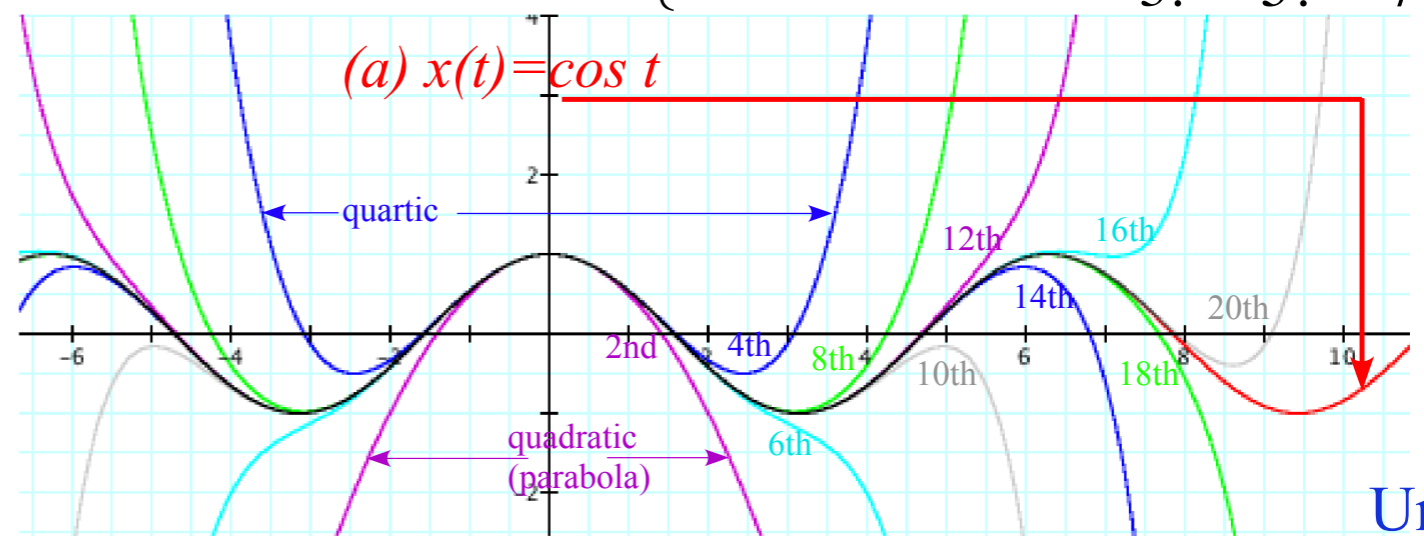
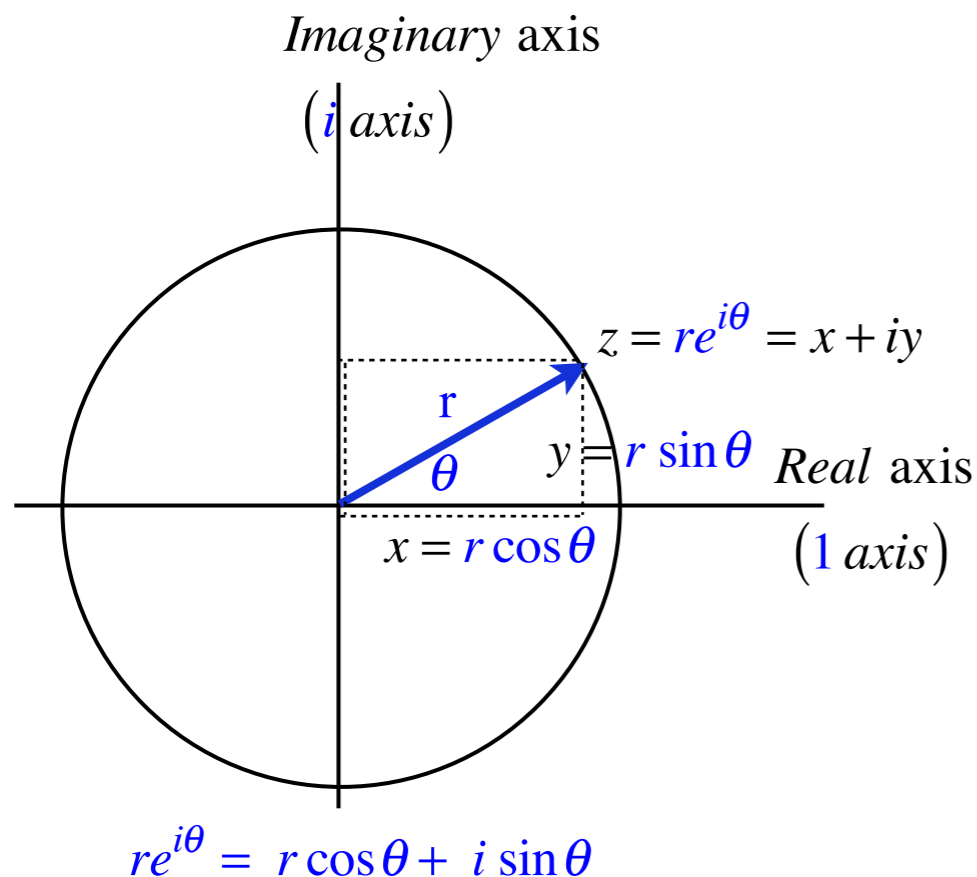
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \dots \right)$$

To match series for

$$\begin{cases} \text{cosine : } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \text{sine : } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{cases}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Euler-DeMoivre Theorem



2. *What Good Are Complex Exponentials?*

Easy trig



Easy 2D vector analysis



Easy oscillator phase analysis

Easy rotation and “dot” or “cross” products

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember $\cos(a+b)$ or $\sin(a+b)$? Just factor $e^{i(a+b)} = e^{ia} e^{ib} \dots$

$$\begin{aligned} e^{i(a+b)} &= e^{ia} e^{ib} \\ \cos(a+b) + i \sin(a+b) &= (\cos a + i \sin a) (\cos b + i \sin b) \\ \boxed{\cos(a+b)} + i \boxed{\sin(a+b)} &= \boxed{[\cos a \cos b - \sin a \sin b]} + i \boxed{[\sin a \cos b + \cos a \sin b]} \end{aligned}$$

What Good Are Complex Exponentials?

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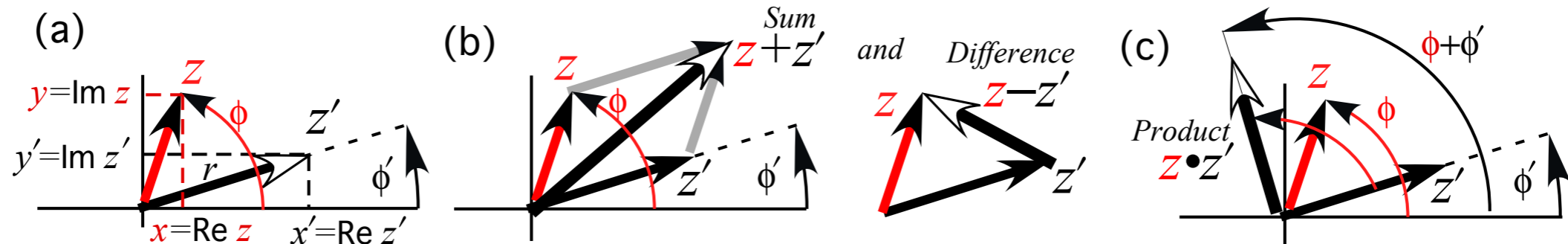
$$e^{i(a+b)} = e^{ia} e^{ib}$$

$$\cos(a+b) + i \sin(a+b) = (\cos a + i \sin a) (\cos b + i \sin b)$$

$$\boxed{\cos(a+b)} + i \boxed{\sin(a+b)} = \boxed{[\cos a \cos b - \sin a \sin b]} + i \boxed{[\sin a \cos b + \cos a \sin b]}$$

2. Complex numbers add like vectors. $z_{sum} = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$

$$z_{diff} = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y')$$



Unit 1
Fig. 10.6

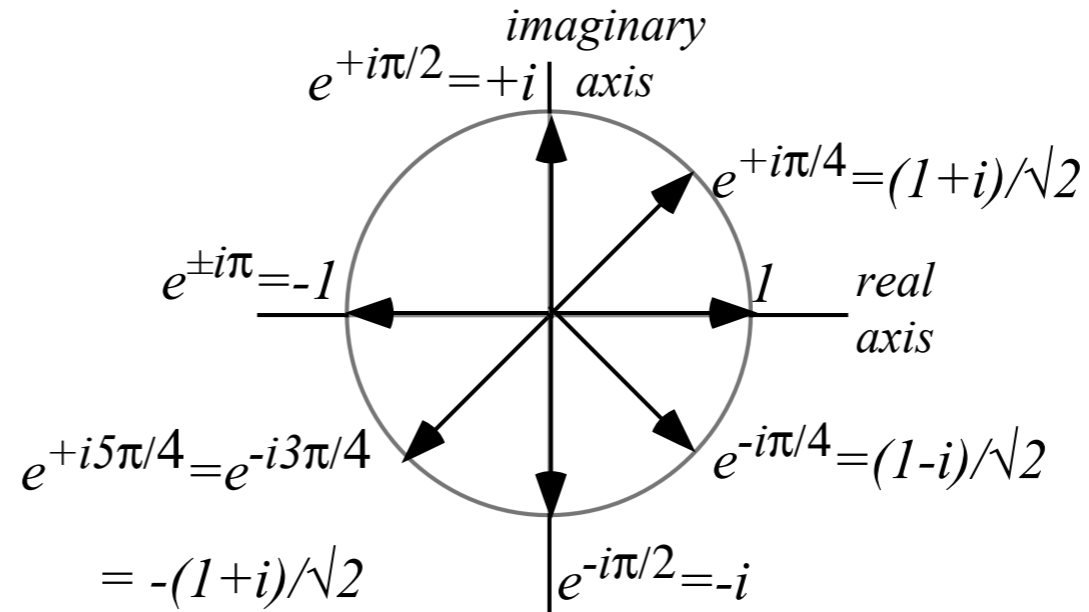
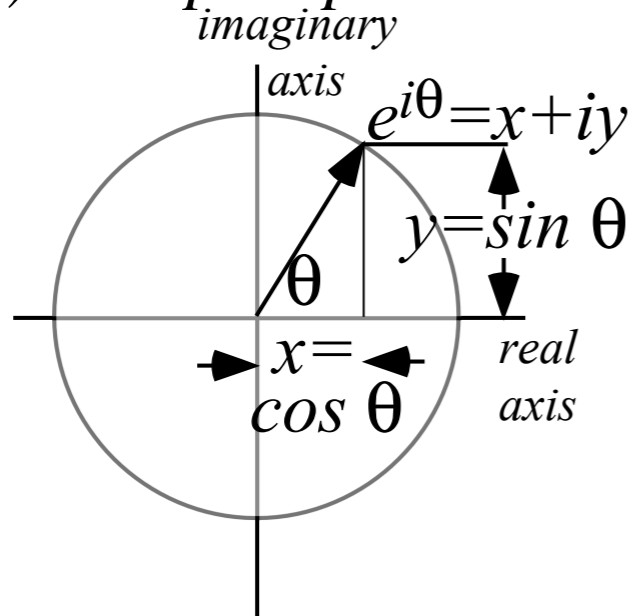
$$|z_{SUM}| = \sqrt{(z+z')^* (z+z')} = \sqrt{(re^{i\phi} + r'e^{i\phi'})^* (re^{i\phi} + r'e^{i\phi'})} = \sqrt{(re^{-i\phi} + r'e^{-i\phi'}) (re^{i\phi} + r'e^{i\phi'})}$$

$$= \sqrt{r^2 + r'^2 + rr'(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')})} = \sqrt{r^2 + r'^2 + 2rr' \cos(\phi - \phi')} \quad (\text{quick derivation of Cosine Law})$$

What Good Are Complex Exponentials? (contd.)

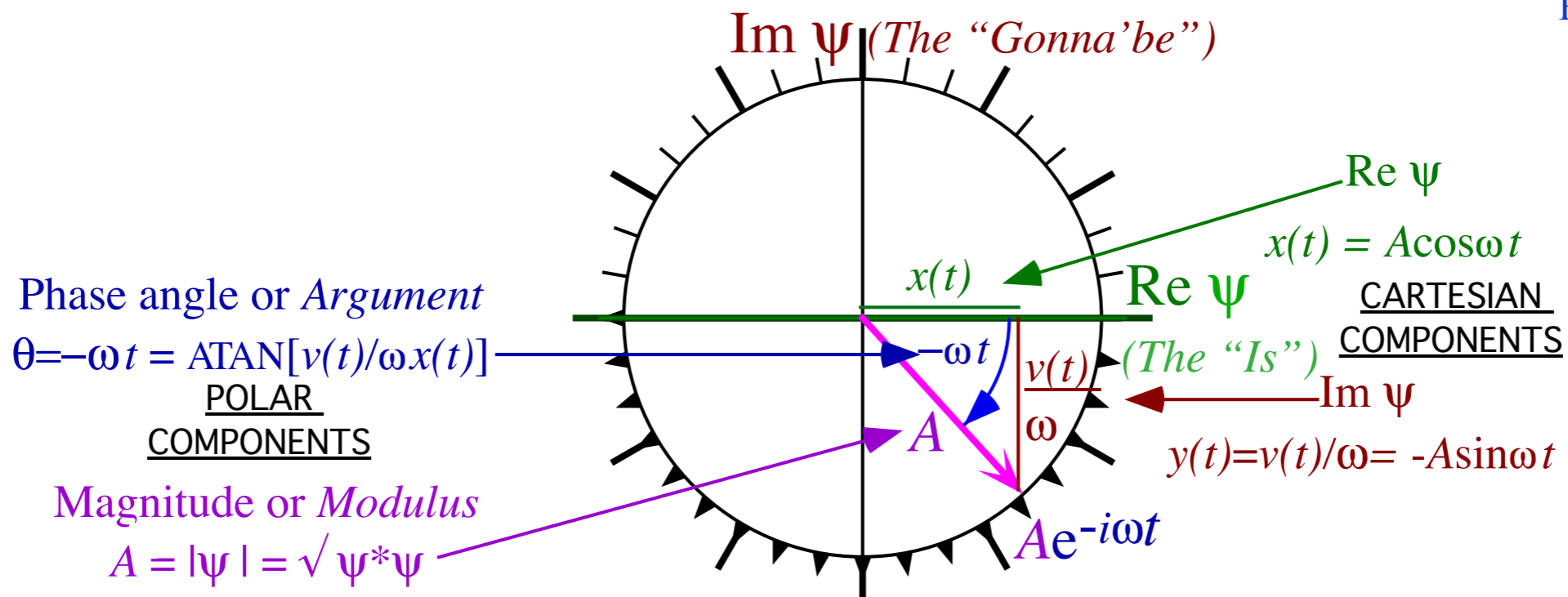
3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - iA\sin\omega t = x + iy$

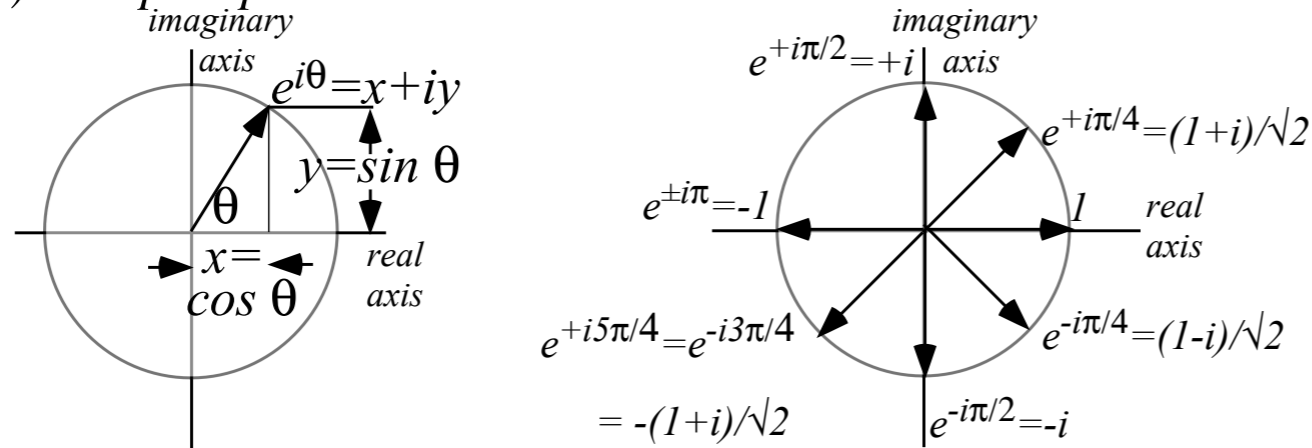
Unit 1
Fig. 10.5



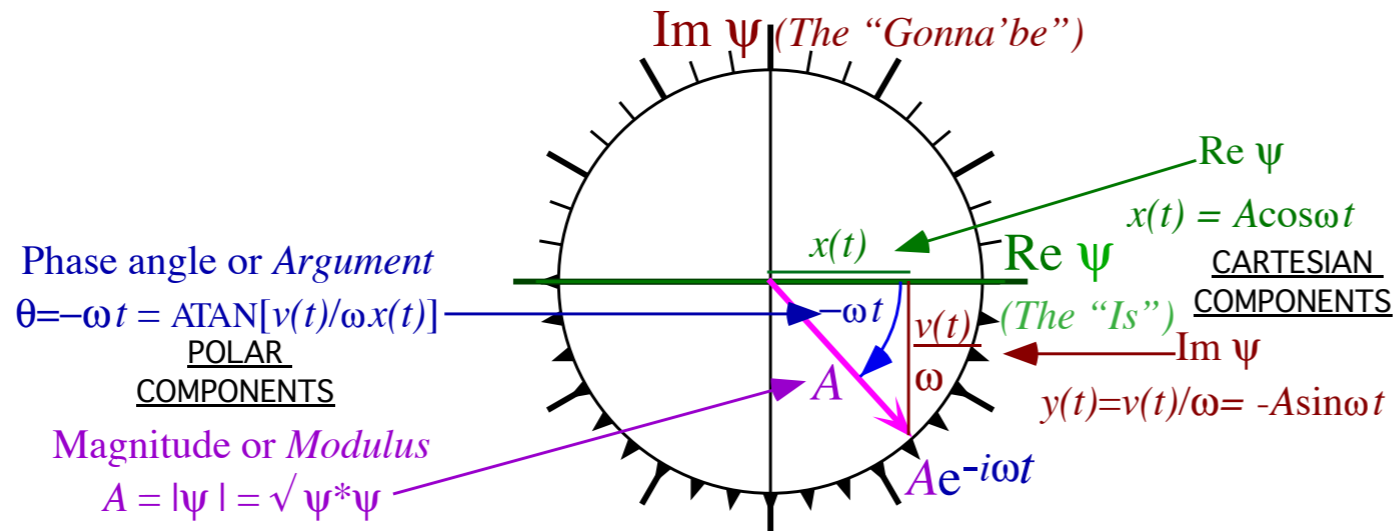
What Good Are Complex Exponentials? (contd.)

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i A\sin\omega t = x + iy$



Unit 1
Fig. 10.5

Some Rect-vs-Polar relations worth remembering

$$\text{Cartesian } (x,y) \text{ form } \begin{cases} \psi_x = \text{Re } \psi(t) = x(t) = A \cos \omega t = \frac{\psi + \psi^*}{2} \\ \psi_y = \text{Im } \psi(t) = \frac{v(t)}{\omega} = -A \sin \omega t = \frac{\psi - \psi^*}{2i} \end{cases}$$

$$\psi = r e^{+i\theta} = r e^{-i\omega t} = r(\cos \omega t - i \sin \omega t)$$

$$\psi^* = r e^{-i\theta} = r e^{+i\omega t} = r(\cos \omega t + i \sin \omega t)$$

$$\text{Polar } (r,\theta) \text{ form } \begin{cases} r = A = |\psi| = \sqrt{\psi_x^2 + \psi_y^2} = \sqrt{\psi^* \psi} \\ \theta = -\omega t = \arctan(\psi_y / \psi_x) \end{cases}$$

$$\cos \theta = \frac{1}{2}(e^{+i\theta} + e^{-i\theta}) \quad \text{Re } \psi = \frac{\psi + \psi^*}{2}$$

$$\sin \theta = \frac{1}{2i}(e^{+i\theta} - e^{-i\theta}) \quad \text{Im } \psi = \frac{\psi - \psi^*}{2i}$$

2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

 *Easy rotation and “dot” or “cross” products*

What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

$$e^{i\phi} \cdot z = (\cos\phi + i \sin\phi) \cdot (x + iy) = x \cos\phi - y \sin\phi + i (x \sin\phi + y \cos\phi)$$

$$\mathbf{R}_{+\phi} \cdot \mathbf{r} = (x \cos\phi - y \sin\phi) \hat{\mathbf{e}}_x + (x \sin\phi + y \cos\phi) \hat{\mathbf{e}}_y$$
$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos\phi - y \sin\phi \\ x \sin\phi + y \cos\phi \end{pmatrix}$$

What Good Are Complex Exponentials? (contd.)

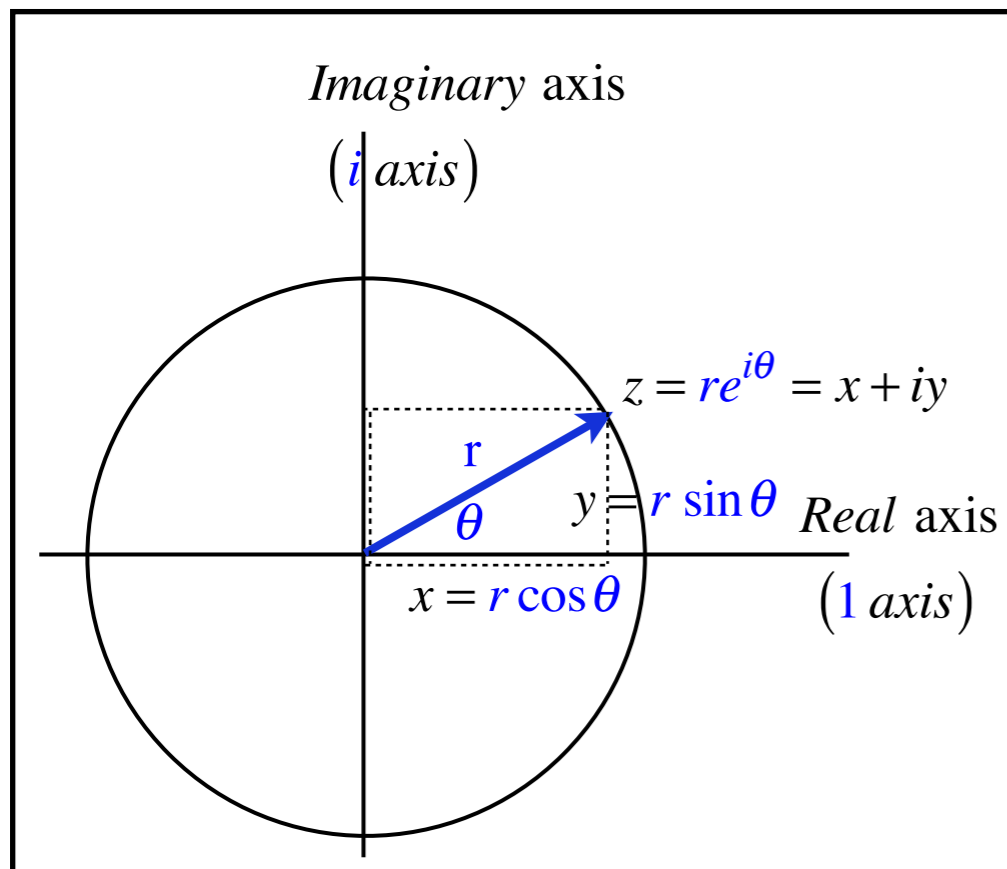
4. Complex products provide 2D rotation operations.

$$e^{i\phi} \cdot z = (\cos\phi + i \sin\phi) \cdot (x + iy) = x \cos\phi - y \sin\phi + i (x \sin\phi + y \cos\phi)$$

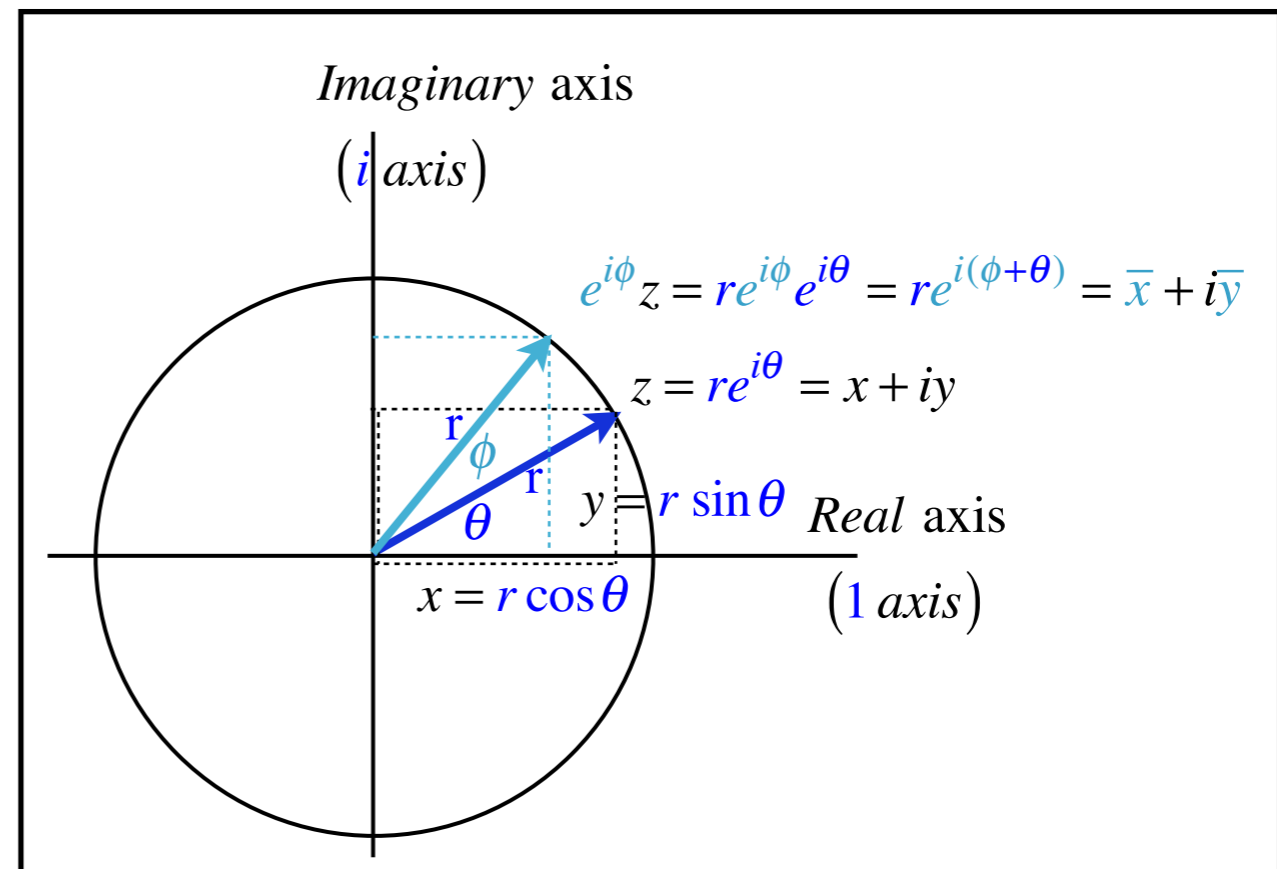
$$\mathbf{R}_{+\phi} \cdot \mathbf{r} = (x \cos\phi - y \sin\phi) \hat{\mathbf{e}}_x + (x \sin\phi + y \cos\phi) \hat{\mathbf{e}}_y$$

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$e^{i\phi}$ acts on this: $z = re^{i\theta}$



to give this: $e^{i\phi} e^{i\theta} z = re^{i\phi} e^{i\theta}$



What Good Are Complex Exponentials? (contd.)

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5. Complex products provide 2D “dot”(•) and “cross”(×) products.

Two complex numbers $A = A_x + iA_y$ and $B = B_x + iB_y$ and their “star” (*)-product $A * B$.

$$\begin{aligned} A * B &= (A_x + iA_y)^* (B_x + iB_y) = (A_x - iA_y)(B_x + iB_y) \\ &= (A_x B_x + A_y B_y) + i(A_x B_y - A_y B_x) = \mathbf{A} \cdot \mathbf{B} + i |\mathbf{A} \times \mathbf{B}|_{Z \perp(x,y)} \end{aligned}$$

Real part is scalar or “dot”(•) product $\mathbf{A} \cdot \mathbf{B}$.

Imaginary part is vector or “cross”(×) product, but just the Z-component normal to xy-plane.

Rewrite $A * B$ in polar form.

$$\begin{aligned} A * B &= (|A| e^{i\theta_A})^* (|B| e^{i\theta_B}) = |A| e^{-i\theta_A} |B| e^{i\theta_B} = |A| |B| e^{i(\theta_B - \theta_A)} \\ &= |A| |B| \cos(\theta_B - \theta_A) + i |A| |B| \sin(\theta_B - \theta_A) = \mathbf{A} \cdot \mathbf{B} + i |\mathbf{A} \times \mathbf{B}|_{Z \perp(x,y)} \end{aligned}$$

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$$\mathbf{A} \cdot \mathbf{B} = |A| |B| \cos(\theta_B - \theta_A)$$

$$= |A| \cos\theta_A |B| \cos\theta_B + |A| \sin\theta_A |B| \sin\theta_B$$

$$= A_x B_x + A_y B_y$$

$$|\mathbf{A} \times \mathbf{B}| = |A| |B| \sin(\theta_B - \theta_A)$$

$$= |A| \cos\theta_A |B| \sin\theta_B - |A| \sin\theta_A |B| \cos\theta_B$$

$$= A_x B_y - A_y B_x$$

What Good are complex variables?



Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence” ($\nabla \cdot \mathbf{F}$) and “curl” ($\nabla \times \mathbf{F}$) of 2D vector field

Relation of (z, z^*) to $(x = \operatorname{Re}z, y = \operatorname{Im}z)$ defines a z -derivative $\frac{df}{dz}$ and “star” z^* -derivative. $\frac{df}{dz^*}$

$$z = x + iy$$

$$x = \frac{1}{2}(z + z^*)$$

$$z^* = x - iy$$

$$y = \frac{1}{2i}(z - z^*)$$

Applying
chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{i}{2} \frac{\partial f}{\partial y}$$

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7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.

Take any function $f(z)$, conjugate it (change all i 's to $-i$) to give $f^*(z^*)$ for which $\frac{df^*}{dz^*} = 0$.

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For example: if $f(z) = a \cdot z$ then $f^*(z^*) = a \cdot z^* = a(x - iy)$ is not function of z so it has zero z -derivative.

$\mathbf{F} = (F_x, F_y) = (f_x^*, f_y^*) = (a \cdot x, -a \cdot y)$ has *zero divergence*: $\nabla \cdot \mathbf{F} = 0$ and has *zero curl*: $|\nabla \times \mathbf{F}| = 0$.

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial(-ay)}{\partial y} = 0 \quad |\nabla \times \mathbf{F}|_{Z \perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial(ax)}{\partial y} = 0$$

A *DFL* field \mathbf{F} (*Divergence-Free-Laminar*)

What Good Are Complex Exponentials? (contd.)

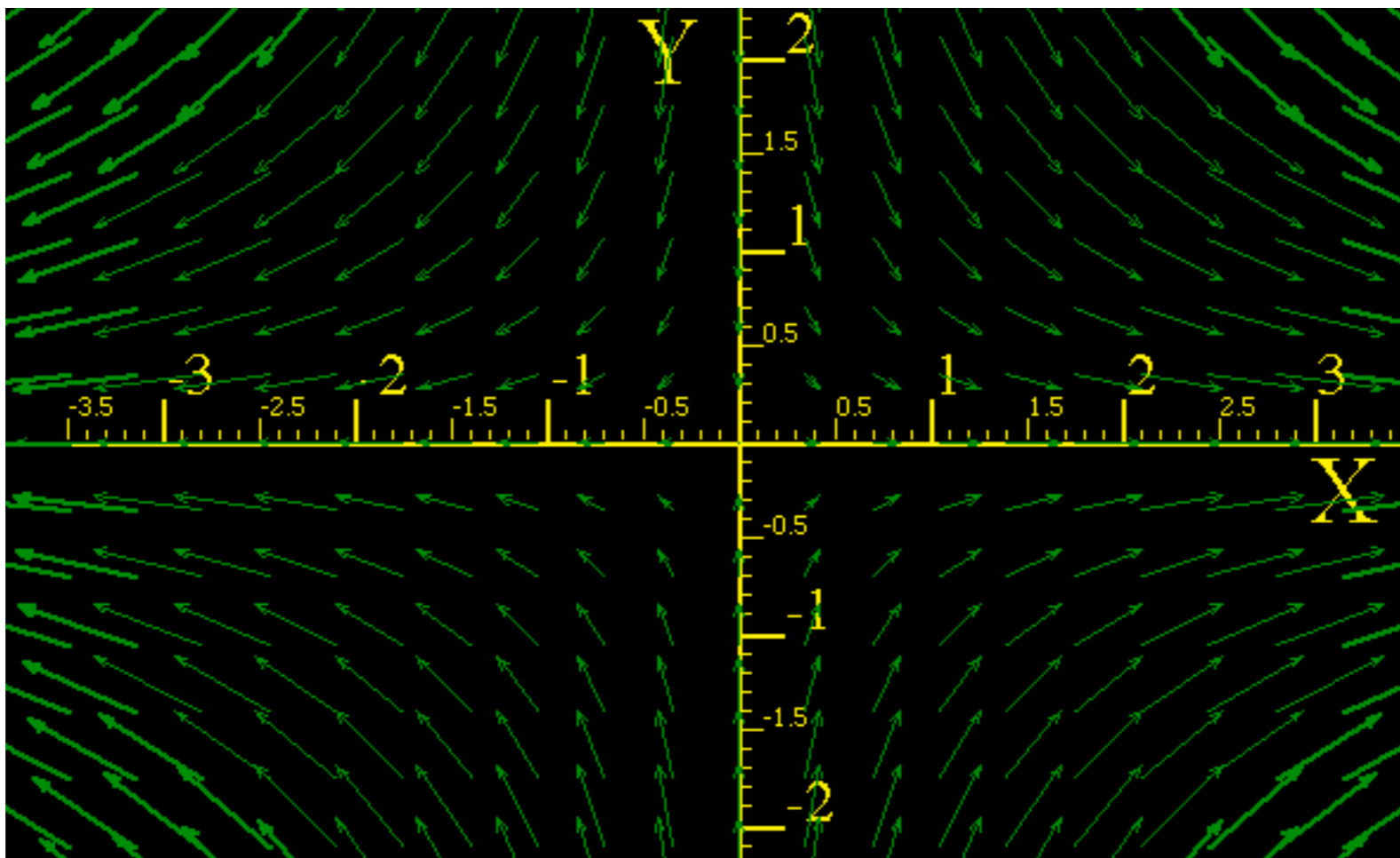
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*precursor to
Unit 1
Fig. 10.7*

$\mathbf{F} = (f_x^*, f_y^*) = (a \cdot x, -a \cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory



Easy 2D vector field-potential theory

What Good Are Complex Exponentials? (contd.)

8. Complex potential ϕ contains “scalar” ($\mathbf{F}=\nabla\Phi$) and “vector” ($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

Any *DFL* field \mathbf{F} is a gradient of a *scalar potential field* Φ or a curl of a *vector potential field* \mathbf{A} .

$$\mathbf{F} = \nabla\Phi$$

$$\mathbf{F} = \nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\Phi(x,y)+i\mathbf{A}(x,y)$ exists whose z -derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\Phi(x,y)-i\mathbf{A}(x,y)$ has z^* -derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field \mathbf{F} .

What Good Are Complex Exponentials? (contd.)

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To find $\phi=\Phi+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\text{Re } \phi = \Phi$) and imaginary ($\text{Im } \phi = \mathbf{A}$) parts.

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$$f(z)=\frac{d\phi}{dz} \Rightarrow \phi = \Phi + i\mathbf{A} = \int f \cdot dz = \int az \cdot dz = \frac{1}{2} az^2$$

What Good Are Complex Exponentials? (contd.)

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What Good Are Complex Exponentials? (contd.)

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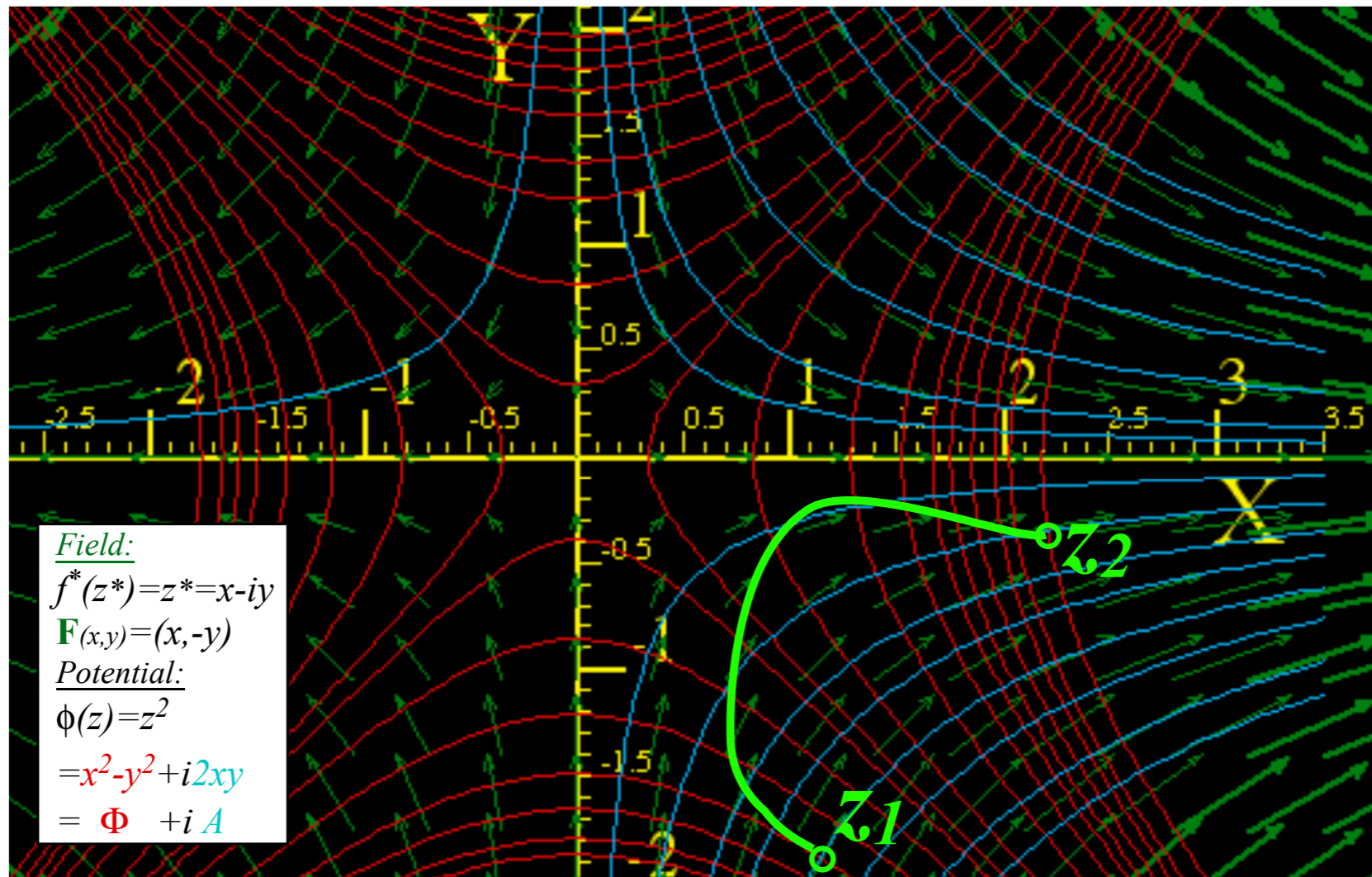
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Unit 1
Fig. 10.7



What Good Are Complex Exponentials? (contd.)

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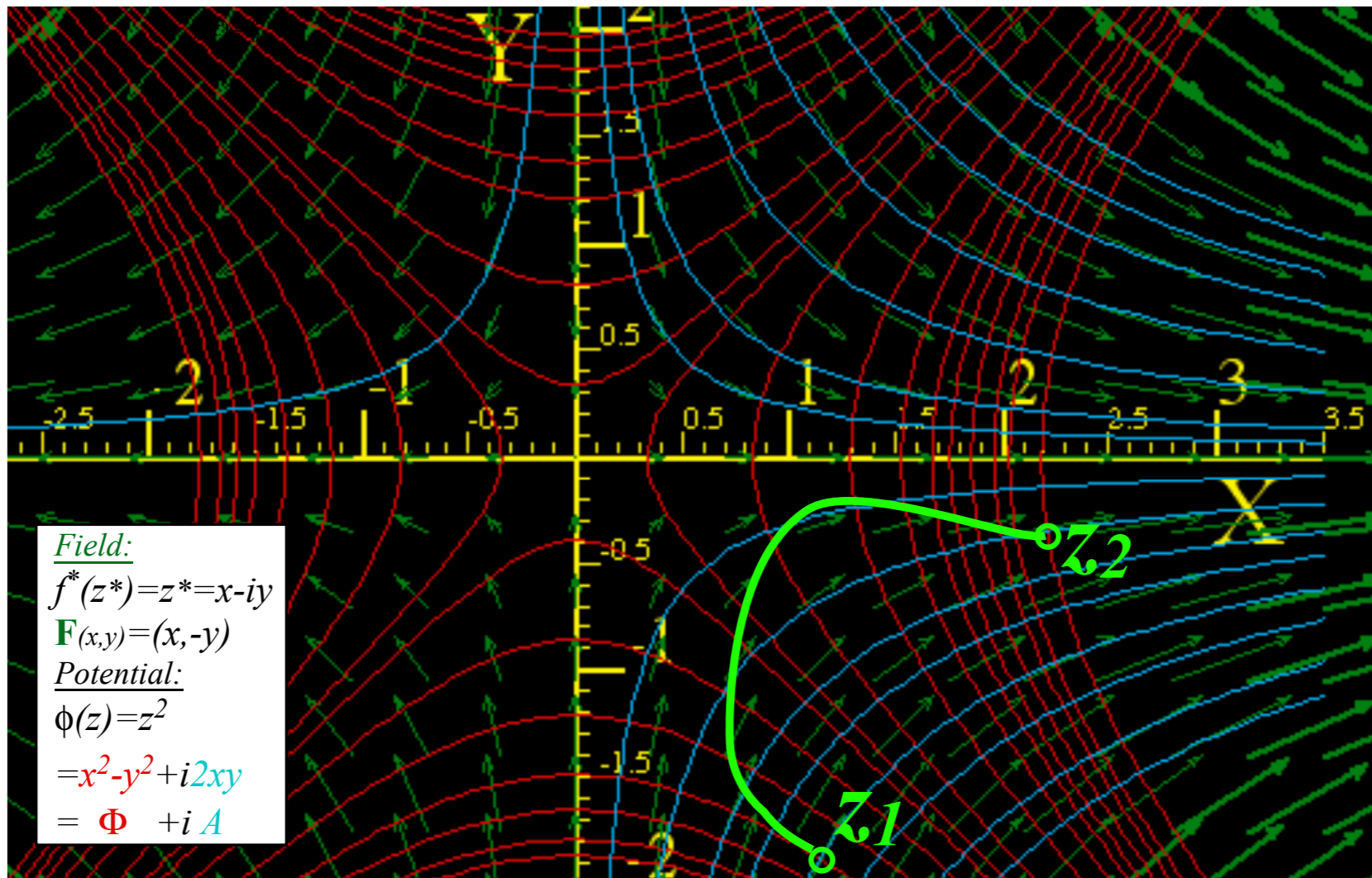
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BONUS!
Get a free
coordinate
system!



Unit 1
Fig. 10.7

The (Φ, \mathbf{A}) grid is a GCC coordinate system*:

$$q^1 = \Phi = (x^2 - y^2)/2 = \text{const.}$$

$$q^2 = \mathbf{A} = (xy) = \text{const.}$$

*Actually it's OCC.


What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

 *Easy 2D vector field-potential theory*

 *The **half-n'-half** results: (Riemann-Cauchy Derivative Relations)*

What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains “scalar” ($\mathbf{F} = \nabla \Phi$) and “vector” ($\mathbf{F} = \nabla \times \mathbf{A}$) potentials
 ...and either one (or *half-n'-half!*) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix}$ of scalar Φ and curl $\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix}$ of vector \mathbf{A} (and they're equal!)

$$f(z) = \frac{d\phi}{dz} \Rightarrow$$

$$\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - i\mathbf{A}) = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\Phi - i\mathbf{A}) = \frac{1}{2} \left(\frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial \mathbf{A}}{\partial y} - i \frac{\partial \mathbf{A}}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times \mathbf{A}$$

$$\frac{d}{dz} = \frac{1}{2} \frac{\partial}{\partial x} - \frac{i}{2} \frac{\partial}{\partial y}$$

$$\frac{d}{dz^*} = \frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y}$$

What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains “scalar” ($\mathbf{F}=\nabla\Phi$) and “vector” ($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
 ...and either one (or *half-n'-half!*) works just as well.

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Note, *mathematician definition* of force field $\mathbf{F} = +\nabla\Phi$ replaces usual physicist's definition $\mathbf{F} = -\nabla\Phi$

What Good Are Complex Exponentials? (contd.)

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Note, *mathematician definition* of force field $\mathbf{F} = +\nabla\Phi$ replaces usual physicist's definition $\mathbf{F} = -\nabla\Phi$

Given ϕ : $\phi = \Phi + i\mathbf{A} = \frac{1}{2} a(x^2 - y^2) + i axy$

find:

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y} \frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial\mathbf{A}}{\partial y} \\ -\frac{\partial\mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The *half-n'-half* result

What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains “scalar” ($\mathbf{F}=\nabla\Phi$) and “vector” ($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
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$$\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - i\mathbf{A}) = \frac{1}{2} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right) (\Phi - i\mathbf{A}) = \frac{1}{2} \left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x} \right) = \frac{1}{2} \nabla\Phi + \frac{1}{2} \nabla\times\mathbf{A}$$

Note, mathematician definition of force field $\mathbf{F} = +\nabla\Phi$ replaces usual physicist's definition $\mathbf{F} = -\nabla\Phi$

Given ϕ :
 $\phi = \Phi + i\mathbf{A}$
 $= \frac{1}{2} a(x^2 - y^2) + i axy$
The *half-n'-half* result

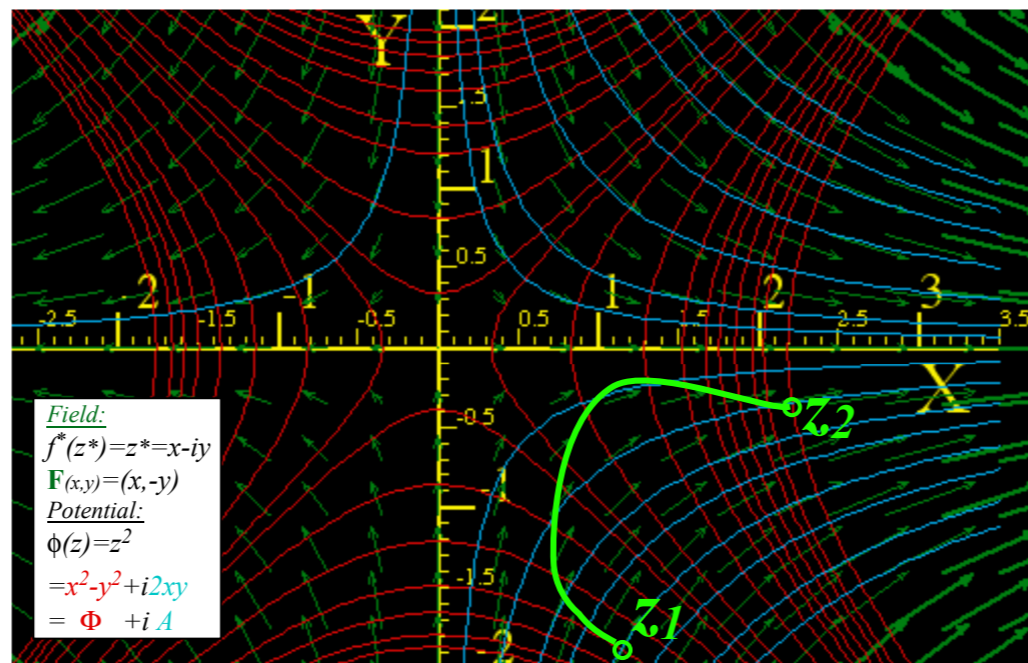
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$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y} \frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$

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Scalar *static potential lines* $\Phi = \text{const.}$ and vector *flux potential lines* $\mathbf{A} = \text{const.}$ define *DFL field-net*.



What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains “scalar” ($\mathbf{F}=\nabla\Phi$) and “vector” ($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
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The *half-n-half* result

$$\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - i\mathbf{A}) = \frac{1}{2} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right) (\Phi - i\mathbf{A}) = \frac{1}{2} \left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x} \right) = \frac{1}{2} \nabla\Phi + \frac{1}{2} \nabla\times\mathbf{A}$$

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Given ϕ : $\phi = \Phi + i\mathbf{A} = \frac{1}{2} a(x^2 - y^2) + i axy$ The *half-n-half* result

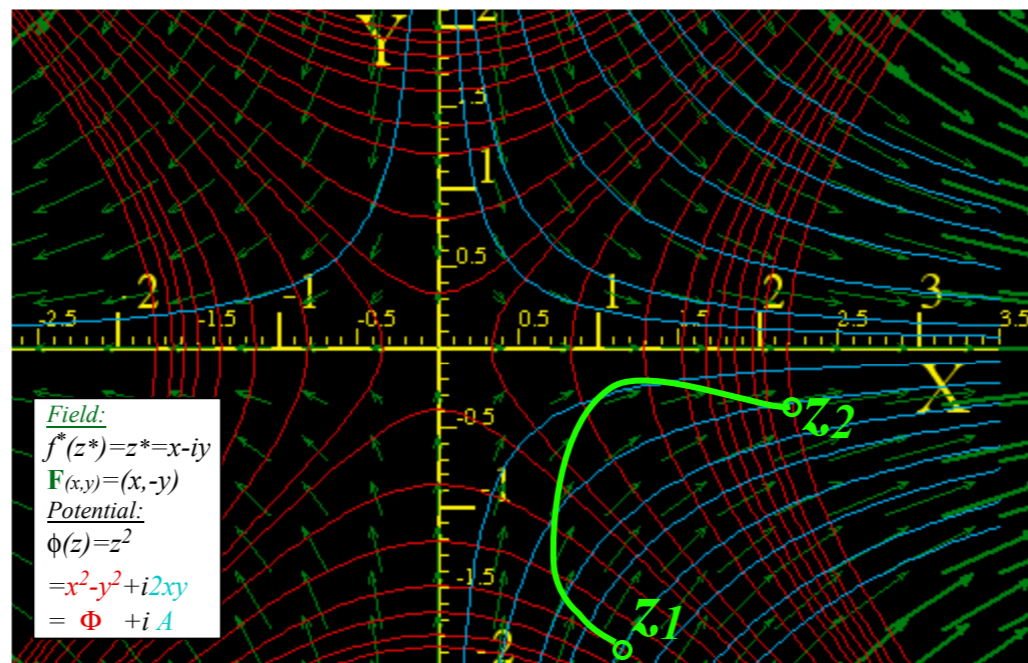
find:

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or find:

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Scalar *static potential lines* $\Phi = \text{const.}$ and vector *flux potential lines* $\mathbf{A} = \text{const.}$ define *DFL field-net*.



The *half-n-half* results
 are called
Riemann-Cauchy
Derivative Relations

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\mathbf{A}}{\partial y} \quad \text{is:} \quad \frac{\partial \text{Re}f(z)}{\partial x} = \frac{\partial \text{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial\mathbf{A}}{\partial x} \quad \text{is:} \quad \frac{\partial \text{Re}f(z)}{\partial y} = -\frac{\partial \text{Im}f(z)}{\partial x}$$

→ *4. Riemann-Cauchy conditions* *What's analytic? (...and what's not?)*

Review (z, z^*) to (x, y) transformation relations

$$\begin{aligned} z &= x + iy & x &= \frac{1}{2}(z + z^*) & \frac{df}{dz} &= \frac{\partial x}{\partial z} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2i} \frac{\partial f}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f \\ z^* &= x - iy & y &= \frac{1}{2i}(z - z^*) & \frac{df}{dz^*} &= \frac{\partial x}{\partial z^*} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{1}{2i} \frac{\partial f}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f \end{aligned}$$

Criteria for a field function $f = f_x(x, y) + i f_y(x, y)$ to be an **analytic function $f(z)$** of $z = x + iy$:

First, $f(z)$ must not be a function of $z^* = x - iy$, that is: $\frac{df}{dz^*} = 0$

This implies $f(z)$ satisfies differential equations known as the **Riemann-Cauchy conditions**

$$\begin{aligned} \frac{df}{dz^*} = 0 &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y} \right) \text{ implies: } \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y} \\ \frac{df}{dz} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) = \frac{\partial f_x}{\partial x} + i \frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x} (f_x + i f_y) = \frac{\partial}{\partial iy} (f_x + i f_y) \end{aligned}$$

Review (z,z) to (x,y) transformation relations*

$$\begin{aligned} z &= x + iy & x &= \frac{1}{2}(z + z^*) & \frac{df}{dz} &= \frac{\partial x}{\partial z} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2i} \frac{\partial f}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f \\ z^* &= x - iy & y &= \frac{1}{2i}(z - z^*) & \frac{df}{dz^*} &= \frac{\partial x}{\partial z^*} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{1}{2i} \frac{\partial f}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f \end{aligned}$$

*Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an **analytic function $f(z)$** of $z=x+iy$:*

First, $f(z)$ must not be a function of $z^=x-iy$, that is: $\frac{df}{dz^*} = 0$*

*This implies $f(z)$ satisfies differential equations known as the **Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y} \right) \text{ implies: } \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}$$

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*Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an **analytic function $f(z^*)$** of $z^*=x-iy$:*

First, $f(z^)$ must not be a function of $z=x+iy$, that is: $\frac{df}{dz} = 0$*

This implies $f(z^)$ satisfies differential equations we call **Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) = \text{implies: } \frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y} \right) = \frac{\partial f_x}{\partial x} + i \frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x} (f_x + i f_y) = -\frac{\partial}{\partial iy} (f_x + i f_y)$$

What's analytic? (...and what's not?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z=x+iy$?

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

$$\begin{aligned} f(x,y) = 2x + i4y &= 2 \frac{(z+z^*)}{2} + i4 \frac{-i(z-z^*)}{2} \\ &= z+z^* + (2z-2z^*) \\ &= 3z-z^* \end{aligned}$$

A: ***NO!*** *It's a function of z and z^* so not analytic for either.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: ***NO!*** *$r(x,y)=z^*z$ is a function of z and z^* so not analytic for either.*

What's analytic? (...and what's not?)

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Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: **NO!** $r(x,y)=z^*z$ is a function of z and z^* so not analytic for either.

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=x+iy$?

A: **YES!** $s(x,y)=(x+iy)^2 = z^2$ is analytic function of z . (Yay!)

4. Riemann-Cauchy conditions What's analytic? (...and what's not?)

 *Easy 2D circulation and flux integrals*

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n -pole analysis

Easy 2^n -multipole field and potential expansion

Easy stereo-projection visualization

What Good Are Complex Exponentials? (contd.)

9. Complex integrals $\int f(z)dz$ count 2D “circulation” ($\int \mathbf{F} \cdot d\mathbf{r}$) and “flux” ($\int \mathbf{F} \times d\mathbf{r}$)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\Phi(x_2, y_2) - \Phi(x_1, y_1)}_{\Delta\Phi} + i \underbrace{[\mathbf{A}(x_2, y_2) - \mathbf{A}(x_1, y_1)]}_{\Delta\mathbf{A}}$$

$\Delta\phi = \quad \Delta\Phi \quad + i \quad \Delta\mathbf{A}$

In *DFL*-field \mathbf{F} , $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2 .

What Good Are Complex Exponentials? (contd.)

9. Complex integrals $\int f(z)dz$ count 2D “circulation” ($\int \mathbf{F} \cdot d\mathbf{r}$) and “flux” ($\int \mathbf{F} \times d\mathbf{r}$)

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$$\Delta\phi = \Delta\Phi + i \Delta\mathbf{A}$$

In *DFL*-field \mathbf{F} , $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2 .

$$\begin{aligned} \int f(z)dz &= \int \left(f^*(z^*) \right)^* dz = \int \left(f^*(z^*) \right)^* (dx + i dy) = \int \left(f_x^* + i f_y^* \right)^* (dx + i dy) = \int \left(f_x^* - i f_y^* \right) (dx + i dy) \\ &= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx) \\ &= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \times d\mathbf{r} \cdot \hat{\mathbf{e}}_Z \\ &= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \cdot d\mathbf{r} \times \hat{\mathbf{e}}_Z \\ &= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \cdot d\mathbf{S} \quad \text{where: } d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_Z \end{aligned}$$

What Good Are Complex Exponentials? (contd.)

9. Complex integrals $\int f(z)dz$ count 2D “circulation” ($\int \mathbf{F} \cdot d\mathbf{r}$) and “flux” ($\int \mathbf{F} \times d\mathbf{r}$)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\Phi(x_2, y_2) - \Phi(x_1, y_1)}_{\Delta\Phi} + i \underbrace{[\mathbf{A}(x_2, y_2) - \mathbf{A}(x_1, y_1)]}_{\Delta\mathbf{A}}$$

$$\Delta\phi = \Delta\Phi + i \Delta\mathbf{A}$$

In *DFL*-field \mathbf{F} , $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2 .

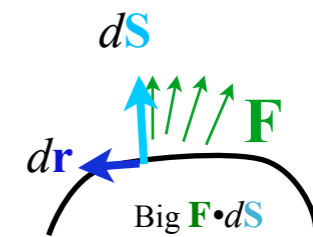
$$\int f(z)dz = \int \left(f^*(z^*) \right)^* dz = \int \left(f^*(z^*) \right)^* (dx + i dy) = \int \left(f_x^* + i f_y^* \right)^* (dx + i dy) = \int \left(f_x^* - i f_y^* \right) (dx + i dy)$$

$$= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx)$$

$$= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \times d\mathbf{r} \cdot \hat{\mathbf{e}}_z$$

$$= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \cdot d\mathbf{r} \times \hat{\mathbf{e}}_z$$

$$= \boxed{\int \mathbf{F} \cdot d\mathbf{r}} + i \boxed{\int \mathbf{F} \cdot d\mathbf{S}} \quad \text{where:} \quad d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_z$$



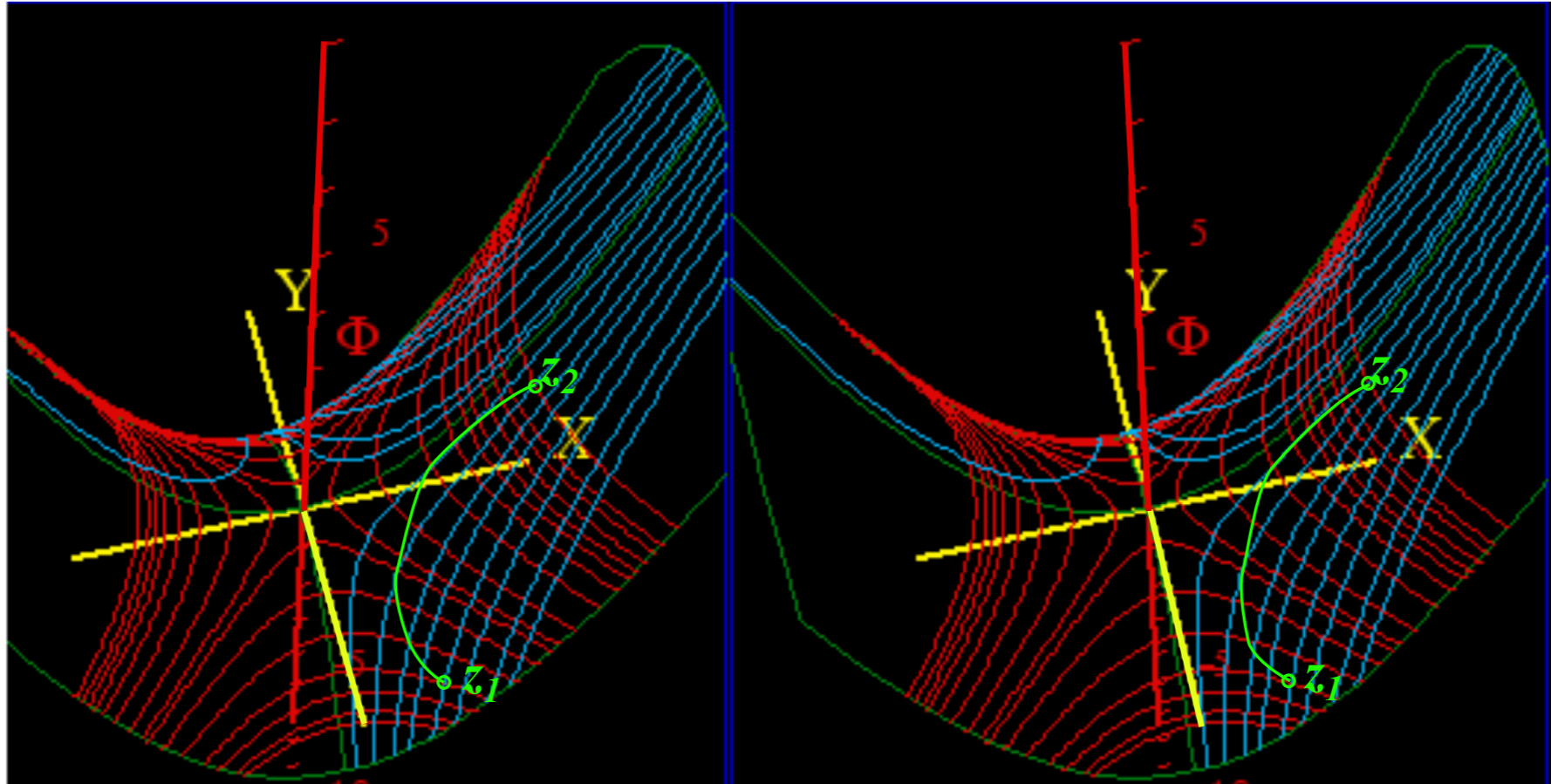
Real part $\int_1^2 \mathbf{F} \cdot d\mathbf{r} = \Delta\Phi$
 sums \mathbf{F} projections *along* path $d\mathbf{r}$ that is, *circulation* on path to get $\Delta\Phi$.

Imaginary part $\int_1^2 \mathbf{F} \cdot d\mathbf{S} = \Delta\mathbf{A}$
 sums \mathbf{F} projection *across* path $d\mathbf{r}$ that is, *flux* thru surface elements $d\mathbf{S} = d\mathbf{r} \times \mathbf{e}_z$ normal to $d\mathbf{r}$ to get $\Delta\mathbf{A}$.

Here the scalar potential $\Phi = (x^2 - y^2)/2$ is stereo-plotted vs. (x, y)

The $\Phi = (x^2 - y^2)/2 = \text{const.}$ curves are topography lines

The $A = (xy) = \text{const.}$ curves are streamlines normal to topography lines



4. Riemann-Cauchy conditions What's analytic? (...and what's not?)

Easy 2D circulation and flux integrals

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What Good Are Complex Exponentials? (contd.)

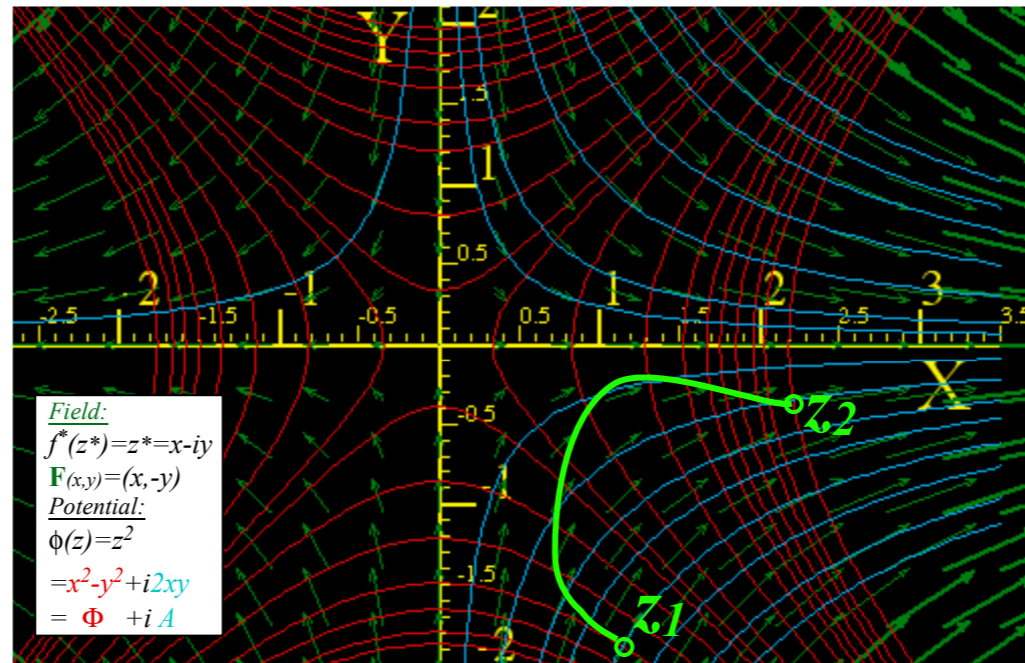
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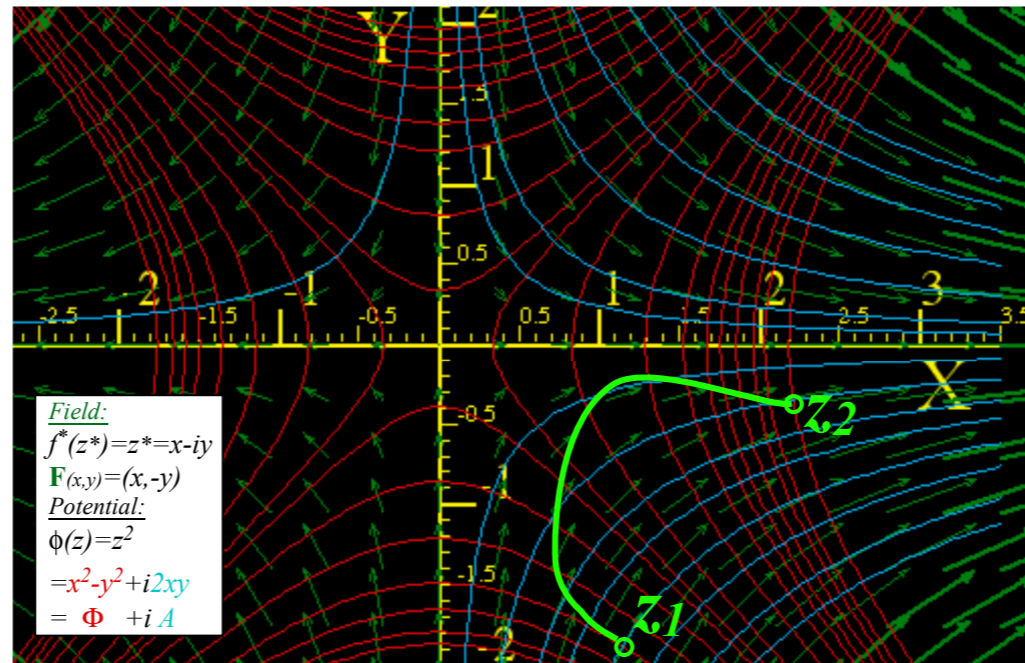
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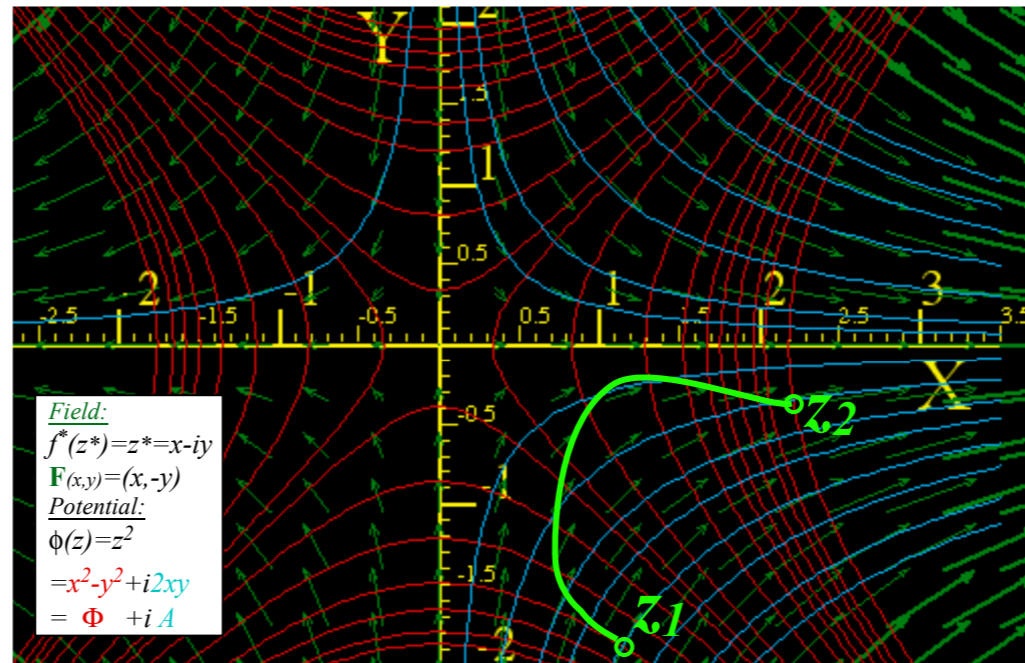
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Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ potential Φ obeys Laplace equation

What Good Are Complex Exponentials? (contd.)

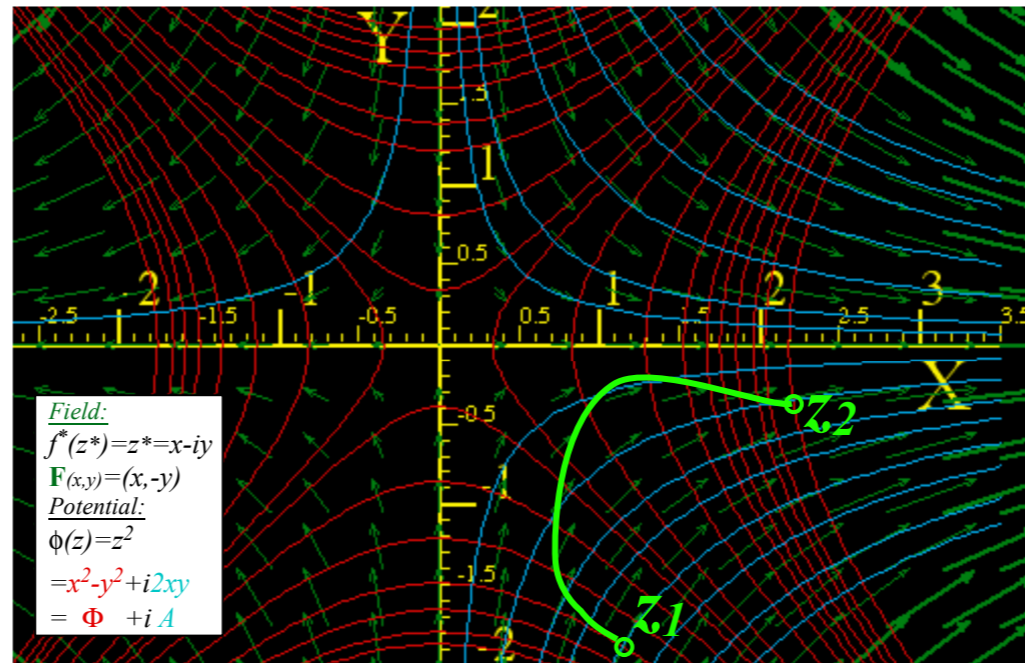
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What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D *monopole* fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit *monopole* field: $f(z)=\frac{1}{z}=z^{-1}$

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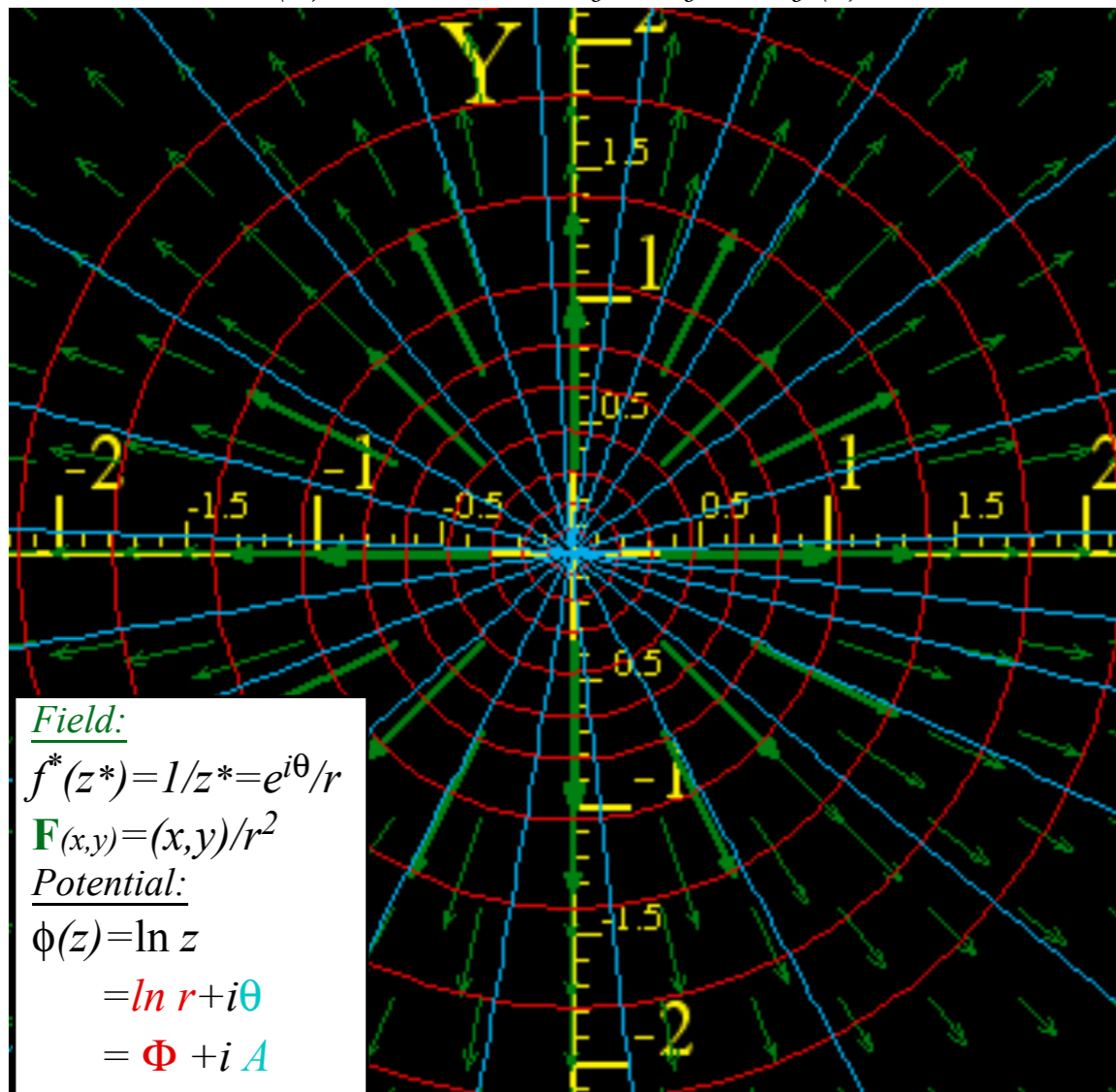
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(a) Unit Z-line-flux field $f(z)=1/z$



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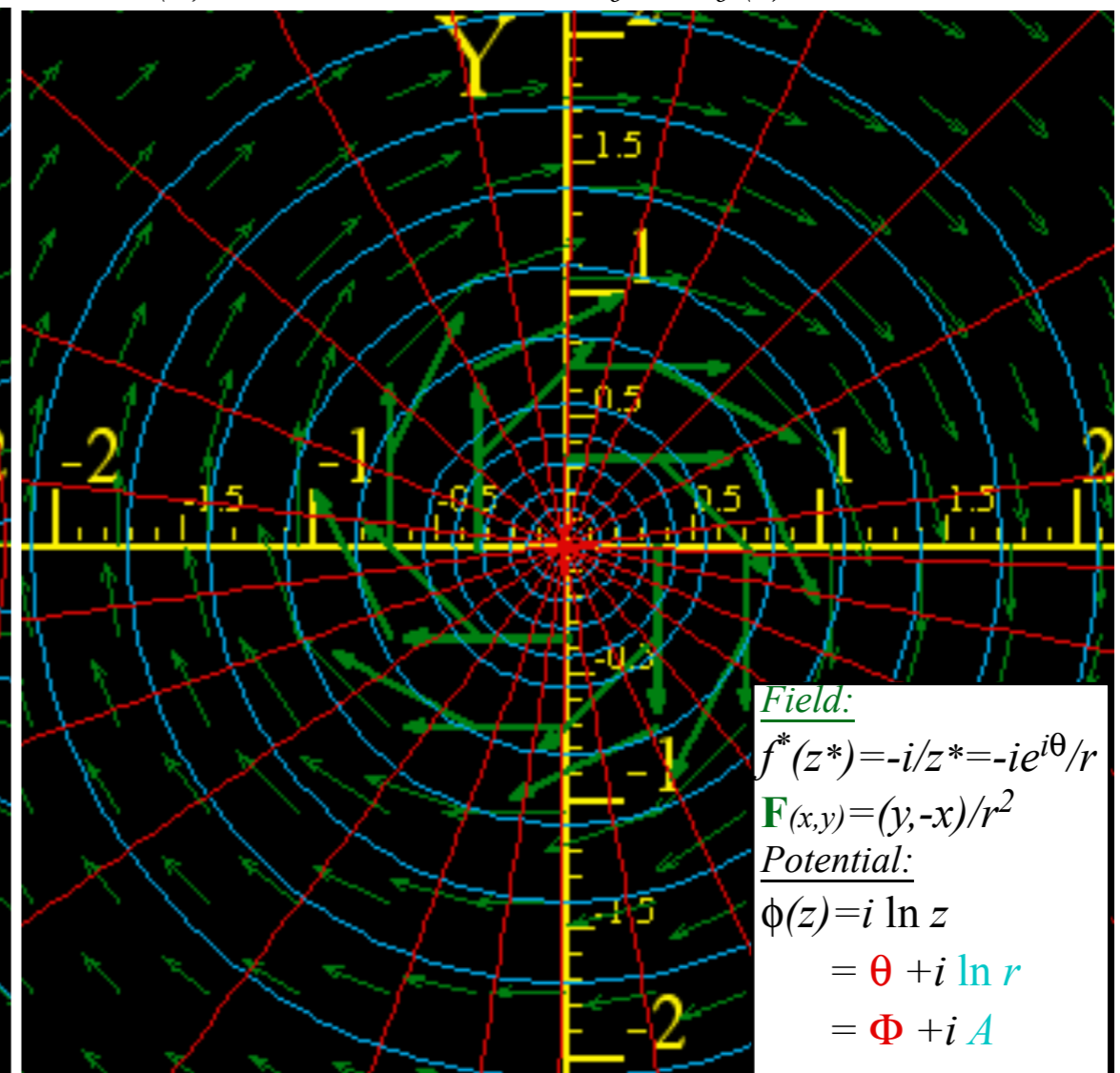
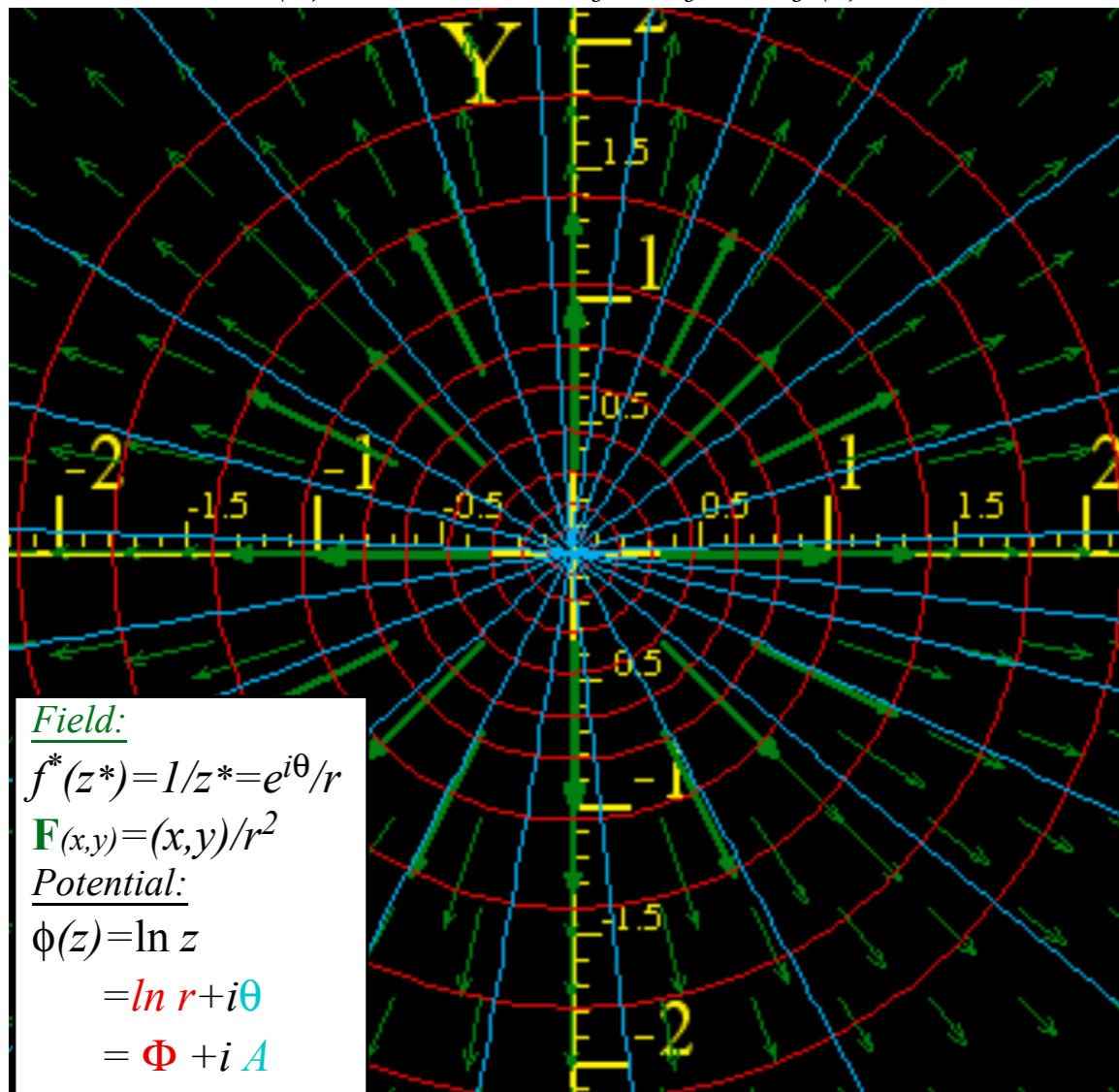
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(a) Unit Z-line-flux field $f(z)=1/z$

(b) Unit Z-line-vortex field $f(z)=i/z$



What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D *monopole* fields and potentials

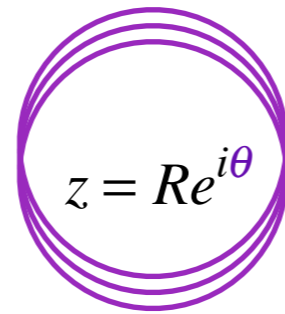
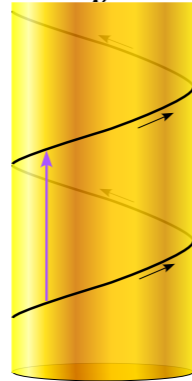
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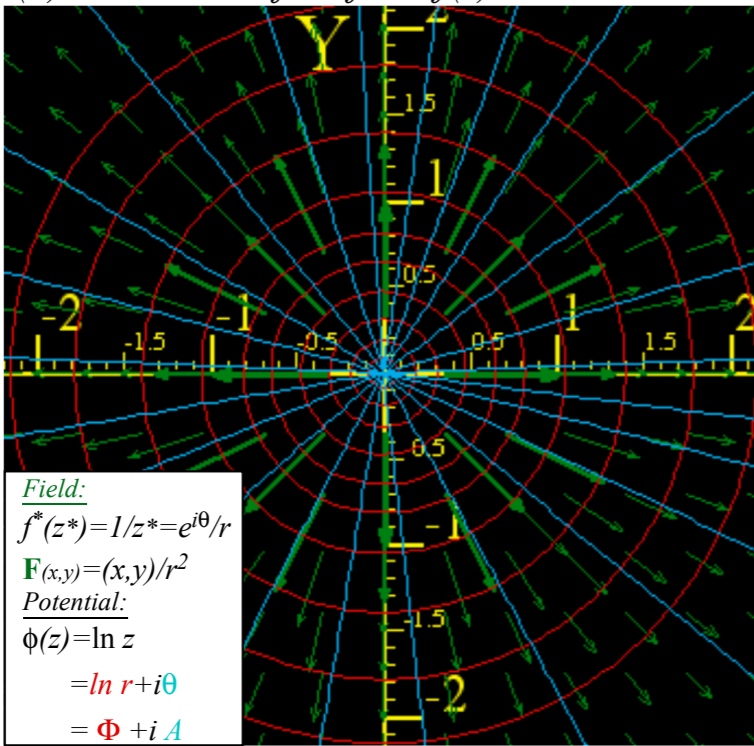
A *monopole* field is the only power-law field whose integral (potential) depends on *path of integration*.



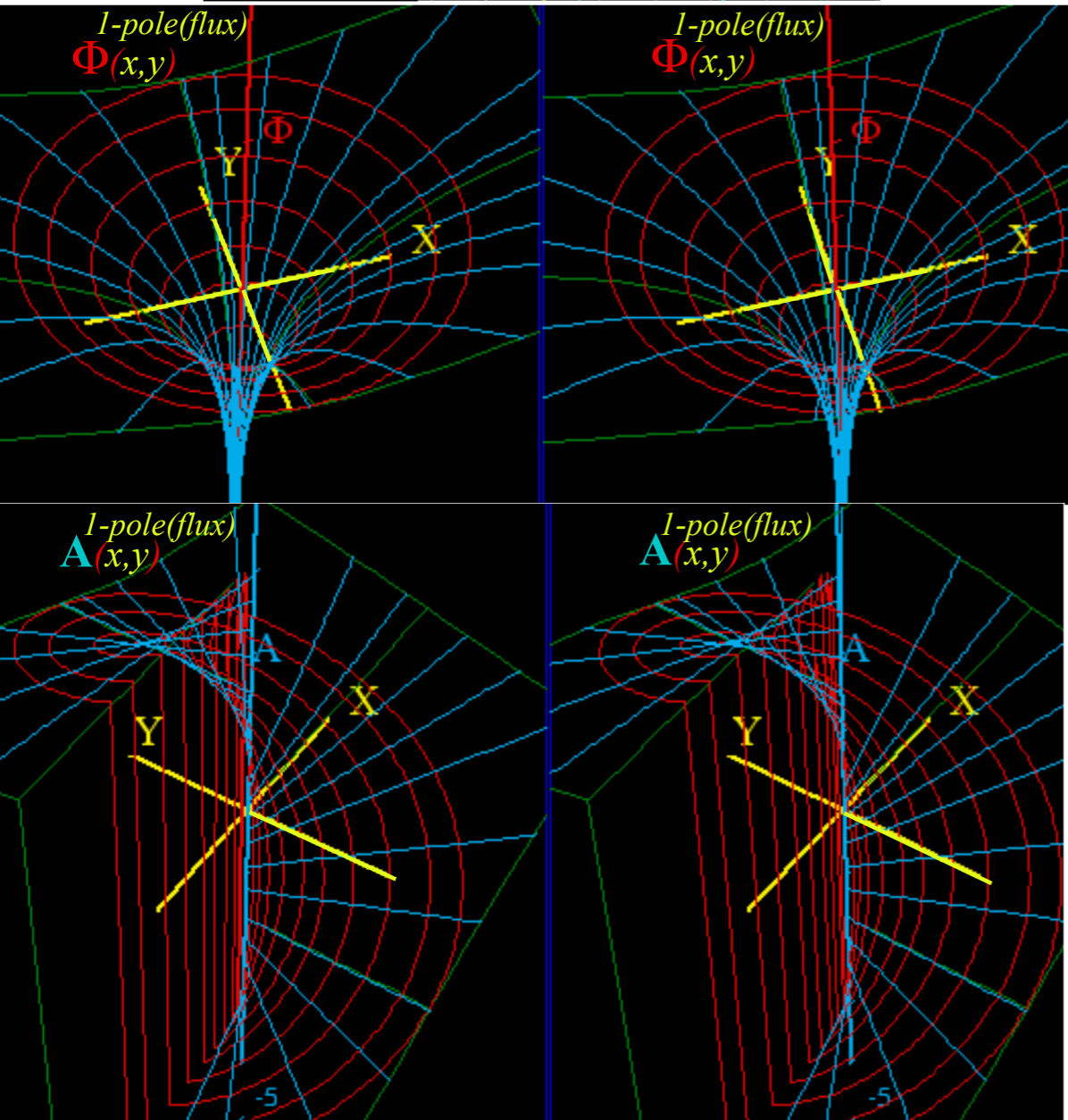
path that goes N times around origin (r=0) at constant r = R.

$$\Delta\phi = \oint f(z)dz = a \oint \frac{dz}{z} = a \int_{\theta=0}^{\theta=2\pi N} \frac{d(Re^{i\theta})}{Re^{i\theta}} = a \int_{\theta=0}^{\theta=2\pi N} i d\theta = ai \theta \Big|_0^{2\pi N} = 2a\pi i N$$

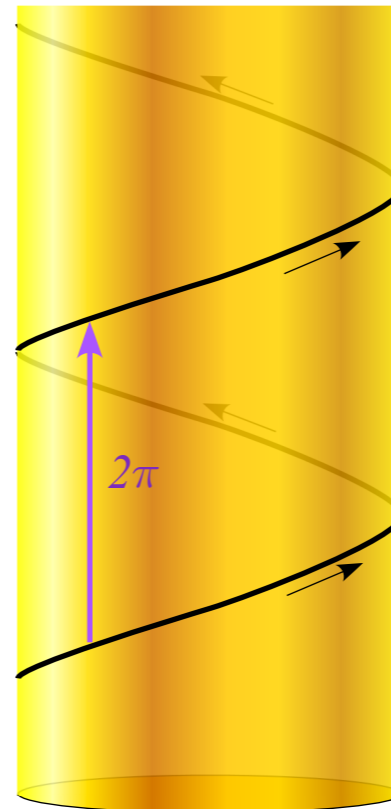
(a) Unit Z-line-flux field $f(z)=1/z$



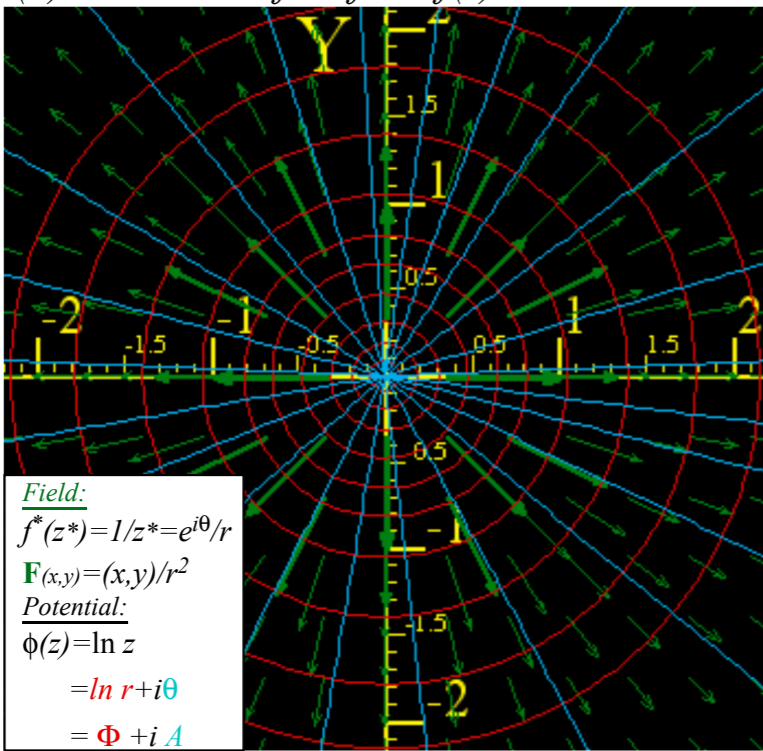
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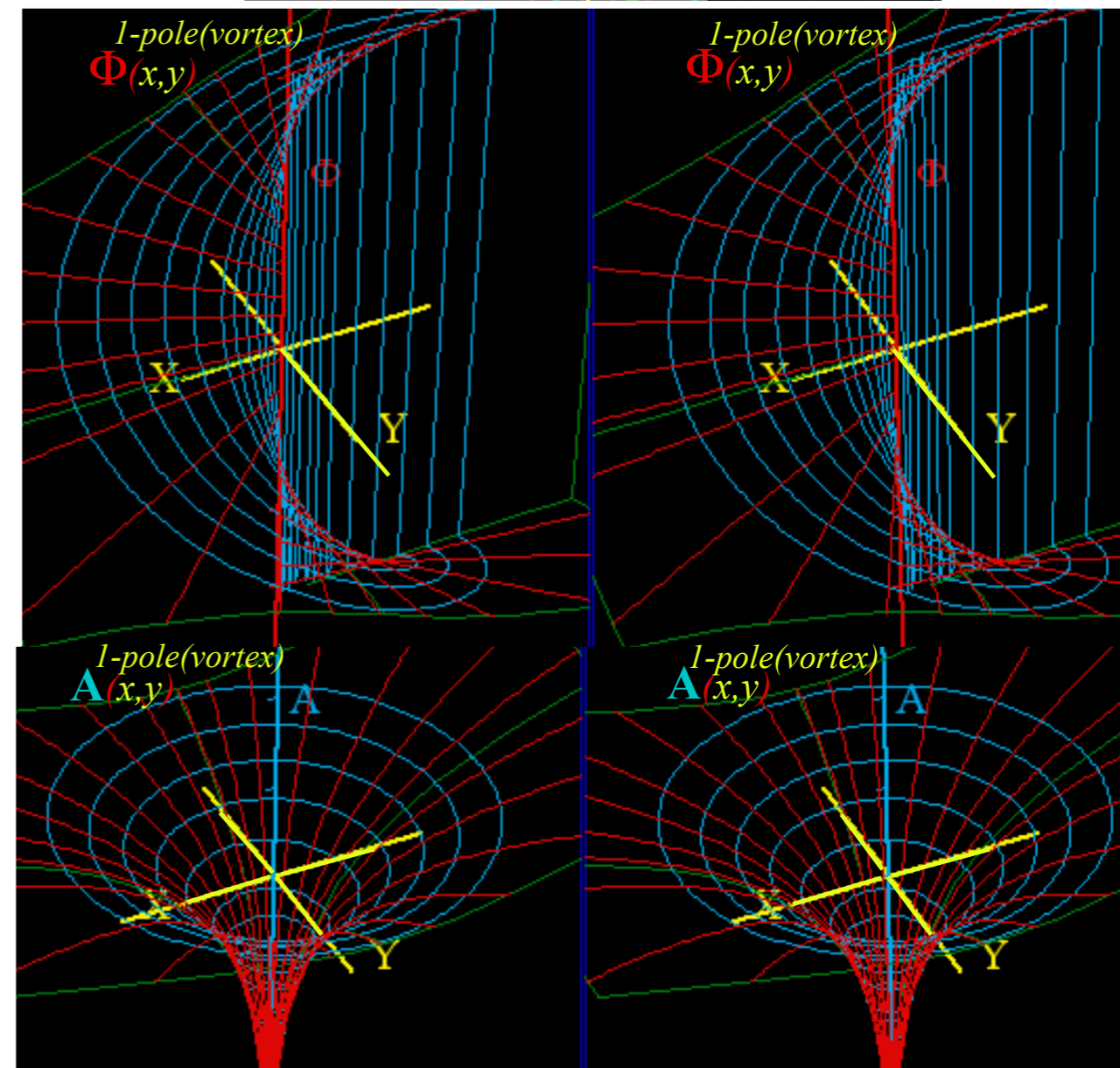
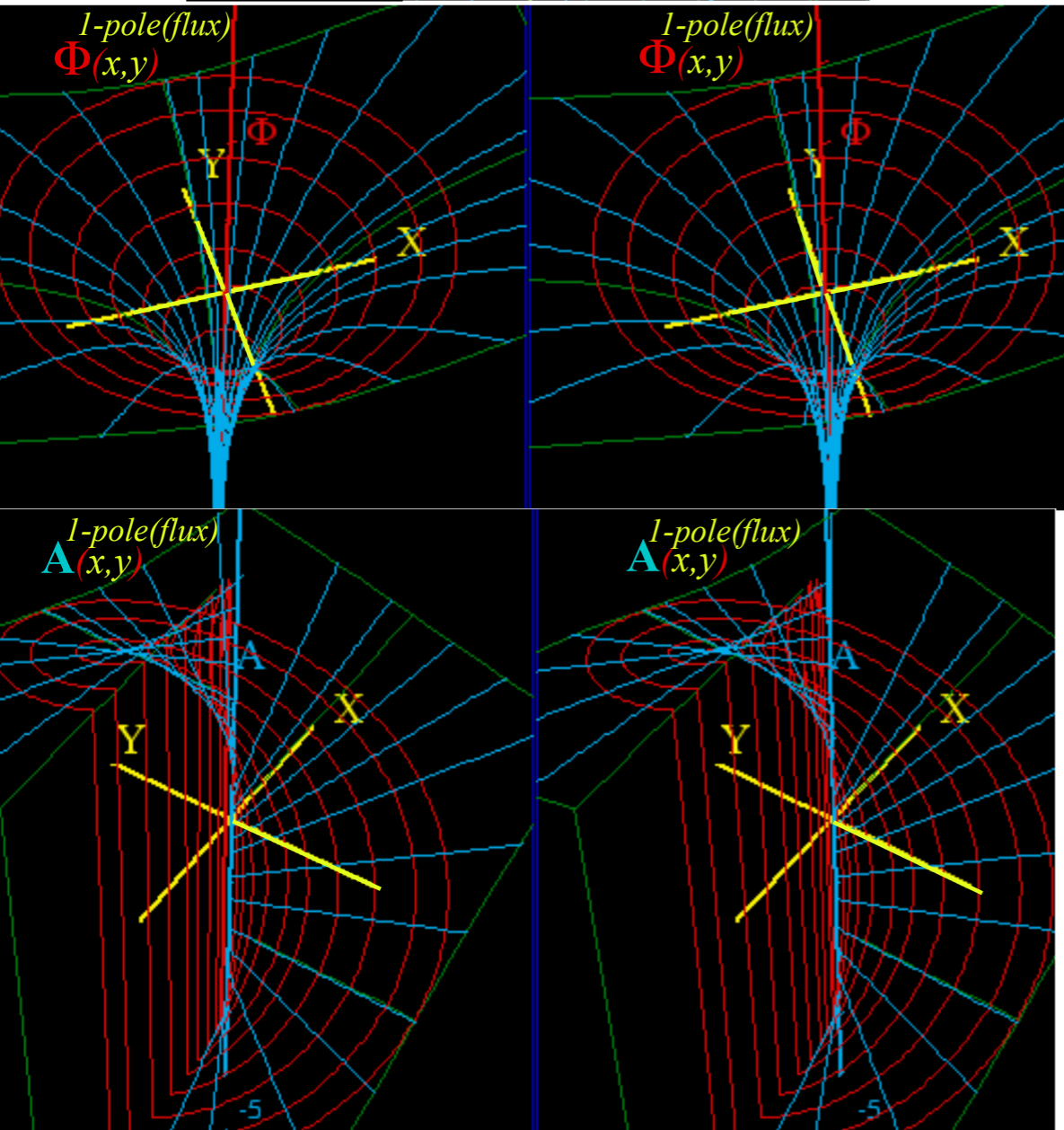
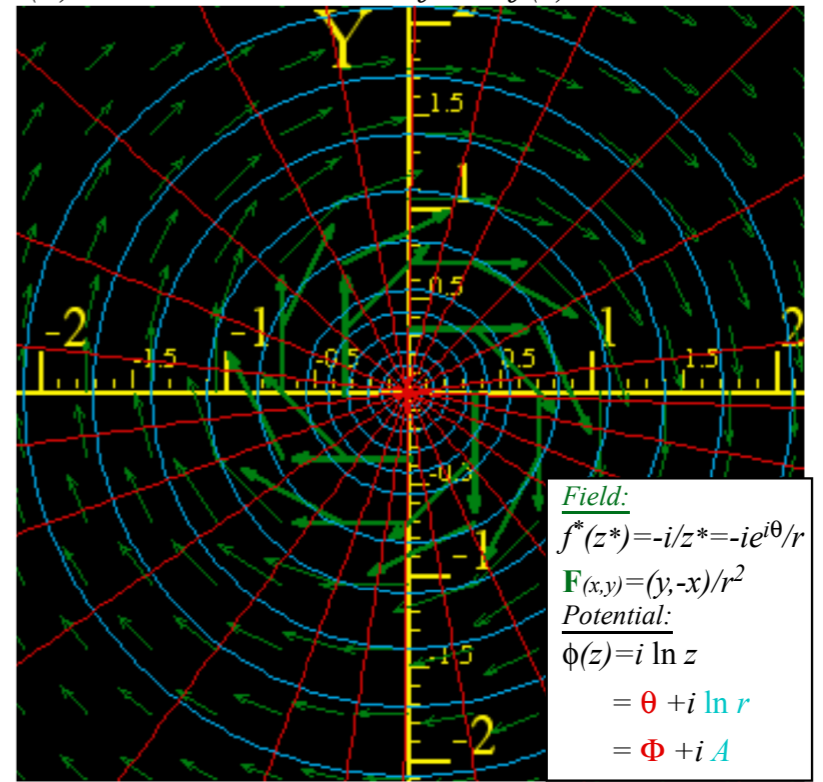
Each turn around origin adds $2\pi i$ to vector potential iA



(a) Unit Z-line-flux field $f(z)=1/z$



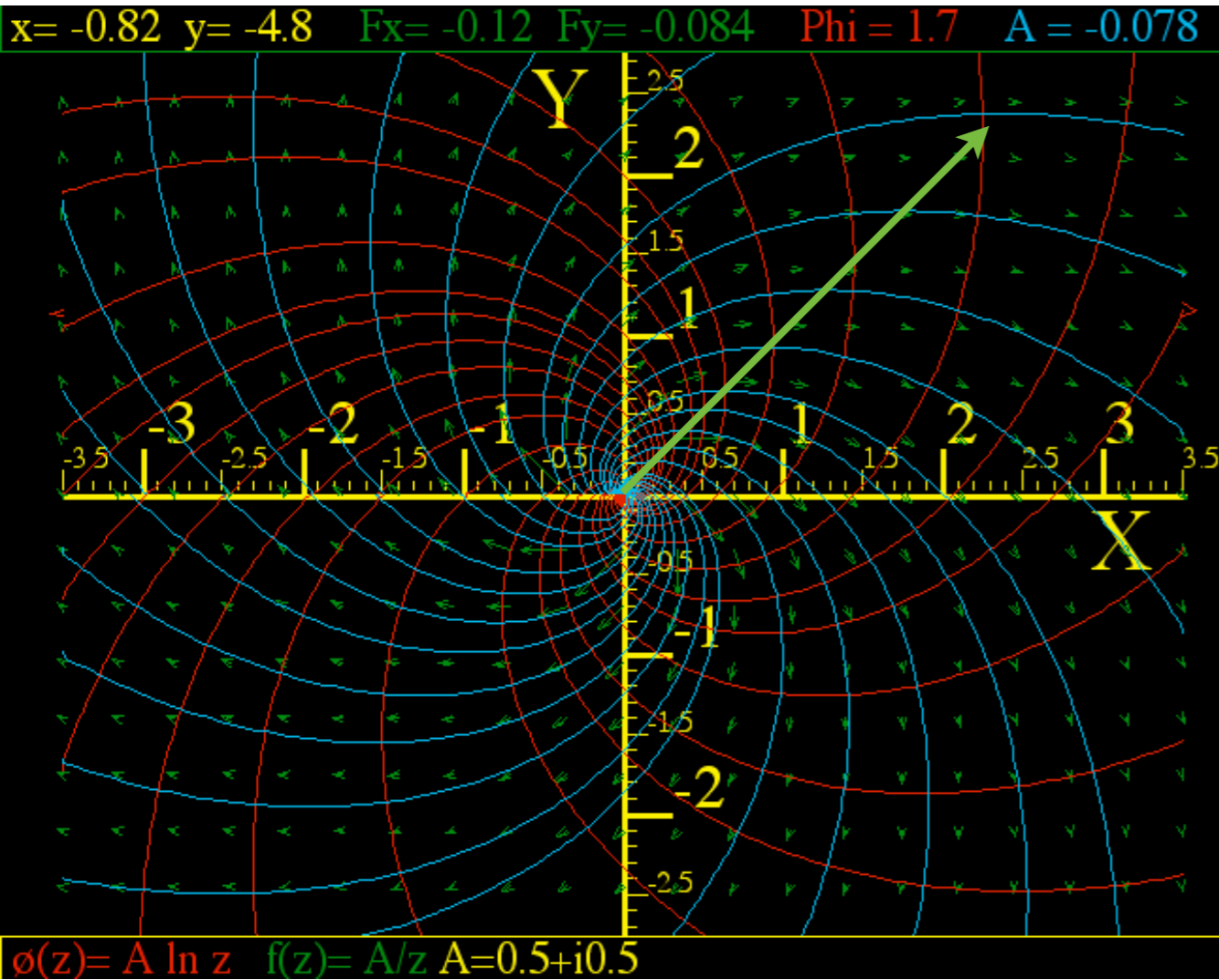
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What Good Are Complex Exponentials? (contd.)

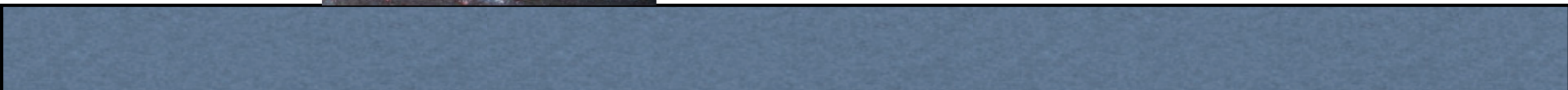
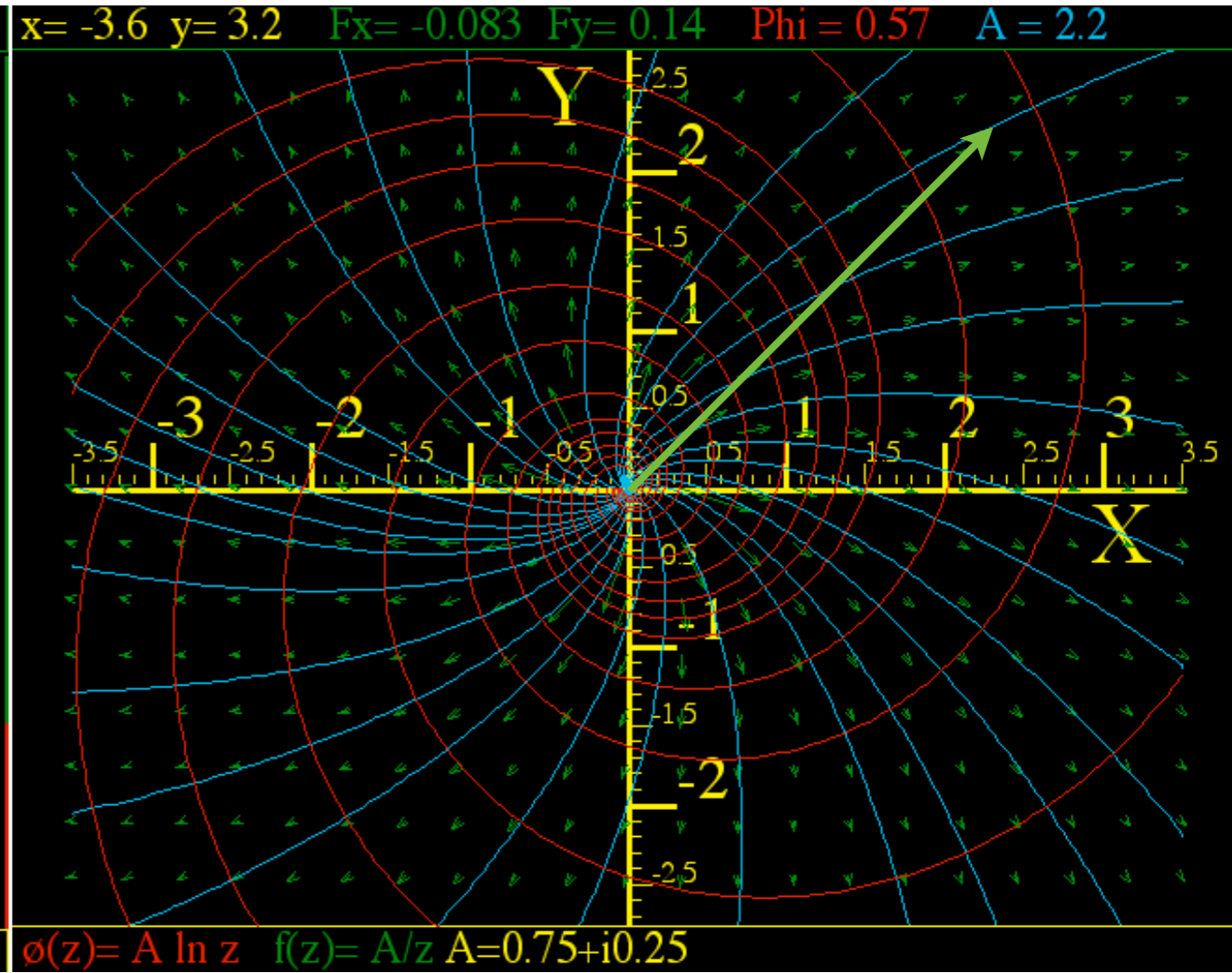
$$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$$

“Vortex”



$$f(z) = (0.75 + i0.25)/z = e^{i18^\circ}/z\sqrt{n}$$


“Hurricane”



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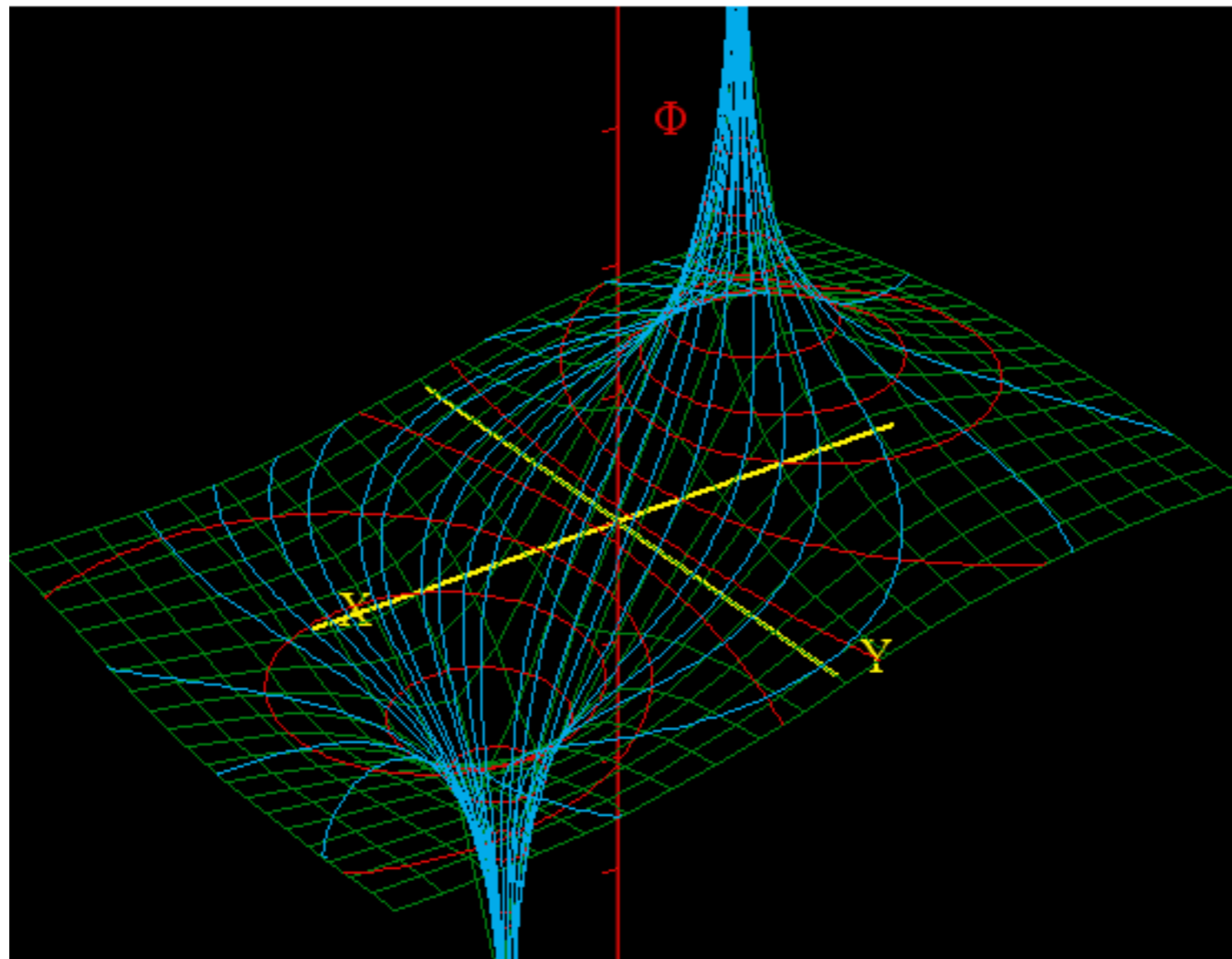
12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a \ln z$ of source strength a .

$$f^{1-pole}(z) = \frac{a}{z} = \frac{d\phi^{1-pole}}{dz} \quad \phi^{1-pole}(z) = a \ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ . This sum (actually difference) of f^{1-pole} -fields is called a *dipole field*.

$$f^{dipole}(z) = \frac{a}{z + \frac{\Delta}{2}} - \frac{a}{z - \frac{\Delta}{2}} = \frac{-a \cdot \Delta}{z^2 - \frac{\Delta^2}{4}} \quad \phi^{dipole}(z) = a \ln\left(z - \frac{\Delta}{2}\right) - a \ln\left(z + \frac{\Delta}{2}\right) = a \ln \frac{z - \frac{\Delta}{2}}{z + \frac{\Delta}{2}}$$



*So-called
“physical dipole”
has finite Δ
(+)(-) separation*

What Good Are Complex Exponentials? (2D monopole, dipole, and 2ⁿ-pole analysis)

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* f^{2-pole} that is the z -derivative of f^{1-pole} .

$$f^{2-pole} = \frac{-a}{z^2} = \frac{df^{1-pole}}{dz} = \frac{d\phi^{2-pole}}{dz} \quad \phi^{2-pole} = \frac{a}{z} = \frac{d\phi^{1-pole}}{dz}$$

What Good Are Complex Exponentials? (2D monopole, dipole, and 2^n -pole analysis)

12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a \ln z$ of source strength a .

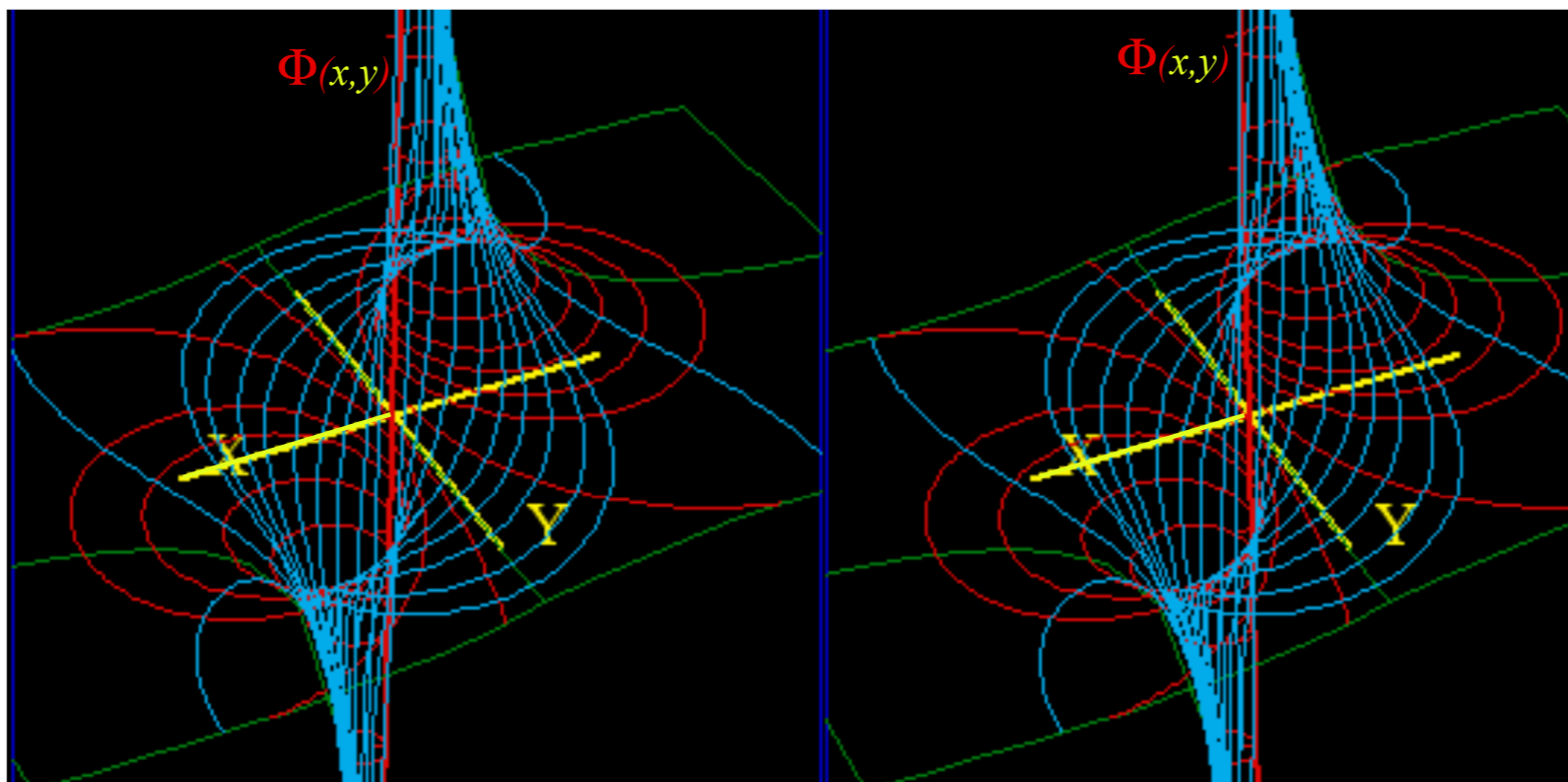
$$f^{1-pole}(z) = \frac{a}{z} = \frac{d\phi^{1-pole}}{dz} \quad \phi^{1-pole}(z) = a \ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ . This sum (actually difference) of f^{1-pole} -fields is called a *dipole field*.

$$f^{dipole}(z) = \frac{a}{z+\frac{\Delta}{2}} - \frac{a}{z-\frac{\Delta}{2}} = \frac{-a \cdot \Delta}{z^2 - \frac{\Delta^2}{4}} \quad \phi^{dipole}(z) = a \ln\left(z - \frac{\Delta}{2}\right) - a \ln\left(z + \frac{\Delta}{2}\right) = a \ln \frac{z - \frac{\Delta}{2}}{z + \frac{\Delta}{2}}$$

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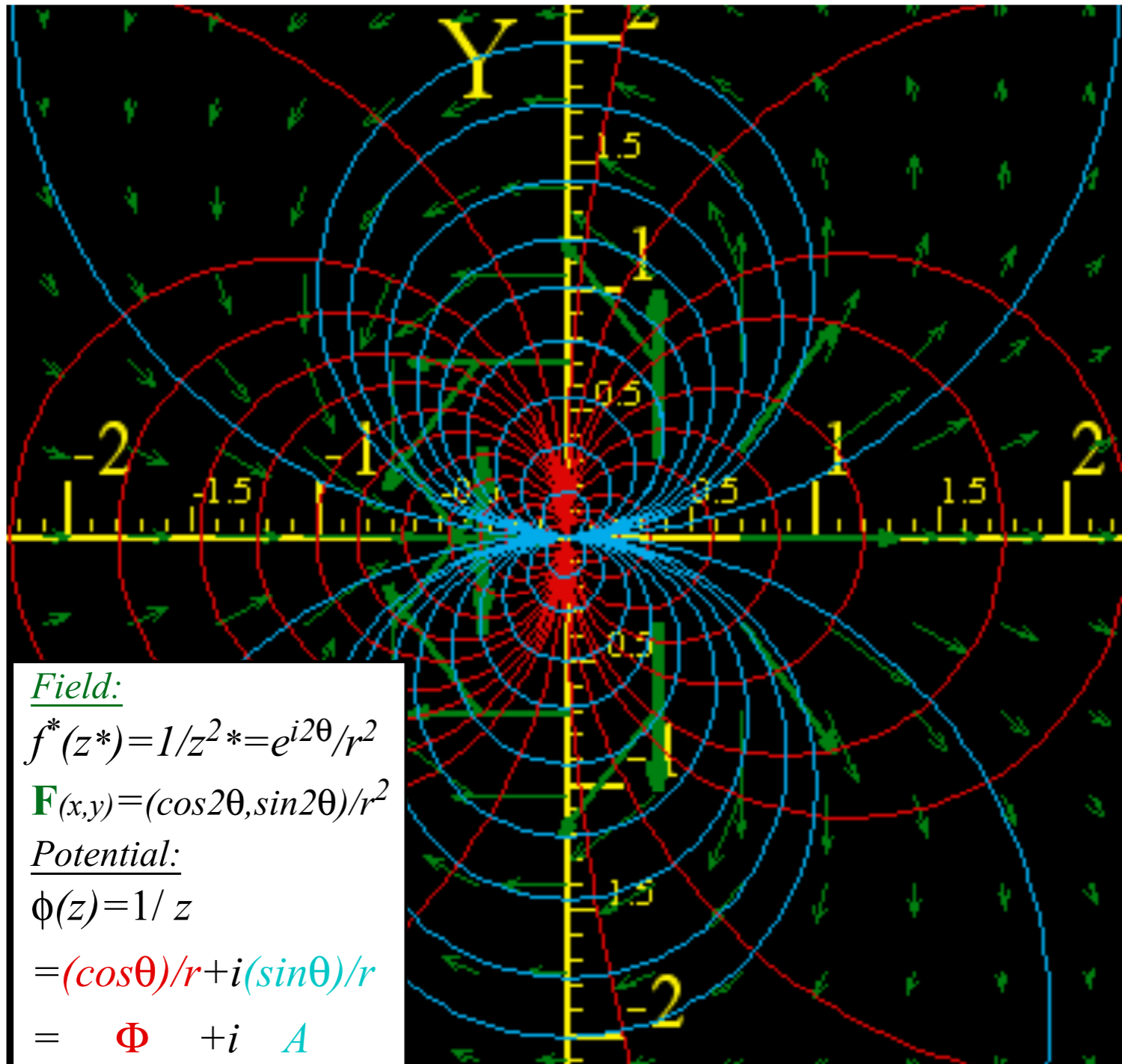
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A *point-dipole potential* ϕ^{2-pole} (whose z -derivative is f^{2-pole}) is a z -derivative of ϕ^{1-pole} .

$$\begin{aligned} \phi^{2-pole} &= \frac{a}{z} = \frac{a}{x+iy} = \frac{a}{x+iy} \frac{x-iy}{x-iy} = \frac{ax}{x^2+y^2} + i \frac{-ay}{x^2+y^2} = \frac{a}{r} \cos \theta - i \frac{a}{r} \sin \theta \\ &= \Phi^{2-pole} + i \mathbf{A}^{2-pole} \end{aligned}$$

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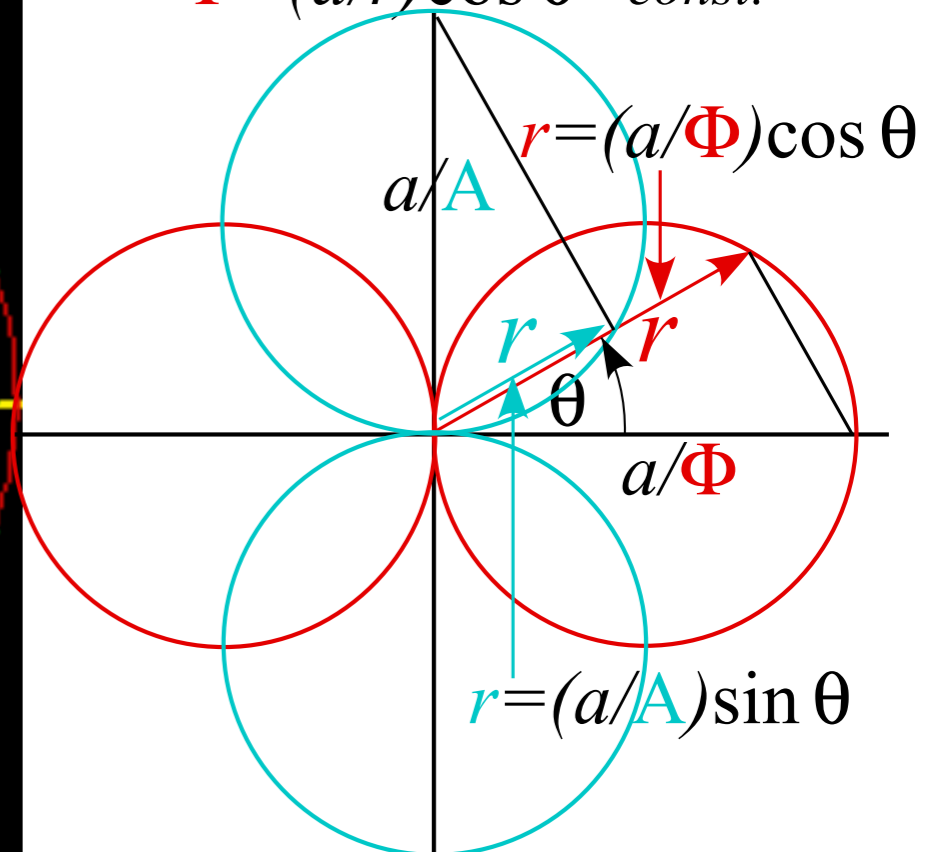
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Field:
 $f^*(z^*) = 1/z^{2*} = e^{i2\theta}/r^2$
 $\mathbf{F}(x,y) = (\cos 2\theta, \sin 2\theta)/r^2$
Potential:
 $\phi(z) = 1/z$
 $= (\cos \theta)/r + i(\sin \theta)/r$
 $= \Phi + i A$

Scalar potentials

$\Phi = (a/r) \cos \theta = const.$



Vector potentials

$A = (a/r) \sin \theta = const.$

2^n -pole analysis (quadrupole: $2^2=4$ -pole, octapole: $2^3=8$ -pole, ..., pole dancer,

What if we put a (-)copy of a 2-pole near its original?

Well, the result is 4-pole or *quadrupole* field f^{4-pole} and potential ϕ^{4-pole} .

Each a z-derivative of f^{2-pole} and ϕ^{2-pole} .

$$f^{4-pole} = \frac{a}{z^3} = \frac{1}{2} \frac{df^{2-pole}}{dz} = \frac{d\phi^{4-pole}}{dz}$$

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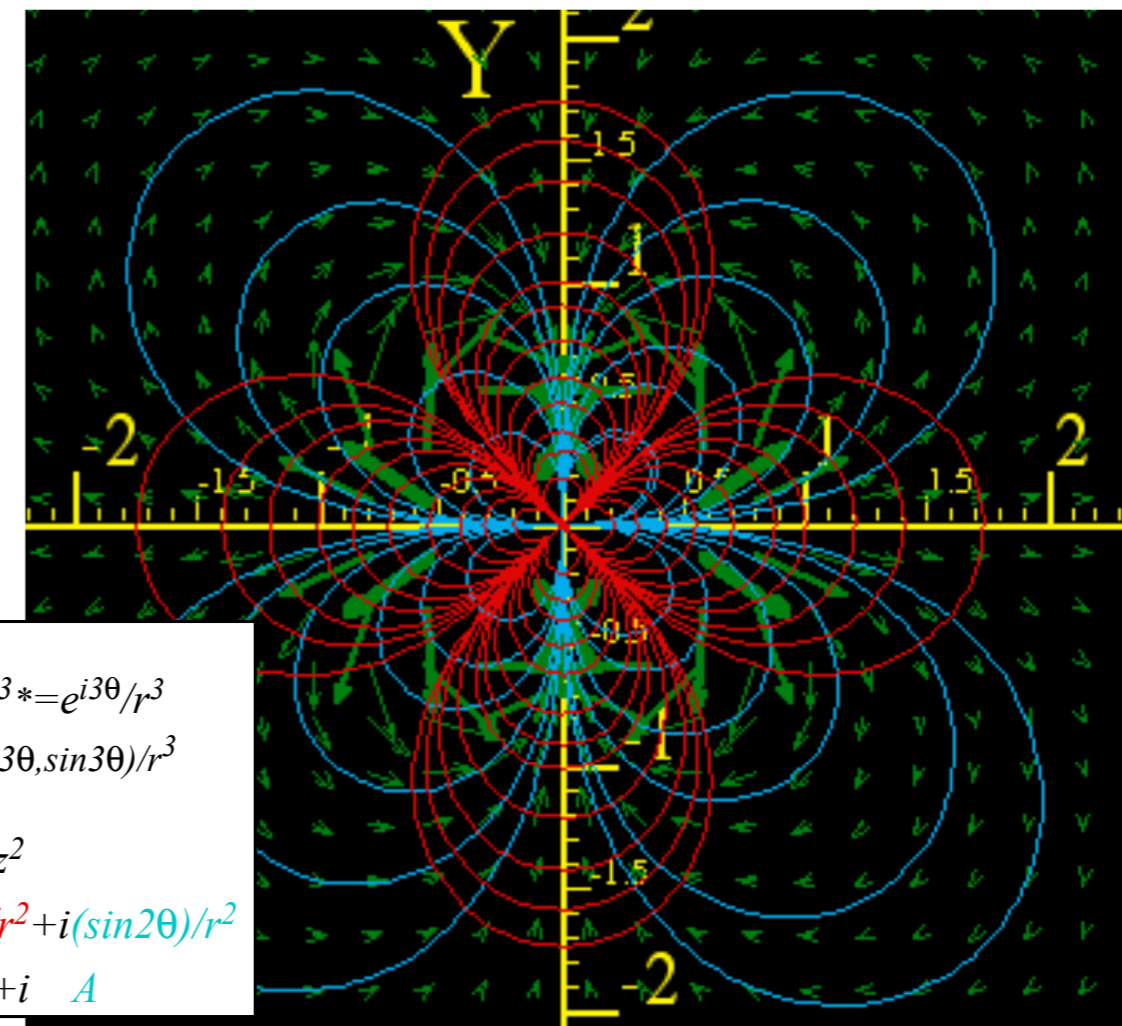
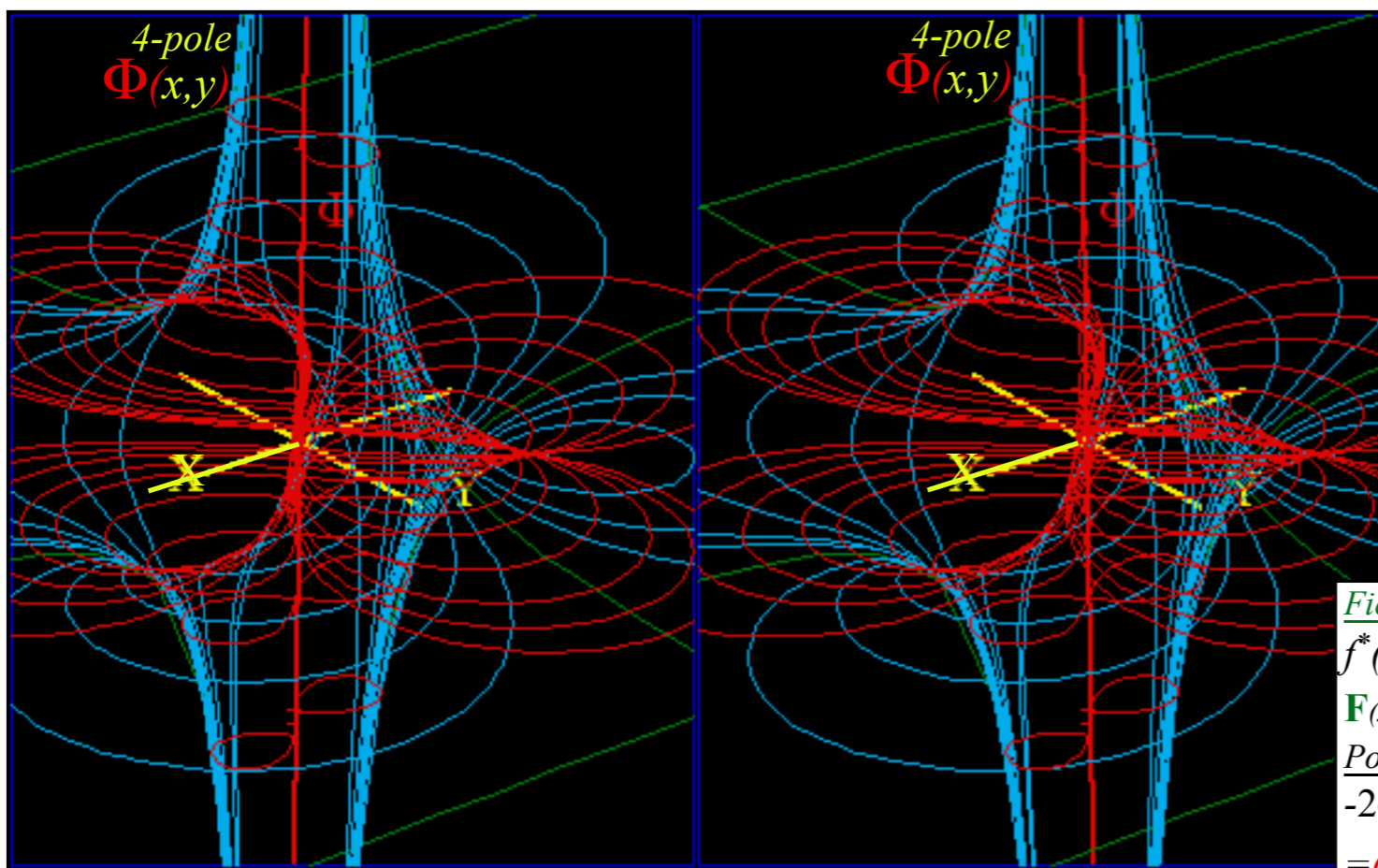
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Field:
 $f^*(z^*) = 1/z^3 = e^{i3\theta}/r^3$
 $\mathbf{F}(x,y) = (\cos 3\theta, \sin 3\theta)/r^3$
Potential:
 $-2\phi(z) = 1/z^2$
 $= (\cos 2\theta)/r^2 + i(\sin 2\theta)/r^2$
 $= \Phi + iA$

4. Riemann-Cauchy conditions What's analytic? (...and what's not?)

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n -pole analysis

Easy 2^n -multipole field and potential expansion

Easy stereo-projection visualization



2^n -pole analysis: Laurent series (Generalization of Maclaurin-Taylor series)

Laurent series or *multipole expansion* of a given complex field function $f(z)$ around $z=0$.

$$\begin{aligned} \frac{d\phi}{dz} = f(z) &= \dots a_{-3}z^{-3} + a_{-2}z^{-2} + \mathbf{a_{-1}z^{-1}} + a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + \dots \\ &\quad \dots \begin{array}{l} 2^2\text{-pole} \\ \text{(quadrupole)} \\ \text{at } z=0 \end{array} + \begin{array}{l} 2^1\text{-pole} \\ \text{(dipole)} \\ \text{at } z=0 \end{array} + \begin{array}{l} 2^0\text{-pole} \\ \text{(monopole)} \\ \text{at } z=0 \end{array} + \begin{array}{l} 2^1\text{-pole} \\ \text{(dipole)} \\ \text{at } z=\infty \end{array} + \begin{array}{l} 2^2\text{-pole} \\ \text{(quadrupole)} \\ \text{at } z=\infty \end{array} + \begin{array}{l} 2^3\text{-pole} \\ \text{(octapole)} \\ \text{at } z=\infty \end{array} + \begin{array}{l} 2^4\text{-pole} \\ \text{(hexadecapole)} \\ \text{at } z=\infty \end{array} + \begin{array}{l} 2^5\text{-pole} \\ \text{at } z=\infty \end{array} + \begin{array}{l} 2^6\text{-pole} \\ \text{at } z=\infty \end{array} \dots \\ \int f dz = \\ \phi(z) &= \dots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + \mathbf{a_{-1} \ln z} + a_0z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \dots \end{aligned}$$

All field terms $a_{m-1}z^{m-1}$ except $1\text{-pole } \frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a 2^m -pole.

These are located at $z=0$ for $m < 0$ and at $z=\infty$ for $m > 0$.

$$\phi(z) = \dots \begin{array}{l} \text{(octapole)}_0 \\ \frac{a_{-4}}{-3}z^{-3} \end{array} + \begin{array}{l} \text{(quadrupole)}_0 \\ \frac{a_{-3}}{-2}z^{-2} \end{array} + \begin{array}{l} \text{(dipole)}_0 \\ \frac{a_{-2}}{-1}z^{-1} \end{array} + \mathbf{a_{-1} \ln z} + \begin{array}{l} \text{(monopole)} \\ a_0z \end{array} + \begin{array}{l} \text{(dipole)}_\infty \\ \frac{a_1}{2}z^2 \end{array} + \begin{array}{l} \text{(quadrupole)}_\infty \\ \frac{a_2}{3}z^3 \end{array} + \dots$$

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Laurent series or *multipole expansion* of a given complex field function $f(z)$ around $z=0$.

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	...	2 ² -pole <i>(quadrupole)</i> at $z=0$	2 ¹ -pole <i>(dipole)</i> at $z=0$	2 ⁰ -pole <i>(monopole)</i> at $z=0$	2 ¹ -pole <i>(dipole)</i> at $z=\infty$	2 ² -pole <i>(quadrupole)</i> at $z=\infty$	2 ³ -pole <i>(octapole)</i> at $z=\infty$	2 ⁴ -pole <i>(hexadecapole)</i> at $z=\infty$	2 ⁵ -pole at $z=\infty$	2 ⁶ -pole at $z=\infty$...
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$$\int f dz = \phi(z) = \dots \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-2}}{-1} z^{-1} + a_{-1} \ln z + a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \frac{a_3}{4} z^4 + \frac{a_4}{5} z^5 + \frac{a_5}{6} z^6 + \dots$$

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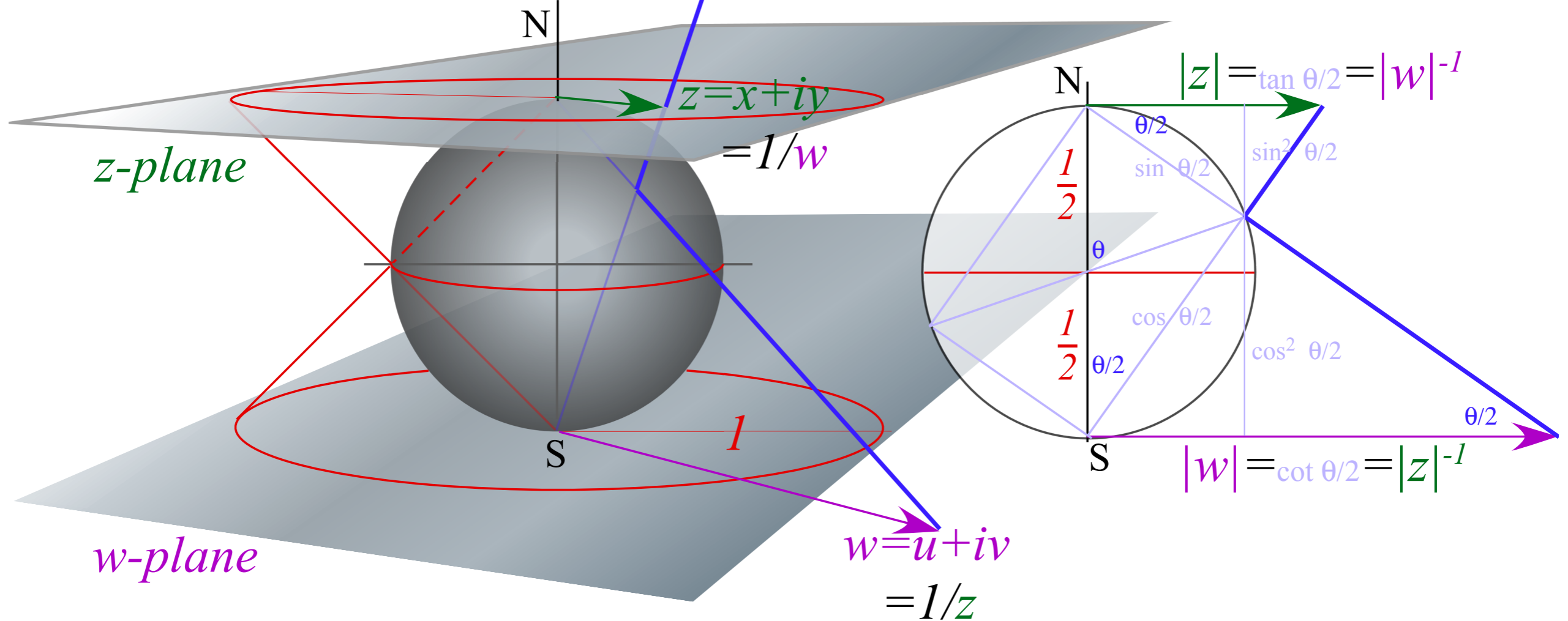
(octapole)₀
(quadrupole)₀
(dipole)₀
(monopole)
(dipole)_∞
(quadrupole)_∞
(octapole)_∞

$$\phi(w) = \dots \frac{a_{-4}}{-3} w^{-3} + \frac{a_{-3}}{-2} w^{-2} + \frac{a_{-2}}{-1} w^{-1} + a_{-1} \ln w + a_0 w + \frac{a_1}{2} w^2 + \frac{a_2}{3} w^3 + \dots$$

(with $z \rightarrow w$)

$$= \dots \frac{a_2}{3} z^{-3} + \frac{a_1}{2} z^{-2} + a_0 z^{-1} - a_{-1} \ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^2 + \frac{a_{-4}}{-3} z^3 + \dots$$

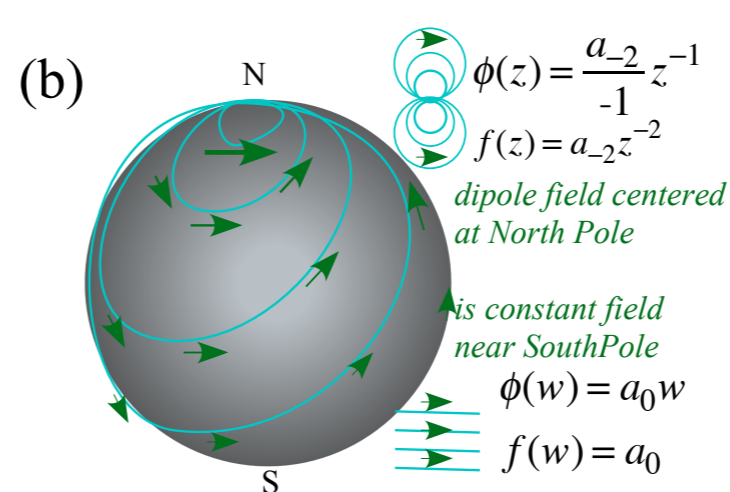
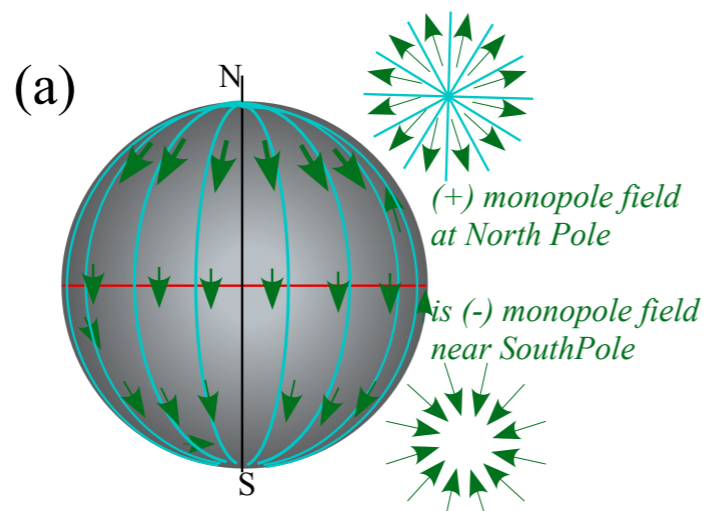
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(with $z \rightarrow w$)

(with $w = z^{-1}$)



$\phi(z) = \frac{a_{-3}}{-2} z^{-2}$

$f(z) = a_{-3} z^{-3}$

quadrupole field centered at North Pole

is quadratic field near South Pole

$\phi(w) = a_0 w^2$

$f(w) = a_1 w$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + \dots$$

Of all 2^m -pole field terms $a_{m-l}z^{m-l}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \frac{1}{2\pi i} \oint f(z)dz$$

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This $m=1$ -pole constant- a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\dots a_{-3} = \frac{1}{2\pi i} \oint z^2 f(z)dz, \quad a_{-2} = \frac{1}{2\pi i} \oint z^1 f(z)dz, \quad a_{-1} = \frac{1}{2\pi i} \oint f(z)dz, \quad a_0 = \frac{1}{2\pi i} \oint \frac{f(z)}{z} dz, \quad a_1 = \frac{1}{2\pi i} \oint \frac{f(z)}{z^2} dz, \dots$$

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Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

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Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point- a . Function $f(z)$ is just $f(a)$ on a *tiny* circle around point- a .

(assume *tiny* circle around $z=a$)

$$\oint \frac{f(z)}{z-a} dz = \oint \frac{f(a)}{z-a} dz = f(a) \oint \frac{1}{z-a} dz = 2\pi i f(a)$$

(but any contour that doesn't "touch a gives same answer)

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + \dots$$

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(quadrupole)₀ (dipole)₀ (monopole) (dipole)_∞ (quadrupole)_∞ (octapole)_∞ (hexadecapole)_∞ ...

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5. Mapping and Non-analytic 2D source field analysis

The *half-n'-half* results
are called
Riemann-Cauchy
Derivative Relations

$\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y}$	is:	$\frac{\partial \operatorname{Re}\phi(z)}{\partial x} = \frac{\partial \operatorname{Im}\phi(z)}{\partial y}$	or:	$\frac{\partial \operatorname{Re}f(z)}{\partial x} = \frac{\partial \operatorname{Im}f(z)}{\partial y}$	is:	$\frac{\partial f_x(z)}{\partial x} = \frac{\partial f_y(z)}{\partial y}$
$\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x}$	is:	$\frac{\partial \operatorname{Re}\phi(z)}{\partial y} = -\frac{\partial \operatorname{Im}\phi(z)}{\partial x}$	or:	$\frac{\partial \operatorname{Re}f(z)}{\partial y} = -\frac{\partial \operatorname{Im}f(z)}{\partial x}$	is:	$\frac{\partial f_x(z)}{\partial y} = -\frac{\partial f_y(z)}{\partial x}$

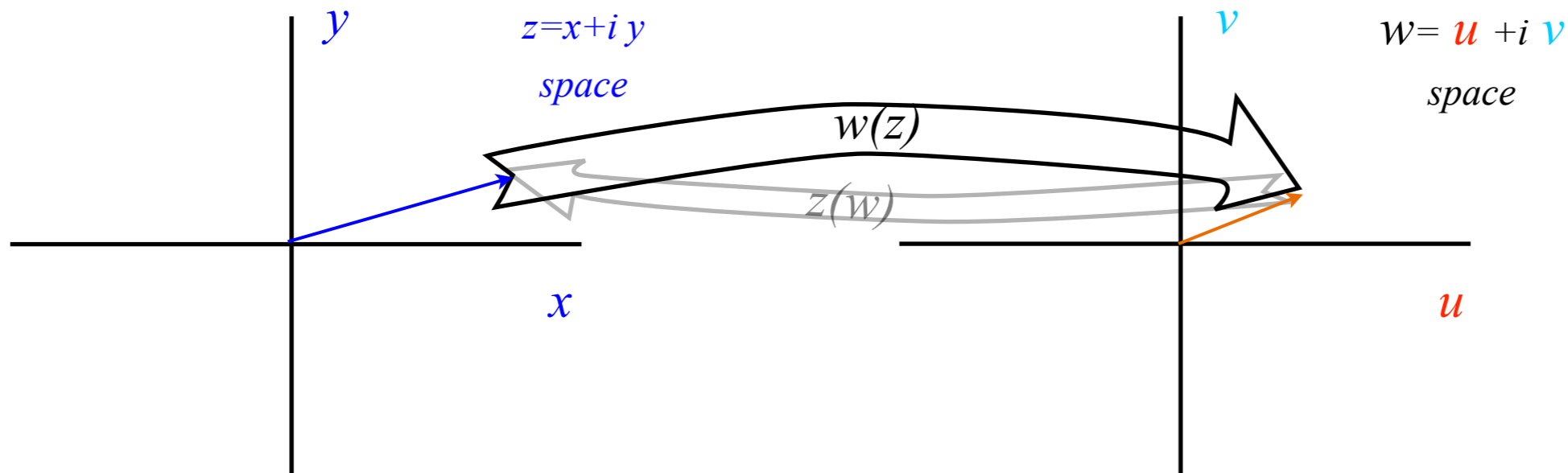
RC applies to analytic potential $\phi(z) = \Phi + iA$ and analytic field $f(z) = f_x + if_y$ and any analytic function

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Common notation for mapping: $w(z) = u + iv$

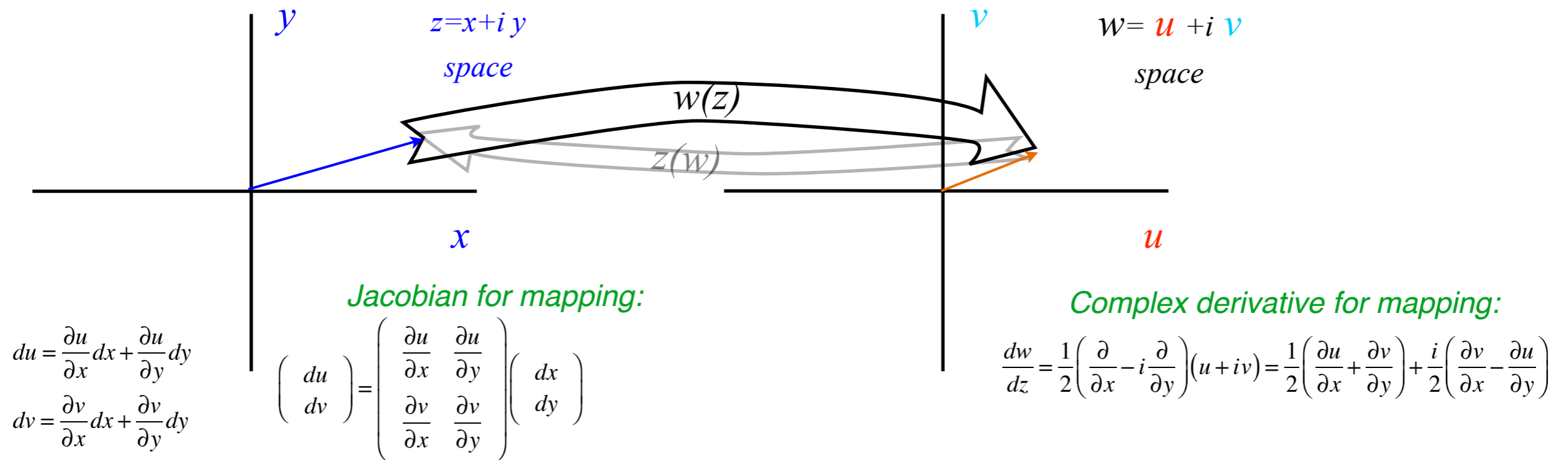


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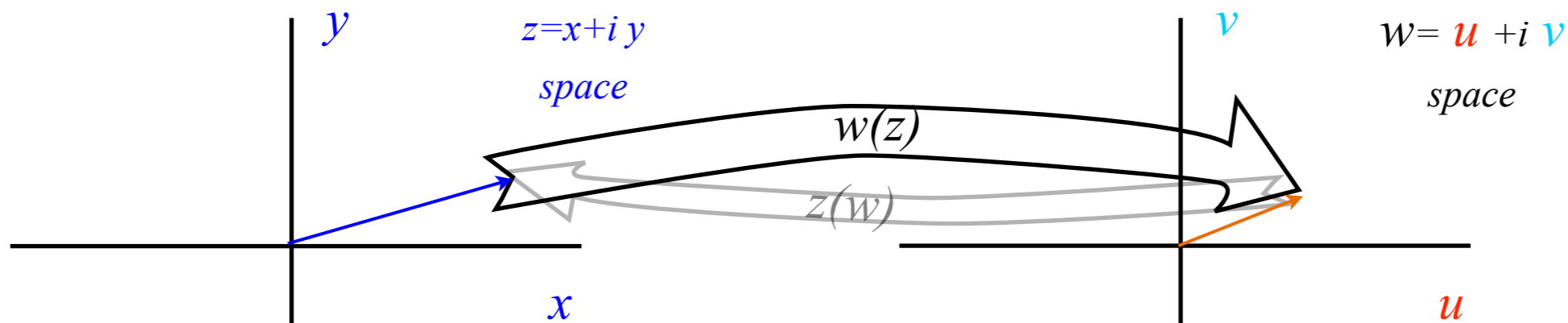


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Jacobian for mapping:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + iv) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left| \frac{dw}{dz} \right|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2$$

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Derivative Relations*

$\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y}$	is:	$\frac{\partial \text{Re}\phi(z)}{\partial x} = \frac{\partial \text{Im}\phi(z)}{\partial y}$	or:	$\frac{\partial \text{Re}f(z)}{\partial x} = \frac{\partial \text{Im}f(z)}{\partial y}$	is:	$\frac{\partial f_x(z)}{\partial x} = \frac{\partial f_y(z)}{\partial y}$
$\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x}$	is:	$\frac{\partial \text{Re}\phi(z)}{\partial y} = -\frac{\partial \text{Im}\phi(z)}{\partial x}$	or:	$\frac{\partial \text{Re}f(z)}{\partial y} = -\frac{\partial \text{Im}f(z)}{\partial x}$	is:	$\frac{\partial f_x(z)}{\partial y} = -\frac{\partial f_y(z)}{\partial x}$

RC applies to analytic potential $\phi(z) = \Phi + iA$ and analytic field $f(z) = f_x + if_y$ and any analytic function

Common notation for mapping: $w(z) = u + iv$

Jacobian for mapping:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\begin{aligned} \frac{dw}{dz} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + iv) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \end{aligned}$$

Complex derivative abs-square:

$$\left| \frac{dw}{dz} \right|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \det|J|$$

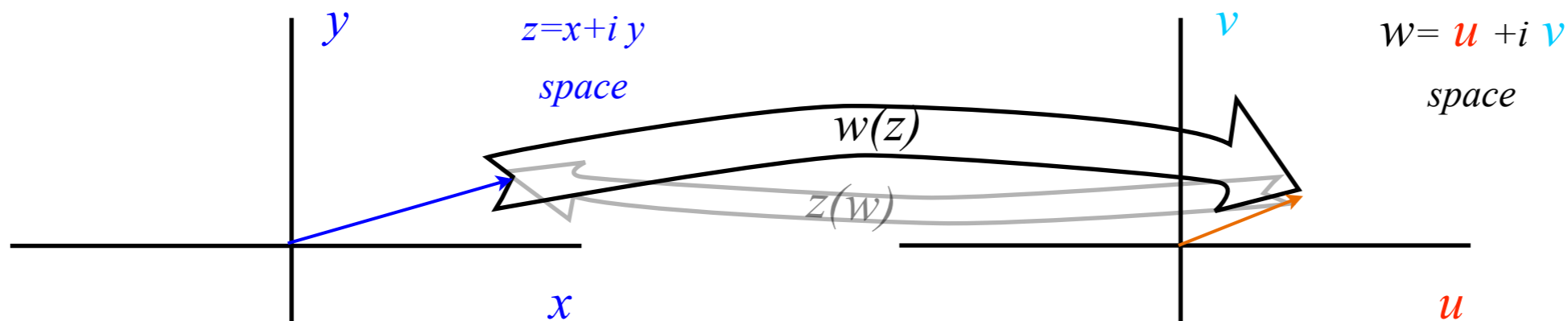
...equals Jacobian Determinant

The *half-n'-half* results
are called
*Riemann-Cauchy
Derivative Relations*

$\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y}$	is:	$\frac{\partial \text{Re}\phi(z)}{\partial x} = \frac{\partial \text{Im}\phi(z)}{\partial y}$	or:	$\frac{\partial \text{Re}f(z)}{\partial x} = \frac{\partial \text{Im}f(z)}{\partial y}$	is:	$\frac{\partial f_x(z)}{\partial x} = \frac{\partial f_y(z)}{\partial y}$
$\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x}$	is:	$\frac{\partial \text{Re}\phi(z)}{\partial y} = -\frac{\partial \text{Im}\phi(z)}{\partial x}$	or:	$\frac{\partial \text{Re}f(z)}{\partial y} = -\frac{\partial \text{Im}f(z)}{\partial x}$	is:	$\frac{\partial f_x(z)}{\partial y} = -\frac{\partial f_y(z)}{\partial x}$

RC applies to analytic potential $\phi(z) = \Phi + iA$ and analytic field $f(z) = f_x + if_y$ and any analytic function

Common notation for mapping: $w(z) = u + iv$



Important result:

$$dw = \sqrt{J} \cdot e^{i\theta} \cdot dz$$

is scaled rotation of dz.

Jacobian for mapping is scaled rotation:

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\ dv &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \end{aligned}$$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \sqrt{\det J} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

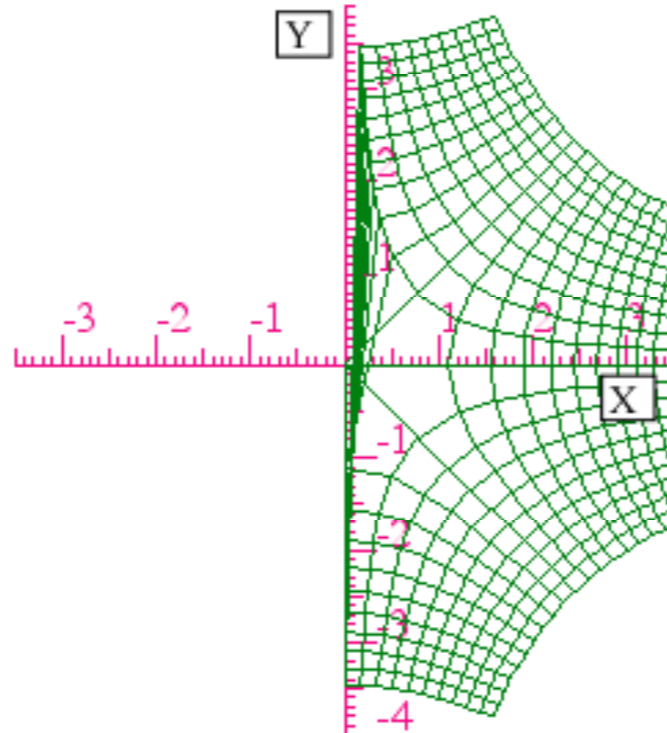
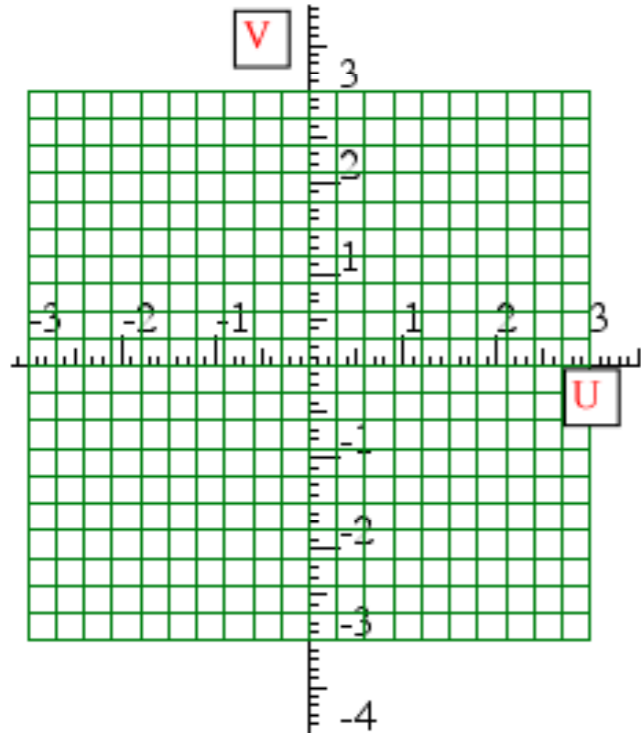
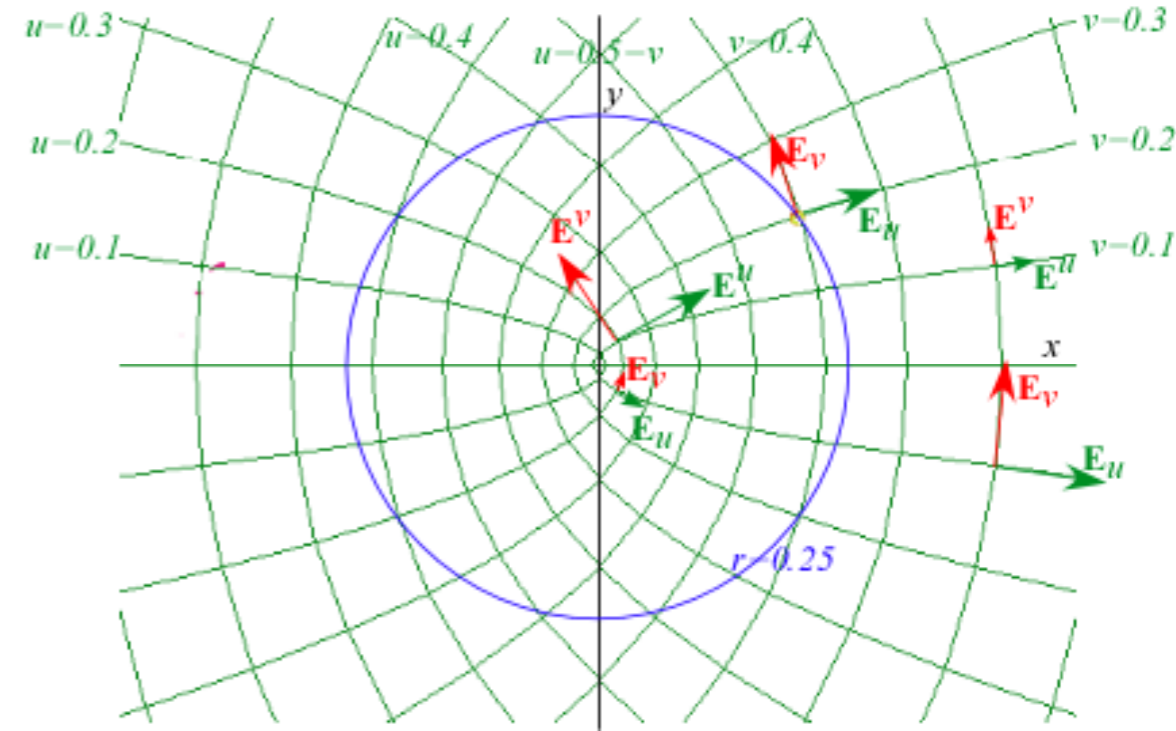
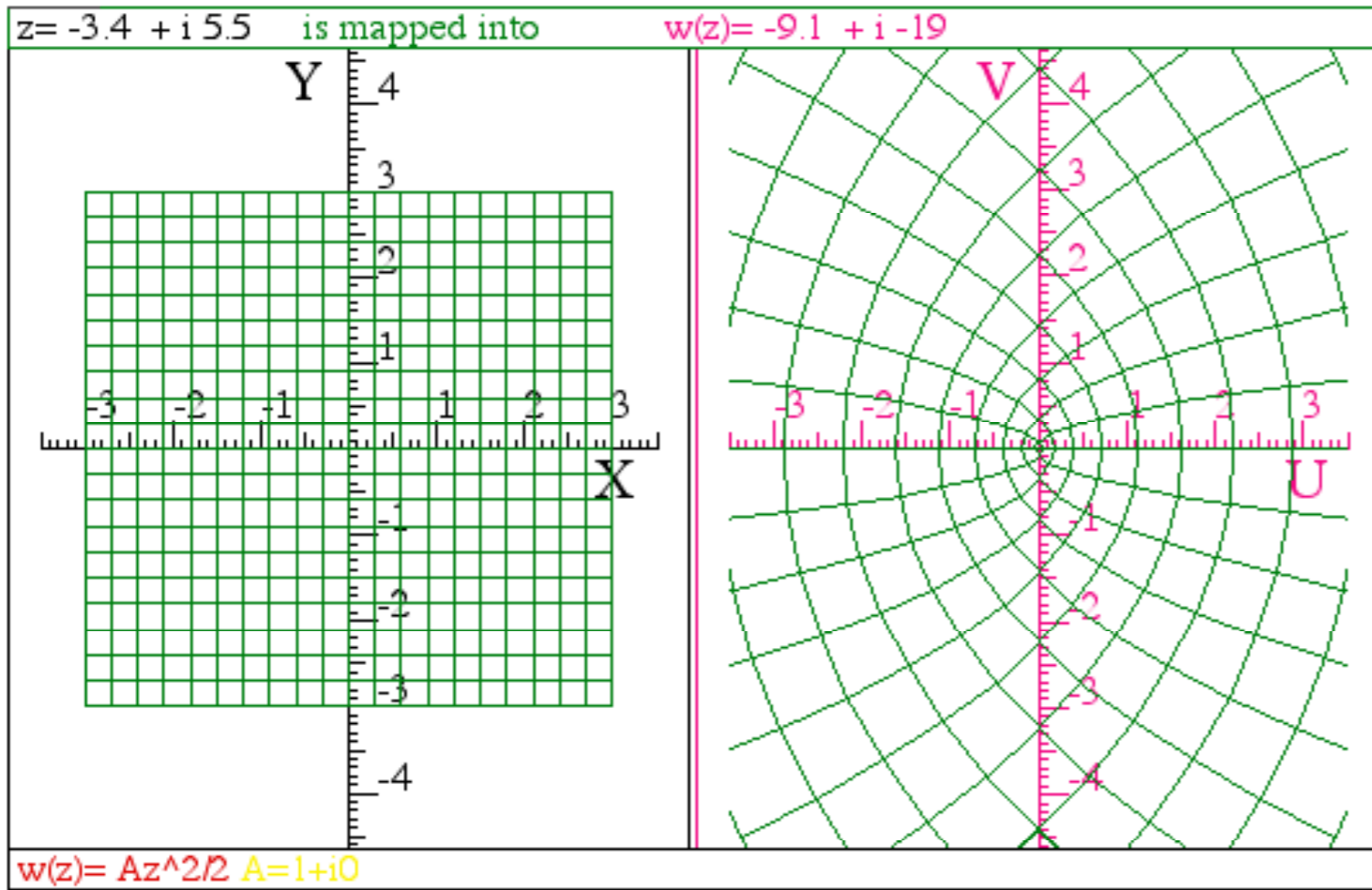
$$\begin{aligned} \frac{dw}{dz} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + iv) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \end{aligned}$$

Complex derivative abs-square:

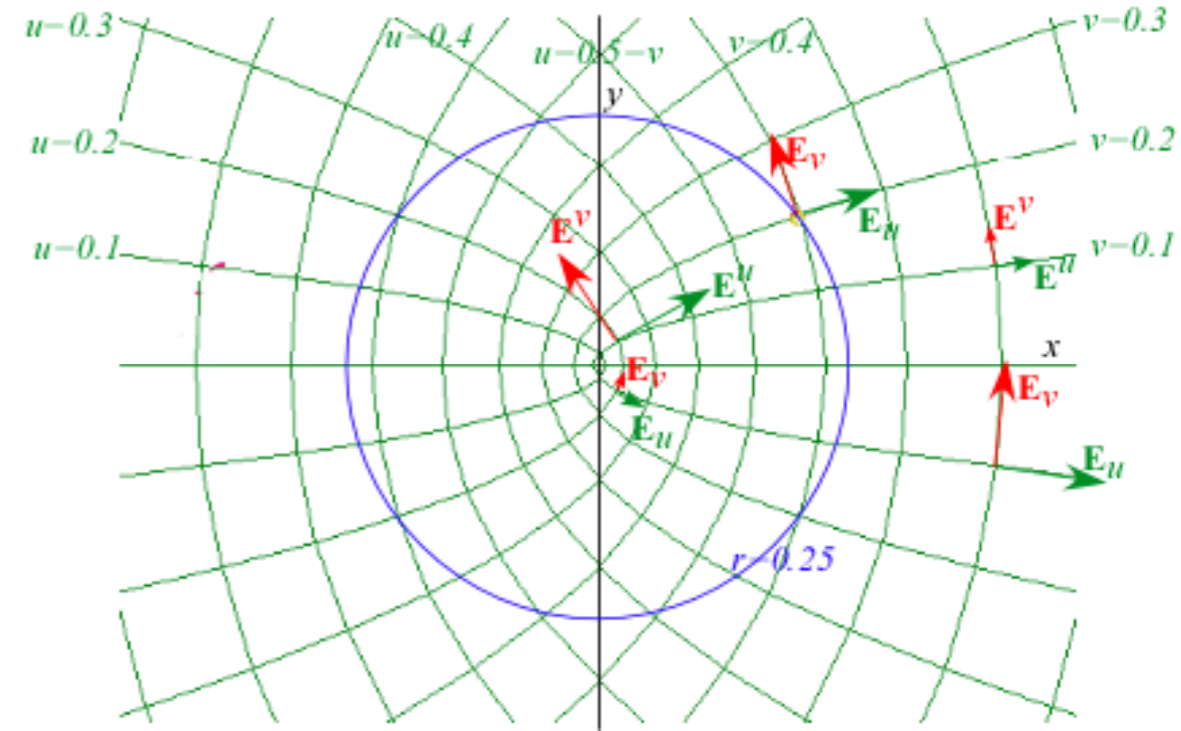
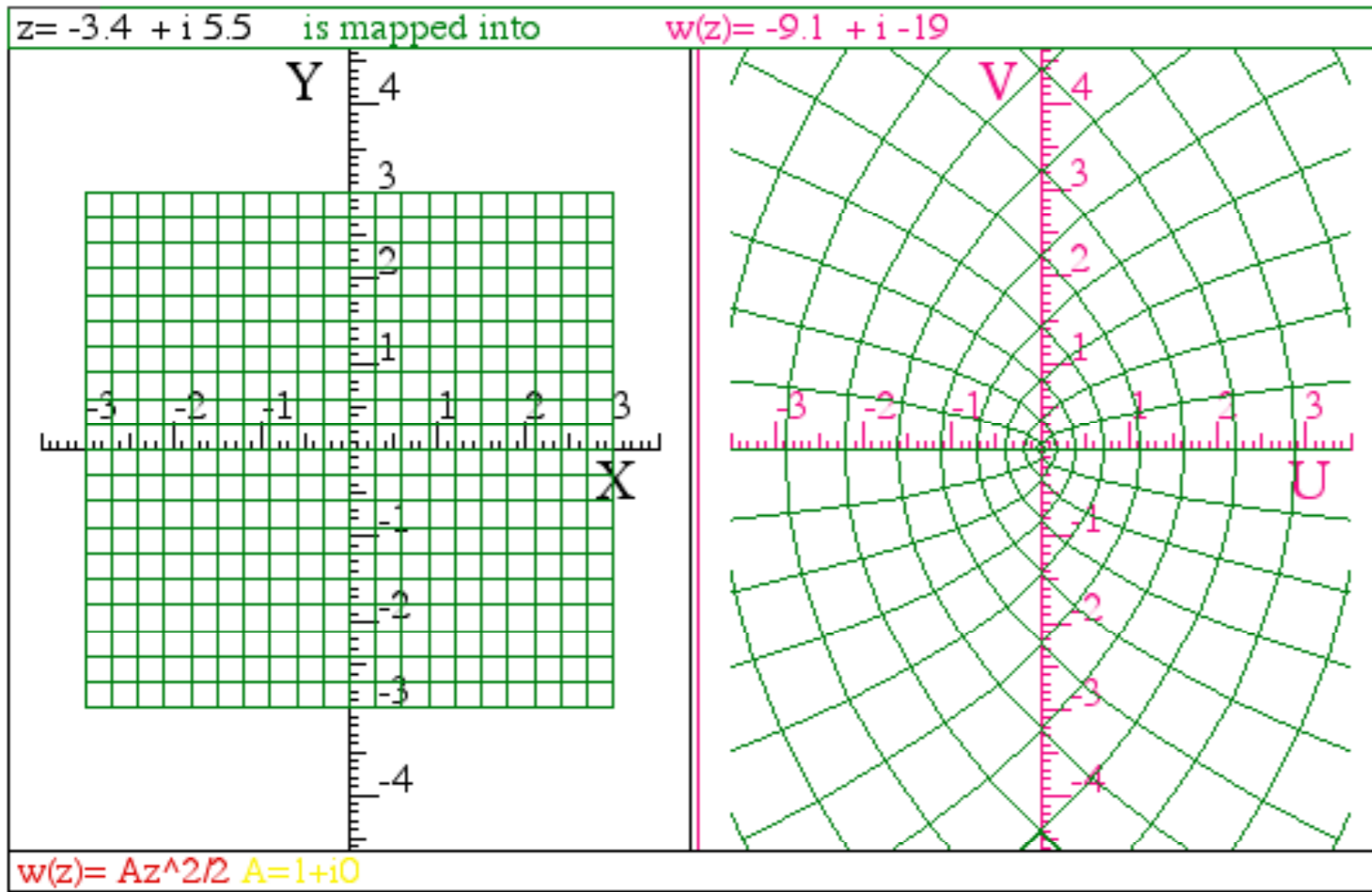
$$\left| \frac{dw}{dz} \right|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \det|J|$$

...equals Jacobian Determinant

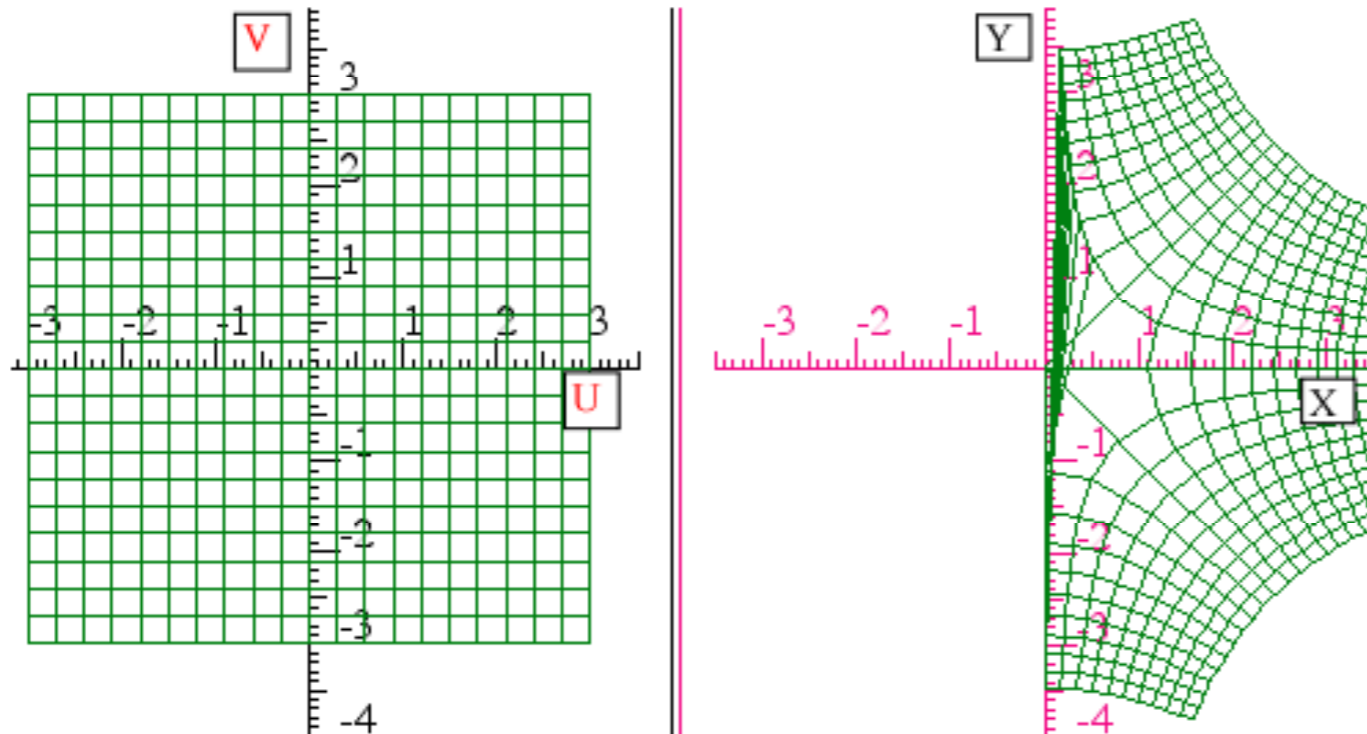
$w(z) = z^2$ gives parabolic OCC



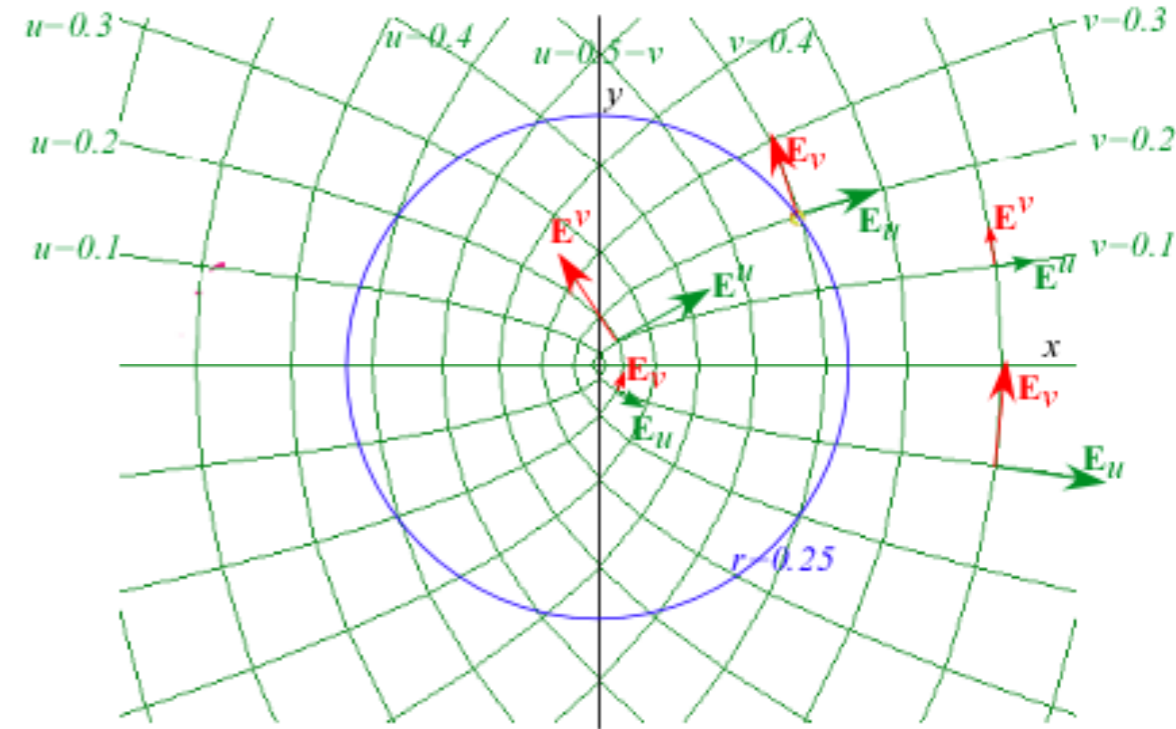
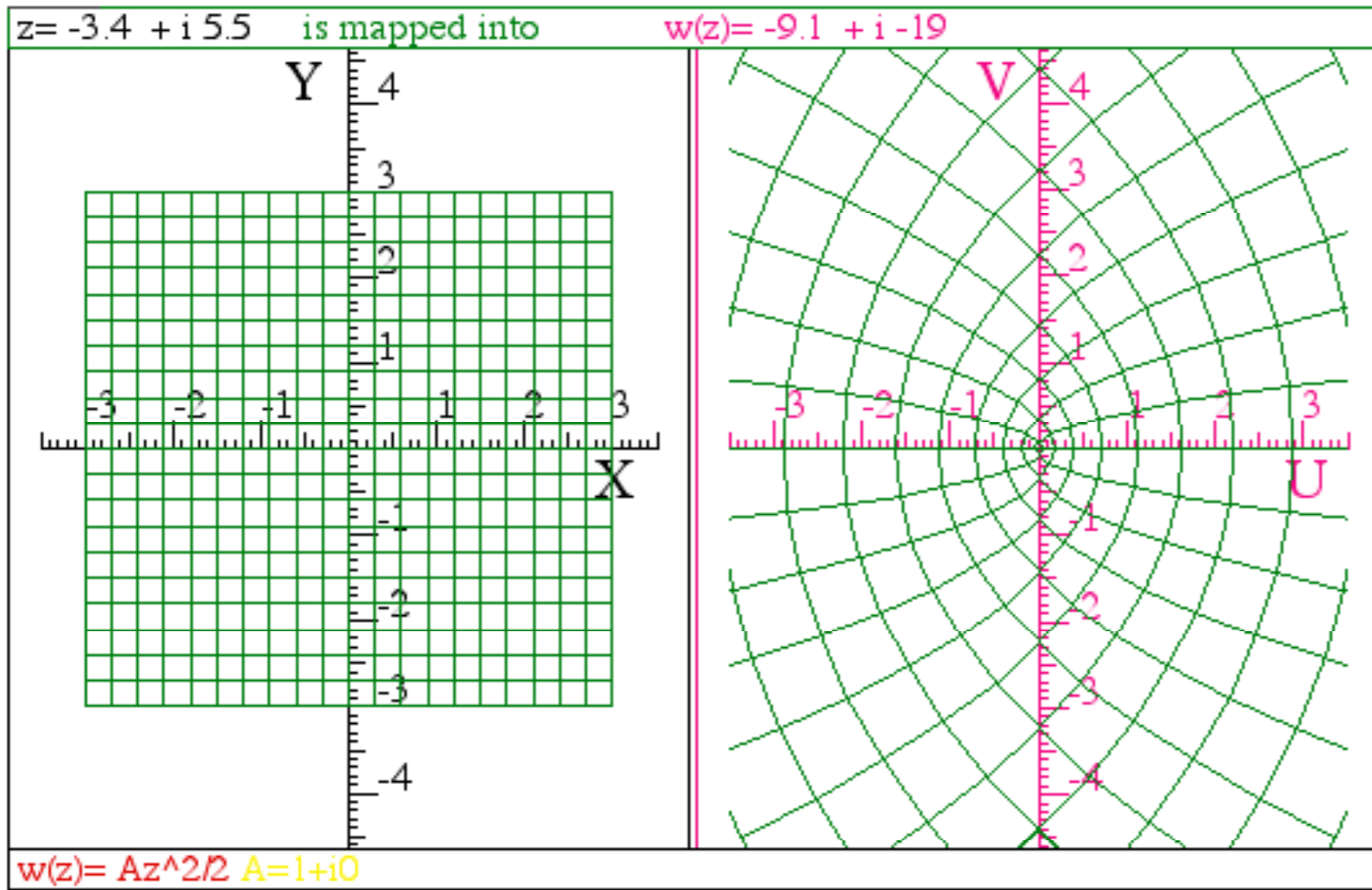
$w(z) = z^2$ gives parabolic OCC



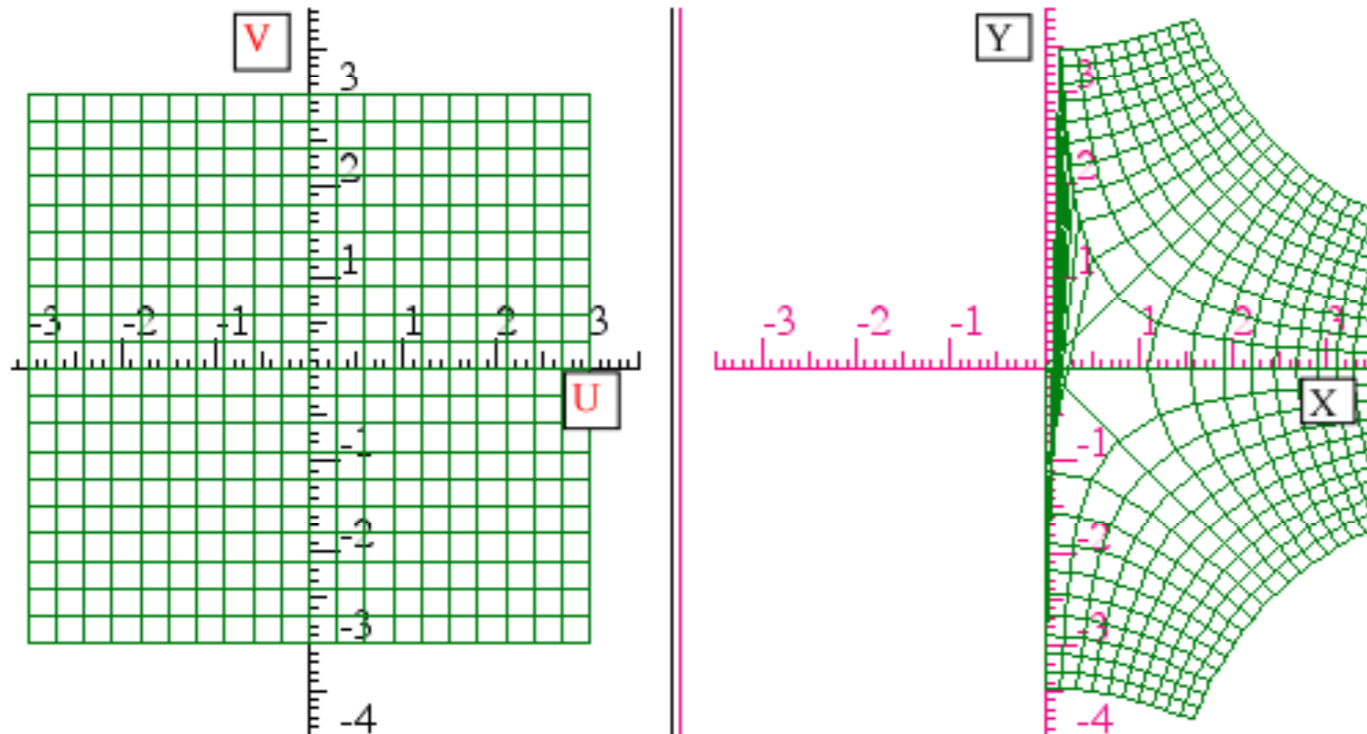
Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



$w(z) = z^2$ gives parabolic OCC



Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



$w = (u + iv) = z^2 = (x + iy)^2$ is analytic function of z and w

Expansion: $u = x^2 - y^2$ and $v = 2xy$ may be solved using $|w| = |z^2| = |z|^2$

Expansion: $|w| = \sqrt{u^2 + v^2} = x^2 + y^2 = |z|^2$

Solution: $x^2 = \frac{u + \sqrt{u^2 + v^2}}{2}$ $y^2 = \frac{-u + \sqrt{u^2 + v^2}}{2}$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}^u \\ \bar{\mathbf{E}}^v \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ +2y & 2x \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}_u & \bar{\mathbf{E}}_v \end{pmatrix} = \begin{pmatrix} 2x & +2y \\ -2y & 2x \end{pmatrix} / 4(x^2 + y^2)$$

Non-analytic potential, force, and source field functions

A general 2D complex field may have:

1. non-analytic *potential field function* $\phi(z, z^*) = \Phi(x, y) + iA(x, y)$,
2. non-analytic *force field function* $f(z, z^*) = f_x(x, y) + if_y(x, y)$,
3. non-analytic *source distribution function* $s(z, z^*) = \rho(x, y) + iI(x, y)$.

Source definitions are made to generalize the \mathbf{f}^* field equations (10.33) based on relations (10.31) and (10.32).

$$2 \frac{df^*}{dz} = s^*(z, z^*) \qquad 2 \frac{df}{dz^*} = s(z, z^*)$$

Field equations for the potentials are like (10.33) with an extra factor of 2.

$$2 \frac{d\phi}{dz} = f(z, z^*) \qquad 2 \frac{d\phi^*}{dz^*} = f^*(z, z^*)$$

Source equations (10.46) expand like (10.32) into a real and imaginary parts of divergence and curl terms.

$$\begin{aligned} s^*(z, z^*) = 2 \frac{df^*}{dz} &= \left[\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right] \left[f_x^*(x, y) + if_y^*(x, y) \right] = \rho - iI, \quad \text{where: } f_x^* = f_x, \text{ and: } f_y^* = -f_y \\ &= \left[\frac{\partial f_x^*}{\partial x} + \frac{\partial f_y^*}{\partial y} \right] + i \left[\frac{\partial f_y^*}{\partial x} - \frac{\partial f_x^*}{\partial y} \right] = \left[\nabla \cdot \mathbf{f}^* \right] + i \left[\nabla \times \mathbf{f}^* \right]_z \end{aligned}$$

Real part: *Poisson scalar source equation (charge density ρ)*. Imaginary part: *Biot-Savart vector source equation (current density I)*
 $\nabla \cdot \mathbf{f}^* = \rho$ $\nabla \times \mathbf{f}^* = -I$

Field equations (10.47) expand into Re and Im parts; x and y components of $\text{grad } \Phi$ and $\text{curl } A_z$ from potential $\phi = \Phi + iA$ or $\phi^* = \Phi - iA$.

$$\begin{aligned} f^*(z, z^*) = 2 \frac{d\phi^*}{dz^*} &= \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] (\Phi - iA) = f_x^* + if_y^* \\ &= \left[\frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right] + \left[\frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right] = \left[\nabla \Phi \right] + \left[\nabla \times \mathbf{A}_z \right] \end{aligned}$$

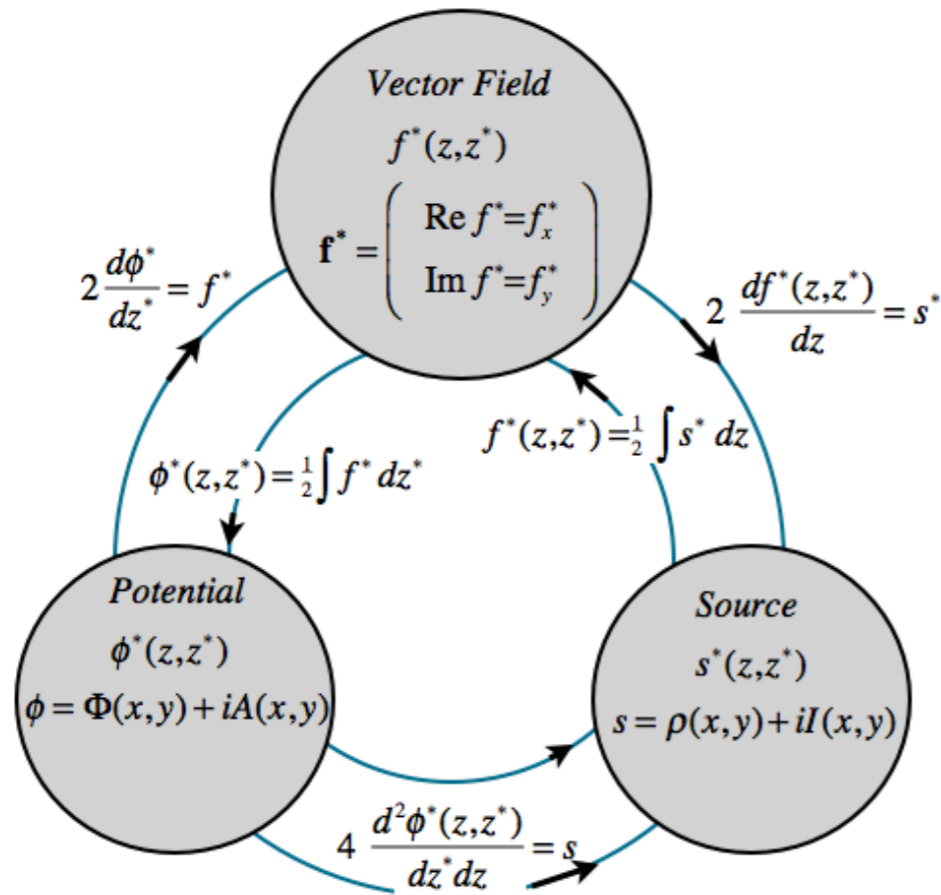
Two parts: gradient of scalar potential called the *longitudinal field* \mathbf{f}_L^* and curl of a vector potential called the *transverse field* \mathbf{f}_T^* .

$$\mathbf{f}^* = \mathbf{f}_L^* + \mathbf{f}_T^* \qquad \mathbf{f}_L^* = \nabla \Phi \qquad \mathbf{f}_T^* = \nabla \times \mathbf{A}$$

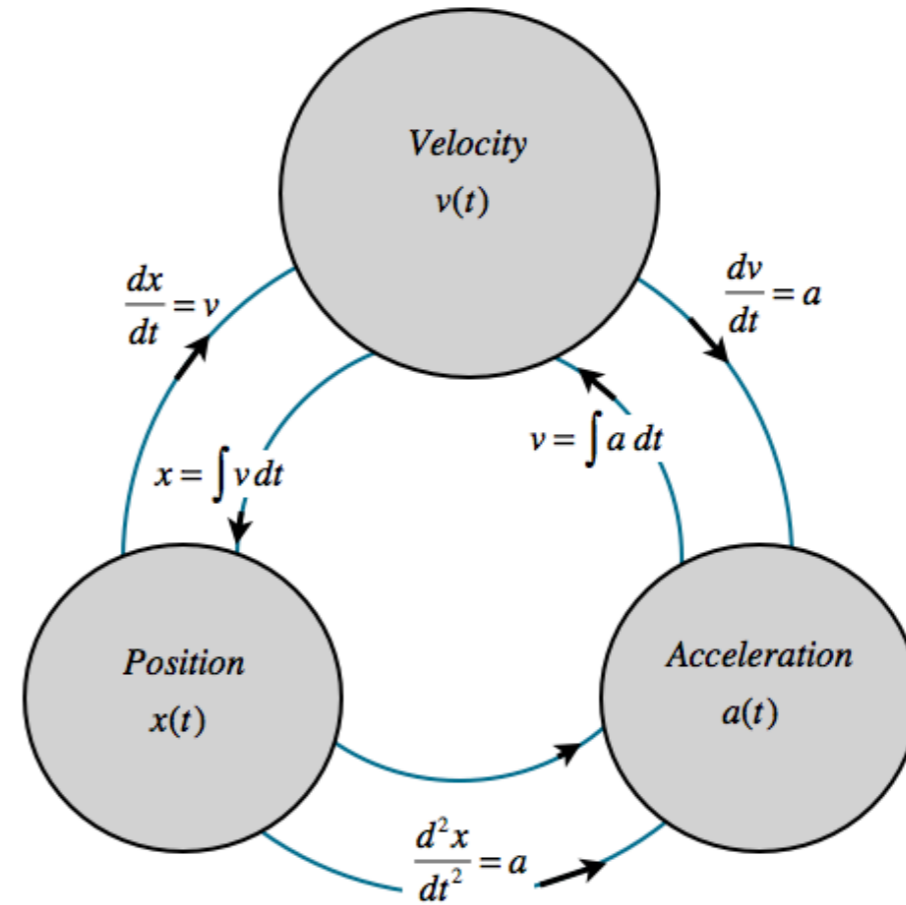
(For source-free analytic functions these two fields are identical.)

Potential, force, and source field equations vs. position, velocity, and acceleration equations

Field equations



Newton equations



Example 1

Consider a non-analytic field $f(z) = (z^*)^2$ or $f^*(z) = z^2$.

The non-analytic potential function follows by integrating

$$s^*(z, z^*) = 2 \frac{df^*}{dz} = 4z = 4x + i4y,$$

$$\text{or: } \rho = 4x, \quad \text{and: } I = -4y.$$

$$\phi(z, z^*) = \frac{1}{2} \int f(z) dz = \frac{1}{2} \int (z^*)^2 dz = \frac{z(z^*)^2}{2} = \frac{(x+iy)(x^2-y^2-i2xy)}{2},$$

$$\text{or: } \Phi = \frac{x^3 + xy^2}{2}, \quad \text{and: } A = \frac{-y^3 - yx^2}{2}.$$

The longitudinal field \mathbf{f}_L^* is quite different from the transverse field \mathbf{f}_T^* .

$$\mathbf{f}_L^* = \nabla \Phi = \nabla \left(\frac{x^3 + xy^2}{2} \right) = \left(\frac{3x^2 + y^2}{2}, xy \right), \quad \mathbf{f}_T^* = \nabla \times \mathbf{A} = \nabla \times \left(\frac{-y^3 - yx^2}{2} \mathbf{e}_z \right) = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x} \right) = \left(\frac{-3y^2 - x^2}{2}, xy \right).$$

The longitudinal field \mathbf{f}_L^* has no curl and the transverse field \mathbf{f}_T^* has no divergence. The sum field has both making a violent storm, indeed, as shown by a plot of in Fig. 10.17.

$$\mathbf{f}^* = \mathbf{f}_L^* + \mathbf{f}_T^* = \left(\frac{3x^2 + y^2}{2}, xy \right) + \left(\frac{-3y^2 - x^2}{2}, xy \right) = \left(\frac{x^2 - y^2}{2}, 2xy \right), \quad \nabla \cdot \mathbf{f}^* = \nabla \cdot \mathbf{f}_L^* = 4x = \rho, \quad \nabla \times \mathbf{f}^* = \nabla \times \mathbf{f}_T^* = 4y = -I.$$

