Equations of Lagrange and Hamilton mechanics in Generalized Curvilinear Coordinates (GCC)

(Ch. 12 of Unit 1 and Ch. 1-5 of Unit 2 and Ch. 1-5 of Unit 3)

Quick Review of Lagrange Relations in Lectures 8-9

Using differential chain-rules for coordinate transformations
  Polar coordinate example of Generalized Curvilinear Coordinates (GCC)
  Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
  Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2

How to say Newton’s “F=ma” in Generalized Curvilinear Coords.
  Use Cartesian KE quadratic form $KE=T=\frac{1}{2}v\cdot M\cdot v$ and $F=M\cdot a$ to get GCC force
  Lagrange GCC trickery gives Lagrange force equations
  Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)

GCC Cells, base vectors, and metric tensors
  Polar coordinate examples: \textbf{Covariant} $E_m$ vs. \textbf{Contra}variant $E^m$
  \textbf{Covariant} $g_{mn}$ vs. \textbf{In}variant $\delta_m^n$ vs. \textbf{Contra}variant $g^{mn}$

Lagrange prefers \textbf{Covariant} $g_{mn}$ with \textbf{Contra}variant velocity
  GCC Lagrangian definition
  GCC “canonical” momentum $p_m$ definition
  GCC “canonical” force $F_m$ definition
  Coriolis “fictitious” forces (... and weather effects)
Quick Review of Lagrange Relations in Lectures 9-10

$0^{th}$ and $1^{st}$ equations of Lagrange and Hamilton
Quick Review of Lagrange Relations in Lectures 9-10

0th and 1st equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and Estrangian have no explicit dependence on momentum \( p \)

\[
\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}
\]

Hamiltonian and Estrangian have no explicit dependence on velocity \( v \)

\[
\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}
\]

Lagrangian and Hamiltonian have no explicit dependence on speedinum \( V \)

\[
\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}
\]

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections

\[
\nabla_v L = \frac{\partial L}{\partial v} = \frac{\partial}{\partial v} \left( v \cdot M \cdot v \right) / 2 = M \cdot v = p
\]

\[
\begin{pmatrix}
\frac{\partial L}{\partial v_1} \\
\frac{\partial L}{\partial v_2}
\end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}
\]

Lagrange’s 1st equation(s)

\[
\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial v} = p
\]

\[
\nabla_p H = v = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \left( p \cdot M^{-1} \cdot p \right) / 2 = M^{-1} \cdot p = v
\]

\[
\begin{pmatrix}
\frac{\partial H}{\partial p_1} \\
\frac{\partial H}{\partial p_2}
\end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]

Hamilton’s 1st equation(s)

\[
\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial p} = v
\]

p. 25 of Lecture 9
(a) **Lagrangian plot**

\[ L(v) = \text{const.} = v \cdot M \cdot v/2 \]

\[ v_2 = \frac{p_2}{m_2} \]

\[ L = \text{const.} = E \]

(b) **Hamiltonian plot**

\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p/2 \]

\[ p_2 = m_2 v_2 \]

\[ H = \text{const.} = E \]

(c) **Overlapping plots**

**Lagrangian tangent at velocity** \( v \)

\[ p = \nabla_v L = M \cdot v \]

**Hamiltonian tangent at momentum** \( p \)

\[ p = \nabla_p H = M^{-1} \cdot p \]

\[ v = \nabla_p H = \frac{M^{-1} \cdot p}{M} \]

(d) **Less mass**

(e) **More mass**
(a) Lagrangian plot
\[ L(v) = \text{const.} = v \cdot M \cdot v/2 \]

(b) Hamiltonian plot
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p/2 \]

\[ p_2 = m_2 v_2 \]

\[ p_1 = m_1 v_1 \]

(c) Overlapping plots

1st equation of Lagrange
\[ L = \text{const.} = E \]

1st equation of Hamilton
\[ H = \text{const.} = E \]

(d) Less mass

(e) More mass

\[ p = \nabla_v L = M \cdot v \]

\[ p = \nabla_p H = M^{-1} \cdot p \]

\[ \text{Lagrangian tangent at velocity } v \]

\[ \text{is normal to momentum } p \]

\[ \text{Hamiltonian tangent at momentum } p \]

\[ \text{is normal to velocity } v \]
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions $f(x,y)$ and $g(x,y)$ can define a coordinate system on $(x,y)$-space, for example: polar coordinates

$$r^2(x,y)=x^2+y^2 \text{ and } \theta(x,y)=\text{atan2}(y,x)$$

d$r(x,y)=\frac{\partial r}{\partial x}dx+\frac{\partial r}{\partial y}dy$

d$\theta(x,y)=\frac{\partial \theta}{\partial x}dx+\frac{\partial \theta}{\partial y}dy$ (Not in text. Recall Lecture 9 p. 15-19)
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space for example: polar coordinates

\[
df(x,y) = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]

\[
dg(x,y) = \frac{\partial g}{\partial x} \, dx + \frac{\partial g}{\partial y} \, dy
\]

\[
dr(x,y) = \frac{\partial r}{\partial x} \, dx + \frac{\partial r}{\partial y} \, dy \quad \text{(Not in text. Recall Lecture 9 p. 15-19)}
\]

\[
d\theta(x,y) = \frac{\partial \theta}{\partial x} \, dx + \frac{\partial \theta}{\partial y} \, dy
\]

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
dx = \frac{\partial x}{\partial f} \, df + \frac{\partial y}{\partial g} \, dg
\]

\[
dy = \frac{\partial y}{\partial f} \, df + \frac{\partial y}{\partial g} \, dg
\]

\[
x = r \cos \theta
\]

\[
y = r \sin \theta
\]

\[
\frac{dx}{dx} \, dr + \frac{dy}{dr} \, d\theta
\]

\[
\frac{dy}{dx} \, dr + \frac{dy}{d\theta} \, d\theta
\]

\[
\begin{pmatrix}
  dx \\
  dy
\end{pmatrix} =
\begin{pmatrix}
  \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
  \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix}
\begin{pmatrix}
  dr \\
  d\theta
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta & -r \sin \theta \\
  \sin \theta & r \cos \theta
\end{pmatrix}
\begin{pmatrix}
  dr \\
  d\theta
\end{pmatrix}
\]
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space, for example: polar coordinates \( r^2(x,y) = x^2 + y^2 \) and \( \theta(x,y) = \text{atan2}(y,x) \) \( dr(x,y) = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \)
\( d\theta(x,y) = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy \)

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
\begin{align*}
dx &= \frac{\partial x}{\partial f} df + \frac{\partial y}{\partial g} dg \\
dy &= \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg
\end{align*}
\]
\( x = r \cos \theta \)
\( y = r \sin \theta \)

\[
\begin{pmatrix}
dx \\ dy
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix}
\begin{pmatrix}
dr \\ d\theta
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta
\end{pmatrix}
\begin{pmatrix}
dr \\ d\theta
\end{pmatrix}
\]

Notation for differential GCC (Generalized Curvilinear Coordinates \( \{q^1, q^2, q^3, \ldots \} \))

\[
dx^j = \frac{\partial x^j}{\partial q^m} dq^m \equiv \sum_{m=1}^{N} \frac{\partial x^j}{\partial q^m} dq^m \quad \{ \text{Defining a shorthand dummy-index } m \text{-sum} \}
\]

These \( x^j \) are plain old CC (Cartesian Coordinates \( \{dx^1=dx, dx^2=dy, dx^3=dx, dx^4=dt\} \))

† (Not in text. Recall Lecture 9 p. 15-19)

What does “\( q \)” stand for?
One guess: “Queer”
And they do get pretty queer!
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x, y) \) and \( g(x, y) \) can define a coordinate system on \((x, y)\)-space for example: polar coordinates

\[
dr(x, y) = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy
\]

\[
d\theta(x, y) = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
\]

(Not in text. Recall Lecture 9 p. 15-19)

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
dx = \frac{\partial x}{\partial f} df + \frac{\partial x}{\partial g} dg
\]

\[
dy = \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg
\]

\[
x = r \cos \theta
\]

\[
y = r \sin \theta
\]

Notation for differential GCC (Generalized Curvilinear Coordinates \(\{q^1, q^2, q^3,...\}\))

\[
dx^j = \frac{\partial x^j}{\partial q^m} dq^m \equiv \sum_{m=1}^{N} \frac{\partial x^j}{\partial q^m} dq^m \quad \text{Defining a shorthand dummy-index} \ m\text{-sum}
\]

Connection lines may help to indicate summation (OK on scratch paper...Difficult in text)

These \(x\)s are plain old CC (Cartesian Coordinates \(\{dx^1=dx, \ dx^2=dy, \ dx^3=dx, \ dx^4=dt\}\) )
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Getting the GCC ready for mechanics:

*Generalized velocity relation follows from GCC chain rule*

\[
\begin{align*}
 dx^j &= \frac{\partial x^j}{\partial q^m} dq^m \\
 \dot{x}^j &= \frac{\partial x^j}{\partial q^m} \dot{q}^m
\end{align*}
\]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( \nu^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \).
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv q^m \equiv \frac{dq^m}{dt} \)

This is a key “lemma-1” for setting up mechanics:
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule:

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \):

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

This is a key "lemma-1" for setting up mechanics:

\[ \frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \]

or:

\[ \frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \]

Jacobian \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ J^j_m = \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \]

Defining Jacobian matrix component:

Recall polar coordinate transformation matrix:

\[ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]
**Getting the GCC ready for mechanics:**

Generalized velocity relation follows from GCC chain rule:

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( \nu^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \):

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

This is a key “lemma-1” for setting up mechanics:

\[ \frac{\partial x^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \]

**Jacobian** \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ J^j_m \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \quad \text{Defining Jacobian} \]

\[ \text{Recall polar coordinate transformation matrix:} \]

\[ \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

Inverse (so-called) **Kajobian** \( K^m_j \) matrix is flipped partial derivatives of \( J^j_m \).

\[ K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial x^j} \quad \text{Defining "Kajobian" (inverse to Jacobian)} \]

\[ \text{Polar coordinate inverse transformation matrix:} \]

\[ \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

\[ \text{Defining 2x2 matrix inverse: (always test inverse matrices!)} \]

\[ \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \]

\[ \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

\[ = \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

\[ \text{(det} J = r) \]

\[ = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

\[ \text{Defining 2x2 matrix inverse: (always test inverse matrices!)} \]

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \]

\[ \text{AD} - BC \]
Getting the GCC ready for mechanics:
Generalized velocity relation follows from GCC chain rule

$$dx^j = \frac{\partial x^j}{\partial q^m} dq^m$$

Same kind of linear relation exists between CC velocity and GCC velocity

$$v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \quad \text{and GCC velocity} \quad v^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$$

This is a key "lemma-1" for setting up mechanics:

$$\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \quad \text{or:} \quad \frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}$$

Jacobian \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

$$J^j_m \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \{ \text{Defining Jacobian matrix component} \}$$

Recall polar coordinate transformation matrix:

$$\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}$$

Inverse (so-called) Kajobian \( K^m_j \) matrix is flipped partial derivatives of \( J^j_m \).

$$K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial \dot{x}^j} \quad \{ \text{Defining "Kajobian" (inverse to Jacobian)} \}$$

$$\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix}^{-1} =
\begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\
\frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-sin \theta & \cos \theta
\end{pmatrix}$$

Polar coordinate inverse transformation matrix:

$$\begin{pmatrix}
r \cos \theta & r \sin \theta \\
-sin \theta & \cos \theta
\end{pmatrix} =
\begin{pmatrix}
\frac{\cos \theta}{r} & \frac{\sin \theta}{r} \\
\frac{\sin \theta}{r} & \frac{\cos \theta}{r}
\end{pmatrix}$$

Defining 2x2 matrix inverse: (always test inverse matrices!)

$$\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} =
\begin{pmatrix}
D & -B \\
-C & A
\end{pmatrix} =
\begin{pmatrix}
D & -B \\
AD-BC & A
\end{pmatrix}$$

$$\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \left( \begin{pmatrix}
D & -B \\
-C & A
\end{pmatrix} \right) =
\begin{pmatrix}
AD-BC & 0 \\
0 & AD-BC
\end{pmatrix}$$
Getting the GCC ready for mechanics:
Generalized velocity relation follows from GCC chain rule

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( \dot{v}^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \)

This is a key “lemma-1” for setting up mechanics:

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]
\[ \frac{\partial x^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \]

\[ \frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \]

Jacobian \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ J^j_m \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial q^m} \] \{ Defining Jacobian matrix component \}

Recall polar coordinate transformation matrix:

\[ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

Inverse (so-called) Kajobian \( K^m_j \) matrix is flipped partial derivatives of \( J^j_m \).

\[ K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial \dot{x}^j} \] \{ Defining "Kajobian" \}
\{ inverse to Jacobian \}

Product of matrix \( J^j_m \) and \( K^m_j \) is a unit matrix by definition of partial derivatives.

\[ K^m_j \cdot J^j_n \equiv \frac{\partial q^m}{\partial x^j} \cdot \frac{\partial x^j}{\partial q^n} = \frac{\partial q^m}{\partial q^n} = \delta^m_n = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases} \]

Always test inverse matrices!
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1

Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule:

\[
\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m
\]
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt} (u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m$$

Apply derivative chain sum to Jacobian.

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt}$$
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule:

$$
\dot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m
$$

(Not in text. Recall Lecture 9 p. 15-19)$^\dagger$

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^m \partial q^n} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \frac{dq^n}{dt} \right)
$$
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m$$


Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right) = \frac{\partial}{\partial q^m} (\dot{x}^j)$$

By chain-rule def. of CC velocity:

By chain-rule def. of CC velocity:
Getting the GCC ready for mechanics (2nd part)
Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\dot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m$$

(Not in text. Recall Lecture 9 p. 15-19)

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial^m_n \partial_n^m = \partial_m^n \partial_n^m$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \frac{d}{\partial q^n} \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{d}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right) = \frac{\partial}{\partial q^m} \left( \dot{x}^j \right)$$

By chain-rule def. of CC velocity:

This is the key "lemma-2" for setting up Lagrangian mechanics.

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m} \text{ lemma}$$
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Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule:

$$\dot{x}^j = \frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$$

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^n}\left(\frac{\partial^2 x^j}{\partial q^m \partial q^n}\right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt}\right)$$

By chain-rule def. of CC velocity:

$$\frac{\partial}{\partial q^m}(\dot{x}^j)$$

The “lemma-1” was in the GCC velocity analysis just before this one for acceleration.

This is the key “lemma-2” for setting up Lagrangian mechanics.

$$\frac{\partial x^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}$$  \(\text{lemma 1}\)

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m}$$  \(\text{lemma 2}\)
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE = T = \frac{1}{2} v \cdot M \cdot v$ and $F = M \cdot a$ to get GCC force.

Lagrange GCC trickery gives Lagrange force equations.

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2).
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II
Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy $\frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$

\[ T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \ddot{x}^j \ddot{x}^k \]  

where: $M_{jk}$ are CC inertia constants

Multidimensional CC version of Newt-II ($\mathbf{F} = \mathbf{M} \cdot \mathbf{a}$) using $M_{jk}$ constants

\[ f_j = M_{jk} a^k = M_{jk} \ddot{x}^k \]
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy $\frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$

$$T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$$

where: $M_{jk}$ are inertia constants that are symmetric: $M_{jk} = M_{kj}$

Multidimensional CC version of Newt-II ($F = \mathbf{M} \cdot \mathbf{a}$) using $M_{jk}$ constants

$$f_j = M_{jk} a^k = M_{jk} \dot{x}^k$$

Multidimensional CC version of work-energy differential ($dW = F \cdot dx$). Insert GCC differentials $dq^m$

$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \dot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

(It’s time to bring in the queer $q^m$ !)
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\[
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\]

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = \mathbf{F} \cdot d\mathbf{x} \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

(\( dq^m \) are independent so \( dq^m \)-sum is true term-by-term. (Still holds if all \( dq^m \) are zero but one.)

\[
dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m
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$\left(\text{It's time to bring in the queer } q^m!\right)$

$dq^m$ are independent so $dq^m$-sum is true term-by-term. (Still holds if all $dq^m$ are zero but one.)

$$dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \implies F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}$$
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Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$

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$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

(It’s time to bring in the queer $q^m$ !)

$dq^m$ are independent so $dq^m$-sum is true term-by-term. (Still holds if all $dq^m$ are zero but one.)

$$dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \Rightarrow F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}$$

Here generalized GCC force component $F_m$ is defined:

$\text{where: } F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}$
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE=T=\frac{1}{2}v\cdot M\cdot v$ and $F=M\cdot a$ to get GCC force

Lagrange GCC trickery gives Lagrange force equations
Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)
Now Lagrange GCC trickery begins
Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set \( A = M_{jk} \ddot{x}^k \) and \( B = \frac{\partial x^j}{\partial q^m} \) with calc. formula:

\[
\ddot{A}B = \frac{d}{dt}(\dot{A}B) - \dot{A}\dot{B}
\]

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt}\left(M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}\right) - M_{jk} \ddot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)
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$$\ddot{AB} = \frac{d}{dt}(\dot{AB}) - \dot{A} \dot{B}$$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$$

Cartesian $M_{jk}$ must be constant for this to work
(Bye, Bye relativistic mechanics or QM!)
Now Lagrange GCC trickery begins
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Cartesian \( M_{jk} \) must be constant for this to work
(Bye, Bye relativistic mechanics or QM!)

\[
F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)
\]

Then convert \( \partial x^j \) to \( \partial \dot{x}^j \) by Lemma 1 and Lemma 2 on 2\textsuperscript{nd} term.

\[
F_m = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \left( \frac{\partial \dot{x}^j}{\partial q^m} \right)
\]
Now Lagrange GCC trickery begins

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Lagrange’s clever end game: First set $A = M_{jk} \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\ddot{AB} = \frac{d}{dt}(\dot{AB}) - \dot{AB}$

Then convert $\partial x^j$ to $\partial \dot{x}^j$ by Lemma 1 and Lemma 2 on 2nd term.

Simplify using: $M_{ij} v^j \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial}{\partial q} \left( \frac{v^j v^j}{2} \right)$ where $q$ may be $q^m$ or $q^m$
Now Lagrange GCC trickery begins
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Lagrange’s clever end game: First set $A = M_{jk} \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $$\ddot{AB} = \frac{d}{dt}(\dot{AB}) - \dot{AB}$$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$$

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$$F_m = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \right) - M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial q^m}$$

Simplify using: $$M_{ij} v^i \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial}{\partial q} \left( \frac{v^i v^j}{2} \right)$$ where $q$ may be $q^m$ or $q^m$

$$F_m = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^m} \left( M_{jk} \dot{x}^k \dot{x}^j \right) - \frac{\partial}{\partial q^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right)$$

The result is Lagrange’s GCC force equation in terms of kinetic energy: $$T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$$

$$F_m = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$ or: $$F = \frac{d}{dt} \frac{\partial T}{\partial v} - \frac{\partial T}{\partial r}$$
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE = \frac{1}{2}v \cdot M \cdot v$ and $F = M \cdot a$ to get GCC force

Lagrange GCC trickery gives Lagrange force equations

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)

If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$  

In GCC:  \[ F_m = -\frac{\partial U}{\partial q^m} \]

\[
F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}
\]
But, Lagrange GCC trickery is not yet done...
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If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$
In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes Lagrange’s GCC potential equation with a new definition for the Lagrangian: $L=T-U$.

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}$$

$L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$

$U(r)$ has NO explicit velocity dependence!
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)
If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$

In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

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<table>
<thead>
<tr>
<th>Lagrange’s 1st GCC equation (Defining GCC momentum)</th>
<th>d \frac{\partial L}{dt \partial \dot{q}^m} = \frac{\partial L}{\partial q^m}</th>
</tr>
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<tbody>
<tr>
<td>$p_m = \frac{\partial L}{\partial \dot{q}^m}$</td>
<td>Recall: $p = \frac{\partial L}{\partial v}$</td>
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<table>
<thead>
<tr>
<th>Lagrange’s 2nd GCC equation (Change of GCC momentum)</th>
<th>d \frac{p_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dp_m}{dt} = \frac{\partial L}{\partial q^m}$</td>
<td></td>
</tr>
</tbody>
</table>
GCC Cells, base vectors, and metric tensors

Polar coordinate examples: **Covariant** $E_m$ vs. **Contravariant** $E^m$

**Covariant** $g_{mn}$ vs. **Invariant** $\delta_m^n$ vs. **Contravariant** $g^{mn}$
A dual set of quasi-unit vectors show up in Jacobian $J$ and Kajobian $K$.

J-Columns are covariant vectors \( \{ E_1 = E_r, \ E_2 = E_\phi \} \)

K-Rows are contravariant vectors \( \{ E^1 = E^r, E^2 = E^\phi \} \)

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} \\
\frac{\partial x^2}{\partial q^2} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial y^1}{\partial q^1} & \frac{\partial y^2}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix} \leftarrow E^r = E^1
\]

\[
\begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix} \leftarrow E^\phi = E^2
\]

**Inverse polar definition:**

\( r^2 = x^2 + y^2 \) and \( \phi = \arctan(\frac{y}{x}) \)

**Derived from polar definition:** \( x = r \cos \phi \) and \( y = r \sin \phi \)

**a) Polar coordinate bases**
A dual set of quasi-unit vectors show up in Jacobian $J$ and Kajobian $K$.

$J$-Columns are covariant vectors $\{E_1 = E_r, \ E_2 = E_\phi\}$

K-Rows are contravariant vectors $\{E^1 = E^r, \ E^2 = E^\phi\}$

\[
\langle J \rangle = \left( \begin{array}{cc}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{array} \right) = \left( \begin{array}{cc}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{array} \right)
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\[
\left( \begin{array}{cc}
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\end{array} \right) \leftarrow E^\phi = E^2
\]

Derived from polar definition: $x = r \cos \phi$ and $y = r \sin \phi$

---

(a) Polar coordinate bases

(b) Covariant bases $\{E_1 E_2\} (Tangent)$

\[
dr = E_1 dq^1 + E_2 dq^2
\]

(c) Contravariant bases $\{E^1 E^2\} (Normal)$

\[
F = F_1 E^1 + F_2 E^2
\]

**Inverse polar definition:**

\[
r^2 = x^2 + y^2 \text{ and } \phi = \text{atan2}(y, x)
\]

**NOTE:** These are 2D drawings!  
**No 3D perspective**
Comparison: **Covariant** $E_m = \frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

Covariant bases $\{E_1, E_2\}$ match cell walls

$$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$$

is based on chain rule:

$$dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$$

NOTE: These are 2D drawings! No 3D perspective
**Comparison:** Covariant $E_m = -\frac{\partial r}{\partial q^m}$ vs. Contravariant $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

**Covariant bases** $\{E_1, E_2\}$ match cell walls

$$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$$

is based on chain rule: $dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$

$E_1$ follows tangent to $q^2 = \text{const.}$ ...

since only $q^1$ varies in $\frac{\partial r}{\partial q^1}$

while $q^2, q^3, ...$ remain constant

**NOTE:** These are 2D drawings!

No 3D perspective
Comparison: **Covariant** \( \mathbf{E}_m = -\frac{\partial \mathbf{r}}{\partial q^m} \) vs. **Contravariant** \( \mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m \)

**Covariant bases** \( \{ \mathbf{E}_1, \mathbf{E}_2 \} \) match cell walls

\[ \Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2 \]

is based on chain rule:

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2 \]

\( \mathbf{E}_1 \) follows **tangent** to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial \mathbf{r}}{\partial q^1} \) while \( q^2, q^3, \ldots \) remain constant

\( \mathbf{E}_m \) are convenient bases for extensive quantities like distance and velocity.

\[ \mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2} \]

*NOTE:* These are 2D drawings! *No 3D perspective*
Comparison: **Covariant**  \( E_m = \frac{\partial r}{\partial q^m} \) vs. **Contravariant**  \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

**Covariant bases** \( \{ E_1, E_2 \} \) match cell walls  

(\text{Tangent})  

\[ \Delta r = E_1 \Delta q^1 + E_2 \Delta q^2 \]

is based on chain rule:  

\[ dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2 \]

\( E_1 \) follows \text{tangent} to \( q^2 = \text{const.} \) ... since only \( q^1 \) varies in  

\( \frac{\partial r}{\partial q^1} \)  

while \( q^2, q^3, \ldots \) remain constant

**Contravariant** \( \{ E^1, E^2 \} \) match reciprocal cells  

(\text{Normal})  

\[ \frac{\partial q^2}{\partial r} = \nabla q^2 = E^2 \]

\[ F = F_1 E^1 + F_2 E^2 \]

\( E^1 \) is \text{normal} to \( q^1 = \text{const.} \) since  

gradient of \( q^1 \) \text{is vector sum}  

\[ \nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix} \]

\( E_m \) are convenient bases for \text{extensive} quantities like distance and velocity.  

\[ \mathbf{V} = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial r}{\partial q^1} + V^2 \frac{\partial r}{\partial q^2} \]

\text{NOTE: These are 2D drawings!  
No 3D perspective}
Comparison: **Covariant** $E_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

Covariant bases $\{\mathbf{E}_1, \mathbf{E}_2\}$ match cell walls

\[ \Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2 \]

is based on chain rule:

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2 \]

$E_1$ follows tangent to $q^2 = \text{const.}$ ...

since only $q^1$ varies in $\frac{\partial \mathbf{r}}{\partial q^1}$

while $q^2$, $q^3$, ... remain constant

$E_m$ are convenient bases for extensive quantities like distance and velocity.

\[ \mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2} \]

Contravariant $\{\mathbf{E}^1, \mathbf{E}^2\}$ match reciprocal cells

\[ \frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}^2 \]

\[ \mathbf{F} = F^1 \mathbf{E}^1 + F^2 \mathbf{E}^2 \]

$\mathbf{E}^1$ is normal to $q^1 = \text{const.}$ since

gradient of $q^1$ is vector sum $\nabla q^1 = \left( \frac{\partial q^1}{\partial x}, \frac{\partial q^1}{\partial y} \right)$

$E^m$ are convenient bases for intensive quantities like force and momentum.

\[ \mathbf{F} = F^1 \mathbf{E}^1 + F^2 \mathbf{E}^2 = F^1 \frac{\partial q^1}{\partial \mathbf{r}} + F^2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2 \]

NOTE: These are 2D drawings!

No 3D perspective
Comparison: \textbf{Covariant} $\mathbf{E}_m = -\frac{\partial \mathbf{r}}{\partial q^m}$ \textbf{vs.} \textbf{Contravariant} $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

\textit{Covariant bases} $\{ \mathbf{E}_1, \mathbf{E}_2 \}$ match cell walls

(Tangent)

$\Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$

is based on chain rule: $d \mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$

$\mathbf{E}_1$ follows \textit{tangent} to $q^2 = \text{const.}$ ...

since only $q^1$ varies in $\frac{\partial \mathbf{r}}{\partial q^1}$

while $q^2, q^3, \ldots$ remain constant

$\mathbf{E}_m$ are convenient bases for \textit{extensive} quantities like distance and velocity.

$\mathbf{v} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$

\textit{Contravariant} $\{ \mathbf{E}_1, \mathbf{E}_2 \}$ match reciprocal cells

(Normal)

$\frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}_2$

$\mathbf{F} = F_1 \mathbf{E}_1 + F_2 \mathbf{E}_2$

$\mathbf{E}^1$ is \textit{normal} to $q^1 = \text{const.}$ since

gradient of $q^1$ is vector sum $\nabla q^1 = \left( \frac{\partial q^1}{\partial x}, \frac{\partial q^1}{\partial y} \right)$

$\mathbf{E}^m$ are convenient bases for \textit{intensive} quantities like force and momentum.

$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$

\textit{Co-Contra} dot products $\mathbf{E}_m \cdot \mathbf{E}^n$ are \textit{orthonormal}:

$\mathbf{E}_m \cdot \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial q^n}{\partial \mathbf{r}} = \delta^n_m$
GCC Cells, base vectors, and metric tensors

Polar coordinate examples: **Covariant** $E_m$ vs. **Contravariant** $E^m$

**Covariant** $g_{mn}$ vs. **Invariant** $\delta_m^n$ vs. **Contravariant** $g^{mn}$
Covariant $g_{mn}$ vs. Invariant $\delta_m^n$ vs. Contravariant $g^{mn}$

$$E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}$$

Covariant metric tensor $g_{mn}$

$$E_m \cdot E^n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta_m^n$$

Invariant Kroneker unit tensor $\delta_m^n$

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}$$

Contravariant metric tensor $g^{mn}$
Co\textit{variant} $g_{mn}$ \textit{vs.} \textit{Invariant} $\delta_{m}^{n}$ \textit{vs.} \textit{Contravariant} $g^{mn}$

$$E_{m} \cdot E_{n} = \frac{\partial r}{\partial q^{m}} \cdot \frac{\partial r}{\partial q^{n}} \equiv g_{mn}$$

$$E_{m} \cdot E^{n} = \frac{\partial r}{\partial q^{m}} \cdot \frac{\partial q^{n}}{\partial r} = \delta_{m}^{n}$$

$$E^{m} \cdot E^{n} = \frac{\partial q^{m}}{\partial r} \cdot \frac{\partial q^{n}}{\partial r} \equiv g^{mn}$$

\textit{Covariant metric tensor} $g_{mn}$

\textit{Invariant Kroneker unit tensor} $\delta_{m}^{n}$

$$\delta_{m}^{n} \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

\textit{Contravariant metric tensor} $g^{mn}$

\textbf{Polar coordinate examples (again):}

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^{1}}{\partial q^{1}} & \frac{\partial x^{1}}{\partial q^{2}} \\ \frac{\partial x^{2}}{\partial q^{1}} & \frac{\partial x^{2}}{\partial q^{2}} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\ \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix} \leftarrow E^{r} = E^{1}$$

$$\uparrow E_{1} \quad \uparrow E_{2} \quad \uparrow E_{r} \quad \uparrow E_{\phi}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix} \leftarrow E^{r} = E^{1}$$

$$\uparrow E_{1} \quad \uparrow E_{2} \quad \uparrow E_{r} \quad \uparrow E_{\phi}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix} \leftarrow E^{r} = E^{1}$$

$$\uparrow E_{1} \quad \uparrow E_{2} \quad \uparrow E_{r} \quad \uparrow E_{\phi}$$
**Covariant** $g_{mn}$ \( \text{vs.} \) **Invariant** $\delta^m_n$ \( \text{vs.} \) **Contravariant** $g^{mn}$

\[
E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}
\]

\[
E^m \cdot E^n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta^m_n
\]

\[
E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}
\]

**Covariant** metric tensor $g_{mn}$

**Invariant** Kroneker unit tensor $\delta^m_n$

\[
\delta^m_n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\]

**Contravariant** metric tensor $g^{mn}$

Polar coordinate examples (again):

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi / r & \frac{\partial \phi}{\partial y} = \cos \phi / r
\end{pmatrix} \iff E^r = E^1 \quad \text{and} \quad E^\phi = E^2
\]

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix}
\]

\[
\begin{pmatrix}
\delta^r_r & \delta^\phi_r \\
\delta^r_\phi & \delta^\phi_\phi
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E^r & E_r \cdot E^\phi \\
E_\phi \cdot E^r & E_\phi \cdot E^\phi
\end{pmatrix}
\]

\[
\begin{pmatrix}
g^{rr} & g^{r\phi} \\
g^{\phi r} & g^{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
E^r \cdot E^r & E^r \cdot E^\phi \\
E^\phi \cdot E^r & E^\phi \cdot E^\phi
\end{pmatrix}
\]
Lagrange prefers **Covariant** $g_{mn}$ with **Contra**variant velocity $\dot{q}^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition

Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian $L=KE-U$ is supposed to be explicit function of velocity.

$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$
Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \ddot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = \dot{L}(\dot{q})$$

Use polar coordinate Covariant $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix}
= \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}$$
Lagrange prefers \textbf{Covariant} $g_{mn}$ with \textbf{Contravariant} velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$

Use polar coordinate \textbf{Covariant} $g_{mn}$ metric (page 53)

$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{r\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$
Lagrange prefers **Covariant** $g_{mn}$ with **Contra**variant velocity $\dot{q}^m$

GCC Lagrangian definition

GCC “canonical” momentum $p_m$ definition

GCC “canonical” force $F_m$ definition

Coriolis “fictitious” forces (... and weather effects)
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\[
L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})
\]

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)
\]

*(From preceding page)*
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix}
= \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}$$

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$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$: 

$$p_r = \frac{\partial L}{\partial \dot{r}} = M \ g_{rr} \dot{r} = M \ \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \ddot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$  Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \ddot{\phi}$$  Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.  

Wednesday, September 23, 2015
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity $\dot{q}^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\left( \begin{array}{cc} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{array} \right) = \left( \begin{array}{cc} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & r^2 \end{array} \right)$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. **1st L-equation** is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $M r^2$ automatically for the angular momentum $p_\phi = M r^2 \omega$.

*(From preceding page)*
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

\[
L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m)(E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})
\]

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} =
\begin{pmatrix}
E_r & E_r \\
E_\phi & E_\phi
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)
\]

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

\[
p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}
\]

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

\[
p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = Mr^2 \dot{\phi}
\]

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

2nd $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

\[
\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}
\]

Centrifugal force $Mr \omega^2$

\[
\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}
\]

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence.
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\mathbf{\dot{r}}, \mathbf{\dot{\phi}}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi \phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi \phi}$ gives moment-of-inertia factor $M r^2$ automatically for the angular momentum $p_\phi = M r^2 \omega$.

2nd $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi \phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \ddot{\phi}^2 - \frac{\partial U}{\partial r} \quad \text{Centrifugal force } M r \omega^2$$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from 1st $L$-equation: $\dot{p}_m \equiv \frac{d p_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \ddot{q}^n + g_{mn} \dot{q}^n) \quad \text{Equate it to } \dot{p}_m$ in 2nd $L$-equation:
Lagrange prefers **Covariant** $g_{mn}$ with **Contra**variant velocity $\dot{q}^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-U is supposed to be explicit function of velocity.

\[
L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (E_m \dot{q}^m)(E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})
\]

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} =
\begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)
\]

GCC Lagrange equations follow. **1st** $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

\[
p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}
\]

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot \mathbf{v}$ form

\[
p_\phi = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi} \dot{\phi} = Mr^2 \dot{\phi}
\]

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

**2nd** $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

\[
\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}
\]

Centrifugal force $Mr\omega^2$

\[
\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}
\]

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from **1st** $L$-equation:

\[
\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (\dot{q}_m \dot{q}^m + g_{mn} \ddot{q}^n)
\]

Equate it to $\dot{p}_m$ in **2nd** $L$-equation:

\[
(\text{From preceding page})
\]
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (\mathbf{E}_m \dot{q}^m) \cdot (\mathbf{E}_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
\mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\
\mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & r^2 \end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1\textsuperscript{st} $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r} \quad \text{Nothing too surprising; radial momentum } p_r \text{ has the usual linear } M \cdot v \text{ form}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi} \quad \text{Wow! } g_{\phi\phi} \text{ gives moment-of-inertia factor } M r^2 \text{ automatically for the angular momentum } p_\phi = M r^2 \omega.$$

2\textsuperscript{nd} $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \quad \text{Centrifugal force } M r \omega^2$$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi} \quad \text{Angular momentum } p_\phi \text{ is conserved if potential } U \text{ has no explicit } \phi \text{-dependence}$$

Find $\dot{p}_m$ directly from 1\textsuperscript{st} $L$-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \dot{q}^n + g_{mn} \ddot{q}^n) \quad \text{Equate it to } \dot{p}_m \text{ in 2\textsuperscript{nd} } L$-equation:

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M \dot{r} \quad \text{Centrifugal (center-fleeing) force equals total}$$

$$= M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \quad \text{Centripetal (center-pulling) force}$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (E_m \ddot{q}^m) \cdot (E_n \ddot{q}^n) - U = \frac{1}{2} M (g_{mn} \ddot{q}^m \ddot{q}^n) - U = L(\dot{q})$

Use polar coordinate Covariant $g_{mn}$ metric (page 53)

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$

GCC Lagrange equations follow. 1\textsuperscript{st} $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$

**Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form**

2\textsuperscript{nd} $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$\dot{p}_r = \frac{\partial L}{\partial \dot{r}} = \frac{M}{2} \frac{\partial g_{\phi \phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$

**Centrifugal force $Mr^2 \omega^2$**

Find $\dot{p}_m$ directly from 1\textsuperscript{st} $L$-equation: $\dot{p}_m = \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \ddot{q}^n) = M (g_{mn} \ddot{q}^n + g_{mn} \ddot{q}^m)$

Equate it to $\dot{p}_m$ in 2\textsuperscript{nd} $L$-equation:

$\dot{p}_\phi = \frac{\partial L}{\partial \dot{\phi}} = 0 - \frac{\partial U}{\partial \phi}$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

$\dot{p}_r = \frac{dp_r}{dt} = M \dot{r}$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$\dot{p}_\phi = \frac{dp_\phi}{dt} = 2 M r \dot{\phi} + M r^2 \ddot{\phi}$

**Torque relates to two distinct parts: Coriolis and angular acceleration**

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence
Rewriting GCC Lagrange equations:

\[ \dot{p}_r = \frac{dp_r}{dt} = M \ddot{r} \]

Centrifugal (center-fleeing) force equals total

\[ = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

Centripetal (center-pulling) force

\[ \dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2 M r \ddot{\phi} + M r^2 \dddot{\phi} \]

Torque relates to two distinct parts:

Coriolis and angular acceleration

\[ = 0 - \frac{\partial U}{\partial \phi} \]

Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence

Conventional forms

radial force: \( M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \)

Field-free (\( U=0 \))

radial acceleration: \( \ddot{r} = r \dot{\phi}^2 \)

angular force or torque: \( M r^2 \dddot{\phi} = -2 M r \ddot{\phi} - \frac{\partial U}{\partial \phi} \)

angular acceleration: \( \dddot{\phi} = -2 \frac{\ddot{r} \phi}{r} \)

Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Coriolis acceleration with \( \dot{\phi} > 0 \) and \( \ddot{r} < 0 \)

\( \dddot{\phi} = -2 \frac{\ddot{r} \phi}{r} \) (makes \( \phi \) positive)

Inward flow to pressure low

\( \dddot{r} < 0 \)

...makes wind turn to the right

Northern hemisphere rotation

\( \phi > 0 \)

Northern, September 23, 2015