Geometry and Symmetry of Coulomb Orbital Dynamics
(Ch. 2-4 of Unit 5  12.10.14)

Review and added: Rutherford scattering and differential scattering cross-sections
Parabolic “kite” and envelope geometry

Eccentricity vector $\mathbf{e}$ and $(\epsilon, \lambda)$-geometry of orbital mechanics

$\mathbf{e}$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\mathbf{e}$-vector and Coulomb $p=mv$ geometry

Analytic geometry derivation of $\mathbf{e}$-construction

Algebra of $\mathbf{e}$-construction geometry

Ruler & compass construction of $\mathbf{e}$-vector and orbits

$(R=-0.375$ elliptic orbit$)$

$(R=+0.5$ hyperbolic orbit$)$

Properties of Coulomb trajectory families and envelopes

Graphical $\mathbf{e}$-development of orbits

Launch angle fixed-Varied launch energy
Launch energy fixed-Varied launch angle
Review and added: Rutherford scattering and differential scattering cross-sections
Parabolic “kite” and envelope geometry
Rutherford scattering of $\alpha^+^2$ particles from $Au^{+79}$ nucleus at $O$.

Assume “Dead-On” closest approach $2a$.

(E=$k/2a$) \quad a \approx 10^{-11}m \gg 7.3 \cdot 10^{-15}m$

Pick an “impact parameter” line $y = b$.

Draw circle of radius $a$ around center point $C=(-a,b)$ tangent to $y$-axis.

Draw “focus-locus” line $OCF$.

Copy angle $\angle BCF$ (equal to $\Theta/2$) to make angle $\angle FCB'$ (also equal to $\Theta/2$).

Resulting line $CB'$ is outgoing asymptote at scattering angle $\Theta$.

Locate secondary focus $O'$ by drawing circle around point $C$ of diameter $CO$ thru point $O$. Diameter $O'C'O$ is $2a$. Hyperbolic orbit points $P$ now found using constant $2a=PO-PO'$.
Smaller impact $b$-parameter

Larger Rutherford back-scattering angle $\Theta$

\[ \frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2} \]

$\Theta = \text{scattering angle}$

$\Phi/2 \quad \Theta/2$

$\Phi/2 \quad \Theta/2$

$b = \text{impact parameter}$
Smaller impact b-parameter
Larger Rutherford back-scattering angle $\Theta$

Larger impact b-parameter
Smaller Rutherford back-scattering angle $\Theta$

Larger forward-scattering angle $\Phi = \pi - \Theta$

$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$

$\Theta = \text{scattering angle}$

$x$

$y$

$F'$

$F$

$C$

$a$

$b$

$\phi$

$\rho_-$

$\Phi$

$\theta$

$\frac{\Phi}{2}$

$\frac{\Theta}{2}$

$\frac{\pi}{2}$

$\pi$

Wednesday, December 10, 2014
Review: Coulomb scattering geometry

Review and added: Rutherford scattering and differential scattering cross-sections

Parabolic “kite” and envelope geometry
Rutherford scattering geometry

$2a$

$b =\ \text{impact parameter}$

$\Theta(b)$
Rutherford scattering geometry

Envelope parabola
2nd focus F'
is (-\infty, +2b)

Hyperbola
2nd focus F'
is (-2a, +2b)

"Kite" geometry of envelope parabola

2a = r - r'

2a

hyperbolic parabola directrix
tangent-kite

\Theta(b)

incoming asymptote

hyperbolic impact parameter

b =
impact parameter

\Theta(\theta)

Recall parabolic "kite" geometry

(Unit 1 Chapter 9)

Parabola
4py = x^2 = 2\lambda y
Rutherford scattering geometry

"Kite" geometry of envelope parabola

Envelope parabola
2nd focus $F'$
is $(-\infty, +2b)$

Hyperbola
2nd focus $F'$
is $(-2a, +2b)$

2$a = r-r'$

$\Theta(b)$

incoming asymptote
hyperbolic impact parameter

$b$

$tangent-kite$

$\alpha \varepsilon$

envelope parabola directrix

$F$

$2a$

$p = 2a$

contact point $P$

$r$ and $r'$

$r-r' = 2a$ applies to each point on upper hyperbola and to the contact point $P$ for the envelope parabola

circle of curvature at min-point of parabola
Rutherford scattering geometry

Hyperbola
2nd focus F'
is \((-\infty, +2b)\)

Envelope parabola
2nd focus F'
is \((-2a, +2b)\)

"Kite" geometry of envelope parabola

\[ 2a = r - r' \]

\[ 2a \]

\[ \text{contact point} \ P \]

\[ r' \]

\[ r \]

Recall parabolic "kite" geometry

Envelope parabola

\[ b = \text{impact parameter} \]

\[ d \theta \]

\[ \Theta(b) \]

\[ \text{incoming asymptote} \]

\[ \text{tangent-kite} \]

\[ \text{focal-directrix} \]

\[ \text{parabola} \]

\[ 4p'y = x^2 = 2\lambda y \]

\[ \text{circle of curvature at min-point of parabola} \]

\[ \text{hyperbolic center locus} \]

\[ \text{parabola} \]

\[ p = 2a \]

\[ F \]

\[ 2a \]

\[ 2a \]

\[ \text{2nd focus} \ F' \]

\[ \text{2nd focus} \ F' \]

\[ \text{is} \ (-\infty, +2b) \]

\[ \text{a} \rightarrow \infty \]

\[ \text{contact point} \ P \]

\[ r' \]

\[ r \]

\[ \text{2nd focus} \ F' \]

\[ \text{is} \ (-2a, +2b) \]

\[ \text{envelope parabola-directrix} \]

\[ \Theta(b) \]

\[ \text{asymptote} \]

\[ \text{2nd focus} \ F' \]

\[ \text{is} \ (-\infty, +2b) \]

\[ \text{2nd focus} \ F' \]

\[ \text{is} \ (-2a, +2b) \]

\[ \text{envelope parabola-directrix} \]

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\[ \text{asymptote} \]

\[ \text{tangent-kite} \]

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\[ \text{parabola} \]

\[ 4p'y = x^2 = 2\lambda y \]

\[ \text{circle of curvature at min-point of parabola} \]

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\[ p = 2a \]

\[ F \]

\[ 2a \]

\[ 2a \]

\[ \text{2nd focus} \ F' \]

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\[ \text{2nd focus} \ F' \]

\[ \text{is} \ (-2a, +2b) \]

\[ \text{envelope parabola-directrix} \]

\[ \Theta(b) \]

\[ \text{asymptote} \]

\[ \text{tangent-kite} \]

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\[ 4p'y = x^2 = 2\lambda y \]

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\[ p = 2a \]

\[ F \]

\[ 2a \]

\[ 2a \]

\[ \text{2nd focus} \ F' \]

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\[ \text{2nd focus} \ F' \]

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\[ \text{envelope parabola-directrix} \]

\[ \Theta(b) \]

\[ \text{asymptote} \]

\[ \text{tangent-kite} \]

\[ \text{focal-directrix} \]

\[ \text{parabola} \]

\[ 4p'y = x^2 = 2\lambda y \]

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\[ \text{parabola} \]

\[ p = 2a \]

\[ F \]

\[ 2a \]

\[ 2a \]

\[ \text{2nd focus} \ F' \]

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\[ \text{envelope parabola-directrix} \]

\[ \Theta(b) \]

\[ \text{asymptote} \]

\[ \text{tangent-kite} \]

\[ \text{focal-directrix} \]

\[ \text{parabola} \]

\[ 4p'y = x^2 = 2\lambda y \]

\[ \text{circle of curvature at min-point of parabola} \]

\[ \text{hyperbolic center locus} \]

\[ \text{parabola} \]

\[ p = 2a \]
Rutherford scattering geometry

**Envelope parabola**
- 2\textsuperscript{nd} focus F'
- is \(-\infty,+2b\)

**Hyperbola**
- 2\textsuperscript{nd} focus F'
- is \((-2a,+2b\)

**“Kite” geometry of envelope parabola**
- $2a = r-r'$

**Hyperbolic center locus**

**Hyperbolic focal-directrix**

**Hyperbolic directrix**

**Circle of curvature at min-point of parabola**

**Parabola**
- contacts
- Rutherford
- Hyperbolas of various $b$
- at the point
- where they intersect with equal slope

**Special case: $b=2a$**
- Contact tangent has unit slope

**General case:**
- $b=\text{impact parameter}$
- $\Theta(b)$ incoming asymptote
- $\Theta'(b)$ outgoing asymptote
- $2a \quad 2a$
- $F' \quad F$
- $p=2a$
- $r=r'$
- Hyperbolic impact parameter

**Envelope parabola directrix**
- $r = r' + 2b$
Rutherford scattering geometry

Envelope parabola

Hyperbola

"Kite" geometry of envelope parabola

Parabola

Recall parabolic "kite" geometry

(Unit 1 Chapter 9)
Rutherford scattering geometry

Incremental window $d\sigma = b \cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin \Theta d\Theta d\phi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where $\Theta$ is called the incremental solid angle $d\Omega = \sin \Theta d\Theta d\phi$

Ratio $\frac{d\sigma}{d\Omega} = \frac{b \cdot db \cdot d\phi}{\sin \Theta d\Theta d\phi} = \frac{b}{\sin \Theta} \frac{db}{d\Theta}$ is called the differential scattering crosssection (DSC)

Geometry $b = a \cot \frac{\Theta}{2} = \frac{k}{2E} \cot \frac{\Theta}{2}$ gives the Rutherford DSC. $\frac{d\sigma}{d\Omega} = \frac{k^4}{16E^2 \sin^4 \frac{\Theta}{2}}$

Agrees exactly with 1st Born approximation to quantum Coulomb DSC!

Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-$\alpha$ shock wave found just inside Mars orbital radius $2a \sim 1.2 \text{Au.}$
Rutherford scattering geometry

Two Extremes:

Rutherford (Coulomb) scattering has infinite ($\infty$) total cross section

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \frac{k^4}{16E^2 \sin^4 \Theta} = \infty$$

Hard-sphere scattering has finite ($2\pi r^2$ here) total cross section

\[ b = \text{impact parameter} \]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\varepsilon$-vector and Coulomb $p = m v$ geometry

Analytic geometry derivation of $\varepsilon$-construction

Algebra of $\varepsilon$-construction geometry

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
Eccentricity vector \( \varepsilon \) and \((\varepsilon, \lambda)\) geometry of orbital mechanics

Isotropic field \( V = V(r) \) guarantees conservation of angular momentum vector \( \mathbf{L} \)

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}
\]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation angular momentum vector $L$

$$L = r \times p = m r \times \dot{r}$$

Coulomb $V=-k/r$ also conserves eccentricity vector $\varepsilon$

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = \frac{r}{r} - \frac{p \times (r \times p)}{km}$$
**Eccentricity vector** $\varepsilon$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation **angular momentum vector** $\mathbf{L}$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves **eccentricity vector** $\varepsilon$

$$\varepsilon = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} - \mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{r}$$

$IHO$ $V=(k/2)r^2$ also conserves **Stokes vector** $\mathbf{S}$

- $S_A = \frac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)$
- $S_B = x_1p_1 + x_2p_2$
- $S_C = x_1p_2 - x_2p_1$

$A = km\cdot\varepsilon$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Generate symmetry groups: $U(2) \subset U(2)$ or: $R(3) \subset R(3) \times R(3) \subset O(4)$
Eccentricity vector \( \varepsilon \) and (\( \varepsilon \),\( \lambda \)) geometry of orbital mechanics

Isotropic field \( V=V(r) \) guarantees conservation \textit{angular momentum vector} \( \mathbf{L} \)

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Coulomb \( V=-k/r \) also conserves \textit{eccentricity vector} \( \varepsilon \)

\[
\varepsilon = \hat{r} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} - \mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}
\]

\( \mathbf{A} = km \cdot \varepsilon \) is known as the \textit{Laplace-Hamilton-Gibbs-Runge-Lenz vector}.

Consider dot product of \( \varepsilon \) with a radial vector \( \mathbf{r} \):

\[
\varepsilon \cdot \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{r}}{r} - \frac{\mathbf{r} \cdot \mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} \cdot \mathbf{p} \cdot \mathbf{L}}{km} = \frac{\mathbf{L} \cdot \mathbf{L}}{km}
\]

IHO \( V=(k/2)r^2 \) also conserves \textit{Stokes vector} \( \mathbf{S} \)

\[
S_A = \frac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)
\]

\[
S_B = x_1 p_1 + x_2 p_2
\]

\[
S_C = x_1 p_2 - x_2 p_1
\]

Generate symmetry groups:

\[U(2) \subset U(2)\]

or:

\[R(3) \subset R(3) \times R(3) \subset O(4)\]

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\[\text{Wednesday, December 10, 2014}\]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation of angular momentum vector $\mathbf{L}$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves eccentricity vector $\varepsilon$

$$\varepsilon = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

$A = km \cdot \varepsilon$ is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of $\varepsilon$ with a radial vector $\mathbf{r}$:

$$\varepsilon \cdot \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{r}}{r} - \frac{\mathbf{r} \cdot \mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} \times \mathbf{p} \cdot \mathbf{L}}{km} = \frac{\mathbf{L} \cdot \mathbf{L}}{km}$$

Consider dot product of $\varepsilon$ with momentum vector $\mathbf{p}$:

$$\varepsilon \cdot \mathbf{p} = \frac{\mathbf{p} \cdot \mathbf{r}}{r} - \frac{\mathbf{p} \cdot \mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{p} \cdot \dot{\mathbf{r}}}{km} = p_r$$

IHO $V=(k/2)r^2$ also conserves Stokes vector $\mathbf{S}$

- $S_A = \frac{1}{2} (x_1^2 + p_1^2 - x_2^2 - p_2^2)$
- $S_B = x_1p_1 + x_2p_2$
- $S_C = x_1p_2 - x_2p_1$

Generate symmetry groups:

- $U(2) \subset U(2)$
- $R(3) \subset R(3) \times R(3) \subset O(4)$

...for sake of comparison...
Eccentricity vector \( \varepsilon \) and \((\varepsilon, \lambda)\) geometry of orbital mechanics

Isotropic field \( V=V(r) \) guarantees conservation angular momentum vector \( \mathbf{L} \)

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}
\]

Coulomb \( V=-k/r \) also conserves eccentricity vector \( \mathbf{\varepsilon} \)

\[
\mathbf{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \mathbf{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}
\]

\( \mathbf{A} = km \mathbf{\varepsilon} \) is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of \( \mathbf{\varepsilon} \) with a radial vector \( \mathbf{r} \):

\[
\mathbf{\varepsilon} \cdot \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{r}}{r} - \frac{\mathbf{r} \cdot \mathbf{p} \times \mathbf{L}}{km} = \mathbf{r} - \frac{\mathbf{r} \times \mathbf{p} \cdot \mathbf{L}}{km} = \mathbf{r} - \frac{\mathbf{L} \cdot \mathbf{L}}{km}
\]

Let angle \( \phi \) be angle between \( \mathbf{\varepsilon} \) and radial vector \( \mathbf{r} \)

\[
\varepsilon r \cos \phi = r - \frac{L^2}{km}
\]

Also enjoys conservation eccentricity vector \( \mathbf{\varepsilon} \)

\[
\mathbf{\varepsilon} \cdot \mathbf{p} = \frac{\mathbf{p} \cdot \mathbf{r}}{r} - \frac{\mathbf{p} \cdot \mathbf{p} \times \mathbf{L}}{km} = \mathbf{p} \cdot \dot{\mathbf{r}} = p_r
\]

\[\text{(...for sake of comparison...)}\]

IHO \( V=(k/2)r^2 \) also conserves Stokes vector \( \mathbf{S} \)

\[
\begin{align*}
S_A &= \frac{1}{2} (x_1^2 + p_1^2 - x_2^2 - p_2^2) \\
S_B &= x_1 p_1 + x_2 p_2 \\
S_C &= x_1 p_2 - x_2 p_1
\end{align*}
\]

\[\text{Generate symmetry groups:} U(2) \subset U(2) \text{ or:} R(3) \subset R(3) \times R(3) \subset O(4)\]

\[
\text{...or of} \; \mathbf{\varepsilon} \; \text{with momentum vector} \; \mathbf{p}:
\]

\[
\mathbf{\varepsilon} \cdot \mathbf{p} = \frac{\mathbf{p} \cdot \mathbf{r}}{r} - \frac{\mathbf{p} \cdot \mathbf{p} \times \mathbf{L}}{km} = \mathbf{p} \cdot \dot{\mathbf{r}} = p_r
\]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation angular momentum vector $L$

$$L = r \times p = m \frac{r \times \dot{r}}{km}$$

Coulomb $V=-k/r$ also conserves eccentricity vector $\varepsilon$

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = \frac{r - p \times (r \times p)}{km}$$

$IHO$ $V=(k/2)r^2$ also conserves Stokes vector $S$

$$SA = \frac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$SB = x_1p_1 + x_2p_2$$
$$SC = x_1p_2 - x_2p_1$$

$A = km \cdot \varepsilon$ is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of $\varepsilon$ with a radial vector $r$:

$$\varepsilon \cdot r = \frac{r \cdot r - r \cdot p \times L}{km} = \frac{r - r \times p \cdot L}{km} = \frac{r - L \cdot L}{km}$$

Let angle $\phi$ be angle between $\varepsilon$ and radial vector $r$

$$\varepsilon \cdot p = \frac{p \cdot r - p \cdot p \times L}{km} = p \cdot \hat{r} = p_r$$

$$\varepsilon \cdot p = \frac{L^2}{km}$$

or:

$$r = \frac{L^2}{km(1-\varepsilon \cos \phi)}$$
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation angular momentum vector $L$

$$L = r \times p = m r \times \dot{r}$$

Coulomb $V=-k/r$ also conserves eccentricity vector $\varepsilon$

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = r - \frac{p \times (r \times p)}{km}$$

$A = km \varepsilon$ is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of $\varepsilon$ with a radial vector $r$:

$$\varepsilon \cdot r = \frac{r \cdot r}{r} - \frac{r \cdot p \times L}{km} = r - \frac{r \times p \times L}{km}$$

Let angle $\phi$ be angle between $\varepsilon$ and radial vector $r$

$$\varepsilon r \cos \phi = r - \frac{L^2}{km} \quad \text{or: } r = \frac{L^2 / km}{1 - \varepsilon \cos \phi}$$

For $\lambda=L^2/km$ that matches: $r = \frac{\lambda}{1 - \varepsilon \cos \phi}$

$IHO$ $V=(k/2)r^2$ also conserves Stokes vector $S$

$$S_A = \frac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)$$

$$S_B = x_1 p_1 + x_2 p_2$$

$$S_C = x_1 p_2 - x_2 p_1$$

Generate symmetry groups: $U(2) \subset U(2)$

$or: R(3) \subset R(3) \times R(3) \subset O(4)$

...or of $\varepsilon$ with momentum vector $p$:

$$\varepsilon \cdot p = \frac{p \cdot r}{r} - \frac{p \cdot p \times L}{km} = p \cdot \dot{r} = p_r$$

$$\begin{cases} 
\frac{\lambda}{l - \varepsilon} & \text{if: } \phi = 0 \quad \text{apogee} \\
\frac{\lambda}{l} & \text{if: } \phi = \frac{\pi}{2} \quad \text{zenith} \\
\frac{\lambda}{l + \varepsilon} & \text{if: } \phi = \pi \quad \text{perigee}
\end{cases}$$

(attractive force center)
**Eccentricity vector \( \mathbf{\varepsilon} \) and \((\varepsilon, \lambda)\) geometry of orbital mechanics**

Isotropic field \( V=V(r) \) guarantees conservation of the angular momentum vector \( \mathbf{L} \)

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}
\]

Coulomb \( V=-k/r \) also conserves the eccentricity vector \( \mathbf{\varepsilon} \)

\[
\mathbf{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} - \mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}
\]

\( A = km \cdot \mathbf{\varepsilon} \) is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of \( \mathbf{\varepsilon} \) with a radial vector \( \mathbf{r} \):

\[
\mathbf{\varepsilon} \cdot \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \times \mathbf{p} \cdot \mathbf{L}}{km} = \frac{\mathbf{r} \cdot \mathbf{L} \cdot \mathbf{L}}{km}
\]

Let angle \( \phi \) be angle between \( \mathbf{\varepsilon} \) and radial vector \( \mathbf{r} \)

\[
\varepsilon r \cos \phi = r - \frac{L^2}{km} \quad \text{or:} \quad r = \frac{L^2 / km}{1 - \varepsilon \cos \phi}
\]

For \( \lambda = L^2 / km \) that matches:

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi} = \begin{cases} \frac{\lambda}{1 - \varepsilon} & \text{if: } \phi = 0 \quad \text{apogee} \\ \frac{\lambda}{1 + \varepsilon} & \text{if: } \phi = \pi \quad \text{perigee} \\ \frac{\lambda}{1 + \varepsilon} & \text{if: } \phi = \pi \quad \text{zenith} \end{cases}
\]

(Rotational momentum \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \) is normal to the orbit plane.)

\( \mathbf{p} \leftrightarrow \mathbf{p} \times \mathbf{L} \) (Nothing here)

(attractive force center)

(attractive force center)
Eccentricity vector $\epsilon$ and $(\epsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation angular momentum vector $L$

$$L = r \times p = m r \times \dot{r}$$

Coulomb $V=-k/r$ also conserves eccentricity vector $\epsilon$

$$\epsilon = \frac{\hat{r} - p \times L}{km} = \frac{r - p \times (r \times p)}{km}$$

$IHO V=(k/2)r^2$ also conserves Stokes vector $S$

$$S_A = \frac{1}{2} (x_1^2 + p_1^2 - x_2^2 - p_2^2)$$
$$S_B = x_1 p_1 + x_2 p_2$$
$$S_C = x_1 p_2 - x_2 p_1$$

$A = km \epsilon$ is known as the **Laplace-Hamilton-Gibbs-Runge-Lenz vector**.

Consider dot product of $\epsilon$ with a radial vector $r$:

$$\epsilon \cdot r = \frac{r \cdot r - r \cdot p \times L}{km} = \frac{r - r \times p \cdot L}{km} = r - \frac{L \cdot L}{km}$$

Let angle $\phi$ be angle between $\epsilon$ and radial vector $r$

$$\epsilon r \cos \phi = r - \frac{L^2}{km}$$

or:

$$r = \frac{L^2}{km} \frac{\epsilon \cos \phi}{1 - \epsilon \cos \phi}$$

For $\lambda=L^2/km$ that matches:

$$r = \frac{\lambda}{1 - \epsilon \cos \phi}$$

\[\begin{align*}
\lambda & = \frac{1}{1-\epsilon} & \text{apogee} \\
\lambda & = \frac{\pi}{2} & \text{zenith} \\
\lambda & = \frac{1+\epsilon}{1-\epsilon} & \text{perigee}
\end{align*}\]

(Nothing here)

(a) Attractive ($k>0$) Elliptic ($E<0$)

(b) Attractive ($k>0$) Hyperbolic ($E>0$)

(Rotational momentum $L = r \times p$ is normal to the orbit plane.)

(force center)

(attractive)

(perigee)

(apogee)

(perhelion)

(aphelion)

(latus)

(radius)

(zenith)
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V=V(r)$ guarantees conservation of angular momentum vector $\mathbf{L}$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves eccentricity vector $\varepsilon$

$$\varepsilon = \hat{r} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} - \mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

$A = km \varepsilon$ is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of $\varepsilon$ with a radial vector $\mathbf{r}$:

$$\varepsilon \cdot \mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r} \times \mathbf{p} \cdot \mathbf{L}}{km} = r - \frac{\mathbf{L} \cdot \mathbf{L}}{km}$$

Let angle $\phi$ be angle between $\varepsilon$ and radial vector $\mathbf{r}$

$$\varepsilon r \cos \phi = r - \frac{L^2}{km} \quad \text{or:} \quad r = \frac{L^2}{km} \frac{1}{1 - \varepsilon \cos \phi}$$

For $\lambda=L^2/km$ that matches:

$$r = \frac{\lambda}{1 - \varepsilon \cos \phi}$$

- (a) Attractive ($k>0$)
  - Elliptic ($E<0$)
  
- (b) Attractive ($k>0$)
  - Hyperbolic ($E>0$)

- (c) Repulsive ($k<0$)
  - Hyperbolic ($E>0$)

(Rotational momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is normal to the orbit plane.)

$IHO \ V=(k/2)r^2$ also conserves Stokes vector $\mathbf{S}$

$$S_A = \frac{1}{2} (x_1^2 + p_1^2 - x_2^2 - p_2^2)$$

$$S_B = x_1 p_1 + x_2 p_2$$

$$S_C = x_1 p_2 - x_2 p_1$$

Generate symmetry groups:

- $U(2) \subset U(2)$
- $O(3) \subset R(3) \times R(3) \subset O(4)$

...or of $\varepsilon$ with momentum vector $\mathbf{p}$:

$$\varepsilon \cdot \mathbf{p} = \frac{\mathbf{p} \cdot \mathbf{r} - \mathbf{p} \cdot \mathbf{p} \times \mathbf{L}}{km} = \mathbf{p} \cdot \dot{\mathbf{r}} = p_r$$

$$\begin{cases}
\lambda & \text{if } \phi = 0 \text{ apogee} \\
\frac{\lambda}{1-\varepsilon} & \text{if } \phi = \frac{\pi}{2} \text{ zenith} \\
\frac{\lambda}{1+\varepsilon} & \text{if } \phi = \pi \text{ perigee}
\end{cases}$$
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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$(R=-0.375$ elliptic orbit$)$

$(R=+0.5$ hyperbolic orbit$)$
Dot product of $\varepsilon$ with momentum vector $p$:

$$\varepsilon \cdot p = \frac{p \cdot r}{r} - p \cdot p \times L$$

$$= p \cdot \hat{r} = p_r = \varepsilon p_x$$

This says:

"Projection of $p$ onto $r$ is eccentricity $\varepsilon$ times projection of $p$ onto $\hat{x}$-axis"

($\hat{x} = \hat{\varepsilon}$)

Ellipse has eccentricity $\varepsilon < 1$

($Here: \varepsilon = \sqrt{3}/2 = 0.866$)
Dot product of $\epsilon$ with momentum vector $p$:

$$\epsilon \cdot p = \frac{p \cdot r}{r} - \frac{p \cdot p \times L}{km} = p \cdot \hat{r} = p_r = \epsilon p_x$$

This says:

"Projection of $p$ onto $r$ is eccentricity $\epsilon$ times projection of $p$ onto $\hat{x}$-axis"  
($\hat{x} = \hat{\epsilon}$)

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This says:
"Projection of $p$ onto $r$ is eccentricity $\varepsilon$ times projection of $p$ onto $\hat{x}$-axis"

($\hat{x} = \hat{e}$)

Ellipse has eccentricity $\varepsilon < 1$

(Here: $\varepsilon = \sqrt{3}/2 = 0.866$)
Dot product of $\varepsilon$ with momentum vector $p$:

$$\varepsilon \cdot p = \frac{p \cdot r}{r} - \frac{p \cdot p \times L}{km} = p \cdot \hat{r} = p_r = \varepsilon p_x$$

This says:
"Projection of $p$ onto $r$ is eccentricity $\varepsilon$ times projection of $p$ onto $\hat{x}$-axis"

($\hat{x} = \hat{\varepsilon}$)

Slope of $p$ over focus equals $\varepsilon$

Ellipse has eccentricity $\varepsilon < 1$

(Here: $\varepsilon = \sqrt{3}/2 = 0.866$)
Dot product of $\epsilon$ with momentum vector $p$:

$$\epsilon \cdot p = \frac{p \cdot r - p \cdot p \times L}{r \overline{km}} = p \cdot \hat{r} = p_r = \epsilon p_x$$

This says:

"Projection of $p$ onto $r$ is eccentricity $\epsilon$ times projection of $p$ onto $\hat{x}$-axis"

($\hat{x} = \hat{\epsilon}$)

Slope of $p$ over focus equals $\epsilon$

Projection of $p$ onto radius $\hat{x}$-axis: $p_x = p \cdot \hat{x} = p \cdot \hat{\epsilon}$

Ellipse has eccentricity $\epsilon < 1$

(Here: $\epsilon = \sqrt{3}/2 = 0.866$)
Dot product of $\varepsilon$ with momentum vector $p$:

$$\varepsilon \cdot p = \frac{p \cdot r}{r} - \frac{p \cdot p \times L}{km} = p \cdot \hat{r} = p_r = \varepsilon p_x$$

This says:

"Projection of $p$ onto $r$ is eccentricity $\varepsilon$ times projection of $p$ onto $\hat{x}$-axis"

($\hat{x} = \hat{\varepsilon}$)

Hyperbola has eccentricity $\varepsilon > 1$

(Here: $\varepsilon = 5/4 = 1.25$)
Dot product of $\varepsilon$ with momentum vector $p$:

$$\varepsilon \cdot p = \frac{p \cdot r - p \cdot p \times L}{r \ km} = p \cdot \hat{r} = p_r = \varepsilon p_x$$

This says:

"Projection of $p$ onto $r$ is eccentricity $\varepsilon$ times projection of $p$ onto $\hat{x}$-axis"

($\hat{x} = \hat{\varepsilon}$)

Hyperbola has eccentricity $\varepsilon > 1$

(Here: $\varepsilon = 5/4 = 1.25$)
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$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
Geometry of Coulomb orbits (Let: \( r = \rho \) here)

\[
\frac{r}{\varepsilon} = \frac{\lambda}{\varepsilon} + r \cos \phi
\]

\[
r = \lambda + r \varepsilon \cos \phi
\]

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi}
\]

\[
1 = \frac{1 - \varepsilon \cos \phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon \cos \phi}{\lambda}
\]

\[
\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m} \cos \phi
\]

**All conics defined by:**

*Defining eccentricity* \( \varepsilon \)

Distance to Focal-point = \( \varepsilon \cdot \) Distance to Directrix-line

(From Lecture 28 p. 66)
(From Lecture 26 p. 66) *Geometry of Coulomb orbits (Let: \( r = \rho \) here)*

\[
\frac{r}{\varepsilon} = \frac{\lambda}{\varepsilon} + r \cos \phi \\
\frac{r'}{\varepsilon} = \frac{\lambda}{\varepsilon} + r' \cos \phi \\
\frac{r}{\varepsilon} = \lambda + r \varepsilon \cos \phi \\
\frac{r}{\varepsilon} = \frac{\lambda}{1 - \varepsilon \cos \phi}
\]

\[
1 = \frac{1 - \varepsilon \cos \phi}{\lambda} = 1 - \frac{\varepsilon}{\lambda} \cos \phi
\]

\[
\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \sqrt{\frac{k^2 + 2E\mu^2/m}{\mu^2/m}} \cos \phi
\]

All conics defined by:

*Defining eccentricity* \( \varepsilon \)

Distance to Focal-point = \( \varepsilon \cdot \) Distance to Directrix-line

**Perihelion** \( \rho_- = \lambda / (1 + \varepsilon) \)

**Aphelion** \( \rho_+ = \lambda / (1 - \varepsilon) \)**
Geometry of Coulomb orbits (Let: \( r = \rho \) here)

\[
\frac{r}{\epsilon} = \frac{\lambda}{\epsilon} + \frac{r \cos \phi}{\epsilon}
\]

\[
r = \lambda + r \epsilon \cos \phi
\]

\[
r = \frac{\lambda}{1 - \epsilon \cos \phi}
\]

\[
\frac{1}{r} = \frac{1 - \epsilon \cos \phi}{\lambda} = \frac{1}{\lambda} - \frac{\epsilon}{\lambda} \cos \phi
\]

All conics defined by:

**Defining eccentricity** \( \epsilon \)

Distance to Focus-point = \( \epsilon \cdot \) Distance to Directrix-line

Major axis: \( \rho_+ + \rho_- = 2a \)

\[
\rho_+ + \rho_- = \frac{\lambda (1 + \epsilon) + \lambda (1 - \epsilon)}{(1 - \epsilon^2)} = 2 \frac{\lambda}{|1 - \epsilon^2|}
\]
(From Lecture 26 p. 66) **Geometry of Coulomb orbits (Let: \( r = \rho \) here)**

\[
\frac{r}{\varepsilon} = \frac{\lambda}{\varepsilon} + r \cos \phi \quad r = \lambda + r \varepsilon \cos \phi \quad r = \frac{\lambda}{1 - \varepsilon \cos \phi}
\]

\[
\frac{1}{r} = \frac{1}{\lambda} - \varepsilon \cos \phi
\]

\[
\rho = \frac{\lambda}{1 + \varepsilon} \text{ perihelion}
\]

\[
\rho = \frac{\lambda}{1 - \varepsilon} \text{ aphelion}
\]

**All conics defined by:**

- Defining eccentricity \( \varepsilon \)
- Distance to Focal-point = \( \varepsilon \cdot \text{Distance to Directrix-line} \)

\[
2a = \frac{\lambda}{1 - \varepsilon^2}
\]

- Major axis: \( \rho_+ + \rho_- = 2a \)
  \[
  \rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|
  \]
- Focal axis: \( \rho_+ - \rho_- = 2a\varepsilon \)
  \[
  \rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|
  \]
Geometry of Coulomb orbits (Let: \( r = \rho \) here)

\[
\frac{\rho}{\epsilon} = \frac{\lambda}{\epsilon} + \rho \cos \phi
\]

\[
r = \lambda + r \epsilon \cos \phi
\]

\[
r = \frac{\lambda}{1 - \epsilon \cos \phi}
\]

\[
\frac{1}{r} = \frac{1 - \epsilon \cos \phi}{\lambda} = \frac{1}{\lambda} - \frac{\epsilon}{\lambda} \cos \phi
\]

\[
\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2 E \mu^2/m}}{\mu^2/m} \cos \phi
\]

All conics defined by:

*Defining eccentricity* \( \epsilon \)

Distance to Focal-point = \( \epsilon \cdot \) Distance to Directrix-line

\[
\rho_- = \frac{\lambda}{(1+\epsilon)} \text{ perihelion}
\]

\[
\rho_+ = \frac{\lambda}{(1-\epsilon)} \text{ aphelion}
\]

Major axis: \( \rho_+ + \rho_- = 2a \)

\[
\rho_+ + \rho_- = \left[ \lambda (1+\epsilon) + \lambda (1-\epsilon) \right] / (1-\epsilon^2) = 2\lambda / |1-\epsilon^2|
\]

Focal axis: \( \rho_+ - \rho_- = 2a\epsilon \)

\[
\rho_+ - \rho_- = \left[ \lambda (1+\epsilon) - \lambda (1-\epsilon) \right] / (1-\epsilon^2) = 2\lambda \epsilon / |1-\epsilon^2|
\]

Minor radius: \( b = \sqrt{a^2 - a^2\epsilon^2} = \sqrt{(a\lambda)} \) (ellipse: \( \epsilon < 1 \))

Minor radius: \( b = \sqrt{a^2 \epsilon^2 - a^2} = \sqrt{(\lambda a)} \) (hyperb: \( \epsilon > 1 \))
**Geometry of Coulomb orbits (Let: \( r = \rho \) here)**

\[
\frac{r}{\varepsilon} = \frac{\lambda}{\varepsilon} + r \cos \phi
\]

\[
r = \lambda + r \varepsilon \cos \phi
\]

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi}
\]

\[
1 = \frac{1 - \varepsilon \cos \phi}{\lambda} = 1 - \frac{\varepsilon}{\lambda} \cos \phi
\]

\[
\frac{1}{r} = \frac{-k}{\mu^2/m} + \sqrt{\frac{k^2 + 2E\mu^2/m}{\mu^2/m}} \cos \phi
\]

*All conics defined by:*

\[
\text{Distance to Focal-point } = \varepsilon \cdot \text{Distance to Directrix-line}
\]

**Defining eccentricity \( \varepsilon \)**

<table>
<thead>
<tr>
<th>((x,y)) parameters</th>
<th>physical constants</th>
<th>((r,\phi)) parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a = \frac{k}{2E})</td>
<td>(E = \frac{k}{2a})</td>
<td>(\varepsilon = \sqrt{\frac{k^2 m + 2L^2 E}{k^2 m}} = \sqrt{1 + \frac{b^2}{a^2}})</td>
</tr>
<tr>
<td>(b = \frac{L}{\sqrt{2m</td>
<td>E</td>
<td>}})</td>
</tr>
</tbody>
</table>

*Major axis: \(\rho_+ + \rho_- = 2a\)*

\[
\rho_+ - \rho_- = \frac{\lambda(1+\varepsilon) + \lambda(1-\varepsilon)}{(1-\varepsilon^2)} = 2\lambda/|1-\varepsilon^2|
\]

*Focal axis: \(\rho_+ - \rho_- = 2\varepsilon a\)*

\[
\rho_+ - \rho_- = \frac{\lambda(1+\varepsilon) - \lambda(1-\varepsilon)}{(1-\varepsilon^2)} = 2\varepsilon \lambda/|1-\varepsilon^2|
\]

*Minor radius: \(b = \sqrt{(a^2 - a^2 \varepsilon^2)} = \sqrt{(a\lambda)}\) (ellipse: \(\varepsilon < 1\))*

*Minor radius: \(b = \sqrt{(a^2 \varepsilon^2 - a^2)} = \sqrt{(\lambda a)}\) (hyperb: \(\varepsilon > 1\))*

\[
\varepsilon^2 = 1 - \frac{b^2}{a^2} \quad (\text{ellipse: } \varepsilon < 1) \quad \frac{b^2}{a^2} = \sqrt{1 - \varepsilon^2}
\]

\[
\varepsilon^2 = 1 + \frac{b^2}{a^2} \quad (\text{hyperbola: } \varepsilon > 1) \quad \frac{b^2}{a^2} = \sqrt{\varepsilon^2 - 1}
\]

\[
\lambda = a(1-\varepsilon^2) \quad (\text{ellipse: } \varepsilon < 1) \quad \lambda = a(\varepsilon^2-1) \quad (\text{hyperb: } \varepsilon > 1)
\]

*Wednesday, December 10, 2014*
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

**$\varepsilon$-vector and Coulomb $r$-orbit geometry**

**Review and connection to standard development**

**$\varepsilon$-vector and Coulomb $p=mv$ geometry**

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**Ruler & compass construction of $\varepsilon$-vector and orbits**

$(R = -0.375 \text{ elliptic orbit})$

$(R = +0.5 \text{ hyperbolic orbit})$
\textbf{ε-vector and Coulomb }\textbf{p=mv geometry}

Finding time derivatives of orbital coordinates \( r, \phi, x, y, \) and eventually velocity \( \mathbf{v} \) or momentum \( \mathbf{p}=m\mathbf{v} \)

Radius \( r \):
\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi} = \frac{L^2}{km} \]

Polar angle \( \phi \) using:
\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}
\]
Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p=mv$

**Radius $r$:**

$$ r = \frac{\lambda}{1 - \epsilon \cos \phi} = \frac{L^2}{km} $$

**Polar angle $\phi$ using:**

$$ L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi} $$

$$ \dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} $$
**ε-vector and Coulomb p=mv geometry**

Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p=mv$

**Radius $r$:**

$$r = \frac{\lambda}{1 - \epsilon \cos \phi} = \frac{L^2}{km}$$

**Polar angle $\phi$ using:**

$$L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}$$

$$\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \left( \frac{1}{r^2} \right) = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1 - \epsilon \cos \phi)^2$$

**using:**

$$\frac{1}{r^2} = \left( \frac{km}{L^2} \right)^2 (1 - \epsilon \cos \phi)^2$$
**ε-vector and Coulomb \( p=mv \) geometry**

Finding time derivatives of orbital coordinates \( r, \phi, x, y \), and eventually velocity \( v \) or momentum \( p=mv \)

Radius \( r \):

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi} = \frac{L^2}{km} \left(1 - \varepsilon \cos \phi\right)
\]

Velocity \( \dot{r} \):

\[
\dot{r} = \frac{L^2}{km} \frac{-d}{dt} \left(\varepsilon \cos \phi\right)
\]

Polar angle \( \phi \) using:

\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}
\]

\[
\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left(\frac{km}{L^2}\right)^2 (1 - \varepsilon \cos \phi)^2
\]

Using:

\[
\frac{1}{r^2} = \left(\frac{km}{L^2}\right)^2 (1 - \varepsilon \cos \phi)^2
\]
ε-vector and Coulomb $p=mv$ geometry

Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p=mv$

Radius $r$:
\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi} = \frac{L^2}{km(1 - \varepsilon \cos \phi)}
\]
\[
\dot{r} = \frac{d}{dt} \frac{L^2}{km(1 - \varepsilon \cos \phi)^2}
\]

Polar angle $\phi$ using:
\[
L = m r^2 \frac{d\phi}{dt} = m r^2 \dot{\phi}
\]
\[
\dot{\phi} = \frac{L}{m r^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left(\frac{km}{L^2}\right)^2 (1 - \varepsilon \cos \phi)^2
\]
\[
r \dot{\phi} = \frac{L}{mr}
\]

using:
\[
\frac{1}{r^2} = \left(\frac{km}{L^2}\right)^2 (1 - \varepsilon \cos \phi)^2
\]
Finding time derivatives of orbital coordinates \( r, \phi, x, y \), and eventually velocity \( \mathbf{v} \) or momentum \( \mathbf{p}=m\mathbf{v} \)

**Radius \( r \):**

\[
r = \frac{\lambda}{1-\epsilon \cos \phi} = \frac{L^2}{km(1-\epsilon \cos \phi)^2}
\]

**Polar angle \( \phi \) using:**

\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}
\]

\[
\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1-\epsilon \cos \phi)^2
\]

\[
r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left( \frac{km}{L^2} \right) (1-\epsilon \cos \phi) = \frac{k}{L} (1-\epsilon \cos \phi)
\]

using:

\[
\frac{1}{r} = \left( \frac{km}{L^2} \right) (1-\epsilon \cos \phi)
\]
Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p=mv$

Radius $r$:

$$ r = \frac{\lambda}{1 - \epsilon \cos \phi} = \frac{L^2/km}{1 - \epsilon \cos \phi} $$

$$ \dot{r} = \frac{d}{dt} \left( \frac{L^2}{km} \right) \frac{(-\epsilon \cos \phi)}{(1 - \epsilon \cos \phi)^2} $$

$$ \dot{r} = \frac{L^2}{km} \frac{-\epsilon \sin \phi \dot{\phi}}{(1 - \epsilon \cos \phi)^2} $$

Polar angle $\phi$ using:

$$ L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi} $$

$$ \dot{\phi} = \frac{L}{mr^2} = \frac{L}{m \cdot r^2} = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1 - \epsilon \cos \phi)^2 $$

$$ \dot{r} \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left( \frac{km}{L^2} \right) (1 - \epsilon \cos \phi) = \frac{k}{L} (1 - \epsilon \cos \phi) $$

Using:

$$ \frac{1}{r^2} = \left( \frac{km}{L^2} \right)^2 (1 - \epsilon \cos \phi)^2 $$
**ε-vector and Coulomb p=mv geometry**

Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p=mv$.

**Radius $r$:**

\[
r = \frac{\lambda}{1 - \epsilon \cos \phi} = \frac{L^2}{km (1 - \epsilon \cos \phi)}
\]

\[
\dot{r} = \frac{L^2}{km} \frac{d}{dt} (-\epsilon \cos \phi)
\]

\[
\dot{r} = \frac{L^2}{km} \frac{-\epsilon \sin \phi \dot{\phi}}{(1 - \epsilon \cos \phi)^2}
\]

\[
\dot{r} = -\frac{L^2}{km} \left( \frac{km}{L^2} \right)^2 r^2 \epsilon \sin \phi
\]

**Polar angle $\phi$ using:**

\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}
\]

\[
\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m \frac{L^2}{km}} = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1 - \epsilon \cos \phi)^2
\]

\[
r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left( \frac{km}{L^2} \right) (1 - \epsilon \cos \phi) = \frac{k}{L} (1 - \epsilon \cos \phi)
\]

**using:**

\[
\frac{1}{r^2} = \left( \frac{km}{L^2} \right)^2 (1 - \epsilon \cos \phi)^2
\]

**using:**

\[
\frac{1}{(1 - \epsilon \cos \phi)^2} = \left( \frac{km}{L^2} \right)^2 r^2
\]
Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p = mv$

**Radius $r$:**

$$r = \frac{\lambda}{1 - \epsilon \cos \phi} = \frac{L^2/km}{1 - \epsilon \cos \phi}$$

$$\dot{r} = \frac{dr}{dt} = \frac{L^2}{km} \frac{d}{dt}(-\epsilon \cos \phi)$$

$$\dot{r} = \frac{L^2}{km} \frac{-\epsilon \sin \phi \dot{\phi}}{(1 - \epsilon \cos \phi)^2}$$

$$\dot{r} = -\frac{L^2}{km} \left(\frac{km}{L^2}\right)^2 r^2 \dot{\phi} \epsilon \sin \phi$$

$$\dot{r} = -\frac{k}{L^2} \frac{mr^2 \dot{\phi} \epsilon \sin \phi}{\epsilon \sin \phi} = -\frac{k}{L}$$

**Polar angle $\phi$ using:**

$$L = m r^2 \frac{d\phi}{dt} = m r^2 \dot{\phi}$$

$$\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left(\frac{km}{L^2}\right)^2 (1 - \epsilon \cos \phi)^2$$

$$r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} \left(\frac{km}{L^2}\right) (1 - \epsilon \cos \phi) = \frac{k}{L} (1 - \epsilon \cos \phi)$$

using: $$\frac{1}{r^2} = \left(\frac{km}{L^2}\right)^2 (1 - \epsilon \cos \phi)^2$$

using: $$\frac{1}{(1 - \epsilon \cos \phi)^2} = \left(\frac{km}{L^2}\right)^2 r^2$$
**ε-vector and Coulomb p=mv geometry**

Finding time derivatives of orbital coordinates \( r, \phi, x, y, \) and eventually velocity \( v \) or momentum \( p=mv \)

**Radius \( r \):**

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi} = \frac{L^2/km}{1 - \varepsilon \cos \phi}
\]

\[
\dot{r} = \frac{L^2}{km} \frac{-\varepsilon \sin \phi \dot{\phi}}{(1 - \varepsilon \cos \phi)^2}
\]

\[
\ddot{r} = -\frac{L^2}{km} \left( \frac{km}{L^2} \right)^2 r^2 \dot{\phi} \varepsilon \sin \phi
\]

\[
\dddot{r} = -\frac{k}{L^2} \frac{mr^2 \dot{\phi} \varepsilon \sin \phi = -\frac{k}{L} \varepsilon \sin \phi}
\]

**Cartesian \( x = r \cos \phi \):**

\[
\dot{x} = \frac{dx}{dt} = \dot{r} \cos \phi - \sin \phi \ r \dot{\phi}
\]

**Cartesian \( y = r \sin \phi \):**

\[
\dot{y} = \frac{dy}{dt} = \dot{r} \sin \phi + \cos \phi \ r \dot{\phi}
\]

**Polar angle \( \phi \) using:**

\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}
\]

\[
\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1 - \varepsilon \cos \phi)^2
\]

\[
r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left( \frac{km}{L^2} \right) (1 - \varepsilon \cos \phi) = \frac{k}{L} (1 - \varepsilon \cos \phi)
\]

**using:**

\[
\frac{1}{r^2} = \left( \frac{km}{L^2} \right)^2 (1 - \varepsilon \cos \phi)^2
\]

**using:**

\[
\frac{1}{(1 - \varepsilon \cos \phi)^2} = \left( \frac{km}{L^2} \right)^2 r^2
\]
ε-vector and Coulomb p = mv geometry

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity v or momentum p = mv

Radius r:
\[
r = \frac{\lambda}{1 - \epsilon \cos \phi} = \frac{L^2}{km}\]
\[
\frac{dr}{dt} = \frac{L^2}{km} \frac{d}{dt}(-\epsilon \cos \phi)\]
\[
\dot{r} = \frac{L^2}{km} \frac{-\epsilon \sin \phi \dot{\phi}}{(1 - \epsilon \cos \phi)^2}\]
\[
\dot{r} = \frac{L^2}{km} \left(\frac{km}{L^2}\right)^2 r^2 \dot{\phi} \epsilon \sin \phi\]
\[
\dot{r} = -\frac{k}{L^2} mr^2 \dot{\phi} \epsilon \sin \phi = -\frac{k}{L} \epsilon \sin \phi\]

Polar angle φ using:
\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}\]
\[
\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left(\frac{km}{L^2}\right)^2 (1 - \epsilon \cos \phi)^2\]
\[
r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left(\frac{km}{L^2}\right)(1 - \epsilon \cos \phi) = \frac{k}{L} (1 - \epsilon \cos \phi)\]
\[
using: \frac{1}{r^2} = \left(\frac{km}{L^2}\right)^2 (1 - \epsilon \cos \phi)^2\]
\[
using: \frac{1}{(1 - \epsilon \cos \phi)^2} = \left(\frac{km}{L^2}\right)^2 r^2\]

Cartesian x = r cos φ:
\[
\dot{x} = \frac{dx}{dt} = \dot{r} \cos \phi - \sin \phi \ r \dot{\phi}\]
\[
= -\frac{k}{L} \epsilon \sin \phi \cos \phi - \sin \frac{k}{L} (1 - \epsilon \cos \phi)\]

Cartesian y = r sin φ:
\[
\dot{y} = \frac{dy}{dt} = \dot{r} \sin \phi + \cos \phi \ r \dot{\phi}\]
\[
= -\frac{k}{L} \epsilon \sin \phi \sin \phi + \cos \phi \frac{k}{L} (1 - \epsilon \cos \phi)\]
Finding time derivatives of orbital coordinates $r$, $\phi$, $x$, $y$, and eventually velocity $v$ or momentum $p=mv$

**Radius $r$:**

\[
r = \frac{\lambda}{1 - \varepsilon \cos \phi} = \frac{L^2}{\lambda \cdot km} \\
\dot{r} = \frac{L^2}{km} \cdot \frac{d}{dt} \left( -\varepsilon \cos \phi \right)
\]

\[
\dot{r} = \frac{L^2}{km} \left( \frac{km}{L^2} \right)^2 r^2 \dot{\phi} \varepsilon \sin \phi
\]

\[
\dot{r} = -\frac{k}{L^2} mr^2 \dot{\phi} \varepsilon \sin \phi = -\frac{k}{L} \varepsilon \sin \phi
\]

**Polar angle $\phi$ using:**

\[
L = mr^2 \frac{d\phi}{dt} = mr^2 \dot{\phi}
\]

\[
\dot{\phi} = \frac{L}{mr^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1 - \varepsilon \cos \phi)^2
\]

\[
r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left( \frac{km}{L^2} \right) (1 - \varepsilon \cos \phi) = \frac{k}{L} (1 - \varepsilon \cos \phi)
\]

\[
\text{using: } \frac{1}{r^2} = \left( \frac{km}{L^2} \right)^2 (1 - \varepsilon \cos \phi)^2
\]

\[
\text{using: } \frac{1}{(1 - \varepsilon \cos \phi)^2} = \left( \frac{km}{L^2} \right)^2 r^2
\]

**Cartesian $x = r \cos \phi$:**

\[
\dot{x} = \frac{dx}{dt} = \dot{r} \cos \phi - \sin \phi r \dot{\phi}
\]

\[
= -\frac{k}{L} \varepsilon \sin \phi \cos \phi - \sin \phi \frac{k}{L} (1 - \varepsilon \cos \phi)
\]

\[
= -\frac{k}{L} \varepsilon \sin \phi
\]

**Cartesian $y = r \sin \phi$:**

\[
\dot{y} = \frac{dy}{dt} = \dot{r} \sin \phi + \cos \phi r \dot{\phi}
\]

\[
= -\frac{k}{L} \varepsilon \sin \phi \sin \phi + \cos \phi \frac{k}{L} (1 - \varepsilon \cos \phi)
\]

\[
= \frac{k}{L} (\cos \phi - \varepsilon)
\]
\[ \frac{\lambda}{\epsilon} = 4 \]
\[ b = 2 \]
\[ = \sqrt{3/2} \]
\[ = 1 \]
\[
\frac{\lambda}{\varepsilon} = 4
\]
\[
b = 2
\]
\[
\varepsilon = \sqrt{3}/2
\]
\[
\lambda = 1
\]
\[ \frac{\lambda}{\varepsilon} = \frac{4}{2} = \frac{\sqrt{3}}{2} = 1 \]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\varepsilon$-vector and Coulomb $p=mv$ geometry

Analytic geometry derivation of $\varepsilon$-construction

Algebra of $\varepsilon$-construction geometry

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375$ elliptic orbit$)$

$(R=+0.5$ hyperbolic orbit$)$
Next several pages give step-by-step constructions of \( \varepsilon \)-vector and Coulomb orbit and trajectory physics.

Fig. 5.4.2 Construction of eccentricity vector \( \varepsilon \) and orbit from initial \( \mathbf{r}, \mathbf{p} \) with \( KE/PE=-3/8 \).
**ε-vector and Coulomb orbit construction steps**

Pick launch point $P$ (radius vector $\mathbf{r}$) and elevation angle $\gamma$ from radius (momentum initial $\mathbf{p}$ direction).

Next several pages give step-by-step constructions of $\varepsilon$-vector and Coulomb orbit and trajectory physics.
**ε-vector and Coulomb orbit construction steps**

Pick launch point \( P \)
(radius vector \( r \))
and elevation angle \( \gamma \) from radius
(momentum initial \( p \) direction)

Copy \( F \)-center circle around launch point \( P \)
Copy elevation angle \( \gamma (\angle FPP') \) onto \( \angle P'PQ \)
Extend resulting line \( QPQ' \) to make **focus locus**

Reason for **focus locus**:
Line \( r \) from 1st focus \( F \) "reflects" off line \( p \) (or \( P'P \)) toward 2nd focus \( F' \) somewhere
so incident-angle \( \gamma \) equals reflected-angle \( \gamma \)

Next several pages give step-by-step constructions
of **ε-vector** and Coulomb orbit and trajectory physics
**ε-vector** and Coulomb orbit construction steps

- Pick launch point $P$ (radius vector $\mathbf{r}$)
- Copy $F$-center circle around launch point $P$
- Copy elevation angle $\gamma$ (angle between $FPP'$) onto $\angle P'PQ$
- Extend resulting line $QPQ'$ to make focus locus
- Copy double angle $2\gamma(\angle FPQ)$ onto $\angle PFT$
- Extend $\angle PFT$ chord $PT$ to make $R$-ratio scale line
- Label chord $PT$ with $R=0$ at $P$ and $R=-1.0$ at $T$.
- Mark $R$-line fractions $R=0, +1/4, +1/2, ...$ above $P$ and $R=0, -1/8, -1/4, -1/2, ..., -3/4$ below $P$.
- Copy $\epsilon$-vector and Coulomb orbit construction steps

**Reason for focus locus:**
- Line $r$ from 1st focus $F$ “reflects” off line $p$ (or $PP'$) toward 2nd focus $F'$ somewhere so incident-angle $\gamma$ equals reflected-angle $\gamma$
\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} - \frac{k}{r(0)} \]

\[ = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]
**ε-vector and Coulomb orbit construction steps**

1. **Pick launch point** \(P\) (radius vector \(r\) )
2. **Copy F-center circle around launch point** \(P\)
3. **Copy elevation angle** \(\gamma(\angle FP'P')\) onto \(\angle P'PQ\)
4. **Extend resulting line** \(QPQ'\) to make **focus locus**

**Reason for focus locus:**
- **Line** \(r\) from 1st focus \(F\) “reflects” off line \(p\) (or \(P'P\)) toward 2nd focus \(F'\) somewhere so incident-angle \(\gamma\) equals reflected-angle \(\gamma\)

**Pick initial** \(R=KE/PE\) **value**
- **Draw** \(\epsilon\)-vector
- **Label chord** \(PT\) with \(R=0, +1/4, +1/2, \ldots\) **above** \(P\) and \(R=0, -1/8, -1/4, -1/2, \ldots, -3/4\) **below** \(P\) and \(-5/4, -3/2, \ldots\) **below** \(T\).

**Copy double angle** \(2\gamma(\angle FPQ)\) onto \(\angle PFT\)

**Mark R-line fractions** \(R=0, +1/4, +1/2, \ldots\) **above** \(P\) and \(R=0, -1/8, -1/4, -1/2, \ldots, -3/4\) **below** \(P\) and \(-5/4, -3/2, \ldots\) **below** \(T\).

**Eccentricity vector**
- **Focus** \(F\) and 2nd focus \(F'\) allow final construction of orbital trajectory.
- Here it is an \(R=-3/8\) ellipse.

(Detailed Analytic geometry of \(\epsilon\)-vector follows.)
**ε-vector and Coulomb orbit construction steps**

Pick launch point P (radius vector r)

and elevation angle γ from radius (momentum initial p direction)

Copy F-center circle around launch point P

Copy elevation angle γ (∠FPP') onto ∠P'PQ

Extend resulting line QPQ' to make focus locus

Copy double angle 2γ (∠FPQ) onto ∠PFT

Extend ∠PFT chord PT to make R-ratio scale line

Label chord with R=0 at P and R=-1.0 at T.

Mark R-line fractions R=0, +1/4, +1/2, ... above P and R=0, -1/8, -1/4, -1/2, ..., -3/4 below P and -5/4, -3/2, ... below T.

Reason for focus locus:

Line r from 1st focus F "reflects" off line p (or P'P) toward 2nd focus F', somewhere so incident-angle γ equals reflected-angle γ

Here it intersects 2nd focus F' somewhere

Mark R-line fractions above P and below T.

Pick initial R=KE/PE value

(Here R=+1/2)

Draw ε-vector from focus F to R-point

(Here it intersects 2nd focus F')

Focus F and 2nd focus F' allow final construction of orbital trajectory.

Here it is an R=+1/2 hyperbola.

(Detailed Analytic geometry of ε-vector follows.)
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb orbit construction steps

Connection to standard development

Analytic geometry derivation of $\varepsilon$-constructions

Algebra of $\varepsilon$-construction geometry

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
Analytic geometry derivation of $\varepsilon$-construction

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \frac{(mv_0)(mv_0r_0)}{km} \sin \gamma \hat{L}_p \times$$

where: $L_p \equiv p \times L$

Fig. 5.4.3
Construction of eccentricity vector $\varepsilon$ and orbit from initial $r, p$ with $KE/PE=+1/2$. 

$$R = \frac{Initial\ KE}{Initial\ PE} = \frac{mv^2(0)}{2} / \left(-k / r(0)\right)$$

$$= \pm \left( \frac{Initial\ velocity}{Escape\ velocity} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$
Analytic geometry derivation of $\varepsilon$-construction

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km} \hat{L}_p$$

where: $L_p \equiv p \times L$

$$\varepsilon = \hat{r} + 2\sin\gamma \frac{mv_0^2/2}{-k/r_0} \hat{L}_p = \hat{r} + 2\sin\gamma \frac{KE}{PE} \hat{L}_p$$

Fig. 5.4.3
Construction of eccentricity vector $\varepsilon$
and orbit from initial $r, p$ with $KE/PE=+1/2$.

$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)/2}{-k/r(0)}$$

$$= \pm \left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$
Fig. 5.4.3 Construction of eccentricity vector \( \mathbf{\varepsilon} \) and orbit from initial \( r \), \( p \) with KE/PE=+1/2.

\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0) / 2}{-k / r(0)} \]

\[ = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]

\[ \mathbf{\varepsilon} = \mathbf{\hat{r}} - \frac{p \times L}{km} = \mathbf{\hat{r}} - \frac{(mv_0)(mv_0r_0')\sin\gamma}{km} \mathbf{\hat{L}_{p\times}} \]

where: \( L_{p\times} = p \times L \)

The eccentricity vector is:

\[ \mathbf{\varepsilon} = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} + 2 \sin\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ R = \begin{pmatrix} \cos\gamma \\ (2R+1)\sin\gamma \end{pmatrix} \]
Analytic geometry derivation of $\varepsilon$-construction

$\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \frac{(mv_0)(mv_0r_0) \sin \gamma}{km} \hat{L}_{p\times}$

where: $L_{p\times} = p \times L$

$\varepsilon = \hat{r} + 2 \sin \gamma \frac{mv_0^2}{-k/r_0} \hat{L}_{p\times} = \hat{r} + 2 \sin \gamma \frac{KE}{PE} \hat{L}_{p\times}$

The eccentricity vector is:

$\varepsilon = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2 \sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$R = \begin{pmatrix} \cos \gamma \\ (2R+1) \sin \gamma \end{pmatrix}$

For: $\gamma = 45^\circ$ and: $R = +\frac{1}{2}$

$\varepsilon = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}(2R+1) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 2/\sqrt{2} \end{pmatrix}$

$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)/2}{-k/r(0)}$

$= \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$
Analytic geometry derivation of $\varepsilon$-constructions

$\mathbf{\varepsilon} = \hat{r} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \hat{r} - \left(\frac{m\mathbf{v}_0}{m\mathbf{v}_0 r_0}\right) \sin \gamma \hat{L}_{p\times}$

where: $\hat{L}_{p\times} \equiv \mathbf{p} \times \mathbf{L}$

$\varepsilon = \hat{r} + 2\sin \gamma \frac{m\mathbf{v}_0^2 / 2}{-k/r_0} \hat{L}_{p\times} = \hat{r} + 2\sin \gamma \frac{KE}{PE} \hat{L}_{p\times}$

The eccentricity vector is:

$\varepsilon = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2\sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} (2R+1)\sin \gamma$

For: $\gamma = 45^\circ$ and: $R = \pm \frac{1}{2}$

$\varepsilon = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 2/\sqrt{2} \end{pmatrix}$

The eccentricity parameter defined by:

$\varepsilon^2 = \cos^2 \gamma + (2R+1)^2 \sin^2 \gamma = 1 \pm \frac{a^2}{b^2}$

$= 1 + 4R(R+1)\sin^2 \gamma = \frac{5}{2}$

$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m\mathbf{v}^2(0)}{2} / \left(-k / r(0)\right)$

$= \pm \left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm \frac{\mathbf{v}^2(0)}{\mathbf{v}^2(\infty)}$
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb orbit construction steps

Connection to standard development

Analytic geometry derivation of constructions

$\rightarrow$ Algebra of $\varepsilon$-construction geometry

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375\text{ elliptic orbit})$

$(R=+0.5\text{ hyperbolic orbit})$
Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse } (\varepsilon < 1)$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola } (\varepsilon > 1)$$
Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2 \gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1) \quad \text{where:} \quad 4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1) \quad \text{where:} \quad 4R(R+1)\sin^2 \gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$$
Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \text{ for ellipse (} \varepsilon < 1 \text{) where: } 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ implying: } R(R+1) < 0$$

$$= 1 + \frac{b^2}{a^2} \text{ for hyperbola (} \varepsilon > 1 \text{) where: } 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ implying: } R(R+1) > 0$$
Algebra of \( \varepsilon \)-construction geometry

The *eccentricity* parameter relates ratios \( R = \frac{KE}{PE} \) and \( \frac{b^2}{a^2} \):

\[
\varepsilon^2 = 1 + 4R(R+1)\sin^2 \gamma
\]

\[
= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1) \quad \text{where:} \quad 4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1 \quad \text{implying:} \quad R(R+1) < 0 \quad \text{(or:} \quad R^2 < R) \\
= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1) \quad \text{where:} \quad 4R(R+1)\sin^2 \gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1 \quad \text{implying:} \quad R(R+1) > 0 \quad \text{(or:} \quad R^2 > R)
\]
Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$.

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

- for ellipse ($\varepsilon < 1$) where: $4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $R^2 < R$) (or: $R < 1$)

- for hyperbola ($\varepsilon > 1$) where: $4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $R^2 > R$) (or: $R > 1$)

Total $\frac{-k}{2a} = E = energy = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

$$-\frac{k}{2a} = E = KE + PE = R \cdot PE + PE = (R+1)PE = \frac{R+1}{r} \frac{-k}{r} = \frac{1}{2a} = \frac{(R+1)}{r} = (R+1)$$
Algebra of ε-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

\[\epsilon^2 = 1 + 4R(R+1)\sin^2\gamma\]

- For ellipse ($\epsilon < 1$) where: $4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \epsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $R^2 < R$)

- For hyperbola ($\epsilon > 1$) where: $4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \epsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $R^2 > R$)

Total $\frac{-k}{2a} = E = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

\[-\frac{k}{2a} = E = KE + PE = R \cdot PE + PE = (R+1)PE = (R+1) \frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a} = (R+1) \frac{1}{r} = (R+1)\]

$a = \frac{1}{2(R+1)}$ assuming *unit* initial radius ($r = 1$).
### Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$.

\[
\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma
\]

- For ellipse ($\varepsilon < 1$) where: $4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $R^2 < R$)

- For hyperbola ($\varepsilon > 1$) where: $4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $R^2 > R$)

Total $\frac{-k}{2a} = E = \text{energy} = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

\[
\frac{-k}{2a} = E = KE + PE = R \cdot PE + PE = (R + l)PE = (R + l)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a} = (R + l)\frac{1}{r}
\]

\[
a = \frac{1}{2(R + l)} \quad \text{assuming unit initial radius (r≡1)}.
\]

\[
4R(R+1)\sin^2\gamma = \pm \frac{b^2}{a^2} \quad \text{implies:} \quad 2\sqrt{R(R+1)}\sin\gamma = \frac{b}{a} \quad \text{or:} \quad b = 2a\sqrt{R(R+1)\sin\gamma}
\]

\[
b = \sqrt{\frac{R}{R+1}}\sin\gamma
\]
Algebra of $\epsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\epsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

- $= 1 - \frac{b^2}{a^2}$ for ellipse ($\epsilon < 1$) where: $4R(R+1)\sin^2\gamma = - \frac{b^2}{a^2} = \epsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $R^2 < R$)

- $= 1 + \frac{b^2}{a^2}$ for hyperbola ($\epsilon > 1$) where: $4R(R+1)\sin^2\gamma = + \frac{b^2}{a^2} = \epsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $R^2 > R$)

Total $\frac{-k}{2a} = E = energy = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

$$\frac{-k}{2a} = E = KE + PE = R \cdot PE + PE = (R+1)PE = (R+1)\frac{-k}{r}$$

or: $$\frac{1}{2a} = (R+1)\frac{1}{r}$$

$$a = \frac{1}{2(R+1)}$$ assuming unit initial radius ($r=1$).

$$4R(R+1)\sin^2\gamma = \pm \frac{b^2}{a^2}$$ implies: $$2\sqrt{R(R+1)}\sin\gamma = \frac{b}{a}$$ or: $$b = 2a\sqrt{R(R+1)\sin\gamma}$$

$$b = \sqrt{\frac{R}{R+1}}\sin\gamma$$

*Latus radius* is similarly related:

$$\lambda = \frac{b^2}{a} = 2R\sin^2\gamma$$
Algebra of $\varepsilon$-construction geometry

The **eccentricity** parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

\[
\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma
\]

\[
\varepsilon^2 = 1 - \frac{b^2}{a^2} \quad \text{ellipse ($\varepsilon < 1$)}
\]

\[
4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2}
\]

\[
= 1 + \frac{b^2}{a^2} \quad \text{hyperbola ($\varepsilon > 1$)}
\]

\[
4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2}
\]

\[
a = \frac{1}{2(R+1)} \quad \text{assuming **unit** initial radius ($r=1$)}.
\]

\[
b = \sqrt{\frac{R}{R+1}} \sin\gamma
\]

\[
\lambda = \frac{b^2}{a} = 2R\sin^2\gamma
\]

\[
\frac{b}{a} = 2\sqrt{R(R+1)}\sin\gamma
\]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb orbit construction steps

Connection to standard development

Analytic geometry derivation of construction

Algebra of $\varepsilon$-construction geometry

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
Extend FP to make major axis sum
FPP': \((r + r' = 2a)\) at intersect of \(r'-arc\)

\(R = -3/8\) elliptic orbit
construction

\(\gamma = 45^\circ\)
Strike radius-\( r \) arc about point \( P' \) to intersect original radius-\( r \) circle about focus \( F \) at ends of bisection line \( BB' \). Draw radius-\( a \) circle at \( F \) tangent to bisection line \( BB' \).

\( R = -3/8 \) elliptic orbit construction

\( R = -3/8 \)  
\( \gamma = 45^\circ \)
Strike radius-\(r\) arc about point \(P'\) to intersect original radius-\(r\) circle about focus \(F\) at ends of bisection line \(BB'\). Draw radius-\(a\) circle at tangent to bisection line \(BB'\).

Draw radius-\(a\) circle at \(F'\).

Draw radius-\(a\) and radius-\(b\) circles at \(O\) (Center of bisection line \((\pm b)\).
Strike radius-$r$ arc about point $P'$ to intersect original radius-$r$ circle about focus at ends of bisection line $BB'$. Draw radius-$a$ circle at tangent to bisection line $BB'$. Extend $FP$ to make major axis sum $FPP'(r+r'=2a)$ at $P'$, intersect of $r'$-arc of $r'$-arc. $R=\frac{-3}{8}$ elliptic orbit construction $\gamma=45^\circ$.

Draw radius-$a$ circle at $F'$.

Draw radius-$a$ and radius-$b$ circles at $O$ (Center of bisection line $(\pm b)$).
\[
\varepsilon = \sqrt{1 + 4R(R+1)\sin^2 \gamma} = \frac{\sqrt{34}}{8} = .73
\]

\[
a = \frac{1}{2(R+1)} = \frac{4}{5}
\]

\[
b = \sqrt{\frac{R}{R+1} \sin \gamma} = \frac{\sqrt{3}}{\sqrt{10}} = .54
\]

\[
\lambda = \frac{b^2}{a} = 2R \sin^2 \gamma = \frac{3}{8} = .375
\]

\[
\frac{b}{a} = 2\sqrt{R(R+1)\sin \gamma} = \tan 34^\circ
\]

\[
\gamma = 45^\circ
\]

**Draw radius-a circle at F**

**Draw radius-a and radius-b circles at O**

(Center of bisection line (±b).) Do (a,b)-ellipse construction.
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb orbit construction steps

Connection to standard development

Analytic geometry derivation of construction

Algebra of $\varepsilon$-construction geometry

Ruler & compass construction of $\varepsilon$-vector and orbits

(R = -0.375 elliptic orbit)

(R = +0.5 hyperbolic orbit)
Major diameter $2a$ is difference $(r-r'=2a)$.
Major radius $a$ is half of difference $(r-r')/2=a$
Major diameter $2a$ needs to be centered on $F-F'$ focal axis

$$R=+1/2 \text{ hyperbolic orbit construction}$$

$$\gamma=45^\circ$$
Major diameter $2a$ is difference $(r-r'=2a)$.
Major radius $a$ is half of difference $(r-r'/2=a)$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.
1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.

$R=+1/2$ hyperbolic orbit construction

$R=+1/2$

$\gamma=45^\circ$
Major diameter 2a is difference (r-r' = 2a).
Major radius a is half of difference (r-r')/2 = a.
Major diameter 2a needs to be centered on F-F' focal axis.

1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F' focal axis using F-F' circle intersections to locate orbit center C.

\[ R = +1/2 \] hyperbolic orbit construction

\[ \gamma = 45^\circ \]
Major diameter 2a is difference (r-r'\(=2a\)).

Major radius a is half of difference (r-r')/2=a.

Major diameter 2a needs to be centered on F-F' focal axis.

1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F' focal axis using F-F' circle intersections to locate orbit center C.

\[
\text{Hyperbolic orbit construction:} \quad R=+1/2
\]

\[
\gamma=45^\circ
\]
Major diameter $2a$ is difference $(r-r')=2a$.
Major radius $a$ is half of difference $(r-r')/2=a$.

Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$. 

$R=+1/2$ hyperbolic orbit construction

$\gamma=45^\circ$
Major diameter $2a$ is difference $(r-r') = 2a$.
Major radius $a$ is half of difference $(r-r')/2 = a$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a = (r-r')/2$.
5. Copy circle of major radius $a = (r-r')/2$ about orbit center $C$.

\[ R = +1/2 \text{ hyperbolic orbit construction} \]
\[ \gamma = 45^\circ \]
Major diameter 2a is difference (r-r' = 2a).
Major radius a is half of difference (r-r')/2 = a.
Major diameter 2a needs to be centered on F-F' focal axis.
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F' focal axis using F-F' circle intersections to locate orbit center C.
4. Swing radius r'/2 onto r/2 section to make major radius a = (r-r')/2.
5. Copy circle of major radius a = (r-r')/2 about orbit center C.
6. Draw focal circle of diameter 2ae about orbit center C.
Major diameter $2a$ is difference $(r-r' = 2a)$.
Major radius $a$ is half of difference $(r-r')/2 = a$
Major diameter $2a$ needs to be centered on $F-F'$ focal axis
1. Bisect $F$-$P$ radius $r$ using $F$-$P$ circle intersections to define $r/2$ sections.
2. Bisect $F$-$F'$ focal axis using $F$-$F'$ circle intersections to locate orbit center $C$.
3. Bisect $F'$-$P$ radius $r'$ using $F'$-$P$ circle intersections.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$.
5. Copy circle of major radius $a=(r-r')/2$ about orbit center $C$.
6. Draw focal circle of diameter $2ae$ about orbit center $C$.
7. Erect minor radius $b$ tangent to $a$-circle from point $a$ on $C\varepsilon$-axis to point $b$ on focal circle.
Major diameter $2a$ is difference $(r-r'=2a)$.  
Major radius $a$ is half of difference $(r-r')/2=a$. 
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.
1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections. 
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$. 
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$. 
5. Copy circle of major radius $a=(r-r')/2$ about orbit center $C$. 
6. Draw focal circle of diameter $2aE$ about orbit center $C$. 
7. Erect minor radius $b$ tangent to $a$-circle from point $a$ on $Cε$-axis to point $b$ on focal circle. 
8. Complete orbit $a-x-b$ box between focal circle and $a$-circle and its diagonal asymptotes.
9. Draw section of hyperbolic orbit.

\[ \gamma = 45^\circ \]

\[ R = +1/2 \]

Hyperbolic orbit construction
9. Draw section of hyperbolic orbit.

Construction based on: \( r - r' = 2a \) or: \( r' = r - 2a \)

1st draw an \( r \)-arc about focus \( F \).
9. Draw section of hyperbolic orbit.

Construction based on: \( r-r'=2a \) or \( r'=r-2a \)

1st draw an \( r \)-arc about focus \( F \).
2nd set compass to \( (r-2a) \) using \( r \)-arc-minus-\( 2a \) on \( C\varepsilon \)-line.

\( R=+1/2 \) hyperbolic orbit construction

\( R=+1/2 \)

\( \gamma=45^\circ \)
9. Draw section of hyperbolic orbit.

\[ R = +1/2 \] hyperbolic orbit construction

\[ \gamma = 45^\circ \]

Construction based on: \( r - r' = 2a \) or: \( r' = r - 2a \)

1\textsuperscript{st} draw an \( r \)-arc about focus \( F \).
2\textsuperscript{nd} set compass to \( (r - 2a) \) using
3\textsuperscript{rd} draw \( (r - 2a) \)-arc about focus \( F' \).
9. Draw section of hyperbolic orbit.

Construction based on: $r-r' = 2a$ or: $r' = r-2a$

1st draw an $r$-arc about focus $F$.
2nd set compass to $(r-2a)$ using $r$-arc-minus-$2a$ on Cε-line.
3rd draw $(r-2a)$-arc about focus $F'$.

Orbit points at intersections.

$R = +1/2$ hyperbolic orbit construction

$R = +1/2$

$\gamma = 45^\circ$
9. Draw section of hyperbolic orbit.
9. Draw section of hyperbolic orbit.
9. Draw section of hyperbolic orbit.

$R=+1/2$ hyperbolic orbit construction

$R=+1/2$

$\gamma=45^\circ$
9. Draw section of hyperbolic orbit.

\[ \varepsilon = \sqrt{1 + 4R(R+1)\sin^2\gamma} = \sqrt{\frac{3}{2}} = 1.58 \]

\[ a = \frac{1}{2(R+1)} = \frac{1}{3} = 0.33 \]

\[ b = \sqrt{\frac{R}{R+1}} \sin \gamma = \frac{1}{\sqrt{6}} = 0.408 \]

\[ \lambda = \frac{b^2}{a} = 2R \sin^2 \gamma = \frac{1}{2} = 0.5 \]

\[ \frac{b}{a} = 2\sqrt{R(R+1)} \sin \gamma = \tan 50.7^\circ \]
Properties of Coulomb trajectory families and envelopes

Graphical ε-development of orbits
- Launch angle fixed - Varied launch energy
- Launch energy fixed - Varied launch angle
- Launch optimization
**Start with initial velocity**

\( \mathbf{v}(0) \) or \( -\mathbf{v}(0) \)

- Label Main Focus \( F \)
- Construct focus for 2\(^{nd} \) foci \( F' \)

Launch Elevation Angle

\( \gamma = 70^\circ \)

\( \alpha = 20^\circ \)

Range Longitude
Start with initial velocity \( v(0) \) or \(-v(0)\)

Label Main Focus \( F \)
Construct focus \( F \) for 2\(^{nd} \) foci \( F' \)
Construct \( R \)-line normal to initial velocity \( \pm v(0) \) line

Launch Elevation Angle

\[ \gamma = 70^\circ \]

\[ \alpha = 20^\circ \]
Start with initial velocity \( v(0) \) or \(-v(0)\)

Label Main Focus \( F \)

Construct focus \( F' \) for 2nd foci \( F' \)

Construct \( R \)-line normal to initial velocity \( \pm v(0) \) line

\((N=8)\)-sect \( R \)-line normal to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8 \ldots \)

for eccentricity \( \varepsilon \)-vector scale
Start with initial velocity \( \mathbf{v}(0) \) or \(-\mathbf{v}(0)\)

Label Main Focus \( F \)

Construct \textit{R-line normal} to initial velocity \( \mathbf{v}(0) \) line

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)\)-sect \textit{R-line normal} to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8 \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \textit{R-line}-point

Range Longitude
Start with initial velocity $v(0)$ or $-v(0)$

- Label Main Focus $F$
- Construct $R$-line normal to initial velocity $v(0)$ line
- Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE, PE=0, \pm 1/8, \pm 2/8, \pm 3/8$ for eccentricity $\varepsilon$-vector scale

- Extend eccentricity $\varepsilon$-vectors from the main Focus $F$ to each $R$-line point and beyond to prime foci $F'$

Range Longitude
This (R = -9/8) \( \varepsilon \)-line hits focus-locus far away.

This (R = \( \pm \infty \)) \( \varepsilon \)-line intersects focus-locus on unit circle. \([ (R = \pm \infty) \varepsilon \)-line parallel to R-scale line.\]

This (R = -1) \( \varepsilon \)-line intersects focus-locus at \( \infty \).

Label Main Focus F

Construct R-line normal to initial velocity \( v(0) \) line

Construct focus locus for prime foci \( F' \)

(N = 8)-sec R-line normal to \( F \) mark R = KEDE = \( \pm 1/8 = -1/8, 8/8 \) for eccentricity - vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus F to each R-line-point and beyond to prime foci \( F' \)

\[ \varepsilon \]-line parallel to focus-locus.
Properties of Coulomb trajectory families and envelopes

Graphical $\epsilon$-development of orbits

- Launch angle fixed - Varied launch energy
- Launch energy fixed - Varied launch angle

Launch optimization
Start with initial velocity \( \mathbf{v}(0) \) or \(-\mathbf{v}(0)\)

Label Main Focus \( F \)

Construct \( R\)-line normal to initial velocity \( \mathbf{v}(0) \) line

Construct focus locus for prime foci \( F' \)

\((N=8)\)-sect \( R\)-line normal to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \( R\)-line-point and beyond to prime foci \( F' \)

Range bisection circles (these are not orbits) indicate reentry ranges
Start with initial velocity $v(0)$ or $-v(0)$

Label Main Focus $F$

Construct $R$-line normal to initial velocity $v(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8 ...$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$

to each $R$-line point and beyond to prime foci $F'$

Origin

Range bisection circles (these are not orbits) indicate reentry ranges

$\mathbf{v}(0)$ Same arc centered on unit circle measures “string length”

$2a=r'+r=r'+1$

$29.2$
Start with initial velocity \( \mathbf{v}(0) \) or \(-\mathbf{v}(0)\)

Label Main Focus \( F \)

Construct \( R \)-line normal to initial velocity \( \mathbf{v}(0) \) line

Construct focus locus for prime foci \( F' \)

\((N=8)\)-sect \( R \)-line normal to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8 \ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \( R \)-line-point and beyond to prime foci \( F' \)

\( \mathbf{v}(0) \) Same arc centered on unit circle measures “string length” \( 2a=r'+r=r'+1 \)

Range bisection circles (these are not orbits) indicate reentry ranges

Construct ellipse point by point
Construct focus loci for prime focus F'

Label Main Focus F

Construct R-line normal to initial velocity V(0) line

(N=8) sect R-line normal to mark R=KE/PE=0\pm18\pm78\pm38...

(e) focus loci for prime focus F'

Mark focus loci for eccentricity e-vector scale

Construct focus loci for prime focus F'

This puts 2nd focus at \( \infty \).

\( r = 1 \)

(\( R_0 = 1 \) e = 1 line parallel to [focus loci])

This marks range limit for this elevation angle and e=1 (parabola)

\( \alpha = 20^\circ \) has \( R = 1 \)

Maximum range limit for this elevation angle

Range bisection indicates re-entry ranges
Label Main Focus F

Construct **R-line normal** to initial velocity \( \mathbf{v}(0) \) line

Construct **focus locus** for prime foci \( F' \)

\( (N=8) \)-sect **R-line normal** to mark \( R=KE/PE=0,\pm1/8,\pm2/8,\pm3/8, \ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \)
to each **R-line**-point and
beyond to prime foci \( F' \)

**Maximum** range limit for this elevation angle \( \alpha=20^\circ \) is range \( \phi=280^\circ \)

\( \varepsilon = 1 \)-line parallel to focus-locus

This puts 2nd focus at \( \infty \).
Label Main Focus \( F \)

Construct \textit{R-line normal} to initial velocity \( \mathbf{v}(0) \) line

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)\)-sect \textit{R-line normal} to mark \( R=KE/PE=0,\pm1/8,\pm2/8,\pm3/8,\ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \textit{R-line}-point and beyond to prime foci \( F' \)

\textbf{Maximum} range limit for this elevation angle \( \alpha=20^\circ \) is range \( \phi=280^\circ \)

\textbf{Maximum} range limit for this elevation angle \( \alpha=20^\circ \) has \( R=-1 \) and \( \varepsilon=1 \) (parabola)

\( (R=-1) \ \varepsilon = 1 \)-line parallel to focus-locus

This puts 2nd focus at \( \infty \).
This puts 2nd focus at $\infty$. 

Revu: geometry of parabola “kites”

Maximum range limit for this elevation angle $\alpha = 20^\circ$ is range $\phi = 280^\circ$.

Maximum range limit for this elevation angle $\alpha = 20^\circ$ has $R = -1$ and $\varepsilon = 1$ (parabola)

$\[ (R = -1) \varepsilon = 1 \text{-line parallel to focus-locus} \]$

Wednesday, December 10, 2014
Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

- Launch angle fixed-Varied launch energy
- $\Rightarrow$ Launch energy fixed-Varied launch angle
- Launch optimization
Start with initial velocity \( v(0) \) or \(-v(0)\)

Label Main Focus \( F \)

Construct \( R\)-line normal to initial velocity \( v(0) \) line

Construct focus locus for prime foci \( F' \)

\((N=8)\)-sect \( R\)-line normal to mark \( R=KEIPE=0,\pm 1/8,\pm 2/8,\pm 3/8,\ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \( R\)-line-point and beyond to prime foci \( F' \)

Range Longitude
Label Main Focus $F$

Construct $R$-line normal to initial velocity $v(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8$

for eccentricity $\epsilon$-vector scale

Extend eccentricity $\epsilon$-vectors from the main Focus $F$ to each $R$-line-point and beyond to prime foci $F'$

focus locus for fixed Energy or fixed $R=KE/PE=-5/8$

Focus locus for fixed launch angle $\alpha=20^\circ$
**Coulomb envelope geometry**

(a) Focus locus for KE/PE $= R = -3/8$

(b) Caustic for KE/PE $= R = -3/8$

(c) Diving orbit

**Ideal comet “tails” in solar wind**
Launch optimization

\[ \theta = (\pi/2 - \theta) = 2\theta - \pi/2 \]

Optimum (smallest) focus-focus circle to achieve range \( \rho \)

Optimum energy angle relations:

\[ 2\theta + \rho/2 = \pi/2 \]
\[ \theta = (\pi - \rho)/4 \]
\[ \rho = \pi - 4\theta \]

These angles are equal because of equal-focal reflection angles.

\( \theta \) is \( \alpha \) here