Lecture 25
Tue. 11.18-20.2014

Introduction to coupled oscillation and eigenmodes

(Ch. 2-4 of Unit 4 11.19.13)

Review: Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

Beat, lifetimes, and quality factor $q = \omega_0/2\Gamma$ *and* $Q = \upsilon_0/2\Gamma = q/2\pi$

Review: Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

Common Lorentzian (a.k.a. Witch of Agnesi) and geometry

2D harmonic oscillator equations

Lagrangian and matrix forms

Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues⇒eigenvectors)

Spectral decomposition and P-operator expansions (how projectors⇒eigensolutions)

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ *phase*

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with fluid phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

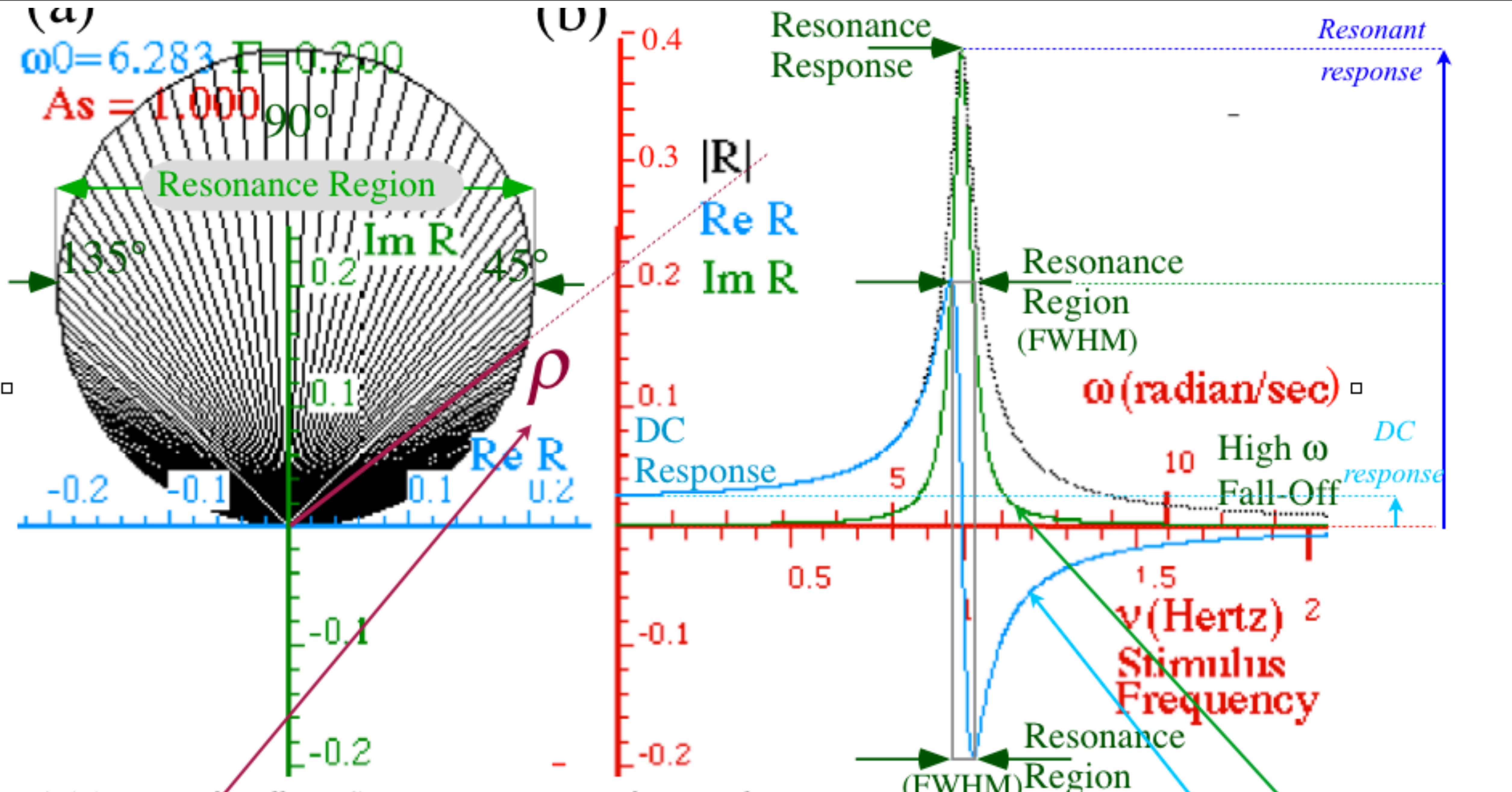


Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\mathrm{Re}\,G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Real part

$$\mathrm{Im}\,G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0}(\omega_s = \omega_0)|}{|G_{\omega_0}(0)|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

Review of Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0}L(\Delta - i\Gamma)$$

Define *complex detuning-decay* $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L \quad + i\,\mathrm{Im}\,L \quad = \frac{\Delta}{\Delta^2 + \Gamma^2} + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

$$= |L|e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$

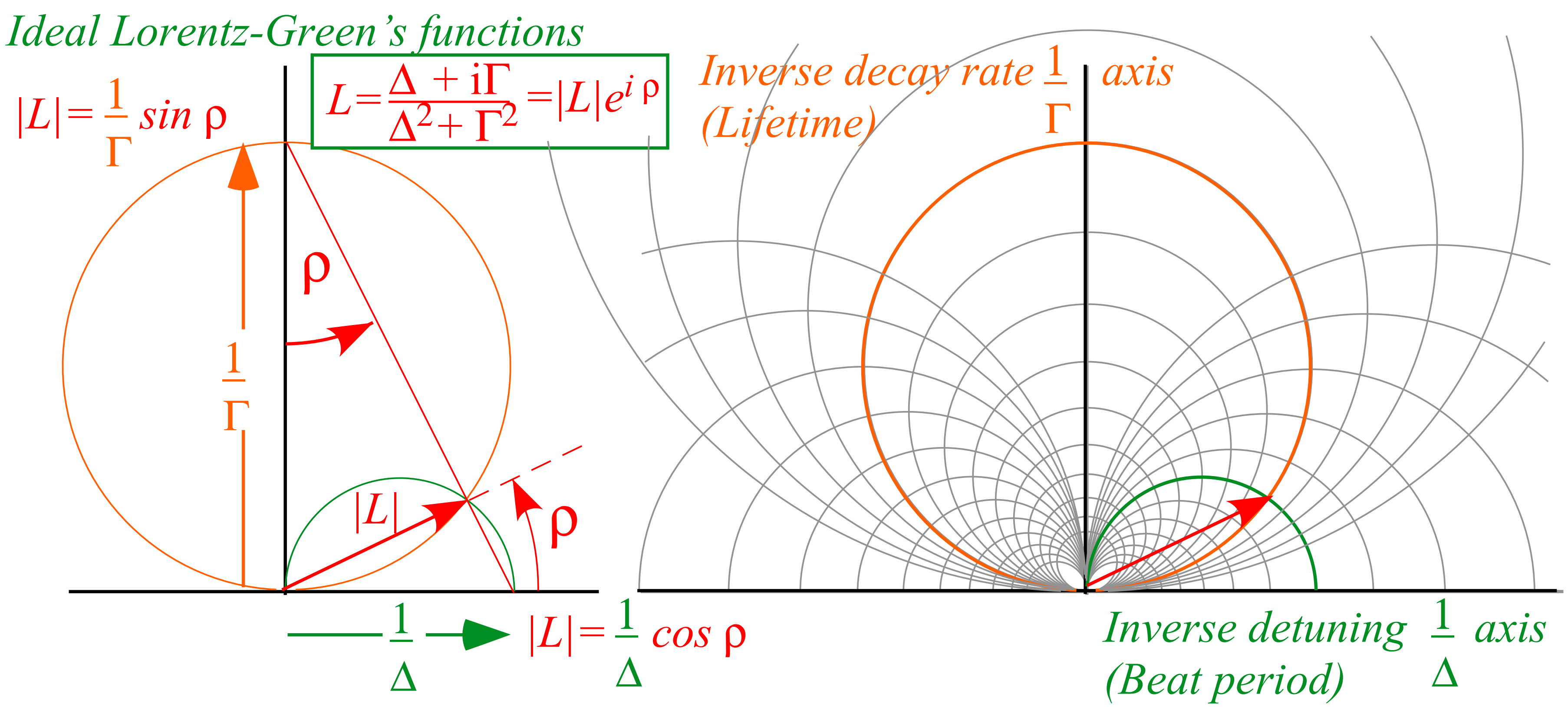


Fig. 4.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ *vs. beat-period* $1/\Delta$ *coordinates)*

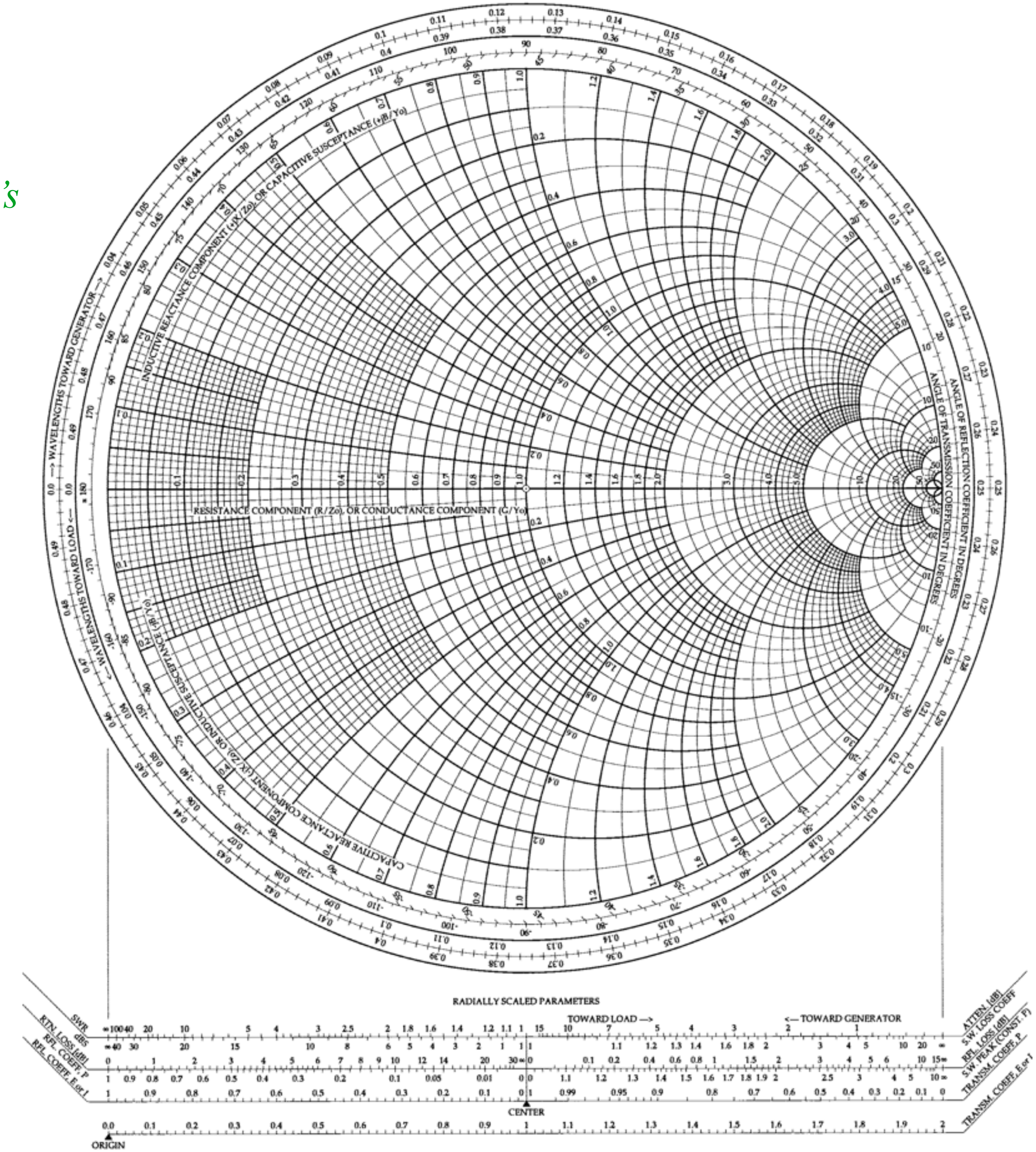
Constant Δ and Γ curves in Fig. 3.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.

SMITH CHART (Invented by Phillip H. Smith 1905-1987)

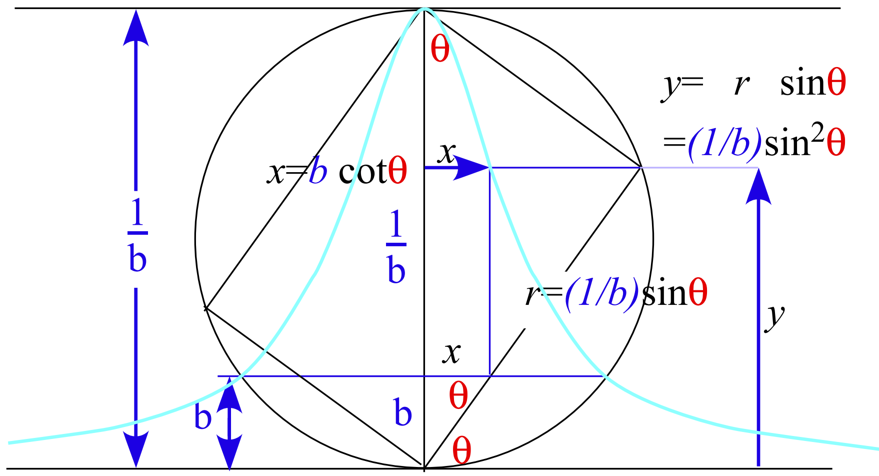
An FDHO Green's Function Slide rule

A plot of f(z) =1/z

For wavy "Ohm's Laws" V =I·Z I=V/Z



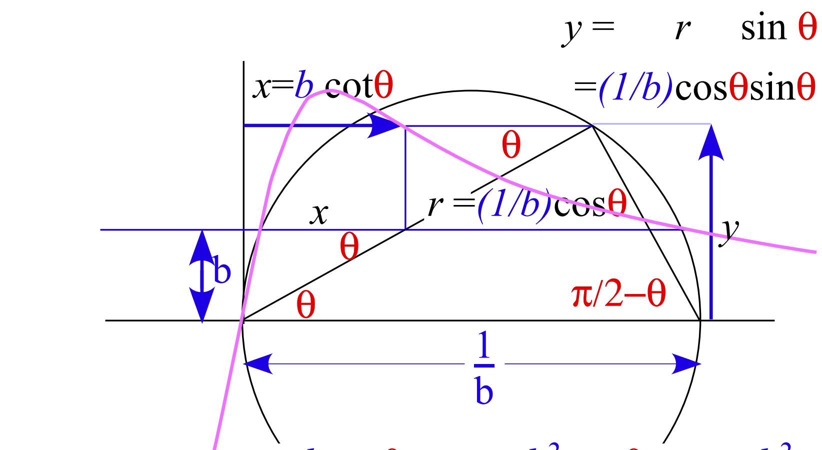
The Common Lorentzian (a.k.a. *The Witch of Agnesi*)



$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$

$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$

$y=\frac{b}{x^2+b^2}$
Common Lorentzian function I.
(imaginary "absorbtive" part)



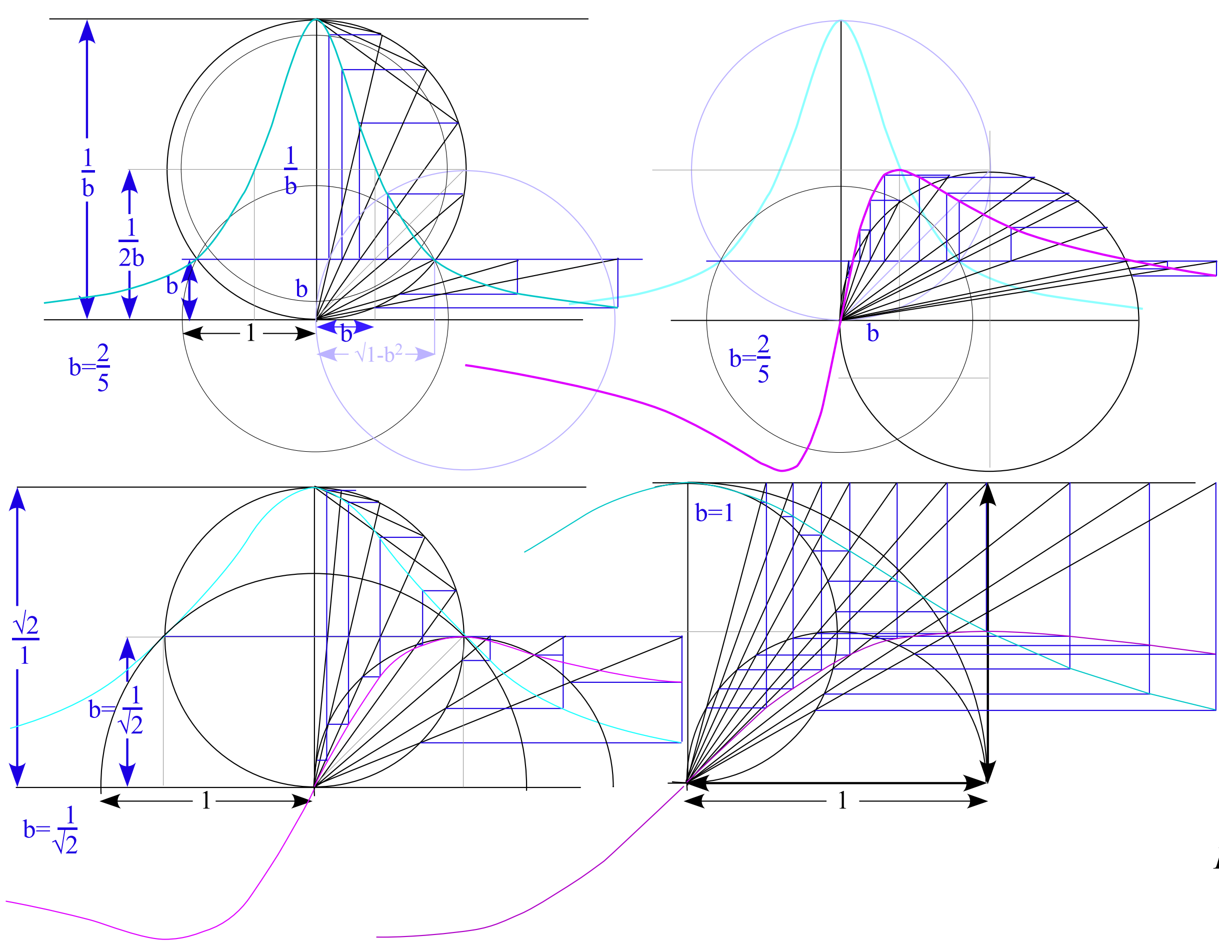
$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$

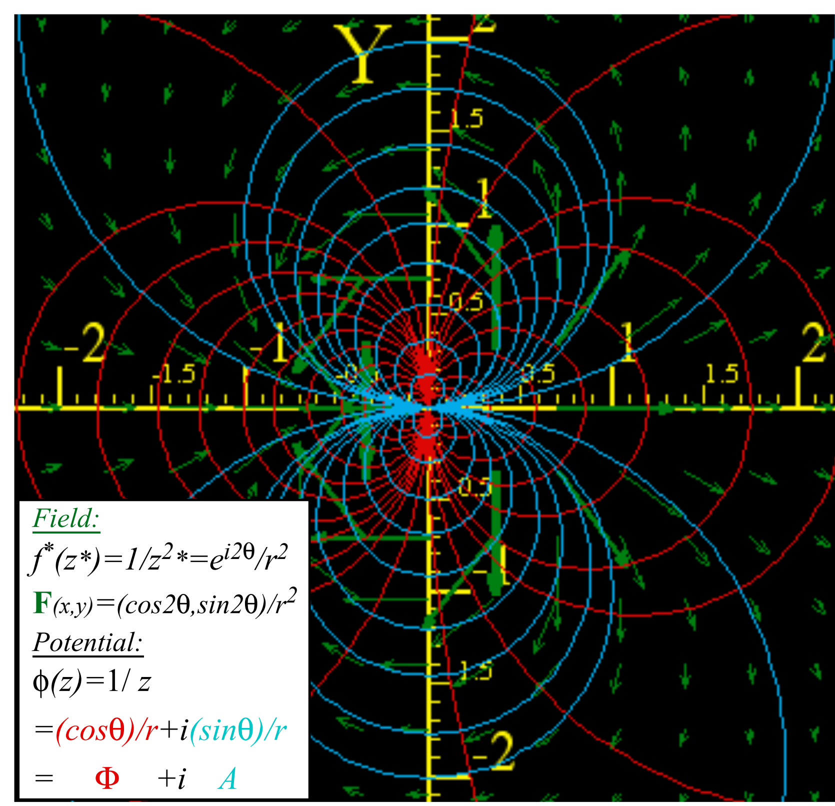
$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$

$y=\frac{x}{x^2+b^2}$
Common Lorentzian function II.
(real "refractory" part)

Maria Gaetana Agnesi

Born	May 16, 1718
Died	January 9, 1799 (aged 80)
Residence	Italy
Nationality	Italy
Fields	Mathematics





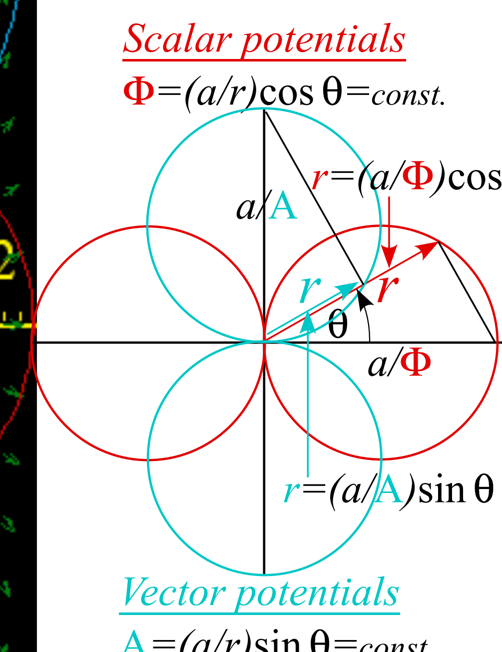


Fig. 10.11 Dipole **F***-field $f(z)=1/z^2$ and scalar potential (Φ=const.)-circles orthogonal to (A=const.)-circles.*

➔ *2D harmonic oscillator equations*

Lagrangian and matrix forms

Reciprocity symmetry

2D harmonic oscillators

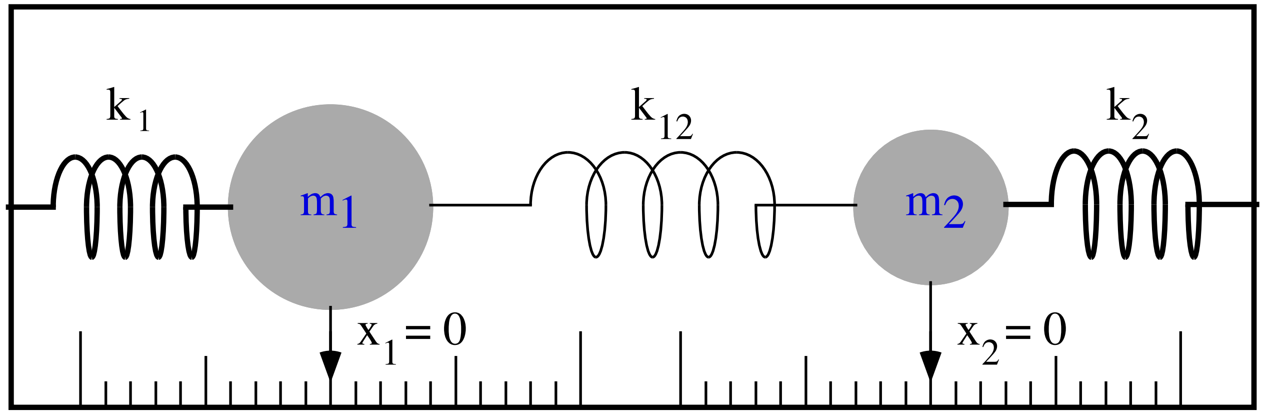


Fig. 3.3.1 Two 1-dimensional coupled oscillators

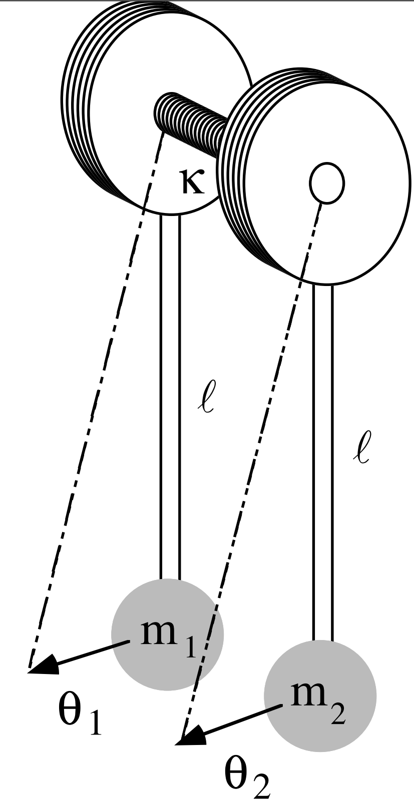


Fig. 3.3.2 Coupled pendulums

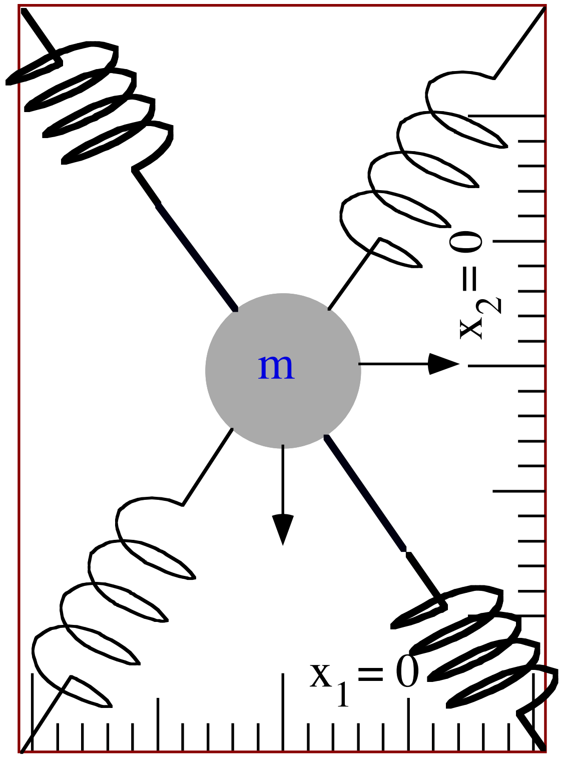
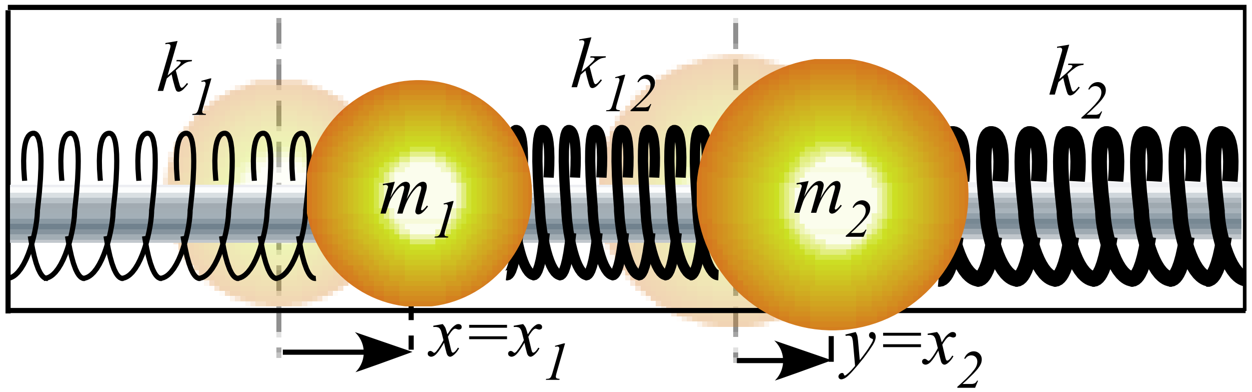
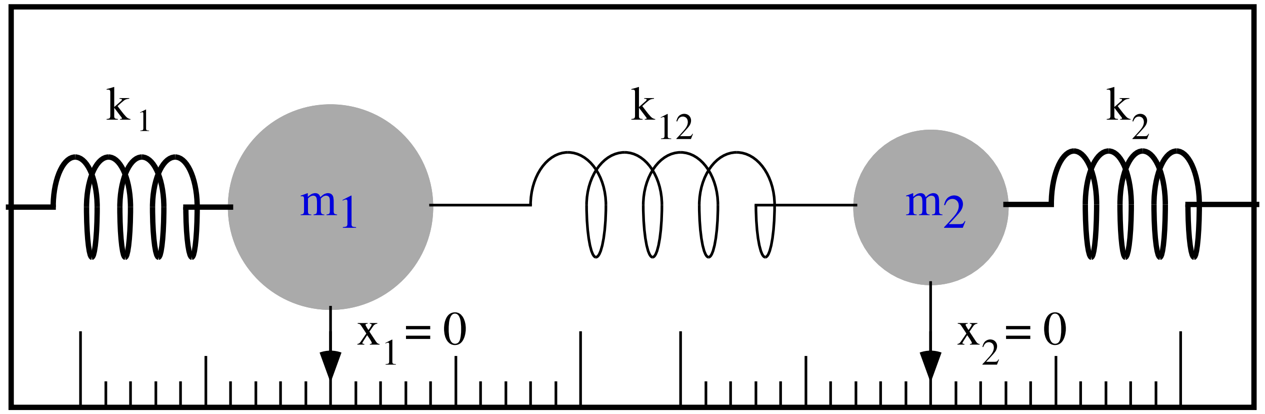


Fig. 3.3.3 One 2-dimensional coupled oscillator



2D harmonic oscillator energy



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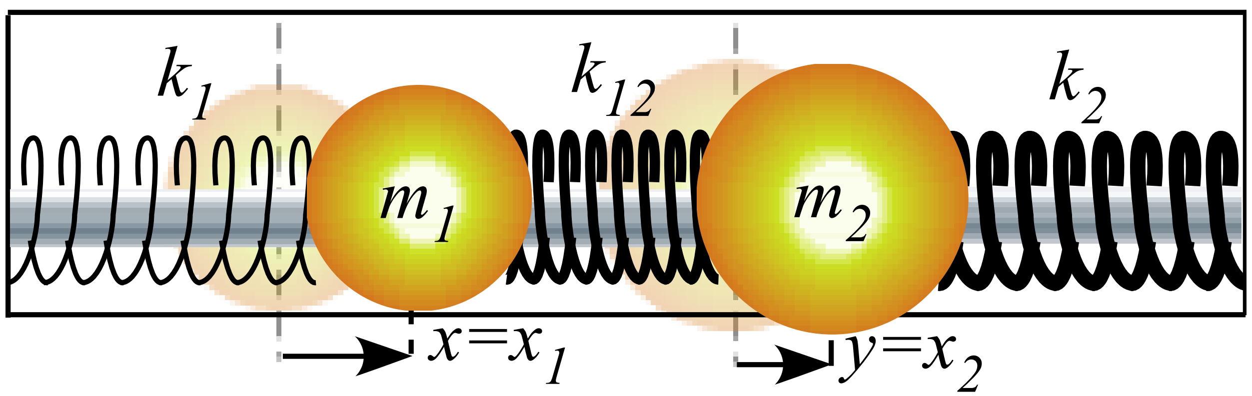


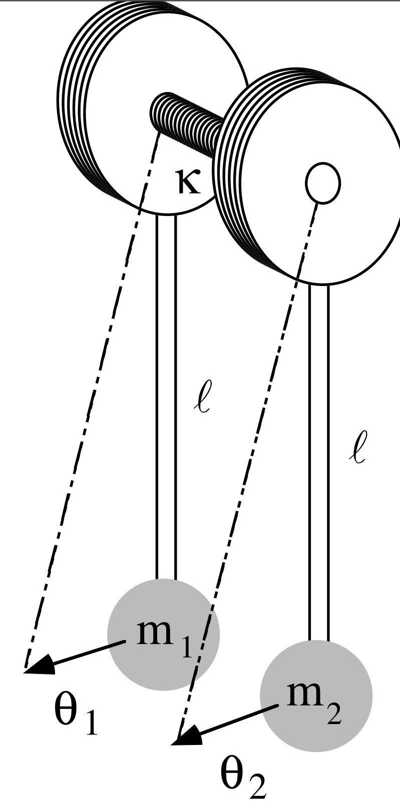


Fig. 3.3.2 Coupled pendulums

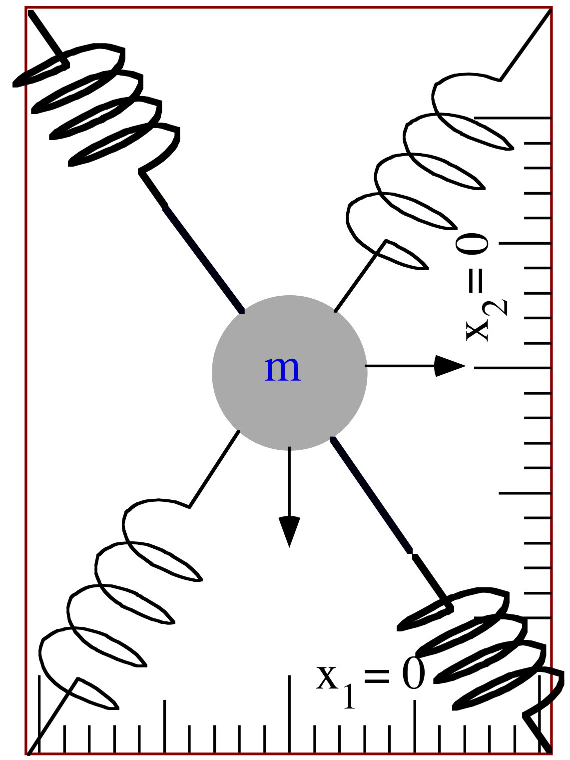


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D harmonic oscillator energy

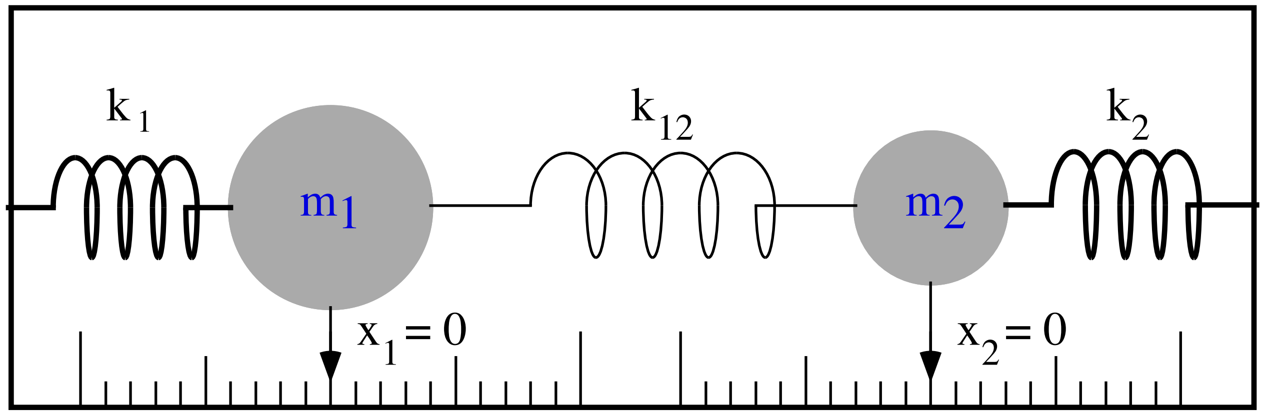


Fig. 3.3.1 Two 1-dimensional coupled oscillators

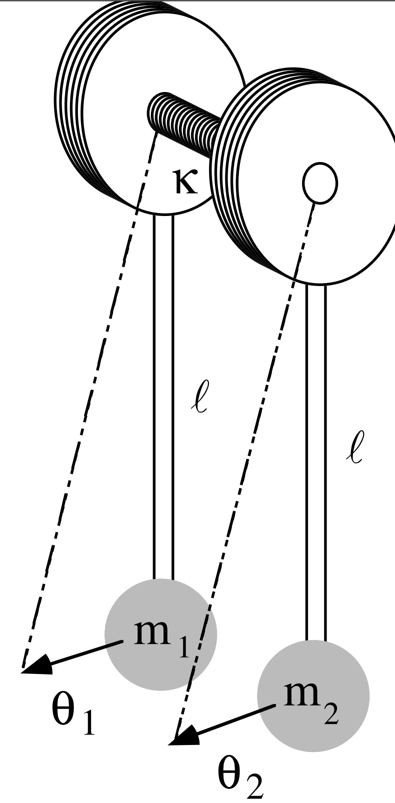


Fig. 3.3.2 Coupled pendulums

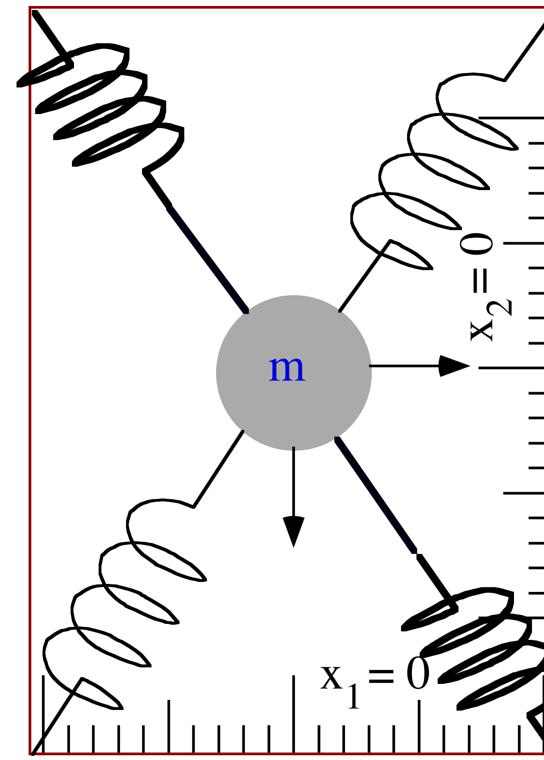
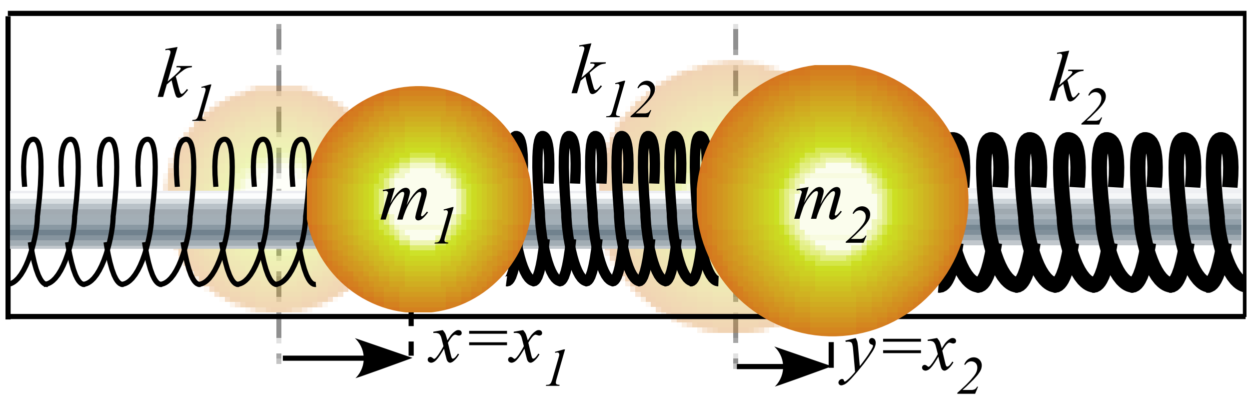


Fig. 3.3.3 One 2-dimensional coupled oscillator



2D HO kinetic energy $T(v_1, v_2)$

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2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} \left(x_1 - x_2 \right)^2$$

$$= \frac{1}{2} \left(k_1 + k_{12} \right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} \left(k_2 + k_{12} \right) x_2^2$$

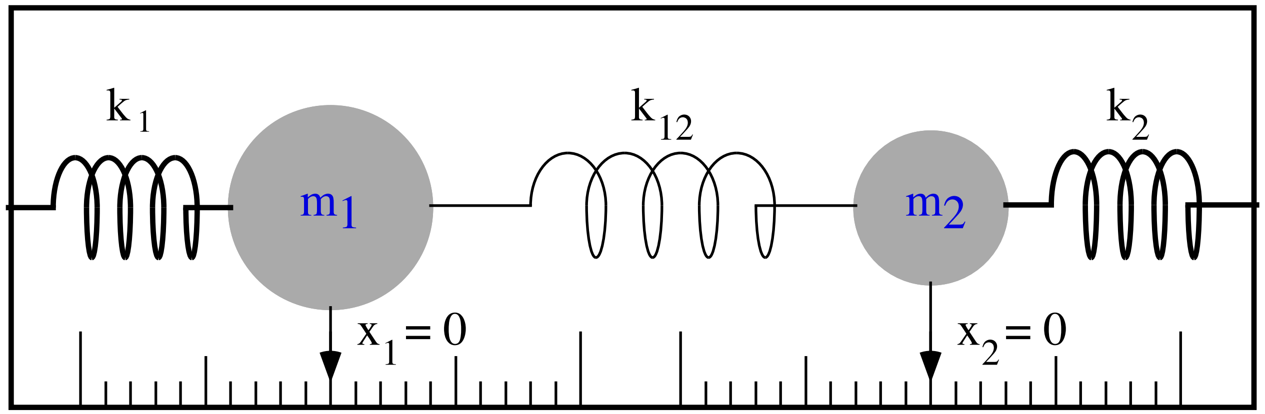
Lagrangian $L = T - V$

2D harmonic oscillator equations

➔ *Lagrangian and matrix forms*

Reciprocity symmetry

2D harmonic oscillator equations



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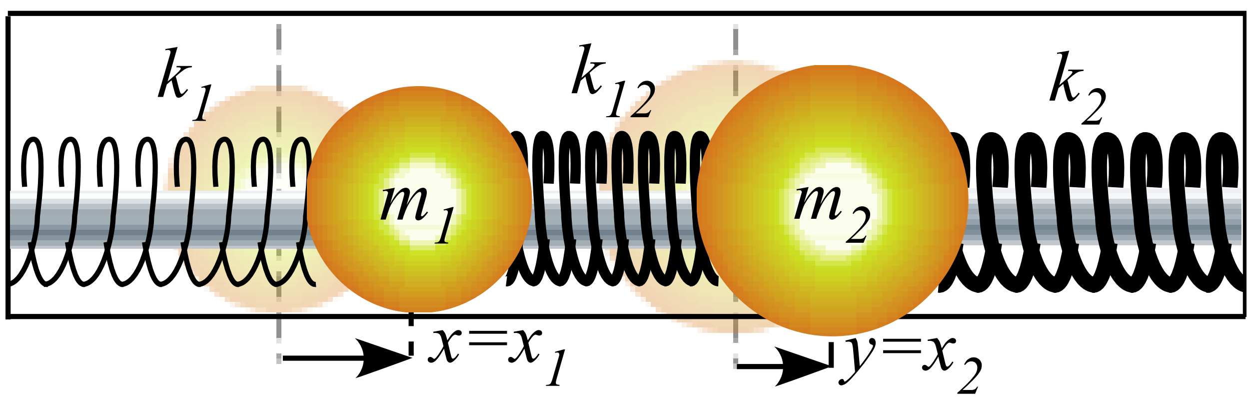

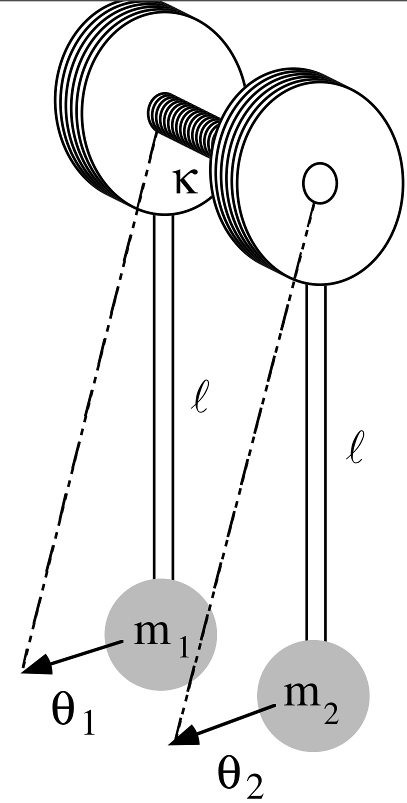

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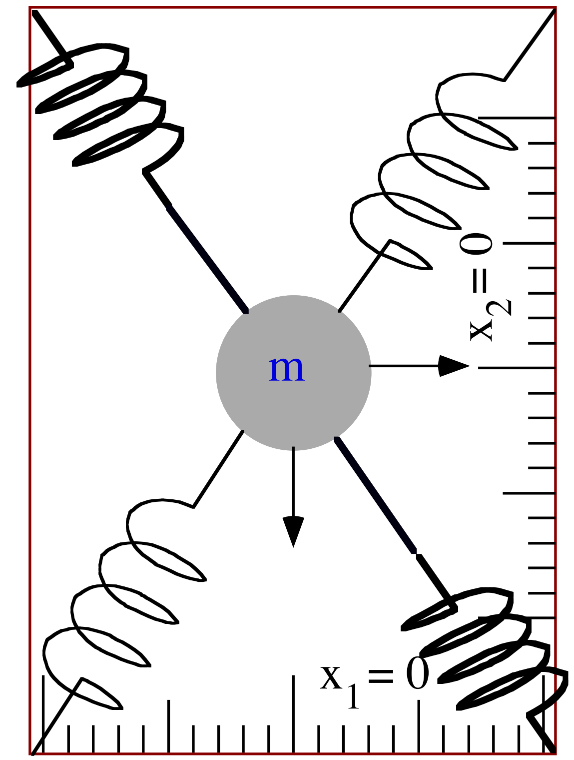

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Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12} \right) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12} \right) x_2$$

2D harmonic oscillator equations

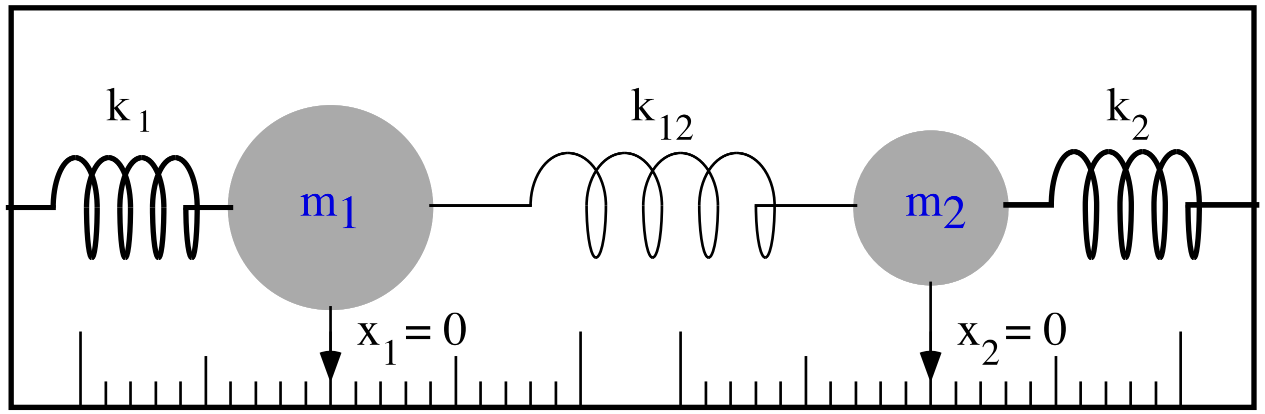
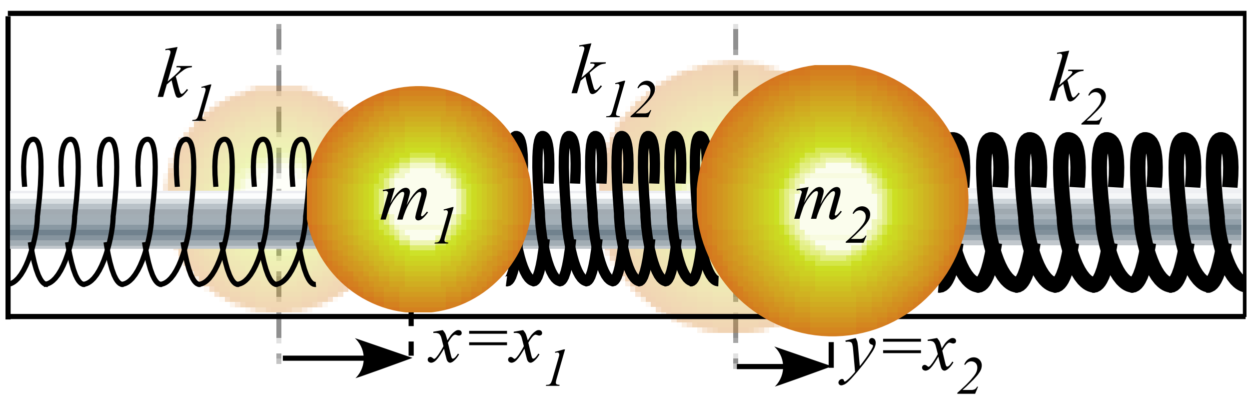


Fig. 3.3.1 Two 1-dimensional coupled oscillators



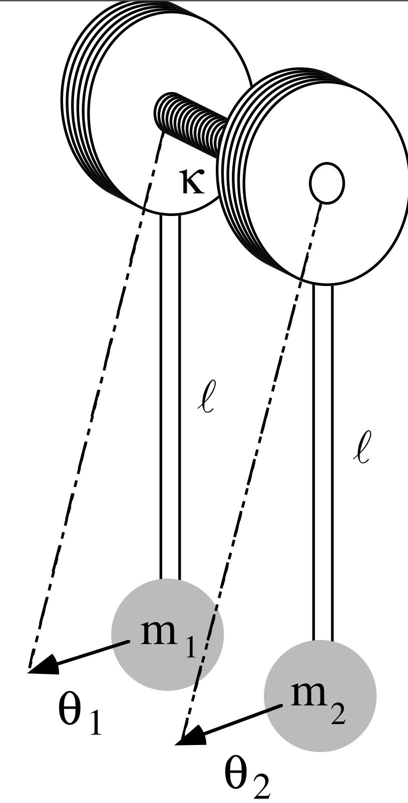


Fig. 3.3.2 Coupled pendulums

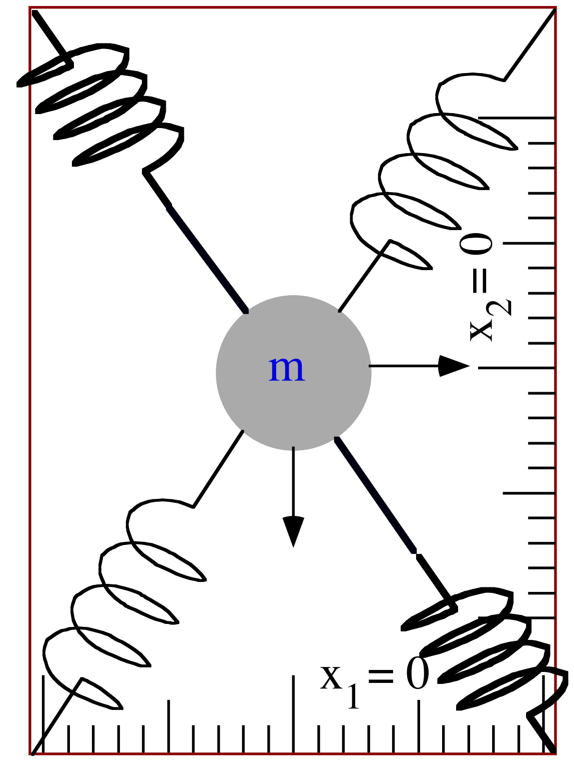


Fig. 3.3.3 One 2-dimensional coupled oscillator

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Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12}\right) x_1 + k_{12} x_2$$

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2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations

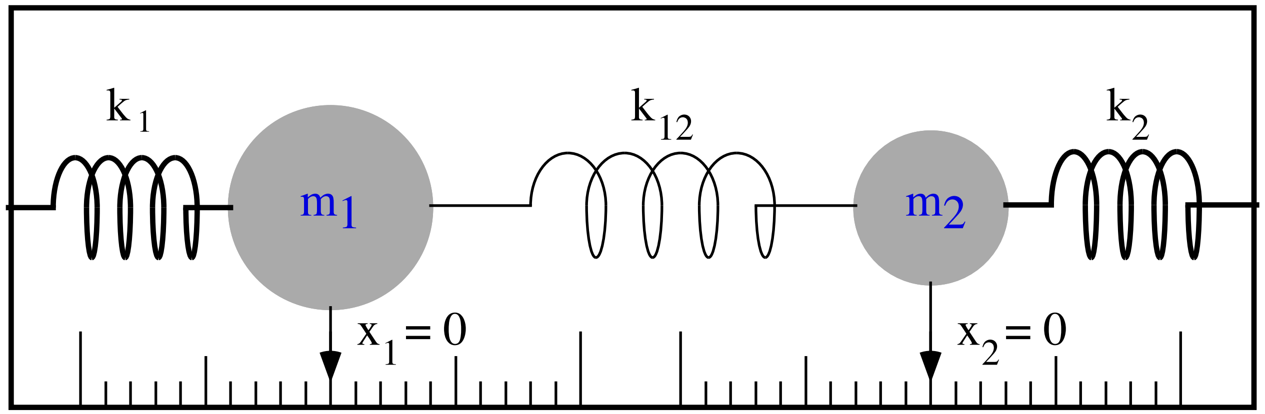
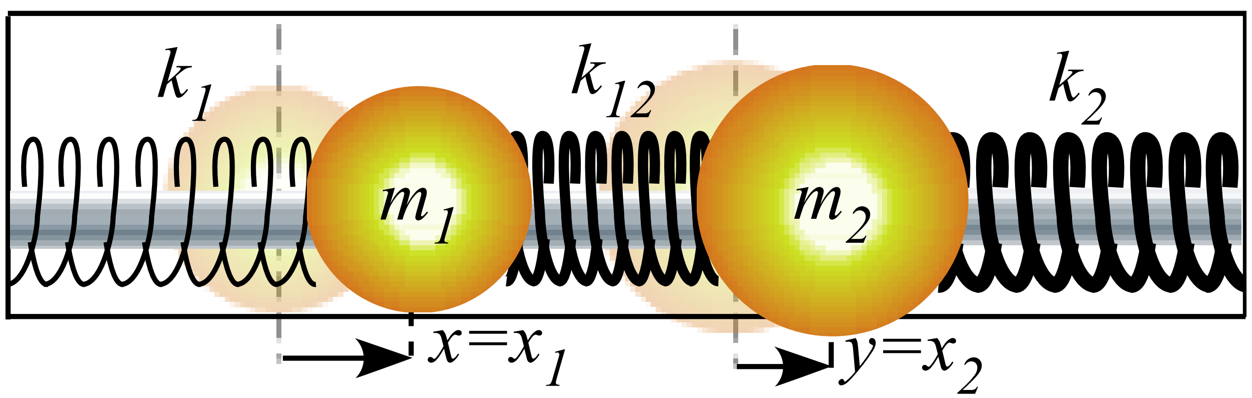


Fig. 3.3.1 Two 1-dimensional coupled oscillators



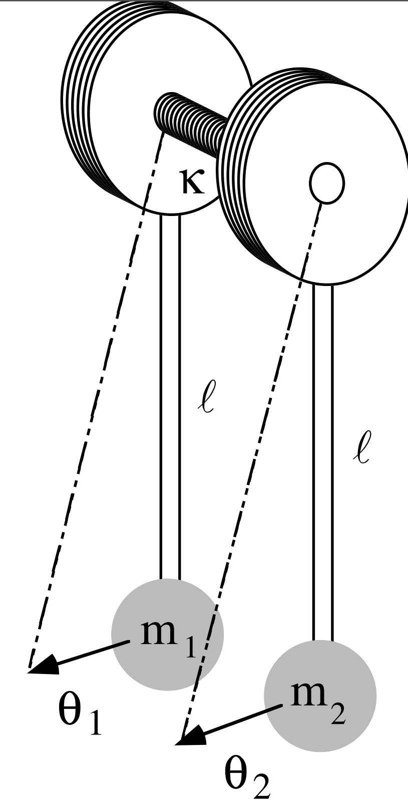


Fig. 3.3.2 Coupled pendulums

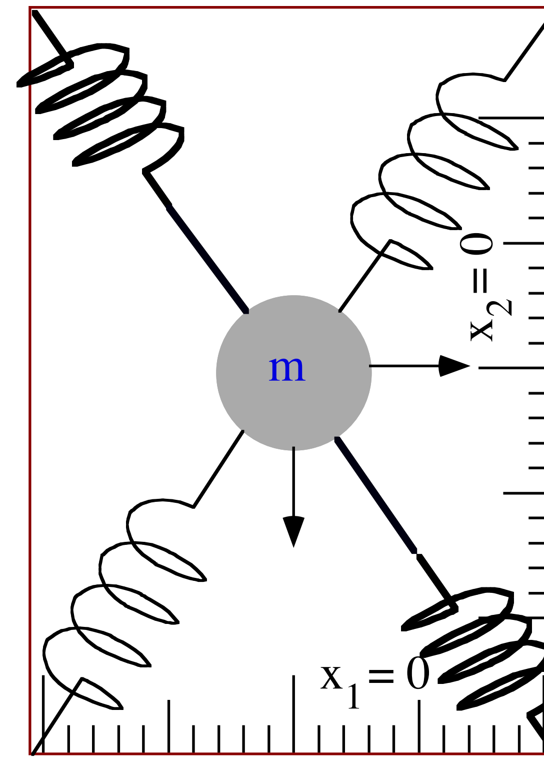


Fig. 3.3.3 One 2-dimensional coupled oscillator

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$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} \left(x_1 - x_2 \right)^2$$

$$= \frac{1}{2} \left(k_1 + k_{12} \right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} \left(k_2 + k_{12} \right) x_2^2$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12} \right) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12} \right) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \bullet \left| \ddot{\mathbf{x}} \right\rangle \quad = \quad - \quad \mathbf{K} \bullet \left| \mathbf{x} \right\rangle$$

2D harmonic oscillator equations

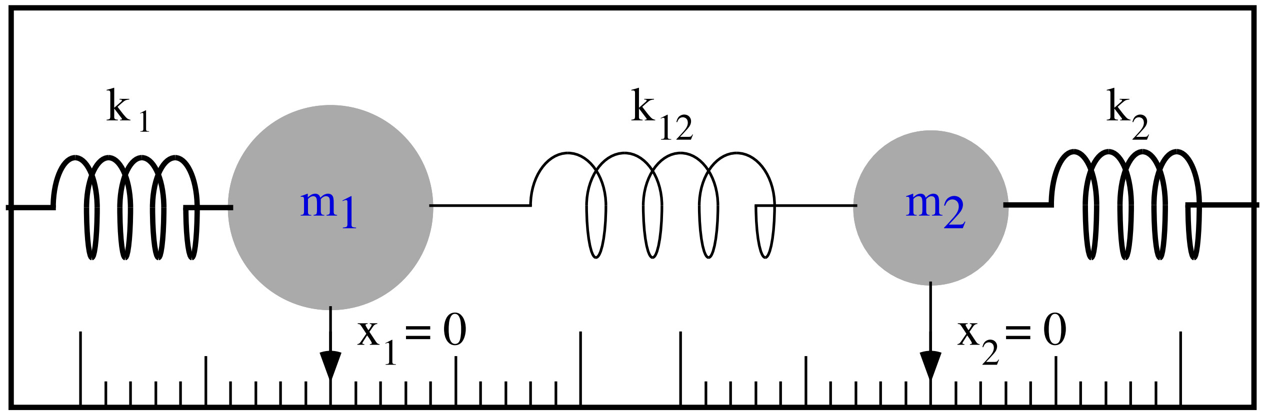
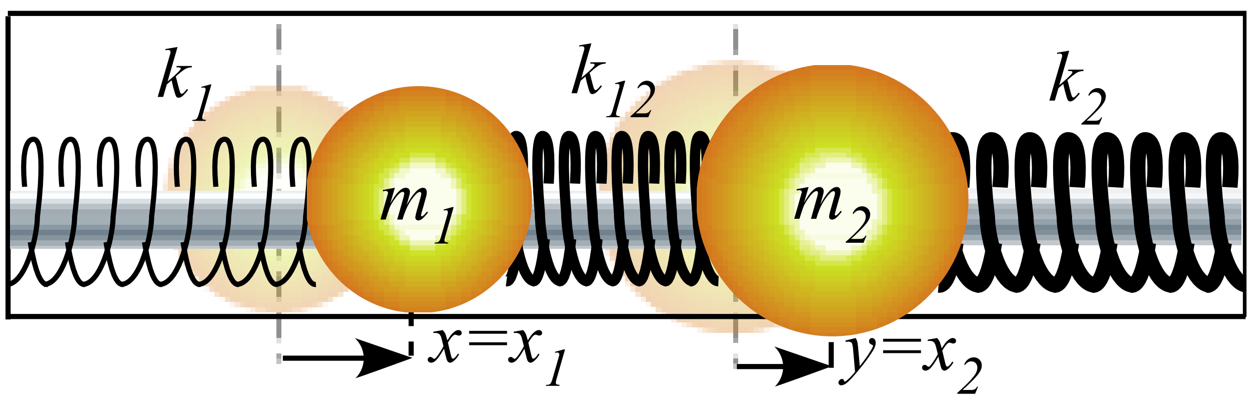


Fig. 3.3.1 Two 1-dimensional coupled oscillators



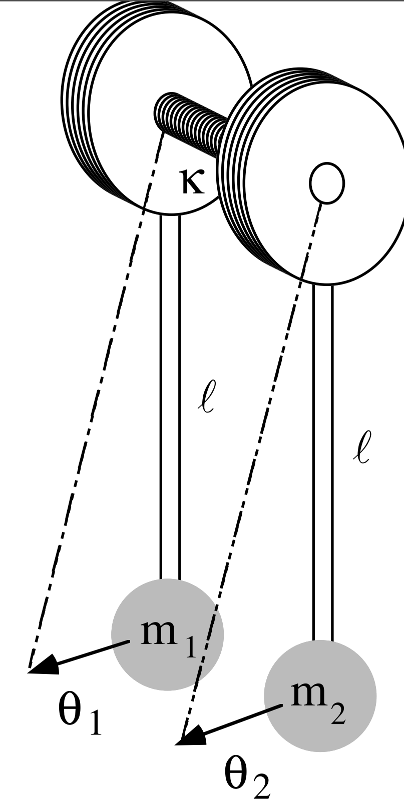


Fig. 3.3.2 Coupled pendulums

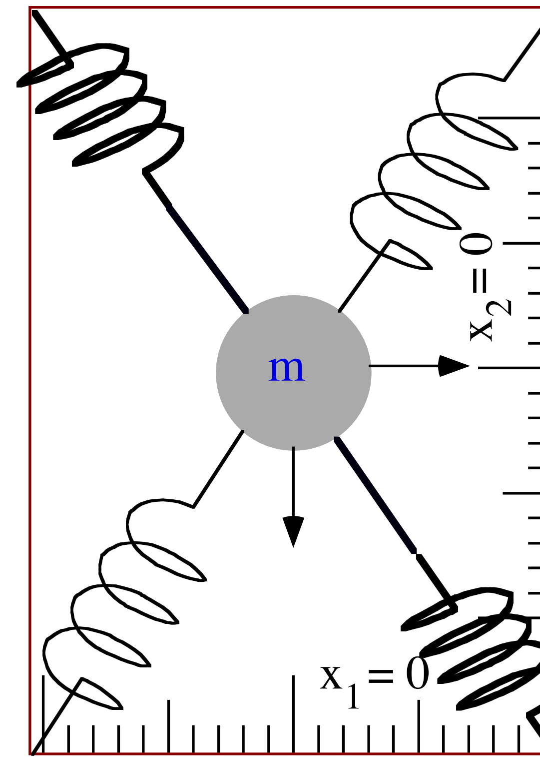


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy T(v₁, v₂)

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy V(x₁, x₂)

$$V = \frac{1}{2}(k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}(k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian L=T-V

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

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Matrix operator notation:

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2D harmonic oscillator equations

Lagrangian and matrix forms

➤ *Reciprocity symmetry*

Matrix equations and reciprocity symmetry

General form of 2D-HO equation of motion has force matrix components: $\kappa_{11} = k_1 + k_{11}, \quad \kappa_{22} = k_2 + k_{22}$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} m_1 \ddot{x}_1 \\ m_2 \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Off-diagonal force constants satisfy *Reciprocity Relations:*

$$-\kappa_{12} = k_{12} = \frac{\partial F_1}{\partial x_2} = -\frac{\partial^2 V}{\partial x_2 \partial x_1} = -\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial F_2}{\partial x_1} = k_{21} = -\kappa_{21}$$

Rescaling and symmetrization

Each coordinate (x_1, x_2) is rescaled $(q_1 = s_1 x_1, q_2 = s_2 x_2)$ to symmetrize mass factors on $\ddot{q}_j$-terms.

$$-\frac{m_1}{s_1}\ddot{q}_1 = \kappa_{11}\frac{q_1}{s_1} + \kappa_{12}\frac{q_2}{s_2} \qquad -\ddot{q}_1 = \frac{\kappa_{11}}{m_1}q_1 + \frac{\kappa_{12}s_1}{m_1 s_2}q_2 \equiv K_{11}q_1 + K_{12}q_2$$

$$-\frac{m_2}{s_2}\ddot{q}_2 = \kappa_{12}\frac{q_1}{s_1} + \kappa_{22}\frac{q_2}{s_2} \qquad -\ddot{q}_2 = \frac{\kappa_{12}s_2}{m_2 s_1}q_1 + \frac{\kappa_{22}}{m_2}q_2 \equiv K_{21}q_1 + K_{22}q_2$$

New constants K_{ij} have pseudo-reciprocity symmetry for a special scale factor ratio: $\frac{s_2}{s_1} = \sqrt{\frac{m_2}{m_1}}$

$$K_{21} = \frac{\kappa_{12}s_2}{m_2 s_1} = K_{12} = \frac{\kappa_{12}s_1}{m_1 s_2} = \frac{-k_{12}}{\sqrt{m_1 m_2}}$$

Diagonal constants K_{jj} are not affected by scaling. To be equal requires: $\frac{\kappa_{11}}{m_1} = \frac{\kappa_{22}}{m_2}$ or: $\frac{\kappa_{11}}{\kappa_{22}} = \frac{m_1}{m_2}$

$$K_{11} = \frac{\kappa_{11}}{m_1} = \frac{k_1 + k_{12}}{m_1} \qquad K_{22} = \frac{\kappa_{22}}{m_2} = \frac{k_2 + k_{12}}{m_2}$$

Caution is advised since such forced symmetry may give modes with imaginary frequency.

➤ *2D harmonic oscillator equation eigensolutions*

Geometric method

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues⇒eigenvectors)

Spectral decomposition and P-operator expansions (how projectors⇒eigensolutions)

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle=-\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A}=\mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}^{-1}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=-\begin{pmatrix}\frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1}\\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle,|\mathbf{e}_2\rangle,...$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle=\varepsilon_n|\mathbf{e}_n\rangle=\omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle=-\mathbf{A}|\mathbf{e}_n\rangle=-\varepsilon_n|\mathbf{e}_n\rangle=-\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

To introduce eigensolutions we take a simple case of unit masses *($m_1=1=m_2$)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

2D harmonic oscillator equation eigensolutions

➡ *Geometric method*

Matrix-algebraic method with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues⇒eigenvectors)

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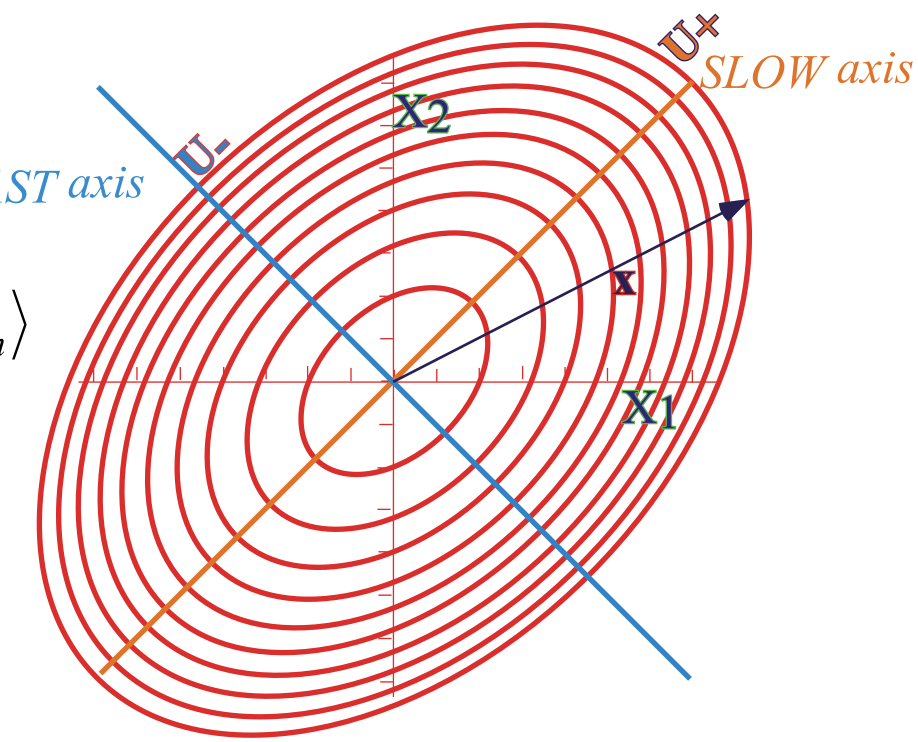
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??



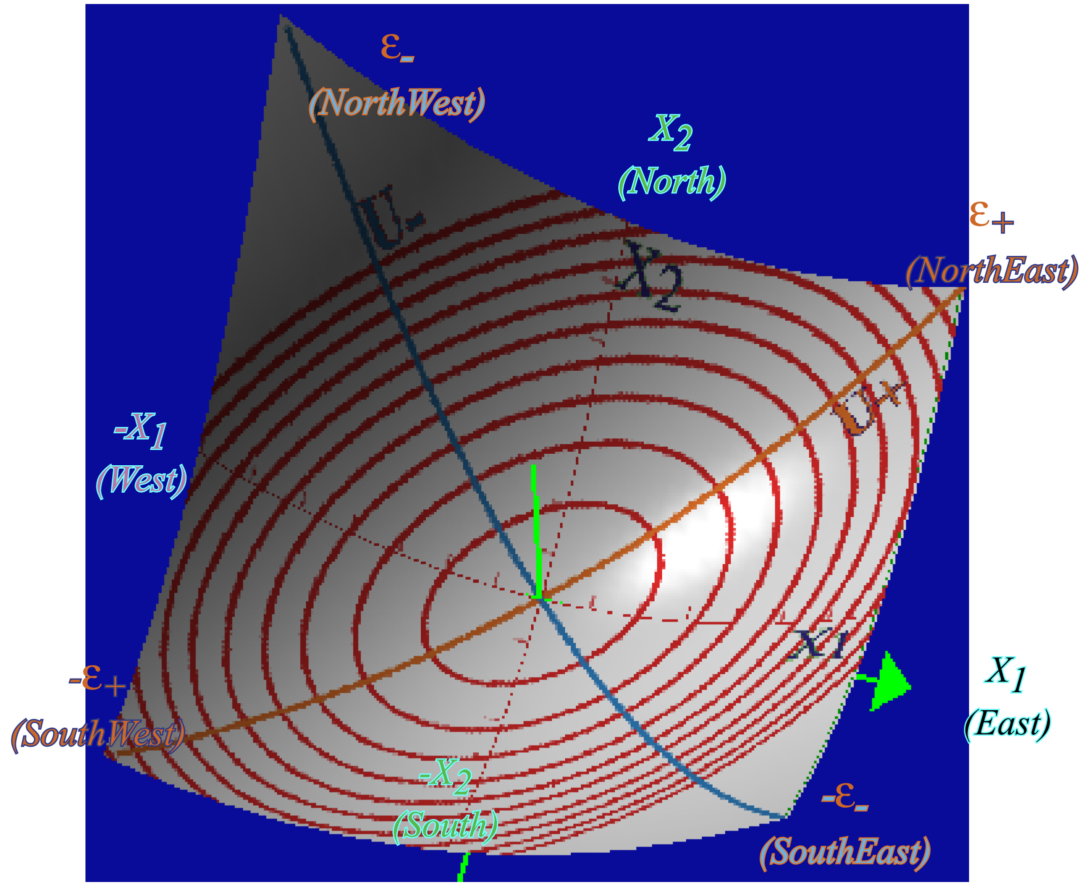
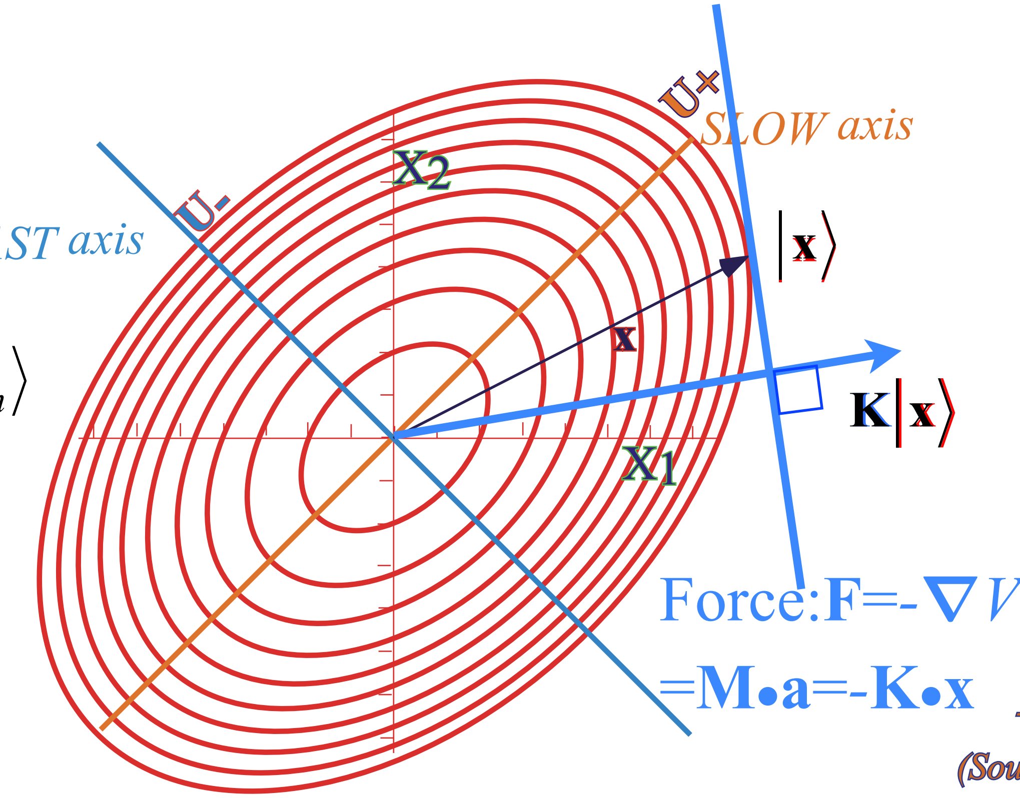


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$
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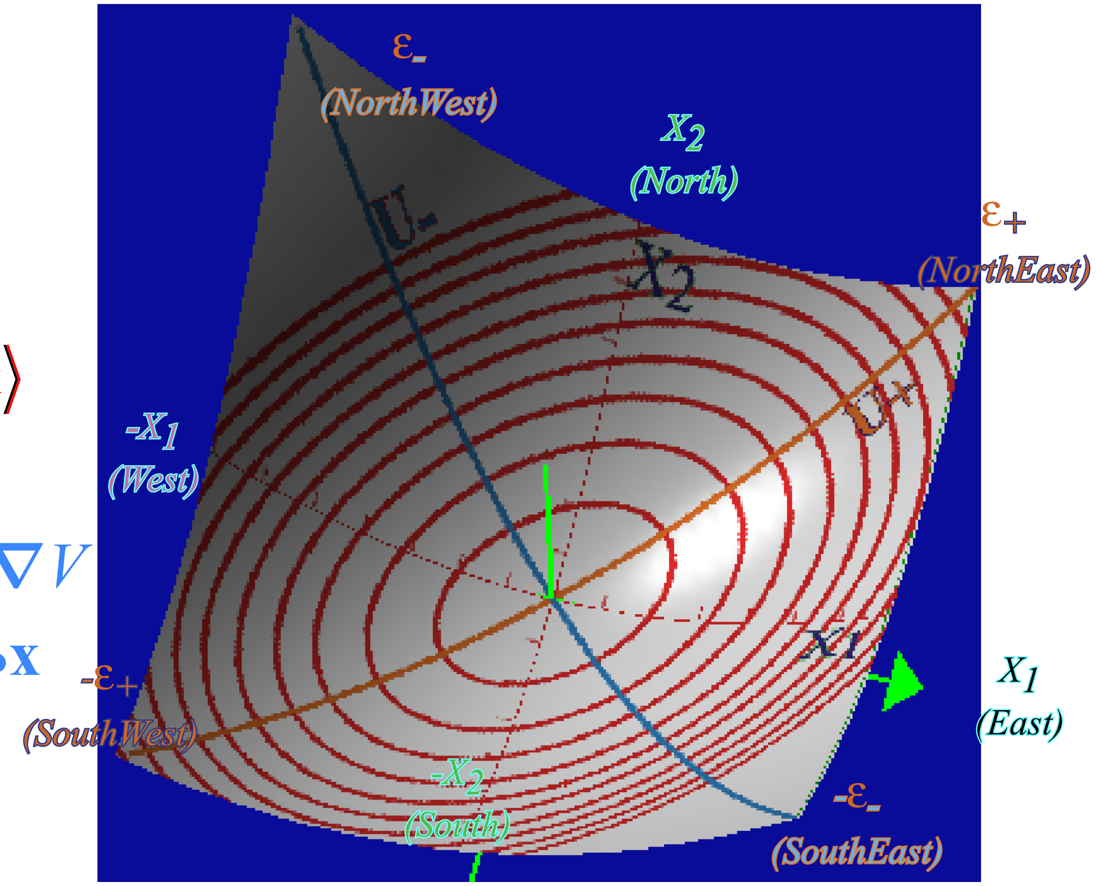
Not <u>most</u> directions!



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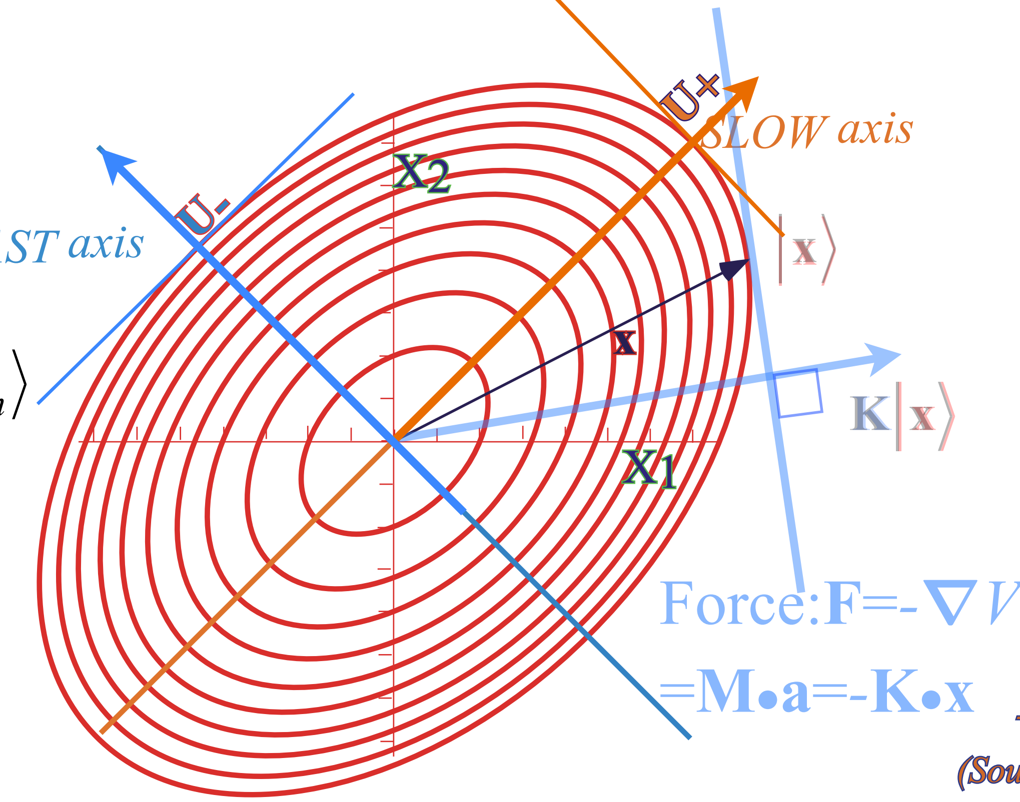


What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$?? Not <u>most</u> directions! Only extremal axes work. (major or minor axes)

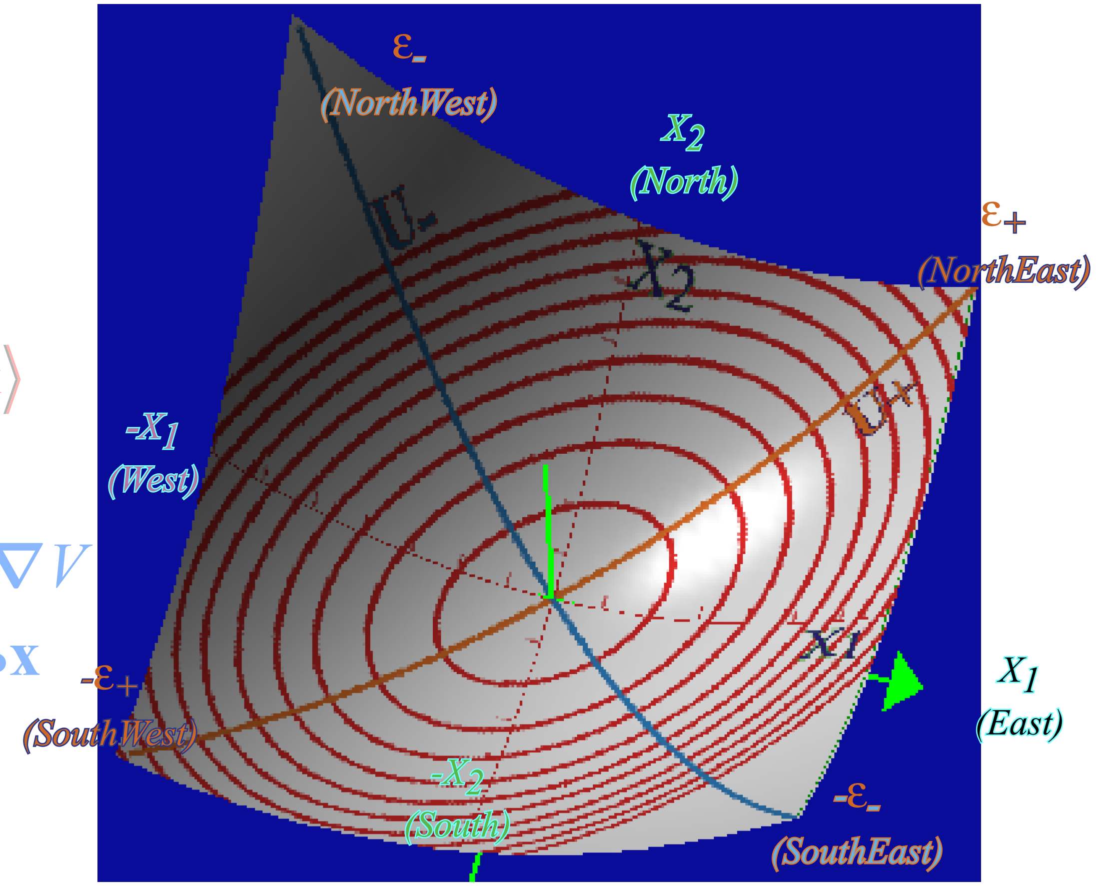
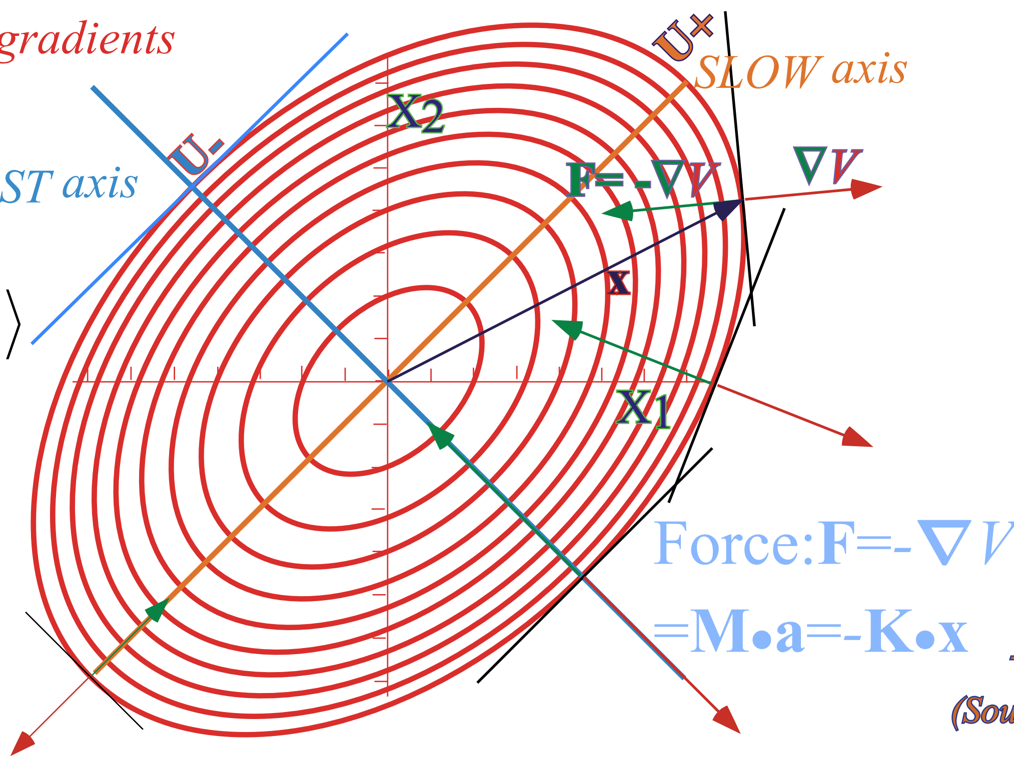


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



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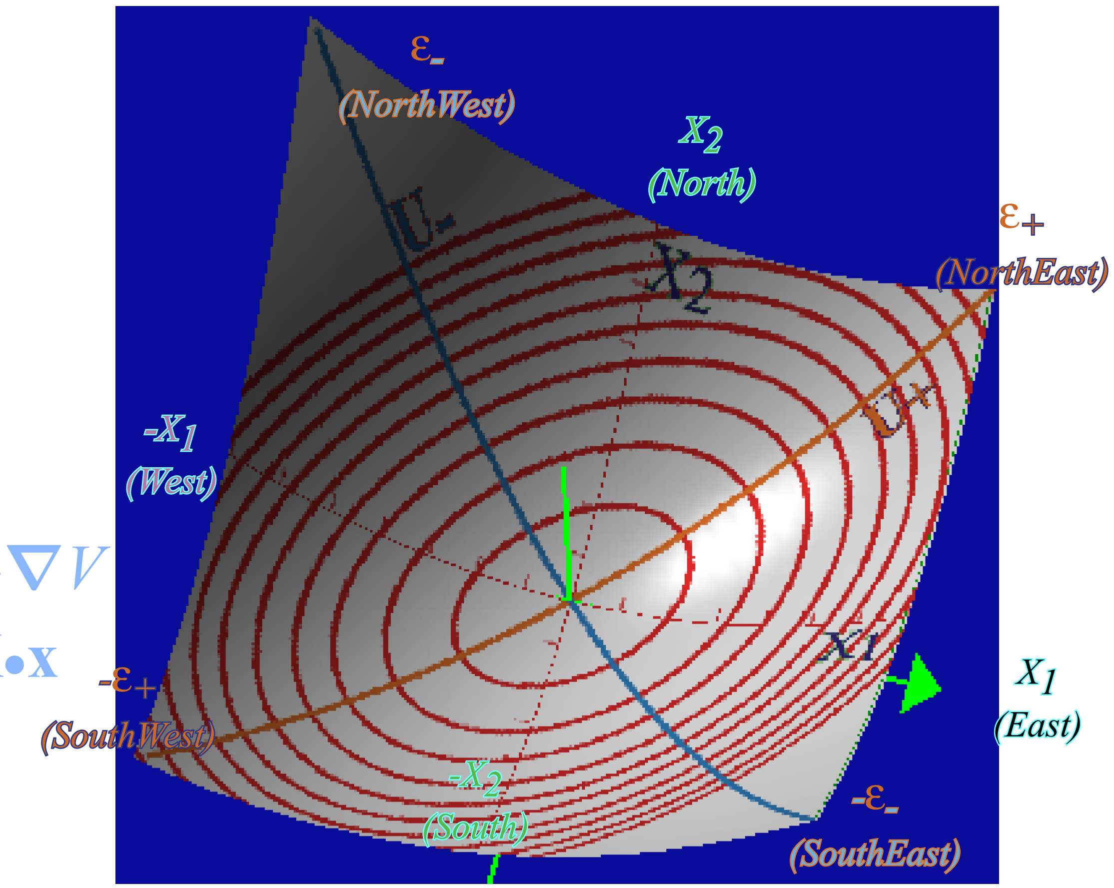
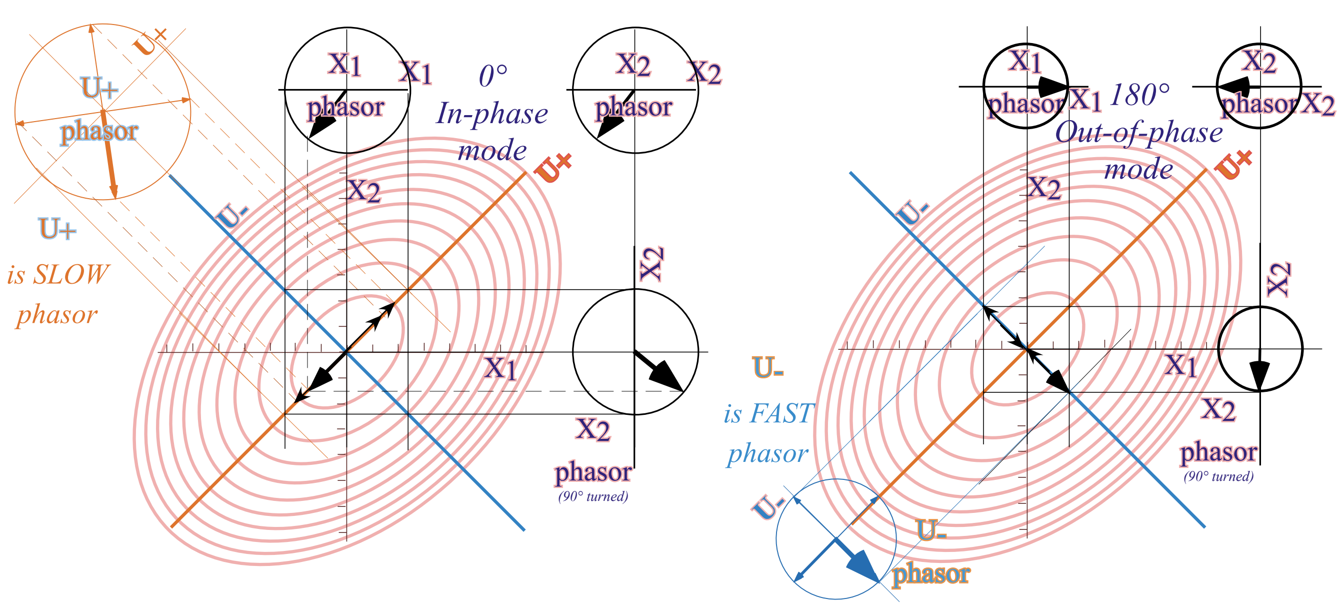


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode *(c) Anti-symmetric U- Coordinate FAST Mode*



2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

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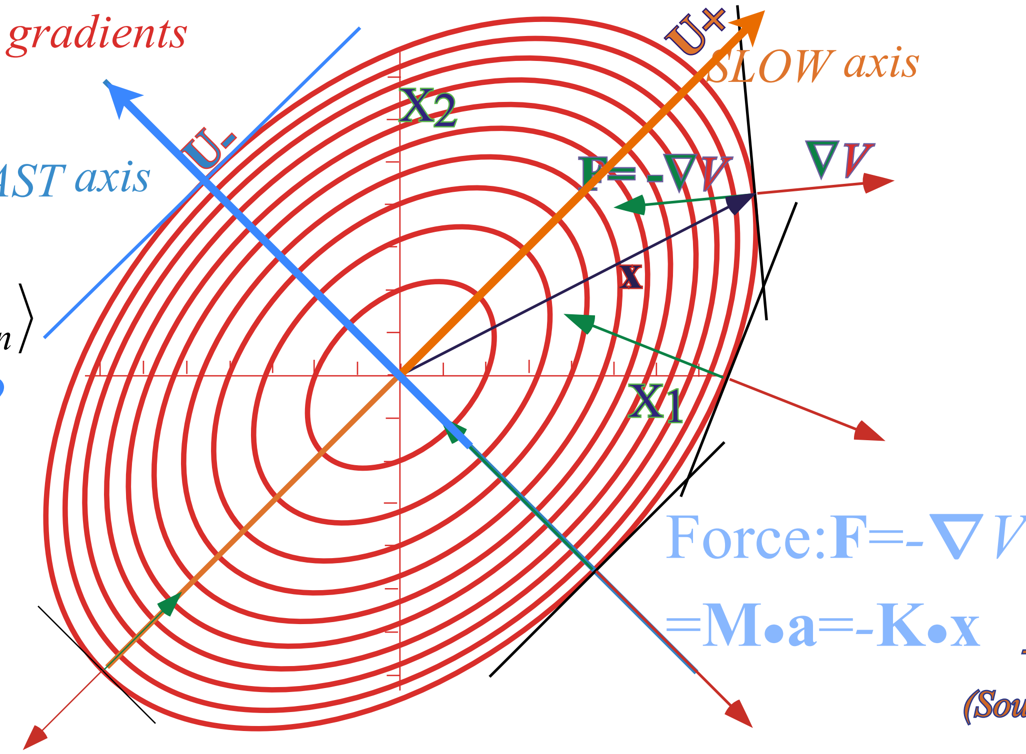

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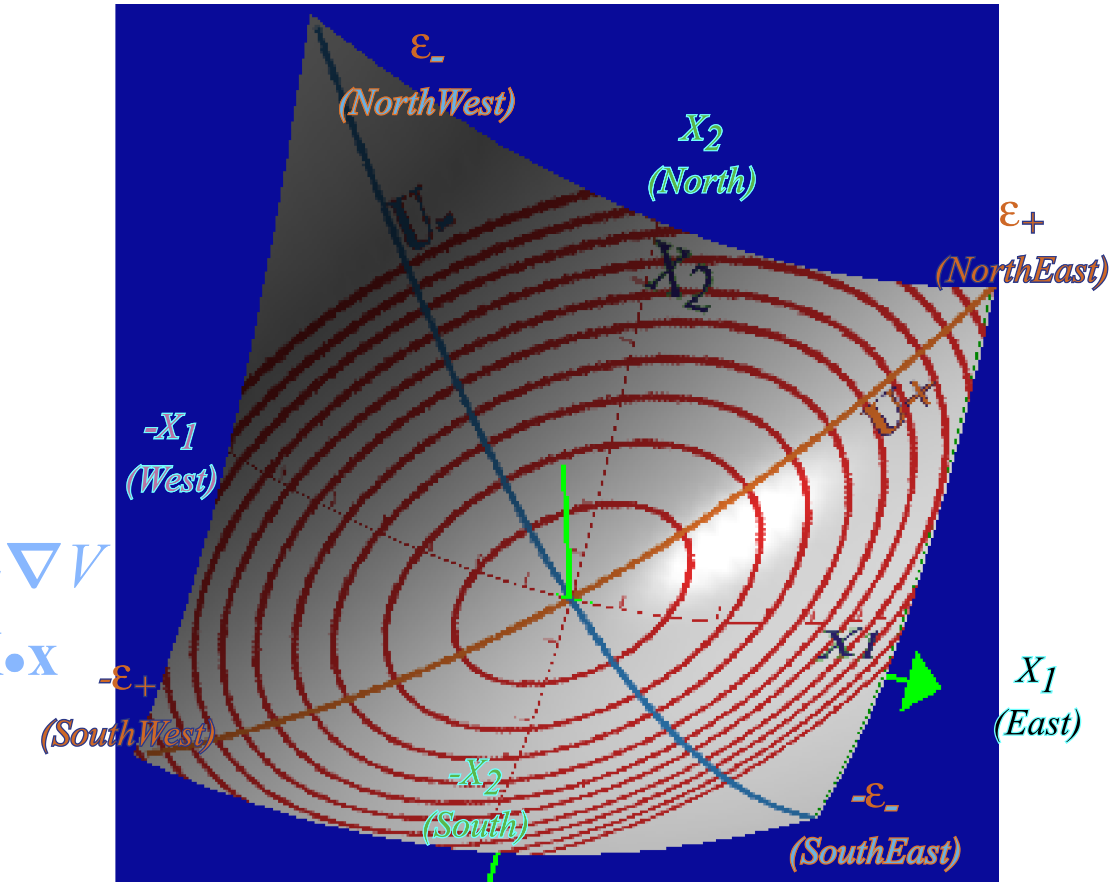

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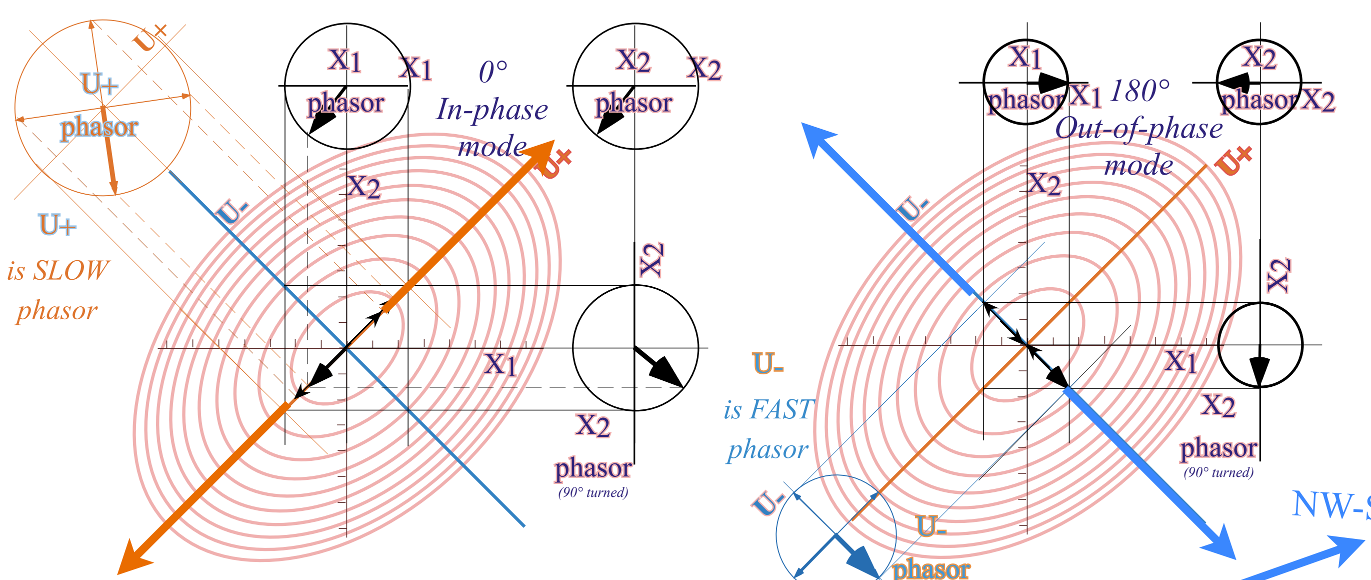

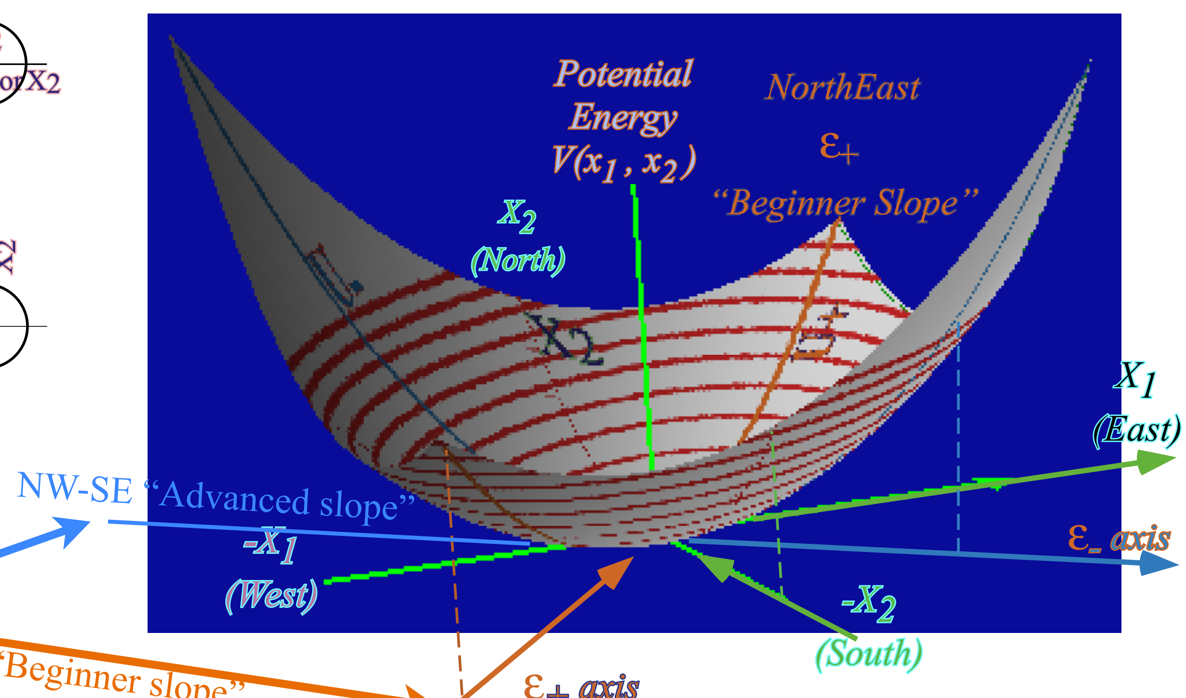

With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

2D harmonic oscillator equation eigensolutions

Geometric method

➤ *Matrix-algebraic method with example* $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues⇒eigenvectors)

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Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle, \text{ or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix}=\begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

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$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{or: }\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ a_n=(-1)^n\det|\mathbf{M}|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})$$

2D harmonic oscillator equation eigensolutions

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First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

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$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

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$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

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Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle = \varepsilon_k|\varepsilon_k\rangle, \text{ or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle = \mathbf{0}$$

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First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}| = 0 = (-1)^n\left(\varepsilon^n + a_1\varepsilon^{n-1} + a_2\varepsilon^{n-2} + \ldots + a_{n-1}\varepsilon + a_n\right)$$

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Trying to solve by Kramer's inversion:

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Notice $\mathbf{p}_k$ *commutes with* **M**,..
since $\mathbf{M}^1$, $\mathbf{M}^2$,..*commute with* **M**.

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$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{M}\mathbf{p}_1=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\mathbf{p}_1$$

$$\mathbf{M}\mathbf{p}_2=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\mathbf{p}_2$$

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic method with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

➡ *Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues⇒eigenvectors)*

Spectral decomposition and P-operator expansions (how projectors⇒eigensolutions)

Idempotent means: **P·P=P**

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

$$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

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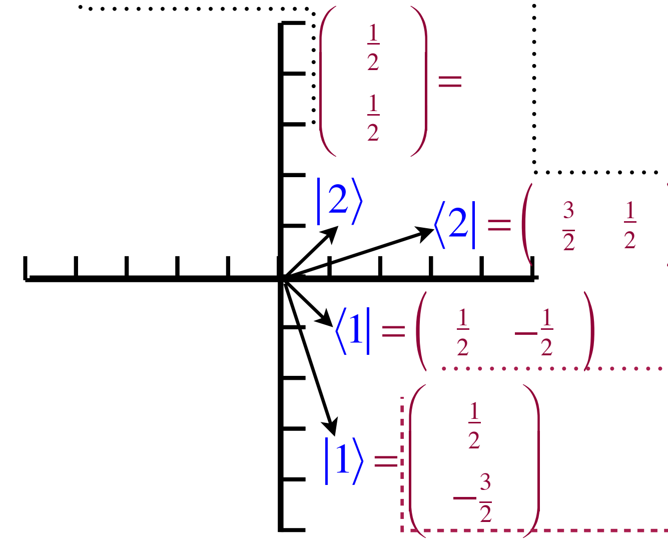
The $\mathbf{P}_j$ are *Mutually Ortho-Normal*



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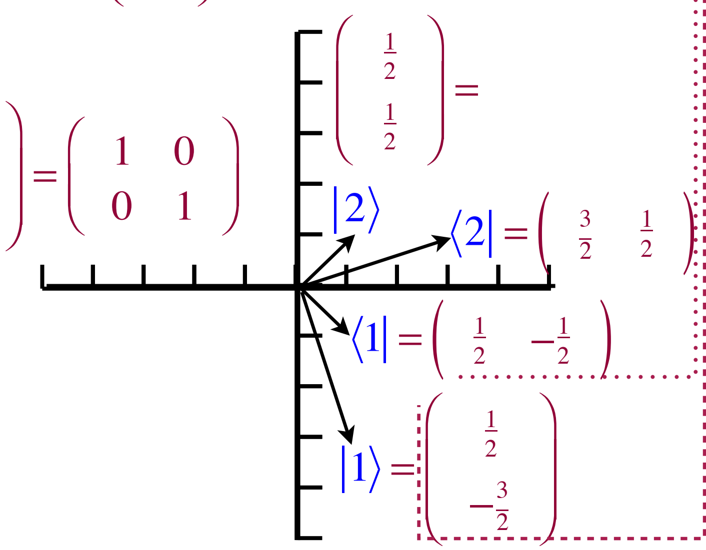

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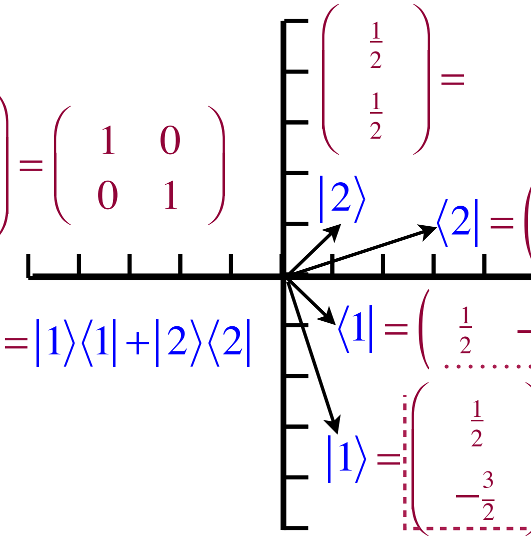

$$|2\rangle=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\qquad \langle 2|=\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}\qquad \langle 1|=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\qquad |1\rangle=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}$$

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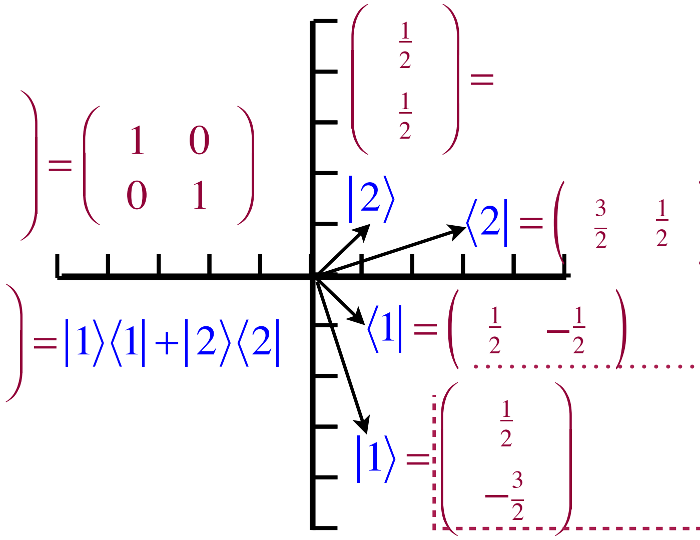

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$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|1\rangle,|2\rangle\}$-completeness

$$\langle x|y\rangle=\delta_{x,y}=\langle x|\mathbf{1}|y\rangle=\langle x|1\rangle\langle 1|y\rangle+\langle x|2\rangle\langle 2|y\rangle$$

Vector-tensor analysis
vs.
wave-functional analysis

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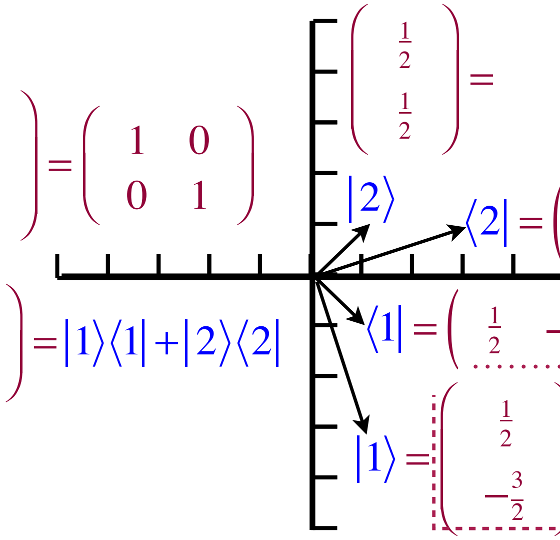

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|1\rangle,|2\rangle\}$-completeness

$$\langle x|y\rangle=\delta_{x,y}=\langle x|\mathbf{1}|y\rangle=\langle x|1\rangle\langle 1|y\rangle+\langle x|2\rangle\langle 2|y\rangle$$

$$\langle x|y\rangle=\delta(x,y)=\quad \psi_1(x)\psi_1^*(y)+\psi_2(x)\psi_2^*(y)+\ldots$$

Vector-tensor analysis

vs.

wave-functional analysis

$\{|1\rangle,|2\rangle\}$-orthonormality with $\{|x\rangle,|y\rangle\}$-completeness

$$\langle i|j\rangle=\delta_{i,j}=\langle i|\mathbf{1}|j\rangle=\langle i|x\rangle\langle x|j\rangle+\langle i|y\rangle\langle y|j\rangle$$

$$\langle i|j\rangle=\delta_{i,j}=\quad \psi_i^*(x)\psi_j(x)+\psi_2(y)\psi_2^*(y)+\ldots.=\int dx\,\psi_i^*(x)\psi_j(x)$$

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic method with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues⇒eigenvectors)

➤ *Spectral decomposition and P-operator expansions (how projectors⇒eigensolutions)*

Idempotent means: **P·P=P**

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Multiplication properties of $\mathbf{p}_j$:

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Factoring bra-kets into "Ket-Bras:

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k=\dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

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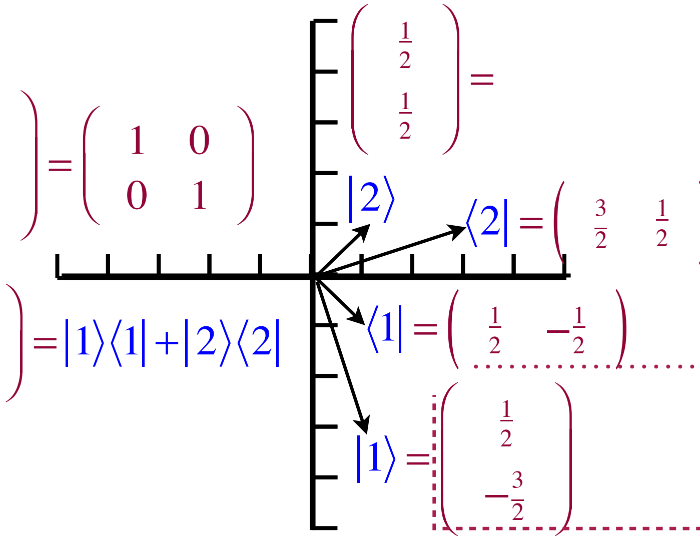
$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases} \qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$



$$\begin{pmatrix}\langle 1|1\rangle&\langle 1|2\rangle\\\langle 2|1\rangle&\langle 2|2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

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Matrix-algebraic method for finding eigenvector and eigenvalues   *With example matrix* $\mathbf{M}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$

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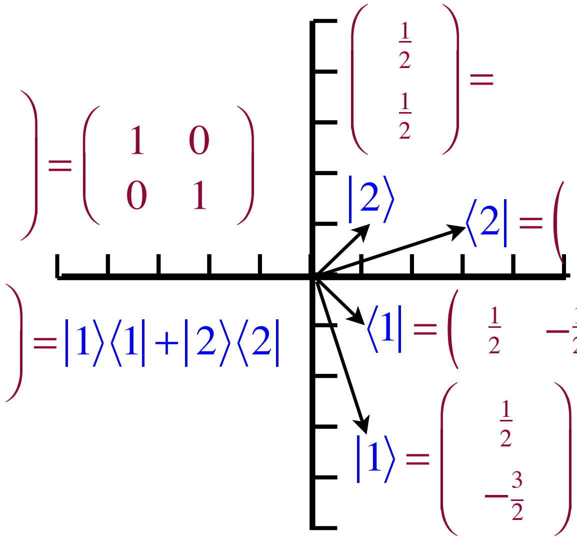

$$\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}=|2\rangle \qquad \langle 2|=\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix} \qquad \langle 1|=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix} \qquad |1\rangle=\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}$$

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Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

$$\mathbf{M}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle 1|+5|2\rangle\langle 2|$$

$$=1\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

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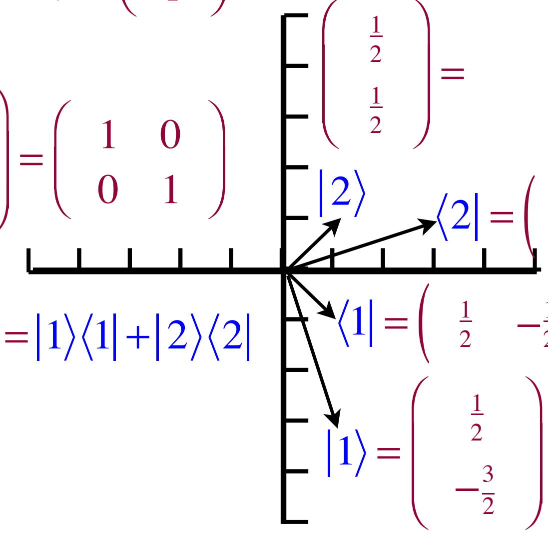
$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}\qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$



$$|2\rangle=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\qquad \langle 2|=\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}\qquad \langle 1|=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\qquad |1\rangle=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}$$

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$$=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

...and *Functional Spectral Decomposition* of any function $f(\mathbf{M})$ of $\mathbf{M}$

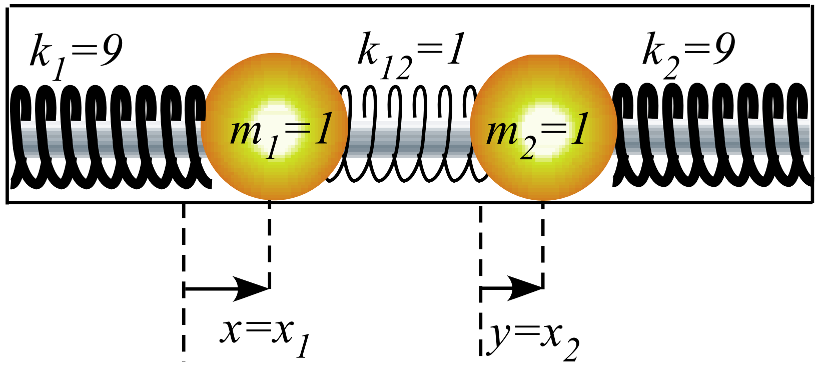
$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n$$

Examples with $\mathbf{M}^{50}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1^{50}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5^{50}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$

$$\sqrt{\mathbf{M}}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=\pm\sqrt{1}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}\pm\sqrt{5}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

➤ *2D-HO eigensolution example with bilateral (B-Type) symmetry*

Mixed mode beat dynamics and fixed π/2 phase



Analyzing 2D-HO beats and mixed mode eigen-solutions

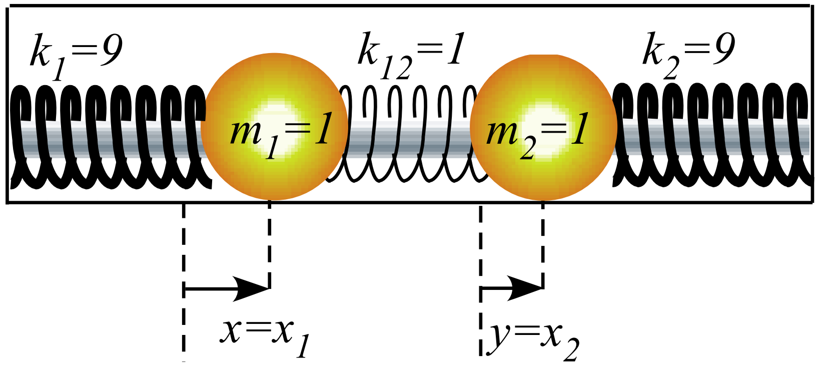
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$

$Trace(\mathbf{K}) = 10 + 10 = 20$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$



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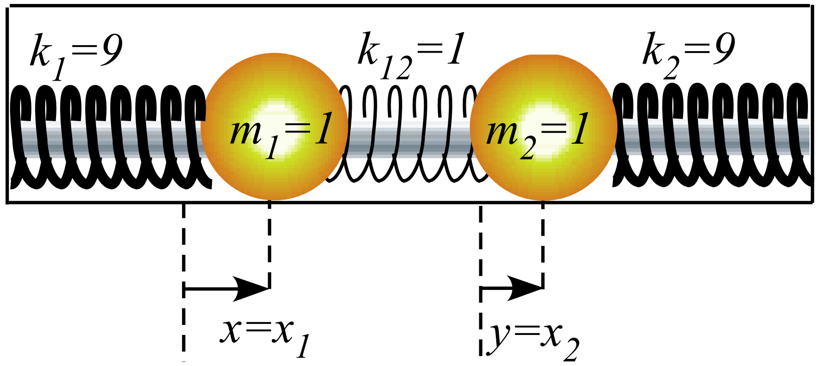
$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

Analyzing 2D-HO beats and mixed mode eigen-solutions



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$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

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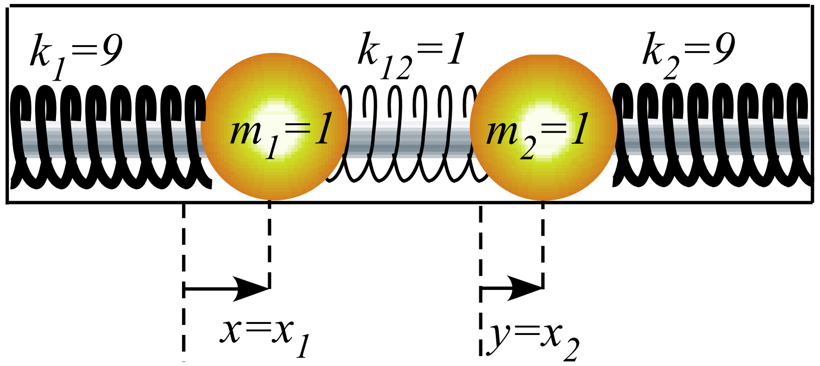
Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$



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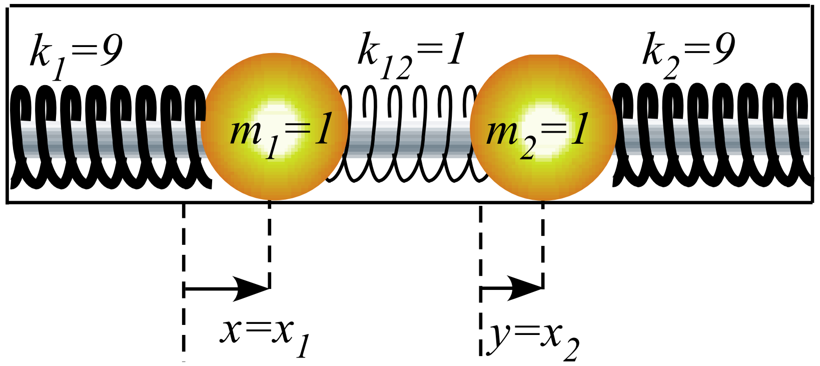
Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



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$$\mathbf{P}_2=\frac{\begin{pmatrix}K_{11}-K_1 & K_{12}\\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix}10-9 & -1\\ -1 & 10-9\end{pmatrix}}{11-9}=\frac{\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}{2}$$

$$=\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix}1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

100% modulation (SWR=0) $\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}\dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2}\\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix}e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t}\\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}$$

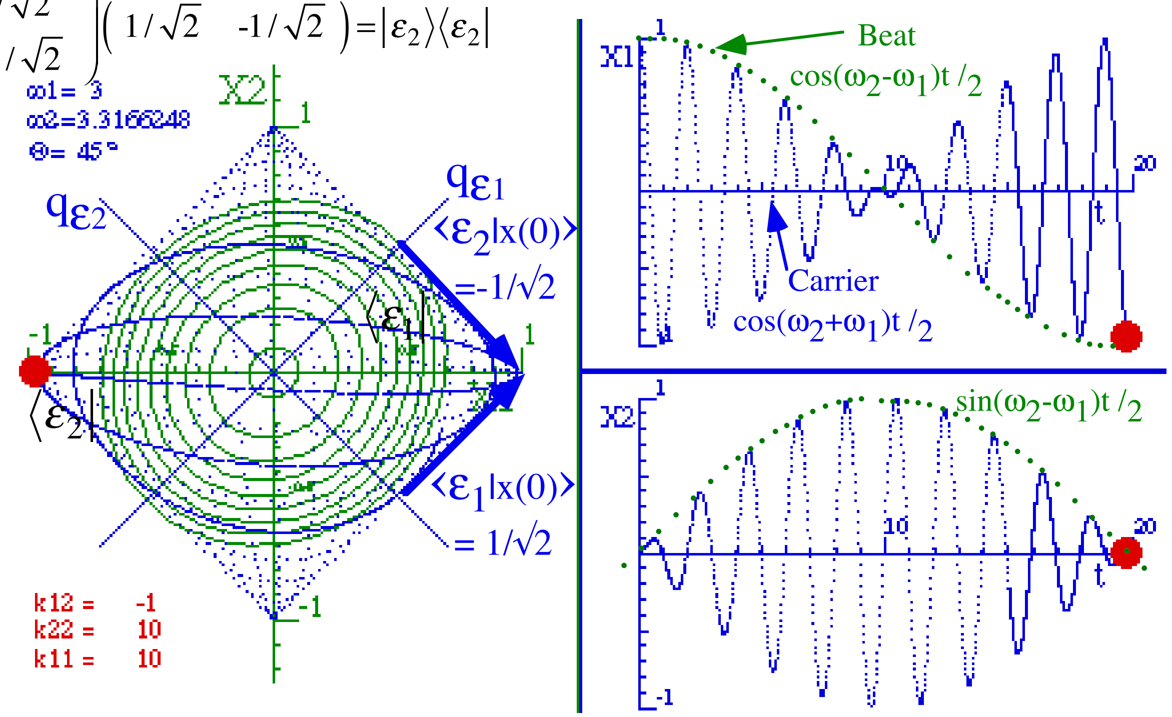
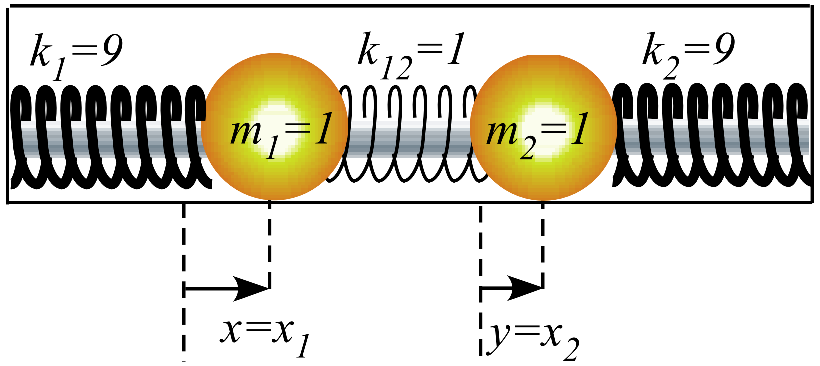


Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10\end{pmatrix}$$

$Det(\mathbf{K}) = 10\cdot 10 - 1 = 99$

$Trace(\mathbf{K}) = 10 + 10 = 20$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11\end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1\end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9\end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}}{2}$$

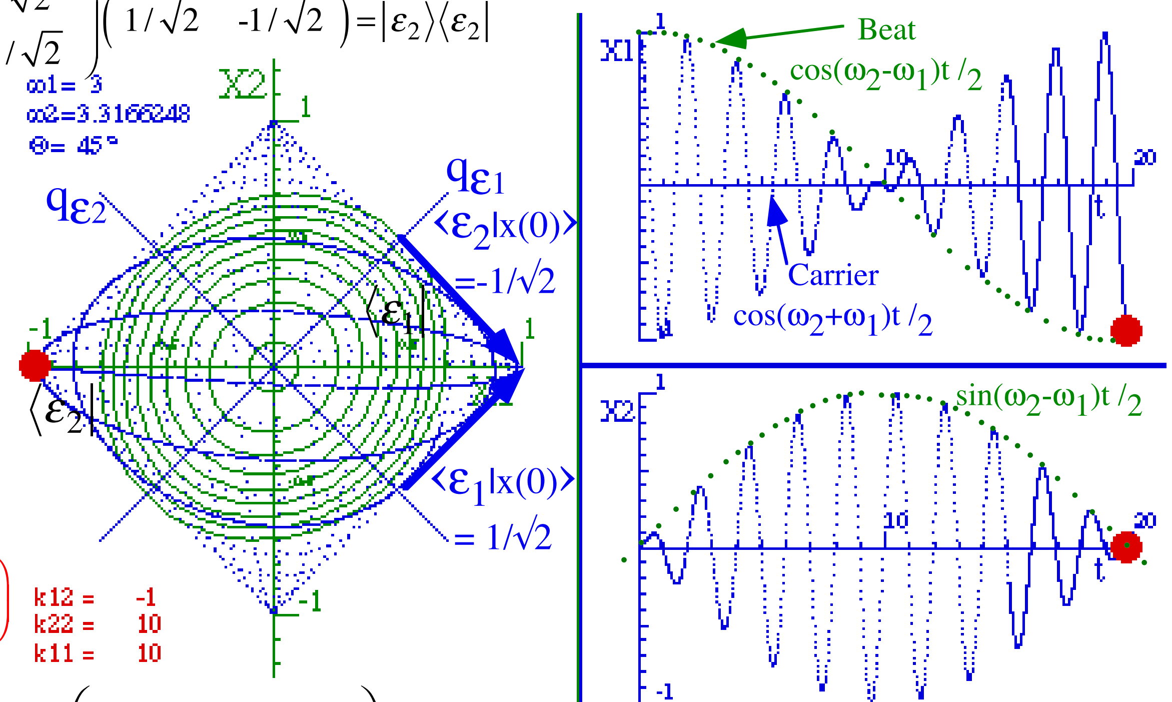
$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



100% modulation (SWR=0)

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2} = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{pmatrix} \dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix} = \frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix} = e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\dfrac{(\omega_2-\omega_1)t}{2} \\ i\sin\dfrac{(\omega_2-\omega_1)t}{2}\end{pmatrix}$$

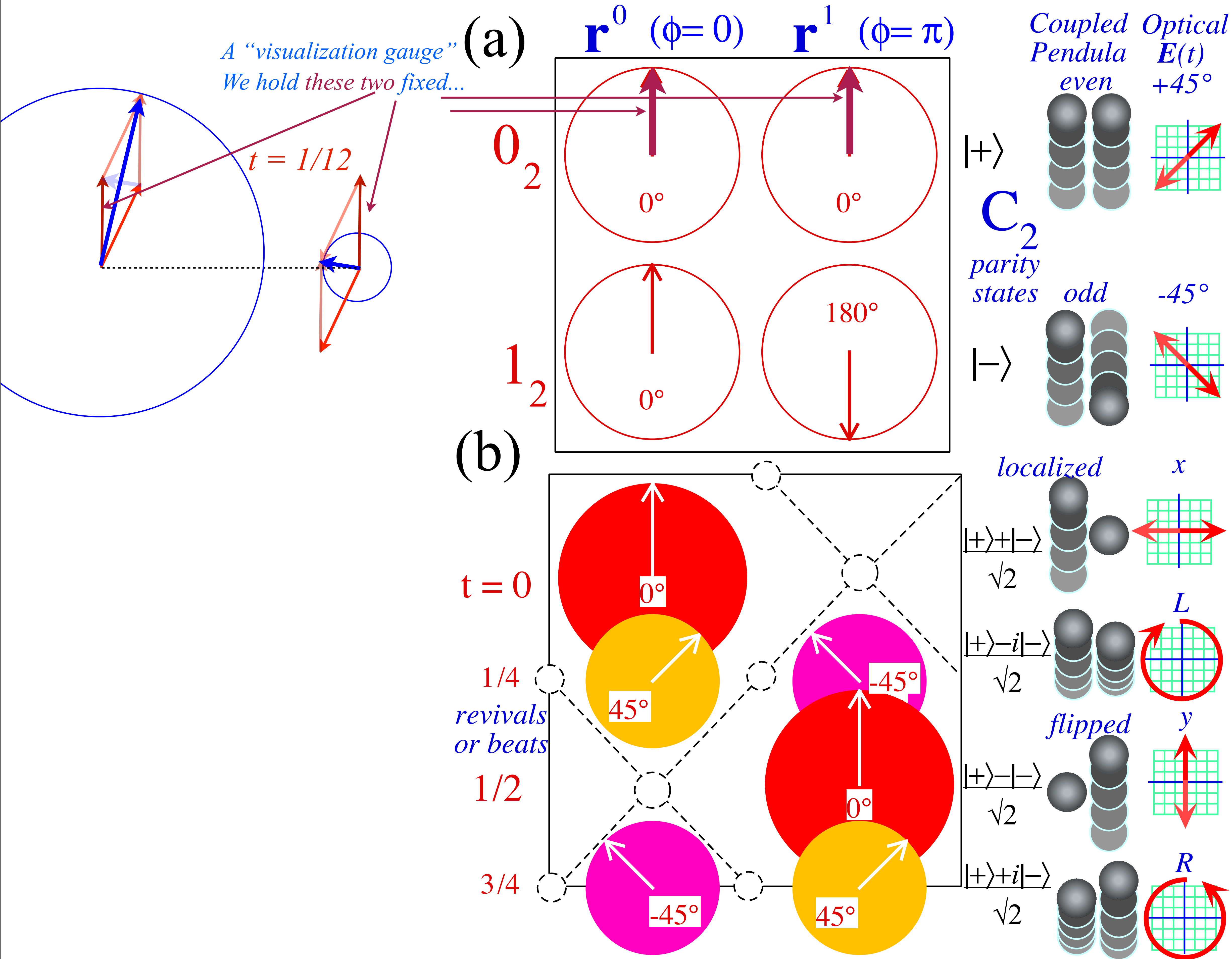
Note the i phase

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

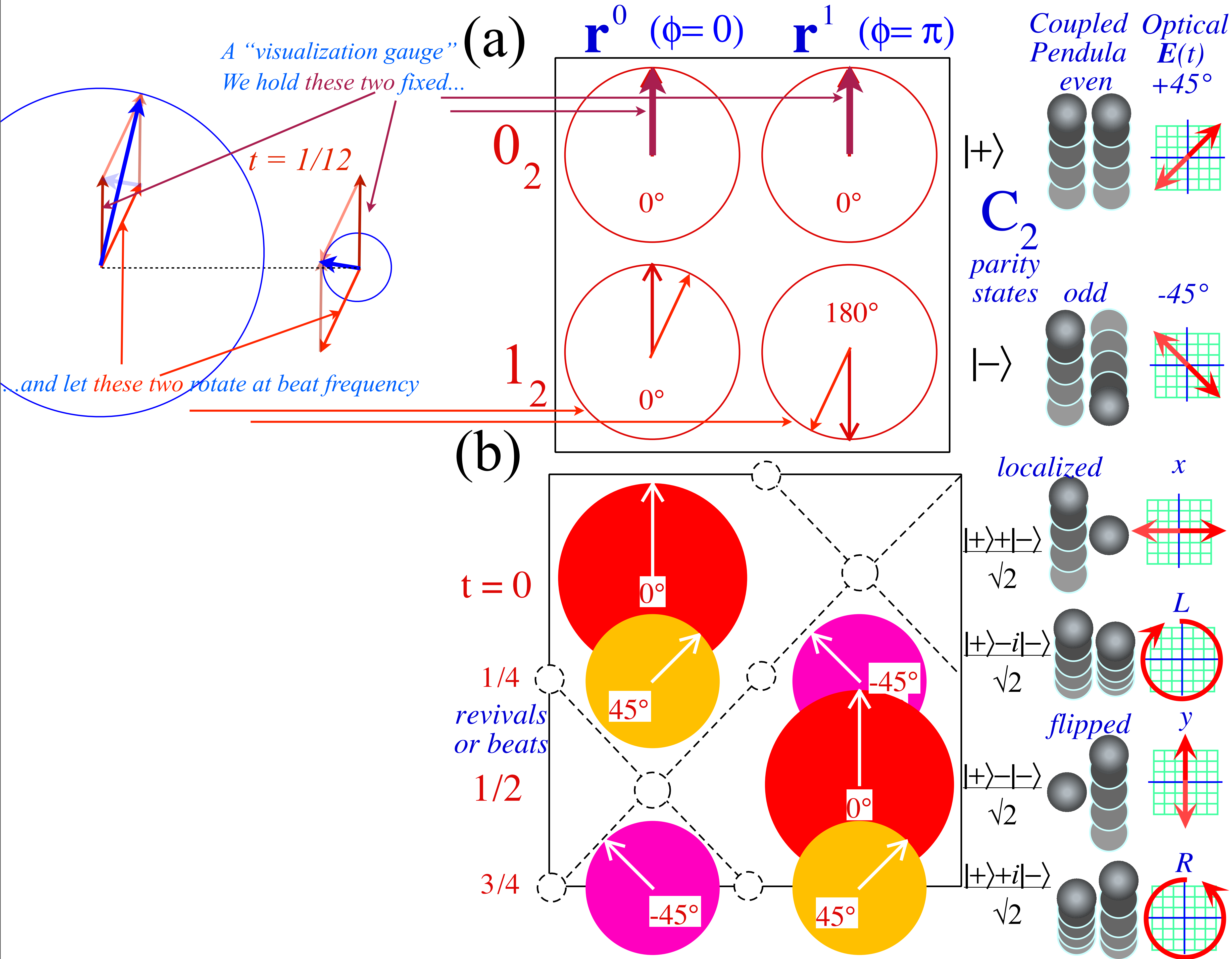
2D-HO eigensolution example with bilateral (B-Type) symmetry

➔ *Mixed mode beat dynamics and fixed π/2 phase*

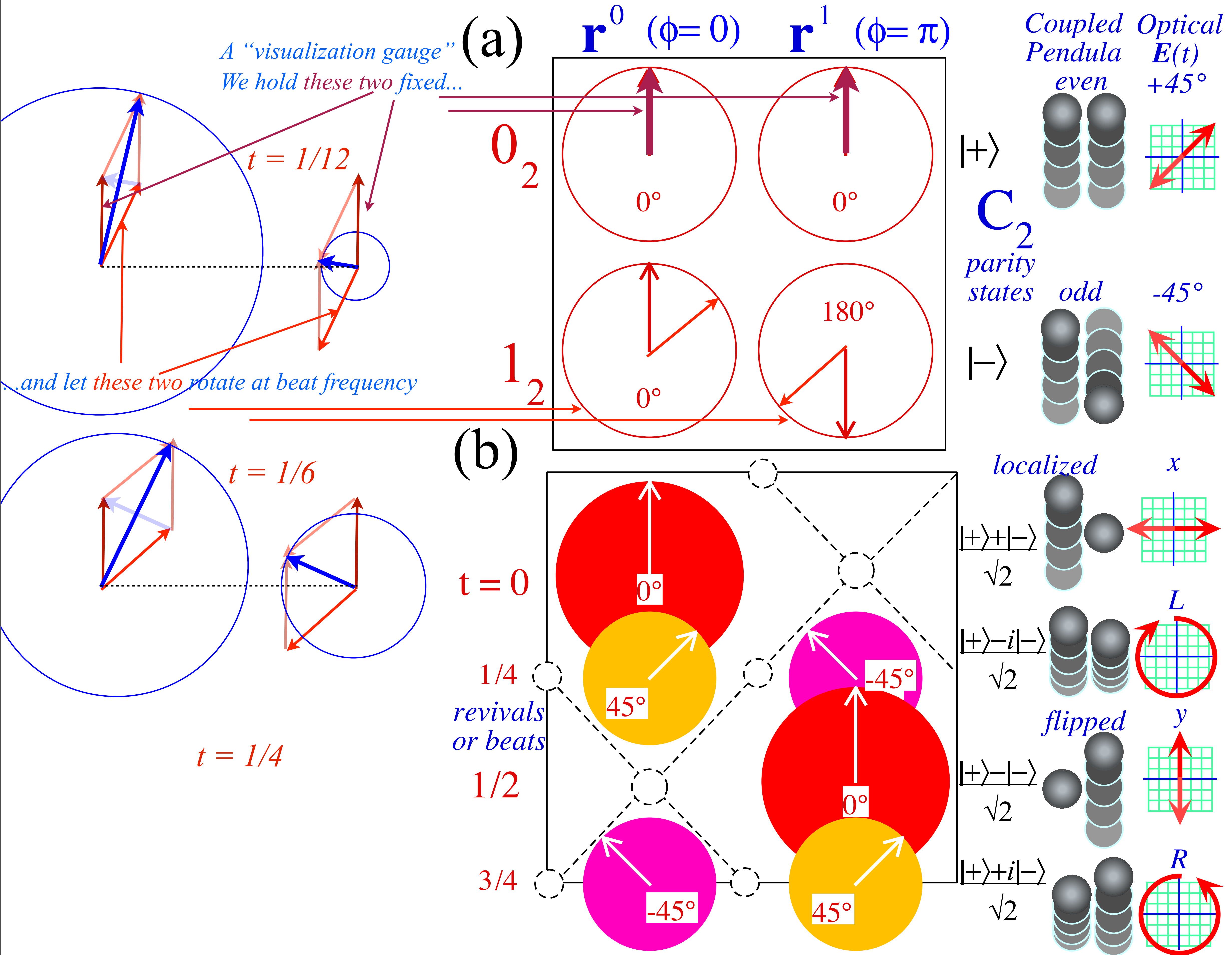
2D-HO beats and mixed mode geometry



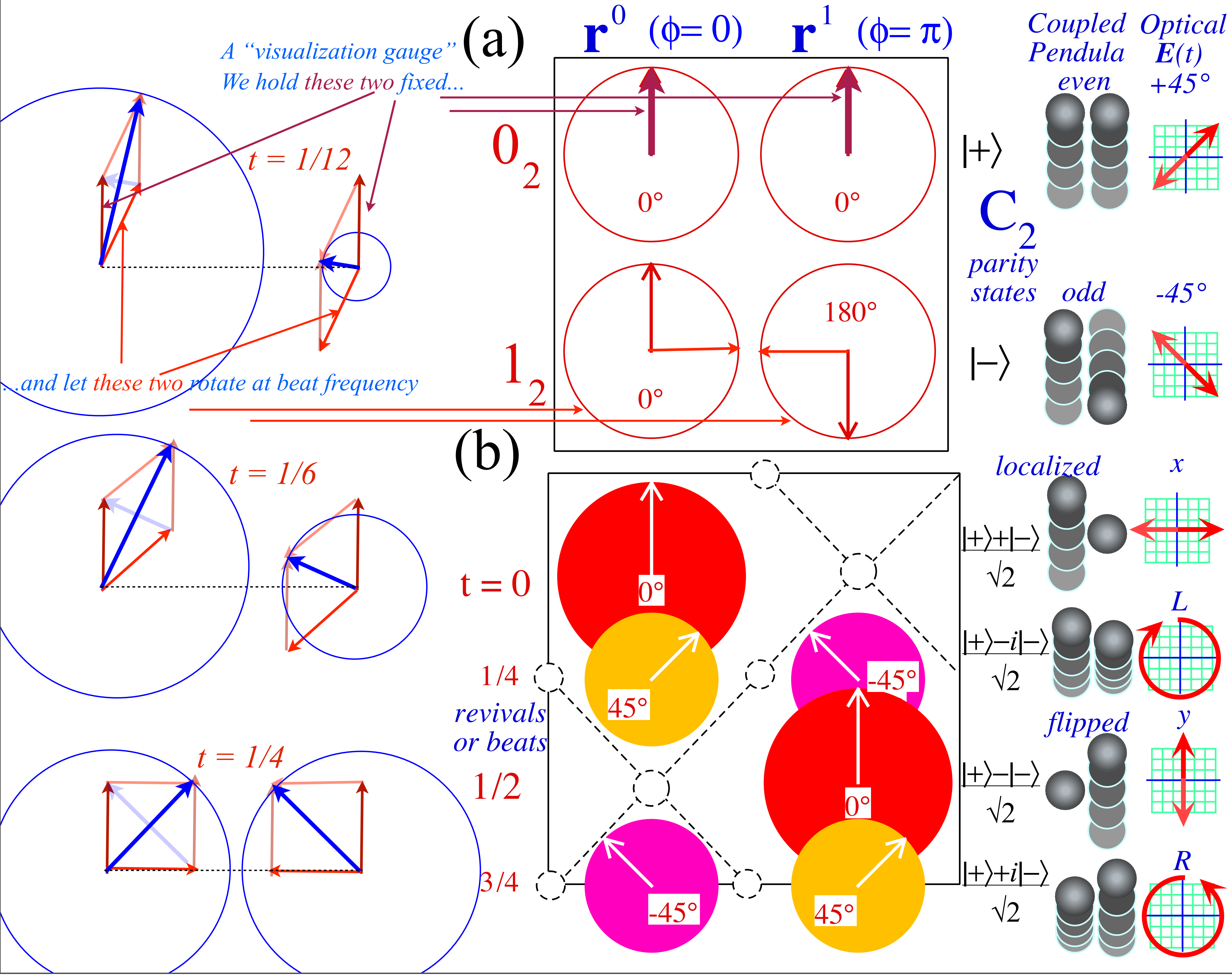
2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



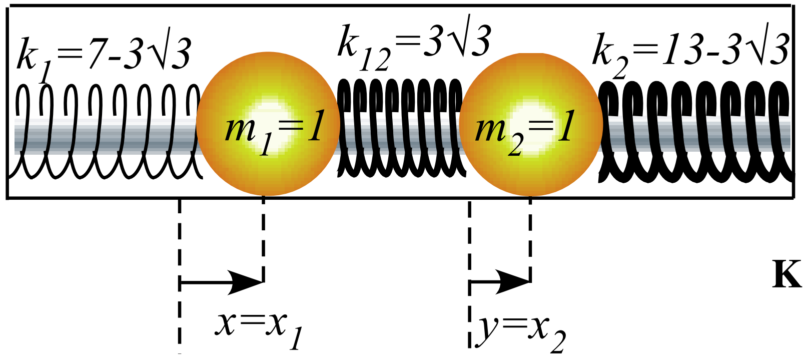
2D-HO beats and mixed mode geometry



➔ *2D-HO eigensolution example with asymmetric (A-Type) symmetry*

Initial state projection, mixed mode beat dynamics with fluid phase

Spectral decomposition of 2D-HO mode dynamics for lower symmetry

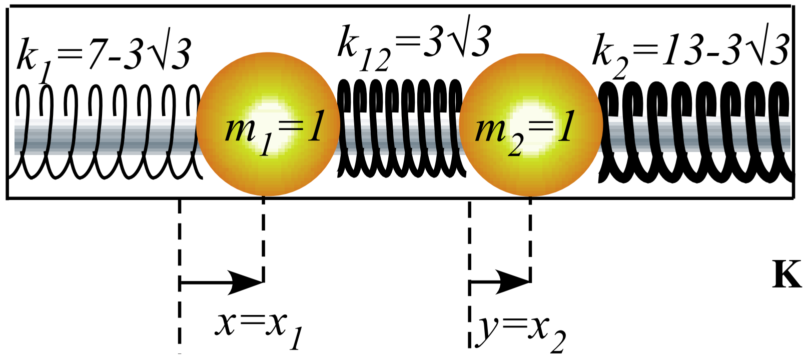


$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$$Det(\mathbf{K})=7\cdot 13-27=91-27=64$$

$$Trace(\mathbf{K})=7+13=20$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



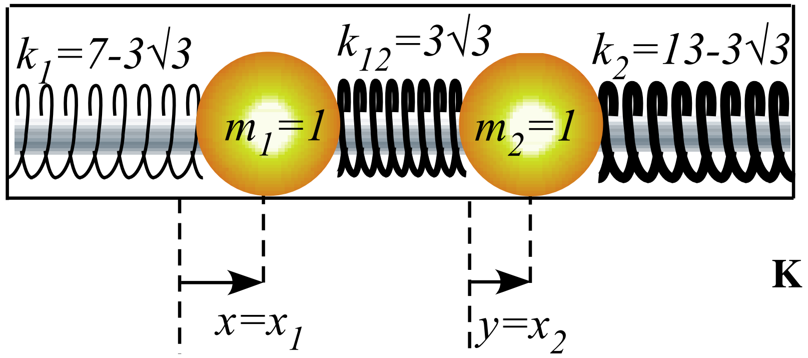
$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$Det(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



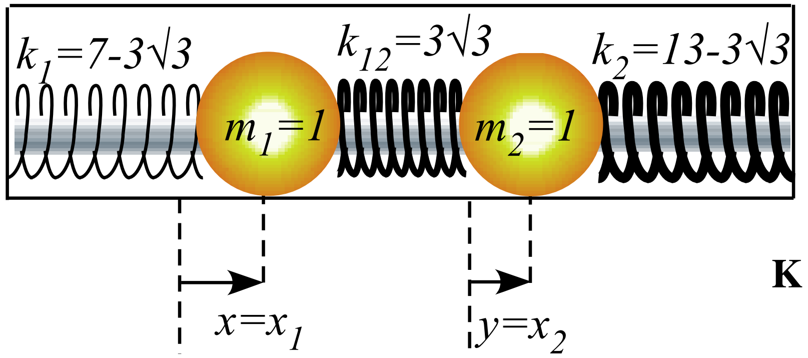
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

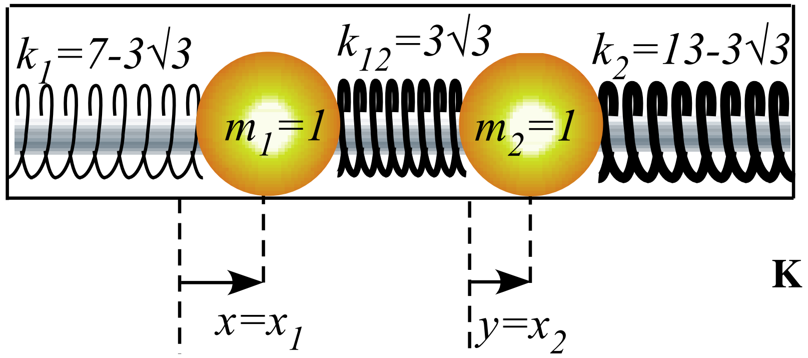
The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

$Trace(\mathbf{K}) = 7 + 13 = 20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix}7 & -3\sqrt{3}\\ -3\sqrt{3} & 13\end{pmatrix}$$

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$Det(\mathbf{K})=7\cdot13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

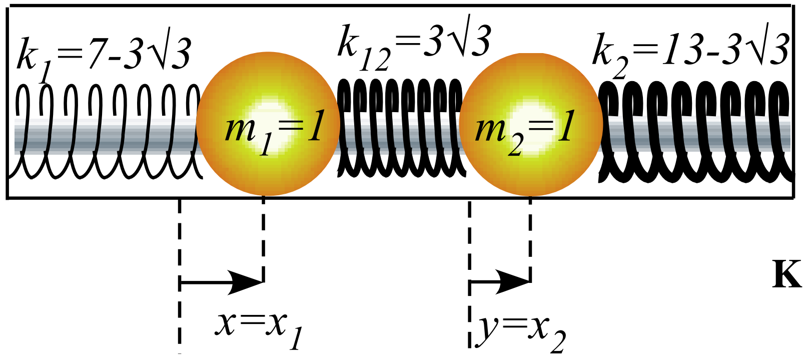
Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4,\quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix}K_{11}-K_2 & K_{12}\\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix}7-16 & -3\sqrt{3}\\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix}9 & +3\sqrt{3}\\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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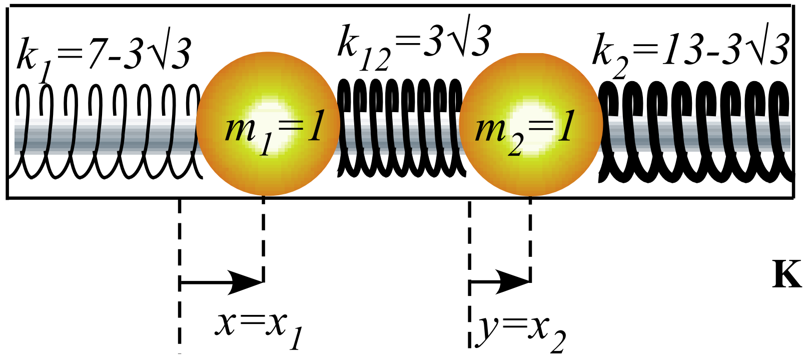
$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

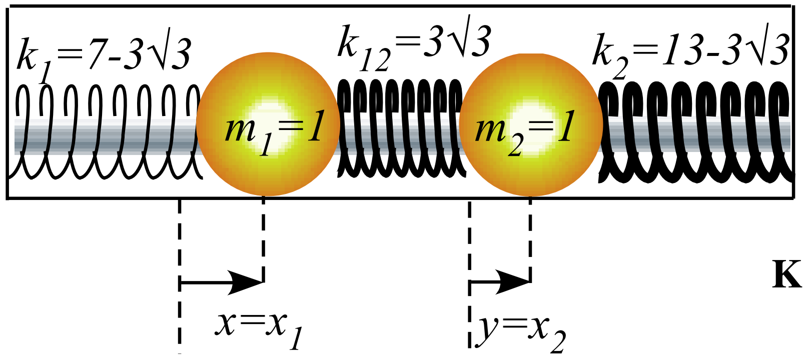
$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

2D-HO eigensolution example with asymmetric (A-Type) symmetry

➤ *Initial state projection, mixed mode beat dynamics with fluid phase*

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

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Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

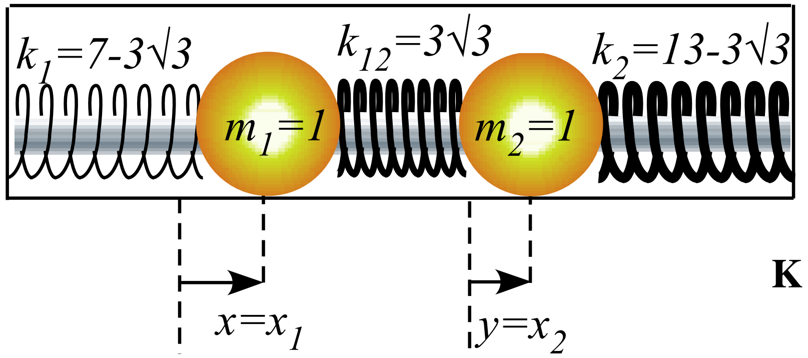
$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state ***x****(0)=(1,0):*

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix}7 & -3\sqrt{3}\\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

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Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix}K_{11}-K_2 & K_{12}\\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix}7-16 & -3\sqrt{3}\\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16} = \frac{\begin{pmatrix}9 & +3\sqrt{3}\\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}}{4} = \begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix}K_{11}-K_1 & K_{12}\\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix}7-4 & -3\sqrt{3}\\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4} = \frac{\begin{pmatrix}3 & -3\sqrt{3}\\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}}{4} = \begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}$

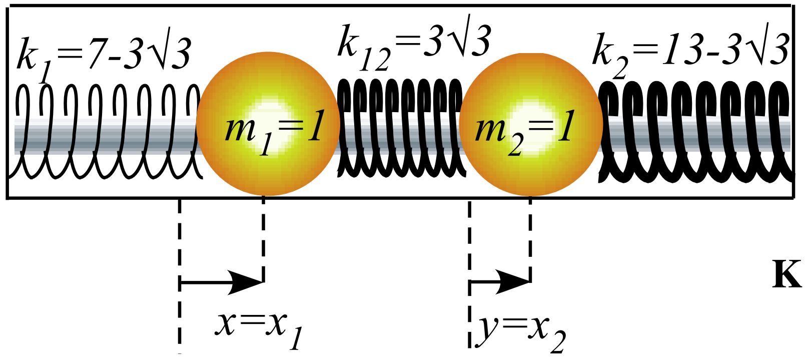
Spectral decomposition of initial state ***x****(0)=(1,0):*

$$\mathbf{1}\cdot\mathbf{x}(0) = (\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2}) + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

$Trace(\mathbf{K})=7+13=20$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4, \quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

Spectral decomposition of initial state **x**(0)=(1,0)*:*

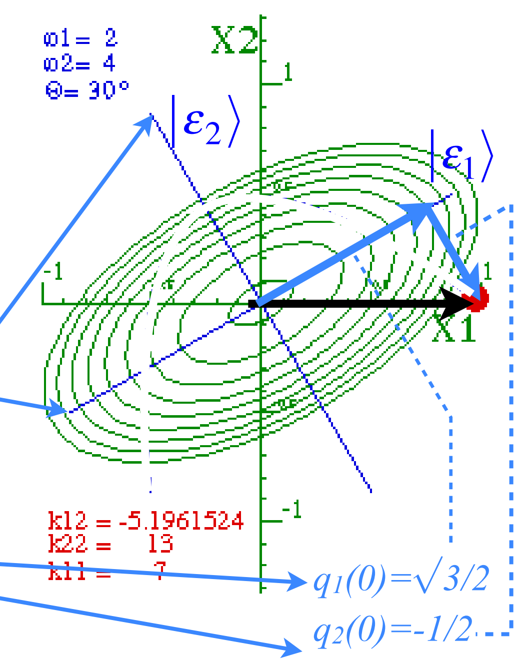
$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\left(\frac{\sqrt{3}}{2}\right)+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\left(-\frac{1}{2}\right)$$

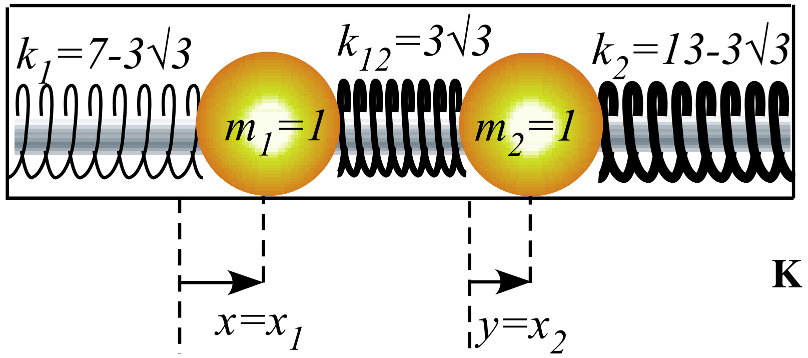
(Note projection of **x**(0) *onto eigen-axes)*

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t)=-\frac{1}{2}\cos 4t \right)$$

$$\mathbf{x}(0)=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

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$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

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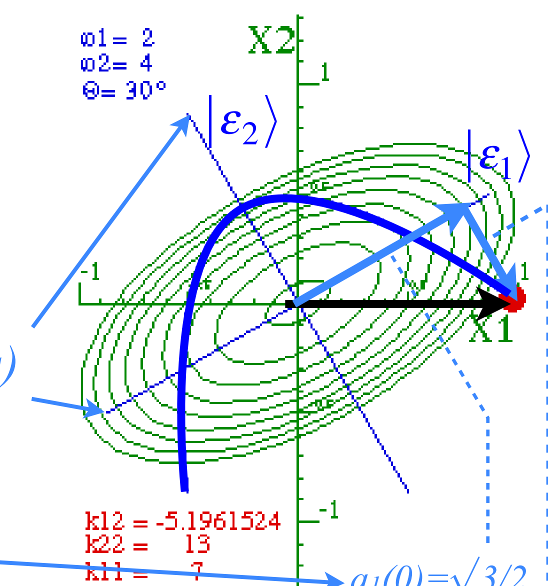
$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection of **x**(0) *onto eigen-axes)*

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

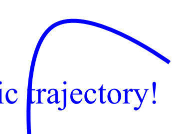


$$\mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$

$q_1(0)=\sqrt{3/2}$

$q_2(0)=-1/2$

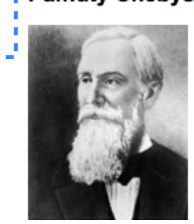
Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!



$$q_2(t)=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}[q_1(t)]^2+\frac{1}{2}$$

Example of a Tschebycheff Polynomial order 2

Pafnuty Chebyshev

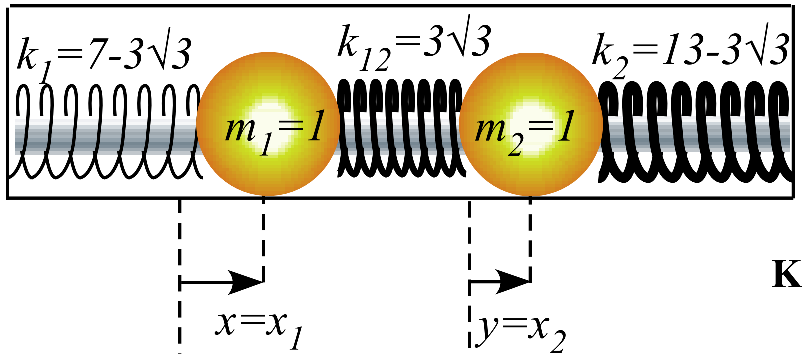


Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshov, Tchebychev or Tchebycheff, or Tschebyschev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk

Died: December 8, 1894, Saint Petersburg

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$=\frac{\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

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$$=\frac{\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\frac{\sqrt{3}}{2}\right)+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\frac{1}{2}\right)$$

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$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

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Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2(t)=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}\left[q_1(t)\right]^2+\frac{1}{2}$$

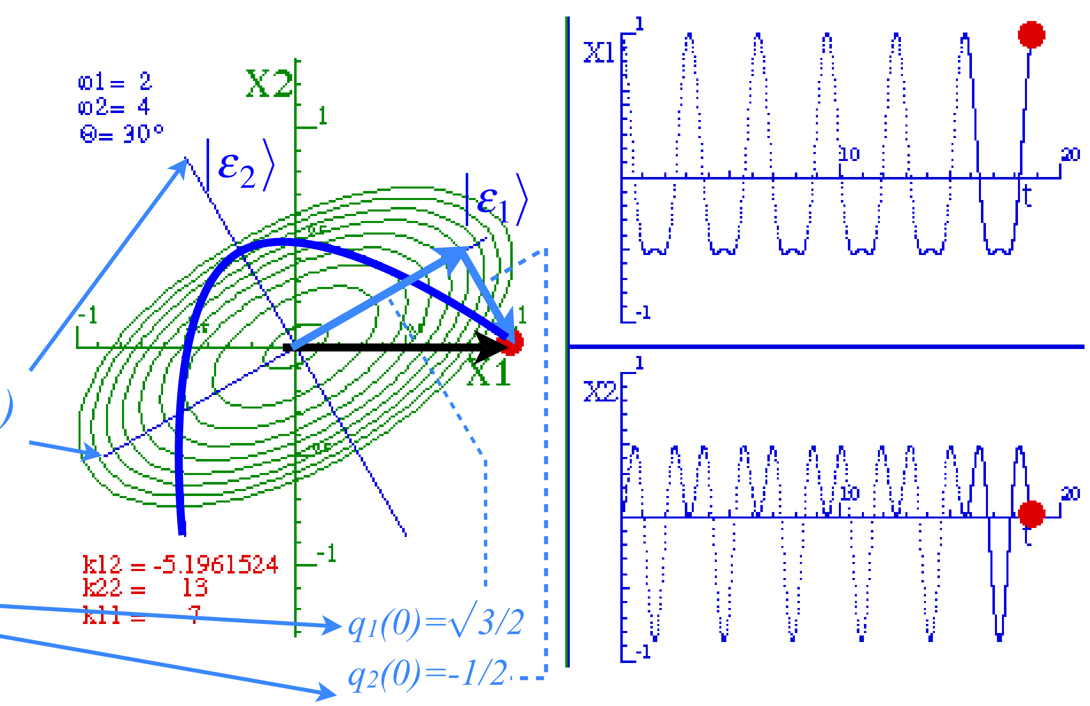

Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.

➤ *ANALOGY: 2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

ANALOGY: *2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

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that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Both have 4 parameters ($2^2 = 2+2$)

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of *real* 1st-order differential equations.

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

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$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

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$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

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$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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$$i\hbar|\dot{\Psi}(t)\rangle=\mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\mathbf{H}^\dagger$$

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$$\dot{x}_2=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-Bx_1-Dx_2+Cp_1$$

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c=\frac{A}{2}\left(p_1^2+x_1^2\right)+B\left(x_1x_2+p_1p_2\right)+C\left(x_1p_2-x_2p_1\right)+\frac{D}{2}\left(p_2^2+x_2^2\right)$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2DHO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

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QM vs. Classical Equations are identical

Then start with classical Hamiltonian. (Designed to give same result.)

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Then Hamilton's equations of motion are the following.

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