Lecture 22
Tue 11.06.2014

Reimann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)

Separation of GCC Equations: Effective Potentials

Small radial oscillations
- *2D Spherical pendulum or "Bowl-Bowling"*

Cycloidal ruler&compass geometry
- *Cycloid as* brachistichrone *with various geometries*
- *Cycloid as* tautochrone
  - *Cycloidulum vs Pendulum*

Cycloidal geometry of flying levers
- *Practical poolhall application*

Separation of GCC Equations: Effective Potentials

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2D Spherical pendulum or "Bowl-Bowling"
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Cycloid as tautochrone
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Cycloidal geometry of flying levers
Practical poolhall application

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

(Numerically correct ONLY!)

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

(Formally **and** Numerically correct)

Separation of GCC Equations: Effective Potentials (For isotropic $H(r,p_r,\phi,p_\phi)$)

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

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(Formally **and** Numerically correct)

Potential V is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)

H has no explicit ϕ–dependence and the ϕ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

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(Numerically correct ONLY!)

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(Formally **and** Numerically correct)

Potential V is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)

H has no explicit φ–dependence and the φ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

If H has no explicit z–dependence then the z–momenta is constant, too.

$$m\dot{z} = p_z = const. = k$$

Separation of GCC Equations: Effective Potentials

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H has no explicit ϕ–dependence and the ϕ–momenta is constant.

If H has no explicit z–dependence then the z–momenta is constant, too.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

$$m\dot{z} = p_z = const. = k$$

$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const.$$

Separation of GCC Equations: Effective Potentials

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(Formally **and** Numerically correct)

Potential V is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)
H has no explicit ϕ–dependence and the ϕ–momenta is constant.

If H has no explicit z–dependence then the z–momenta is constant, too.

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(Let $k = 0$)

Separation of GCC Equations: Effective Potentials

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H has no explicit ϕ–dependence and the ϕ–momenta is constant.

If H has no explicit z–dependence then the z–momenta is constant, too.

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$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const.$$

(Let k =0)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m}p_\rho^2 + V^{eff}(\rho) = E = const.$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

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(Let k =0)

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An *effective potential* $V^{eff}(\rho)$ has a *centrifugal barrier*.

$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$

Separation of GCC Equations: Effective Potentials

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(Formally **and** Numerically correct)

Potential *V* is *isotropic* (cylindrical) function of radius ρ. (*V = V*(ρ))
H has no explicit ϕ–dependence and the ϕ–momenta is constant.

If *H* has no explicit *z*–dependence then the *z*–momenta is constant, too.

$$m\rho^2\dot{\phi}=p_\phi=const.=\mu$$

$$m\dot{z}=p_z=const.=k$$

$$H=\frac{1}{2m}p_\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{k^2}{2m}+V(\rho)=E=const.$$

(Let k =0)

Symmetry reduces problem to a one-dimensional form.

$$H=\frac{1}{2m}p_\rho^2+V^{eff}(\rho)=E=const.$$

An *effective potential* $V_{eff}(\rho)$ has a *centrifugal barrier*.

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+V(\rho)$$

Velocity relations:

$$\dot{\phi}=\mu/\left(m\rho^2\right)$$

$$\dot{\rho}=\frac{d\rho}{dt}=\frac{\partial H}{\partial p_\rho}=\frac{p_\rho}{m}=\pm\sqrt{\frac{2}{m}\left(E-V^{eff}(\rho)\right)}$$

Separation of GCC Equations: Effective Potentials

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$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$

Velocity relations:

$$\dot{\phi} = \mu / \left(m\rho^2\right)$$

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}$$

Equations solved by a *quadrature integral* for time versus radius.

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}} = \left(\text{Travel time } \rho_0 \text{ to } \rho_1\right) = t_1 - t_0$$

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Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_0} = 0 \ , \quad \text{with:} \ \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0 \ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_0) + 0 + \frac{1}{2}(\rho - \rho_0)^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0}$$

Stable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0$ Unstable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} < 0$

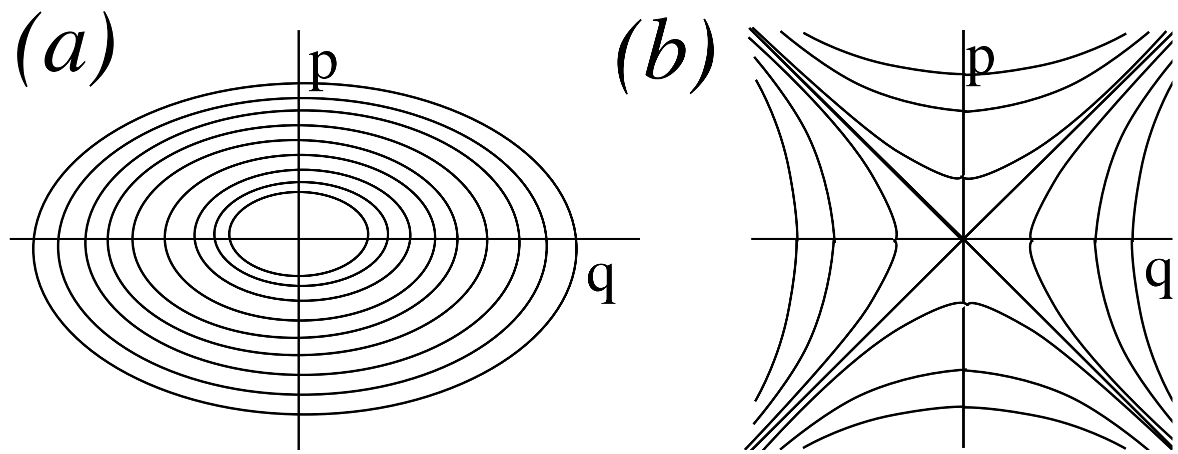

Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point

Small radial oscillations

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$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \qquad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}$$

Small radial oscillations

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Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_\phi} = \frac{\omega_{\rho_{stable}}}{\dot\phi(\rho_{stable})} = \frac{n_\rho}{n_\phi} \Leftrightarrow \text{Orbit is closed-periodic}$$

Small radial oscillations

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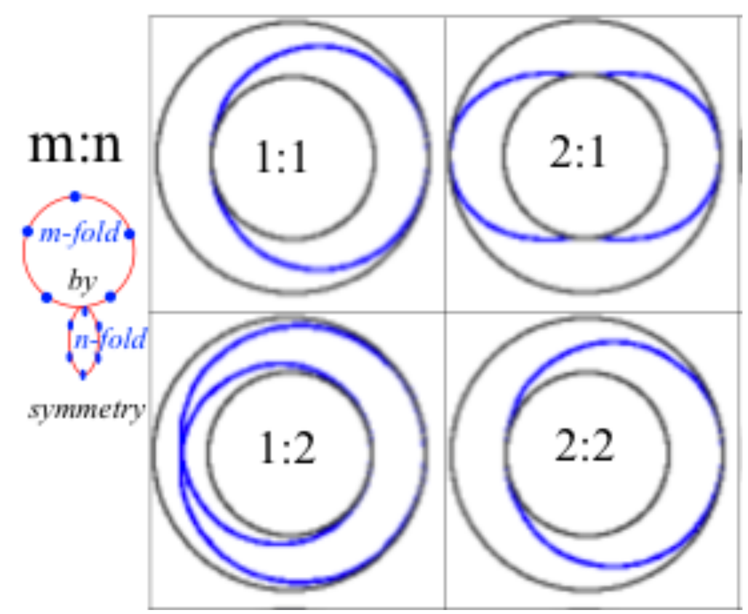
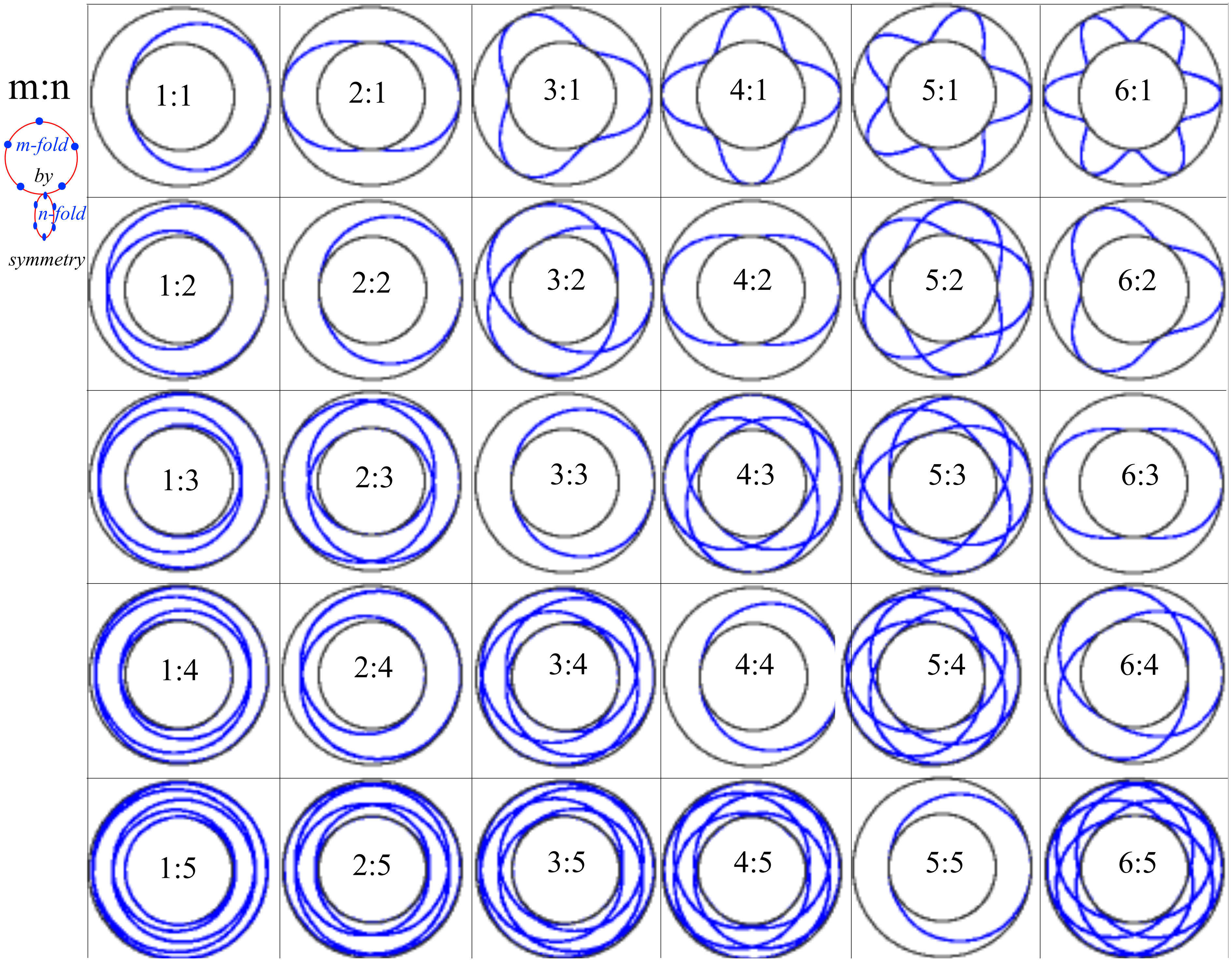
Some generic shapes resulting from various ratios $n\rho : n\phi$




m:n
m-fold
by
n-fold
symmetry
1:1
2:1
3:1
4:1
5:1
6:1
1:2
2:2
3:2
4:2
5:2
6:2
1:3
2:3
3:3
4:3
5:3
6:3
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4:5
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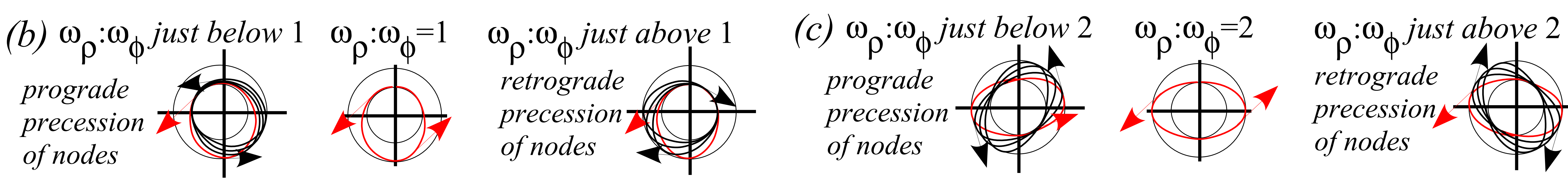
6:5


(b) $\omega_\rho:\omega_\phi$ just below 1
prograde precession of nodes
$\omega_\rho:\omega_\phi=1$
$\omega_\rho:\omega_\phi$ just above 1
retrograde precession of nodes
(c) $\omega_\rho:\omega_\phi$ just below 2
prograde precession of nodes
$\omega_\rho:\omega_\phi=2$
$\omega_\rho:\omega_\phi$ just above 2
retrograde precession of nodes

Separation of GCC Equations: Effective Potentials

Small radial oscillations

2D Spherical pendulum or "Bowl-Bowling"

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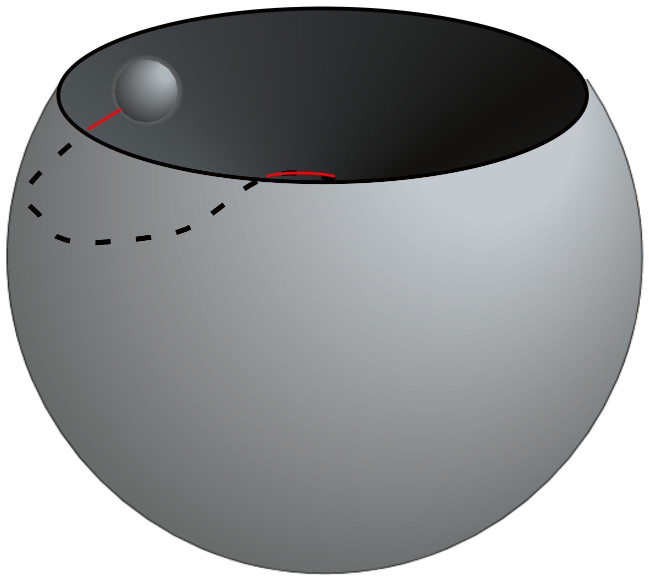
Cycloidal geometry of flying levers

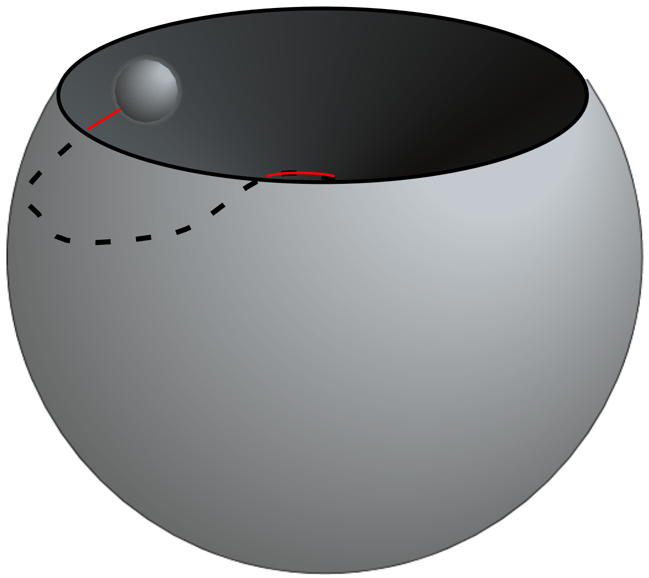
Practical poolhall application

2D Spherical pendulum or "Bowl-Bowling"



Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

$x=x^1=r\sin\theta\cos\phi$, $y=x^2=r\sin\theta\sin\phi$, $z=x^3=r\cos\theta$, *2D Spherical pendulum or "Bowl-Bowling"*



Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$

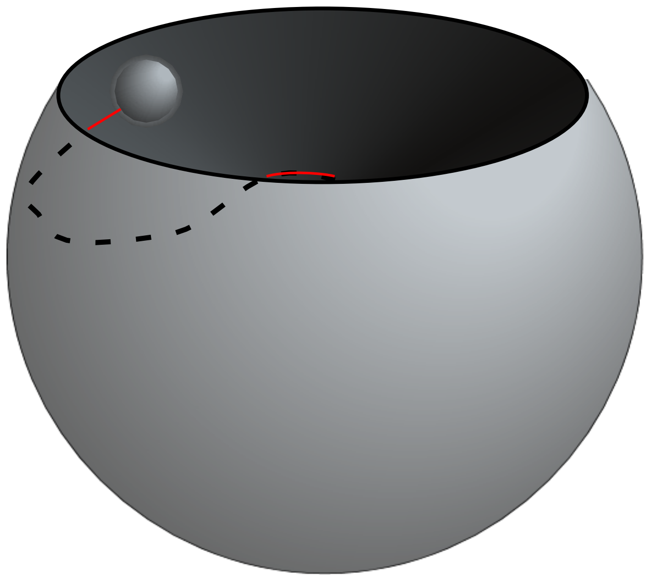
Jacobian matrices and determinants:

Reduced to cylindrical coordinates:

$$J=\begin{array}{c} \begin{array}{ccc} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \end{array} \\ \left(\begin{array}{ccc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{array}\right) \end{array} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[\substack{\theta=\pi/2 \\ r=\rho}]{} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

$$\det J = \det J^{\mathrm{T}} = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2\sin\theta \xrightarrow[\substack{\theta=\pi/2 \\ r=\rho}]{} \rho^2$$

2D Spherical pendulum or "Bowl-Bowling"



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$$
J=\begin{array}{c}
\begin{array}{ccc}\mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi\end{array}\\
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{pmatrix}
\end{array}
=\begin{pmatrix}
\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\
\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\
\cos\theta & -r\sin\theta & 0
\end{pmatrix}
\xrightarrow[r=\rho]{\theta=\pi/2}
\begin{pmatrix}
\cos\phi & 0 & -\rho\sin\phi\\
\sin\phi & 0 & \rho\cos\phi\\
0 & -\rho & 0
\end{pmatrix}
$$

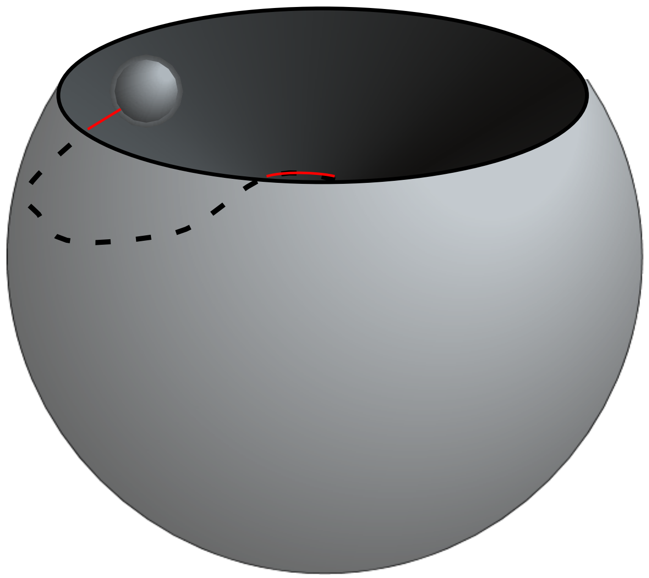
$$\det J=\det J^{\mathrm{T}}=\frac{\partial\{xyz\}}{\partial\{r\theta\phi\}}=r^2\sin\theta\xrightarrow[r=\rho]{\theta=\pi/2}\rho^2$$

Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T\cdot J$ of Jacobian and its transpose. OCC g's are diagonal.

Covariant: $g_{rr}=\mathbf{E}_r\bullet\mathbf{E}_r=1,\quad g_{\theta\theta}=\mathbf{E}_\theta\bullet\mathbf{E}_\theta=r^2,\quad g_{\phi\phi}=\mathbf{E}_\phi\bullet\mathbf{E}_\phi=r^2\sin^2\theta,$

Contravariant: $\quad g^{rr}=1, \qquad g^{\theta\theta}=1/r^2, \qquad g^{\phi\phi}=1/r^2\sin^2\theta.$

2D Spherical pendulum or "Bowl-Bowling"



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Jacobian matrices and determinants:

Reduced to cylindrical coordinates:

$$J=\begin{array}{c}\begin{array}{ccc}\mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi\end{array}\\ \begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{pmatrix}\end{array}=\begin{pmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{pmatrix}\xrightarrow[r=\rho]{\theta=\pi/2}\begin{pmatrix}\cos\phi & 0 & -\rho\sin\phi\\ \sin\phi & 0 & \rho\cos\phi\\ 0 & -\rho & 0\end{pmatrix}$$

$$\det J=\det J^{\mathrm{T}}=\frac{\partial\{xyz\}}{\partial\{r\theta\phi\}}=r^2\sin\theta\xrightarrow[r=\rho]{\theta=\pi/2}\rho^2$$

Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T\cdot J$ of Jacobian and its transpose. OCC g's are diagonal.

Covariant: $g_{rr}=\mathbf{E}_r\bullet\mathbf{E}_r=1,\ \ g_{\theta\theta}=\mathbf{E}_\theta\bullet\mathbf{E}_\theta=r^2,\ \ g_{\phi\phi}=\mathbf{E}_\phi\bullet\mathbf{E}_\phi=r^2\sin^2\theta,$

Contravariant: $\quad g^{rr}=1,\qquad g^{\theta\theta}=1/r^2,\qquad g^{\phi\phi}=1/r^2\sin^2\theta.$

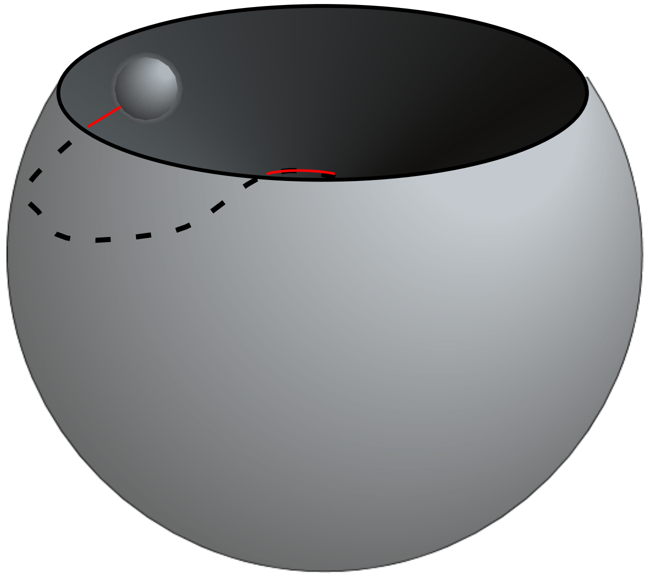
(Lagrangian form) *(Hamiltonian form)*

$$T=\frac{m}{2}(g_{rr}\dot r^2+g_{\theta\theta}\dot\theta^2+g_{\phi\phi}\dot\phi^2)\ =\frac{1}{2m}(g^{rr}p_r{}^2+g^{\theta\theta}p_\theta{}^2+g^{\phi\phi}p_\phi{}^2)$$

$$=\frac{1}{2}(\gamma_{rr}\dot r^2+\gamma_{\theta\theta}\dot\theta^2+\gamma_{\phi\phi}\dot\phi^2)\ =\frac{1}{2}\quad(\gamma^{rr}p_r{}^2+\gamma^{\theta\theta}p_\theta{}^2+\gamma^{\phi\phi}p_\phi{}^2)$$

$$=\frac{m}{2}(\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\dot\phi^2)=\frac{1}{2m}(p_r{}^2+\frac{p_\theta{}^2}{r^2}+\frac{p_\phi{}^2}{r^2\sin^2\theta})$$

2D Spherical pendulum or "Bowl-Bowling"



Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

$x=x^1=r\sin\theta\cos\phi$, $y=x^2=r\sin\theta\sin\phi$, $z=x^3=r\cos\theta$,

Jacobian matrices and determinants:

Reduced to cylindrical coordinates:

$$J=\begin{array}{c}\begin{array}{ccc}\mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi\end{array}\\ \begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{pmatrix}\end{array}=\begin{pmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{pmatrix}\xrightarrow[r=\rho]{\theta=\pi/2}\begin{pmatrix}\cos\phi & 0 & -\rho\sin\phi\\ \sin\phi & 0 & \rho\cos\phi\\ 0 & -\rho & 0\end{pmatrix}$$

$$\det J=\det J^{\mathrm{T}}=\frac{\partial\{xyz\}}{\partial\{r\theta\phi\}}=r^2\sin\theta\xrightarrow[r=\rho]{\theta=\pi/2}\rho^2$$

Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T\cdot J$ of Jacobian and its transpose. OCC g's are diagonal.

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(Lagrangian form) *(Hamiltonian form)*

$$T=\frac{m}{2}(g_{rr}\dot{r}^2+g_{\theta\theta}\dot{\theta}^2+g_{\phi\phi}\dot{\phi}^2)=\frac{1}{2m}(g^{rr}p_r{}^2+g^{\theta\theta}p_\theta{}^2+g^{\phi\phi}p_\phi{}^2)$$

$$=\frac{1}{2}(\gamma_{rr}\dot{r}^2+\gamma_{\theta\theta}\dot{\theta}^2+\gamma_{\phi\phi}\dot{\phi}^2)=\frac{1}{2}\quad(\gamma^{rr}p_r{}^2+\gamma^{\theta\theta}p_\theta{}^2+\gamma^{\phi\phi}p_\phi{}^2)$$

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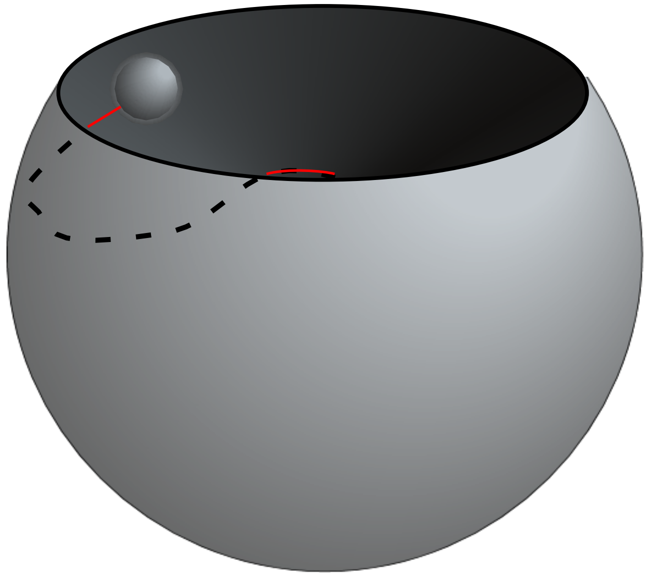
Spherical coordinates with constant radius r implies conserved azimuthal momentum:

$$p_\phi\equiv\frac{\partial T}{\partial\dot{\phi}}=m(R^2\sin^2\theta)\dot{\phi}\ =const.$$

2D Spherical pendulum or "Bowl-Bowling"



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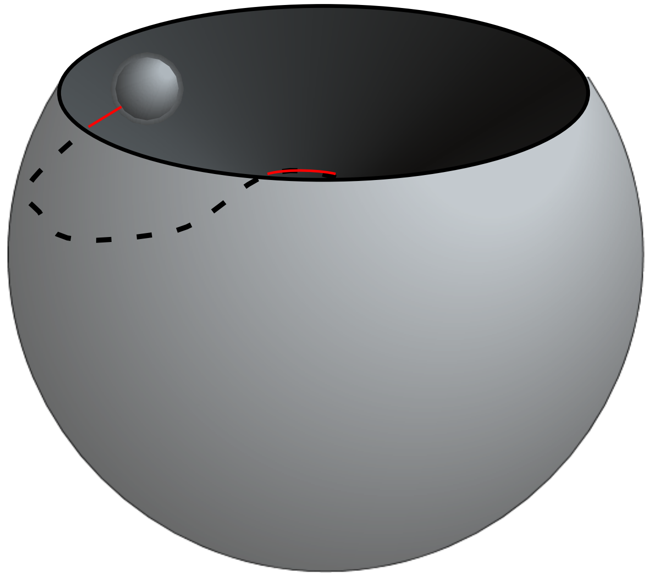
Total Energy from Hamiltonian $E=T+V(gravity)=const.$:

$$E=\frac{mR^2}{2}\dot\theta^2+V^{effective}(\theta)=\frac{mR^2}{2}\dot\theta^2+\frac{p_\phi^2}{2mR^2\sin^2\theta}+mgR\cos\theta \;=\; \alpha\dot\theta^2+\frac{\delta}{\sin^2\theta}+\gamma\cos\theta$$

Let: $\alpha=\frac{mR^2}{2}$, $\delta=\frac{p_\phi^2}{2mR^2}$, $\gamma=mgR$ where: $p_\phi=mR^2\sin^2\theta(\dot\phi\;)$

2D Spherical pendulum or "Bowl-Bowling"



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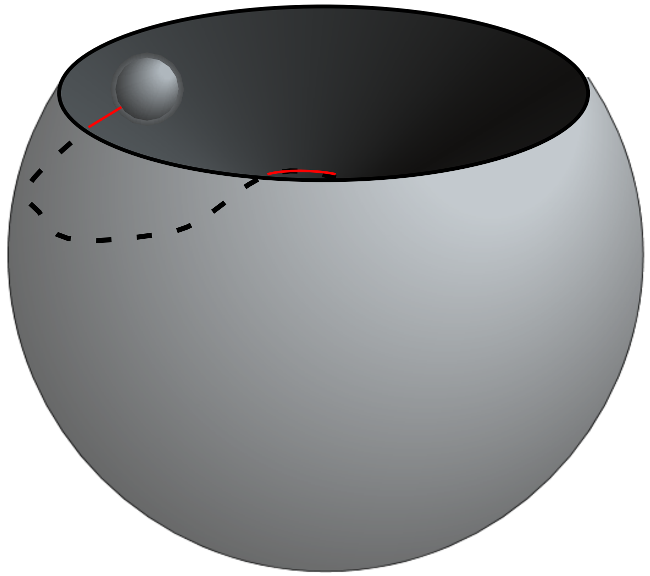

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Equilibrium point of stable orbit

$$\frac{dV^{effective}(\theta)}{d\theta}=\frac{-2\delta\cos\theta}{\sin^3\theta}-\gamma\sin\theta=0=\frac{-2p_\phi^2\cos\theta}{2mR^2\sin^3\theta}-mgR\sin\theta$$

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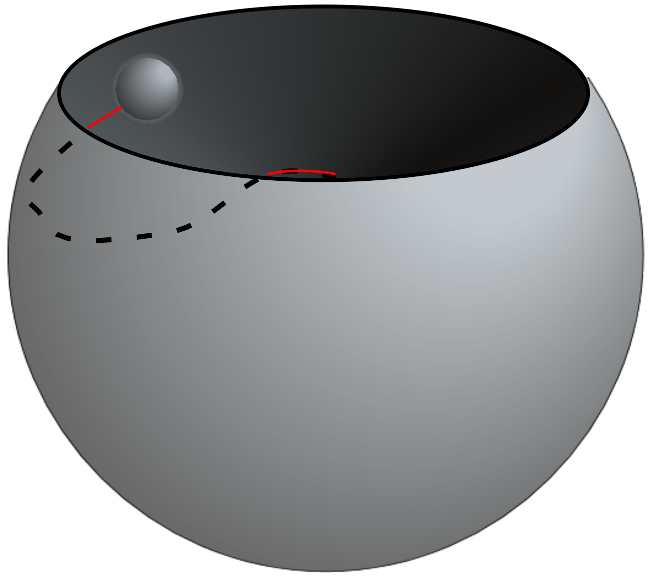

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Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{effective}(\theta)}{d\theta} = \frac{-2\delta\cos\theta}{\sin^3\theta} - \gamma\sin\theta = 0 = \frac{-2p_\phi^2\cos\theta}{2mR^2\sin^3\theta} - mgR\sin\theta$$

$$\left(\omega_\theta^{equil}\right)^2 = \frac{1}{mR^2}\left.\frac{d^2V^{effective}(\theta)}{d\theta^2}\right|_{equil}$$

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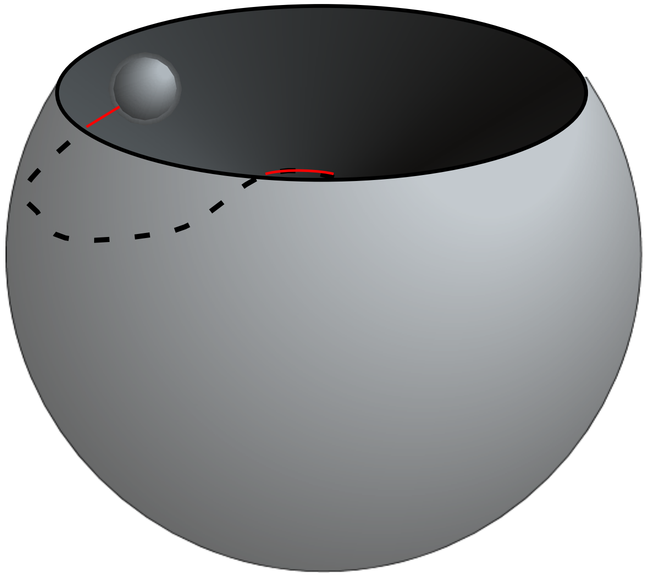

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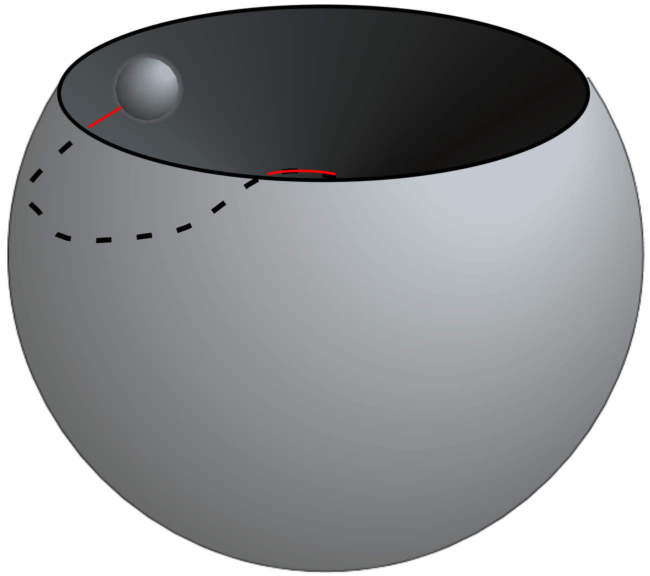
$$0 = (mR^2\sin\theta)\dot{\phi}^2\cos\theta - mgR\sin\theta$$

or: $\dot{\phi}_{equil}^2 = -\frac{g}{R\cos\theta_{equil}}$

(Polar angle librational frequency ω_θ^{equil} is related to azimuthal frequency $\dot{\phi}_{equil}^2$.)

2D Spherical pendulum or "Bowl-Bowling"



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V-Derivative for small oscillation frequency:

$$\frac{d^2V^{effective}(\theta)}{d\theta^2}=-\gamma\cos\theta+\frac{2\delta\sin\theta}{\sin^3\theta}+\frac{3\cdot 2\delta\cos^2\theta}{\sin^4\theta}=-\gamma\cos\theta+2\delta\frac{\sin^2\theta+3\cos^2\theta}{\sin^4\theta}$$

$$=-mgR\cos\theta+\frac{2\left(mR^2\sin^2\theta\quad\dot{\phi}\right)^2}{2mR^2}\frac{1+2\cos^2\theta}{\sin^4\theta}$$

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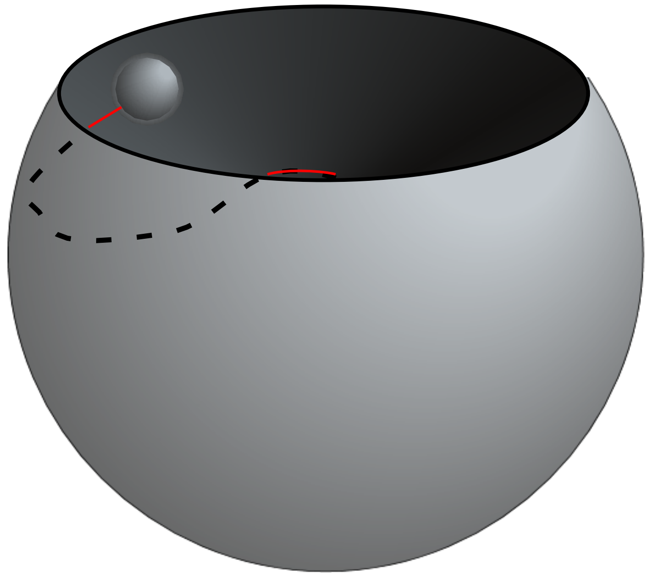

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Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{effective}(\theta)}{d\theta}=\frac{-2\delta\cos\theta}{\sin^3\theta}-\gamma\sin\theta=0=\frac{-2p_\phi^2\cos\theta}{2mR^2\sin^3\theta}-mgR\sin\theta$$

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At equilibrium:

$$\left.\frac{d^2V^{effective}(\theta)}{d\theta^2}\right|_{equil}=-mgR\cos\theta_{equil}+mR^2\left(-\frac{g}{R\cos\theta_{equil}}\right)\left(1+2\cos^2\theta_{equil}\right)$$

$$=-\frac{mgR}{\cos\theta_{equil}}\left(1+3\cos^2\theta_{equil}\right)$$

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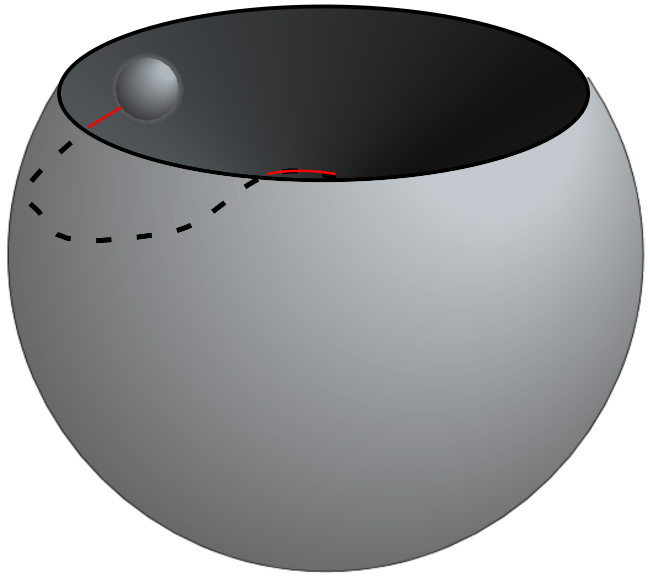

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$$\boxed{\left(\omega_\theta^{equil}\right)^2/(\dot{\phi}_{equil}^2)=\left(1+3\cos^2\theta_{equil}\right)}$$

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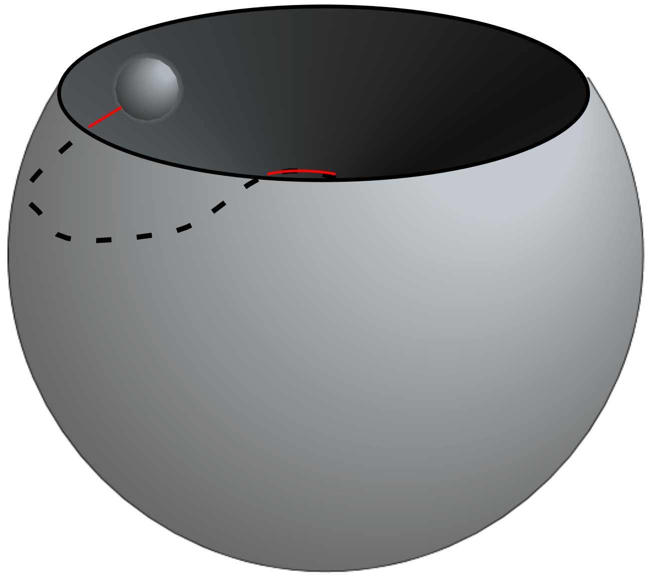

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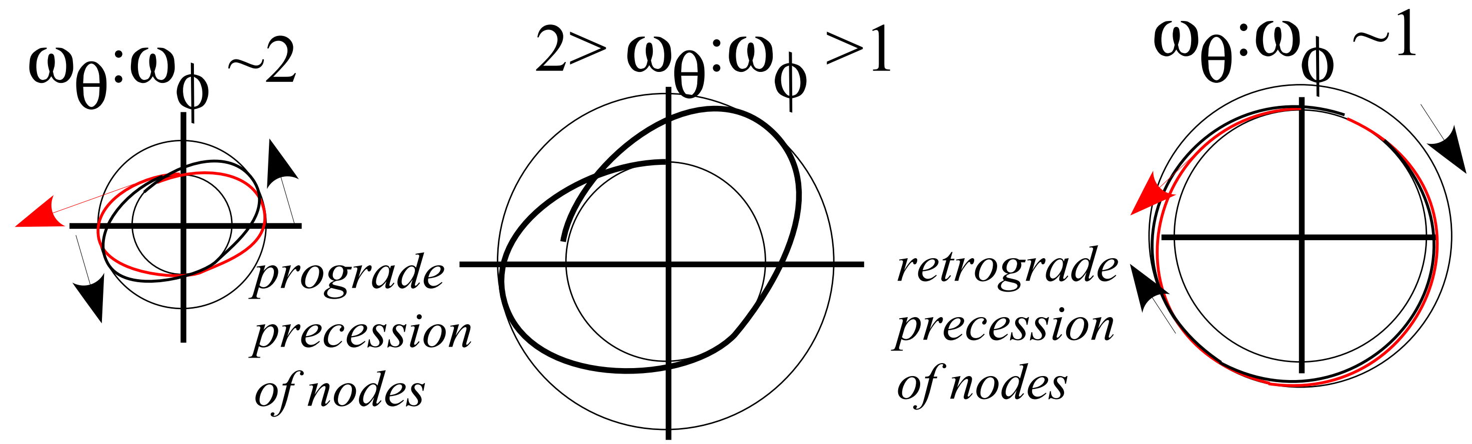
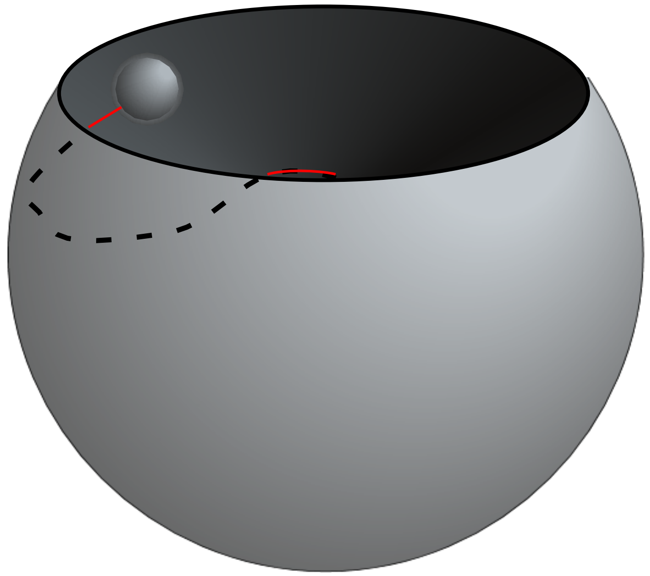
At bottom θ→π the ratio of in-out ω_θ to circle ω_ϕ approaches 2:1

At equator θ→π/2 the ratio approaches 1:1.



2D Spherical pendulum or "Bowl-Bowling"



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$$=-mgR\cos\theta+mR^2\dot{\phi}^2\left(1+2\cos^2\theta\right)$$

At equilibrium:

$$\left.\frac{d^2V^{effective}(\theta)}{d\theta^2}\right|_{equil}=-mgR\cos\theta_{equil}+mR^2\left(-\frac{g}{R\cos\theta_{equil}}\right)\left(1+2\cos^2\theta_{equil}\right)$$

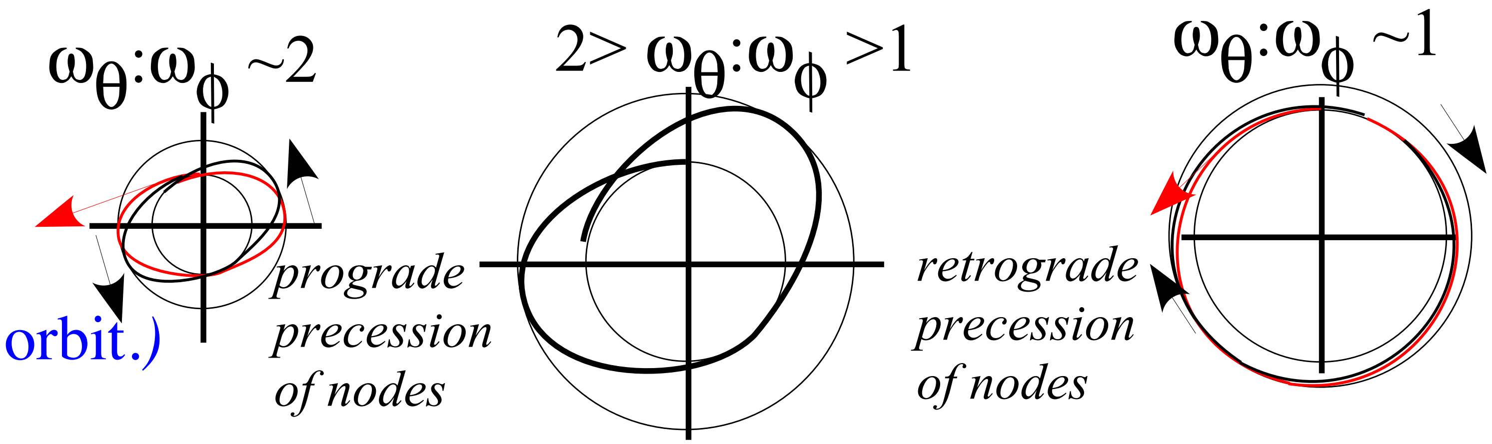
$$=-\frac{mgR}{\cos\theta_{equil}}\left(1+3\cos^2\theta_{equil}\right)$$

$$\boxed{\left(\omega_\theta^{equil}\right)^2/\left(\dot{\phi}_{equil}^2\right)=\left(1+3\cos^2\theta_{equil}\right)}$$

At bottom θ→π the ratio of in-out ω_θ to circle ω_ϕ approaches 2:1

At equator θ→π/2 the ratio approaches 1:1.

Ratio is between 2 and 1
(Usually irrational non-closed orbit).
(2:1 is like 2D IHO, but 1:1 is like coulomb orbit.)



Separation of GCC Equations: Effective Potentials

Small radial oscillations

2D Spherical pendulum or "Bowl-Bowling"

Cycloidal ruler&compass geometry

Cycloid as brachistichrone

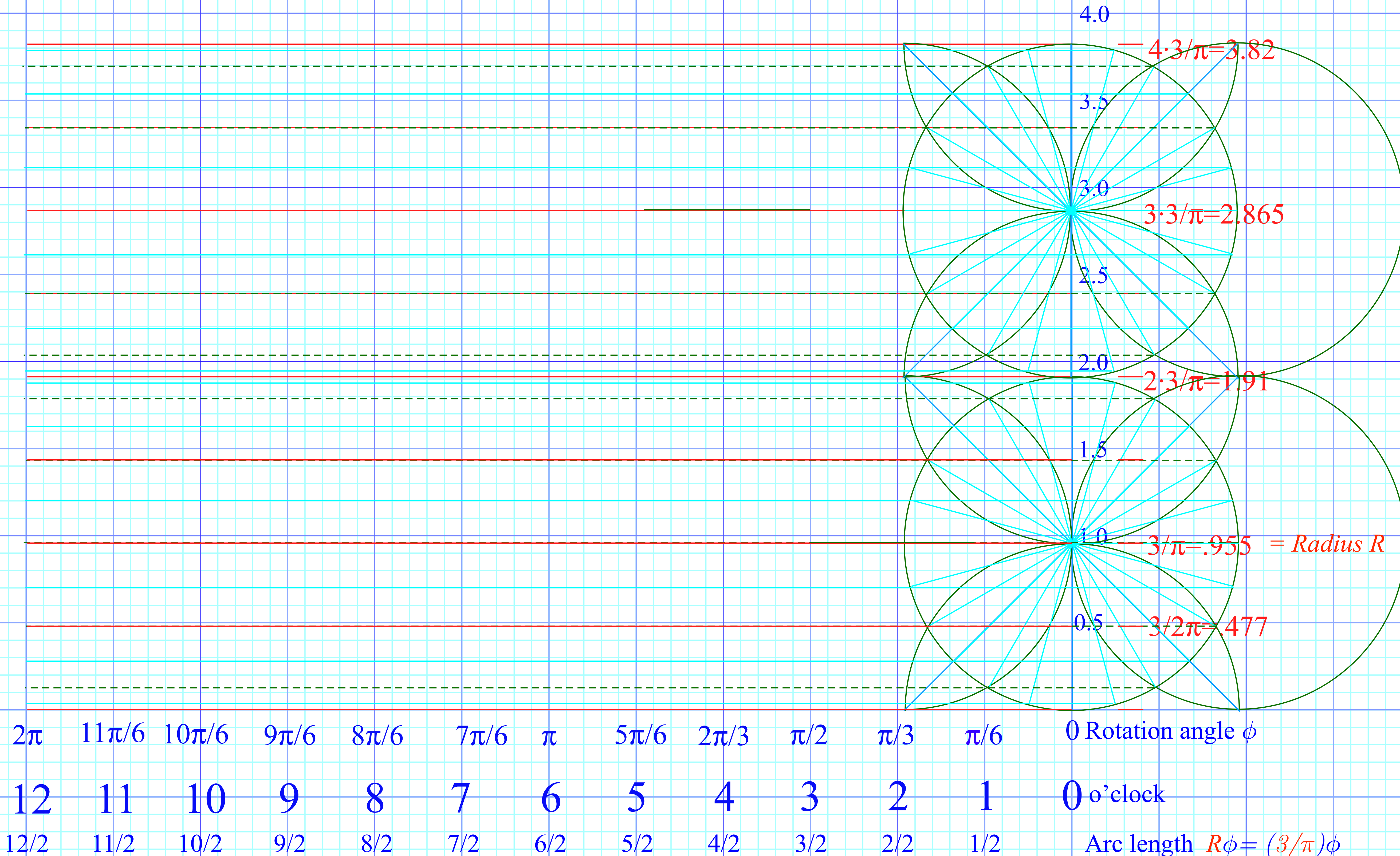
Cycloid as tautochrone

Cycloidulum vs Pendulum

Cycloidal geometry of flying levers

Practical poolhall application

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled -out circumference $R\phi= (3/\pi)m\pi/n=3m/n$.



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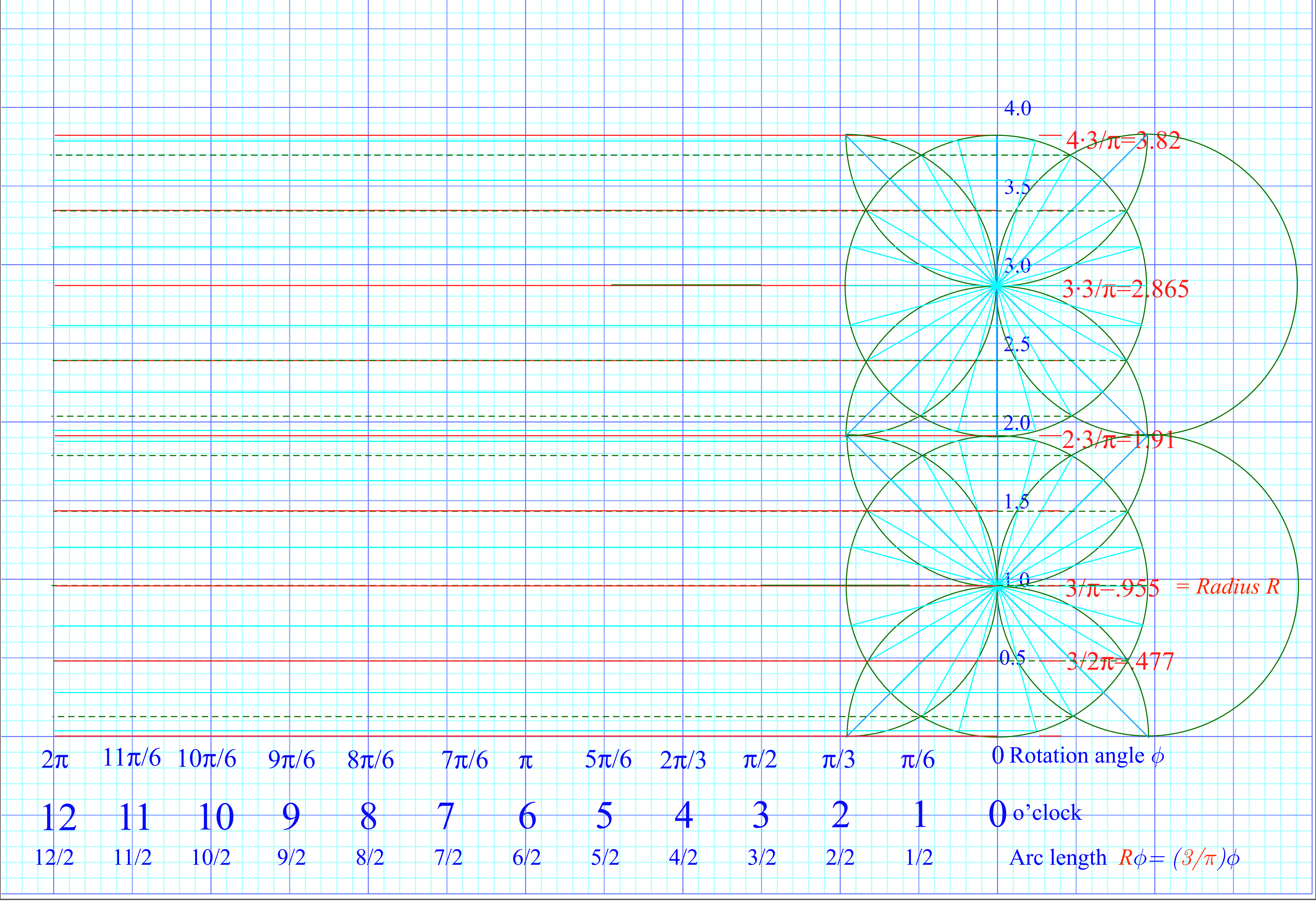

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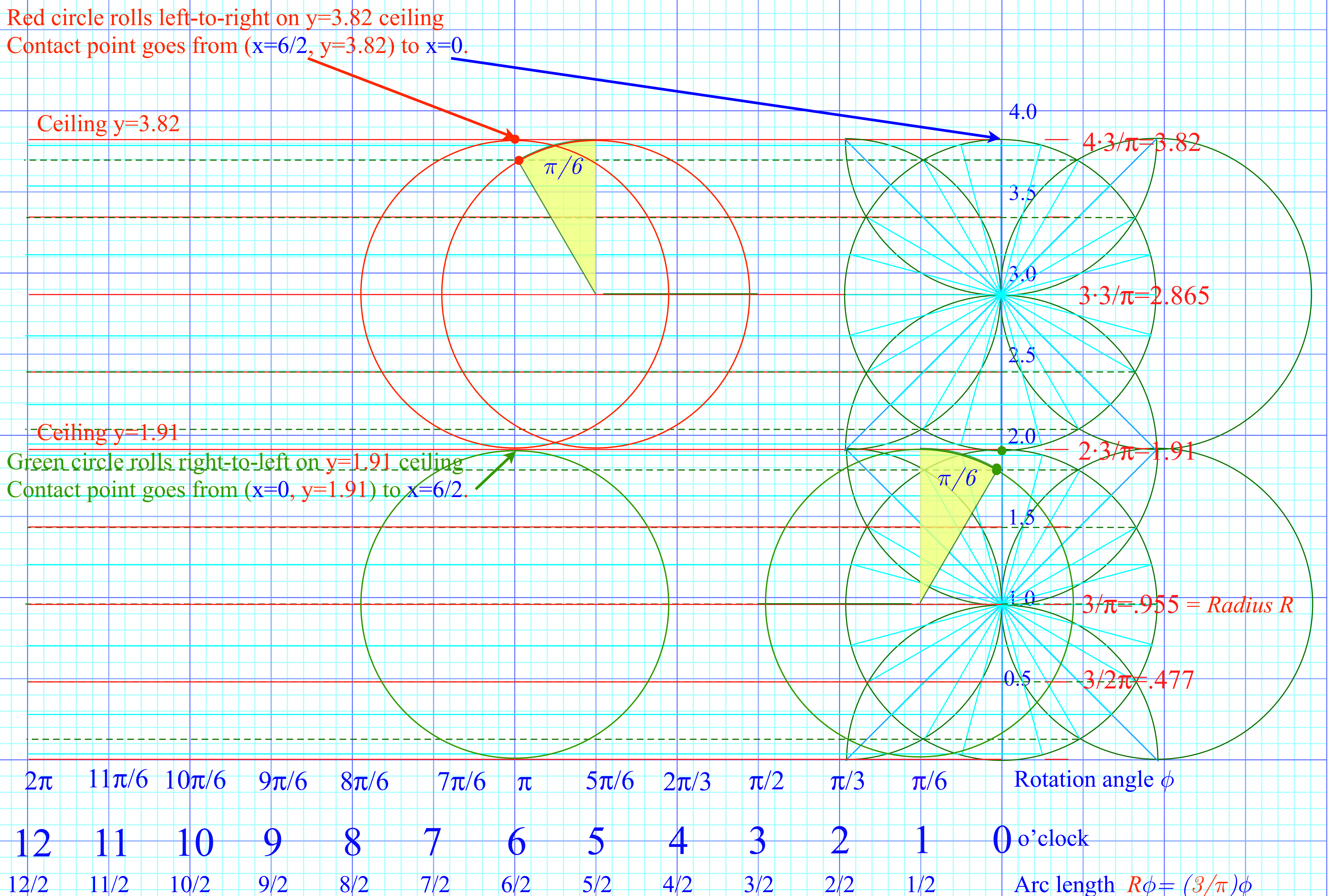

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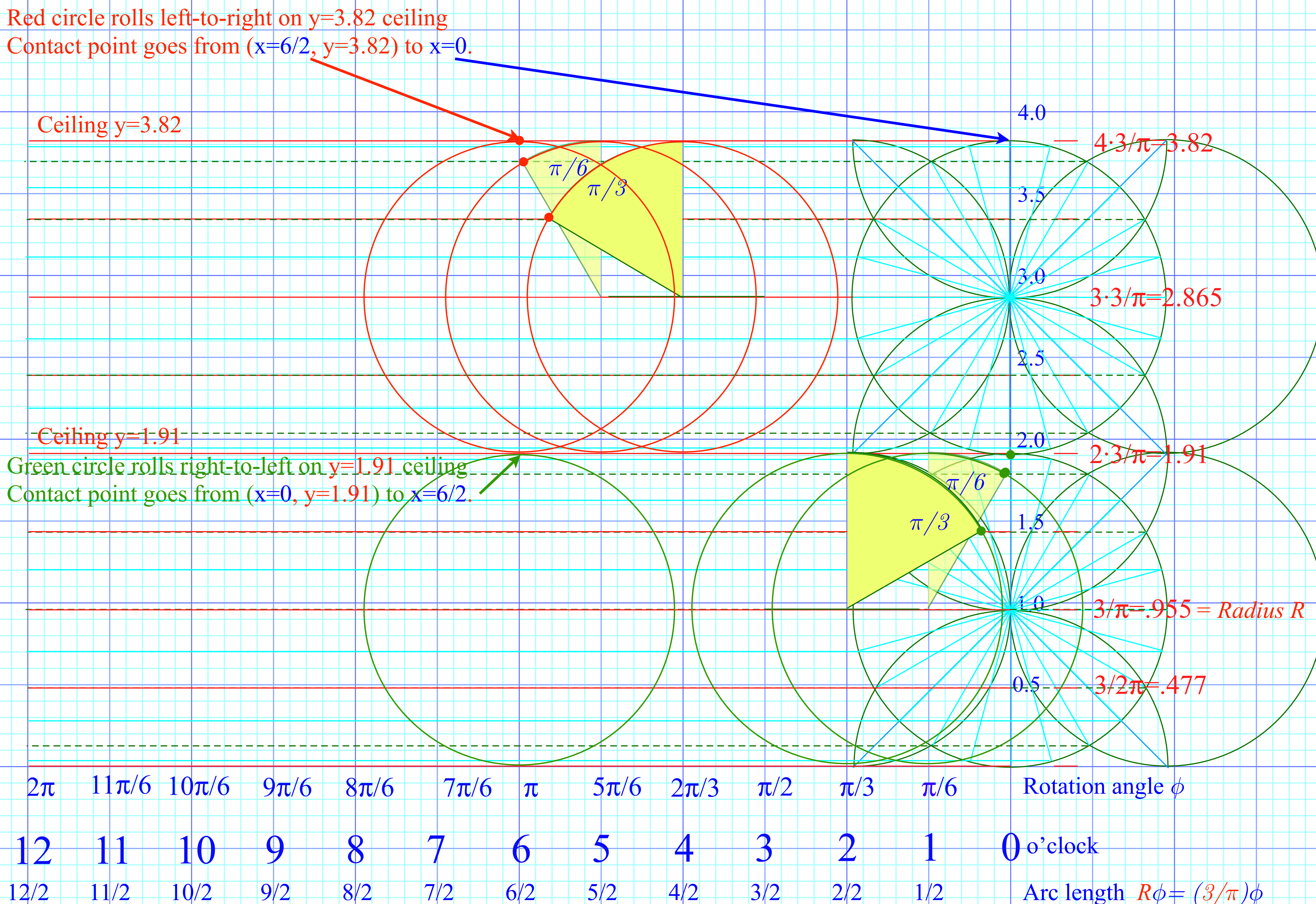

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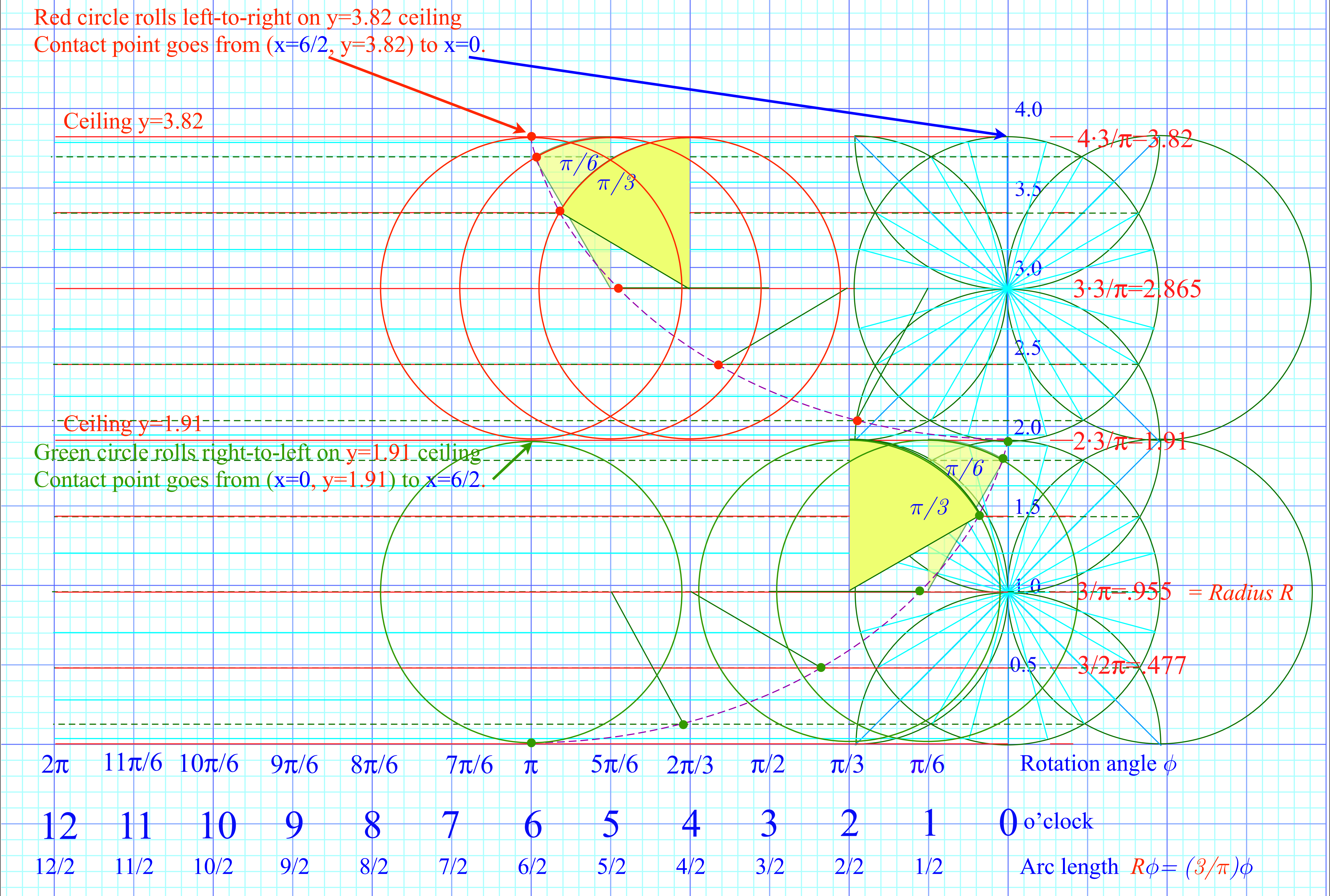

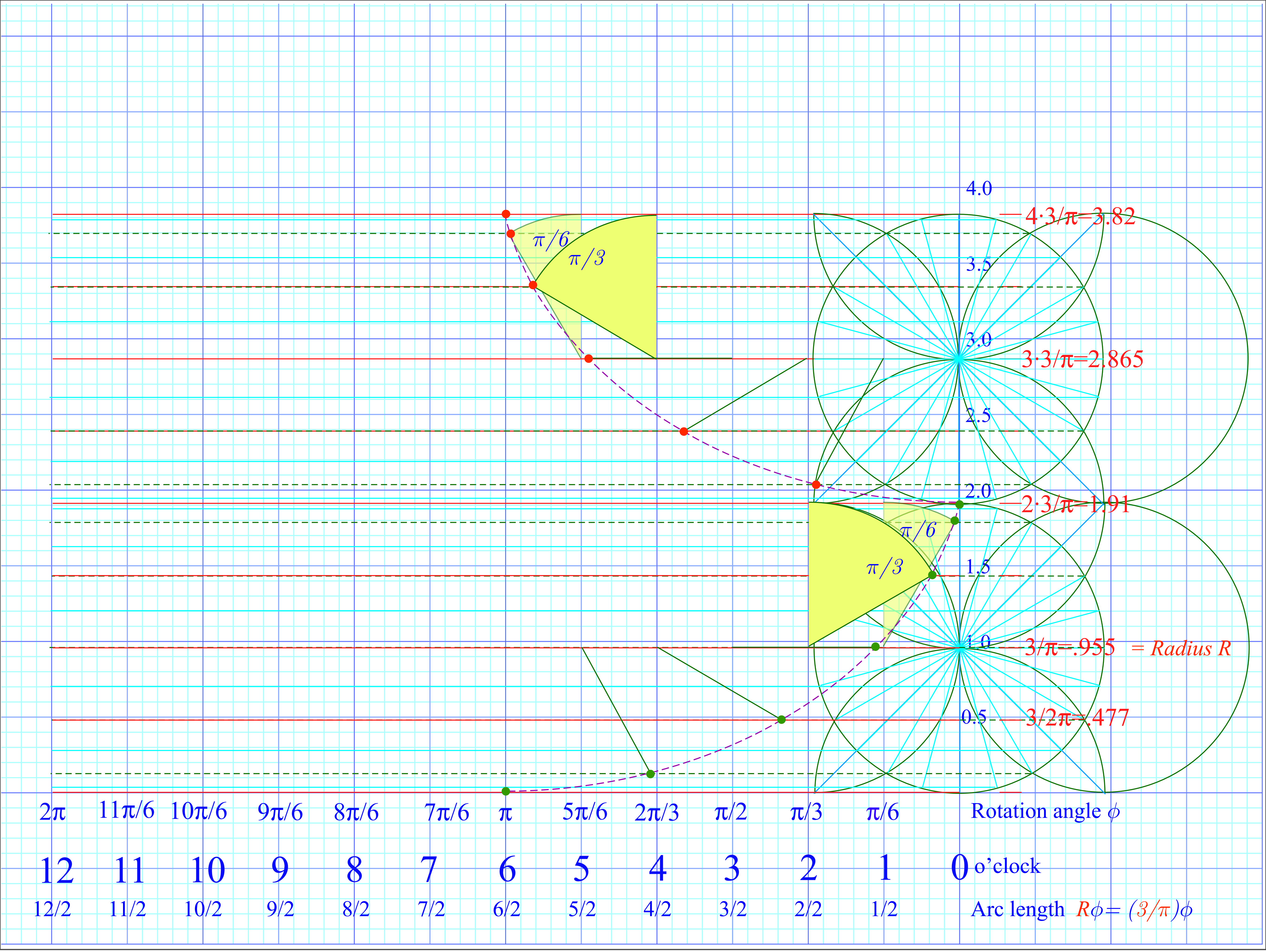
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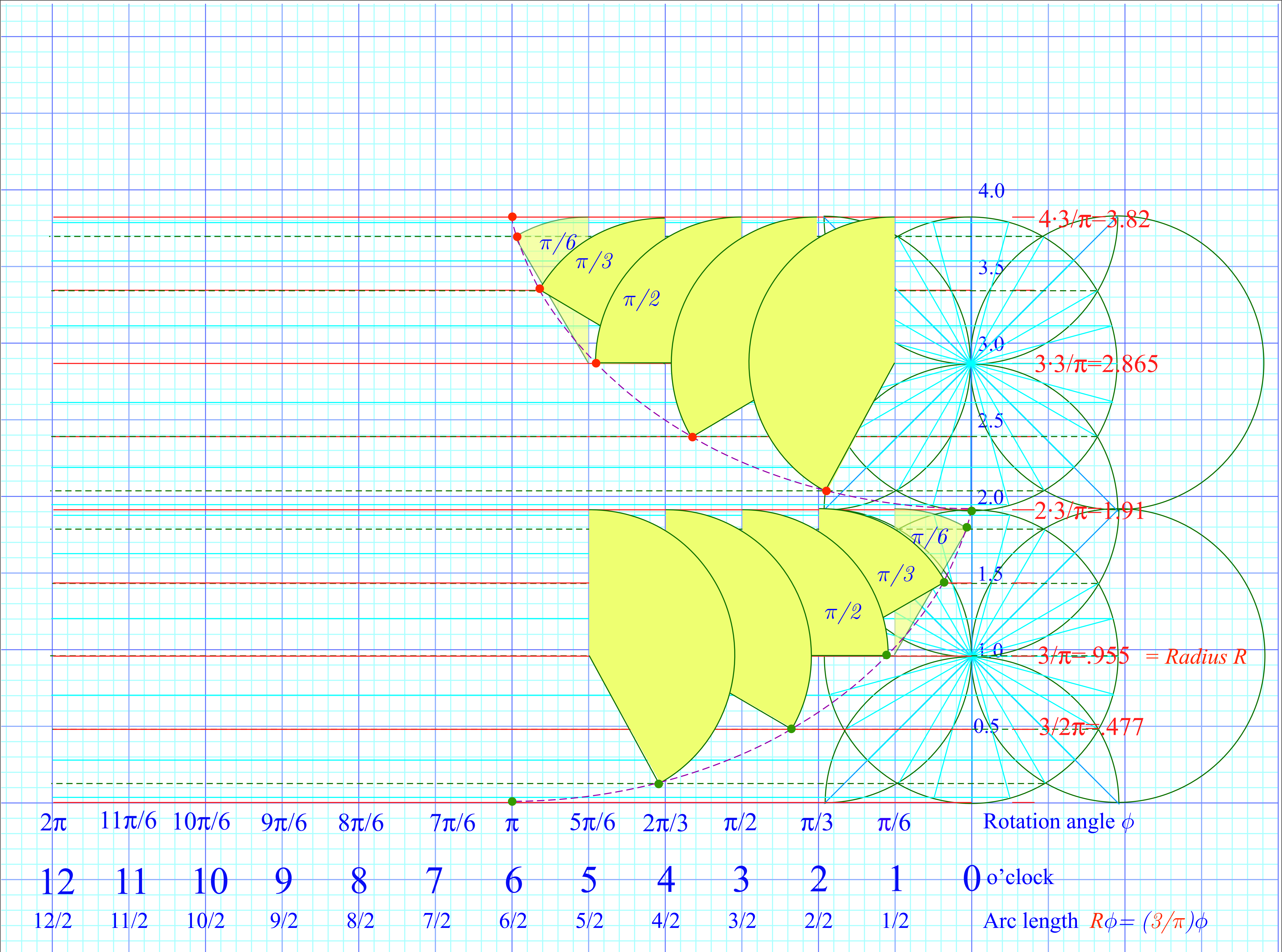
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6
Rotation angle ϕ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2
Arc length Rϕ= (3/π)ϕ



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12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
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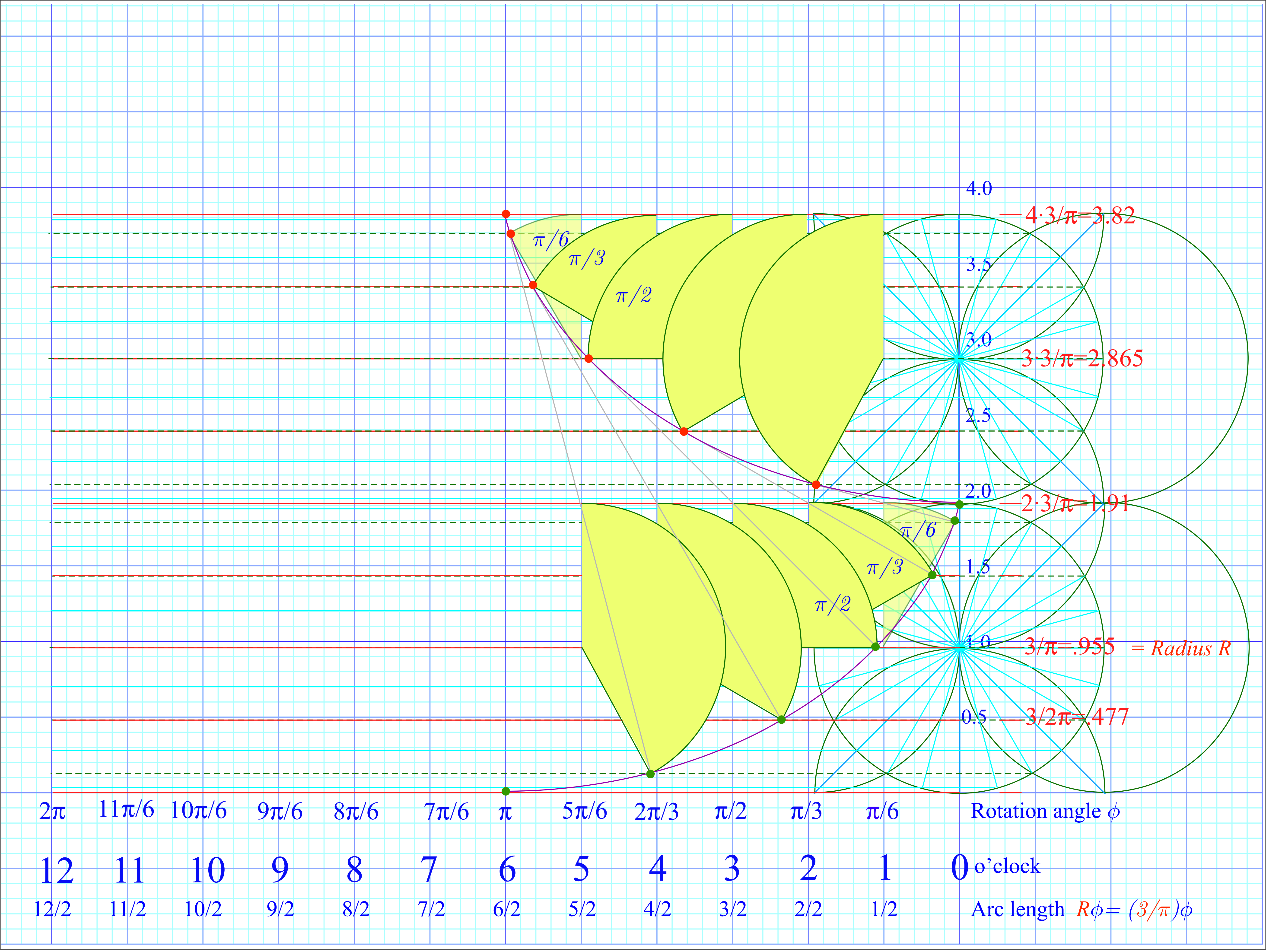
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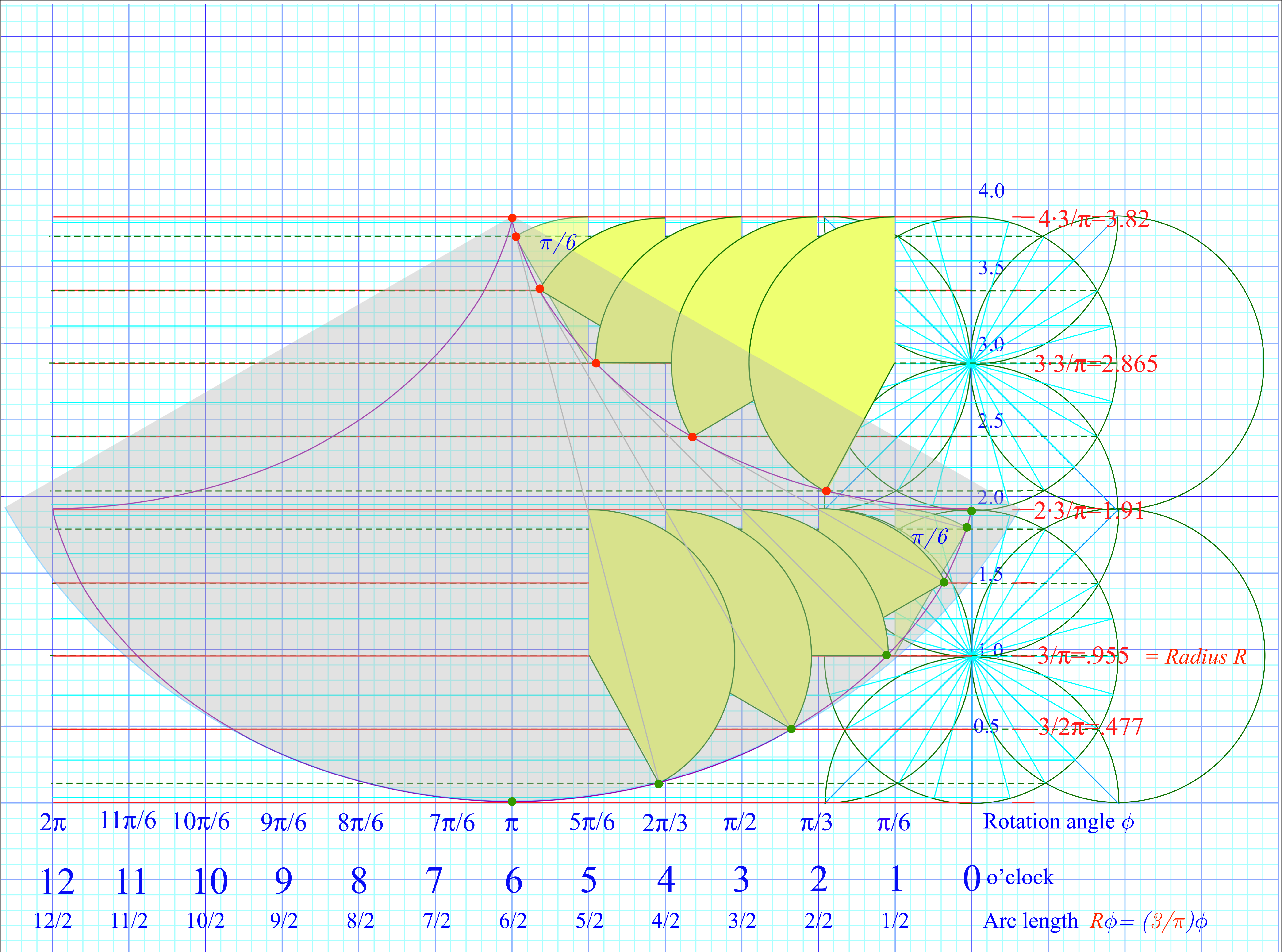

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The *brachistichrone* or minimum-time curve for a particle falling in a uniform gravitational potential.

Its solution gives that of another problem, the *tautochrone* or equal-time period curve of Huygens.

Energy conservation gives velocity v from gravitational g. Elapsed travel time t is to be minimized.

$$\frac{ds}{dt} = v = \sqrt{2gy}$$

$$t = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int dy \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \int L\, dy$$

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A "pseudo-momentum" p_x for "pseudo-Lagrange" L in y-integral is constant if L is x-independent.

$$p_x = const. = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \frac{x'}{\sqrt{2gy}\sqrt{1+x'^2}} = \frac{1}{y'\sqrt{2gy}\sqrt{1+1/y'^2}}$$ where: $x' = \frac{dx}{dy} = \frac{1}{y'}$

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$$\left(\frac{dv}{dx}\right)^2 = \frac{g^2}{v^2}\frac{1-p_x^2v^2}{p_x^2v^2} = \frac{g^2}{v^2}\frac{p_x^{-2}-v^2}{v^2} \text{ becomes: } \frac{dv}{dx} = \frac{g}{v^2}\sqrt{p_x^{-2}-v^2} \text{ and integral:} \int \frac{v^2 dv}{g\sqrt{a^2-v^2}} = \int dx \quad \text{where: } a^2 = p_x^{-2}$$

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An elementary integral results and suggests an elementary substitution $v=a\cos\theta$.

$$\int\frac{a^2\cos^2\theta\, a\sin\theta d\theta}{g a\sin\theta} = \int\frac{a^2}{g}\cos^2\theta\, d\theta = \int dx = \boxed{x = -\int \frac{a^2}{2g}(1+\cos 2\theta)d\theta = -R(2\theta+\sin 2\theta)} \quad \text{where: } R = \frac{a^2}{4g}$$

$$v^2 = 2gy = a^2\cos^2\theta \qquad \text{gives: } \boxed{y = \frac{a^2}{2g}\cos^2\theta = R(1+\cos 2\theta)}$$

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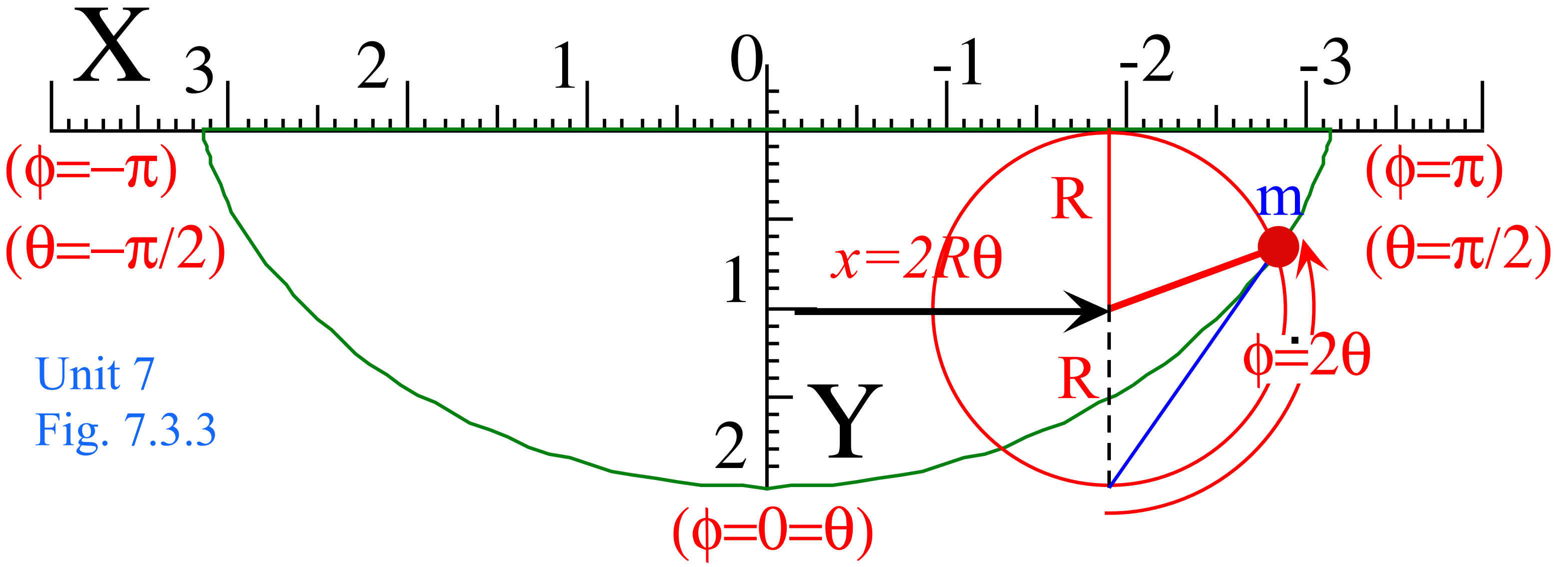


Unit 7
Fig. 7.3.3

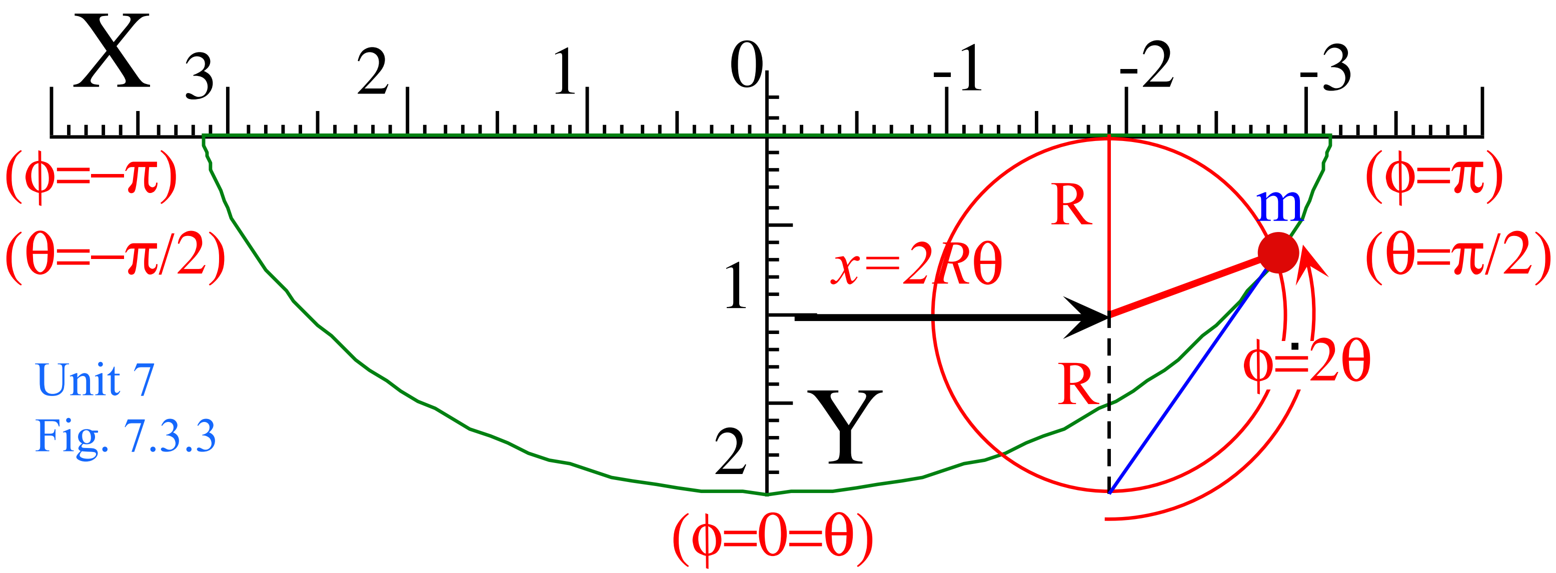
$$x = -R\left(2\theta + \sin 2\theta\right) \text{ where: } R = \frac{a^2}{4g} = \frac{p_x^{-2}}{4g}$$

$$y = R\left(1 + \cos 2\theta\right)$$

Some extraordinary properties of the cycloid are related to the constant p_x (pseudo-momentum)

$$p_x = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'}\frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \frac{x'}{\sqrt{2gy}\sqrt{1+x'^2}} = \frac{1}{\sqrt{2gy}\sqrt{y'^2+1}} \text{ where: } x' = \frac{dx}{dy} = \frac{1}{y'} \text{ and: } p_x^2 = \frac{1}{4Rg}$$

$$\frac{1}{p_x^{\ 2}} = const. = 2gy\left(y'^2+1\right) = v^2 \sec^2\theta = a^2$$



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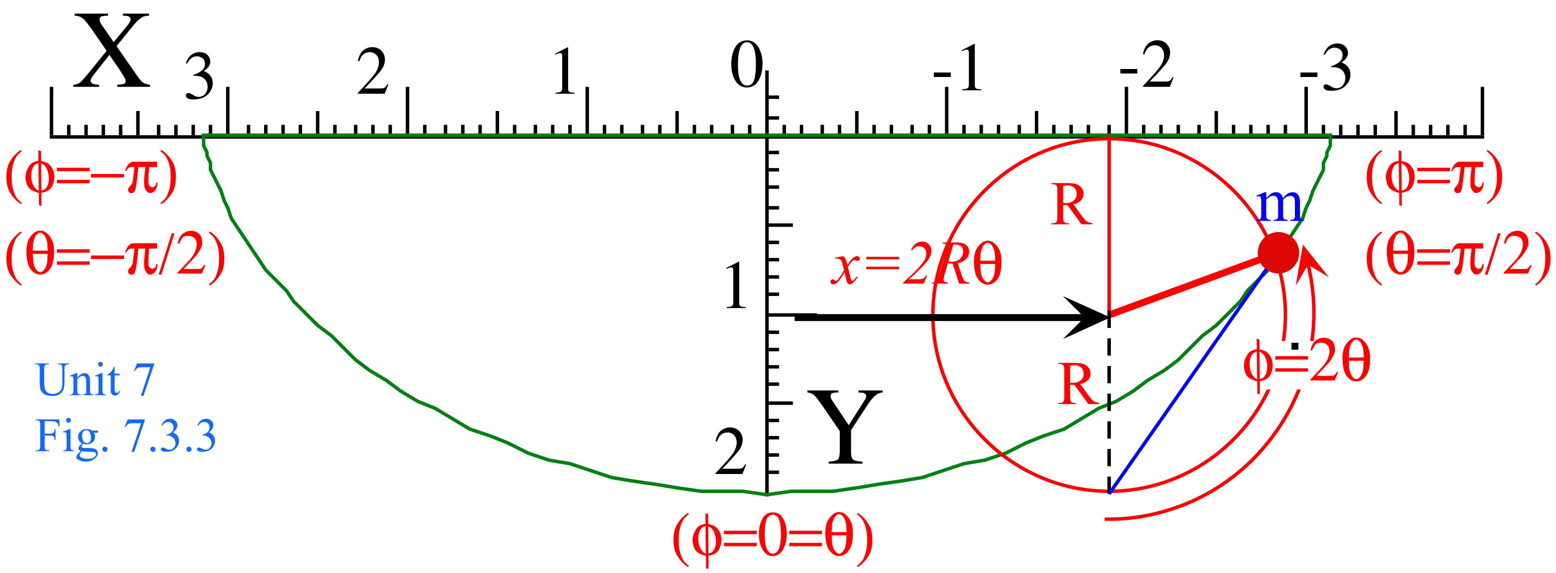
t-derivatives of (x,y) give v vs $\phi = 2\theta$: $v^2 = \dot{x}^2 + \dot{y}^2 = \dot{\phi}^2\left[\left(R + R\cos\phi\right)^2 + \left(-R\sin\phi\right)^2\right] = 2R\dot{\phi}^2\left(1+\cos\phi\right) = 4R^2\dot{\phi}^2\cos^2\theta$



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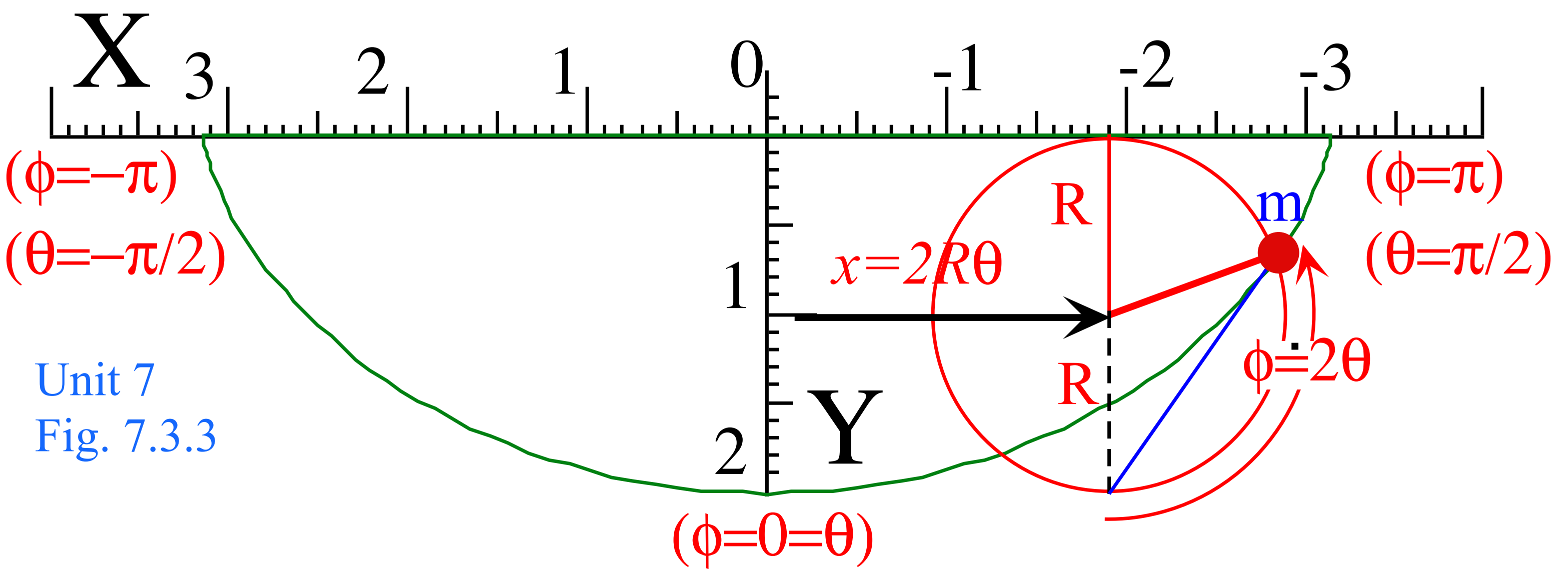
The circle starting at $\phi=\pi=2\theta$ turns at a constant rate $\dot{\phi}=\omega$ and moves at a constant velocity $v=\omega R$.

$$\frac{1}{p_x} = a = \sqrt{4gR} = 4R\dot{\phi} = 8R\dot{\theta} \text{ or: } \omega = \dot{\phi} = \sqrt{\frac{g}{4R}}$$



Unit 7
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$$\frac{1}{p_x} = a = \sqrt{4gR} = 4R\dot{\phi} = 8R\dot{\theta} \quad \text{or:} \quad \omega=\dot{\phi} = \sqrt{\frac{g}{4R}}$$

This relates to the arc length of the cycloid from bottom $(\theta=0)$ to a point at angle $\theta<\pi/2$ or $\phi<\pi$.

$$s = \int_0^t v\,dt = \int_0^t 2R\omega\cos\theta\,dt = \int_0^\theta 2R\left(\omega/\dot{\theta}\right)\cos\theta\,d\theta = 4R\sin\theta$$

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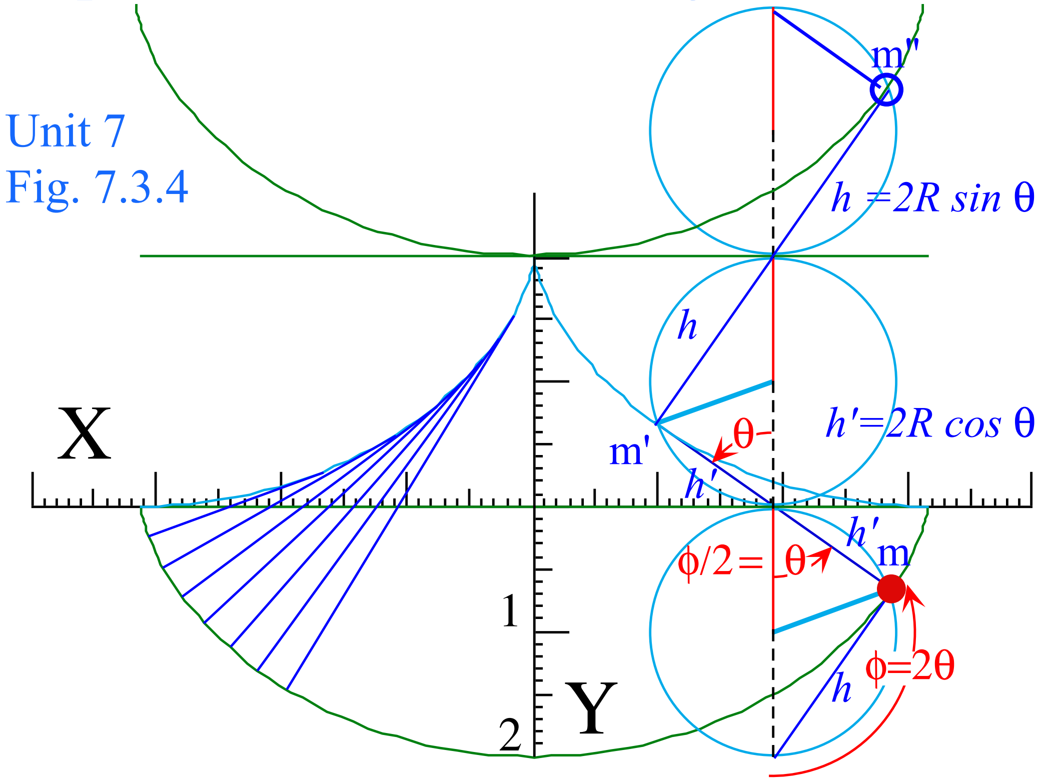
Arc length *s* is indicated by a segment *hh* of length $2h=4R\sin\theta$ left hand Fig. 7.3.4 below. That is precisely the length of unwound string between points m′ and m″, and between points m′ and m, is a segment $h'h'$ of length $2h'=4R\cos\theta$ unwound from middle cycloid.

Unit 7
Fig. 7.3.4



Unit 7
Fig. 7.3.5

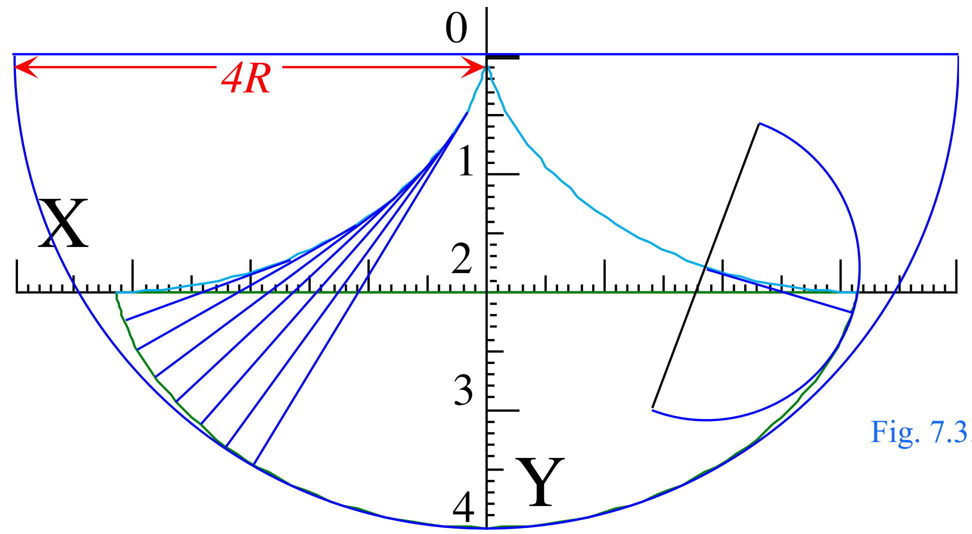
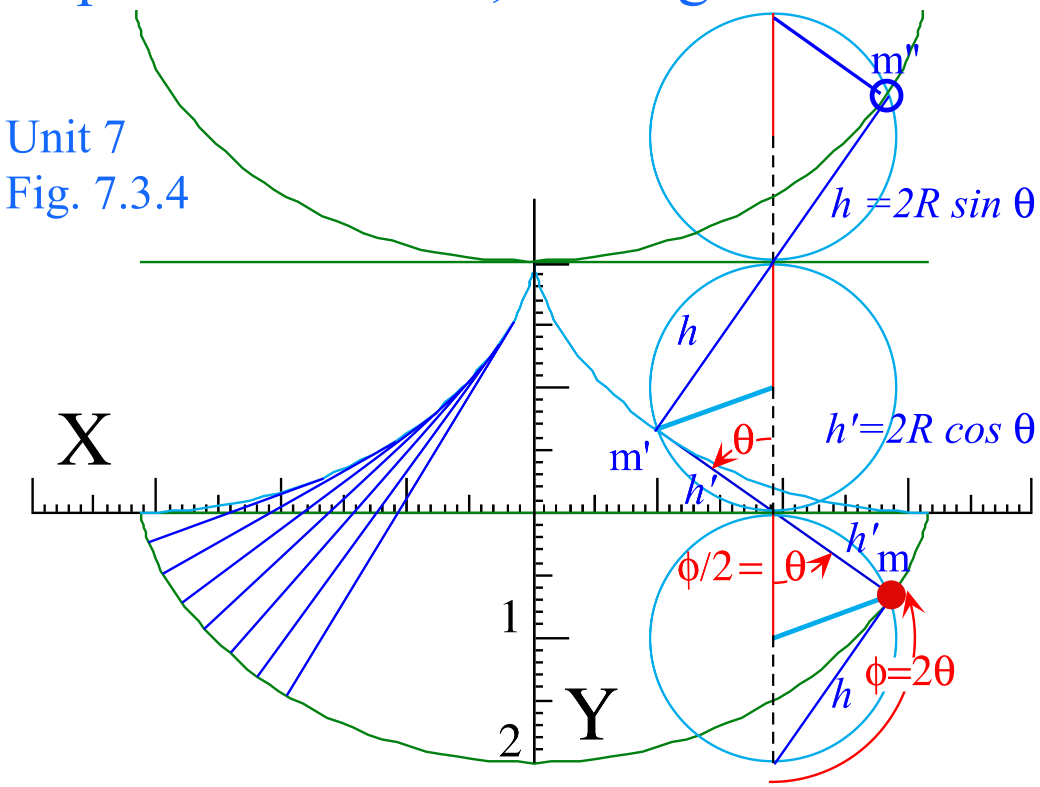


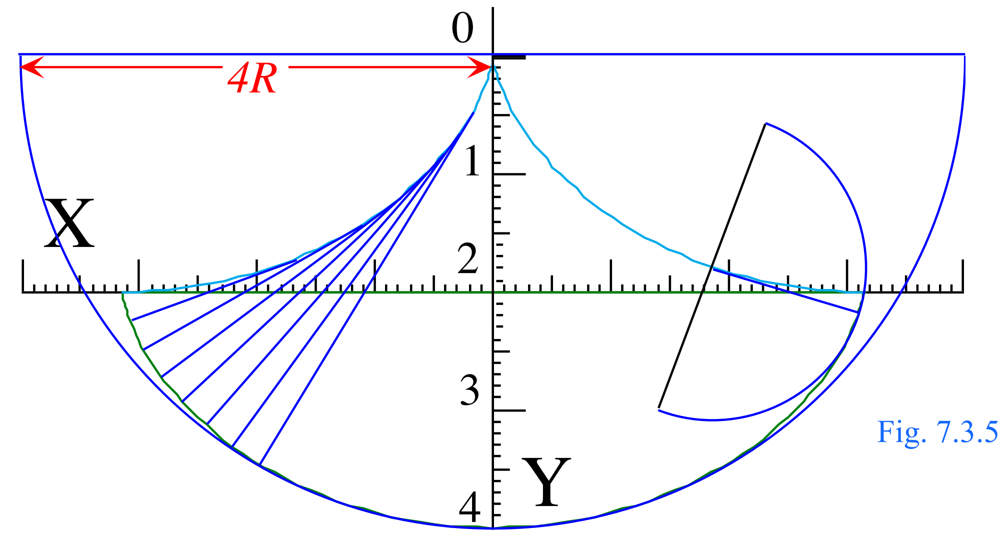
Fig. 7.3.5

Arc length *s* is indicated by a segment *hh* of length $2h=4R\sin\theta$ left hand Fig. 7.3.4 below. That is precisely the length of unwound string between points m′ and m″, and between points m′ and m, is a segment *h′h′* of length $2h'=4R\cos\theta$ unwound from middle cycloid.

Unit 7
Fig. 7.3.4



Unit 7
Fig. 7.3.5



Segment *hh* is the *radius of curvature* $r_C(m') = 2h = 4R\sin\theta$ of the *m′* cycloid and the points *m′* or *m″* are *centers of curvature* for circular arcs around unwinding points *m″* or *m′*, respectively.

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Unit 7
Fig. 7.3.4

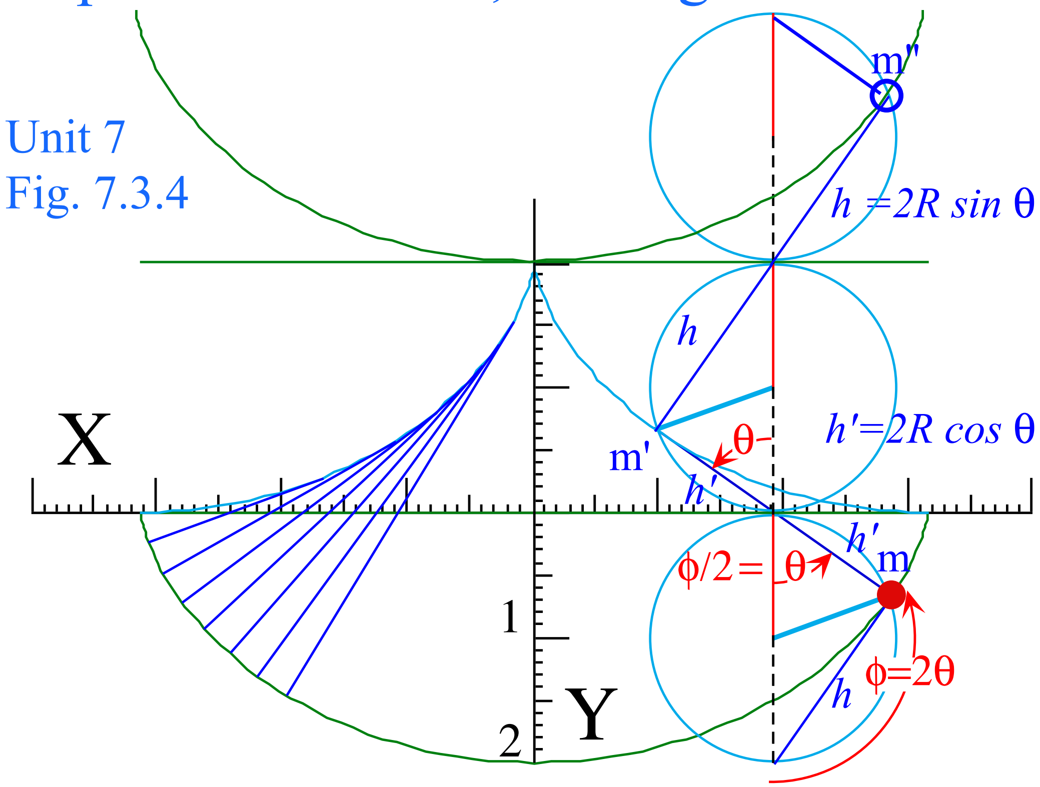

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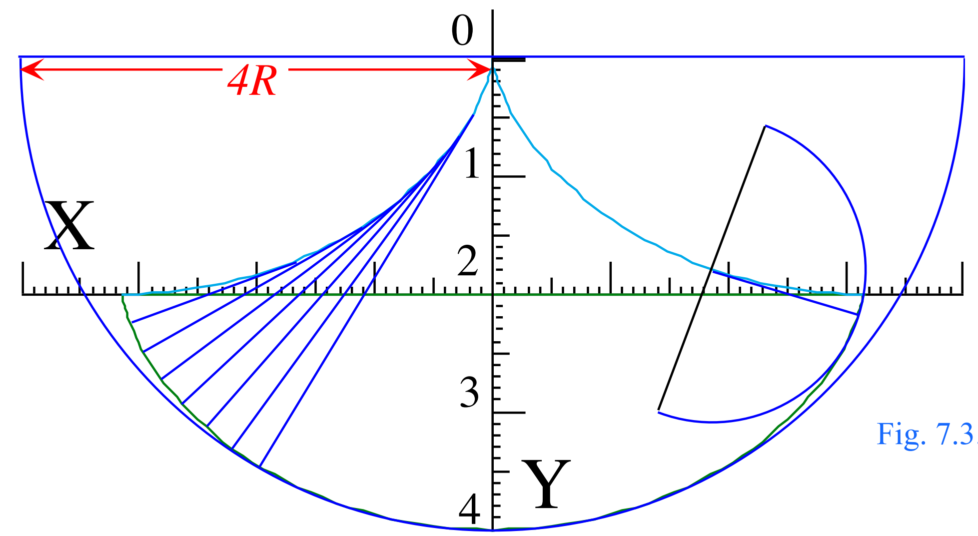
Fig. 7.3.5



Segment hh is the *radius of curvature* $r_C(m') =2h=4R\sin\theta$ of the m' cycloid and the points m' or m'' are *centers of curvature* for circular arcs around unwinding points m'' or m', respectively.

Three wheels roll synchronically on their respective ceilings. As point m approaches the top of its cycloid, point m' approaches m so that curvature becomes infinite.($k=1/r_C\rightarrow\infty$ as $\theta\rightarrow\pi/2$.)

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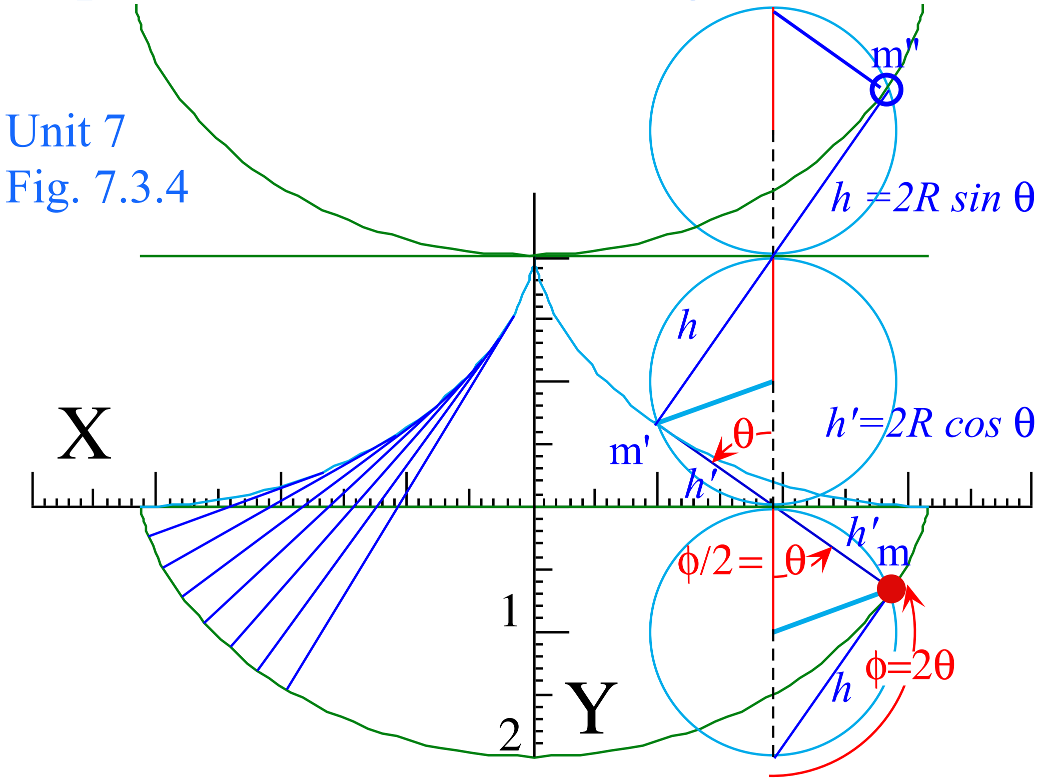

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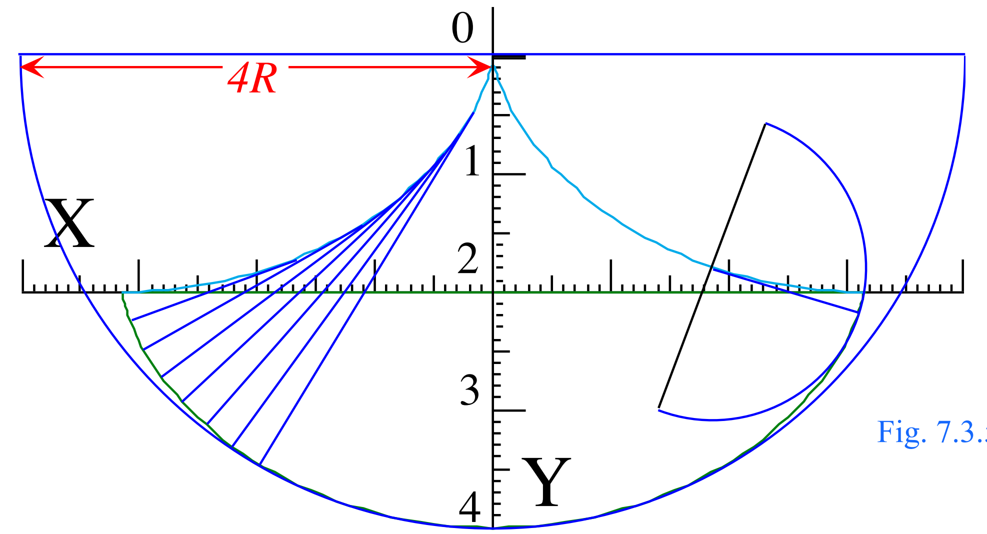

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Figure 7.3.5 shows circular arcs fitting a cycloid. The largest arc and one with the least curvature $k_C = 1/(4R)$ is a circle of radius $r_C = 4R$ that surrounds the entire cycloid. This is the path of a simple circular pendulum. The figure shows that the circle deviates only slightly from the cycloid with the greatest deviation near the tips of the cycloid where curvature blows up.

Separation of GCC Equations: Effective Potentials

- *Small radial oscillations*
  - *2D Spherical pendulum or "Bowl-Bowling"*
- *Cycloidal ruler&compass geometry*
  - *Cycloid as brachistichrone*
  - *Cycloid as tautochrone*
    - *Cycloidulum vs Pendulum*
- *Cycloidal geometry of flying levers*
  - *Practical poolhall application*

The cycloid path has the unique ability to guarantee the same frequency $\omega = \sqrt{(g/4R)}$ for any amplitude θ_0 of oscillation within the range $\{-\pi/2<\theta_0<\pi/2\}$ between cycloid tips.

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The circular pendulum frequency $\omega = \sqrt{(g/\ell)}$ holds only for small amplitudes $\theta<<1$.

The time integral below varies with θ_0 in the range $\{-\pi/2<\theta_0<\pi/2\}$.

$$t_{1/4} = \int_{s_0}^{0} \frac{ds}{\sqrt{2g(y-y_0)}} = \int_0^{\theta_0} \frac{4R\cos\theta\, d\theta}{\sqrt{2gR(\cos 2\theta - \cos 2\theta_0)}} = \sqrt{\frac{4R}{g}} \int_0^{\theta_0} \frac{\cos\theta\, d\theta}{\sqrt{\sin^2\theta_0 - \sin^2\theta}}$$

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Arc length $s=4R\sin\theta$ and cycloid height $y=R(1+\cos 2\theta)$ are used above.
To finish integral for a 1/4-period we set: $\sin\theta = \sin\theta_0 \sin\alpha$ below.

$$t_{1/4} = \sqrt{\frac{4R}{g}} \int_0^{\alpha=\pi/2} \frac{\sin\theta_0 \cos\alpha\, d\alpha}{\sin\theta_0 \sqrt{1-\sin^2\alpha}} = \frac{\pi}{2}\sqrt{\frac{4R}{g}}$$

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A cycloid has a full period of $t_1=2\pi\sqrt{\ell/g}$ for <u>*all*</u> θ_0. Even for large θ_0 the "cycloidulum" matches the period of a simple circular $(\ell=4R)$-pendulum at small θ_0.

SSeparation of GCC Equations: Effective Potentials

Small radial oscillations

2D Spherical pendulum or "Bowl-Bowling"

Cycloidal ruler&compass geometry

Cycloid as brachistichrone

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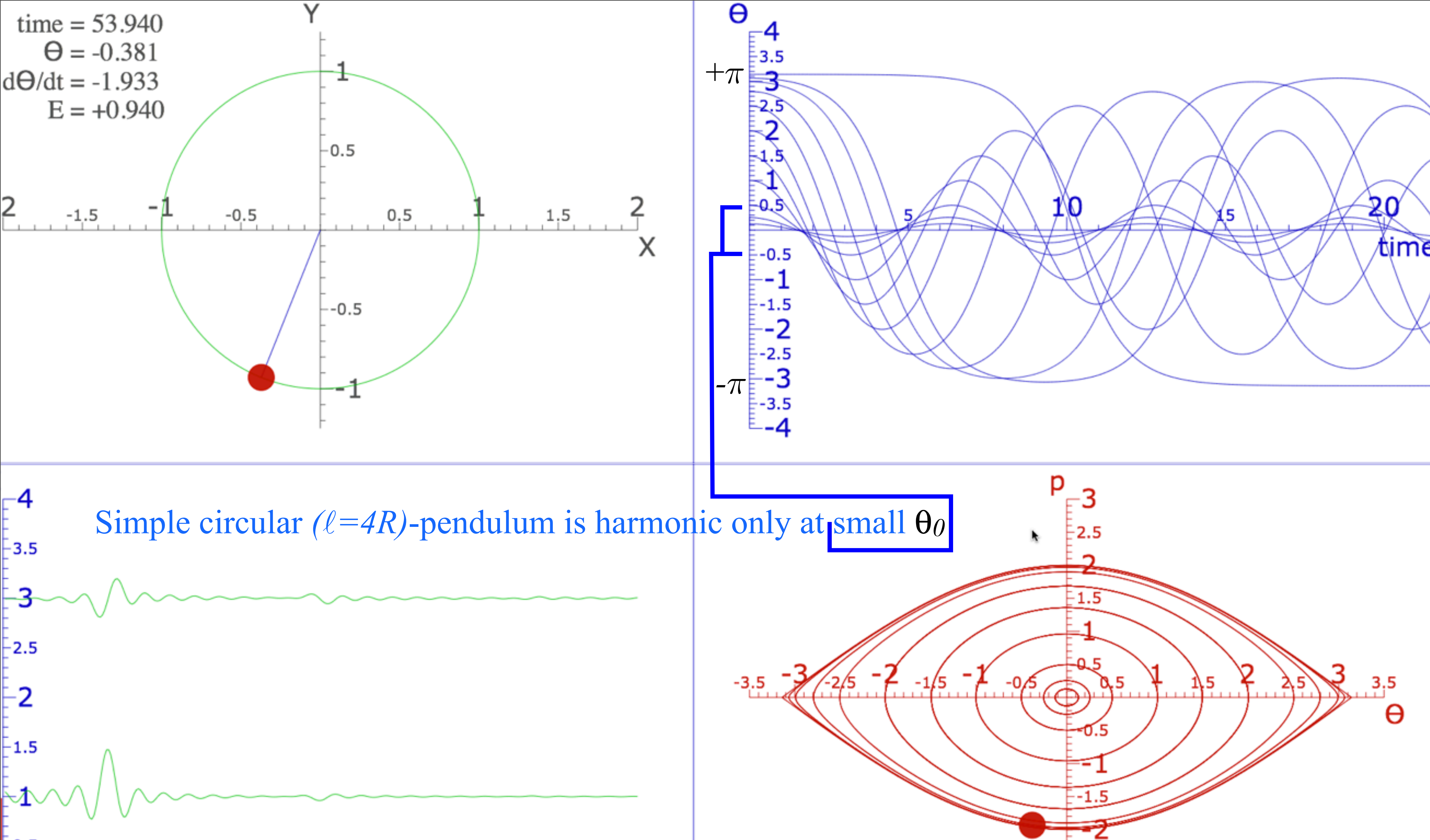
→ *Cycloidulum vs Pendulum*

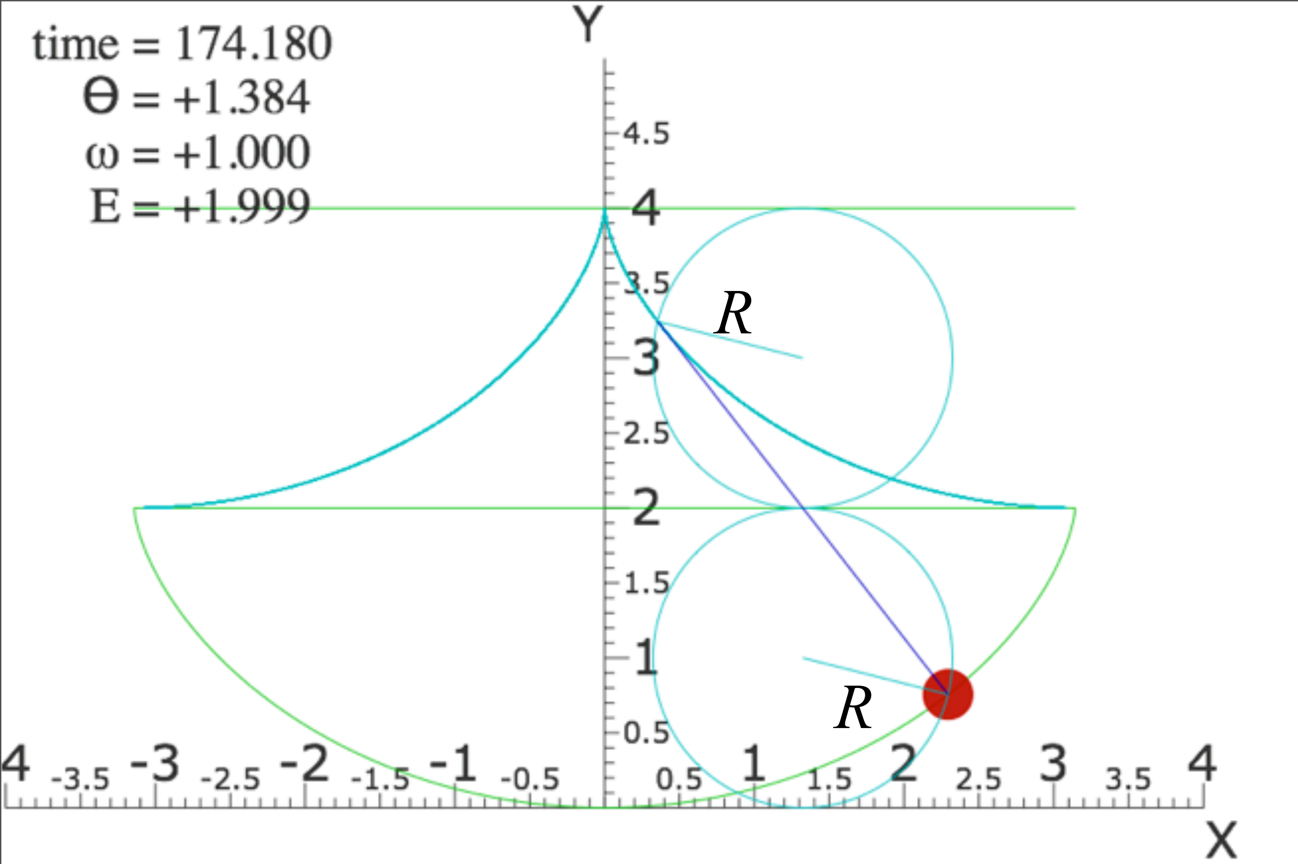
Cycloidal geometry of flying levers

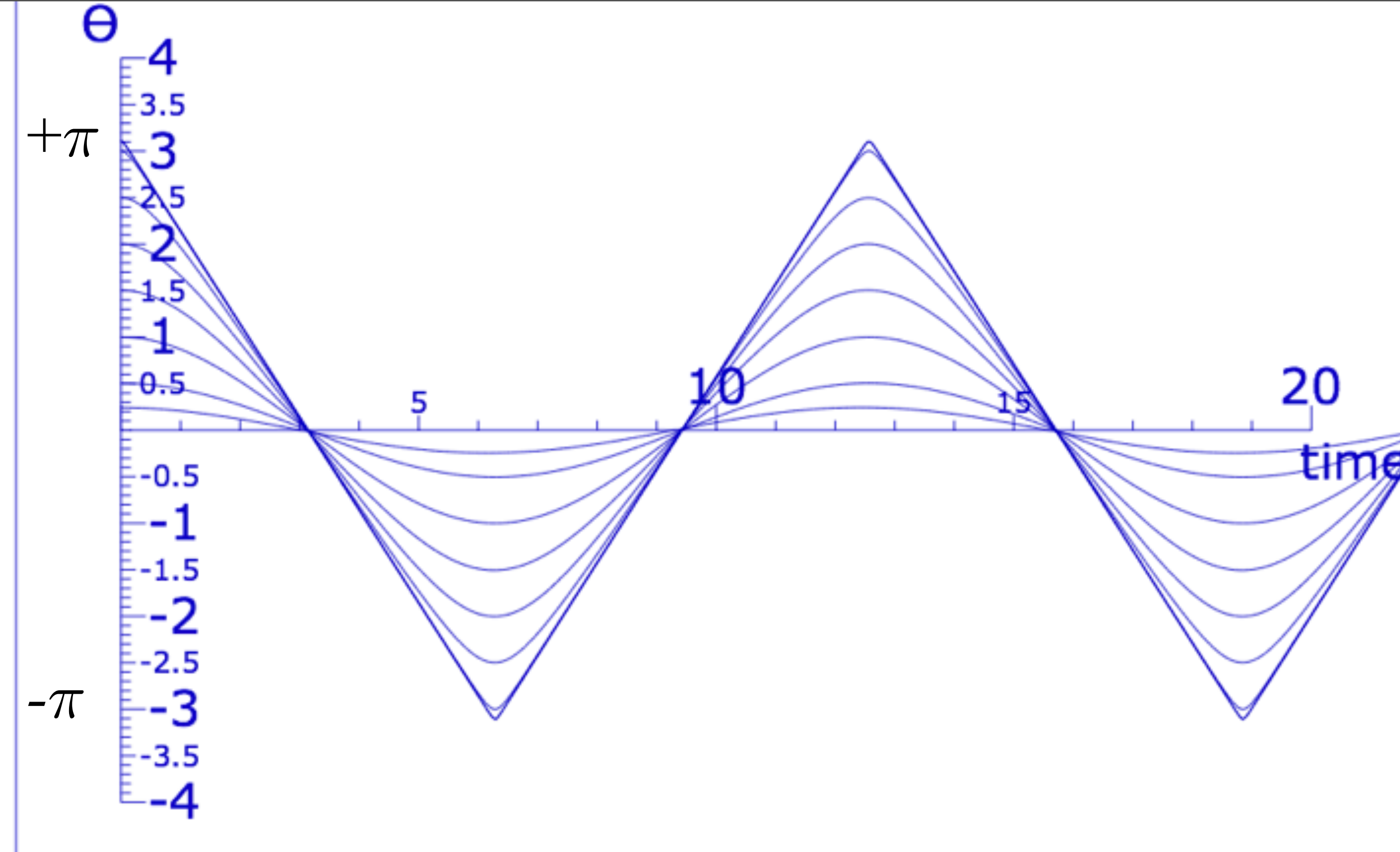
Practical poolhall application



Simple circular *(ℓ=4R)*-pendulum is harmonic only at small θ_0

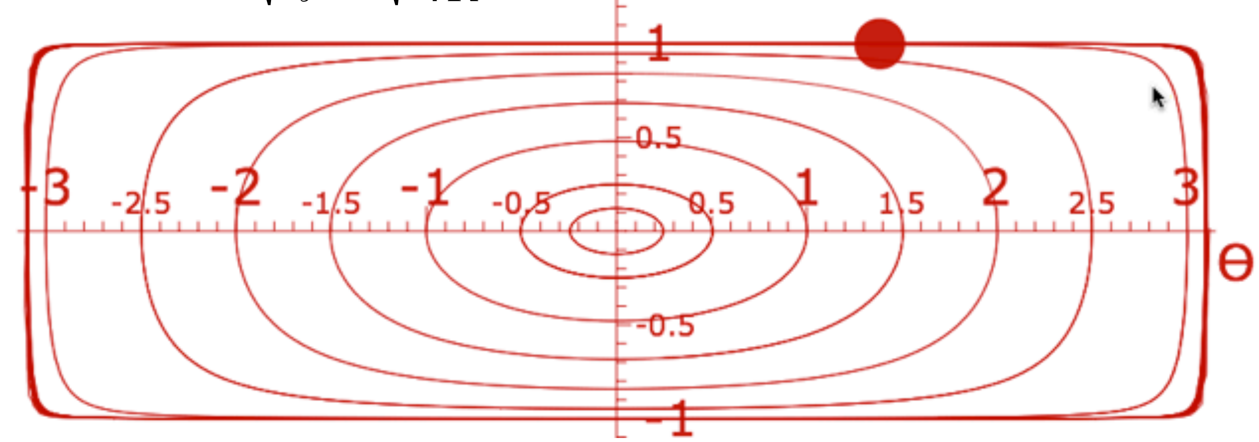
http://www.uark.edu/ua/modphys/markup/PendulumWeb.html





Huygen's *Cycloidulum* ($\ell=4R$)-is harmonic at all θ_0 in range -π to +π.

Angular frequency is exactly: $\omega = \sqrt{\frac{g}{\ell}} = \sqrt{\frac{g}{4R}}$



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Separation of GCC Equations: Effective Potentials

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Practical poolhall application

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h{\cdot}\Pi$

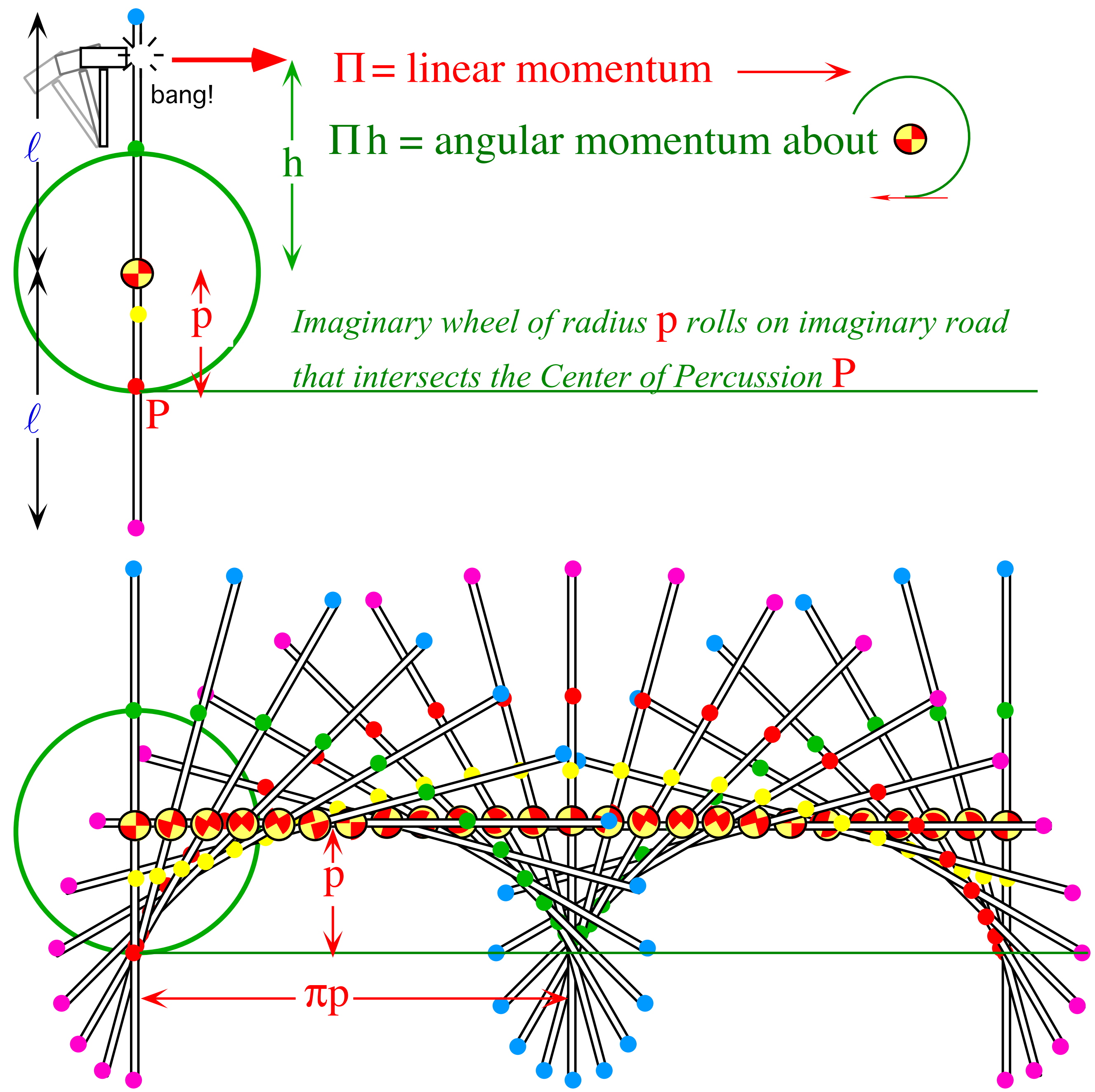

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
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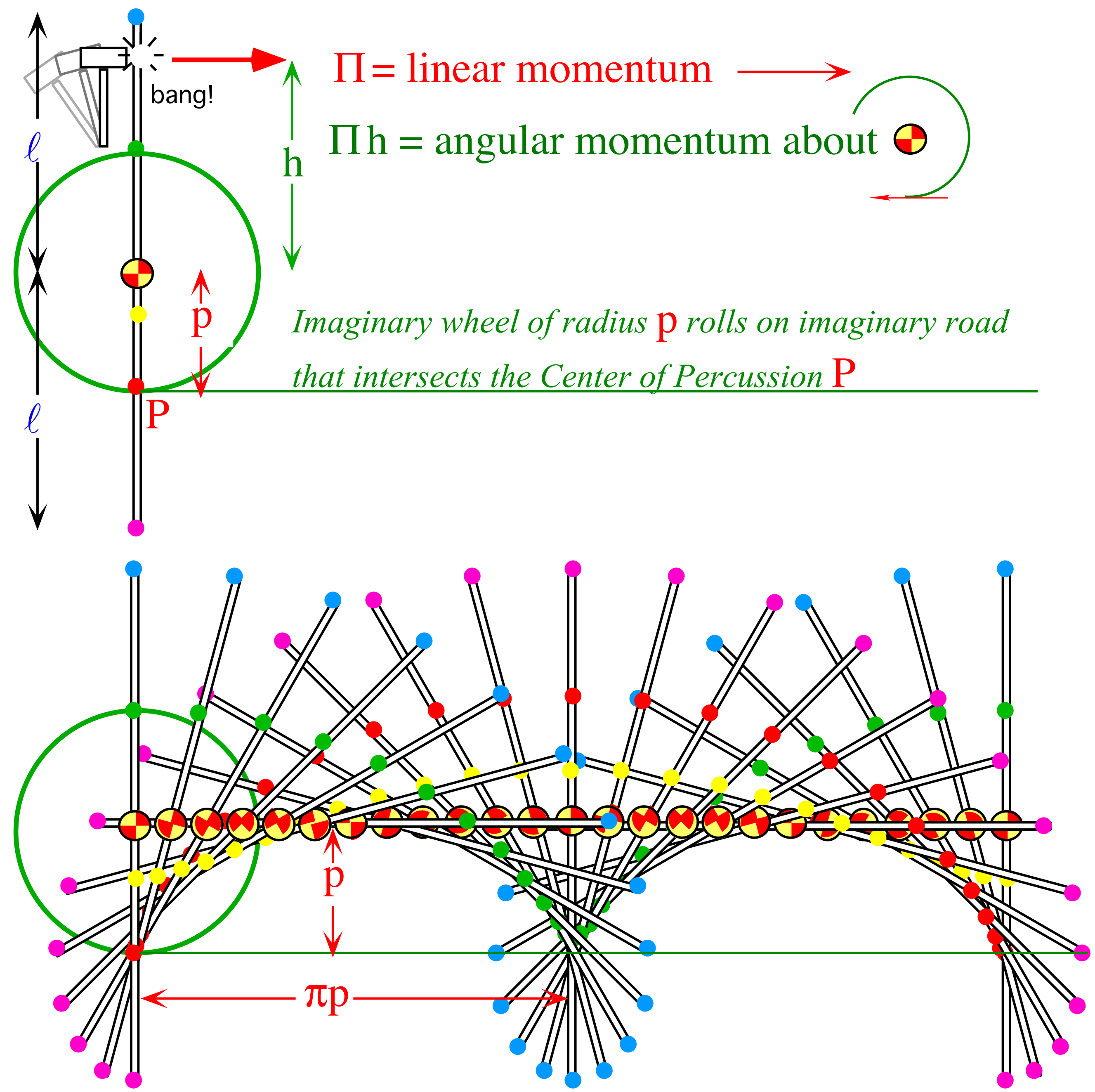
Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.



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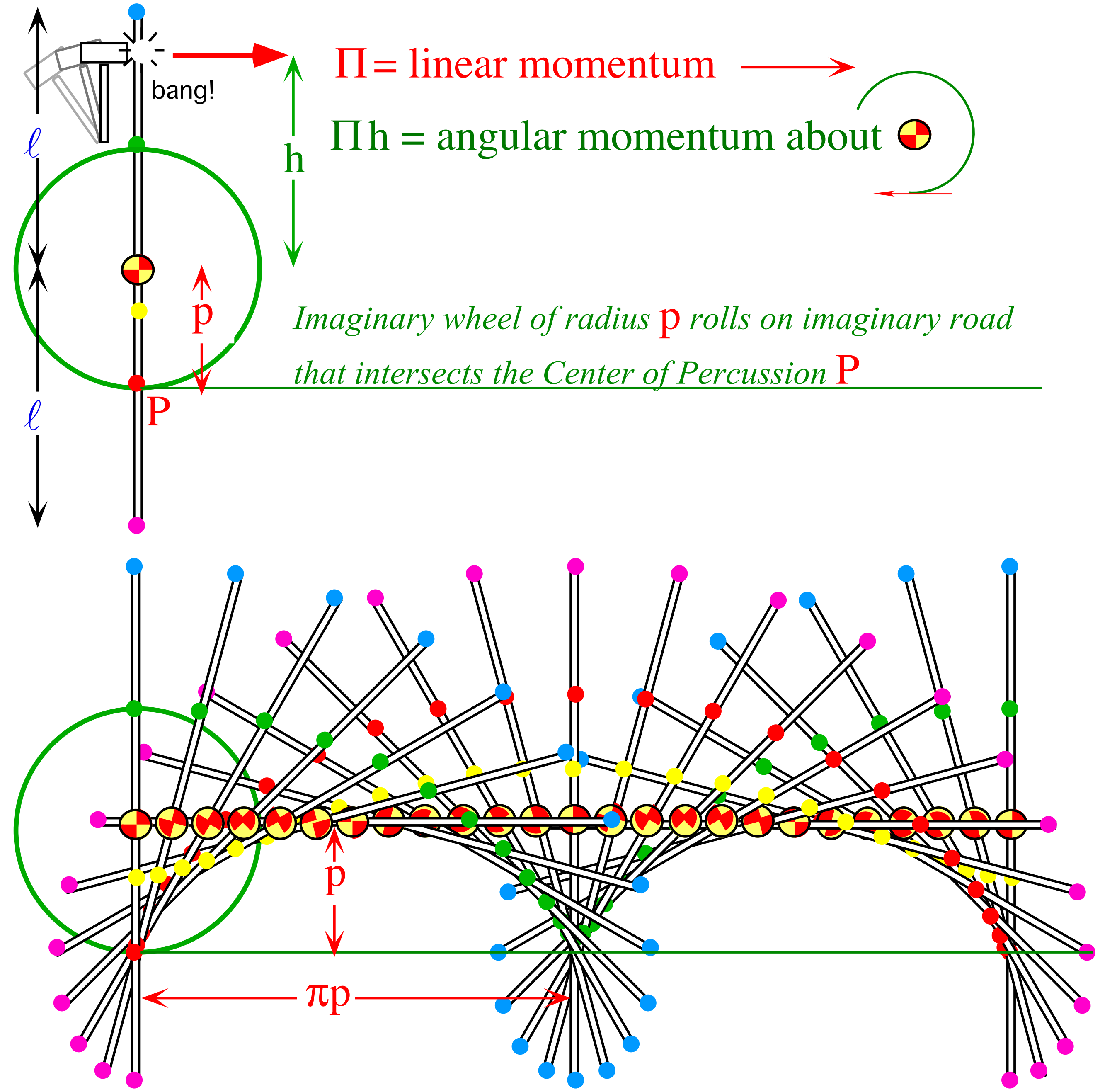

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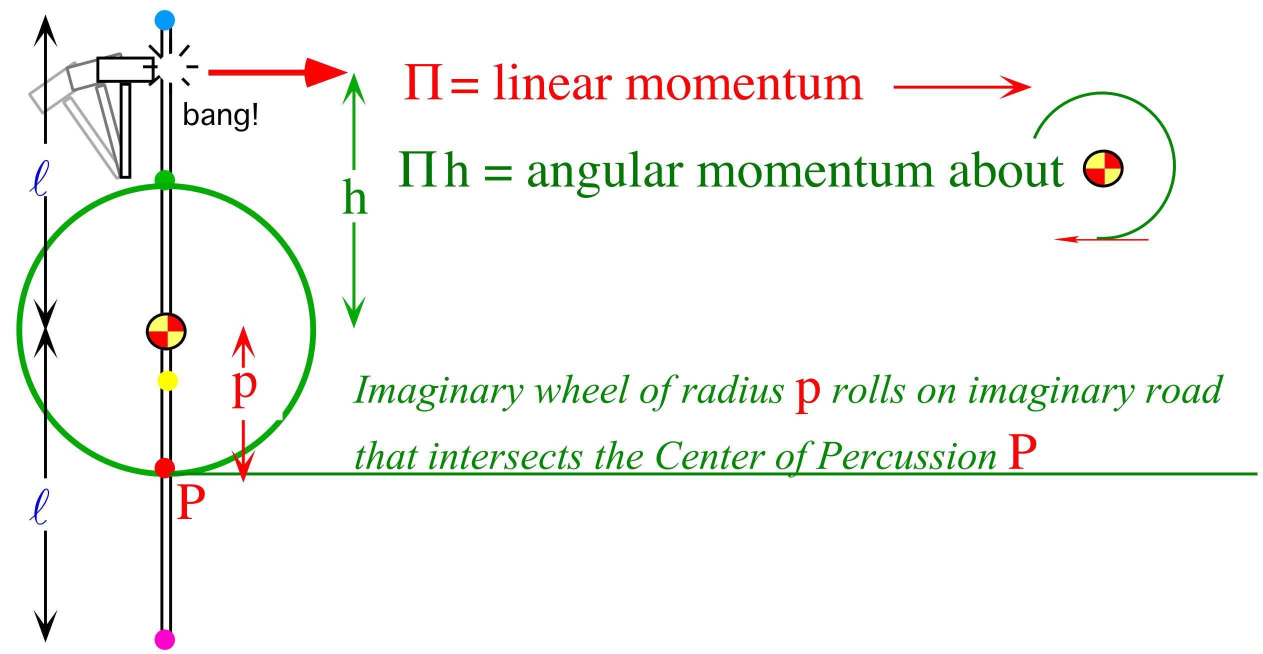

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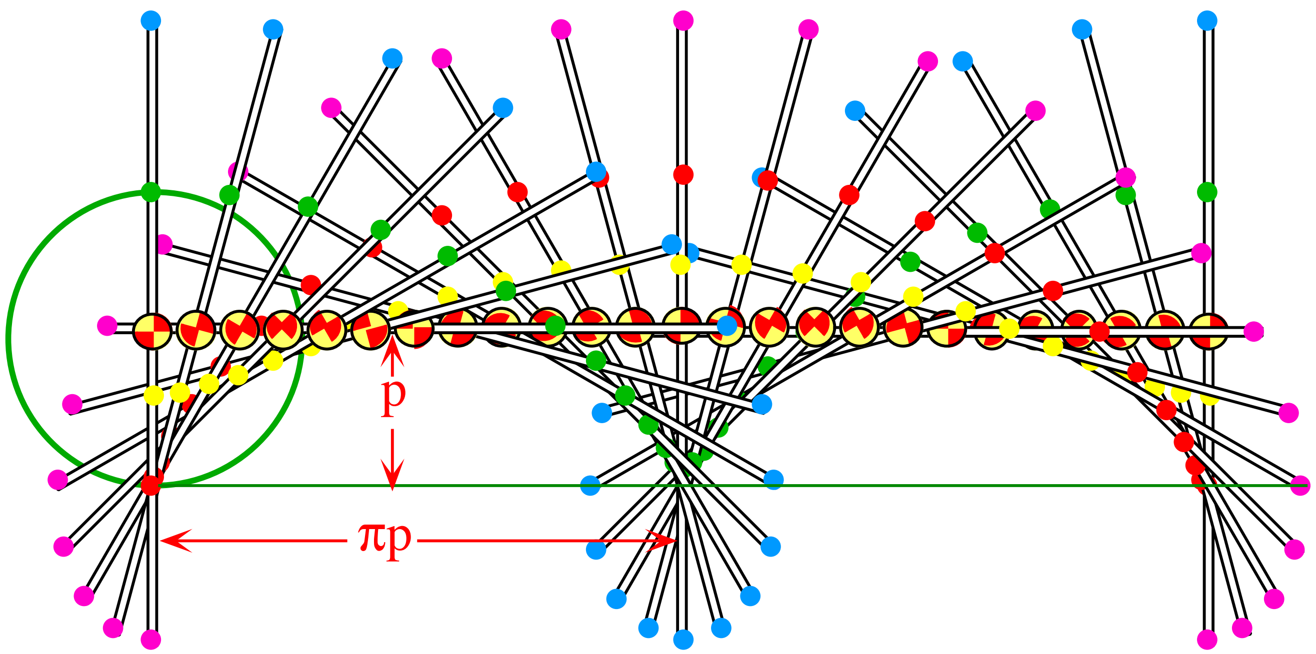
One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.



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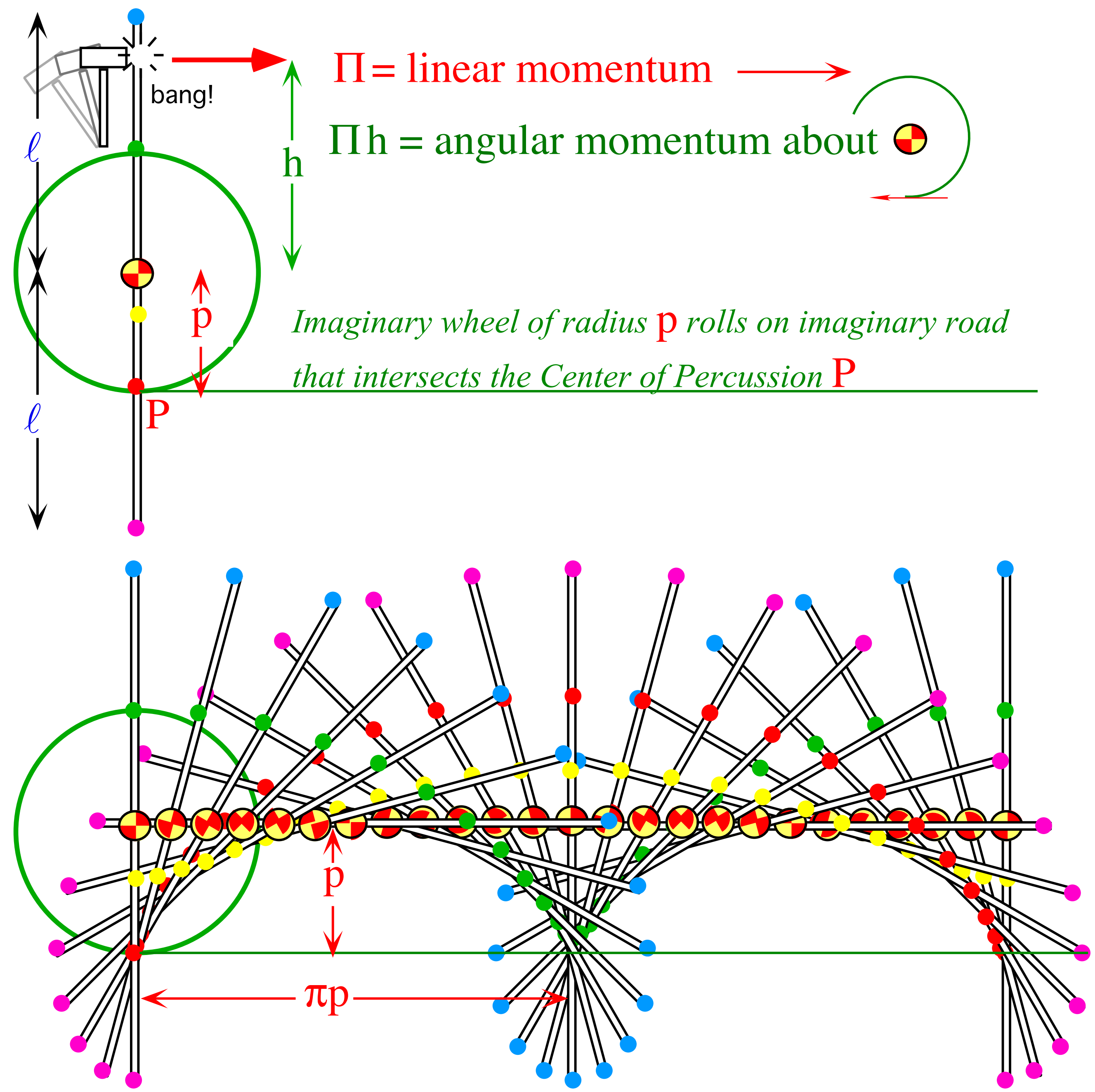

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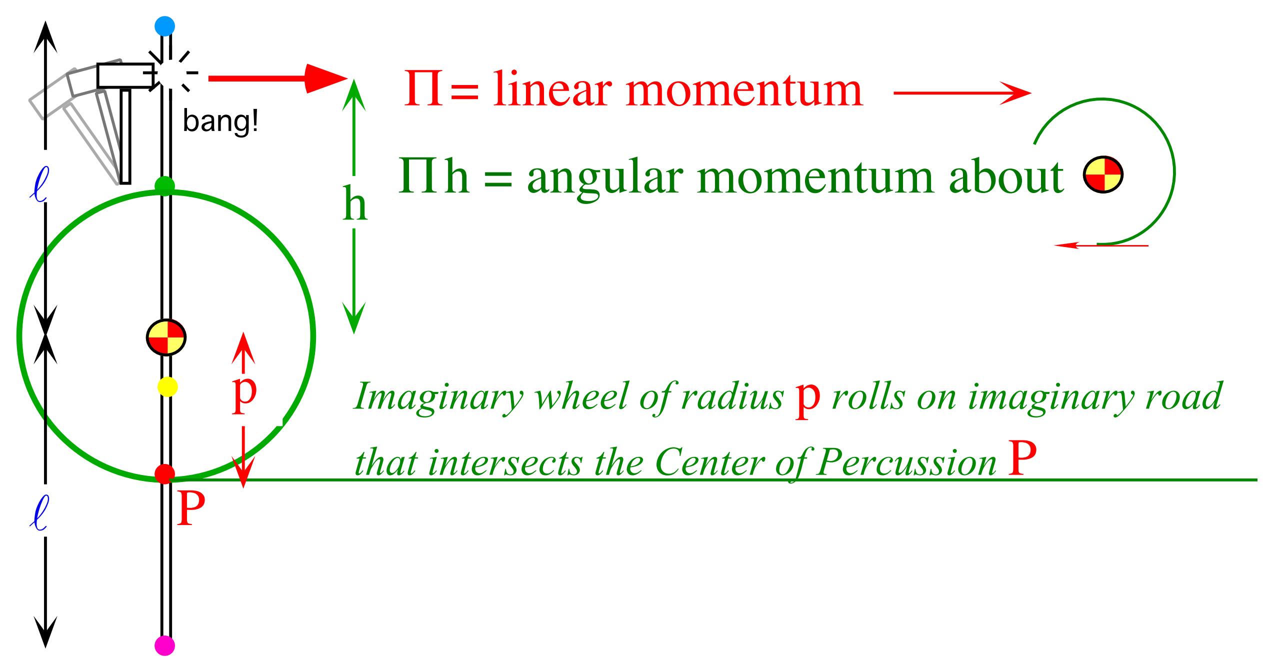

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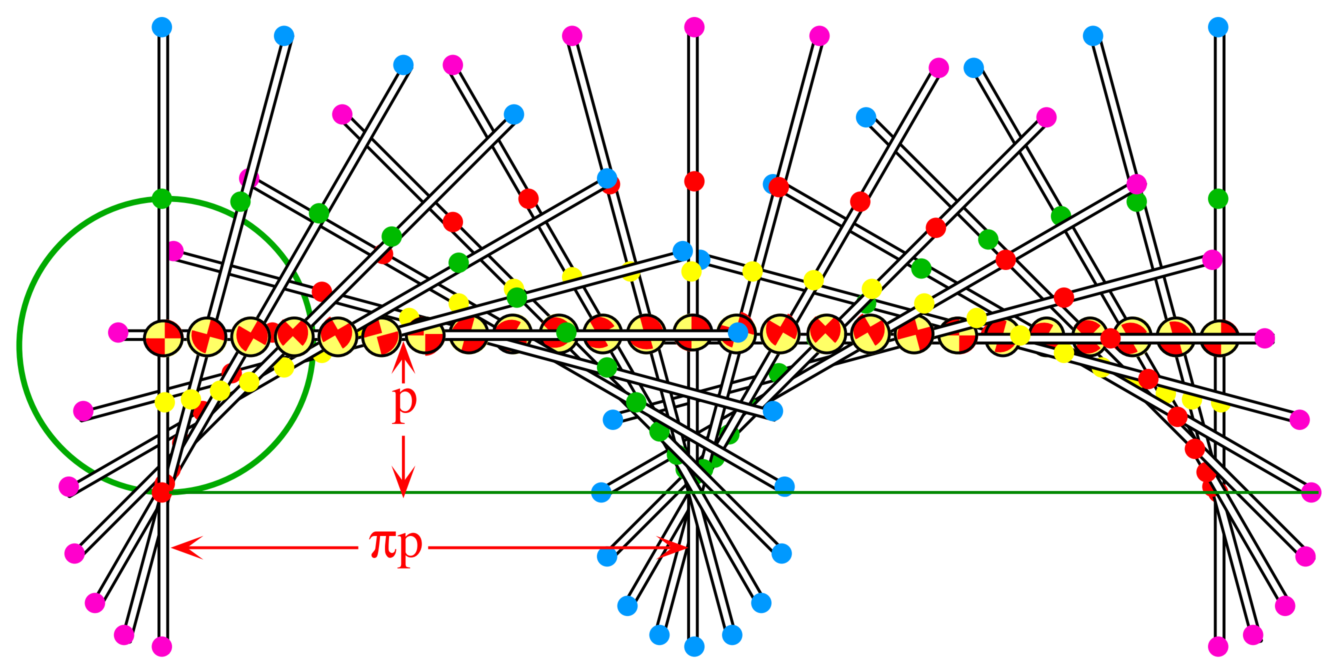

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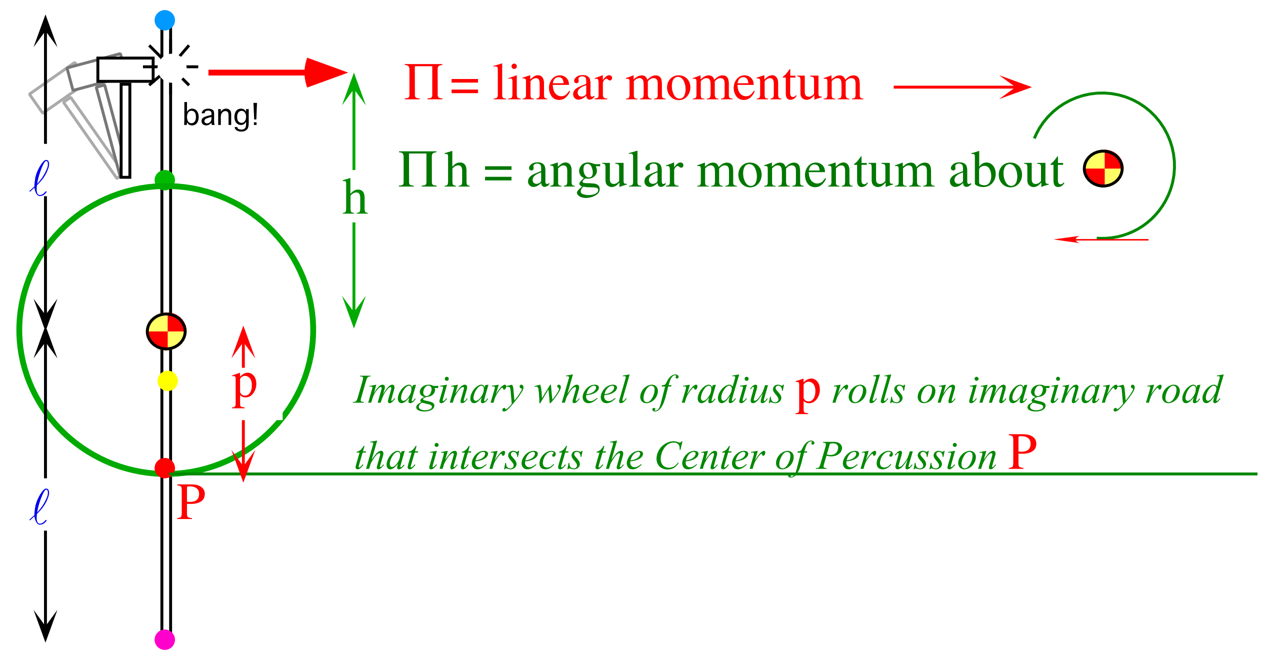

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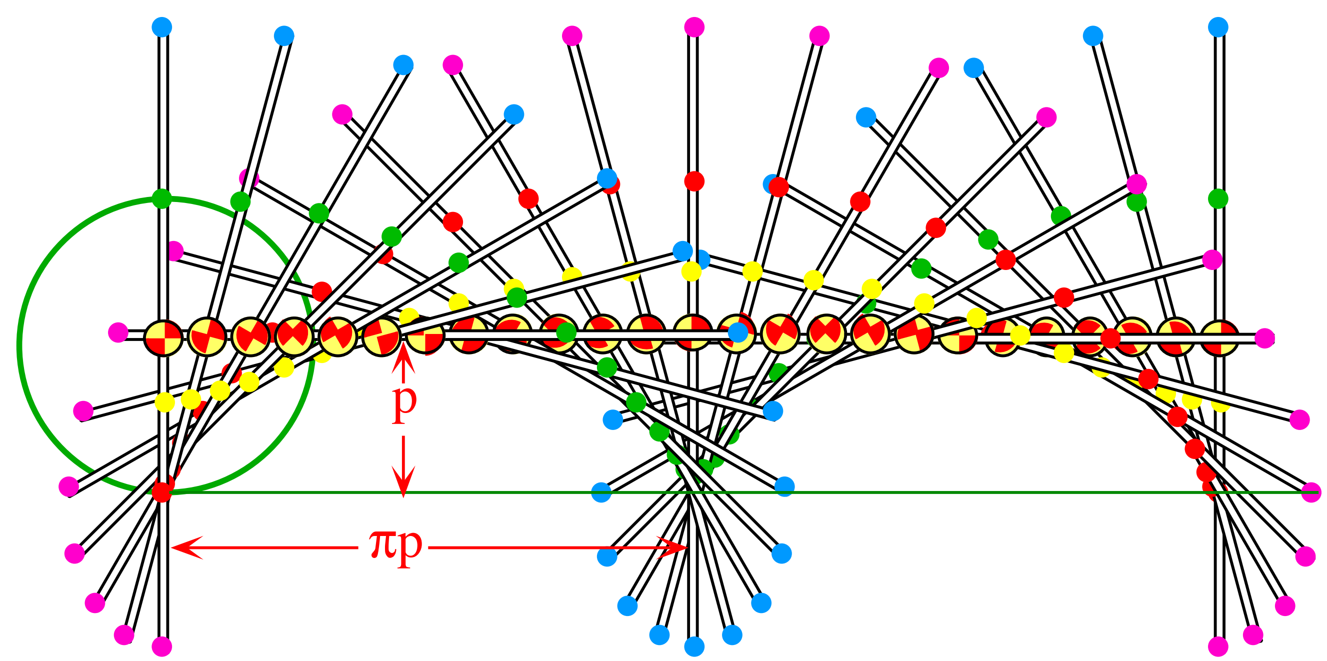

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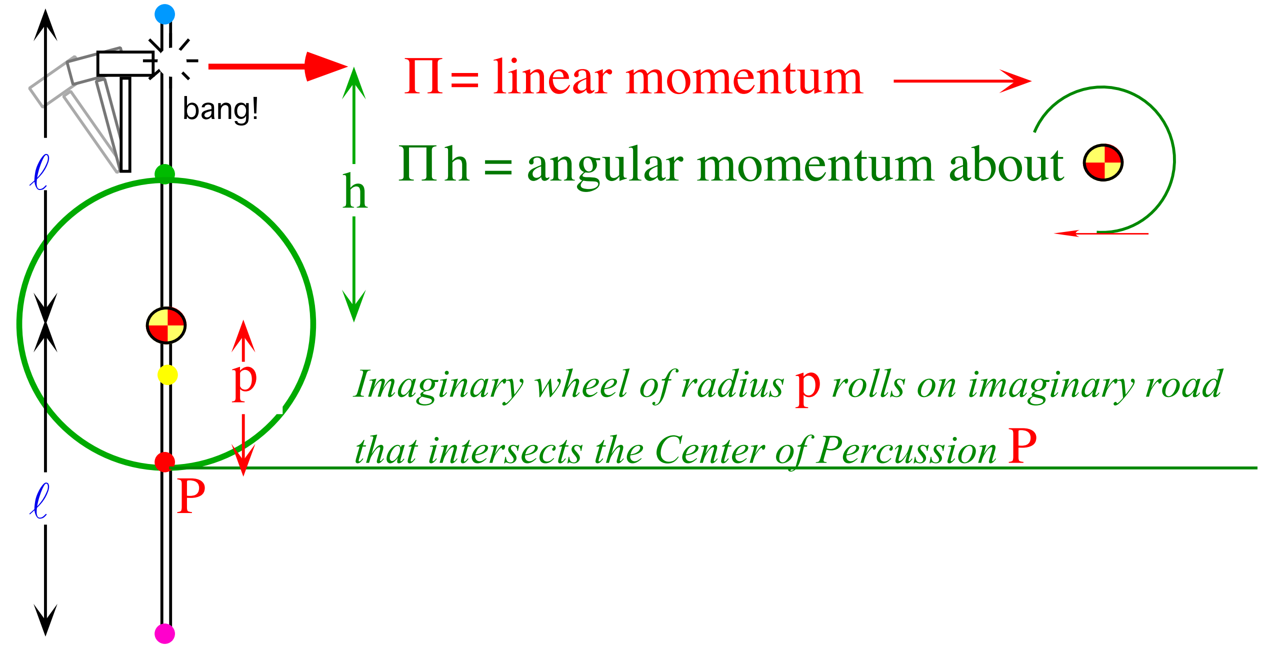

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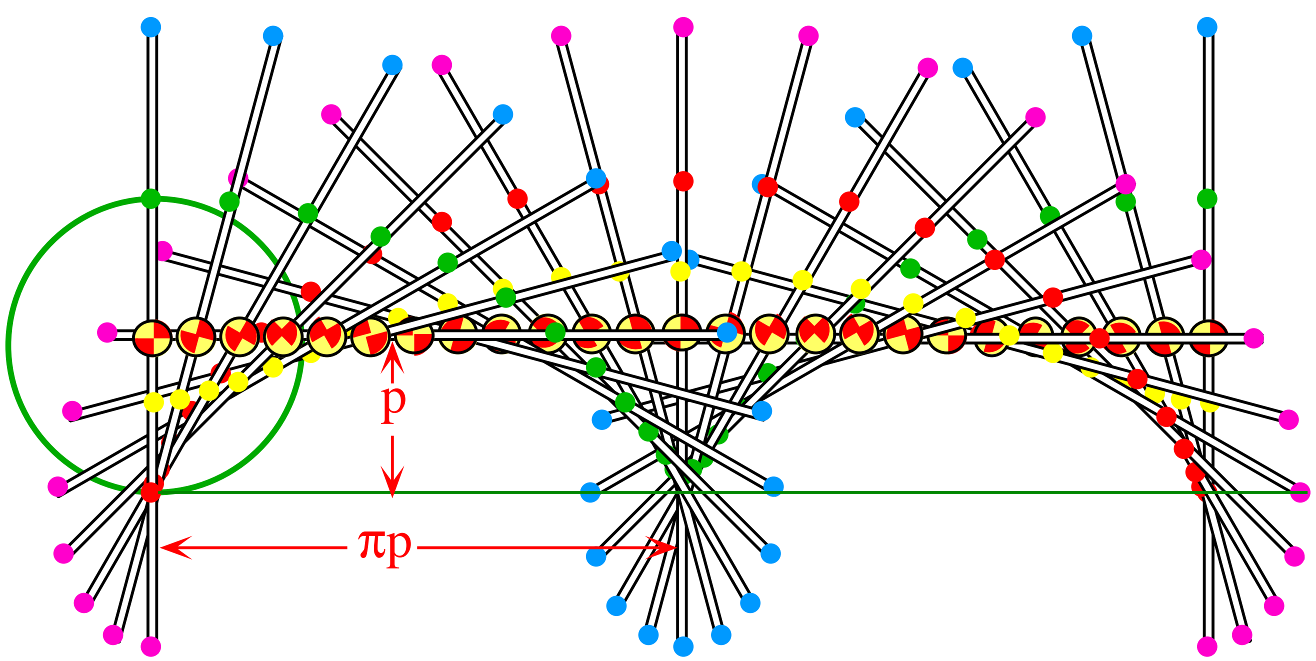
$I / M = \quad = \quad = p \cdot h \qquad$ *or:* $p=I/(Mh)$

P follows a normal cycloid made by a circle
of radius $p=I/(Mh)$ rolling on an imaginary road
thru point P in direction of Π.



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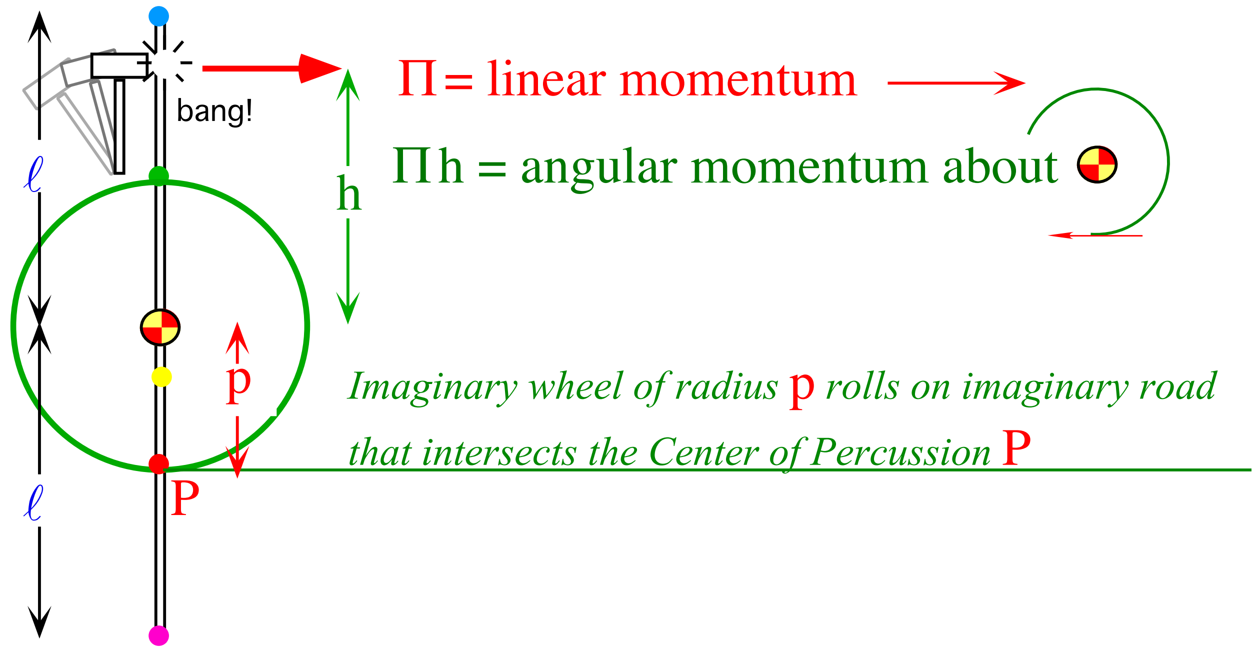

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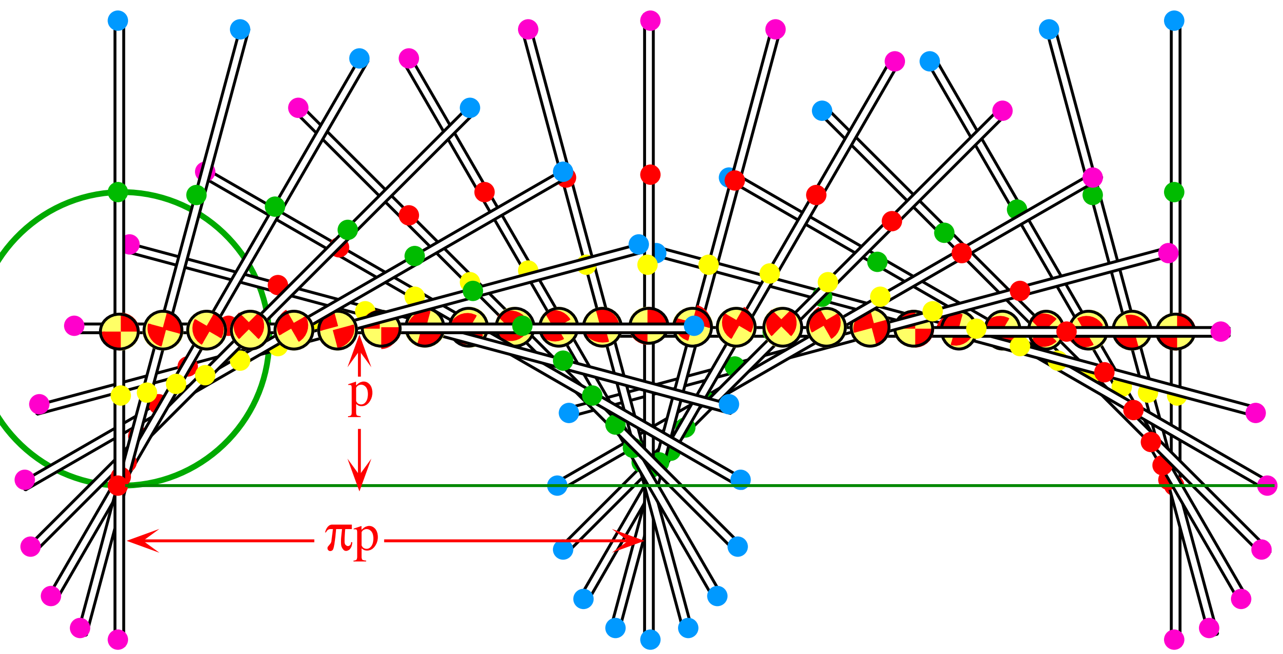

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

The *percussion radius* $p = \ell^2/3h$ is of the CoP point
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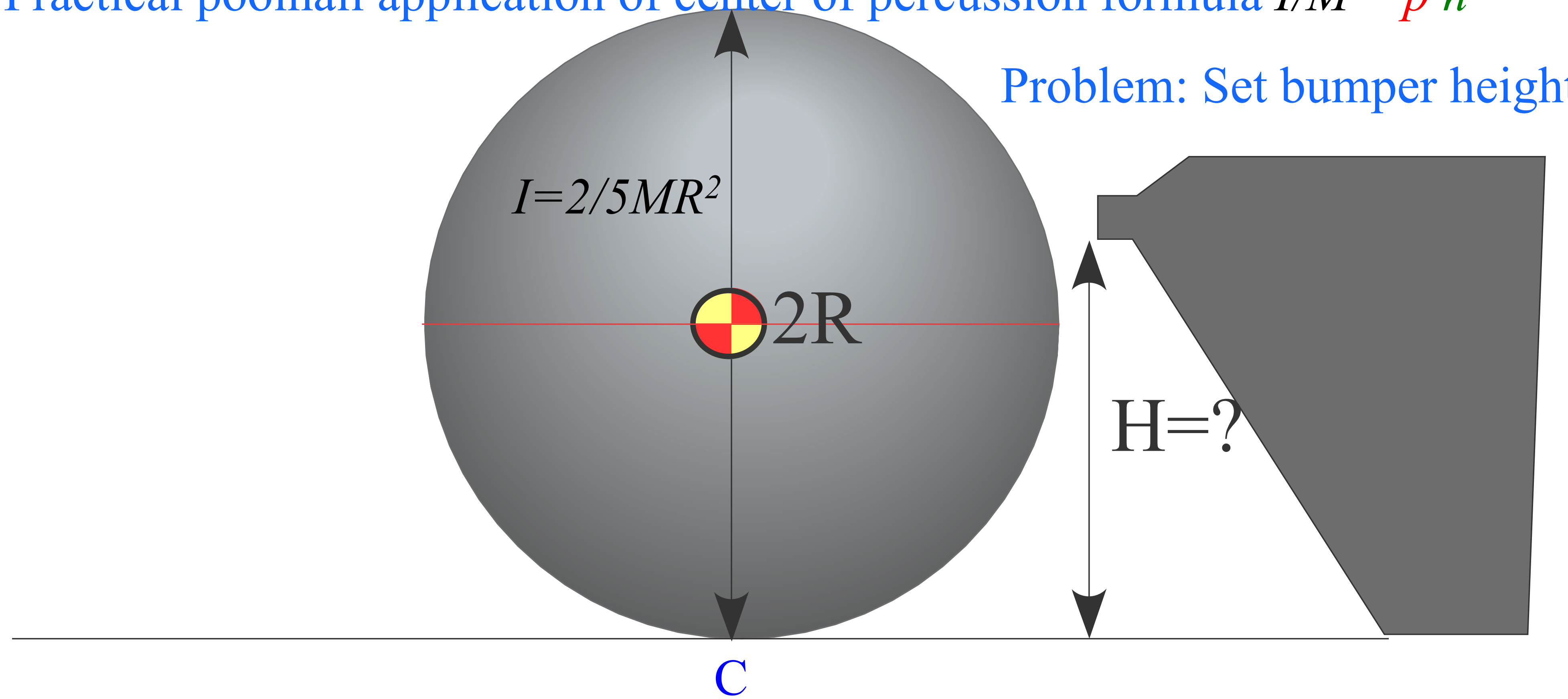
→ *Practical poolhall application*

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



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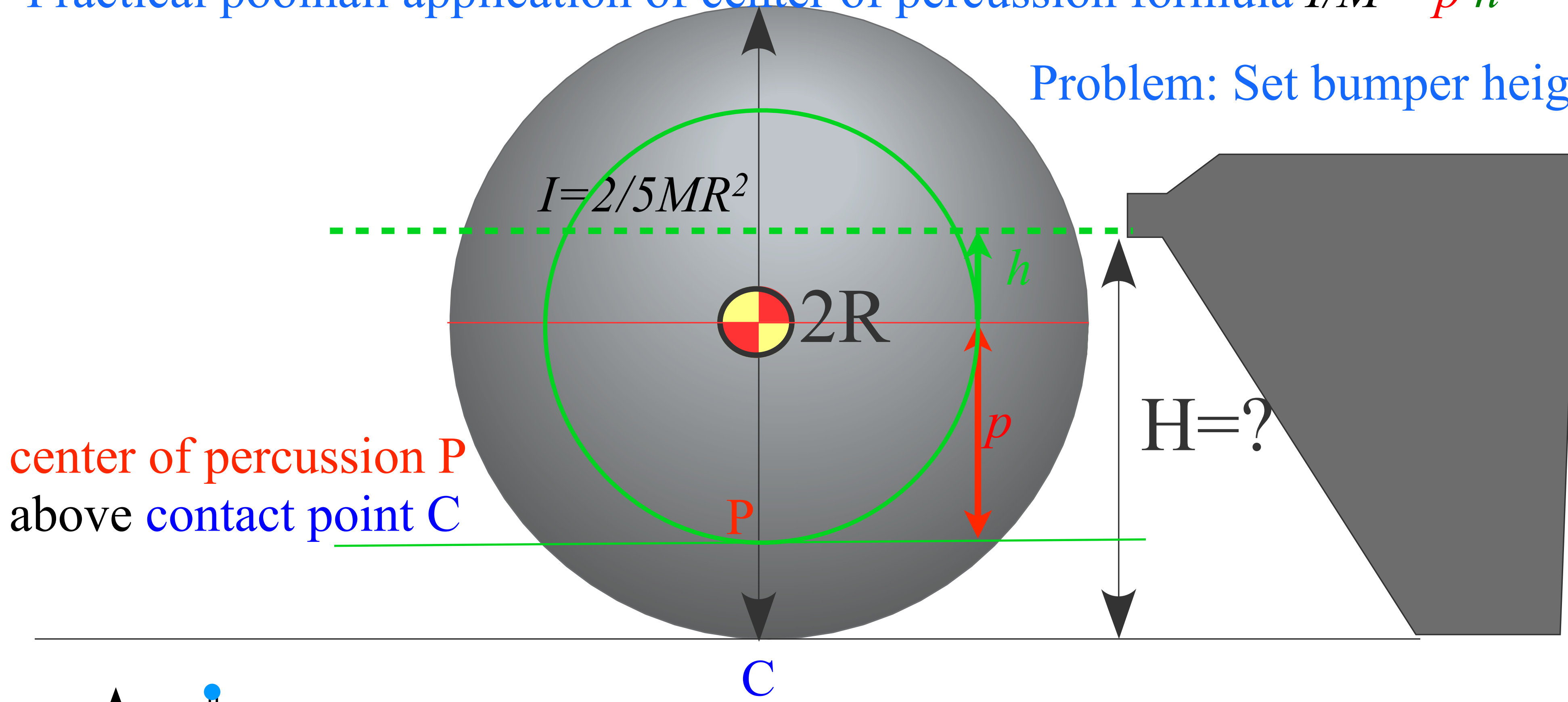
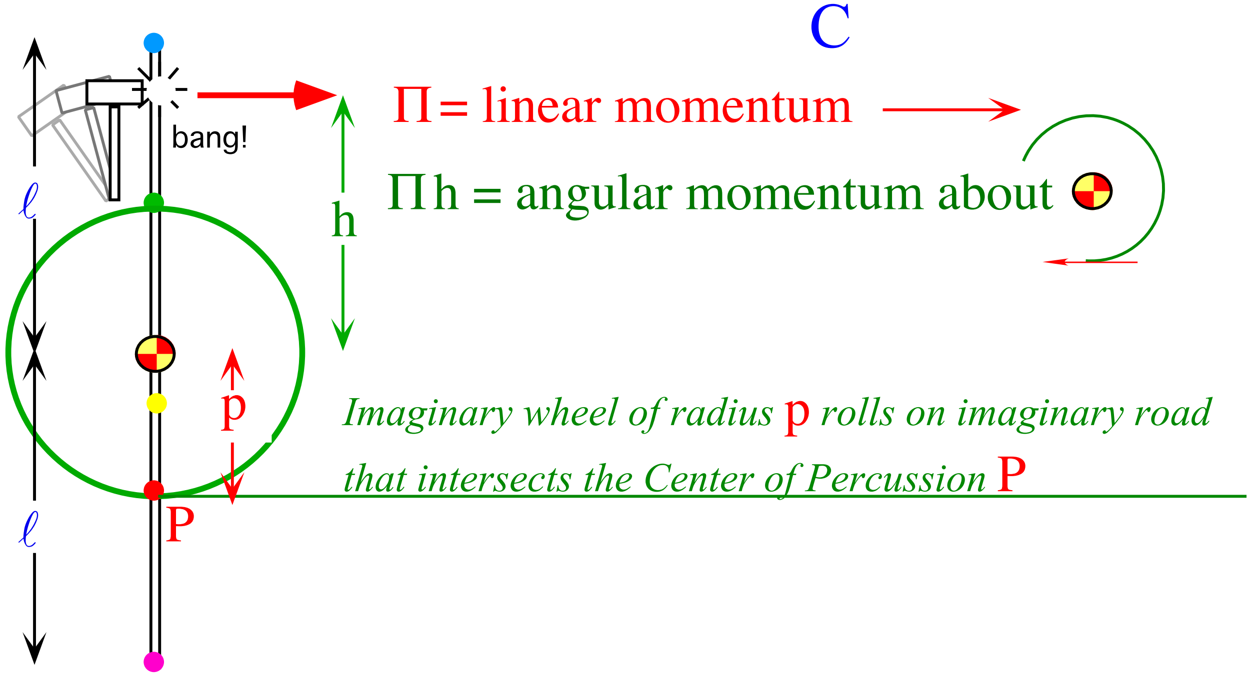


Where should bumper height H be set to make ball contact point C at the center of percussion P?



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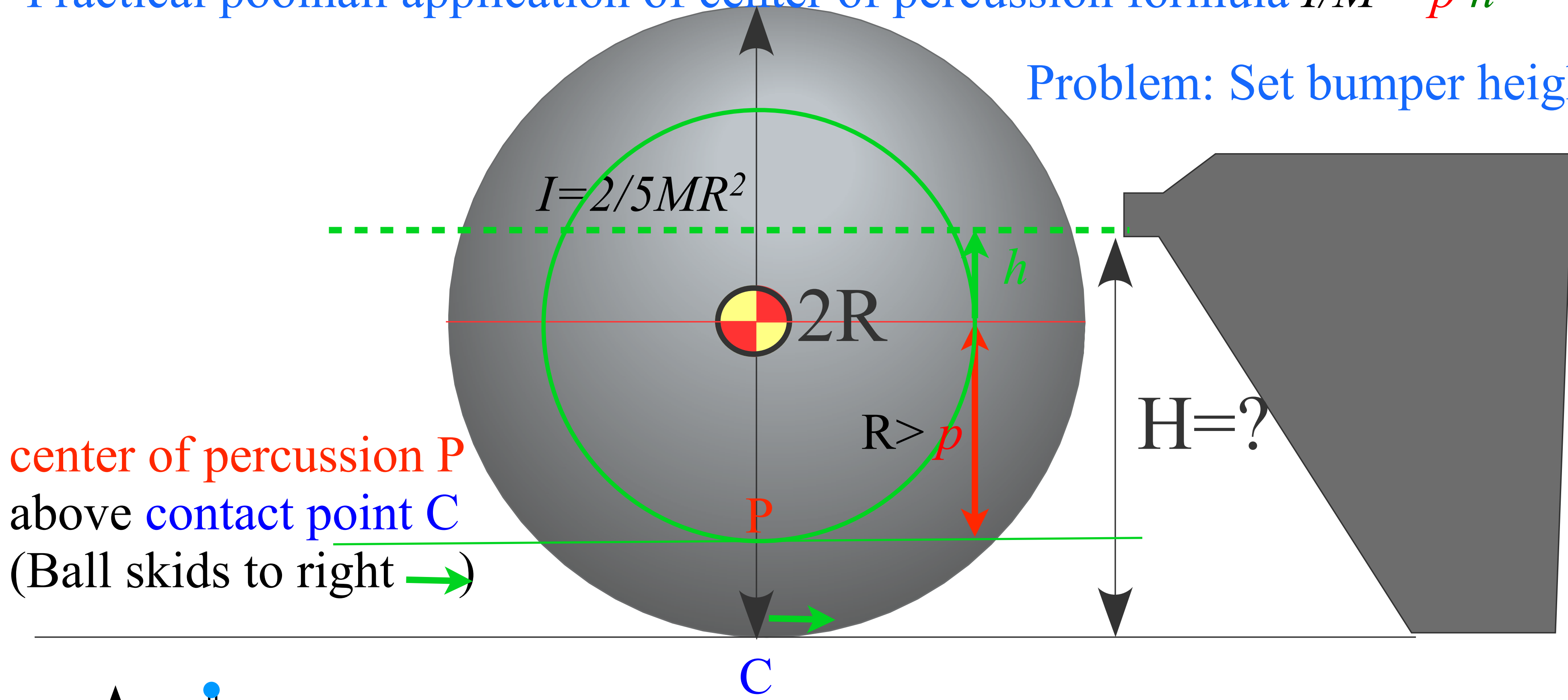

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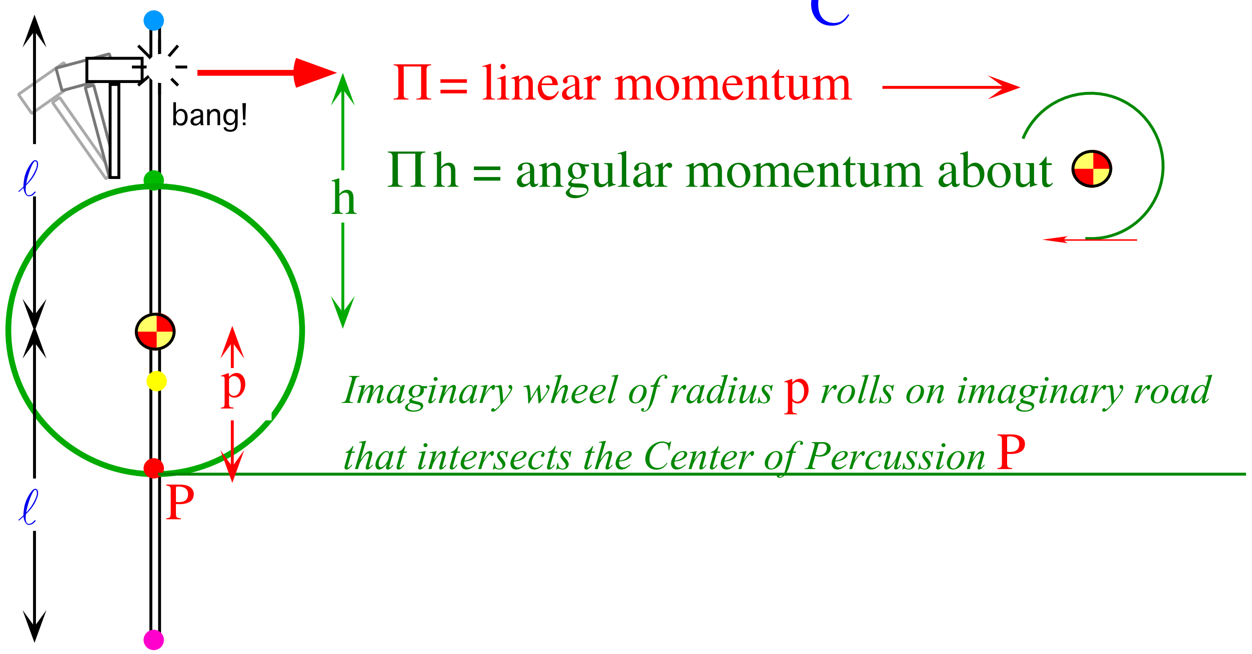

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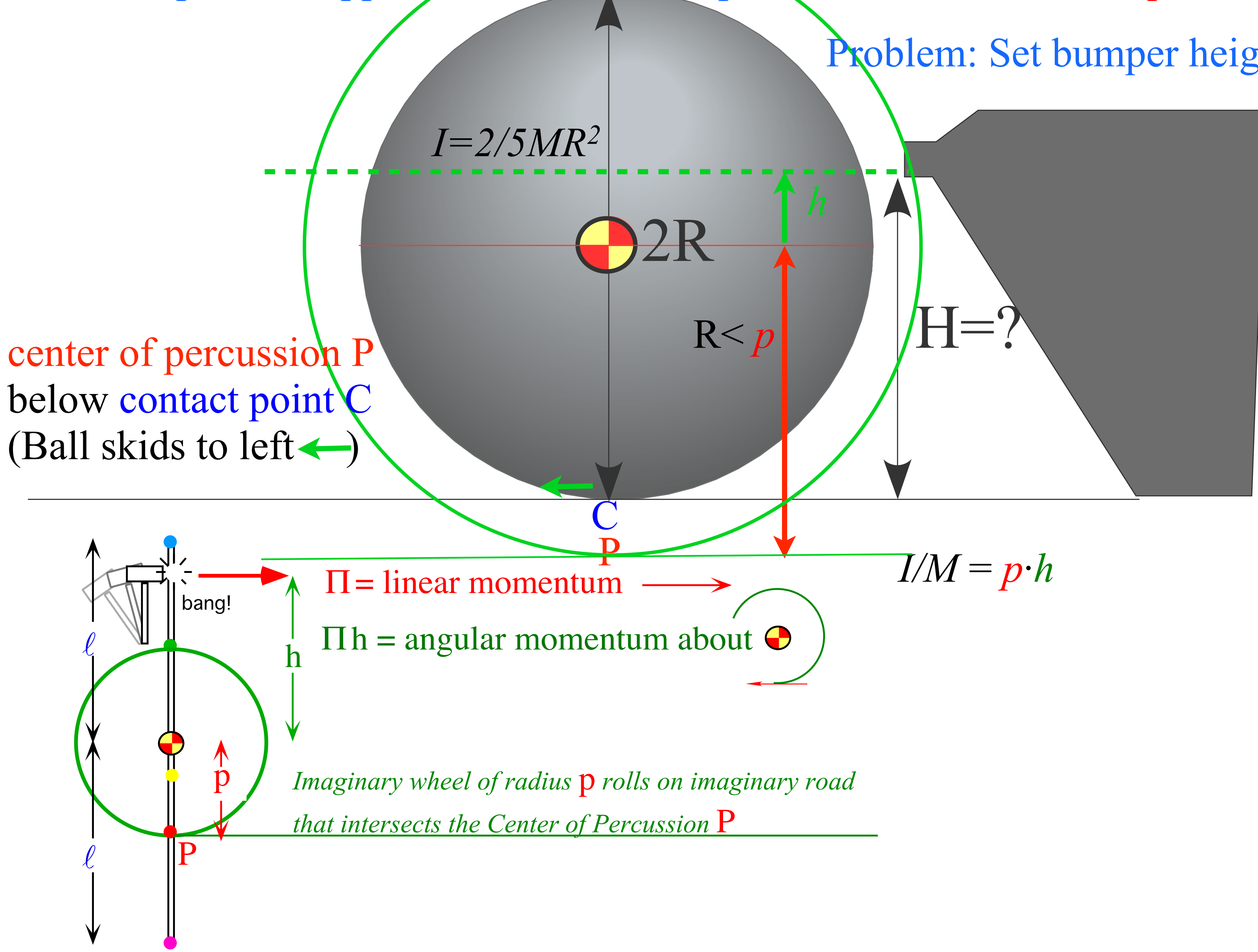

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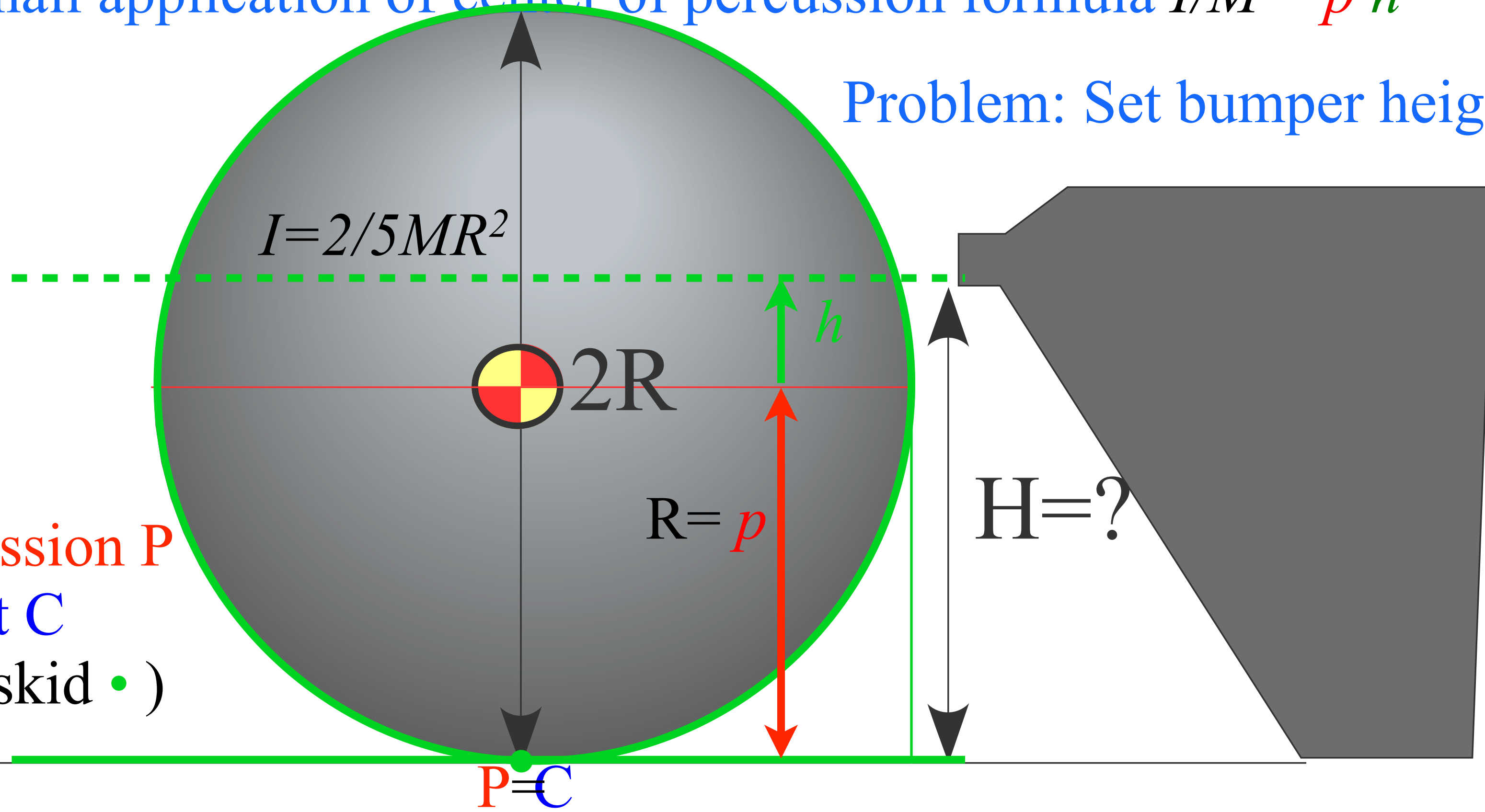

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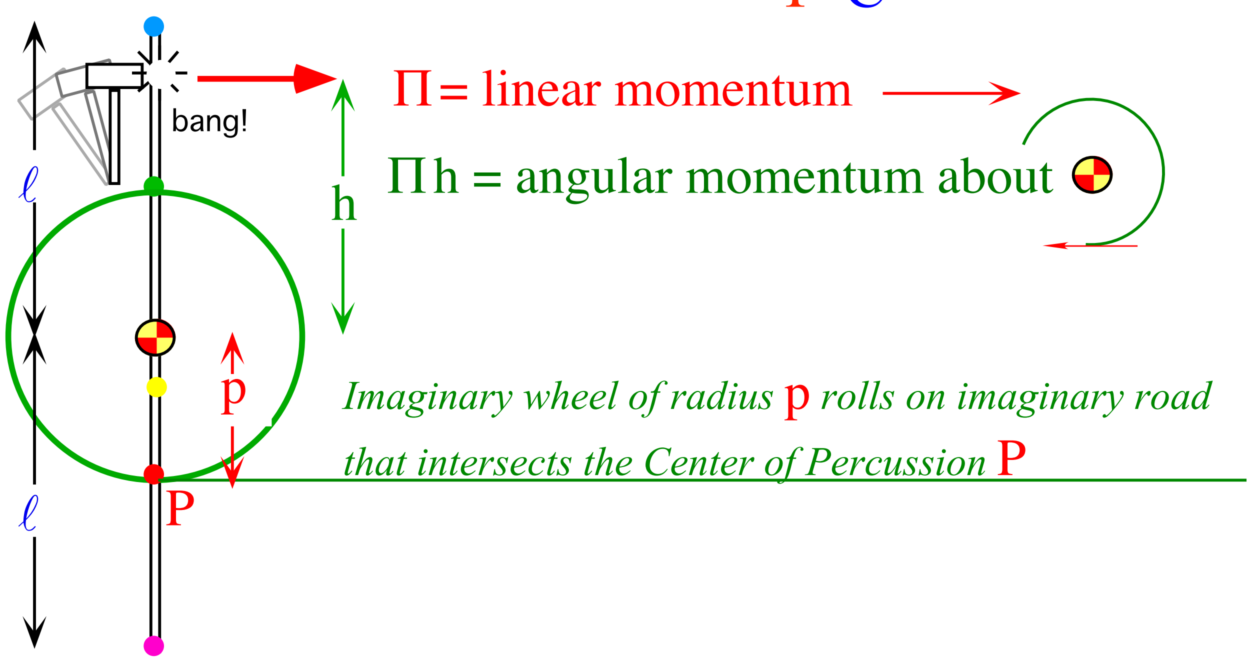
center of percussion P
at contact point C
(Ball does not skid •)

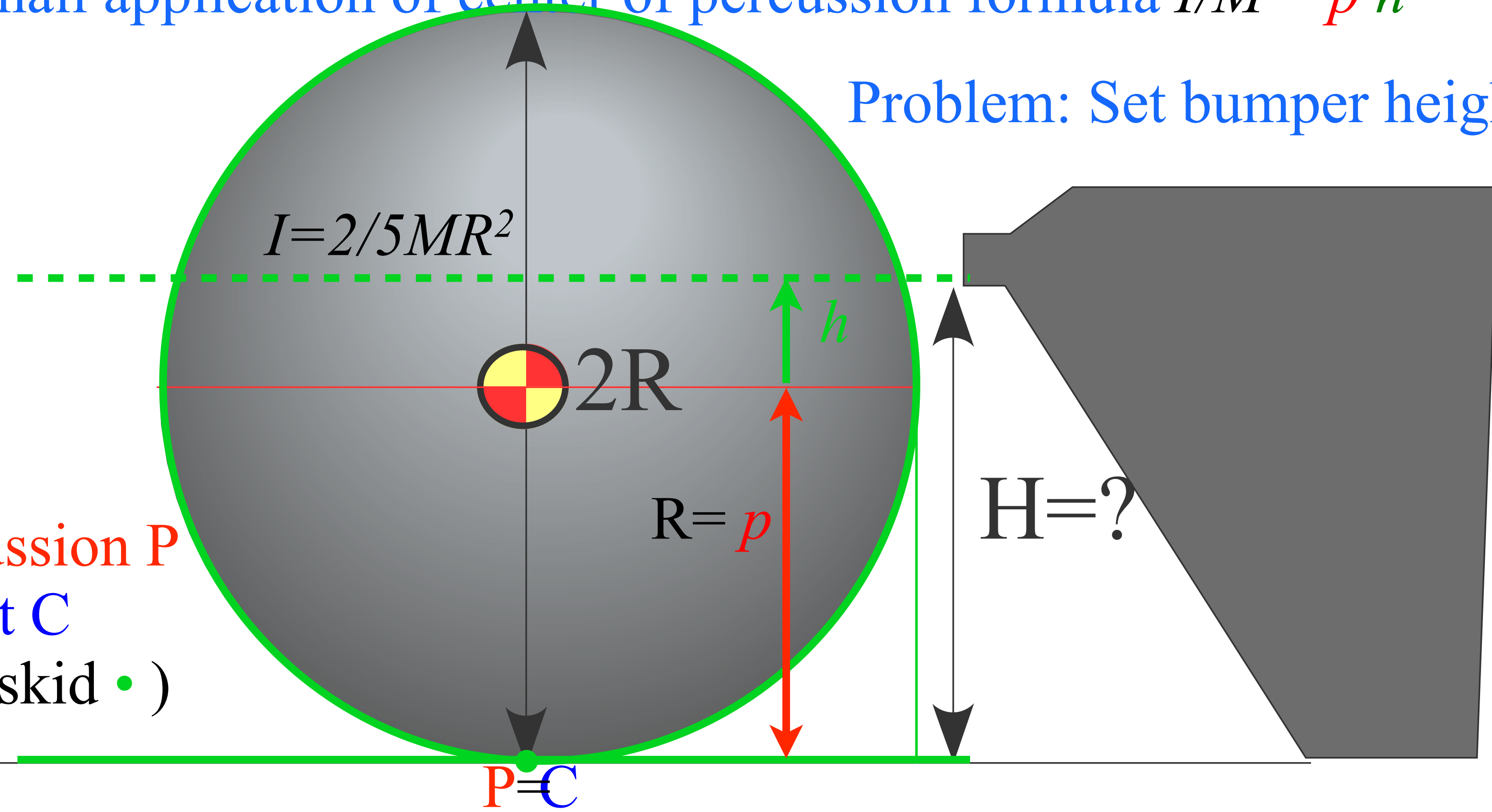


$$I/M = p \cdot h$$

$$h = I/Mp = I/MR \qquad \text{(For R} = p\text{)}$$

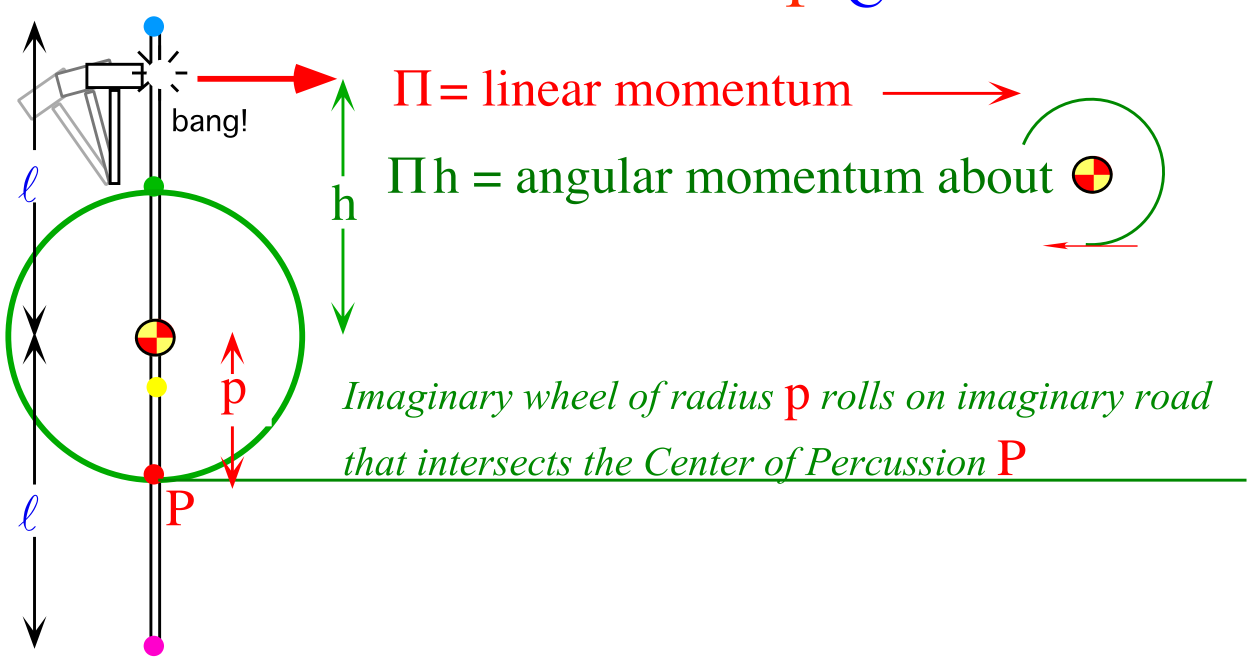
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
at contact point C
(Ball does not skid •)



$I/M = p \cdot h$

$h = I/Mp = I/MR$ (For R= p)

$= 2/5MR^2/MR$

$= 2/5R$

For: $H = R + h = 7/10(2R)$ ball does not skid.

Thats all folks!