Lecture 20
Tue 10.30.2014

# *Reimann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)*

## *Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^{k}$*

*Christoffel g-derivative formula*
*What's a tensor? What's not?*

## *General Riemann equations of motion* *(No explicit t-dependence and fixed GCC)*

*Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

## *Separation of GCC Equations: Effective Potentials*

*Small radial oscillations*
*Cycloid vs Pendulum*

→ *Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$*

*Christoffel g-derivative formula*
*What's a tensor? What's not?*

# *Covariant derivative and Christoffel Coefficients* $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}^{k}$

GCC $q^m$ derivatives of vectors **U** are due to:

*(1)* changing $U^m$ components *(2)* curving GCC vectors $\mathbf{E_n}$.

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(U^j \mathbf{E}_j\right) = \frac{\partial U^m}{\partial q^i}\left(\mathbf{E}_m\right) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

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(Note funny semi-colon ; notation)

Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}^\ell$ or else $\mathbf{E}_m$

$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \Gamma_{in;\ell}\,\mathbf{E}^\ell$$

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*Christoffel coefficients* $\Gamma_{ij;k}$ *of the first kind*

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

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*i,n* to *n,i* symmetry guaranteed here

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$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E_i}}{\partial q^n}$$

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*Christoffel coefficients* $\Gamma_{ij;}{}^{k}$ *the second kind* defined by:

$$\Gamma_{in;}{}^{m} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^{m}$$

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Q: Do we need a third kind of $\Gamma$-coefficient or a $\Lambda$-coefficient?

(to differentiate *contra*variant-$\mathbf{E}^n$ or *co*variant $U_n$)

$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda^n_{im}\mathbf{E}^m \text{ , where: } \Lambda^n_{im} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

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Q: Do we need a third kind of Γ-coefficient or a Λ-coefficient?

(to differentiate *contra*variant-$\mathbf{E}^n$ or *co*variant $U_n$)

$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda^n_{im}\mathbf{E}^m \text{ , where: } \Lambda^n_{im} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

A: NO! That Λ-coefficient is just a Γ-coefficient with a (-).

$$0 = \frac{\partial\left(\delta^n_m\right)}{\partial q^i} = \frac{\partial\left(\mathbf{E}^n \bullet \mathbf{E}_m\right)}{\partial q^i} = \frac{\partial \mathbf{E}^n}{\partial q^i}\bullet \mathbf{E}_m + \mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

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$$\Gamma_{in;}{}^{m} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^{m}$$

Any vector derivative can be expressed using $\Gamma_{ij;}{}^{k}$ in terms of $\mathbf{E}_m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}\right)\mathbf{E}_m$$

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Any vector derivative can be expressed using $\Gamma_{ij;}{}^{k}$ in terms of $\mathbf{E}_m$ or $\mathbf{E}^m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}\right)\mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^{n}\right)\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

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$$= U^m_{\ ;i}\ \mathbf{E}_m = U_{m;i}\ \mathbf{E}^m$$

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$$= \quad U^m_{;i}\ \mathbf{E}_m \quad = \quad U_{m;i}\ \mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

Defining *covariant derivative* $U^m_{;i}$
*of a contravariant component* $U^m$

(Note more funny semi-colon ; notation)

$$U^m_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^{\ \ m}$$

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Any vector derivative can be expressed using $\Gamma_{ij;}{}^{k}$ in terms of $\mathbf{E}_m$ or $\mathbf{E}^m$

$$\frac{\partial \mathbf{U}}{\partial q^i}=\left(\frac{\partial U^m}{\partial q^i}+U^n\Gamma_{in;}{}^{m}\right)\mathbf{E}_m=\left(\frac{\partial U_m}{\partial q^i}-U_n\Gamma_{im;}{}^{n}\right)\mathbf{E}^m$$

$$=\quad U^m{}_{;i}\;\mathbf{E}_m\quad=\quad U_{m;i}\;\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i}\bullet\mathbf{E}_m=-\mathbf{E}^n\bullet\frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im}=-\Gamma^n_{im}$

Defining *covariant derivative* $U^m{}_{;i}$ *of a contravariant component* $U^m$

$$U^m{}_{;i}=\frac{\partial U^m}{\partial q^i}+U^n\Gamma_{in;}{}^{m}$$

...and *covariant derivative* $U_{m;i}$ *of a covariant component* $U_m$

$$U_{m;i}=\frac{\partial U_m}{\partial q^i}-U_n\Gamma_{im;}{}^{n}$$

*Intrinsic derivatives:*

*(Mathematicians being cute)*

Defining *intrinsic derivative of contravariant vector components*.

$$\frac{\delta V^k}{\delta t}=\frac{dV^k}{dt}+\Gamma^k_{mn}V^m\dot{q}^n=\frac{\partial V^k}{\partial q^n}\dot{q}^n+\Gamma^k_{mn}V^m\dot{q}^n=V^k_{;n}\dot{q}^n$$

$$F_k=\frac{\delta p_k}{\delta t}$$

Defining *intrinsic derivative of covariant vector components*.

$$\frac{\delta V_k}{\delta t}=\frac{dV_k}{dt}-\Gamma^m_{kn}V_m\dot{q}^n=\frac{\partial V_k}{\partial q^n}\dot{q}^n-\Gamma^m_{kn}V_m\dot{q}^n=V_{k\,;\,n}\dot{q}^n$$

$$F^k=\frac{\delta p^k}{\delta t}$$

*Tensor chain rules*.

$\frac{\delta V^k}{\delta t}=V^k_{;n}\dot{q}^n$ , replaces: $\frac{dV^k}{dt}=\frac{\partial V^k}{\partial q^n}\dot{q}^n$ where: $V^k_{;n}=\frac{\partial V^k}{\partial q^n}+\Gamma^k_{mn}V^m$

$\frac{\delta V_k}{\delta t}=V_{k\,;\,n}\dot{q}^n$ , replaces: $\frac{dV_k}{dt}=\frac{\partial V_k}{\partial q^n}\dot{q}^n$ where: $V_{k\,;\,n}=\frac{\partial V_k}{\partial q^n}-\Gamma^m_{kn}V_m$

*Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^{k}$*

→ *Christoffel g-derivative formula*

*What's a tensor? What's not?*

## *Christoffel g-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\quad\Gamma_{im;n}\quad+\quad\Gamma_{in;m}$$

# *Christoffel g-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$\frac{\partial g_{mi}}{\partial q^n}=\Gamma_{nm;i}+\Gamma_{in;m}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

## *Christoffel g-derivative formula*

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\Gamma_{nm;i}-\Gamma_{in;m}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

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$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}$$

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# *Christoffel g-derivative formula*

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$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\Gamma_{nm;i}-\Gamma_{in;m}$$ (switched i ↔ n)

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}$$ (switched i ↔ m)

Gives the Christoffel formula

$$\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)$$

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*Christoffel g-derivative formula*

→ *What's a tensor? What's not?*

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Gives the Christoffel formula

$$\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ of covariant derivative $U^m{}_{;n}=\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{E_m}$

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*What's a tensor? What's not?*

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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## *What's a tensor? What's not?*

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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$$\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)$$

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1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

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But, still need to write $\frac{\partial\bar{U}^{\bar{m}}}{\partial q^n}$ in terms of $\frac{\partial U^m}{\partial q^n}$.

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$$\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial}{\partial q^n}\left(\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}U^m\right)=\boxed{\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial U^m}{\partial q^n}}+U^m\frac{\partial}{\partial q^n}\left(\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\right)$$

1[st] term is OK, but 2[nd] term is zero only if Jacobian is constant matrix!

$\frac{\partial\bar{U}^{\bar{m}}}{\partial\bar{q}^{\bar{n}}}=\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial U^m}{\partial q^n}$ holds if and only if $\frac{\partial}{\partial q^n}\left(\frac{\partial\bar{q}^{\bar{m}}}{\partial q^m}\right)\equiv 0$

# What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m)}$$

Gives the Christoffel formula

$$\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}$ of covariant derivative $U^m{}_{;n} = \frac{\partial \mathbf{U}}{\partial q^n}\bullet\mathbf{E_m}$

$$\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \bar{\mathbf{U}}}{\partial q^n}\bullet\bar{\mathbf{E}}_{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \bar{\mathbf{U}}}{\partial q^n}\bullet\frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{E}_m$$

$$\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}U^m{}_{;n}$$

The transformation of $U^m{}_{;n} = \frac{\partial U^m}{\partial q^n} + U^\ell\Gamma_{n\ell;}^{\ m}$ is that of general 2nd-rank tensor $T^m{}_n$

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The transformation of $U^m{}_{,n} = \frac{\partial U^m}{\partial q^n}$ is NOT that simple. At first it looks possible. $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$

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Otherwise, $U^m{}_{,n}$ needs "correction" $U^\ell\Gamma_{n\ell;}^{\ \ m}$. And, that $U^\ell\Gamma_{n\ell;}^{\ \ m}$ *cannot* be a $T^m{}_n$-tensor either!

*Riemann equations of motion* *(No explicit t-dependence and fixed GCC)*

*Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

# *Riemann equations of motion* *(No explicit t-dependence and fixed GCC)*

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$ Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

$$T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left( \frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

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Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell}$$

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1st term involves ***covariant momentum*** $p_\ell$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

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$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\frac{\partial x^j}{\partial t}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\frac{\partial x^k}{\partial t}\right\}\right)$$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}$$

*The "4-wheel-drive garbage truck"*

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$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\frac{\partial x^j}{\partial t}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\frac{\partial x^k}{\partial t}\right\}\right)$$

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Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

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Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\frac{\partial x^j}{\partial t}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\frac{\partial x^k}{\partial t}\right\}\right)$$

*All explicit-t-dependent terms are zero*
*(Time must be included as a dimension)*

1[st] term involves *covariant momentum* $p_\ell$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* $\gamma^{mn}$ gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}$$

*The "4-wheel-drive garbage truck"*

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

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Rearrange to expose Christoffel coefficients:

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)$$

# *Riemann equations of motion* *(No explicit t-dependence and fixed GCC)*

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

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$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\frac{\partial x^j}{\partial t}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\frac{\partial x^k}{\partial t}\right\}\right)$$

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*The "4-wheel-drive garbage truck"*

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$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell m}}{\partial q^n} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

This gives ***covariant Riemann equations***

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

# *Riemann equations of motion* *(No explicit t-dependence and fixed GCC)*

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

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$$T = \frac{1}{2} M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\frac{\partial x^j}{\partial t}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\frac{\partial x^k}{\partial t}\right\}\right)$$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}$$

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Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

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Rearrange to expose Christoffel coefficients:

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$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

This gives ***covariant Riemann equations*** and ***contravariant Riemann equations***.

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n$$

$$F^k = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n$$

## *Riemann equations of motion* *(No explicit t-dependence and fixed GCC)*

→ *Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

# *Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

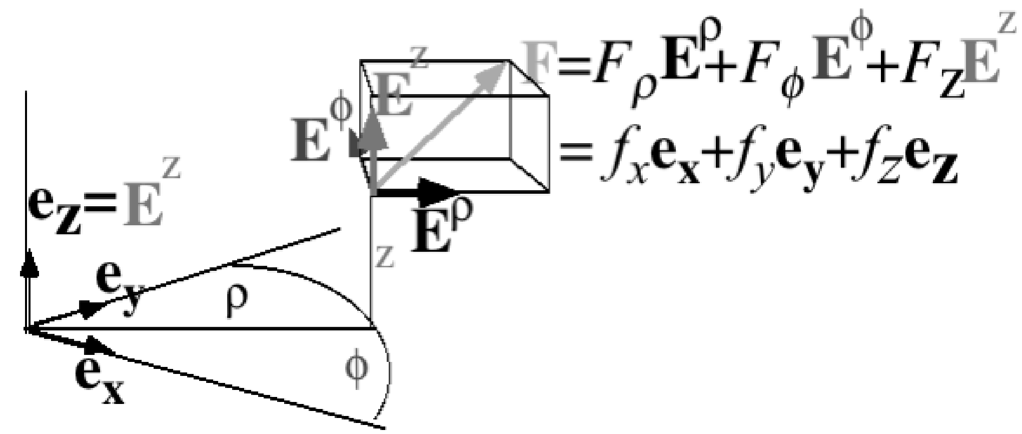
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\rho} \\ \leftarrow \mathbf{E}^{\phi} \\ \leftarrow \mathbf{E}^{z} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_{\rho} & \mathbf{E}_{\phi} & \mathbf{E}_{z} \end{matrix} \qquad\qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



*Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

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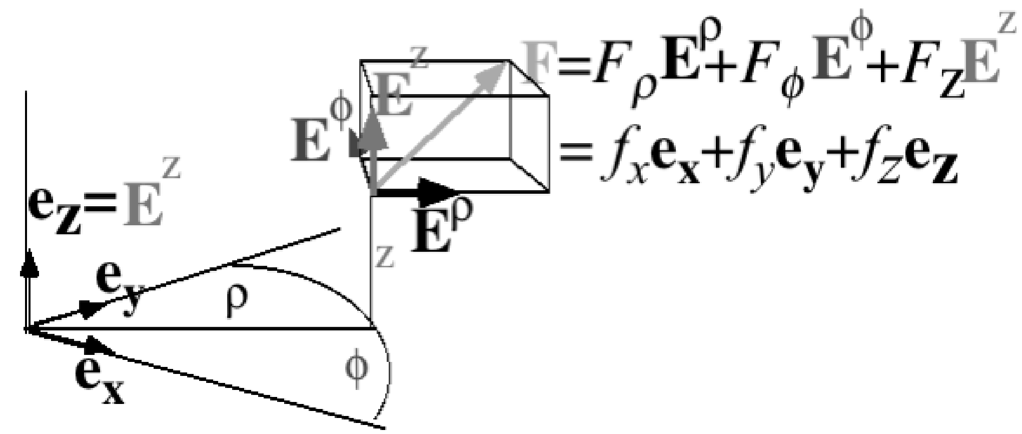


*Covariant forces*

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos\phi + f_y \sin\phi \ + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho\sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

# *Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho, q^2 = \phi, q^3 = z$)*

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

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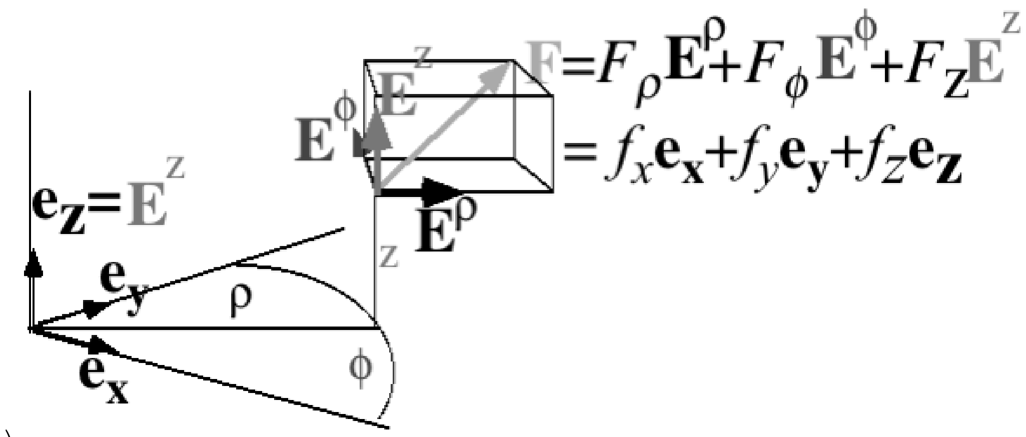


*Covariant forces*

$$F_\rho = f_x\frac{\partial x}{\partial \rho} + f_y\frac{\partial y}{\partial \rho} + f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial \phi} + f_y\frac{\partial y}{\partial \phi} + f_z\frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z} + f_y\frac{\partial y}{\partial z} + f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

*Covariant kinetic metric*

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

# *Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

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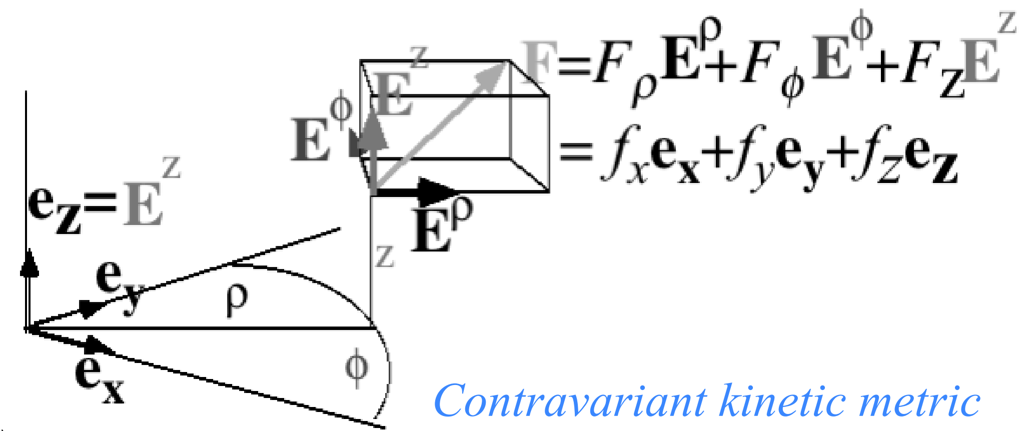


*Covariant forces*

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

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$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

*Contravariant kinetic metric*

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

## Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

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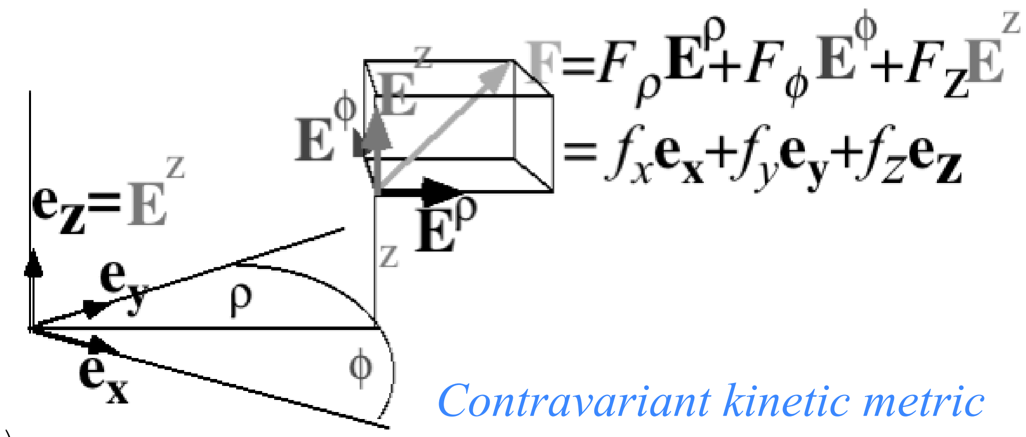


### Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = f_x\cos\phi+f_y\sin\phi+0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

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### Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right)=m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right)=m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

### Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

### Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

## *Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

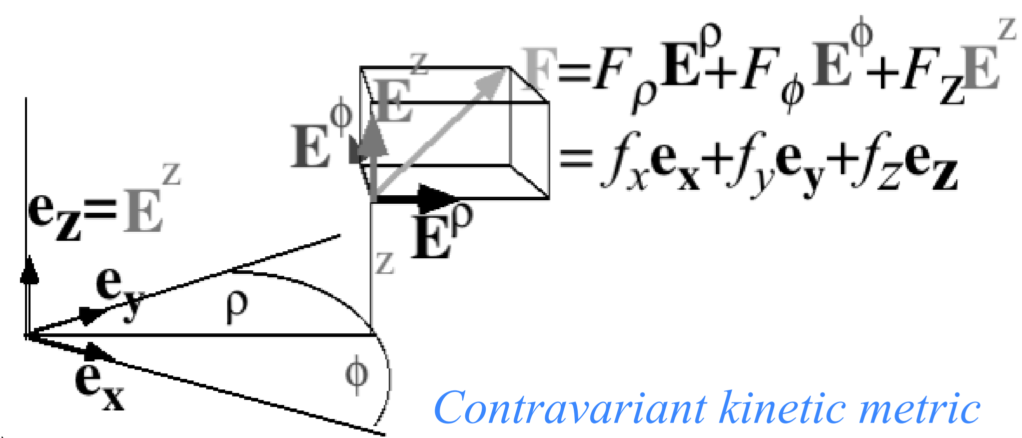
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\rho} \\ \leftarrow \mathbf{E}^{\phi} \\ \leftarrow \mathbf{E}^{z} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_{\rho} & \mathbf{E}_{\phi} & \mathbf{E}_{z} \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$




### *Covariant forces*

$$F_{\rho} = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_{\phi} = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho\cos\phi + 0$$

$$F_{z} = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

### *Covariant kinetic metric*

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_{\rho} \bullet \mathbf{E}_{\rho} = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_{\phi} \bullet \mathbf{E}_{\phi} = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_{z} \bullet \mathbf{E}_{z} = m$$

### *Contravariant kinetic metric*

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

### *Lagrangian*

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

### *Covariant momenta*

$$p_{\rho} = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_{\phi} = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_{z} = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

# Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

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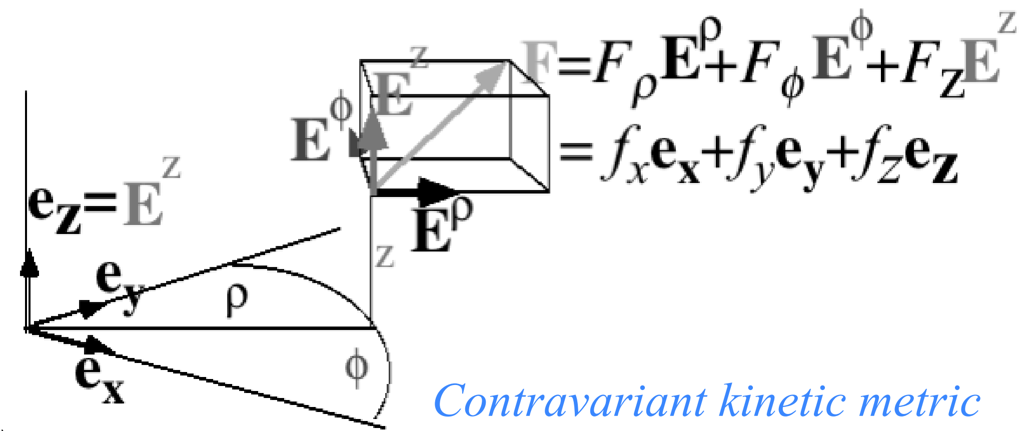


## Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho\sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

## Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

## Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

## Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

## Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

## Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

# Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

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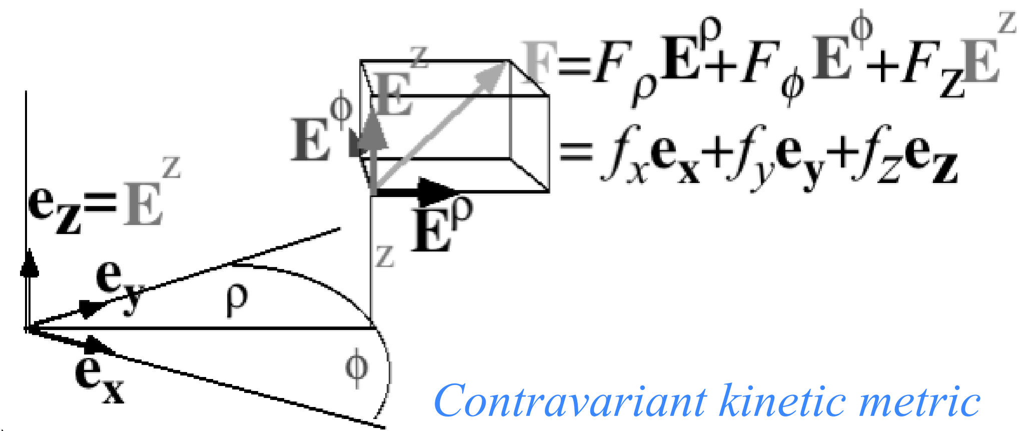


## Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

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$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

## Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

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## Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

## Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

## Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

## Comparing Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

# *Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

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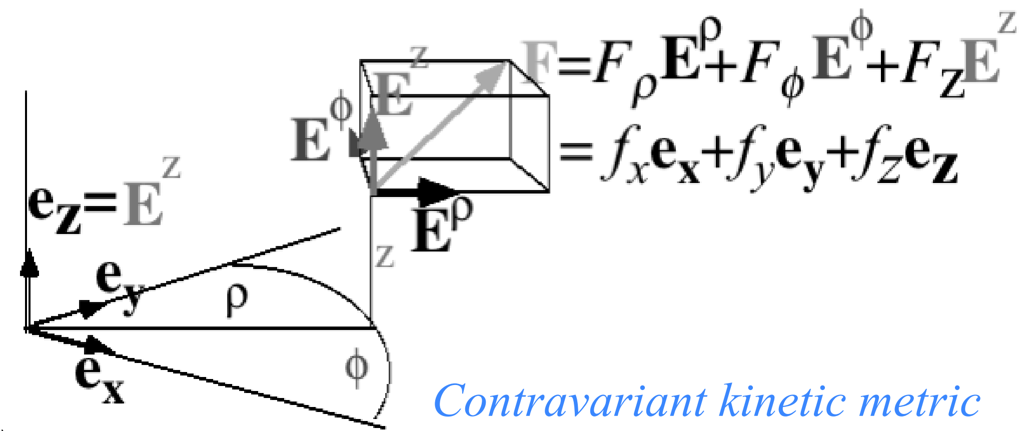


*Covariant forces*

$$F_\rho = f_x \frac{\partial x}{\partial\rho} + f_y \frac{\partial y}{\partial\rho} + f_z \frac{\partial z}{\partial\rho} = f_x \cos\phi + f_y \sin\phi \; + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial\phi} + f_y \frac{\partial y}{\partial\phi} + f_z \frac{\partial z}{\partial\phi} = -f_x \rho\sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 \; + \; 0 \; + f_z$$

*Covariant kinetic metric*

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

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$$\gamma_{zz} = m \frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

*Contravariant kinetic metric*

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

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*Lagrangian*

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

*Covariant momenta*

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

*Contravariant momenta*

$$p^\rho = \dot{\rho}$$

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## Comparing Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Note: This a much more efficient way to derive Γ-coefficients than the g-formula.

## Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot{\rho} + \Gamma_{mn;\rho}\,\dot{q}^m\dot{q}^n \qquad\qquad F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial\phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\,\dot{q}^m\dot{q}^n$$

# Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos\phi$

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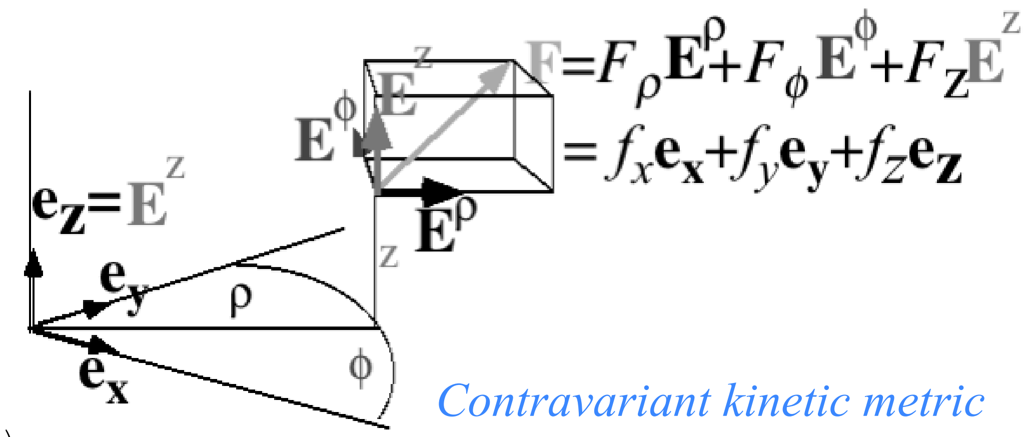


*Covariant forces*

$$F_\rho = f_x\frac{\partial x}{\partial \rho}+f_y\frac{\partial y}{\partial \rho}+f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial \phi}+f_y\frac{\partial y}{\partial \phi}+f_z\frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

*Covariant kinetic metric*

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*Contravariant kinetic metric*

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*Lagrangian*

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

*Covariant momenta*

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

*Contravariant momenta*

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$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n$$

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$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho}\ddot{\rho} + \Gamma_{mn;\rho}\dot{q}^m\dot{q}^n$$

$$= \frac{d\left(m\dot{\rho}\right)}{dt} - \frac{\partial}{\partial \rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right) = m\ddot{\rho} - m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\dot{q}^m\dot{q}^n$$

## Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

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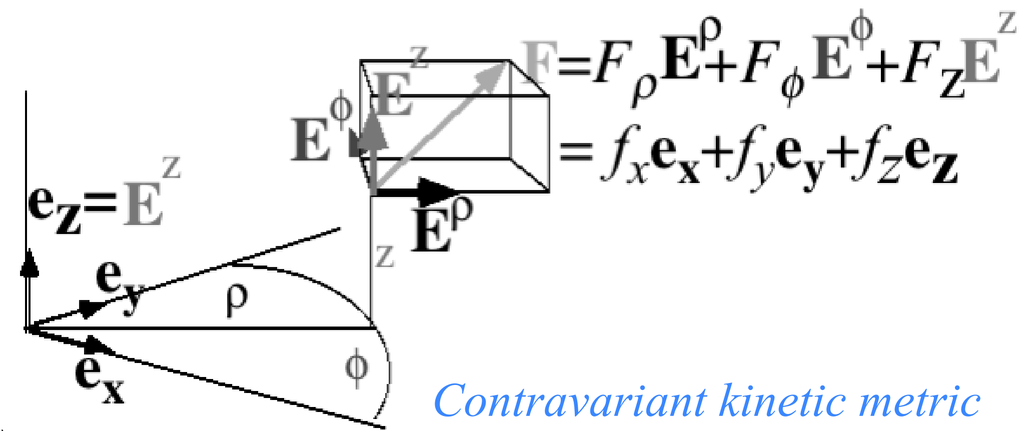


**Covariant forces**

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

**Covariant kinetic metric**

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

**Contravariant kinetic metric**

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

**Lagrangian**

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

**Covariant momenta**

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

**Contravariant momenta**

$$p^\rho = \dot{\rho}$$

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$$p^z = \dot{z}$$

## Comparing Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Note: This a much more efficient way to derive Γ-coefficients than the g-formula.

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*Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ symmetry gives 2 factor for q≠p*

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$$= \frac{d\left(m\dot{\rho}\right)}{dt} - \frac{\partial}{\partial\rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right) = m\ddot{\rho} - m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

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## Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho\cos\phi$

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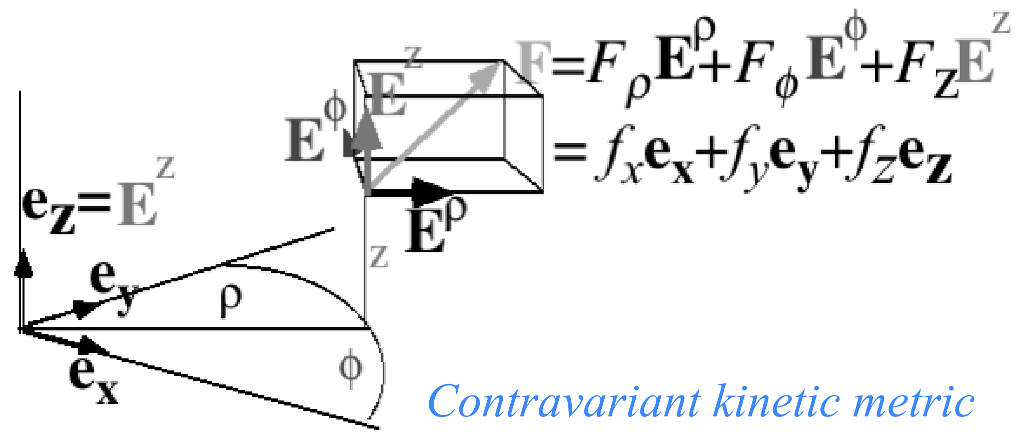


### Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial \rho}+f_y\frac{\partial y}{\partial \rho}+f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi+f_y\sin\phi \;+0$$

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$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \;+\; 0 \;+f_z$$

### Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

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$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

### Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

### Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

### Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

### Contravariant momenta

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## *Contra*variant equations are *acceleration* equations. $F^k = \gamma^{jk}F_j = \ddot{q}^k+\Gamma^k_{mn}\dot{q}^m\dot{q}^n$

$$F^\rho = \gamma^{\rho\rho}F_\rho = \ddot{q}^\rho+\Gamma^\rho_{mn}\dot{q}^m\dot{q}^n \qquad F^\phi = \gamma^{\phi\phi}F_\phi = \ddot{q}^\rho+\Gamma^\rho_{mn}\dot{q}^m\dot{q}^\phi$$

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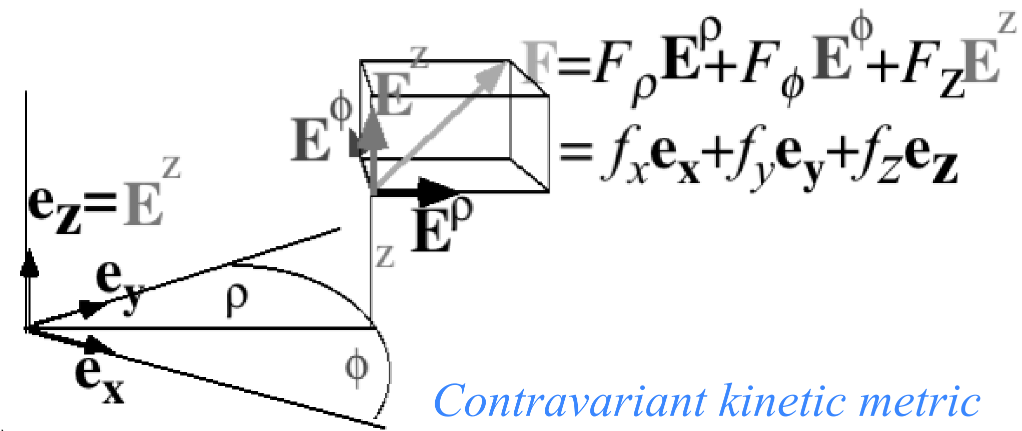


*Covariant forces*

$$F_\rho = f_x\frac{\partial x}{\partial \rho} + f_y\frac{\partial y}{\partial \rho} + f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \ + 0$$

$$F_\phi = f_x\frac{\partial x}{\partial \phi} + f_y\frac{\partial y}{\partial \phi} + f_z\frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

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*Contravariant kinetic metric*

$$\gamma^{\rho\rho} = 1/m$$

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*Lagrangian*

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

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$$F^\phi = \gamma^{\phi\phi}F_\phi = \ddot{q}^\rho + \Gamma^\rho_{mn}\dot{q}^m\dot{q}^n$$

$$= \ddot{\phi} + 2\dot{\rho}\dot{\phi}/\rho \quad \text{so: } \Gamma^\phi_{\rho\phi} = 1/\rho = \Gamma^\phi_{\phi\rho} \qquad \gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

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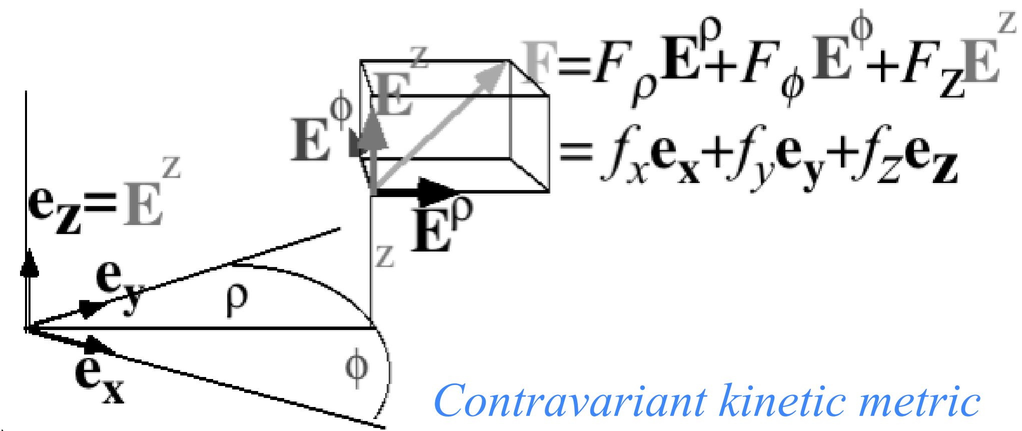


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$$F_\rho = f_x\frac{\partial x}{\partial \rho}+f_y\frac{\partial y}{\partial \rho}+f_z\frac{\partial z}{\partial \rho}= \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

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### Contravariant kinetic metric

$$\gamma^{\rho\rho}=1/m$$

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### Lagrangian

$$T=\frac{1}{2}\gamma_{mn}\dot q^m\dot q^n=\frac{1}{2}m\dot\rho^2+\frac{1}{2}m\rho^2\dot\phi^2+\frac{1}{2}m\dot z^2$$

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$$p_\rho=\frac{\partial T}{\partial\dot\rho}=\gamma_{\rho\rho}\dot\rho \qquad p_\phi=\frac{\partial T}{\partial\dot\phi}=\gamma_{\phi\phi}\dot\phi \qquad p_z=\frac{\partial T}{\partial\dot z}=\gamma_{zz}\dot z$$

$$=m\dot\rho \qquad =m\rho^2\dot\phi \qquad =m\dot z$$

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$$F^\rho=\gamma^{\rho\rho}F_\rho=\ddot q^\rho+\Gamma^\rho_{mn}\dot q^m\dot q^n$$

$$=\ddot\rho-\rho\dot\phi^2 \quad \text{so: } \Gamma^\rho_{\phi\phi}=-\rho \qquad \gamma^{\rho\rho}=1/m$$

$$\ddot\rho=F^\rho+\rho\dot\phi^2 \qquad \textit{(Centrifugal acceleration)}$$

$$F^\phi=\gamma^{\phi\phi}F_\phi=\ddot q^\rho+\Gamma^\rho_{mn}\dot q^m\dot q^n$$

$$=\ddot\phi+2\dot\rho\dot\phi/\rho \quad \text{so: } \Gamma^\phi_{\rho\phi}=1/\rho=\Gamma^\phi_{\phi\rho} \qquad \gamma^{\phi\phi}=1/\left(m\rho^2\right)$$

$$\ddot\phi=F^\phi-2\dot\rho\dot\phi/\rho \qquad \textit{(Coriolis acceleration)}$$

## Rewriting GCC Lagrange equations :

**(Review of Lecture 11)**

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

*Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force*

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

*Torque relates to two distinct parts: Coriolis and angular acceleration*

*Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence*

### Conventional forms

*radial force:* $M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$

*angular force or torque:* $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

### Field-free (U=0)

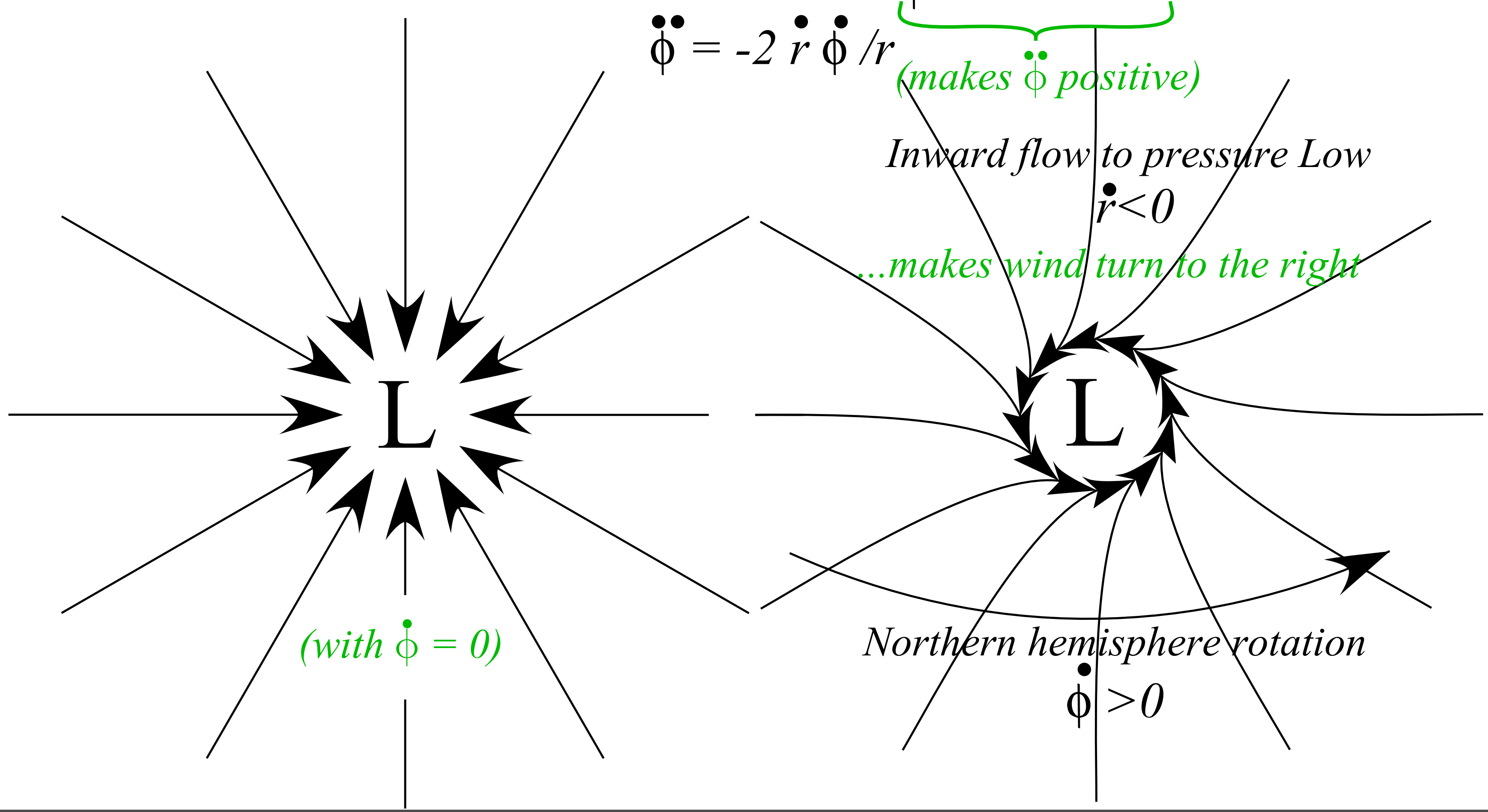
*radial acceleration:* $\ddot{r} = r\,\dot{\phi}^2$

*angular acceleration:* $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

*Because Earth rotation is counter-clockwise (positive) in North*

Coriolis acceleration with $\dot{\phi} > 0$ and $\dot{r} < 0$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}/r$ *(makes $\ddot{\phi}$ positive)*




*Effect on Northern Hemisphere local weather*

*Cyclonic flow around lows*

→ *Separation of GCC Equations: Effective Potentials*

*Small radial oscillations*

*Cycloid vs Pendulum*

# *Separation of GCC Equations: Effective Potentials*

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + V$$

( Numerically correct ONLY! )

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

( Formally **and** Numerically correct )

# *Separation of GCC Equations: Effective Potentials* (For isotropic $H(r,p_r,\phi,p_\phi)$)

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + V$$ ( Numerically correct ONLY! )

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If potential $V$ is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)

$H$ has no explicit φ–dependence and the φ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

# *Separation of GCC Equations: Effective Potentials* (For isotropic $H(r,p_r,\phi,p_\phi)$)

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$$=\frac{1}{2}\gamma^{mn}p_m p_n+V=\frac{1}{2m}p_\rho^2+\frac{1}{2m\rho^2}p_\phi^2+\frac{1}{2m}p_z^2+V$$ ( Formally **and** Numerically correct )

Potential $V$ is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)

$H$ has no explicit ϕ–dependence and the ϕ–momenta is constant.

$$m\rho^2\dot{\phi}=p_\phi=const.=\mu$$

If $H$ has no explicit $z$–dependence then the $z$–momenta is constant, too.

$$m\dot{z}=p_z=const.=k$$

# *Separation of GCC Equations: Effective Potentials*

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

( Numerically correct ONLY! )

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

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If $H$ has no explicit $z$–dependence then the $z$–momenta is constant, too.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

$$m\dot{z} = p_z = const. = k$$

$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const.$$

# *Separation of GCC Equations: Effective Potentials*

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$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const.$$

*(Let $k = 0$)*

# *Separation of GCC Equations: Effective Potentials*

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$ ( Numerically correct ONLY! )

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$ ( Formally **and** Numerically correct )

Potential $V$ is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)
$H$ has no explicit ϕ–dependence and the ϕ–momenta is constant.

If $H$ has no explicit $z$–dependence then the $z$–momenta is constant, too.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

$$m\dot{z} = p_z = const. = k$$

$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const.$$

*(Let k =0)*

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m}p_\rho^2 + V^{eff}(\rho) = E = const.$$

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Velocity relations:

$$\dot{\phi}=\mu/\left(m\rho^2\right)$$

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Equations solved by a *quadrature integral* for time versus radius.

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}} = \left(\text{Travel time } \rho_0 \text{ to } \rho_1\right) = t_1 - t_0$$

## *Separation of GCC Equations: Effective Potentials*

→ *Small radial oscillations*
*Cycloid vs Pendulum*

# *Small radial oscillations*

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_0} = 0 \ , \quad \text{with:} \ \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0 \ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_0) + 0 + \frac{1}{2}(\rho - \rho_0)^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0}$$

Stable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0$ Unstable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} < 0$

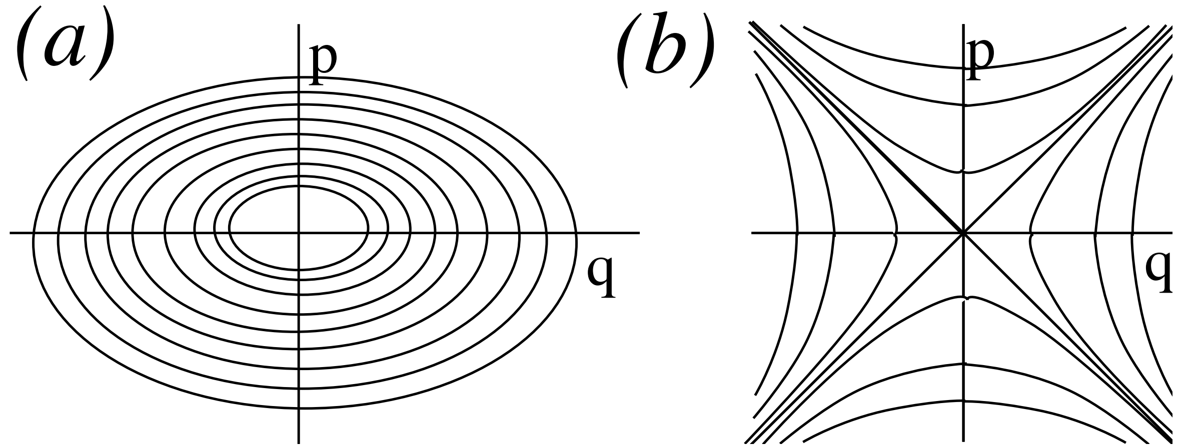


*Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point*

# *Small radial oscillations*

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$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \qquad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}$$

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Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_{\phi}} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_\rho}{n_\phi} \Leftrightarrow \text{Orbit is closed-periodic}$$

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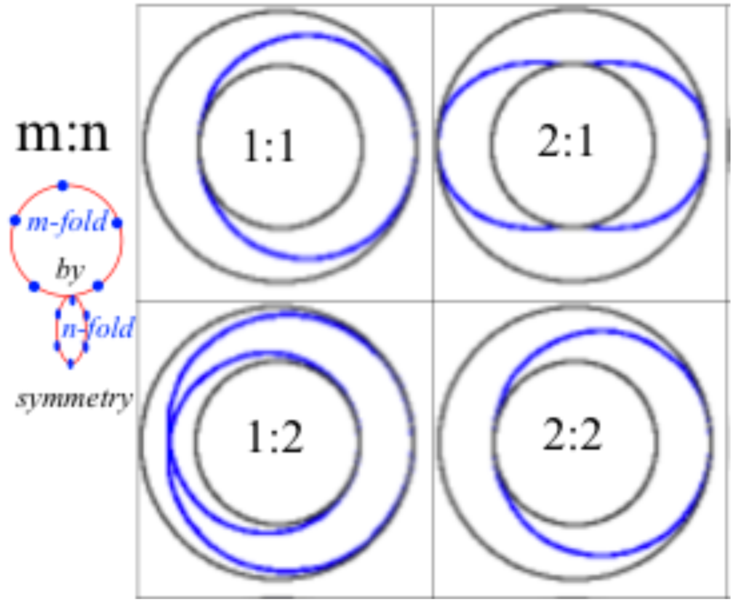
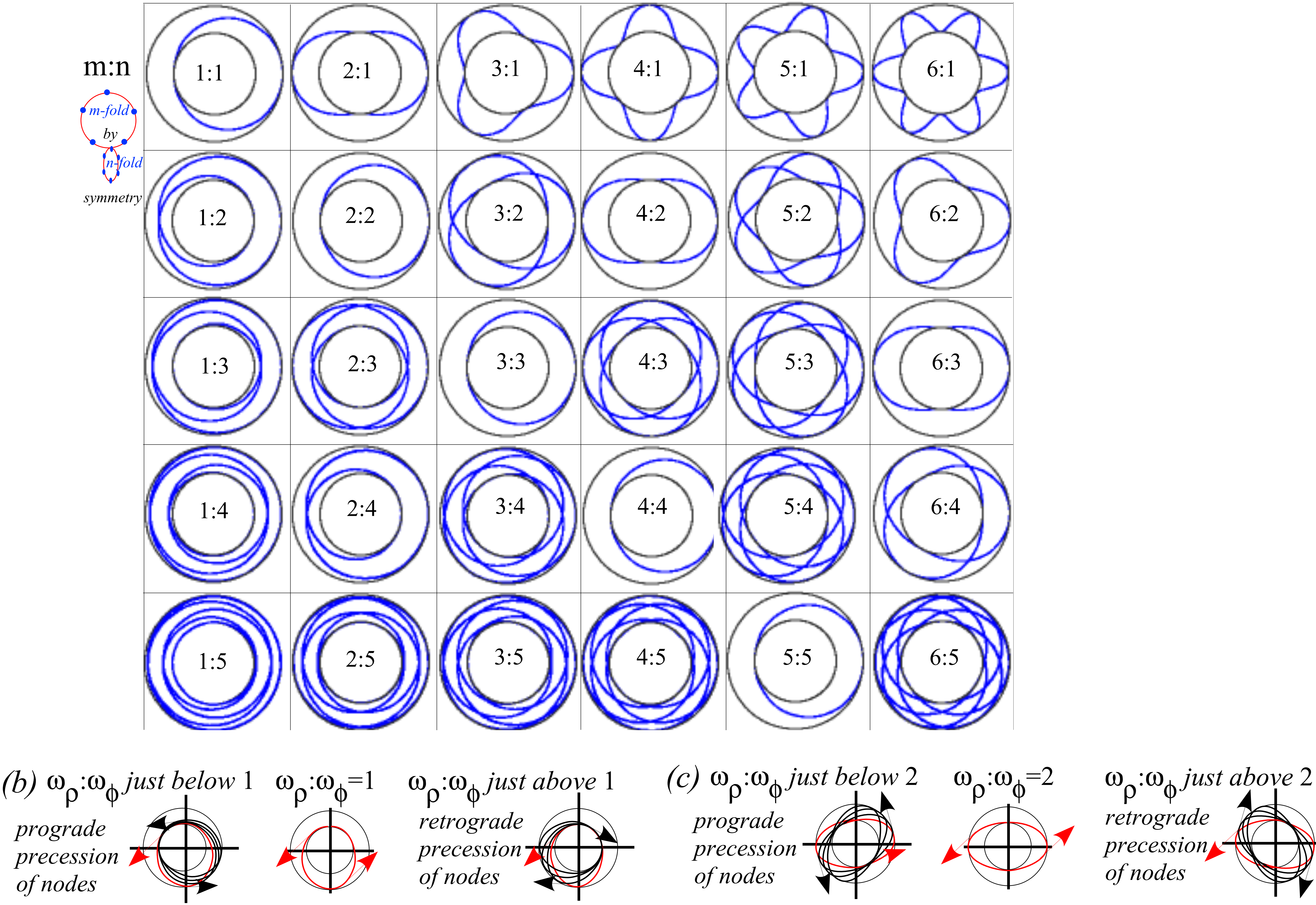
Some generic shapes resulting from various ratios $n\rho : n\phi$

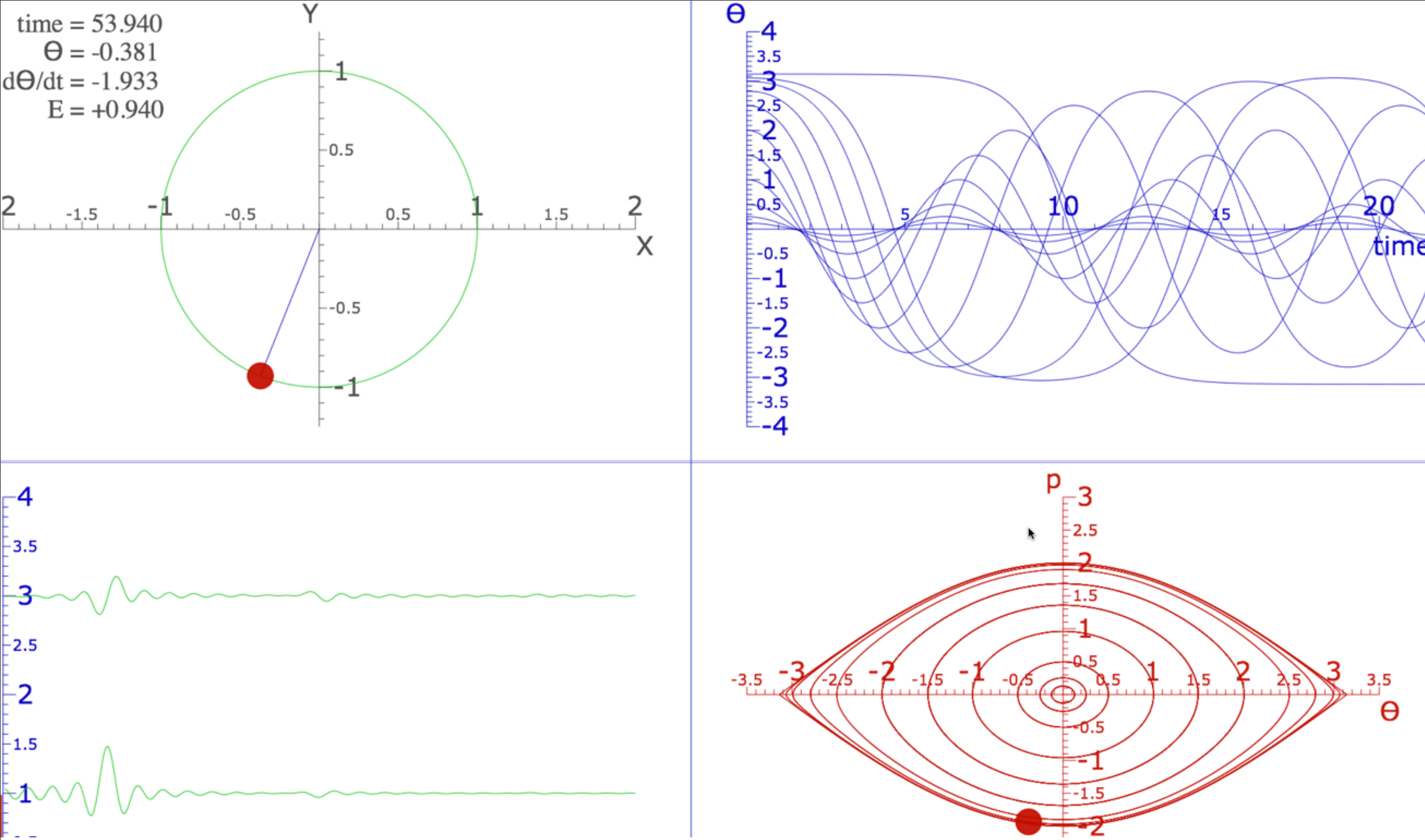



m:n
m-fold
by
n-fold
symmetry
1:1
2:1
3:1
4:1
5:1
6:1
1:2
2:2
3:2
4:2
5:2
6:2
1:3
2:3
3:3
4:3
5:3
6:3
1:4
2:4
3:4
4:4
5:4
6:4
1:5
2:5
3:5
4:5
5:5
6:5
(b) $\omega_\rho:\omega_\phi$ just below 1
prograde precession of nodes
$\omega_\rho:\omega_\phi=1$
$\omega_\rho:\omega_\phi$ just above 1
retrograde precession of nodes
(c) $\omega_\rho:\omega_\phi$ just below 2
prograde precession of nodes
$\omega_\rho:\omega_\phi=2$
$\omega_\rho:\omega_\phi$ just above 2
retrograde precession of nodes

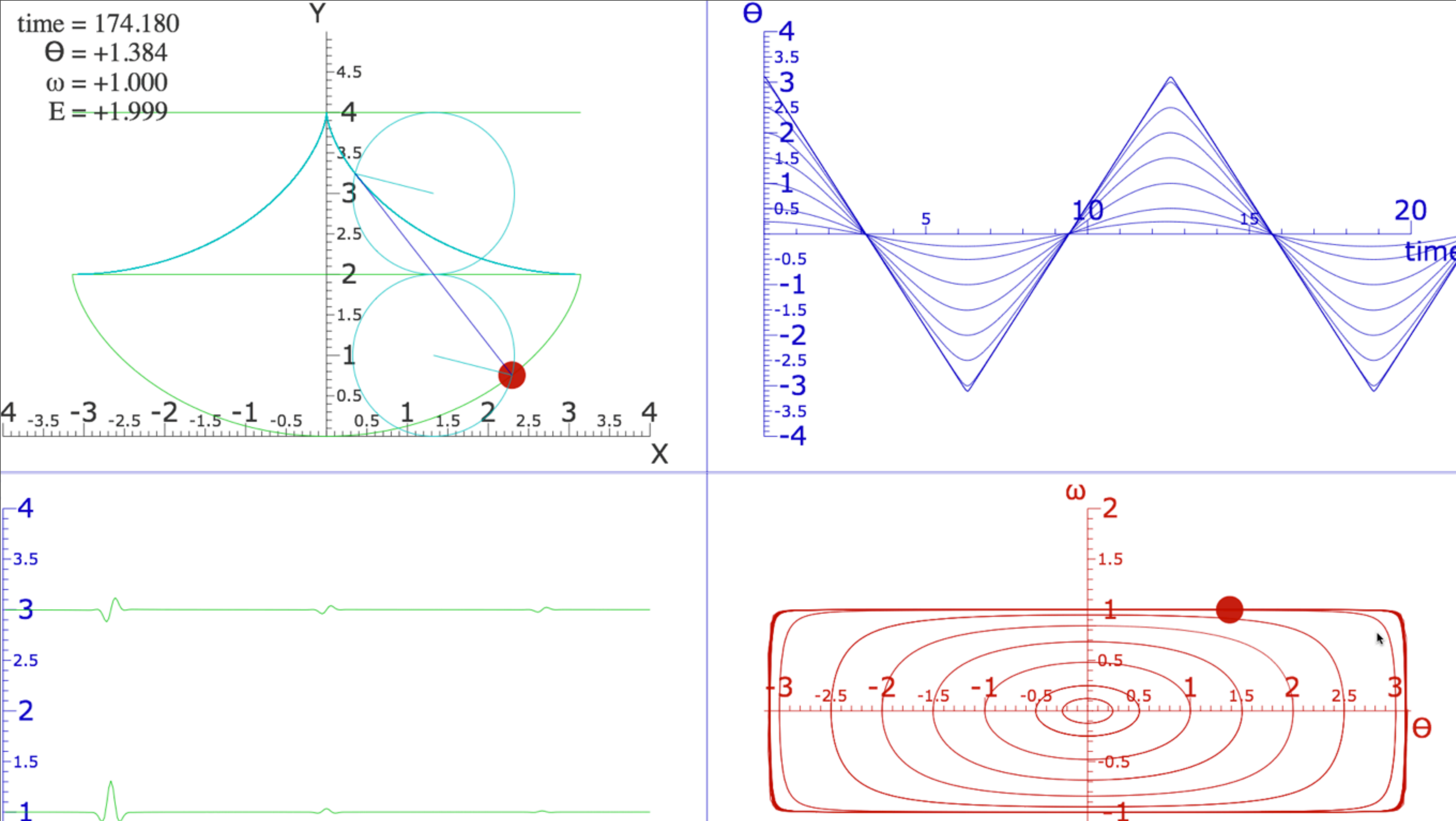
# *Separation of GCC Equations: Effective Potentials*

*Small radial oscillations*

→ *Cycloid vs Pendulum*

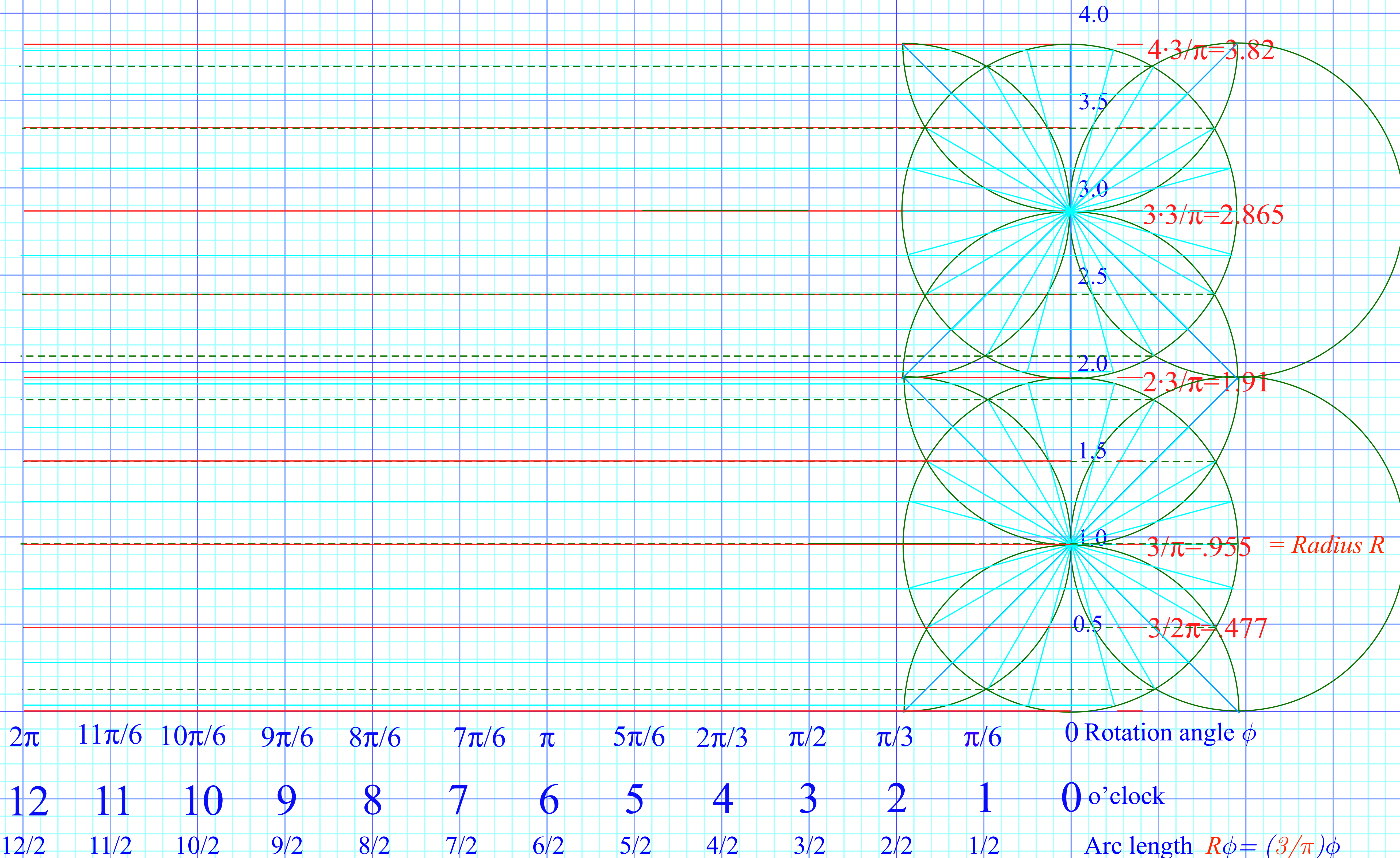



http://www.uark.edu/ua/modphys/markup/PendulumWeb.html




http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled-out circumference $R\phi= (3/\pi)m\pi/n=3m/n$.



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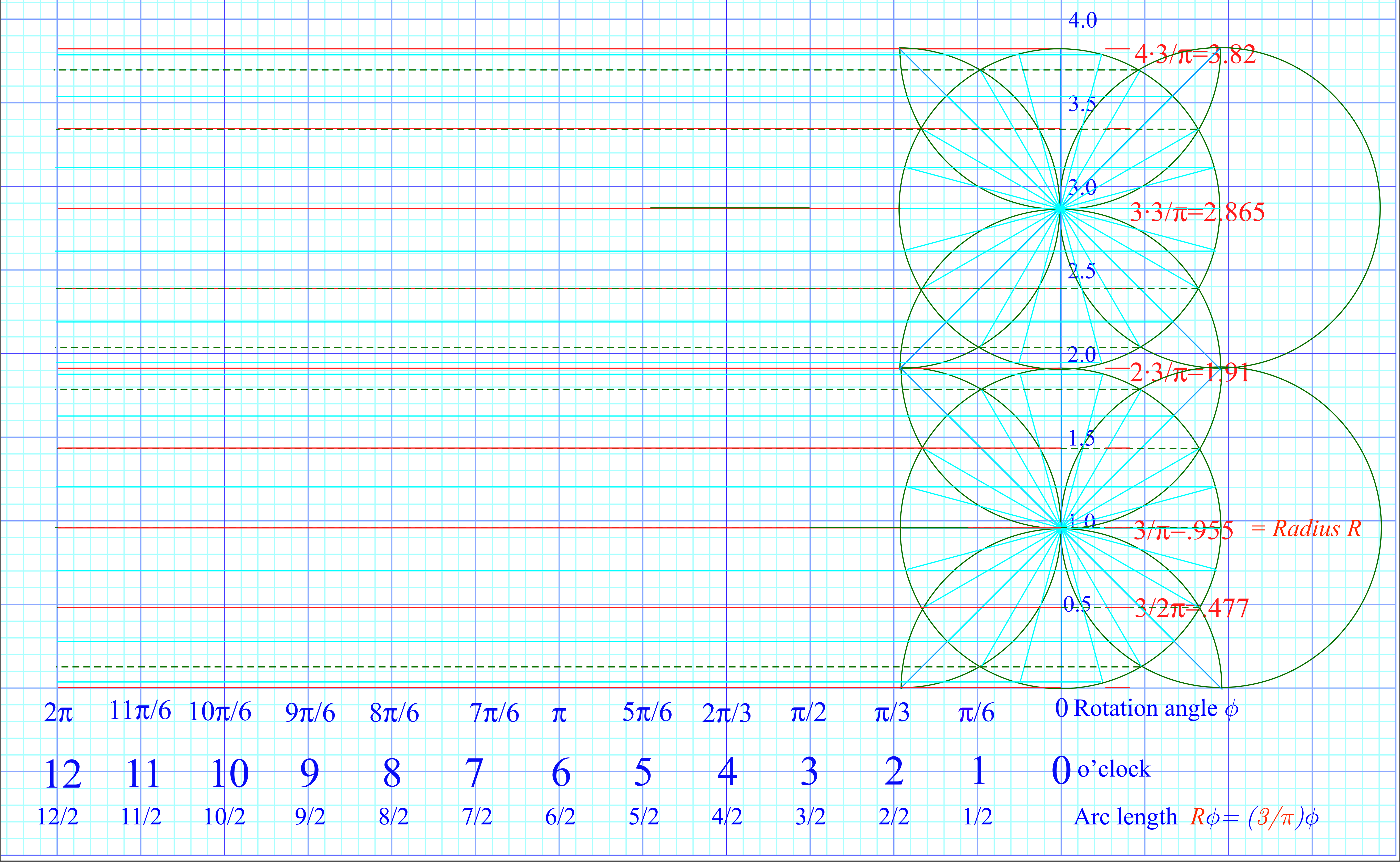

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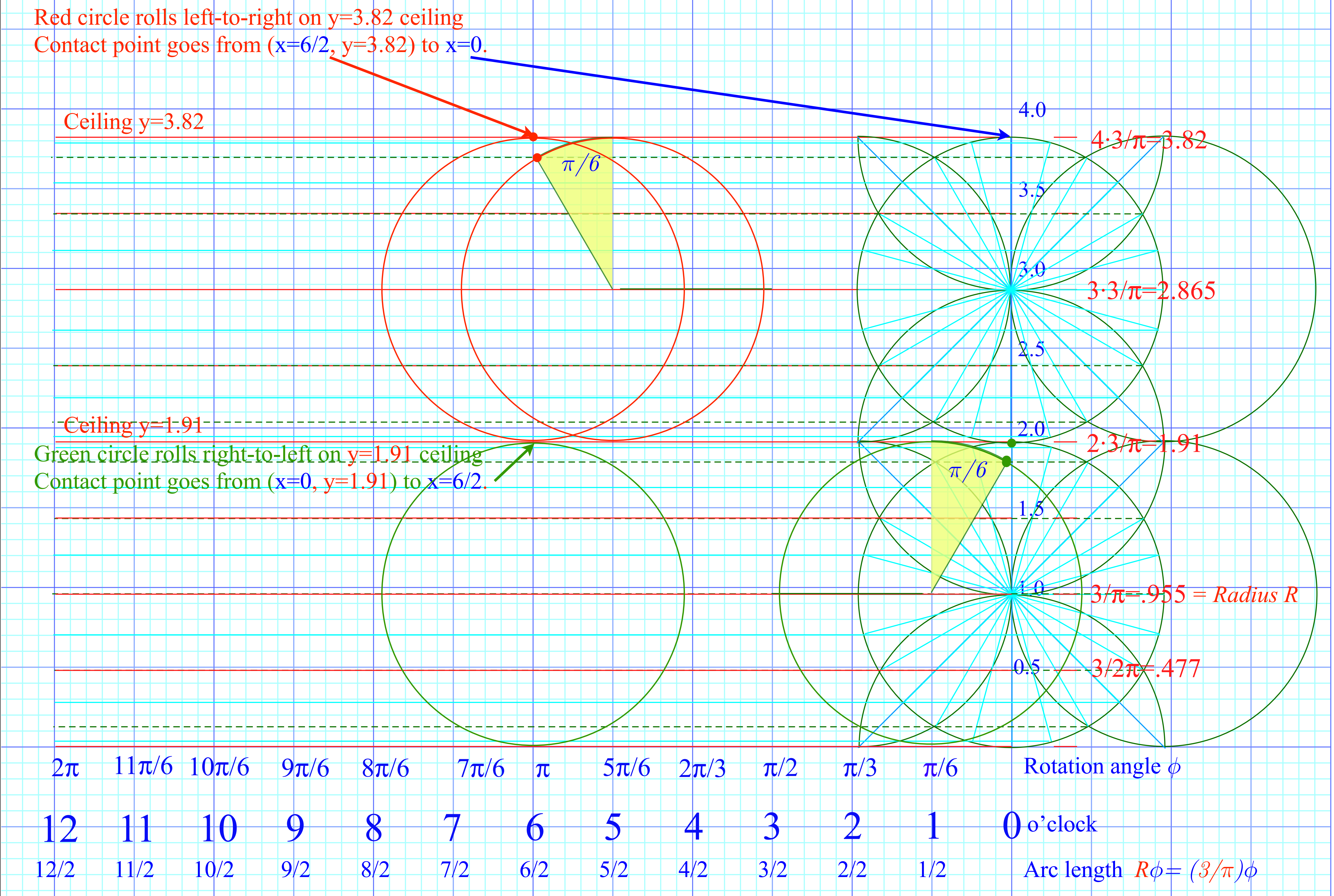

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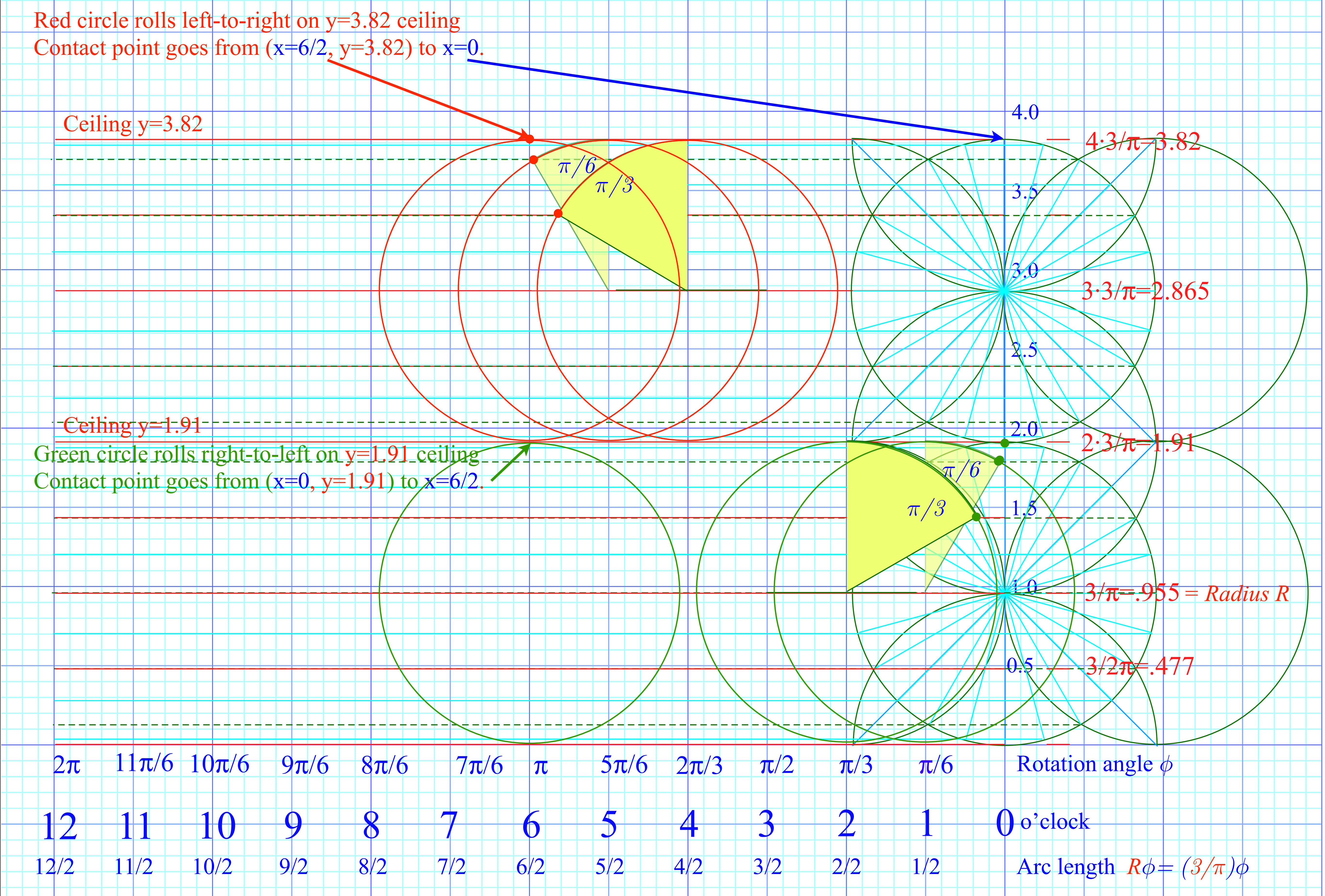

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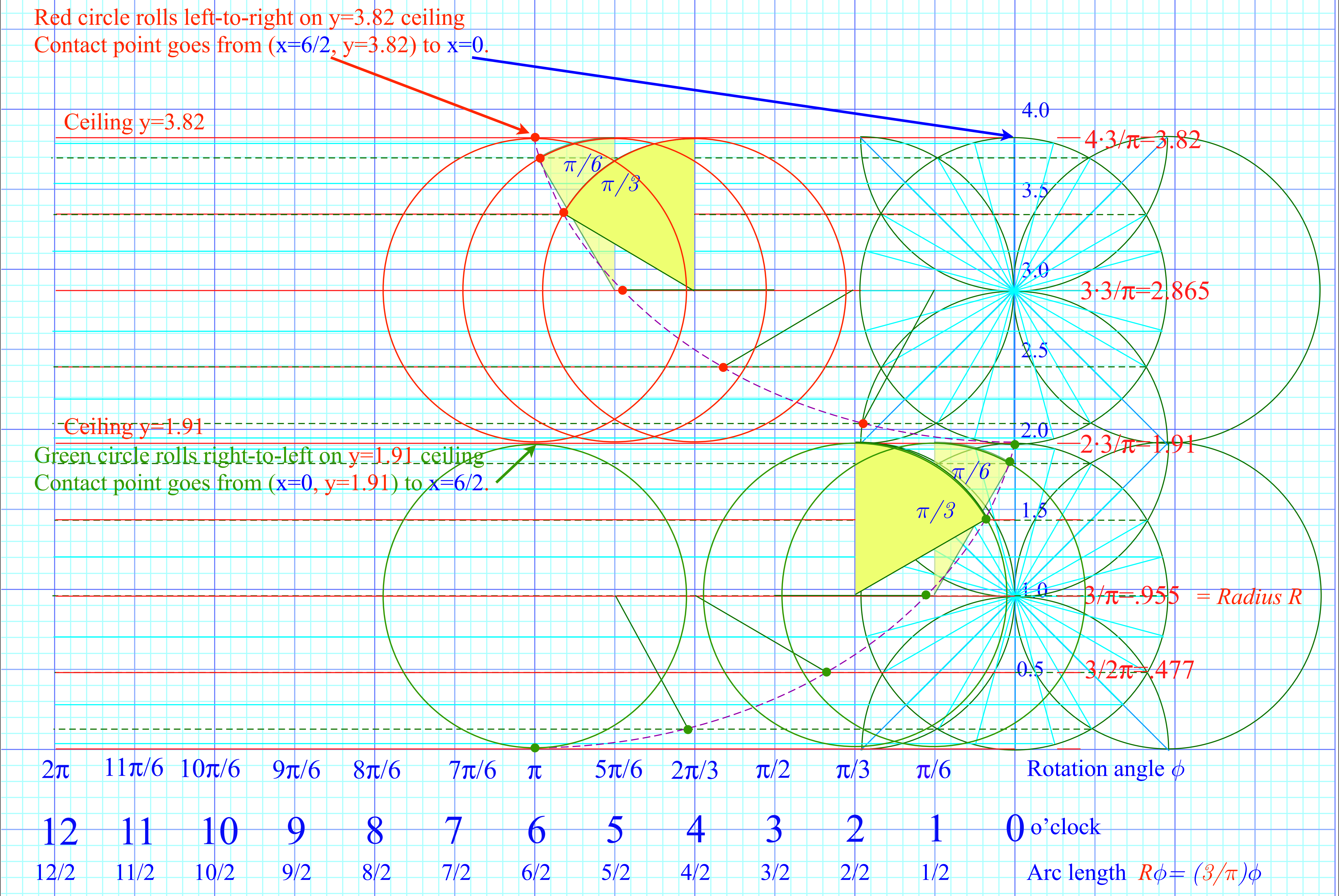
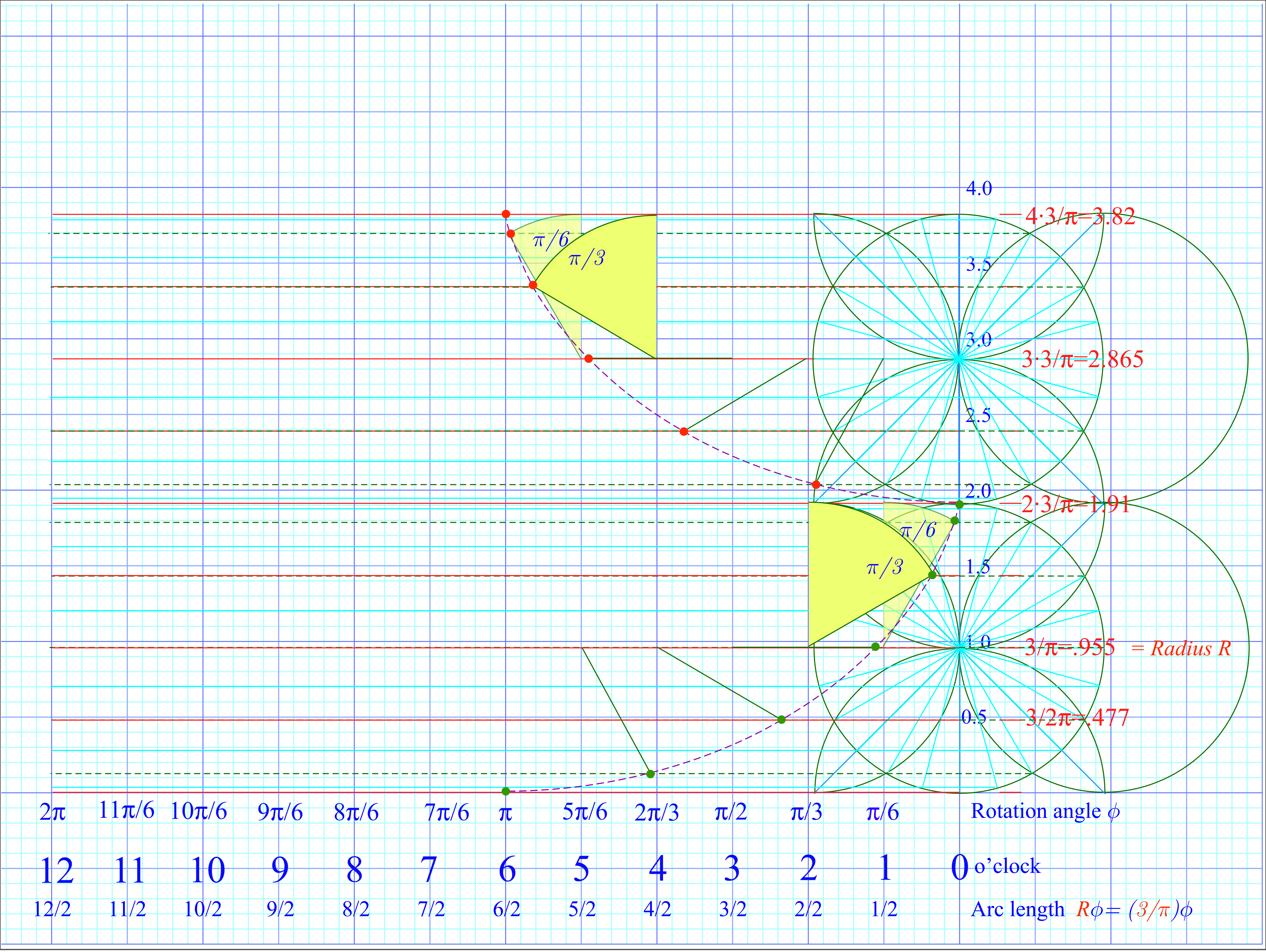
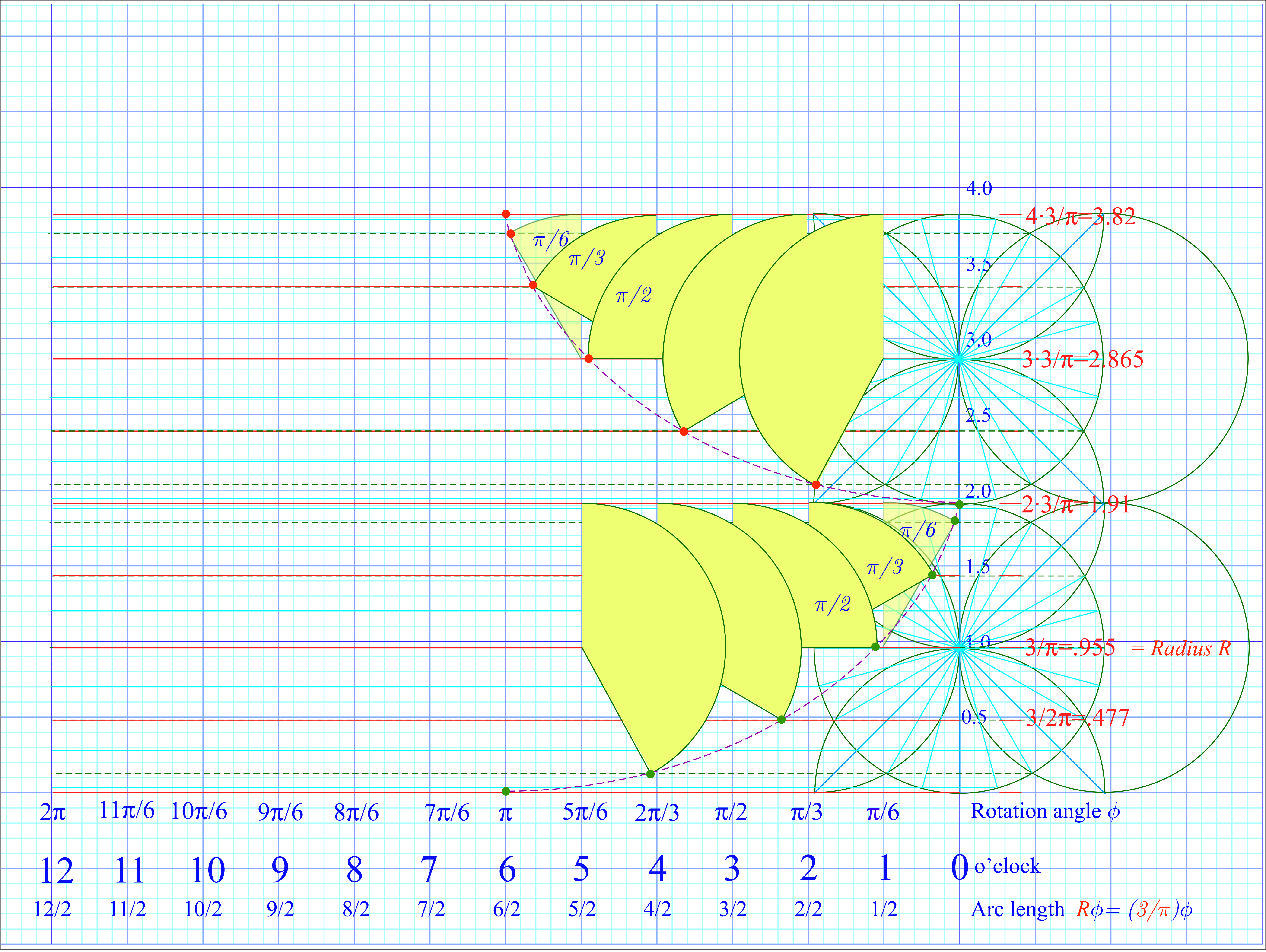




4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/6
π/3
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle φ
12
11
10
9
8
7
6
5
4
3
2
1
0 o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rφ= (3/π)φ



π/6
π/3
π/2
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
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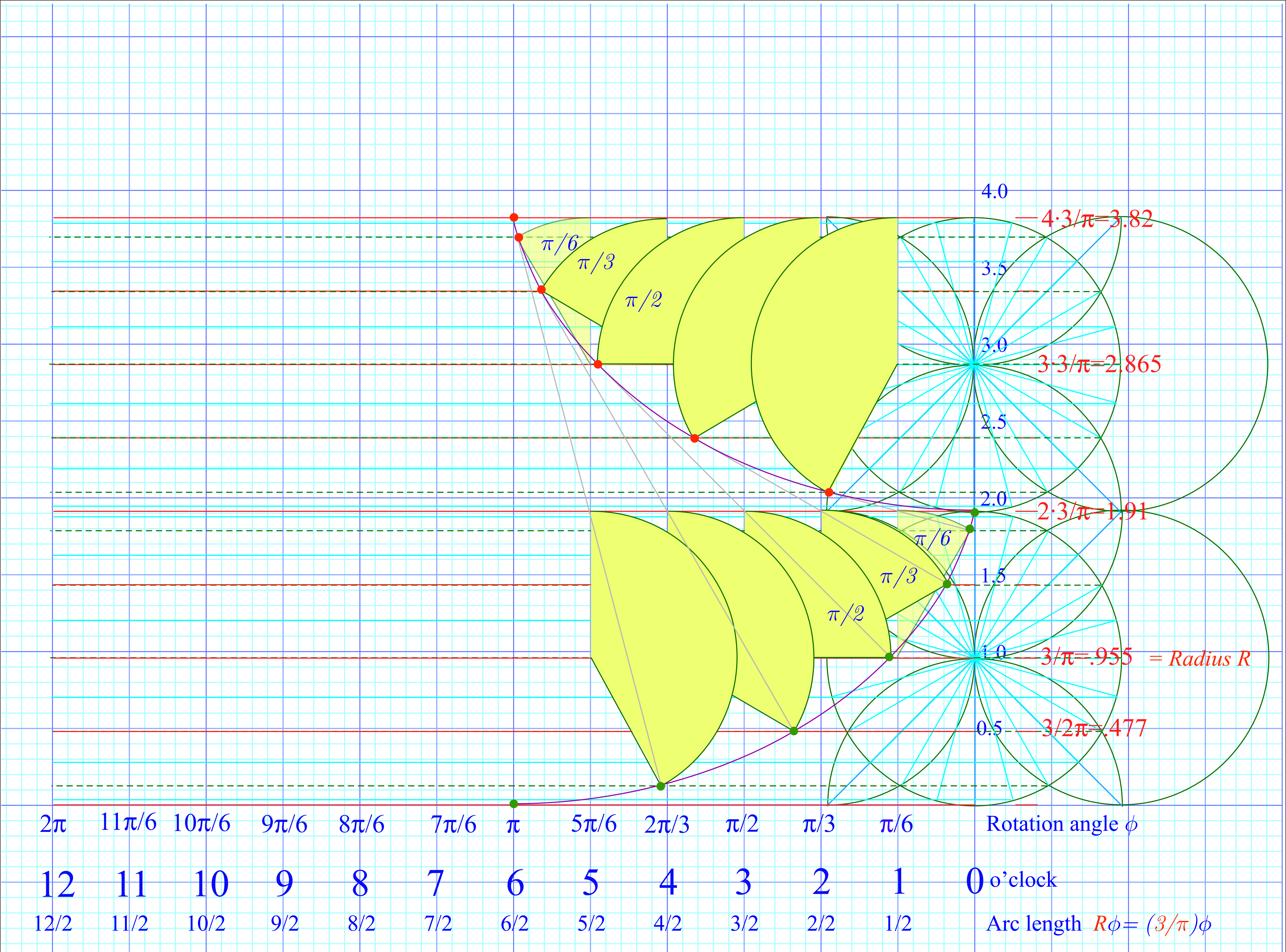
3/2π=.477
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6
Rotation angle ϕ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2
Arc length Rϕ= (3/π)ϕ



4.0
3.5
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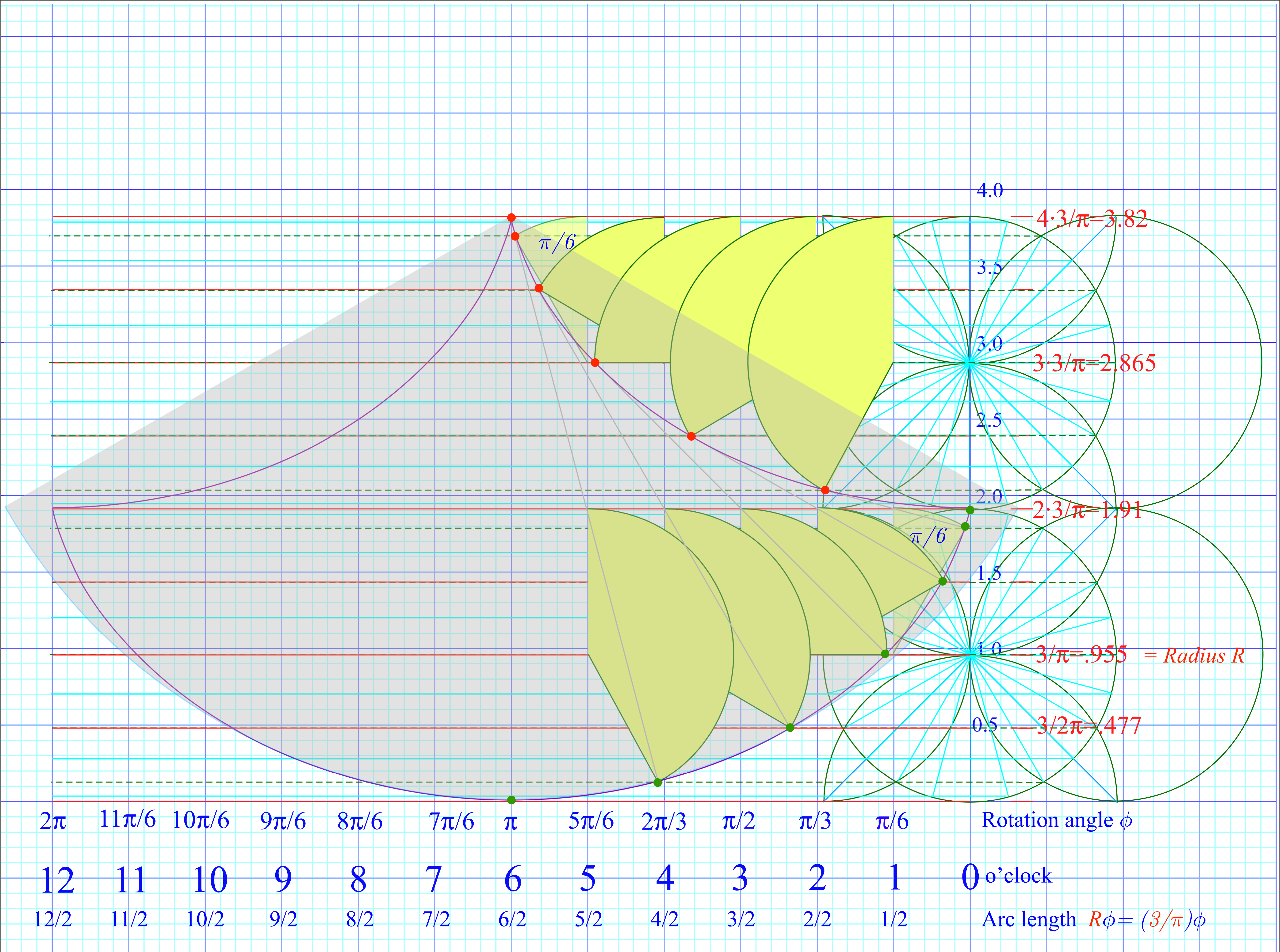
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rϕ= (3/π)ϕ


4.0
3.5
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2.5
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1.0
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π/6
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Rotation angle φ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2
Arc length Rφ= (3/π)φ

If you hammer a stick at a point $h$ meters from its center
you give it some linear momentum $\Pi$
and some angular momentum $\Lambda = h \cdot \Pi$

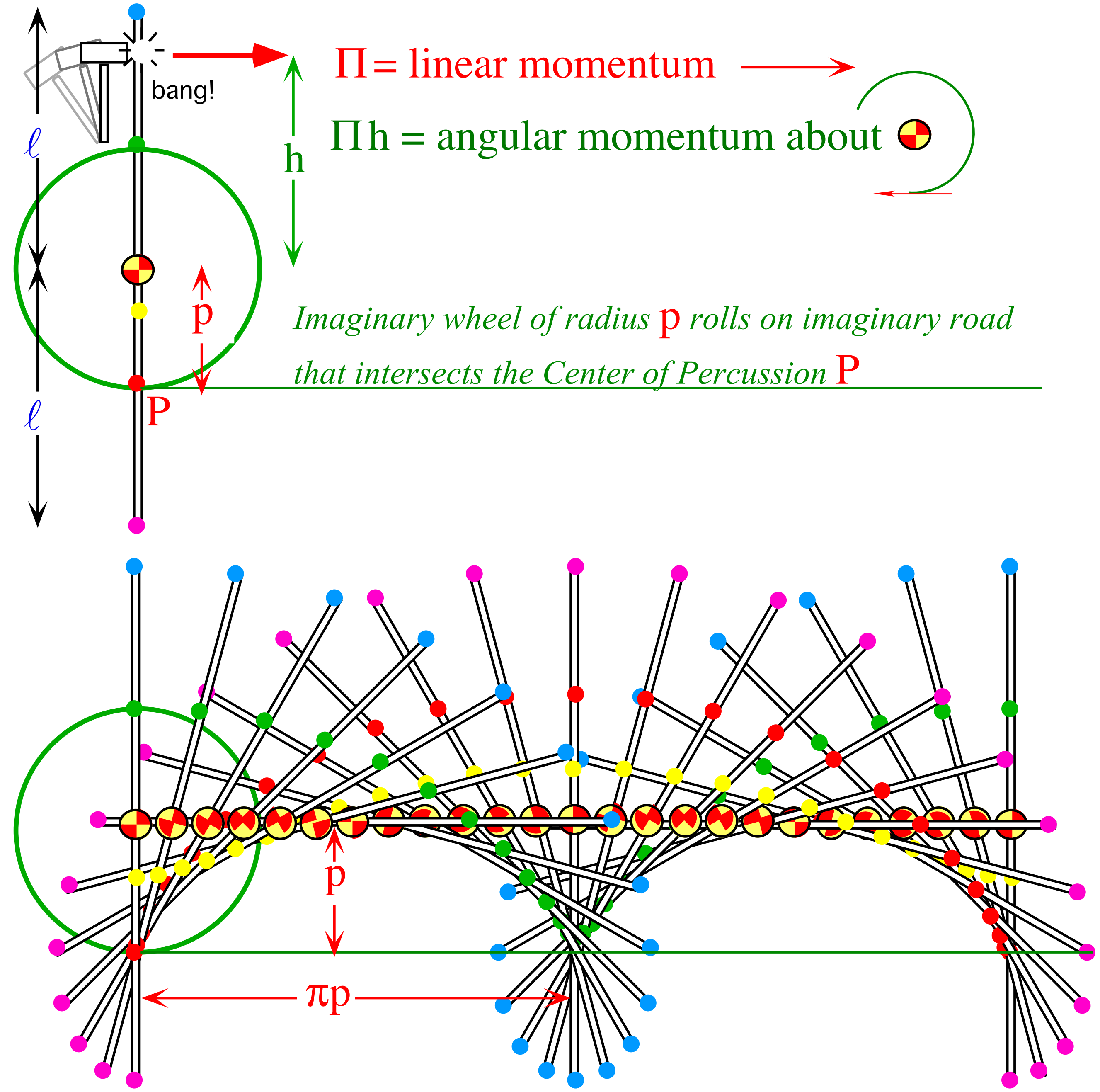
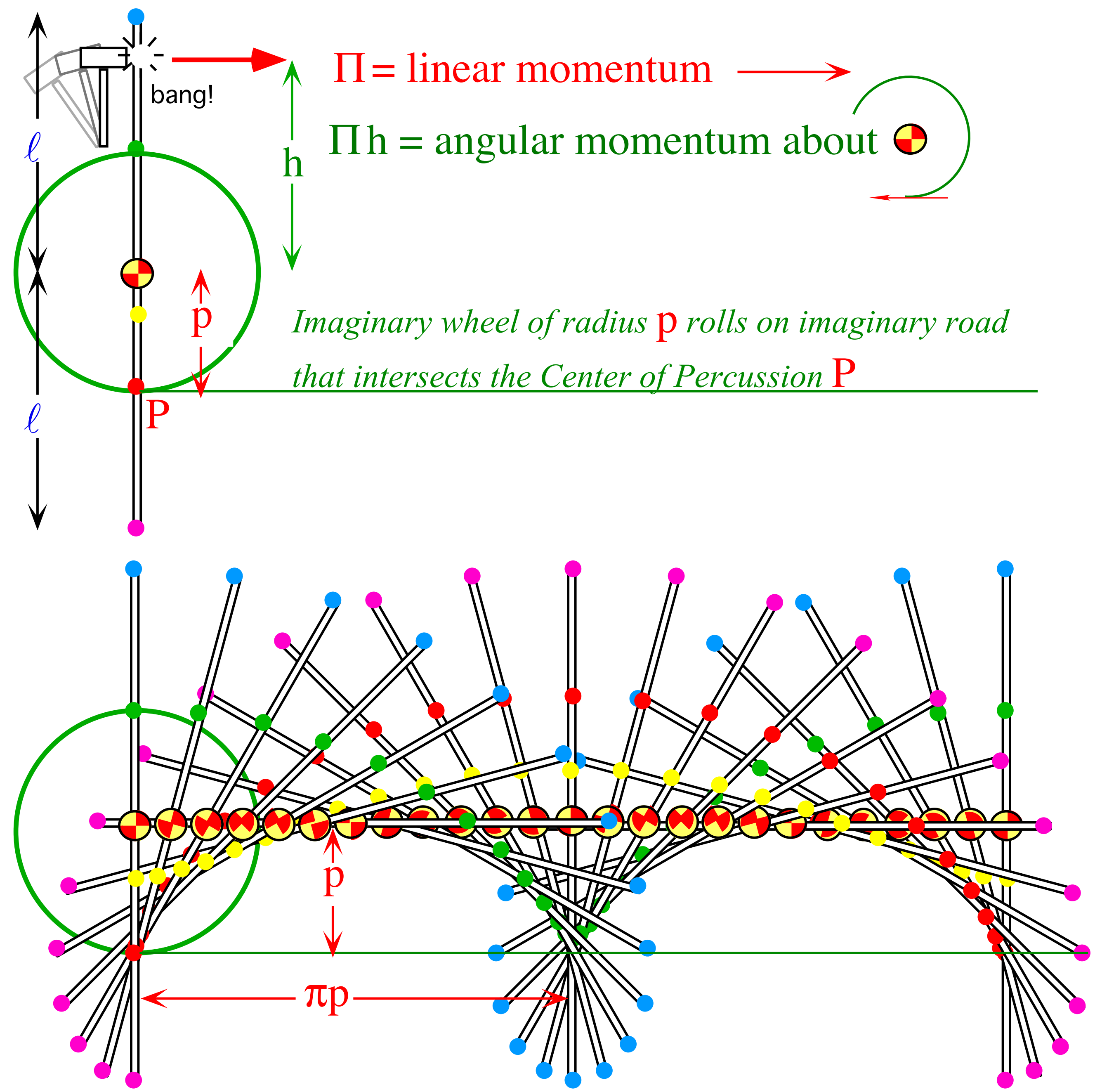



*Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.*

If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h{\cdot}\Pi$

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.

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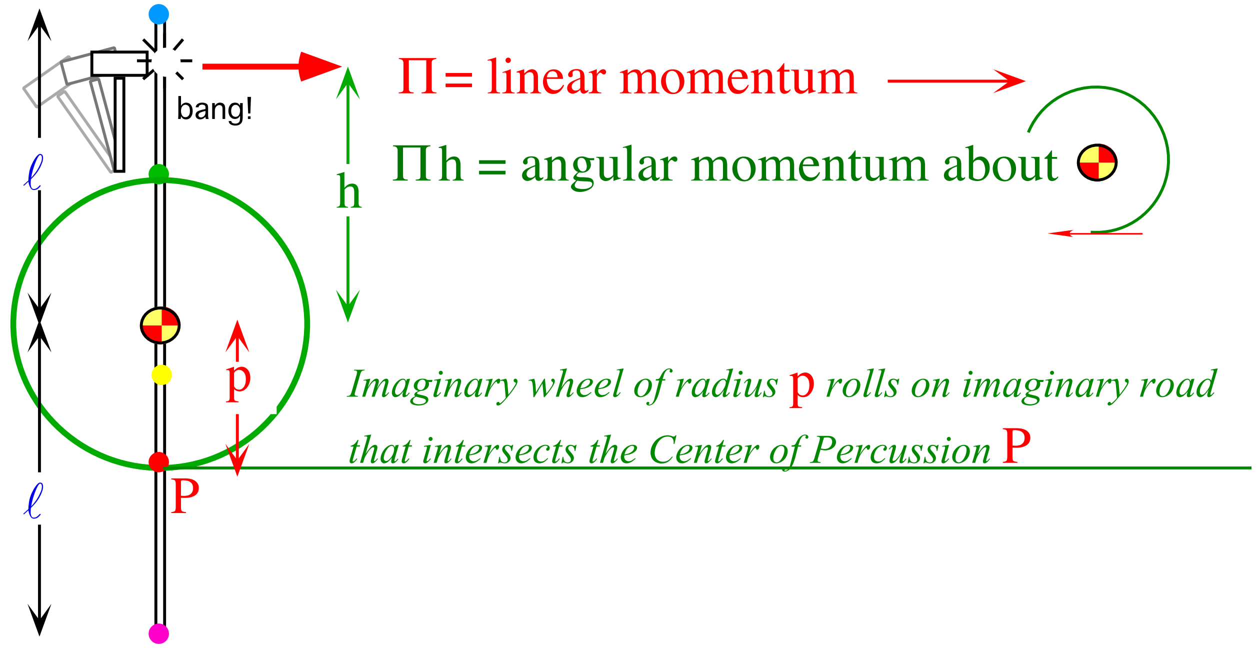


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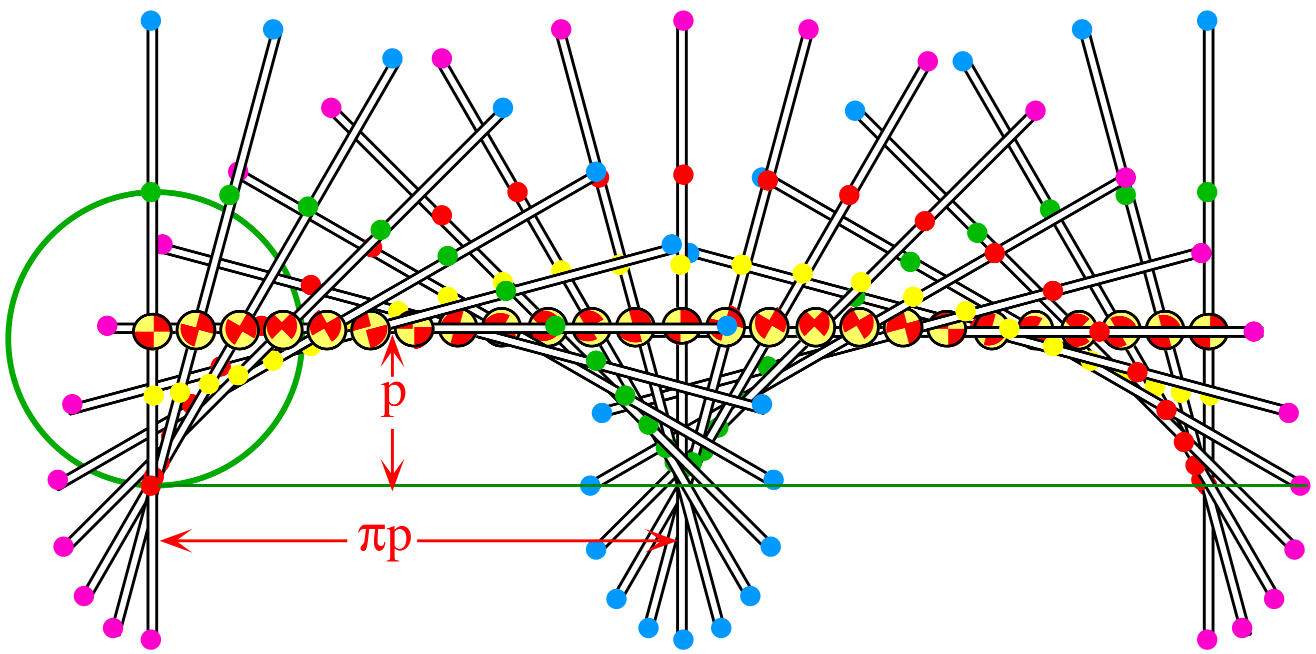
$$\omega = \Lambda / I \qquad (=3\Lambda/(M\ell^2) \text{ for stick})$$
$$= h\Pi / I \quad (=3h\Pi/(M\ell^2) \text{ for stick})$$

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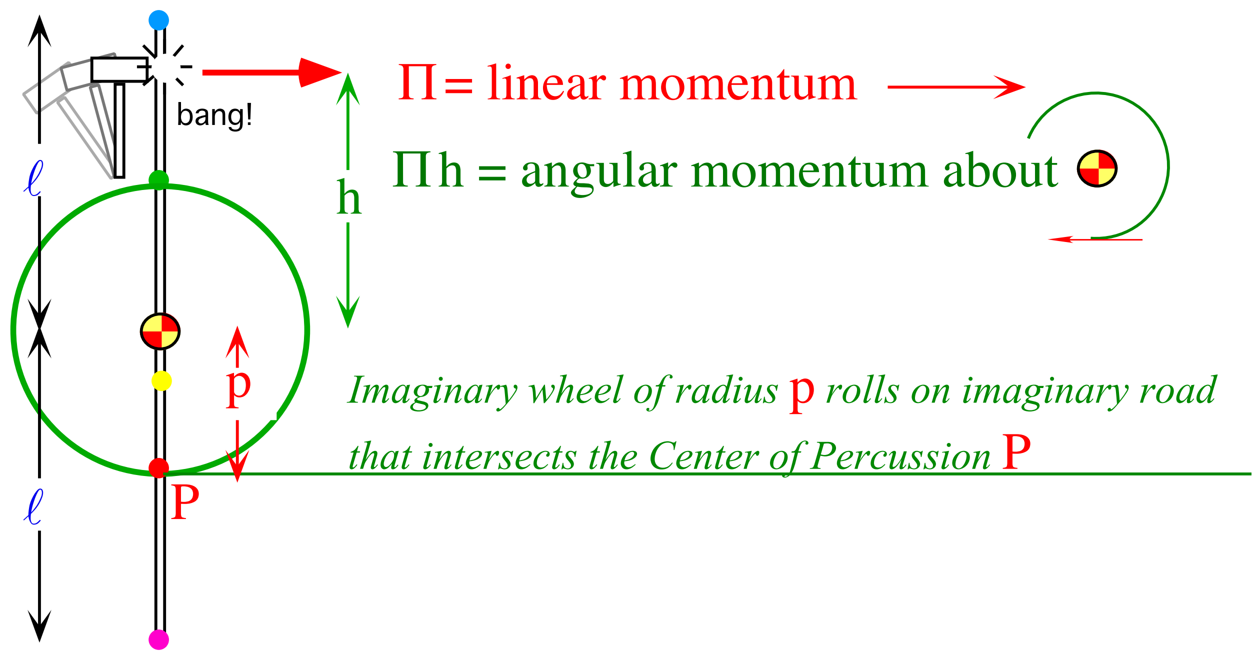


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One point P, or *center of percussion* (CoP), is
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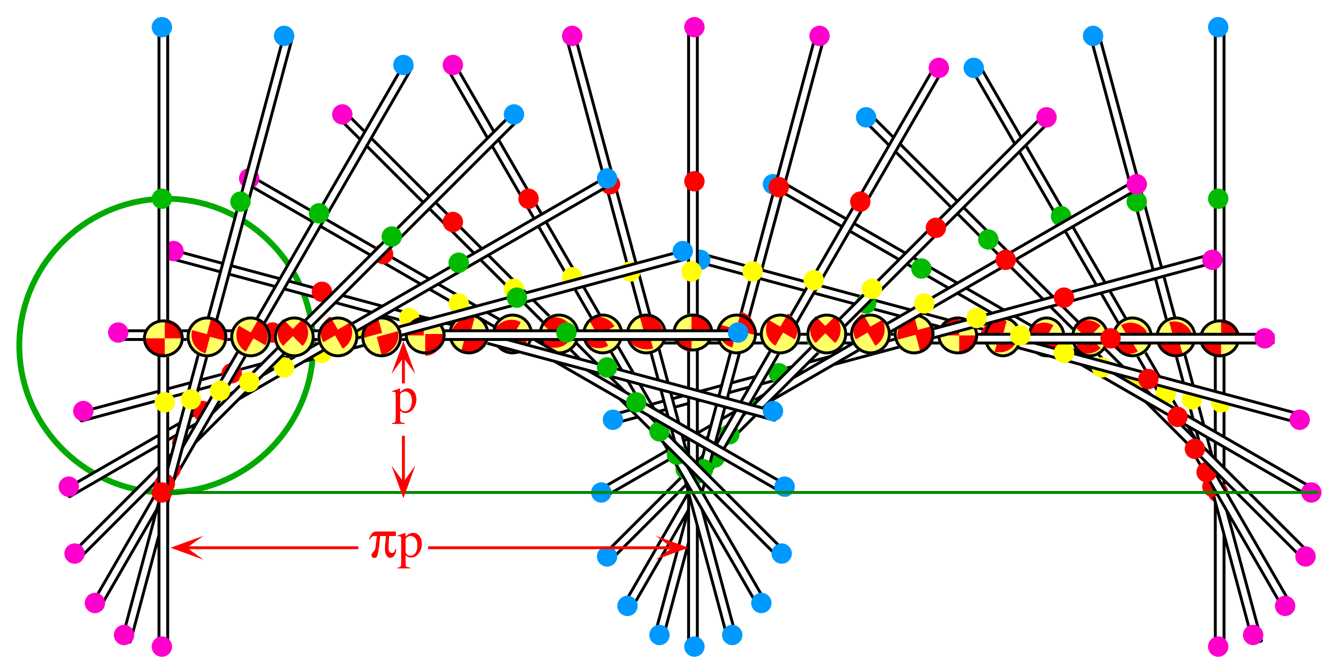


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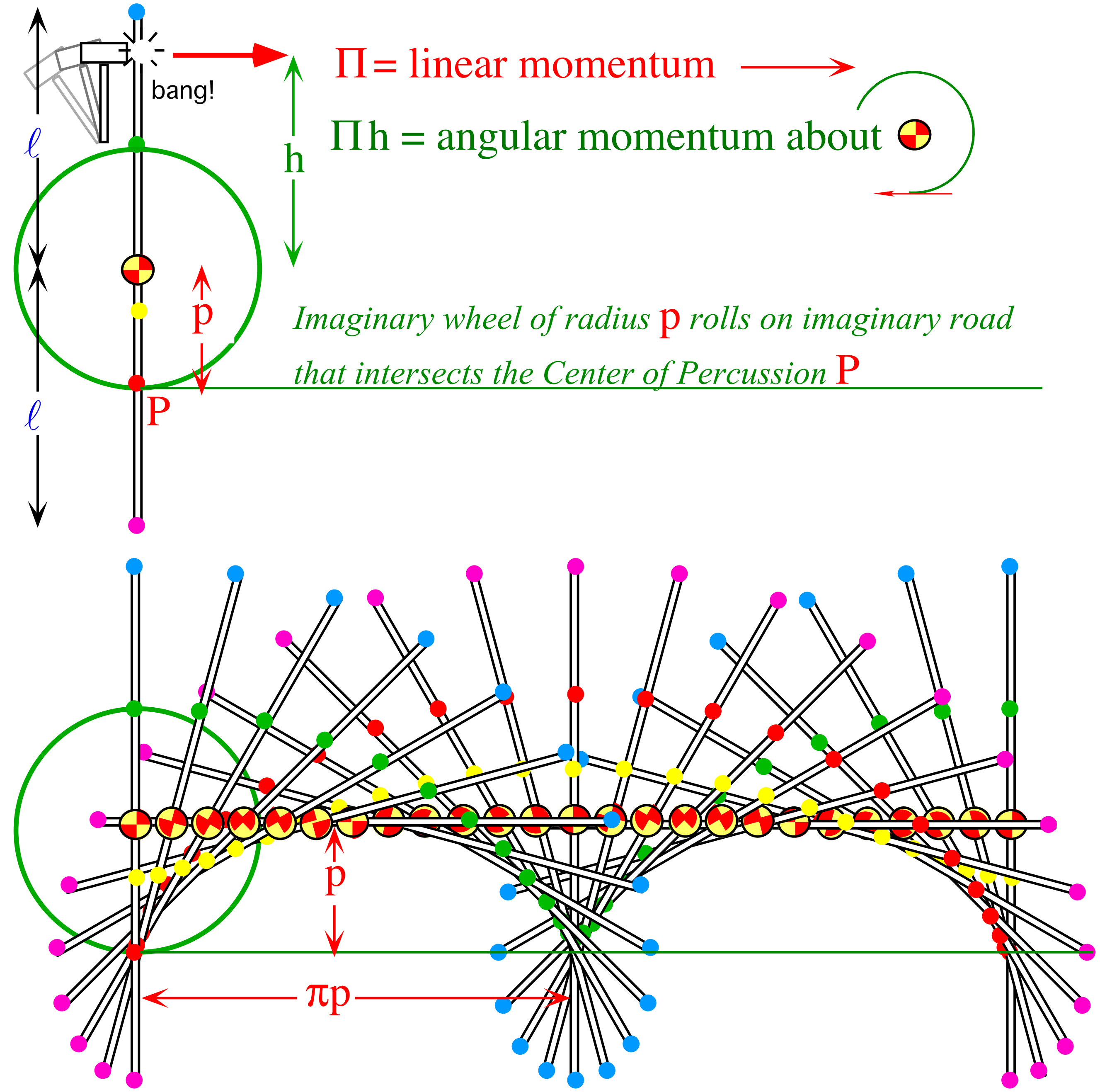
$$\Pi/M = V_{Center} = |p\omega| = p \cdot h\Pi/I$$

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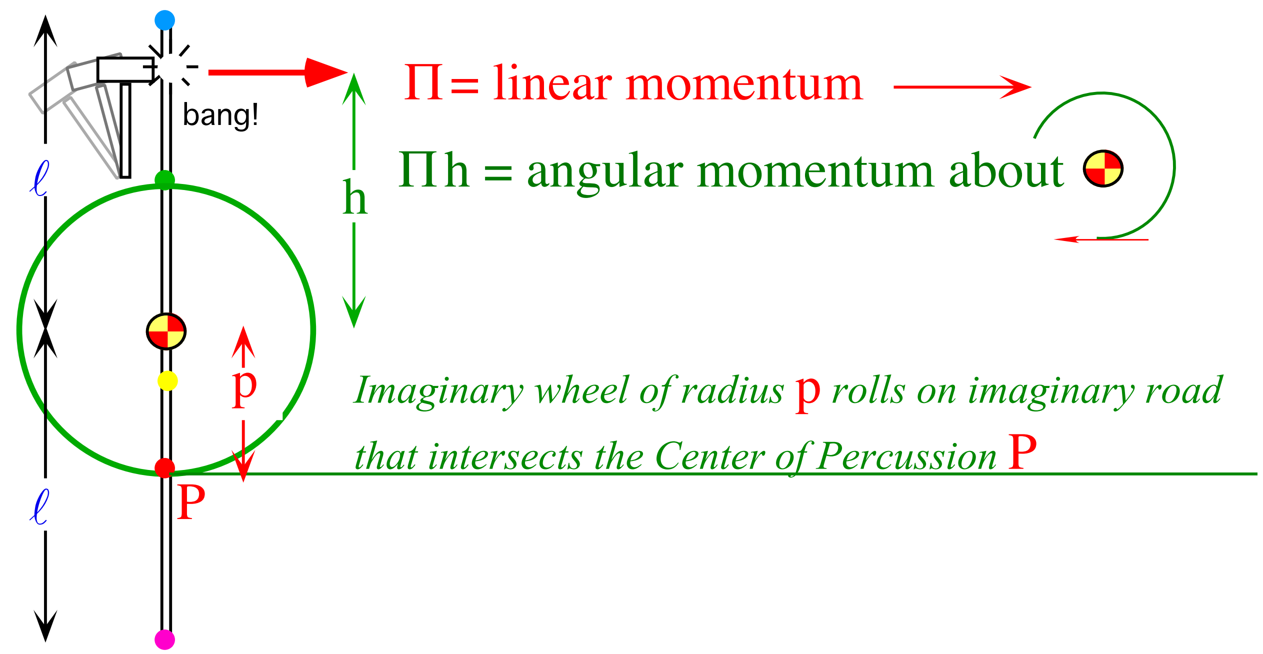


Resulting angular velocity $\omega$ about the center
is angular momentum $\Lambda$ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda\,/\,I \qquad (=3\Lambda\,/(M\,\ell^2)\ \text{for stick})$$
$$\quad = h\Pi\,/\,I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

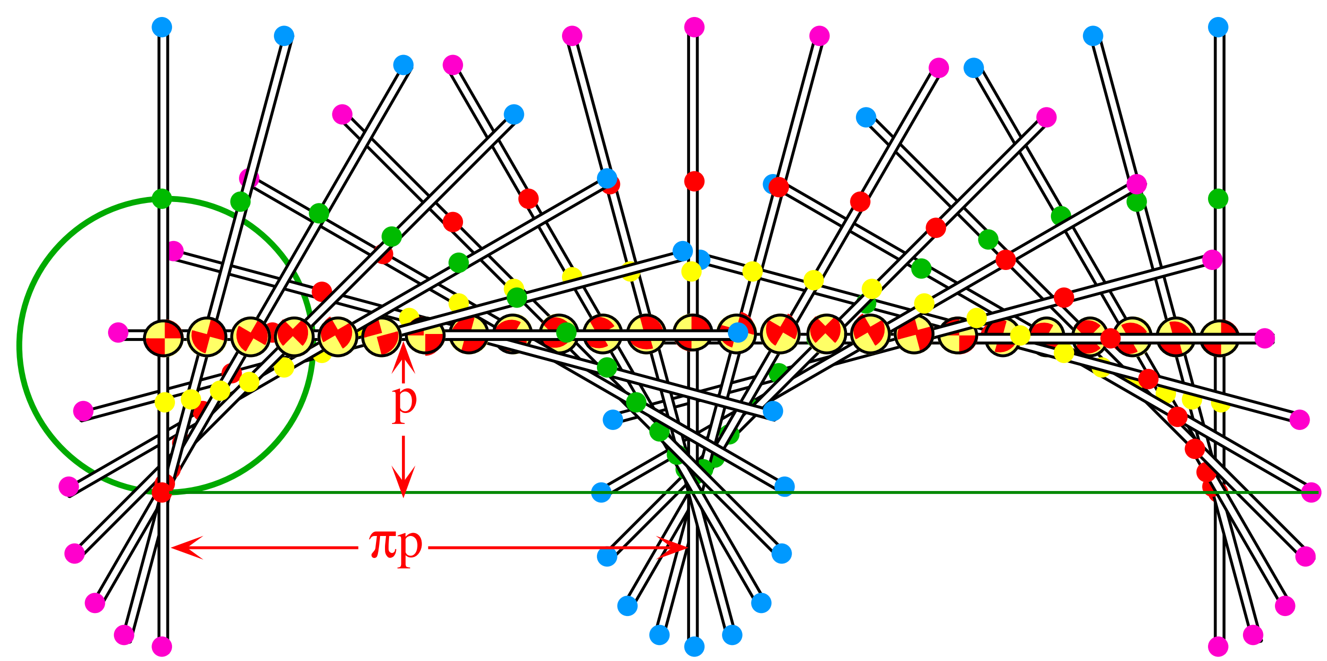


*Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.*

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed $V_{Center}$ of stick.

$$\Pi\,/M = V_{Center} = |p\omega| = p{\cdot}h\Pi/I$$
$$I\,/M = \qquad = \qquad = p{\cdot}h$$

If you hammer a stick at a point $h$ meters from its center
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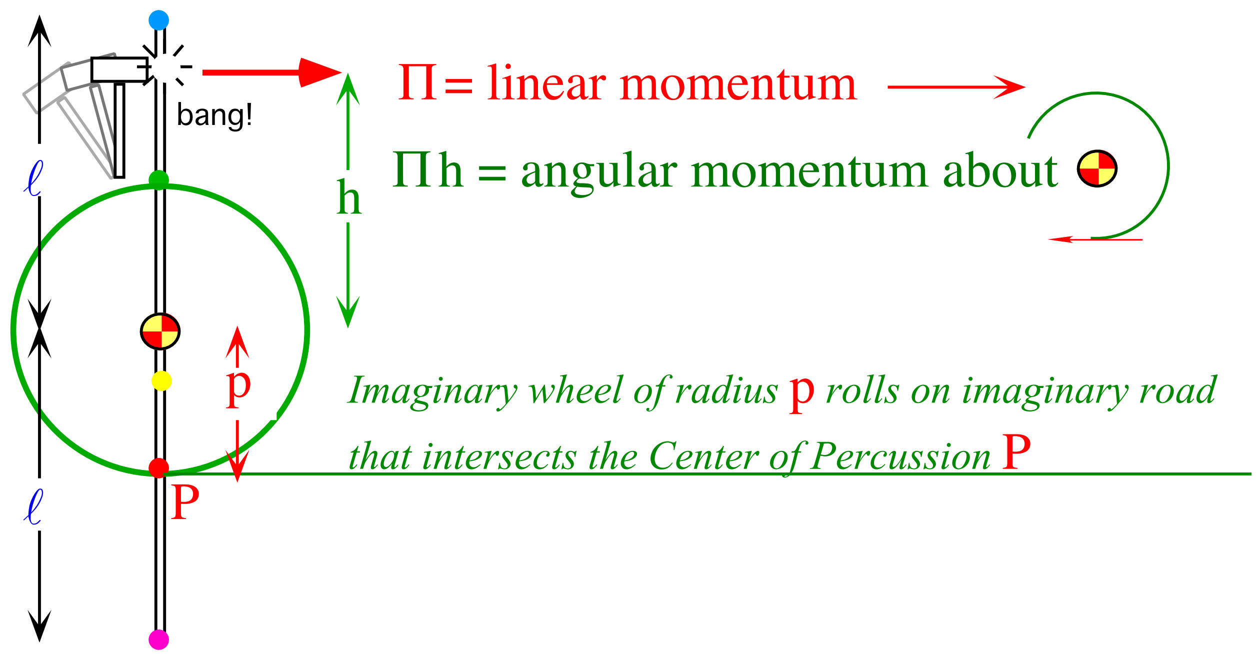
and some angular momentum $\Lambda = h\cdot\Pi$

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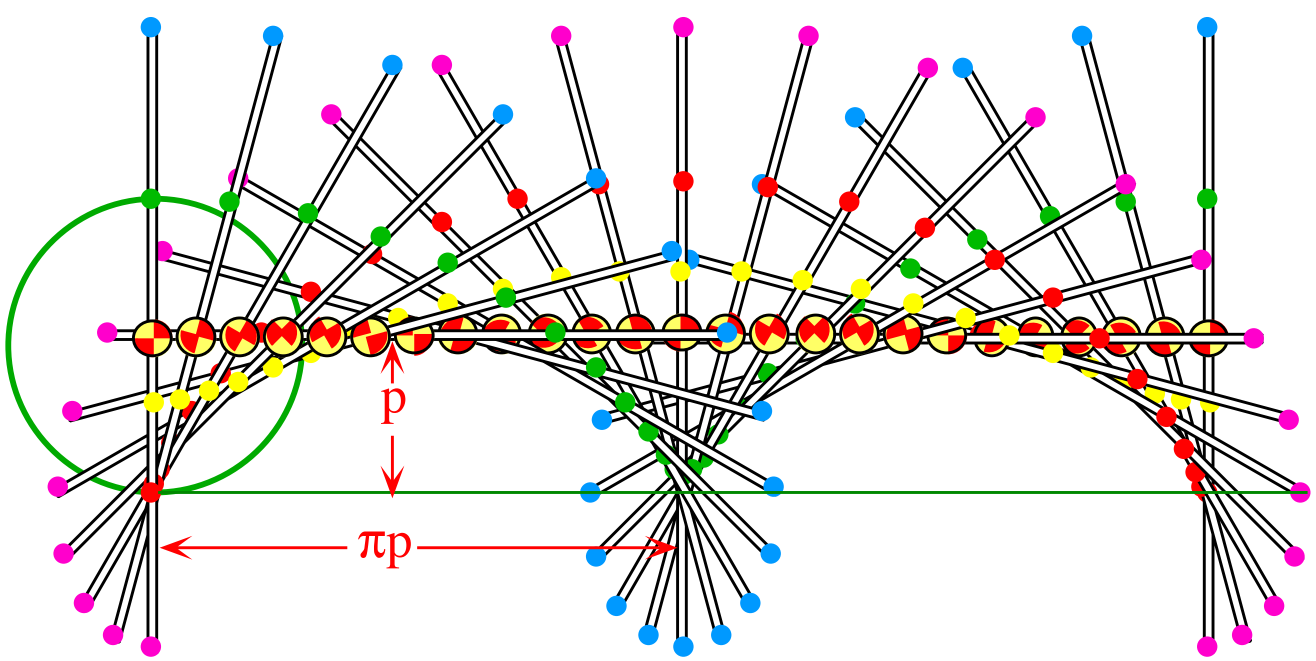


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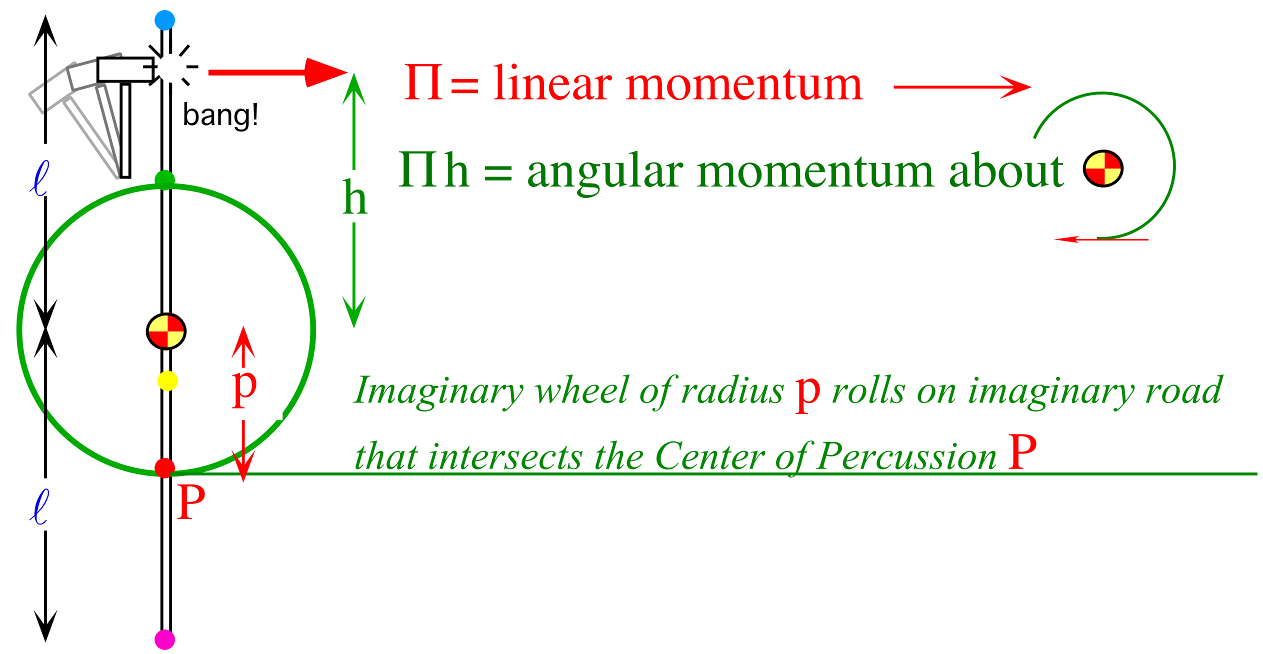


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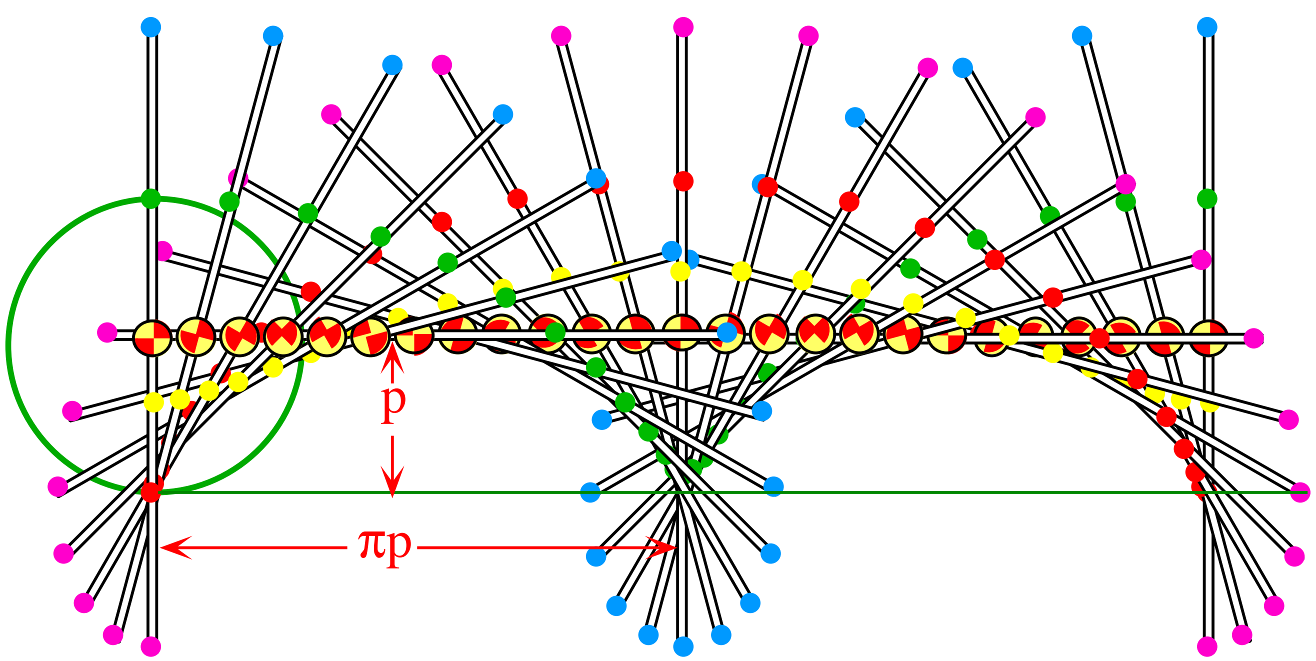
P follows a normal cycloid made by a circle of radius $p=I/(Mh)$ rolling on an imaginary road thru point P in direction of $\Pi$.

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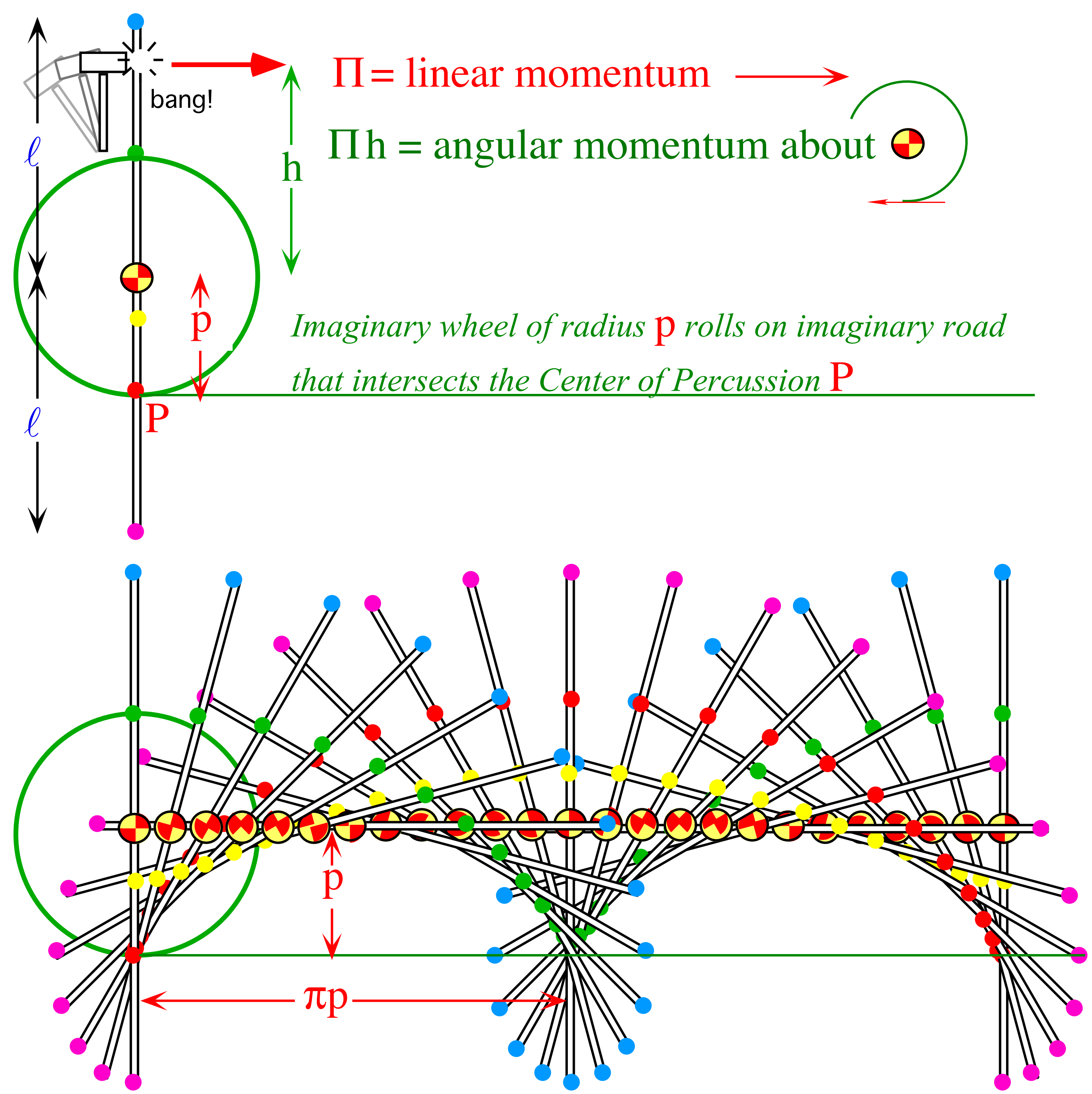


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The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at $h$.