Reimann-Christoffel equations and covariant derivative  
(Ch. 4-7 of Unit 3)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

Christoffel g-derivative formula
What’s a tensor? What’s not?

General Riemann equations of motion (No explicit t-dependence and fixed GCC)

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Separation of GCC Equations: Effective Potentials

Small radial oscillations
Cycloid vs Pendulum
Covariant derivative and Christoffel Coefficients $\Gamma_{ij,k}$ and $\Gamma_{ij;k}$

Christoffel g-derivative formula

What’s a tensor? What’s not?
Covariant derivative and Christoffel Coefficients $\Gamma_{ij; k}$ and $\Gamma_{ij; k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. changing $U^m$ components
2. curving GCC vectors $E_n$.

\[
\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} \left( E_m \right) + U^n \frac{\partial E_n}{\partial q^i}
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Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

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(Note funny semi-colon ; notation)

2. Curving GCC vectors $E_n$

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

$$\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell$$
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

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\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell
\]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind
defined by:

\[
\Gamma_{in;\ell} = \frac{\partial E_n}{\partial q^i} \cdot E_\ell = \Gamma_{ni;\ell}
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

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2. curving GCC vectors $E_n$

(Note funny semi-colon ; notation)

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

$$\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell = \Gamma_{in;}^m E_m$$

**Christoffel coefficients $\Gamma_{ij;}^k$ of the first kind** defined by:

$$\Gamma_{in;}^m = \frac{\partial E_n}{\partial q^i} \cdot E_m = \Gamma_{ni;}^m$$

**Christoffel coefficients $\Gamma_{ij;k}$ the second kind** defined by:

$$\Gamma_{ij;k} = \frac{\partial E_j}{\partial q^i} \cdot E_k = \Gamma_{ni;k}$$
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1) changing $U^m$ components

$$\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^m \frac{\partial E_n}{\partial q^i}$$

2) curving GCC vectors $E_n$

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$$\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell = \Gamma_{in;m} E_m$$

**Christoffel coefficients $\Gamma_{ij;k}$ of the first kind**

defined by:

$$\Gamma_{in;\ell} = \frac{\partial E_n}{\partial q^i} \cdot E_\ell = \Gamma_{ni;\ell}$$

$i,n$ to $n,i$ symmetry guaranteed here

**Christoffel coefficients $\Gamma_{ij;k}$ the second kind**

defined by:

$$\Gamma_{in;m} = \frac{\partial E_n}{\partial q^i} \cdot E_m = \Gamma_{ni;m}$$

$i,n$ to $n,i$ symmetry guaranteed here
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

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1. changing $U^m$ components
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2. curving GCC vectors $E_n$

(Note funny semi-colon ; notation)

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$
\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell = \Gamma_{in;\ell}^m E_m
\]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:
\[
\Gamma_{in;\ell} = \frac{\partial E_n}{\partial q^i} \cdot E^\ell = \Gamma_{ni;\ell}
\]

Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:
\[
\Gamma_{in;\ell}^m = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni;\ell}^m
\]

Q: Do we need a third kind of $\Gamma$-coefficient or a $\Lambda$-coefficient?
(to differentiate contravariant-$E^n$ or covariant $U_n$)
\[
\frac{\partial E^n}{\partial q^i} = \Lambda_{im}^n E^m, \text{ where: } \Lambda_{im}^n = \frac{\partial E^n}{\partial q^i} \cdot E_m
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $\mathbf{U}$ are due to:

1. changing $U^m$ components
   \[
   \frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n \frac{\partial E^n}{\partial q^i}
   \]

2. curving GCC vectors $\mathbf{E}_n$

   Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}\ell$ or else $\mathbf{E}_m$
   \[
   \frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;}^\ell \mathbf{E}\ell = \Gamma_{in;}^m \mathbf{E}_m
   \]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:
\[
\Gamma_{in;}^\ell = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}\ell = \Gamma_{ni;}^\ell
\]

Christoffel coefficients $\Gamma_{ij;}^k$ the second kind defined by:
\[
\Gamma_{in;}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_m = \Gamma_{ni;}^m
\]

Q: Do we need a third kind of $\Gamma$-coefficient or a $\Lambda$-coefficient?
(to differentiate contravariant $\mathbf{E}^n$ or covariant $U_n$)

A: NO! That $\Lambda$-coefficient is just a $\Gamma$-coefficient with a (-).

\[
\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda_{im}^n \mathbf{E}_m, \text{ where: } \Lambda_{im}^n = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_m
\]

So: $\Lambda_{im}^n = -\Gamma_{im}^n$
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $\mathbf{U}$ are due to:

1. changing $U^m$ components
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\]

2. curving GCC vectors $\mathbf{E}_n$
\[
\frac{\partial \mathbf{E}_n}{\partial q^j} = \Gamma_{in;}^j \mathbf{E}_n = \Gamma_{ni;}^j \mathbf{E}_m
\]

(Note funny semi-colon ; notation)

Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}^\ell$ or else $\mathbf{E}_m$
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\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;}^\ell \mathbf{E}_n = \Gamma_{ni;}^\ell \mathbf{E}_m
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Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:
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Christoffel coefficients $\Gamma_{ij;}^k$ the second kind defined by:
\[
\Gamma_{in;}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \mathbf{E}_m = \Gamma_{ni;}^m
\]

Any vector derivative can be expressed using $\Gamma_{ij;}^k$ in terms of $\mathbf{E}_m$
\[
\frac{\partial \mathbf{U}}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) \mathbf{E}_m
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. Changing $U^m$ components
   \[
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\Gamma_{in;m} = \frac{\partial E_n}{\partial q^i} \cdot E_m = \Gamma_{ni;m}
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Any vector derivative can be expressed using $\Gamma_{ij;k}$ in terms of $E_m$ or $E^m$

\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;m} \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im;n} \right) E^m
\]

So:
\[
\Lambda_{im}^n = -\Gamma_{im}^n
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;\;k}$ and $\Gamma_{ij;\;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

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2. curving GCC vectors $E_n$

   (Note funny semi-colon ; notation)

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

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\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\;\ell} E^\ell = \Gamma_{in;\;m} E_m
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**Christoffel coefficients $\Gamma_{ij;\;k}$ of the first kind**

defined by:

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\Gamma_{in;\;\ell} = \frac{\partial E_n}{\partial q^i} \cdot E^\ell = \Gamma_{ni;\;\ell}
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**Christoffel coefficients $\Gamma_{ij;\;k}$ the second kind**

defined by:

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\Gamma_{in;\;m} = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni;\;m}
\]

Any vector derivative can be expressed using $\Gamma_{ij;\;k}$ in terms of $E_m$ or $E^m$

\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;\;m} \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im;\;n} \right) E^m
\]

\[
= U^m_{\;;i} E_m = U_{m;\;i} E^m
\]

(Not more funny semi-colon ; notation)
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

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2. curving GCC vectors $E_n$

   \[
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Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:

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Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:

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\Gamma_{in;}^m = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni;}^m
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Any vector derivative can be expressed using $\Gamma_{ij;k}$ in terms of $E_m$ or $E^m$

\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im;}^n \right) E^m
\]

Defining covariant derivative $U^m_{;i}$ of a contravariant component $U^m$

\[
U^m_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m
\]

(Note more funny semi-colon ; notation)

So:

\[
\Lambda^m_{im} = -\Gamma^n_{im}
\]

(Note more funny semi-colon ; notation)

Tuesday, November 4, 2014
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $U$ are due to:
(1) changing $U^m$ components

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Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

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\frac{\partial E_n}{\partial q^i} = \Gamma_{in;}^m E_m = \Gamma_{in;}^\ell E^\ell
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\[
\Gamma_{in;}^\ell = \frac{\partial E_n}{\partial q^i} \cdot E_\ell = \Gamma_{ni;}^\ell
\]

Christoffel coefficients $\Gamma_{ij;}^k$ the second kind defined by:

\[
\Gamma_{in;}^m = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni;}^m
\]

Any vector derivative can be expressed using $\Gamma_{ij;}^k$ in terms of $E_m$ or $E^m$

\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U_n \Gamma_{im;}^n \right) E^m
\]

Defining covariant derivative $U^m;i$ of a contravariant component $U^m$

\[
U^m;i = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m
\]

...and covariant derivative $U_m;i$ of a covariant component $U_m$

\[
U_m;i = \frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}^n
\]
Intrinsic derivatives:

(Mathematicians being cute)
Defining intrinsic derivative of contravariant vector components.

$$\frac{\delta V^k}{\delta t} = \frac{dV^k}{dt} + \Gamma^k_{mn} V^m \dot{q}^n = \frac{\partial V^k}{\partial q^n} \dot{q}^n + \Gamma^k_{mn} V^m \dot{q}^n = V^k \dot{q}^n$$

$$F_k = \frac{\delta p_k}{\delta t}$$

Tensor chain rules.

$$\frac{\delta V^k}{\delta t} = V^k \dot{q}^n$$, replaces: \( \frac{dV^k}{dt} = \frac{\partial V^k}{\partial q^n} \dot{q}^n \) where: \( V^k = \frac{\partial V^k}{\partial q^n} + \Gamma^k_{mn} V^m \)

Defining intrinsic derivative of covariant vector components.

$$\frac{\delta V_k}{\delta t} = \frac{dV_k}{dt} \Gamma^{mn}_k V_m \dot{q}^n = \frac{\partial V_k}{\partial q^n} \dot{q}^n - \Gamma^{mn}_k V_m \dot{q}^n = V_k \dot{q}^n$$

$$F^k = \frac{\delta p^k}{\delta t}$$

$$\frac{\delta V_k}{\delta t} = V_k \dot{q}^n$$, replaces: \( \frac{dV_k}{dt} = \frac{\partial V_k}{\partial q^n} \dot{q}^n \) where: \( V_k = \frac{\partial V_k}{\partial q^n} - \Gamma^{mn}_k V_m \)
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

Christoffel g-derivative formula
What’s a tensor? What’s not?
Christoffel g-derivative formula

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} + \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]
Christoffel g-derivative formula

\[ \frac{\partial (E_m \bullet E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \bullet E_n + E_m \bullet \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = \Gamma_{nm;i} + \Gamma_{in;m} \quad \text{(switched i ↔ n)} \]

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i ↔ m)} \]
Christoffel g-derivative formula

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^l} = \frac{\partial E_m}{\partial q^l} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^l} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma^{i}_{im;n} + \Gamma^{i}_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = -\Gamma^{n}_{nm;i} - \Gamma^{n}_{in;m} \quad \text{(switched i ↔ n)} \]

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma^{i}_{im;n} + \Gamma^{i}_{mn;n} \quad \text{(switched i ↔ m)} \]
Christoffel g-derivative formula

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)
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\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)
\]
Christoffel g-derivative formula

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{i,m;n} + \Gamma_{i,n;m} \\
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{n,m;i} - \Gamma_{i,n;m} \quad \text{(switched i ↔ n)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{i,m;n} + \Gamma_{m,n;i} \quad \text{(switched i ↔ m)}
\]

Gives the Christoffel formula

\[
\Gamma_{i,m;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

Christoffel g-derivative formula

What’s a tensor? What’s not?
$\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}$

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \\
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{m;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n}) \\
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m})
\]

Chain-saw-sums transform a "bar-frame" view $\bar{U}^m_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m$ of covariant derivative $U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m$

\[
\bar{U}^m_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m = \frac{\partial q^n}{\partial \bar{q}^\bar{n}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m = \frac{\partial q^n}{\partial \bar{q}^\bar{n}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m = \frac{\partial q^m}{\partial \bar{q}^\bar{n}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m
\]

Gives the Christoffel formula

$\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$

\[\begin{bmatrix}
\begin{array}{c}
\frac{\partial g_{mn}}{\partial q^i} \\
\frac{\partial g_{mi}}{\partial q^n} \\
\frac{\partial g_{in}}{\partial q^m}
\end{array}
\end{bmatrix}
\]

What's a tensor? What's not?
What’s a tensor? What’s not?

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = \Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched } i \leftrightarrow n) \]

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Chain-saw-sums transform a "bar-frame" view \( \bar{U}^m;\bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m \) of covariant derivative \( U^m = \frac{\partial U}{\partial q^n} \cdot E_m \)

Gives the Christoffel formula

\[ \Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \]

"U" of covariant derivative

\[ \bar{U}^m;\bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m = \frac{\partial q^n}{\partial \bar{q}^\bar{n}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m = \frac{\partial q^m}{\partial \bar{q}^\bar{m}} \frac{\partial \bar{U}}{\partial q^m} \cdot \bar{E}^m \]

\[ \bar{U}^m;\bar{n} = \frac{\partial \bar{q}^\bar{m}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^\bar{n}} U^m \]
What's a tensor? What's not?

$$\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{n;m;i} - \Gamma_{in;m}$$  \hspace{1cm} \text{(switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}$$  \hspace{1cm} \text{(switched } i \leftrightarrow m)$$

Chain-saw-sums transform a "bar-frame" view

$$\bar{U}^m_{;n} = \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m$$

of covariant derivative

$$U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m$$

Gives the Christoffel formula

$$\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

The transformation of

$$U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{n;\ell}^m$$

is that of general 2nd-rank tensor

$$T^m_{;n} = \frac{\partial q^m}{\partial q^n} \frac{\partial q^n}{\partial q^m} T^m_{;n}$$
What's a tensor? What's not?

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)} \]

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m)} \]

Chain-saw-sums transform a "bar-frame" view \( U_{m;n}^\bar = \frac{\partial U}{\partial q^\bar} \cdot \bar E^m \) of covariant derivative \( U_{m;n}^m = \frac{\partial U}{\partial q^n} \cdot E_m \)

\[ U_{m;n}^\bar = \frac{\partial \bar U}{\partial q^\bar} \cdot \bar E^m = \frac{\partial q^n}{\partial q^\bar} \cdot \frac{\partial U}{\partial q^n} \cdot \bar E^m = \frac{\partial q^n}{\partial q^\bar} \cdot \frac{\partial \bar U}{\partial q^\bar} \cdot \frac{\partial q^m}{\partial q^\bar} E_m \]

The transformation of \( U_{m;n}^m = \frac{\partial U}{\partial q^n} + U_{m}^l \Gamma_{n;l}^m \) is that of general 2nd-rank tensor \( T_{m;n}^m = \frac{\partial q^n}{\partial q^\bar} \frac{\partial q^m}{\partial q^\bar} T_{m}^n \)

The transformation of \( U_{m;n}^m = \frac{\partial U}{\partial q^n} \) is NOT that simple.
What's a tensor? What's not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched } i \leftrightarrow n)\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched } i \leftrightarrow m)\]

Chain-saw-sums transform a "bar-frame" view

\[
\dot{U}^{\bar{m}};\bar{n} = \frac{\partial \dot{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^\bar{m}
\]

The transformation of \( U^m; n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{n \ell;}^m \) is that of general 2nd-rank tensor \( T^m_{\bar{n}} \)

\[
U^m; n = \frac{\partial U^m}{\partial q^n} \quad \text{and} \quad \frac{\partial \bar{U}^\bar{m}}{\partial \bar{q}^\bar{n}} = \frac{\partial \bar{U}^\bar{m}}{\partial \bar{q}^\bar{n}} \cdot \frac{\partial q^n}{\partial \bar{q}^\bar{n}} \cdot E_m
\]

The transformation of \( U^m; n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.
Chain-saw-sums transform a "bar-frame" view \( \bar{U}_{\bar{m} \bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^\bar{m} \) of covariant derivative \( U^m_{\ ;n} = \frac{\partial U}{\partial q^n} \cdot E_m \)

The transformation of \( U^m_{\ ;n} = \frac{\partial U^m}{\partial q^n} + \frac{U^\ell}{\partial q^n} \Gamma^m_{\ell \ ;n} \) is that of general 2nd-rank tensor \( T^m_{\ \bar{n}} \)

The transformation of \( U^m_{\ ;n} = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible. But, still need to write \( \frac{\partial \bar{U}^{\bar{m}}}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

What's a tensor? What's not?
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{m)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

Chain-saw-sums transform a "bar-frame" view \( \bar{U}^m ; n = \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m \) of covariant derivative \( U^m ; n = \frac{\partial U}{\partial q^n} \cdot E_m \)

\[
\bar{U}^m ; n = \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m = \frac{\partial \bar{U}}{\partial q^n} \cdot \frac{\partial q^n}{\partial q^m} \bar{E}^m = \frac{\partial q^n}{\partial q^m} \cdot \frac{\partial U}{\partial q^n} \cdot \bar{E}^m = \frac{\partial q^n}{\partial q^m} \cdot \frac{\partial q^m}{\partial q^n} \cdot E_m
\]

The transformation of \( U^m ; n = \frac{\partial U^m}{\partial q^n} + U^{\ell} \Gamma^m_{\ell ; n} \) is that of general 2nd-rank tensor \( T^m_{n ; \ell} \)

\[
T^m_{n ; \ell} = \frac{\partial q^m}{\partial q^n} \frac{\partial q^n}{\partial q^\ell} T^m_{n}
\]

The transformation of \( U^m ; n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.

\[
\frac{\partial U^m}{\partial q^n} = \frac{\partial q^m}{\partial q^n} \frac{\partial q^n}{\partial q^\ell} \frac{\partial U^m}{\partial q^n}
\]

\[
\frac{\partial U^m}{\partial q^n} = \frac{\partial q^m}{\partial q^n} \frac{\partial q^n}{\partial q^\ell} \frac{\partial U^m}{\partial q^n}
\]

But, still need to write \( \frac{\partial \bar{U}^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[
\frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^m}{\partial q^n} \frac{\partial q^n}{\partial q^\ell} \left( \frac{\partial q^\ell}{\partial q^m} U^m \right)
\]
\[
\frac{\partial}{\partial q^i} \left( E_m \cdot E_n \right) = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

\text{Chain-saw-sums transform a "bar-frame" view } \bar{U}^m_{;n} = \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m \text{ of covariant derivative } U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m

\text{The transformation of } U^m_{;n} = \frac{\partial U^m}{\partial q^n} + \bar{U}^m \Gamma^m_{n\ell} \text{ is that of general 2nd-rank tensor } T^m_{\bar{n};n}

\text{The transformation of } U^m_{;n} = \frac{\partial U^m}{\partial q^n} \text{ is NOT that simple. } \frac{\partial U^m}{\partial q^n} = \frac{\partial U^m}{\partial q^n} = \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial U^m}{\partial q^\bar{n}} \frac{\partial q^\bar{n}}{\partial q^n}

\text{But, still need to write } \frac{\partial \bar{U}^m}{\partial q^n} \text{ in terms of } \frac{\partial U^m}{\partial q^n}.

\text{standard contra-tran: } \bar{U}^m

\text{Gives the Christoffel formula}

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

(switched i \leftrightarrow n)

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

(switched i \leftrightarrow m)

Chain-saw-sums transform a "bar-frame" view

\[
\bar{U}_m^i = \frac{\partial \bar{U}}{\partial q^n} \cdot E^n
\]

of covariant derivative

\[
U^m_i = \frac{\partial U}{\partial q^n} \cdot E^n
\]

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

The transformation of \( U^m_n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{\ell n}^m \) is that of general 2nd-rank tensor \( T^m_n \)

The transformation of \( U^m_n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.

The transformation of \( U^m_n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.

But, still need to write \( \frac{\partial \bar{U}^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[
\frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^n}{\partial q^\bar{m}} \frac{\partial \bar{U}}{\partial q^m} + U^m \frac{\partial}{\partial q^n} \left( \frac{\partial q^m}{\partial q^n} \right)
\]

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

(switched i \leftrightarrow n)

\[
\frac{\partial g_{im}}{\partial q^n} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

(switched i \leftrightarrow m)

Chain-saw-sums transform a "bar-frame" view \( \bar{U}^m ; \bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^n} \cdot \bar{E}^m \) of covariant derivative \( U^m ; n = \frac{\partial U}{\partial q^n} \cdot E_m \)

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

The transformation of \( U^m ; n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{\ell n ; m} \) is that of general 2nd-rank tensor \( T^m_{\bar{n}} \)

The transformation of \( U^m ; n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.

\[
\frac{\partial U^m}{\partial \bar{q}^n} = \frac{\partial q^n}{\partial q^\ell} \frac{\partial U^m}{\partial \bar{q}^\ell} = \frac{\partial q^m}{\partial q^\ell} \frac{\partial q^n}{\partial q^\ell} \frac{\partial U^m}{\partial q^\ell}
\]

But, still need to write \( \frac{\partial U^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial \bar{q}^n} \).

\[
\frac{\partial U^m}{\partial \bar{q}^n} = \frac{\partial q^m}{\partial q^\ell} \frac{\partial q^n}{\partial q^\ell} \frac{\partial U^m}{\partial \bar{q}^\ell}
\]

holds if and only if \( \frac{\partial}{\partial q^n} \left( \frac{\partial \bar{q}^m}{\partial q^m} \right) = 0 \)

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \bullet E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \bullet E_n + E_m \bullet \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{im;m}
\]

\[
- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched } i \leftrightarrow n)\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched } i \leftrightarrow m)\]

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

Chain-saw-sums transform a "bar-frame" view

\[
\bar{U}^m;\bar{n} = \frac{\partial \bar{U}}{\partial q^n} \bullet \bar{E}^m
\]

The transformation of \(U^m;\bar{n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{\bar{n} \ell}^m\) is that of general 2nd-rank tensor \(T^{m}_{\bar{n}}\).

The transformation of \(U^m,n = \frac{\partial U^m}{\partial q^n}\) is NOT that simple. At first it looks possible.

\[
\frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial U^m}{\partial q^n} \quad \text{standard contra-tran: } \bar{U}^m
\]

But, still need to write \(\frac{\partial \bar{U}^m}{\partial q^n}\) in terms of \(\frac{\partial U^m}{\partial q^n}\).  

\[
\frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial \bar{q}^m}{\partial q^n} \frac{\partial q^n}{\partial q^m} \frac{\partial U^m}{\partial q^n} = \frac{\partial U^m}{\partial q^n} \frac{\partial \bar{q}^m}{\partial q^n} + U^m \frac{\partial}{\partial q^n} \left( \frac{\partial \bar{q}^m}{\partial q^n} \right)
\]

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

Otherwise, \(U^m,n\) needs “correction” \(U^\ell \Gamma_{\bar{n} \ell}^m\).
\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im; n} + \Gamma_{in; m}
\]

\[
-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm; i} - \Gamma_{in; m} \quad \text{(switched i} \leftrightarrow \text{n)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im; n} + \Gamma_{mn; i} \quad \text{(switched i} \leftrightarrow \text{m)}
\]

Chain-saw-sums transform a "bar-frame" view \( \tilde{U}^m; \bar{n} = \frac{\partial \tilde{U}}{\partial q^n} \cdot \bar{E}^m \) of covariant derivative \( U^m; n = \frac{\partial U}{\partial q^n} \cdot E_m \)

\[
\tilde{U}^m; \bar{n} = \frac{\partial \tilde{U}}{\partial q^n} \cdot \bar{E}^m = \frac{\partial U^m}{\partial q^n} \cdot E_m
\]

The transformation of \( U^m; n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma^m_{n \ell; } \) is that of general 2nd-rank tensor \( T^m_n \)

The transformation of \( U^m, n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible. \( \frac{\partial \tilde{U}^m}{\partial q^n} = \frac{\partial \tilde{U}^m}{\partial q^n} = \frac{\partial q^m}{\partial q^n} \cdot \frac{\partial U^m}{\partial q^n} \)

But, still need to write \( \frac{\partial \tilde{U}^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[
\frac{\partial \tilde{U}^m}{\partial q^n} = \frac{\partial q^m}{\partial q^n} \cdot \frac{\partial U^m}{\partial q^n}
\]

holds if and only if \( \frac{\partial}{\partial q^n} \left( \frac{\partial q^m}{\partial q^n} \right) = 0 \)

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

Otherwise, \( U^m, n \) needs “correction” \( U^\ell \Gamma^m_{n \ell; } \). And, that \( U^\ell \Gamma^m_{n \ell; } \) \textit{cannot} be a \( T^m_n \)-tensor either!
Riemann equations of motion (No explicit t-dependence and fixed GCC)
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

\[
\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}
\]

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

\[
T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \frac{\partial x^k}{\partial q^n} \dot{q}^n \right) \left( \frac{\partial x^k}{\partial q^m} \dot{q}^m + \frac{\partial x^l}{\partial q^n} \dot{q}^n \right)
\]

All explicit-t-dependent terms are zero
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

\[
\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}
\]

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

\[
T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \frac{\partial x^k}{\partial q^n} \dot{q}^n \right) \left( \frac{\partial x^k}{\partial q^n} \dot{q}^n + \frac{\partial x^j}{\partial q^m} \dot{q}^m \right)
\]

All explicit-t-dependent terms are zero
(Time must be included as a dimension)
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$

$$T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \frac{\partial x^j}{\partial t} \right) \left( \frac{\partial x^k}{\partial q^n} \dot{q}^n + \frac{\partial x^k}{\partial t} \right)$$

All explicit-t-dependent terms are zero (Time must be included as a dimension)
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} \left[ \frac{\partial T}{\partial q^\ell} \right] = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$

$1^{\text{st}}$ term involves covariant momentum $p_\ell$.

$$p_\ell = \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

\[ T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial}{\partial t} \right\} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial q^n} \dot{q}^n + \frac{\partial x^\ell}{\partial q^n} \dot{q}^n \right\} \right) \]
Riemann equations of motion (No explicit $t$-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^\ell} \right) - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$

$1^{st}$ term involves **covariant momentum** $p_\ell$. Inverse **contravariant kinetic metric** $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

All explicit-$t$-dependent terms are zero (Time must be included as a dimension)

$$T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \frac{\partial x^j}{\partial q^n} \dot{q}^n \right) + \left( \frac{\partial x^k}{\partial q^m} \dot{q}^m + \frac{\partial x^k}{\partial q^n} \dot{q}^n \right)$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric \( \gamma_{mn} \) is a covariant tensor transform of an original Cartesian inertia tensor \( M_{ij} \)

\[
\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}
\]

Converts Cartesian kinetic energy \( T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k \) to GCC

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n
\]

Lagrange equations for fixed GCC convert to tensor form

\[
F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)
\]

1st term involves covariant momentum \( p_\ell \).

Inverse contravariant kinetic metric \( \gamma^{mn} \) gives velocity \( \dot{q}^n \)

\[
p_\ell \equiv \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma^{\ell n} \dot{q}^n
\]

Canonical Lagrange equations valid for all GCC, fixed or explicit in time \( t \):

\[
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}
\]

The “4-wheel-drive garbage truck”
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric \(\gamma_{mn}\) is a covariant tensor transform of an original Cartesian inertia tensor \(M_{ij}\)

\[
\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}
\]

Converts Cartesian kinetic energy \(T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k\) to GCC \(T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n\)

Lagrange equations for fixed GCC convert to tensor form

\[
F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn} \dot{q}^m \dot{q}^n}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d \gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn} \dot{q}^m \dot{q}^n}{\partial q^\ell}
\]

1st term involves covariant momentum \(p_\ell\).

Inverse contravariant kinetic metric \(\gamma^{mn}\) gives velocity \(\dot{q}^n\)

\[
p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell} = \gamma^{\ell n} \dot{q}^n
\]

Canonical Lagrange equations valid for all GCC, fixed or explicit in time \(t\):

Following is for fixed GCC only:

\[
F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn} \dot{q}^m \dot{q}^n}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d \gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn} \dot{q}^m \dot{q}^n}{\partial q^\ell}
\]

All explicit-t-dependent terms are zero (Time must be included as a dimension)

The “4-wheel-drive garbage truck”
**Riemann equations of motion** (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^\ell} \right) - \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

1st term involves **covariant momentum** $p_\ell$. Inverse **contravariant kinetic metric** $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$p_\ell = \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma^{mn} \dot{q}^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma^{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma^{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy

$$T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$$
to GCC

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d \gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

$1^{st}$ term involves covariant momentum $p_\ell$.

Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d \gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

The "4-wheel-drive garbage truck"

The time must be included as a dimension.
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

1st term involves covariant momentum $p_\ell$. Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[ \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im:n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{im}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

The “4-wheel-drive garbage truck”

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}$$

Tuesday, November 4, 2014
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy

$$T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$$

to GCC

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial q^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \gamma_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

1st term involves covariant momentum $p_\ell$.

Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d}{dt} \gamma_{\ell n} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \ddot{q}^n \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im:n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{im}}{\partial q^n} - \frac{\partial g_{ni}}{\partial q^m} \right)$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \ddot{q}^n \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

All explicit-t-dependent terms are zero (Time must be included as a dimension)

The “4-wheel-drive garbage truck”
Lagrange equations for fixed GCC convert to tensor form

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \ddot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \dddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

1st term involves covariant momentum $p_\ell$. Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \ddot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \dddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$d\gamma_{\ell n} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^l} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

This gives covariant Riemann equations

$$F_\ell = \gamma_{\ell n} \dddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$
Converting Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $\quad T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

1st term involves covariant momentum $p_\ell$. Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d \gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[ \frac{\partial \gamma_{mn}}{\partial q^m} \frac{\partial q^n}{\partial q^\ell} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{im}}{\partial q^n} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

This gives covariant Riemann equations and contravariant Riemann equations.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

$$F^k = \ddot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)
Example of Riemann-Christoffel forms in cylindrical polar OCC \((q^1 = \rho, q^2 = \phi, q^3 = z)\)

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x}{\partial \rho} &= \cos \phi & \frac{\partial x}{\partial \phi} &= -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} &= \sin \phi & \frac{\partial y}{\partial \phi} &= \rho \cos \phi & 0 \\
0 &= 0 & \frac{\partial z}{\partial \rho} &= 1
\end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix}
\frac{\partial \rho}{\partial x} &= \cos \phi & \frac{\partial \rho}{\partial y} &= \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} &= -\sin \phi & \frac{\partial \phi}{\partial y} &= \cos \phi & 0 \\
\frac{\partial z}{\partial \rho} &= 0 & \frac{\partial z}{\partial \phi} &= 1 & 0
\end{pmatrix}
\]

\[
\leftarrow E^{\rho} \quad x = \rho \cos \phi \\
\leftarrow E^{\phi} \quad y = \rho \sin \phi \\
\leftarrow E^{z} \quad z = z
\]

\[
\uparrow \quad E_{\rho} \quad \uparrow \quad E_{\phi} \quad \uparrow \quad E_{z} = \langle J^{-1} \rangle
\]

\[
E = F_{\rho} E^{\rho} + F_{\phi} E^{\phi} + F_{z} E^{z} = f_{x} e_{x} + f_{y} e_{y} + f_{z} e_{z}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{align*}
\langle J \rangle &= \begin{pmatrix}
\frac{\partial x}{\partial \rho} &= \cos \phi & \frac{\partial y}{\partial \rho} &= \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} &= \sin \phi & \frac{\partial y}{\partial \phi} &= \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} &= 1
\end{pmatrix},
\langle \kappa \rangle &= \begin{pmatrix}
\frac{\partial \rho}{\partial x} &= \cos \phi & \frac{\partial \rho}{\partial y} &= \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} &= -\sin \phi & \frac{\partial \phi}{\partial y} &= \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} &= 1
\end{pmatrix}
\end{align*}
\]

$\leftarrow E^\rho\quad x = \rho \cos \phi$

$\leftarrow E^\phi\quad y = \rho \sin \phi$

$\leftarrow E^z\quad z = z$

Covariant forces

$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$

$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$

$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$

$\mathbf{E} = \mathbf{f}_x \mathbf{e}_x + \mathbf{f}_y \mathbf{e}_y + \mathbf{f}_z \mathbf{e}_z$
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = -\rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[\mathbf{J} = \mathbf{J}^{-1}\]

Covariant forces

\[
\begin{align*}
F_{\rho} &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\
F_{\phi} &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\end{align*}
\]

Covariant kinetic metric

\[
\begin{align*}
\gamma_{\rho\rho} &= m \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m \left( \cos^2 \phi + \sin^2 \phi \right) = m \\
\gamma_{\phi\phi} &= m \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m \left( \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi \right) = m \rho^2 \\
\gamma_{zz} &= m \frac{\partial x}{\partial z} \frac{\partial x}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m
\end{align*}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix},
\begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[
\begin{aligned}
E_\rho &= x = \rho \cos \phi \\
E_\phi &= y = \rho \sin \phi \\
E_z &= z = z
\end{aligned}
\]

Covariant forces

\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]

\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]

\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Covariant kinetic metric

\[
\gamma_{\rho\rho} = m \left( \frac{\partial x_i}{\partial \rho} \frac{\partial x_i}{\partial \rho} \right) = m E_\rho \cdot E_\rho = m \left( \cos^2 \phi + \sin^2 \phi \right) = m
\]

\[
\gamma_{\phi\phi} = m \left( \frac{\partial x_i}{\partial \phi} \frac{\partial x_i}{\partial \phi} \right) = m E_\phi \cdot E_\phi = m \left( \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi \right) = m \rho^2
\]

\[
\gamma_{zz} = m \left( \frac{\partial x_i}{\partial z} \frac{\partial x_i}{\partial z} \right) = m E_z \cdot E_z = m
\]

Contravariant kinetic metric

\[
\gamma^{\rho\rho} = 1 / m
\]

\[
\gamma^{\phi\phi} = 1 / \left( m \rho^2 \right)
\]

\[
\gamma^{zz} = 1 / m
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}, \quad \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{-\sin \phi}{\rho} & \frac{-\cos \phi}{\rho} & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix} = \begin{pmatrix}
x = \rho \cos \phi \\
y = \rho \sin \phi \\
z = z
\end{pmatrix}
\]

Covariant forces
\[F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0\]
\[F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0\]
\[F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z\]

Lagrangian
\[T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} \rho \dot{\phi}^2 + \frac{1}{2} \dot{z}^2\]

Covariant kinetic metric
\[\gamma_{\rho\rho} = m \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \rho} = m E_\rho \cdot E_\rho = m \left( \cos^2 \phi + \sin^2 \phi \right) = m\]
\[\gamma_{\phi\phi} = m \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} = m E_\phi \cdot E_\phi = m \left( \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi \right) = m \rho^2\]
\[\gamma_{zz} = m \frac{\partial x}{\partial z} \frac{\partial x}{\partial z} = m E_z \cdot E_z = m\]

Contravariant kinetic metric
\[\gamma^{\rho\rho} = 1 / m\]
\[\gamma^{\phi\phi} = 1 / \left( m \rho^2 \right)\]
\[\gamma^{zz} = 1 / m\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi \\
\frac{\partial y}{\partial \rho} = \sin \phi \\
0
\end{pmatrix}, \quad \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi \\
\frac{\partial \rho}{\partial y} = \sin \phi \\
0
\end{pmatrix},
\begin{pmatrix}
\frac{\partial \phi}{\partial x} = -\sin \phi \\
\frac{\partial \phi}{\partial y} = \cos \phi \\
\frac{\partial \phi}{\partial z} = 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
x = \rho \cos \phi \\
y = \rho \sin \phi \\
z = z
\end{pmatrix}
\]

**Covariant forces**

\[
\begin{align*}
F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\
F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\end{align*}
\]

**Lagrangian**

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

\[
p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} \\
p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} \\
p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z}
\]

\[
= m \dot{\rho} \\
= m \rho^2 \dot{\phi} \\
= m \dot{z}
\]

**Contravariant kinetic metric**

\[
\gamma^{\rho\rho} = 1 / m \\
\gamma^{\phi\phi} = 1 / \left( m \rho^2 \right) \\
\gamma^{zz} = 1 / m
\]

**Covariant kinematic metric**

\[
\gamma_{\rho\rho} = m \left( \cos^2 \phi + \sin^2 \phi \right) = m \\
\gamma_{\phi\phi} = m \left( \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi \right) = m \rho^2 \\
\gamma_{zz} = m
\]

**Covariant momenta**
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[
\begin{align*}
\rho^x &= f_x, \\
\rho^y &= f_y, \\
\rho^z &= f_z,
\end{align*}
\]

\[
\begin{align*}
Covariant forces \\
F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\
F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\end{align*}
\]

Lagrangian

\[
T = \frac{1}{2} \gamma_{\mu \nu} \dot{q}^\mu \dot{q}^\nu = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\rho}^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

\[
\begin{align*}
p_\rho &= \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho} \\
p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi \phi} \dot{\phi} \\
p_z &= \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z}
\end{align*}
\]

Covariant kinetic metric

\[
\gamma_{\rho \rho} = m \frac{\partial x}{\partial \rho} \cdot \frac{\partial x}{\partial \rho} = m \frac{\partial y}{\partial \rho} \cdot \frac{\partial y}{\partial \rho} = m \frac{\partial z}{\partial \rho} \cdot \frac{\partial z}{\partial \rho} = m
\]

\[
\gamma_{\phi \phi} = m \frac{\partial x}{\partial \phi} \cdot \frac{\partial x}{\partial \phi} = m \frac{\partial y}{\partial \phi} \cdot \frac{\partial y}{\partial \phi} = m \frac{\partial z}{\partial \phi} \cdot \frac{\partial z}{\partial \phi} = m
\]

\[
\gamma_{zz} = \frac{\partial x}{\partial z} \cdot \frac{\partial x}{\partial z} = \frac{\partial y}{\partial z} \cdot \frac{\partial y}{\partial z} = \frac{\partial z}{\partial z} \cdot \frac{\partial z}{\partial z} = 1
\]

Contravariant kinetic metric

\[
\gamma^{\rho \rho} = \frac{1}{m} \\
\gamma^{\phi \phi} = \frac{1}{m \rho^2} \\
\gamma^{zz} = \frac{1}{m}
\]

Contravariant momenta

\[
\begin{align*}
p^{\rho} &= \dot{\rho} \\
p^{\phi} &= \dot{\phi} \\
p^{z} &= \dot{z}
\end{align*}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{bmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{bmatrix} \rightarrow \{ \rho \}, \quad \{ K \} = \begin{bmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial \phi} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial \phi} = \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{bmatrix} \rightarrow E^\rho \quad x = \rho \cos \phi \\
\begin{bmatrix}
0 & 0 & \frac{\partial \phi}{\partial z} = 1
\end{bmatrix} \rightarrow E^\phi \quad y = \rho \sin \phi \\
\begin{bmatrix}
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{bmatrix} \rightarrow E^z \quad z = z
\]

Covariant forces
\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]
\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]
\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Lagrangian
\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

Comparing Lagrange and the Riemann covariant force equations
\[
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \dot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n
\]

Covariant kinetic metric
\[
\gamma_{\rho \rho} = m \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \rho} = m \frac{\partial y}{\partial \rho} \frac{\partial y}{\partial \rho} = m \frac{\partial z}{\partial \rho} \frac{\partial z}{\partial \rho} = m
\]
\[
\gamma_{\phi \phi} = m \left( \cos^2 \phi + \sin^2 \phi \right) = m
\]
\[
\gamma_{zz} = m \delta_{zz} = m
\]

Contravariant kinetic metric
\[
\gamma^{\rho \rho} = 1 / m
\]
\[
\gamma^{\phi \phi} = 1 / \left( m \rho^2 \right)
\]
\[
\gamma^{zz} = 1 / m
\]

Covariant momenta
\[
p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho}
\]
\[
p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi \phi} \dot{\phi}
\]
\[
p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z}
\]

Contravariant momenta
\[
p^\rho = \dot{\rho}
\]
\[
p^\phi = \dot{\phi}
\]
\[
p^z = \dot{z}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\left\{ \frac{\partial}{\partial \rho} = \cos \phi \frac{\partial}{\partial \phi} - \rho \sin \phi \right\}, \quad \left\{ \frac{\partial}{\partial \phi} = \sin \phi \frac{\partial}{\partial \rho} + \rho \cos \phi \right\}, \quad \left\{ \frac{\partial}{\partial z} = \frac{\partial \rho}{\partial z} = 1 \right\}
\]

\[
\frac{\partial}{\partial \rho} \phi = \frac{\partial \rho}{\partial \phi} \phi = m \rho \phi \quad \frac{\partial}{\partial z} \phi = \frac{\partial \rho}{\partial z} \phi = m \rho \phi
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[
F_\rho = \frac{dp_\rho}{dt} = \gamma_{\rho \rho} \rho \dot{\rho} + \Gamma_{\rho \rho; \rho} \dot{\phi} \dot{\phi} \dot{\phi} + \Gamma_{\rho \rho; \rho} \dot{\rho} \dot{\phi}
\]

\[
F_\phi = \frac{dp_\phi}{dt} = \gamma_{\phi \phi} \rho \dot{\rho} + \Gamma_{\rho \phi; \rho} \dot{\rho} \phi \phi + \Gamma_{\rho \phi; \rho} \dot{\phi} \phi
\]

Note: This a much more efficient way to derive $\Gamma$-coefficients than the g-formula.
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Lagrangian

\[
\left\{ \begin{array}{c}
\frac{\partial x}{\partial \rho} = \cos \phi \\
\frac{\partial y}{\partial \rho} = \sin \phi \\
\frac{\partial z}{\partial \rho} = 0
\end{array} \right. \quad \left\{ \begin{array}{c}
\frac{\partial x}{\partial \phi} = -\rho \sin \phi \\
\frac{\partial y}{\partial \phi} = \rho \cos \phi \\
\frac{\partial z}{\partial \phi} = 0
\end{array} \right.
\]

\[\{F\} = \begin{pmatrix}
\frac{\partial \rho}{\partial \rho} = \cos \phi \\
\frac{\partial \phi}{\partial \rho} = -\sin \phi \\
\frac{\partial \rho}{\partial \rho} = \rho
\end{pmatrix} \quad \{\rho\} = \begin{pmatrix}
\frac{\partial \phi}{\partial \rho} = \sin \phi \\
\frac{\partial \phi}{\partial \rho} = \cos \phi \\
\frac{\partial \phi}{\partial \rho} = 0
\end{pmatrix}
\]

Comparing \textbf{Lagrange} and the \textbf{Riemann covariant force equations}

\[
\text{Covariant forces}
\]

\[
F_{\rho} = f_{x,\rho} + f_{y,\rho} + f_{z,\rho} = \rho \cos \phi + f_{y,\rho} \sin \phi + 0
\]

\[
F_{\phi} = f_{x,\phi} + f_{y,\phi} + f_{z,\phi} = -f_{x,\rho} \sin \phi + f_{y,\rho} \cos \phi + 0
\]

\[
F_{z} = f_{x,\rho} + f_{y,\phi} + f_{z,\phi} = 0 + 0 + f_{z}
\]

\textbf{Lagrangian}

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

\textbf{Contravariant kinetic metric}

\[
\gamma^\rho = m \frac{\partial x}{\partial \rho} = m \left( \cos^2 \phi + \sin^2 \phi \right) = m
\]

\[
\gamma^\phi = m \frac{\partial y}{\partial \rho} = m \left( \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi \right) = m \rho^2
\]

\[
\gamma^z = m \frac{\partial z}{\partial \rho} = m
\]

\textbf{Contravariant momenta}

\[
p^\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma^\rho m \rho \dot{\phi}
\]

\[
p^\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma^\phi m \rho^2 \dot{\phi}
\]

\[
p^z = \frac{\partial T}{\partial \dot{z}} = \gamma^z m \dot{z}
\]

\textbf{Note: This a much more efficient way to derive $\Gamma$-coefficients than the g-formula.}

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[
F_{\rho} = \frac{\partial p_{\rho}}{\partial t} - \frac{\partial T}{\partial \dot{\rho}} = \gamma^\rho m \rho \dot{\phi} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n
\]

\[
= m \dot{\rho} - m \rho \ddot{\phi}
\]

so: $\Gamma_{\phi,\rho} = -m \rho$
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{dx}{d\rho} = \cos \phi & \frac{dx}{d\phi} = -\rho \sin \phi & 0 \\
\frac{dy}{d\rho} = \sin \phi & \frac{dy}{d\phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{dz}{d\phi} = 1
\end{pmatrix}, \quad \begin{pmatrix}
\frac{dp}{d\rho} = \cos \phi & \frac{dp}{d\phi} = \sin \phi & 0 \\
\frac{d\phi}{d\rho} = -\sin \phi & \frac{d\phi}{d\phi} = \cos \phi & \rho \\
0 & 0 & \frac{dz}{d\phi} = 1
\end{pmatrix}
\]

\[
\mathbf{F}_\rho = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z
\]

\[
\mathbf{F}_\phi = -\mathbf{E}^\rho
\]

\[
\mathbf{F}_z = -\mathbf{E}^\phi
\]

\[
\mathbf{E}^\rho = \rho \mathbf{e}_z
\]

\[
\mathbf{E}^\phi = \mathbf{e}_\phi
\]

\[
\mathbf{E}^z = \mathbf{e}_z
\]

\[
\begin{aligned}
&x = \rho \cos \phi \\
y = \rho \sin \phi \\
z = z
\end{aligned}
\]

Covariant forces

\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]

\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]

\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Lagrangean

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

Comparing Lagrange and the Riemann covariant force equations

\[
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q_\ell} = \gamma_{\ell \nu} \dot{q}^\nu + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[
F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho \rho} \dot{\rho} + \Gamma_{\rho \rho;\rho} \dot{\rho}^2 = \frac{d(m \dot{\rho})}{dt} - \frac{\partial}{\partial \rho} \left( \frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m \ddot{\rho} - \frac{1}{2} m \rho^2 \ddot{\phi}^2
\]

\[
F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi \phi} \dot{\phi} + \Gamma_{\phi \phi;\phi} \dot{\rho}^2 \dot{\phi} = \frac{d(m \rho^2 \dot{\phi})}{dt} - 0 = m \rho^2 \ddot{\phi} + 2 m \rho \dot{\rho} \dot{\phi}
\]

\[
\Gamma_{\rho \rho;\rho} = -m \rho
\]

\[
\Gamma_{\phi \phi;\phi} = m \rho = \Gamma_{\phi \rho;\phi}
\]

Note: This a much more efficient way to derive $\Gamma$-coefficients than the g-formula.

Contravariant kinetic metric

\[
\gamma^{\rho \rho} = \frac{1}{m}
\]

\[
\gamma^{\phi \phi} = \frac{1}{m \rho^2}
\]

\[
\gamma^{zz} = \frac{1}{m}
\]

Contravariant momenta

\[
p^\rho = \dot{\rho}
\]

\[
p^\phi = \dot{\phi}
\]

\[
p^z = \dot{z}
\]

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$

Symmetry gives 2 factor for $q \neq p$
Example of Riemann-Christoffel forms in cylindrical polar OCC \((q^1 = \rho, q^2 = \phi, q^3 = z)\)

\[
\begin{align*}
\{\rho\} &= \begin{pmatrix} \frac{\partial \rho}{\partial \rho} = \cos \phi & \frac{\partial \rho}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial \phi}{\partial \rho} = \sin \phi & \frac{\partial \phi}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}, \{\phi\} = \begin{pmatrix} \frac{\partial \rho}{\partial \rho} = \cos \phi & \frac{\partial \rho}{\partial \phi} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial \rho} = -\sin \phi & \frac{\partial \phi}{\partial \phi} = -\cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial \phi} = 1
\end{pmatrix} \\
\{\phi\}^{-1} = \begin{pmatrix} \frac{\partial \rho}{\partial \phi} = \rho \cos \phi & \frac{\partial \rho}{\partial \phi} = \rho \sin \phi & 0 \\
\frac{\partial \phi}{\partial \rho} = -\rho \sin \phi & \frac{\partial \phi}{\partial \phi} = -\rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial \phi} = 1
\end{pmatrix} \\
\end{align*}
\]

Covariant forces

\[
F_\rho = f_x \frac{\partial \rho}{\partial \rho} + f_y \frac{\partial \rho}{\partial \phi} + f_z \frac{\partial \rho}{\partial z} = f_x \cos \phi + f_y \sin \phi + 0
\]

\[
F_\phi = f_x \frac{\partial \phi}{\partial \rho} + f_y \frac{\partial \phi}{\partial \phi} + f_z \frac{\partial \phi}{\partial z} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]

\[
F_z = f_x \frac{\partial z}{\partial \rho} + f_y \frac{\partial z}{\partial \phi} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Lagrangian

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \ddot{\rho}^2 + \frac{1}{2} m \dot{\rho}^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

Comparing Lagrange and the Riemann covariant force equations

\[
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q_\ell} = \gamma_{\ell n} \dot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[
F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial q_\rho} = \gamma_{\rho \rho} \dot{\rho} + \Gamma_{\rho \rho} \dot{q}^m \dot{q}^n = \dot{m} \rho - m \rho \dot{\phi}^2
\]

so: \( \Gamma_{\rho \phi} = -m \rho \)

Contravariant equations are acceleration equations.

\[
F^\rho = \gamma^{\rho \rho} F_\rho = \ddot{q}^\rho + \Gamma_{mn}^{\rho} \dot{q}^m \dot{q}^n
\]

Note: This a much more efficient way to derive \(\Gamma\)-coefficients than the g-formula.

\[
F^\phi = \gamma^{\phi \phi} F_\phi = \ddot{q}^\phi + \Gamma_{mn}^{\phi} \dot{q}^m \dot{q}^n
\]

\[
\text{Contravariant kinetic metric} \quad \gamma^{\rho \rho} = \frac{1}{m}
\]

\[
\gamma^{\phi \phi} = \frac{1}{(m \rho^2)}
\]

\[
\gamma^{zz} = \frac{1}{m}
\]

\[
\text{Contravariant momenta} \quad p^\rho = \dot{\rho}
\]

\[
\gamma^{\phi \phi} = \dot{\phi}
\]

\[
\gamma^{zz} = \dot{z}
\]

\[
\text{Note:} \Gamma_{pq;r} = \Gamma_{qp;r} \text{ symmetry gives 2 factor for } q \neq p
\]

\[
\text{Note: } \Gamma_{pq;r} = \Gamma_{qp;r}
\]
Example of Riemann–Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{bmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi \\ 
\frac{\partial x}{\partial \phi} = \frac{\partial y}{\partial \rho} = \rho \cos \phi \\
\end{bmatrix}
= \langle J \rangle
\]
\[
\begin{bmatrix}
\frac{\partial \phi}{\partial x} = \cos \phi \\ 
\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \rho} = \sin \phi \\
\end{bmatrix}
= \langle K \rangle
\]
\[
\begin{bmatrix}
0 \\
0 \\
\frac{\partial z}{\partial \phi} = 1
\end{bmatrix} = \langle J^{-1} \rangle
\]

Covariant forces
\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]
\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]
\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Lagrangian
\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

Comparing Lagrange and the Riemann covariant force equations
\[
F_\ell = \frac{dp_\ell}{dt} - \frac{dT}{dq_\ell} = \gamma_{\ell n} \dot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.
\[
F_\rho = \frac{dp_\rho}{dt} - \frac{dT}{dq_\rho} = \gamma_{\rho \rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n
\]
\[
= \frac{d}{dt} \left( \frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m \ddot{\rho} - m \rho \dot{\phi}^2
\]

Contravariant equations are acceleration equations.
\[
F^\rho = \gamma^{\rho \rho} F_\rho = \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n
\]
\[
= \ddot{\rho} - \rho \dot{\phi}^2
\]

Contravariant kinetic equations
\[
\gamma_{\rho \rho} = m \frac{\partial x}{\partial \rho} = m \rho \left( \cos^2 \phi + \sin^2 \phi \right) = m
\]
\[
\gamma_{\rho \phi} = m \frac{\partial x}{\partial \phi} = m \rho \cos \phi
\]
\[
\gamma_{\phi \phi} = m \frac{\partial y}{\partial \phi} = m \rho \sin \phi
\]
\[
\gamma_{zz} = m \frac{\partial z}{\partial z} = m
\]

Note: This a much more efficient way to derive $\Gamma$-coefficients than the g-formula.

Note: This gives 2 factor for $q^p \neq q^\rho$ symmetry.

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ gives 2 factor for $q^p \neq q^\rho$ symmetry.

Contravariant momenta
\[
p_\rho = \rho
\]
\[
p_\phi = \phi
\]
\[
p_z = z
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC (\(q^1 = \rho, q^2 = \phi, q^3 = z\))

\[
\begin{pmatrix}
\frac{dx}{d\rho} = \cos \phi \\
\frac{dy}{d\rho} = \sin \phi \\
\frac{dz}{d\rho} = 0
\end{pmatrix}, \quad \begin{pmatrix}
\frac{dx}{d\phi} = -\rho \sin \phi \\
\frac{dy}{d\phi} = \cos \phi \\
\frac{dz}{d\phi} = 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{dx}{dz} = 0 \\
\frac{dy}{dz} = 0 \\
\frac{dz}{dz} = 1
\end{pmatrix}
\]

\[
\langle \gamma \rangle = \begin{pmatrix}
\frac{\partial \rho}{\partial \rho} = \cos \phi \\
\frac{\partial \phi}{\partial \rho} = -\sin \phi \\
\frac{\partial z}{\partial \rho} = \rho
\end{pmatrix}, \quad \langle \kappa \rangle = \begin{pmatrix}
\frac{\partial \rho}{\partial \phi} = \cos \phi \\
\frac{\partial \phi}{\partial \phi} = -\sin \phi \\
\frac{\partial z}{\partial \phi} = \rho
\end{pmatrix}
\]

\[
\epsilon^p = x = \rho \cos \phi
\]

\[
\epsilon^\rho = y = \rho \sin \phi
\]

\[
\epsilon^z = z = z
\]

---

Covariant forces

\[
F_\rho = f_x \frac{dx}{d\rho} + f_y \frac{dy}{d\rho} + f_z \frac{dz}{d\rho} = f_x \cos \phi + f_y \sin \phi + 0
\]

\[
F_\phi = f_x \frac{dx}{d\phi} + f_y \frac{dy}{d\phi} + f_z \frac{dz}{d\phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]

\[
F_z = f_x \frac{dx}{dz} + f_y \frac{dy}{dz} + f_z \frac{dz}{dz} = 0 + 0 + f_z
\]

---

Lagrangian

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m\dot{\rho}^2 + \frac{1}{2} m\rho^2 \dot{\phi}^2 + \frac{1}{2} m\dot{z}^2
\]

Comparing \textbf{Lagrange} and the Riemann \textit{covariant} force equations

\[
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q_\ell} = \gamma_{\ell n} \dot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[
F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho \rho} \ddot{\rho} + \Gamma_{\rho \rho \rho} \dot{\rho} \dot{q} \dot{q}
\]

\[
= m\ddot{\rho} - m\rho \dot{\phi}^2
\]

\[
\Gamma_{\rho \phi \rho} = -m\rho
\]

Contravariant equations are \textit{acceleration} equations.

\[
F^\rho = \gamma^{\rho \rho} F_\rho = \ddot{\rho} + \Gamma^\rho_{\rho \rho} \dot{q} \dot{q}
\]

\[
= \ddot{\rho} - \rho \dot{\phi}^2
\]

\[
\ddot{\rho} = F^\rho + \rho \dot{\phi}^2
\]

(Centripetal acceleration)

\[
\gamma^{\rho \rho} = 1/ \rho
\]
Rewriting GCC Lagrange equations:

\[
\dot{p}_r \equiv \frac{dp_r}{dt} = M \ddot{r}
\]

Centrifugal (center-fleeing) force equals total

\[
= M r \dot{\phi}^2 - \frac{\partial U}{\partial r}
\]

Centripetal (center-pulling) force

\[
\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2 M r \dot{r} \dot{\phi} + M r^2 \ddot{\phi}
\]

Torque relates to two distinct parts:

Coriolis and angular acceleration

\[
= 0 - \frac{\partial U}{\partial \phi}
\]

Angular momentum \( p_\phi \) is conserved if

potential \( U \) has no explicit \( \phi \)-dependence

Conventional forms

radial force: \( M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \)

angulor force or torque: \( M r^2 \ddot{\phi} = -2 M r \dot{r} \dot{\phi} - \frac{\partial U}{\partial \phi} \)

Field-free (\( U=0 \))

radial acceleration: \( \ddot{r} = r \dot{\phi}^2 \)

angular acceleration: \( \ddot{\phi} = -2 \frac{\dot{r} \dot{\phi}}{r} \)

Because Earth rotation is counter-clockwise (positive) in North

Coriolis acceleration with \( \dot{\phi} > 0 \) and \( \dot{r} < 0 \)

\( \ddot{\phi} = -2 \frac{\dot{r} \dot{\phi}}{r} \) (makes \( \ddot{\phi} \) positive)

Inward flow to pressure Low \( \dot{r} < 0 \)

...makes wind turn to the right

Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Northern hemisphere rotation \( \dot{\phi} > 0 \)
Separation of GCC Equations: Effective Potentials

Small radial oscillations
Cycloid vs Pendulum
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \]

( Numerically correct ONLY! )

( Formally and Numerically correct )
Separation of GCC Equations: Effective Potentials

For isotropic $H(r,p_r,\phi,p_\phi)$

$$
H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad \text{(Numerically correct ONLY!)}
$$

$$
= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad \text{(Formally and Numerically correct)}
$$

If potential $V$ is isotropic (cylindrical) function of radius $\rho$. ($V = V(\rho)$)

$H$ has no explicit $\phi$--dependence and the $\phi$--momenta is constant.

$$m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$
Separation of GCC Equations: Effective Potentials (For isotropic $H(r,p_r,\phi,p_\phi)$)

\[
H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V
\]

(\text{Numerically correct ONLY!})

\[
= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V
\]

(\text{Formally and Numerically correct})

Potential $V$ is \textit{isotropic} (cylindrical) function of radius $\rho$. ($V = V(\rho)$)

$H$ has no explicit $\phi$–dependence and the $\phi$–momenta is constant.

\[
m \dot{\rho}^2 \phi = p_\phi = \text{const.} = \mu
\]

If $H$ has no explicit $z$–dependence then the $z$–momenta is constant, too.

\[
m \dot{z} = p_z = \text{const.} = k
\]
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]  
\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p^2 + \frac{1}{2m} p^2 + \frac{1}{2m} p^2 + V \]  

(Numerially correct ONLY!)

Formally and Numerically correct

Potential \( V \) is *isotropic* (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

\( H \) has no explicit \( \phi \)--dependence and the \( \phi \)--momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

If \( H \) has no explicit \( z \)--dependence then the \( z \)--momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\rho}^2 \dot{\phi}^2 + \frac{1}{2} m \ddot{z}^2 + V \]  
\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m} p_\phi^2 + \frac{1}{2m} p_z^2 + V \]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

\( H \) has no explicit \( \phi \)-dependence and the \( \phi \)-momenta is constant.

\[ m \dot{\rho}^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

If \( H \) has no explicit \( z \)-dependence then the \( z \)-momenta is constant, too.

\[ m \ddot{z} = p_z = \text{const.} = k \]

\( (\text{Let } k = 0) \)
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad \text{(Numerically correct ONLY!)} \]

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad \text{(Formally and Numerically correct)} \]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)

\( H \) has no explicit \( \phi \)–dependence and the \( \phi \)–momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]

\((\text{Let } k = 0)\)
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\rho}^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \]

(\text{Numerically correct ONLY!})

\[ = \frac{1}{2} \gamma^{m n} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \]

(\text{Formally and Numerically correct})

Potential \( V \) is \textit{isotropic} (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)

\( H \) has no explicit \( \phi \)--dependence and the \( \phi \)--momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

An \textit{effective potential} \( V^{\text{eff}}(\rho) \) has a \textit{centrifugal barrier}.

\[ V^{\text{eff}}(\rho) = \frac{\mu^2}{2m \rho^2} + V(\rho) \]

If \( H \) has no explicit \( z \)--dependence then the \( z \)--momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]

(\text{Let } k = 0)
Separation of GCC Equations: Effective Potentials

\[
H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad \text{ (Numerically correct ONLY!)}
\]

\[
= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2} m p_\rho^2 + \frac{1}{2m} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad \text{ (Formally and Numerically correct)}
\]

Potential \(V\) is isotropic (cylindrical) function of radius \(\rho\). \((V = V(\rho))\)

\(H\) has no explicit \(\phi\)–dependence and the \(\phi\)–momenta is constant.

\[
m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu
\]

\[
H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}
\]

Symmetry reduces problem to a one-dimensional form.

\[
H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.}
\]

An effective potential \(V^{\text{eff}}(\rho)\) has a centrifugal barrier.

\[
V^{\text{eff}}(\rho) = \frac{\mu^2}{2m \rho^2} + V(\rho)
\]

Velocity relations:

\[
\dot{\phi} = \mu / (m \rho^2) \quad \dot{\rho} = \frac{d \rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm \sqrt{\frac{2m}{E - V^{\text{eff}}(\rho)}}
\]

If \(H\) has no explicit \(z\)–dependence then the \(z\)–momenta is constant, too.

\[
m \dot{z} = p_z = \text{const.} = k
\]

(Let \(k = 0\))
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]  

Formally and Numerically correct

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m} p_\phi^2 + \frac{1}{2m} p_z^2 + V \]  

Numerically correct ONLY!

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)

\( H \) has no explicit \( \phi \)–dependence and the \( \phi \)–momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

An effective potential \( V^{\text{eff}}(\rho) \) has a centrifugal barrier.

\[ V^{\text{eff}}(\rho) = \frac{\mu^2}{2m \rho^2} + V(\rho) \]

Velocity relations:

\[ \dot{\phi} = \frac{\mu}{m \rho^2} \]

\[ \dot{\rho} = \frac{dp}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm \sqrt{\frac{2}{m} \left( E - V^{\text{eff}}(\rho) \right)} \]

Equations solved by a quadrature integral for time versus radius.

\[ t \]

\[ \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m} \left( E - V^{\text{eff}}(\rho) \right)}} = \text{(Travel time } \rho_0 \text{ to } \rho_1 \text{)} = t_1 - t_0 \]
Separation of GCC Equations: Effective Potentials

Small radial oscillations
Cycloid vs Pendulum
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[
\left. \frac{dV^{\text{eff}}(\rho)}{d\rho} \right|_{\rho_0} = 0, \quad \text{with: } \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0} > 0.
\]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[
V^{\text{eff}}(\rho) = V^{\text{eff}}(\rho_0) + 0 + \frac{1}{2}(\rho - \rho_0)^2 \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0}
\]

**Stable flat** \( \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0} > 0 \)

**Unstable flat** \( \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0} < 0 \)

Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[ \frac{dV_{\text{eff}}(\rho)}{d\rho} \bigg|_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} > 0. \]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[ V_{\text{eff}}(\rho) = V_{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} \]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[ k_{\text{eff}} = \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} \quad \omega_{\rho_{\text{stable}}} = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}}} \]
**Small radial oscillations**

Stable minimal-energy radius will satisfy a zero-slope equation.

\[
\left. \frac{dV^{\text{eff}}(\rho)}{d\rho} \right|_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} > 0.
\]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[
V^{\text{eff}}(\rho) = V^{\text{eff}}(\rho_{\text{stable}}) + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}
\]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[
k^{\text{eff}} = \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}
\]

\[
\omega_{\rho_{\text{stable}}} = \sqrt{\frac{k^{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}}
\]

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

\[
\frac{\omega_{\rho_{\text{stable}}}}{\omega_{\phi}} = \frac{\omega_{\rho_{\text{stable}}}}{\phi(\rho_{\text{stable}})} = \frac{n_{\rho}}{n_{\phi}} \iff \text{Orbit is closed-periodic}
\]
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[
\left. \frac{dV_{\text{eff}}(\rho)}{d\rho} \right|_{\rho_{\text{stable}}} = 0, \quad \text{with: } \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} > 0.
\]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[
V_{\text{eff}}(\rho) = V_{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2}(\rho - \rho_{\text{stable}})^2 \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}.
\]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[
k_{\text{eff}} = \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}, \quad \omega_{\rho_{\text{stable}}} = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}}.
\]

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

\[
\frac{\omega_{\rho_{\text{stable}}}}{\omega_{\phi}} = \frac{\omega_{\rho_{\text{stable}}}}{\phi(\rho_{\text{stable}})} = \frac{n}{n_{\phi}} \iff \text{Orbit is closed-periodic}
\]

Some generic shapes resulting from various ratios \(n_{\rho} : n_{\phi}\)
(b) $\omega_\rho : \omega_\phi$ just below 1
prograde precession of nodes

$\omega_\rho : \omega_\phi = 1$

$\omega_\rho : \omega_\phi$ just above 1
retrograde precession of nodes

(c) $\omega_\rho : \omega_\phi$ just below 2
prograde precession of nodes

$\omega_\rho : \omega_\phi = 2$

$\omega_\rho : \omega_\phi$ just above 2
retrograde precession of nodes
Separation of GCC Equations: Effective Potentials

Small radial oscillations

Cycloid vs Pendulum
time = 174.180
Θ = +1.384
ω = +1.000
E = +1.999

http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html
Here the radius is plotted as an irrational $R = 3/\pi = 0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled-out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. 

$\phi$-clock $\frac{3\pi}{2}$

Arc length $R\phi = (3/\pi)\phi$
Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$. 

$\frac{\pi}{6} \quad \frac{\pi}{3} \quad \frac{\pi}{2} \quad \frac{2\pi}{3} \quad \frac{5\pi}{6} \quad \pi \quad \frac{7\pi}{6} \quad \frac{\pi}{2} \quad \frac{\pi}{3} \quad \frac{\pi}{6}$

$\frac{2\pi}{12} \quad \frac{2\pi}{11} \quad \frac{2\pi}{10} \quad \frac{2\pi}{9} \quad \frac{2\pi}{8} \quad \frac{2\pi}{7} \quad \frac{2\pi}{6} \quad \frac{2\pi}{5} \quad \frac{2\pi}{4} \quad \frac{2\pi}{3} \quad \frac{2\pi}{2} \quad \frac{2\pi}{1}$

$0^\circ \quad 1^\circ \quad 2^\circ \quad 3^\circ \quad 4^\circ \quad 5^\circ \quad 6^\circ \quad 7^\circ \quad 8^\circ \quad 9^\circ \quad 10^\circ \quad 11^\circ \quad 12^\circ$
Here the radius is plotted as an irrational $R = 3/\pi = 0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled-out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R = 6/\pi = 1.91$.

Red circle rolls left-to-right on $y=3.82$ ceiling.
Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.

Green circle rolls right-to-left on $y=1.91$ ceiling.
Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.

Rotation angle $\phi$ Arc length $R\phi = (3/\pi)\phi$

<table>
<thead>
<tr>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/2</td>
<td>11/2</td>
<td>10/2</td>
<td>9/2</td>
<td>8/2</td>
<td>7/2</td>
<td>6/2</td>
<td>5/2</td>
<td>4/2</td>
<td>3/2</td>
<td>2/2</td>
<td>1/2</td>
<td>Arc length $R\phi = (3/\pi)\phi$</td>
</tr>
</tbody>
</table>
Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$

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Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.
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Red circle rolls left-to-right on $y = 3.82$ ceiling
Contact point goes from $(x = 6/2, y = 3.82)$ to $x = 0$.

Green circle rolls right-to-left on $y = 1.91$ ceiling
Contact point goes from $(x = 0, y = 1.91)$ to $x = 6/2$.

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Rotation angle $\phi$

Arc length $R\phi = (3/\pi)\phi$
Rotation angle $\phi$

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Rotation angle $\phi$

Arc length $R\phi = \frac{3}{\pi} \phi$

$\frac{2\pi}{2} = 0$ o'clock

Radii:
- $\frac{2\pi}{6} = 1.06865$
- $\frac{2\pi}{3} = 1.91386$
- $\pi = 1.57079$
- $\frac{2\pi}{6} = 0.52359$

Radian measures:
- $\frac{\pi}{6} = 0.33983$
- $\frac{\pi}{2} = 0.78539$
- $\frac{2\pi}{3} = 1.04720$
- $\frac{\pi}{3} = 0.52359$
- $\frac{\pi}{2} = 0.62832$

Arc lengths:
- $\frac{2\pi}{2} = 0$ o'clock
- $\frac{2\pi}{3} = 0.95546$
- $\pi = 1.91386$
- $\frac{2\pi}{6} = 1.41372$

Also included:
- Radius $R$
- Arc length $R\phi$
- Rotation angle $\phi$
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$.

Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.
If you hammer a stick at a point $h$ meters from its center, you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$.

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M \ell^2/3$ of the stick.
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$$\omega = \frac{\Lambda}{I} \quad (=3\frac{\Lambda}{(M \ell^2)} \text{ for stick})$$

$$= \frac{h \Pi}{I} \quad (=3\frac{h \Pi}{(M \ell^2)} \text{ for stick})$$

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One point $\text{P}$, or \textit{center of percussion (CoP)}, is on the wheel where speed $p \omega$ due to rotation just cancels translational speed $V_{\text{Center}}$ of stick.

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$$\omega = \Lambda / I \quad (=3\Lambda /(M \ell^2) \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi/(M \ell^2) \text{ for stick})$$

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$$
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$$

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$$
\frac{\Pi}{M} = V_{Center} = |p\omega| = p \cdot h \frac{\Pi}{I}
I / M = \frac{I}{M} = p \cdot h
$$

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\]

\[
I / M = \quad = \quad = p\cdot h \quad \text{or: } p = I / (Mh)
\]
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$$\Pi / M = V_{Center} = |p \omega| = p \cdot h \Pi / I$$

$$I / M = \frac{p \cdot h}{p} \text{ or: } p = I/(Mh)$$

$P$ follows a normal cycloid made by a circle of radius $p = I/(Mh)$ rolling on an imaginary road thru point $P$ in direction of $\Pi$. 

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$$I/M = \frac{\Pi}{V_{Center}} = \frac{p \cdot h}{p\omega}$$

$P$ follows a normal cycloid made by a circle of radius $p = I/(Mh)$ or $p = \ell^2/3h$ rolling on an imaginary road thru point $P$ in direction of $\Pi$.

The percussion radius $p = \ell^2/3h$ is of the CoP point that has no velocity just after hammer hits at $h$. 

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**Fig. 2.A.1** Cycloidal paths due to hitting a stationary stick.