Quadratic form geometry and development of mechanics of Lagrange and Hamilton
(Ch. 12 of Unit 1 and Ch. 4-5 of Unit 7)

Scaling transformation between Lagrangian and Hamiltonian views of KE (Review of Lecture 10)
Introducing 1st Lagrange and Hamilton differential equations of mechanics (Review Of Lecture 10)

Introducing the Poincare’ and Legendre contact transformations
Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

A general contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of ” ” ” ” ” ” ” ”

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html
Three ways to express energy:  Consider kinetic energy (KE) first

1. **Lagrangian** is explicit function of velocity:  
   \[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]
   
   \[ L(v_k ... ) = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2 + ...) = L(\mathbf{v} ...) = \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} + ... = \frac{1}{2} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + ... \]

2. **“Estrangian”** is explicit function of R-rescaled velocity:
   (or l’Estrangian)
   
   \[ \mathbf{v} = \mathbf{R} \cdot \mathbf{v} \quad \text{or:} \quad \textbf{“speedinum” } \mathbf{V} = \mathbf{R} \cdot \mathbf{v} \quad \text{or:} \quad \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]
   
   \[ E(V_k ... ) = \frac{1}{2} (V_1^2 + V_2^2 + ...) = E(\mathbf{V} ...) = \frac{1}{2} \mathbf{V} \cdot \mathbf{1} \cdot \mathbf{V} + ... = \frac{1}{2} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + ... \]

3. **Hamiltonian** is explicit function of M=R²-rescaled velocity:
   
   \[ \text{or: momentum } \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \quad \text{or:} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} m_1 v_1 \\ m_2 v_2 \end{pmatrix} \]
   
   \[ H(\mathbf{p} ... ) = \frac{1}{2} \left( \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + ... \right) = H(\mathbf{p} ...) = \frac{1}{2} \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} + ... = \frac{1}{2} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + ... \]
Scaling transformation between Lagrangian and Hamiltonian views of KE (Review of Lecture 9)
Introducing the (partial) differential equations of mechanics (Review Of Lecture 9)
1st equations of Lagrange and Hamilton
Introducing the (partial $\frac{\partial}{\partial v}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and Estrangian have no explicit dependence on momentum $p = M \cdot v$

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian and Estrangian have no explicit dependence on velocity $v = M^{-1} \cdot p$

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian and Hamiltonian have no explicit dependence on speed $v = M^{1/2} \cdot \dot{v}$

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections

Lagrange’s 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial v} = p$$

$$\nabla_v L = \frac{\partial L}{\partial v} = \frac{\partial}{\partial v} \left( \frac{v \cdot M \cdot v}{2} \right) = M \cdot v = p$$

$\begin{bmatrix}
\frac{\partial L}{\partial v_1} \\
\frac{\partial L}{\partial v_2}
\end{bmatrix} =
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
p_1 \\
p_2
\end{bmatrix}$

Hamilton’s 1st equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial p} = v$$

$$\nabla_p H = v = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \left( \frac{p \cdot M^{-1} \cdot p}{2} \right) = M^{-1} \cdot p = v$$

$$\nabla_p H =
\begin{bmatrix}
\frac{\partial H}{\partial p_1} \\
\frac{\partial H}{\partial p_2}
\end{bmatrix} =
\begin{bmatrix}
m_1^{-1} & 0 \\
0 & m_2^{-1}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2
\end{bmatrix} =
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}$$

Forget Estrangian for now
(But, recall dual ellipse geometry in Lecture 10 p. 44-55)
Unit 1
Fig. 12.2

(a) **Lagrangian plot**
\[ L(v) = \text{const.} = v \cdot M \cdot v / 2 \]

(b) **Hamiltonian plot**
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2 \]

\[ p_2 = m_2 v_2 \]

\[ p_1 = m_1 v_1 \]

\[ H = \text{const.} = E \]

\[ a = \sqrt{2E/m} \]

\[ b = \sqrt{2E/m} \]
(a) Lagrangian plot
\[ L(v) = \text{const.} = v \cdot M \cdot v / 2 \]

(b) Hamiltonian plot
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2 \]

(c) Overlapping plots
Lagrangian tangent at velocity \( v \)
\[ v = \nabla_v L = M \cdot v \]

Hamiltonian tangent at momentum \( p \)
\[ p = \nabla_p H = M^{-1} \cdot p \]
Fig. 12.2

(a) **Lagrangian plot**
\[ L(v) = \text{const.} = v \cdot M \cdot v/2 \]

(b) **Hamiltonian plot**
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p/2 \]

(c) **Overlapping plots**

**1st equation of Lagrange**
\[ L = \text{const.} = E \]
\[ p = \nabla_v L \]
\[ = M \cdot v \]

**1st equation of Hamilton**
\[ H = \text{const.} = E \]
\[ p = \nabla_p H \]
\[ = M^{-1} \cdot p \]

(d) **Less mass**

Hamiltonian tangent at momentum \( p \) is normal to velocity \( v \)

(e) **More mass**

Lagrangian tangent at velocity \( v \) is normal to momentum \( p \)
Introducing the Poincaré' and Legendre contact transformations

Geometry of Legendre contact transformation
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( p = M \cdot v \) or its inverse: \( v = M^{-1} \cdot p \) you might be tempted to rewrite

\[ Q \text{-forms} \quad L(v..) = (1/2)v \cdot M \cdot v \quad \text{or} \quad H(p..) = (1/2)p \cdot M^{-1} \cdot p \quad \text{to be} \quad H = (1/2)p \cdot v \quad \text{or equivalently} \quad L = (1/2)v \cdot p. \]
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( p = M \cdot v \) or its inverse: \( v = M^{-1} \cdot p \) you might be tempted to rewrite

\[ \underline{Q\text{-forms} \quad L(v..) = \frac{1}{2} v \cdot M \cdot v \quad \text{or} \quad H(p..) = \frac{1}{2} p \cdot M^{-1} \cdot p} \] to be \( H = \frac{1}{2} p \cdot v \) or equivalently \( L = \frac{1}{2} v \cdot p \).

Numerically-CORRECT, but Differentially-WRONG!
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\[ Q\text{-forms } L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \text{ or } H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \text{ to be } H = (1/2) \mathbf{p} \cdot \mathbf{v} \text{ or equivalently } L = (1/2) \mathbf{v} \cdot \mathbf{p}. \]

Numerically-CORRECT, but Differentially-WRONG! (In classical physics \( \mathbf{p} \cdot \mathbf{v} \) and \( \mathbf{v} \cdot \mathbf{p} \) are identical)

Instead try: \( H(\mathbf{p}..) = \mathbf{p} \cdot \mathbf{v} - (1/2) \mathbf{v} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}..) \) or else: \( L(\mathbf{v}..) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}..) \)
Introducing the Poincare’ and Legendre contact transformations

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Q-forms \( L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \) or \( H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \) to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).

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That is ... the Legendre contact transformation

\[
L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}) \quad \text{or:} \quad H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})
\]
Introducing the Poincare’ and Legendre contact transformations

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Q-forms $L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$ or $H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p}$ to be $H = (1/2) \mathbf{p} \cdot \mathbf{v}$ or equivalently $L = (1/2) \mathbf{v} \cdot \mathbf{p}$.

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That is ... the Legendre contact transformation

$$L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}) \quad \text{or:} \quad H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})$$

Now explicit dependency (non)-relations give the right derivatives

$$\frac{\partial L(\mathbf{v})}{\partial \mathbf{p}} = \frac{\partial \mathbf{p} \cdot \mathbf{v}}{\partial \mathbf{p}} - \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}}$$
$$0 = \mathbf{v} - \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}}$$

$$\frac{\partial H(\mathbf{p})}{\partial \mathbf{v}} = \frac{\partial \mathbf{p} \cdot \mathbf{v}}{\partial \mathbf{v}} - \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$
$$0 = \mathbf{p} - \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\[ Q\text{-forms} \quad L(\mathbf{v}.) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \quad \text{or} \quad H(\mathbf{p}.) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \quad \text{to be} \quad H = (1/2) \mathbf{p} \cdot \mathbf{v} \quad \text{or equivalently} \quad L = (1/2) \mathbf{v} \cdot \mathbf{p}. \]

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That is ... the **Legendre contact transformation**

\[
L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}) \quad \text{or:} \quad H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v})
\]

Now explicit dependency (non)-relations give the right derivatives

\[
\frac{\partial L(\mathbf{v})}{\partial \mathbf{p}} = \frac{\partial \mathbf{p} \cdot \mathbf{v}}{\partial \mathbf{p}} - \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}}
\]

\[0 = \mathbf{v} - \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}}\]

\[
\frac{\partial H(\mathbf{p})}{\partial \mathbf{v}} = \frac{\partial \mathbf{p} \cdot \mathbf{v}}{\partial \mathbf{v}} - \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}
\]

\[0 = \mathbf{p} - \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}\]

That is **Hamilton’s 1st equation(s)** and **Lagrange’s 1st equation(s)**

\[
\mathbf{v} = \frac{\partial H(\mathbf{p})}{\partial \mathbf{p}}
\]

\[
\mathbf{p} = \frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}
\]
Introducing the Poincaré’ and Legendre contact transformations

Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
**Lagrangian plot**

\[ L(\mathbf{v}) = \mathbf{v} \cdot \mathbf{p} - H(\mathbf{p}) \]

---

**Hamiltonian plot**

\[ H(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}) \]
Preview of Unit 8:
Geometry of Legendre contact transformation persists in relativistic quantum mechanics!
(In fact it is due to the wave mechanics and phase invariance principles.)
How Legendre contact transformations work... (to make $\frac{\partial H}{\partial v} = 0$ or $\frac{\partial L}{\partial p} = 0$)

Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$ and decreasing intercept $-H(v_2) > -H(v_1) > ...$

for increasing velocity $v_2 > v_1 > ... > v_0$

lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has $\max H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$

Unit 1
Fig. 12.4
How Legendre contact transformations work... (to make $\frac{\partial H}{\partial v} = 0$ or $\frac{\partial L}{\partial p} = 0$)

Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$ and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > \ldots$

for increasing velocity $v_{-2} > v_{-1} > \ldots > v_0$

lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has $\max H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$

(Similarly...)

Unit 1

Fig. 12.4

(a) Secant lines: $L(v) = p \cdot v - H$

for fixed slope $p$ and varying $H$

Tangent line points to extreme value $-H(v_0)$ of intercept $-H$ thus: $dH(v)/dv = 0$

(b) Secant lines: $H(p) = p \cdot v - L(v)$

for fixed slope $v$ and varying $L$

Tangent line points to extreme value $-L(p_0)$ of intercept $-L$ thus: $dL(p)/dp = 0$

$dL(p)/dp = 0$ at each point $p = \frac{\partial L}{\partial v}$ of $H(p)$ with slope $v = \frac{\partial H}{\partial p}$
Introducing the Poincaré’ and Legendre contact transformations

Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)

Example from thermodynamics

Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
**Example of Legendre contact transformation in thermodynamics**

*Internal energy* $U(S,V)$ is defined as a function of entropy $S$ and volume $V$.

A new function *enthalpy* $H(S,P)$ depends on entropy and *pressure* $P$.

It is a Legendre transform $H(S,P)=P\cdot V+U$ of energy $U(S,V)$ to new variable $P = -\left(\frac{\partial U}{\partial V}\right)_S$. 
Example of Legendre contact transformation in thermodynamics

\[ L(r, v) \] position \( r \) velocity \( v \)

Internal energy \( U(S, V) \) is defined as a function of entropy \( S \) and volume \( V \).

\[ H(r, p) \] position \( r \) momentum \( p \)

A new function \textit{enthalpy} \( H(S, P) \) depends on entropy and \textit{pressure} \( P \).

\[ H(r, p) = p \cdot v - L \] Lagrangian \( L(r, v) \)

It is a Legendre transform \( H(S, P) = P \cdot V + U \) of energy \( U(S, V) \) to new variable \( P = -\left( \frac{\partial U}{\partial V} \right)_S \).
Example of Legendre contact transformation in thermodynamics

Lagrangian $L(r,v)$  
Hamiltonian $H(r,p)$

<table>
<thead>
<tr>
<th>position $r$</th>
<th>velocity $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>position $r$</td>
<td>momentum $p$</td>
</tr>
</tbody>
</table>

Internal energy $U(S,V)$ is defined as a function of entropy $S$ and volume $V$.

A new function enthalpy $H(S,P)$ depends on entropy and pressure $P$.

$H(r,p)=p \cdot v - L$  
$Lagrangian L(r,v)$  
$p = (\frac{\partial L}{\partial v})_r$

It is a Legendre transform $H(S,P)=P \cdot V + U$ of energy $U(S,V)$ to new variable $P = - (\frac{\partial U}{\partial V})_S$.

Except for ± signs, it’s our Hamiltonian $H(p)=p \cdot v - L(v)$ going from Lagrangian $L(v)$ to use new variable momentum $p = (\frac{\partial L}{\partial v})_x$.
Introducing the Poincare’ and Legendre contact transformations

Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics

Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S = \text{const.} \) curves.

Unit 1
Fig. 12.7
The Legendre transformation does it with contacting straight line tangents.
Legendre transform: special case of General Contact Transformation

**Active-Contact-Transformation Generator or Action function**: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\( Y(X) \) is mapped from \( y(x) \) as an envelope of contacting \( S=\text{const.} \) curves.

...And, Visa-Versa !...

*The Legendre transformation does it with contacting straight line tangents.*
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\[ Y(X) \text{ is mapped from } y(x) \text{ as an envelope of contacting } S = \text{const. \ curves}. \]

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**

Legendre transform: special case of General Contact Transformation

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Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\[ Y(X) \text{ is mapped from } y(x) \text{ as an envelope of contacting } S = \text{const. \ curves}. \]

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\[ Y(X) \text{ is mapped from } y(x) \text{ as an envelope of contacting } S = \text{const. \ curves}. \]

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\[ Y(X) \text{ is mapped from } y(x) \text{ as an envelope of contacting } S = \text{const. \ curves}. \]

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: \( S(x,y;X,Y) = \text{const.} \) does mapping.

\[ Y(X) \text{ is mapped from } y(x) \text{ as an envelope of contacting } S = \text{const. \ curves}. \]

...And, Visa-Versa !...

**The Legendre transformation does it with contacting straight line tangents.**
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: $S(x,y,X,Y) = \text{const.}$ does mapping.

$Y(X)$ is mapped from $y(x)$ as an envelope of contacting $S = \text{const.}$ curves.

...And, Visa-Versa !...

The Legendre transformation does it with contacting straight line tangents.

Poincare's differential action

$$dS = L dt = p \cdot dq dt - H \cdot dt$$

$= p \cdot dq - H \cdot dt$

(Quantum phase differential)
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function: $S(x,y:X,Y)=\text{const.}$ does mapping.

$Y(X)$ is mapped from $y(x)$ as an envelope of contacting $S=\text{const.}$ curves.

...And, Visa-Versa !...

The Legendre transformation does it with contacting straight line tangents.

Poincaré's differential action

$$dS = L\,dt = p \cdot \dot{q}\,dt - H \cdot dt$$

$= p \cdot dq - H \cdot dt$

(Quantum phase differential)

This extraordinary claim needs extraordinary proof!

(...given in Ch. 12 Unit 1 and in Unit 8.)

Tuesday, September 30, 2014
A general contact transformation from sophomore physics

Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”

Intuitive-geometric development of “ ” and “ ”
\( \alpha = 45^\circ \)

(a) Volcanic plumes on Jupiter’s moon Io

(b) Atomic clock controls expanding balls of Cesium atoms rising and falling in Earth gravity

(NIST Boulder Labs)

(c) Trajectory family for fixed \( g \) and \( v_0 \)

Atom ball expands at constant rate \( v_0 \) as center falls at constantly increasing rate \( g-t \) and it maintains two contact points with the envelope after reaching its highest point.
**UP-1 formulas for trajectories in constant gravity g**

\[
x(t) = (v_0 \cos \alpha) t \\
y(t) = (v_0 \sin \alpha) t - \frac{1}{2} gt^2 \\
\dot{x}(0) = v_x(0) = v_0 \cos \alpha \\
\dot{y}(0) = v_y(0) = v_0 \sin \alpha
\]

Substitute time \(t=x/(v_0 \cos \alpha)\) into \(y(t)\)

\[
y(x) = \frac{v_0 \sin \alpha}{v_0 \cos \alpha} x - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]
http://www.uark.edu/ua/modphys/markup/CoulItWeb.html
Convert $y(x)$ solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{becomes:} \quad S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0$$

Figure 12.6
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

**Envelopes** of the \( v_0 \)-trajectory region contain extremal **contact points** with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]
Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha: x, y)$

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha: x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

**Envelopes** of the $v_0$-trajectory region contain extremal *contact points* with each trajectory where:

\[
\frac{\partial S(v_0, \alpha: x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2 \sin \alpha \cos \alpha}{\cos^3 \alpha}
\]
Convert $y(x)$ solution into Active Contact Transformation Generator $S(v_0, \alpha: x, y)$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{becomes:} \quad S(v_0, \alpha: x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0$$

**Envelopes** of the $v_0$-trajectory region contain extremal contact points with each trajectory where:

$$\frac{\partial S(v_0, \alpha: x, y)}{\partial \alpha} = 0$$

$$x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^2 \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \cdot \sin \alpha$$

**gives**: \[ \tan \alpha = \frac{v_0^2}{gx} \quad \text{or} \quad x = \frac{v_0^2}{g \tan \alpha} \]
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

\[
\frac{x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha}}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2\sin \alpha}{\cos^3 \alpha}
\]

\[
\tan \alpha = \frac{v_0^2}{gx} \quad \text{or} \quad x = \frac{v_0^2}{g \tan \alpha}
\]

\[
y_{env}(x) = x \tan \alpha - \frac{gx^2}{2v_0^2} \left(1 + \tan^2 \alpha \right) = \frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2} \left(1 + \frac{v_0^4}{g^2 x^2} \right)
\]

\( \text{Unit 1} \)

Fig. 12.6

Envelopes of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]
Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$

becomes:

$$S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0$$

**Envelopes** of the $v_0$-trajectory region contain extremal contact points with each trajectory where:

$$\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0$$

$$x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2 \cos^3 \alpha} 2 \sin \alpha$$

$$\tan \alpha = \frac{v_0^2}{gx} \quad \text{or} \quad x = \frac{v_0^2}{g \tan \alpha}.$$
The Plumes of Prometheus
NASA-Galileo Project
Jo fly-by on August 18, 1997

Pretty bad sketch of plumes (Las Vegas model of planetary ejecta?)

Do these guys need a geometry lesson?

Go fly a kite?

October 4, 1999: Thirty years ago, before the Voyager probes visited Jupiter, if you had described Io to a literary critic it would have been declared overwrought science fiction. Jupiter's strange moon is literally bursting with volcanoes. Dozens of active vents pepper the landscape which also includes gigantic frosty plains, towering mountains and volcanic rings the size of California. The volcanoes themselves are the hottest spots in the solar system with temperatures exceeding 1800 K (1527 C). The plumes which rise 300 km into space are so large they can be seen from Earth by the Hubble Space Telescope. Confounding common sense, these high-rising ejecta seem to be made up of, not blisteringly hot lava, but frozen sulfur dioxide. And to top it all off, Io bears a striking resemblance to a pepperoni pizza. Simply unbelievable.

Right: Digital Radiance simulation of Pillan Patera just before the Galileo flyby. click for animation →.
...conventional parabolic geometry...carried to extremes...

Recall Lecture 8 p.16 to 18

Unit 1
Fig. 9.4
A general contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of ” ” ” and ” ” ” ”
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises
Q3. How high can $\alpha=45^\circ$ path rise?
Q4. Where on x-axis does $\alpha=45^\circ$ path hit?

Right at the tippy-tip

Time to go to top by $\Delta y$ equals time to fall $\Delta y$

$\Delta y$
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?  
Q2. ...where is the blast wave?  
Q3. How high can $\alpha=45^\circ$ path rise?  
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. How high can $\alpha=45^\circ$ path rise?
Q4. Where on x-axis does $\alpha=45^\circ$ path hit?

This sets $\alpha=45^\circ$ parabolic “kite”
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far out as 90° ball rises.
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high.
Q4. Where on x-axis does $\alpha=45^\circ$ path hit ? $x=2$.
Q5. Where is blast wave then?
Q6 Where is $\alpha=45^\circ$ path focus?
Q7 Guess for all-path envelope? and its focus? directrix?

This sets $\alpha=45^\circ$ parabolic "kite" and focus and range, etc.
Say \( \alpha = 90^\circ \) path rises to 1.0 then drops. When at \( y = 1.0 \)...

Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises.
Q3. How high can \( \alpha = 45^\circ \) path rise? \( \frac{1}{2} \) as high.
Q4. Where on \( x\)-axis does \( \alpha = 45^\circ \) path hit? \( x = 2 \).
Q5. Where is blast wave then? centered on 45° normal.
Q6 Where is \( \alpha = 45^\circ \) path focus?
Q7 Guess for all-path envelope? and its focus? directrix?

\[
\sin \alpha = \sin 45^\circ = \frac{v_0}{\sqrt{2}}
\]

implies:
\[
v_0^2 \sin^2 \alpha = \frac{v_0^2}{2}
\]
so y-coord. KE is \( \frac{1}{2} \) for \( \alpha = 45^\circ \)

So: y-peak PE is \( \frac{1}{2} \) for \( \alpha = 45^\circ \)

This sets \( \alpha = 45^\circ \) parabolic “kite” and focus and range, etc.

\( \alpha = 45^\circ \) Envelope CONTACT POINT
That is maximum horizontal range
so must be tangent to blast circle
and must be tangent to envelope
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rise.
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45° normal.
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope?
   and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? 
directrix?
Say $\alpha=90^\circ$ path rises to 1.0
then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on x-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? Centered on 45° normal
Q6. Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7. Guess for all-path envelope?
and its focus? directrix?

Q7. Where is $\alpha=45^\circ$ “kite” geometry?
Q8. Where is $\alpha=0^\circ$ path focus? directrix?

Where is a=30° path?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on $45^\circ$ normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?

Where is $\alpha=30^\circ$ path? ...and kite structure?
Where is $\alpha=60^\circ$ path? ...and kite structure?

For $\alpha=60^\circ$ parabolic trajectory contact-parabolic envelope, timing ($\alpha=0^\circ$)-parabola, ($\alpha=90^\circ$)-blast-wave-circle, ($\alpha=60^\circ$)-blast-wave-circle.

Lecture 11 ends here | Tue. 9.30.2014