Lecture 31
Tue. 12.12.2014

Multi-particle and Rotational Dynamics

(Ch. 2-7 of Unit 6 12.12.14)

2-Particle orbits
Ptolemetric or LAB view and reduced mass
Copernican or COM view and reduced coupling

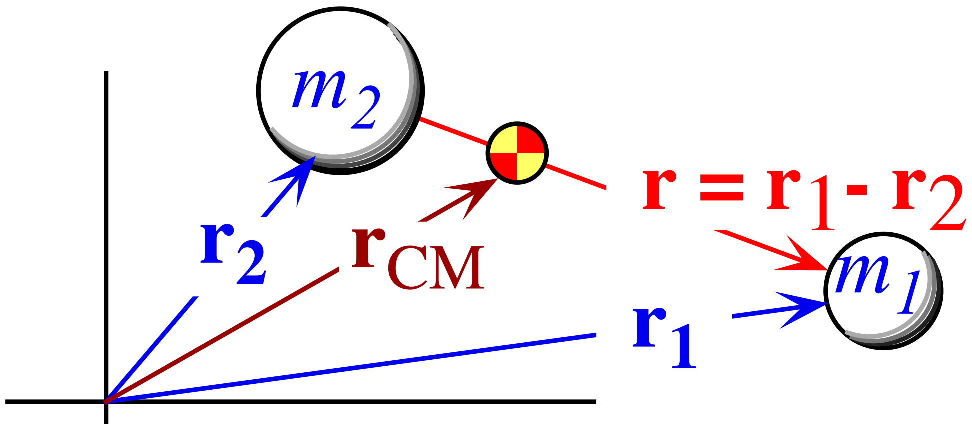
2-Particle orbits and scattering: LAB-vs.-COM frame views
Ruler & compass construction (or not)

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*
How to make my boomerang come back
The gyrocompass and mechanical spin analogy

Rotational momentum and velocity tensor relations
Quadratic form geometry and duality (again)
angular velocity **ω**-*ellipsoid vs. angular momentum* **L**-*ellipsoid*
Lagrangian **ω**-*equations vs. Hamiltonian momentum* **L**-*equation*

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)
Symmetric, asymmetric, and spherical-top dynamics (Constant **L***)*
BOD-frame cone rolling on LAB frame cone
Deformable spherical rotor RES and semi-classical rotational states and spectra

2-Particle orbits and center-of-mass (CM) coordinate frame



$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Defining *relative coordinate vector*

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and *mass-weighted-average* or *center-of-mass coordinate vector* $\mathbf{r}_{CM}$

$$\bar{\mathbf{r}} = \mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

The inverse coordinate transformation.

$$\mathbf{r}_1 = \mathbf{r}_{CM} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_{CM} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

2-Particle orbits

➡ *Ptolemetric or LAB view and reduced mass*

Copernican or COM view and reduced coupling

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

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Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

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Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

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(Why it's reduced)

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Re-scaled force: A Copernican view

relative radius vector

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2-Particle orbits

Ptolemetric view and reduced mass

➡ *Copernican view and reduced coupling*

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(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels
coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

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Reduced mass: Ptolemetric views

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$F(r) = \frac{k}{r^2}$ *(Coulomb)* becomes: $F(\frac{m_1}{\mu}r_1) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$

$$k \to k_1 = k\,\mu^2/m_1^2, \qquad k \to k_2 = k\,\mu^2/m_2^2$$

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$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$
Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\mu = \frac{m_2 m_1}{m_1 + m_2}$ $\quad \ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}$$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\dots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\dots\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels
coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(\frac{m_1}{\mu}r_1)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F(\frac{m_2}{\mu}r_2)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

$F(r) = \frac{k}{r^2}$ becomes: $F(\frac{m_1}{\mu}r_1) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$ *(Coulomb)*

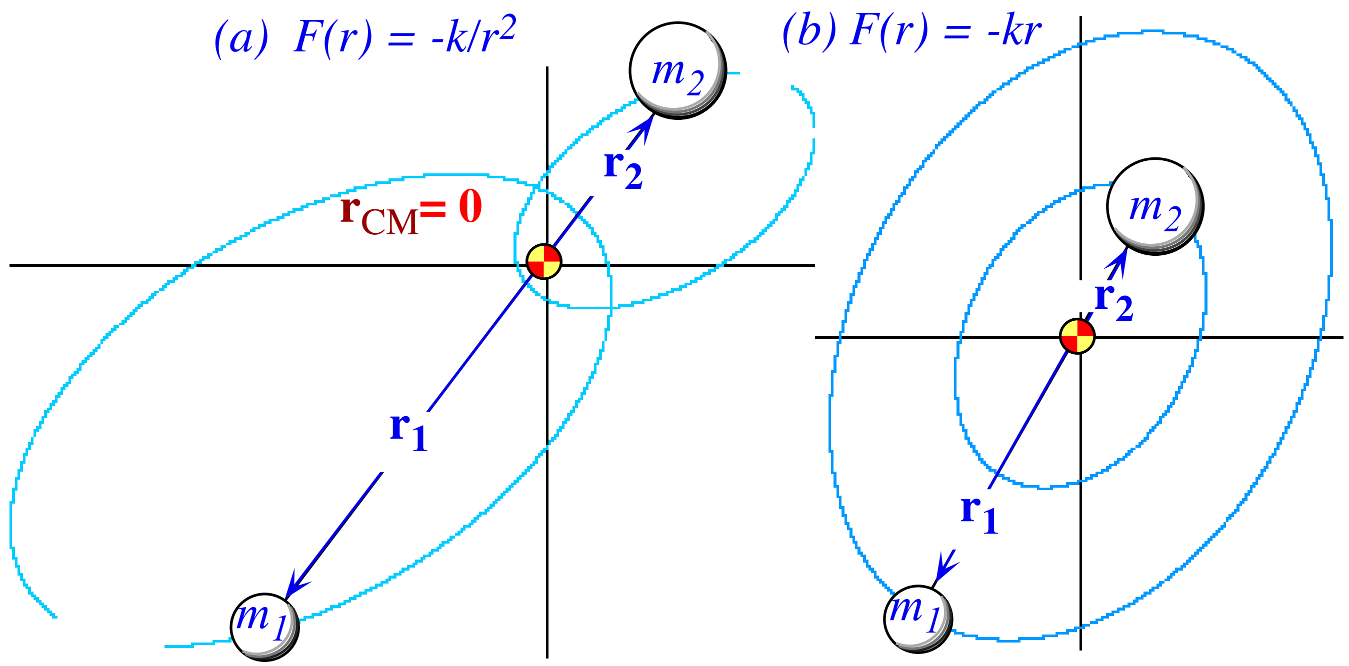
$k \to k_1 = k\mu^2/m_1^2, \quad k \to k_2 = k\mu^2/m_2^2$

$F(r) = -kr$ becomes: $F(\frac{m_1}{\mu}r_1) = -\frac{m_1}{\mu}k\,r_1$ *(Harmonic Oscillator)*

$k \to k_1 = k m_1/\mu, \quad k \to k_2 = k m_2/\mu$

2-Particle orbits and scattering: LAB-vs.-COM frame views
Ruler & compass construction (or not)

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.



Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.

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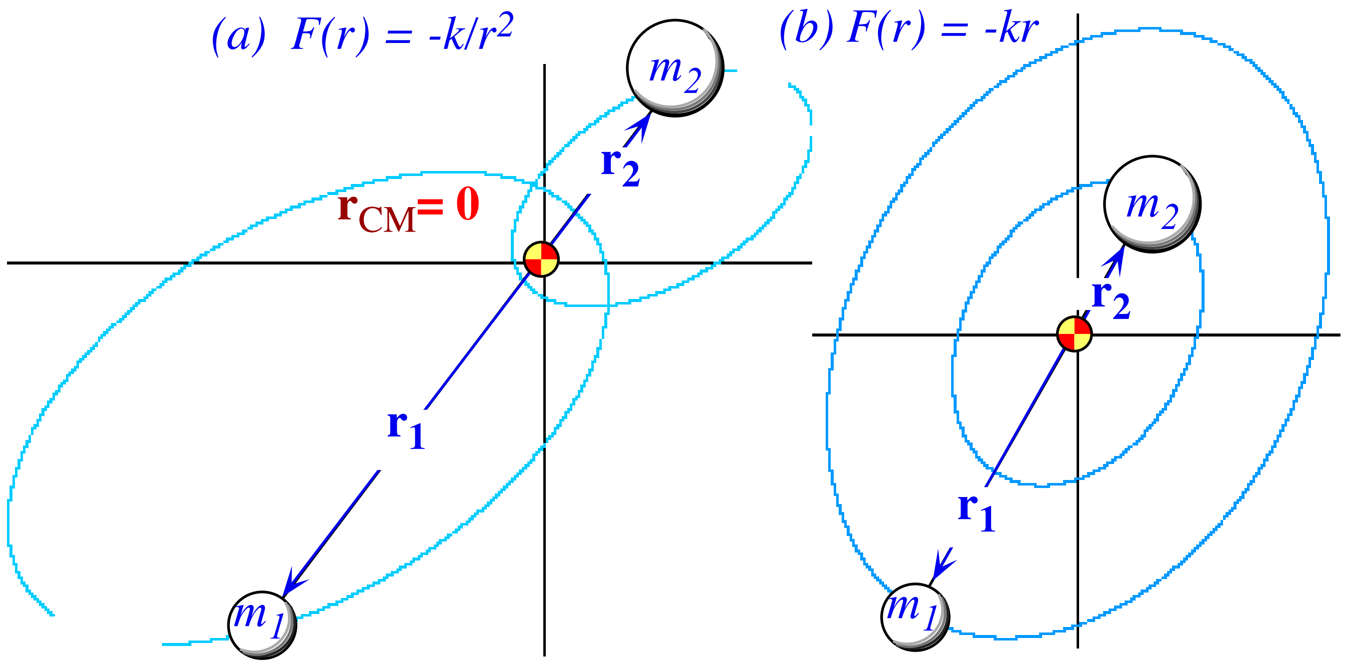

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Orbits differ in size of axes (a_1 , b_1) and (a_2 , b_2)
Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).

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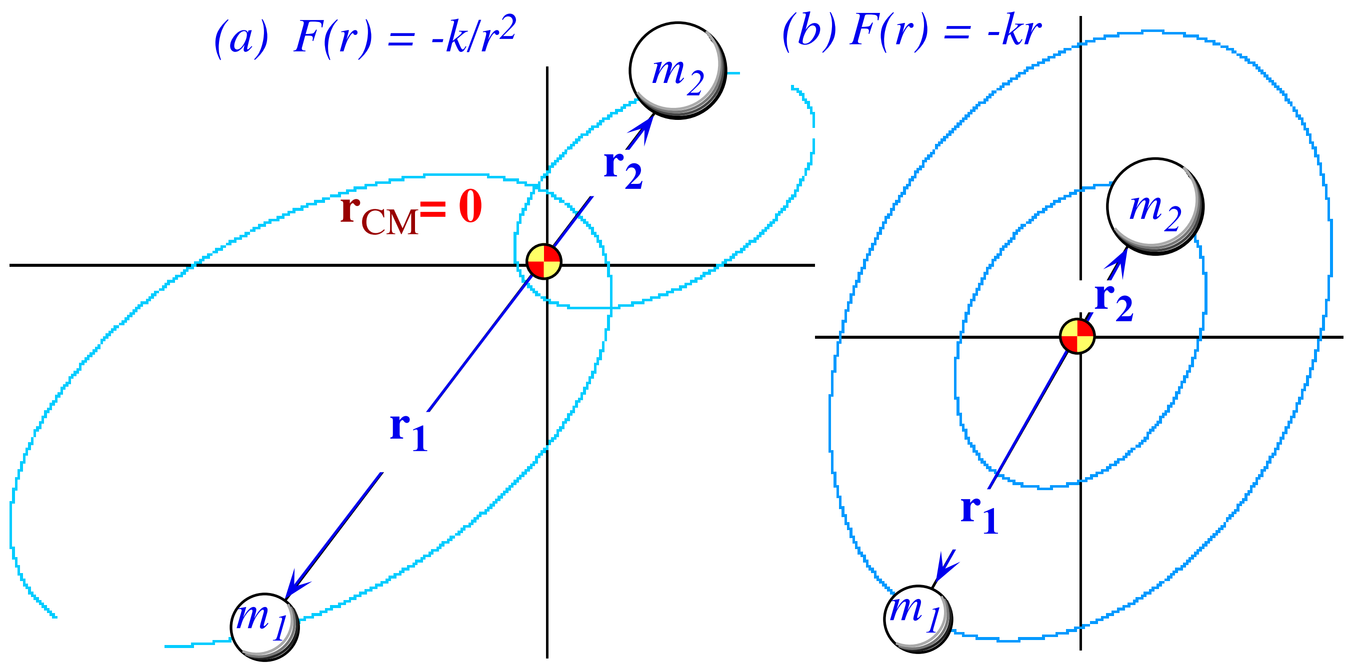

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Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).

Orbit axial dimensions (a_k , b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

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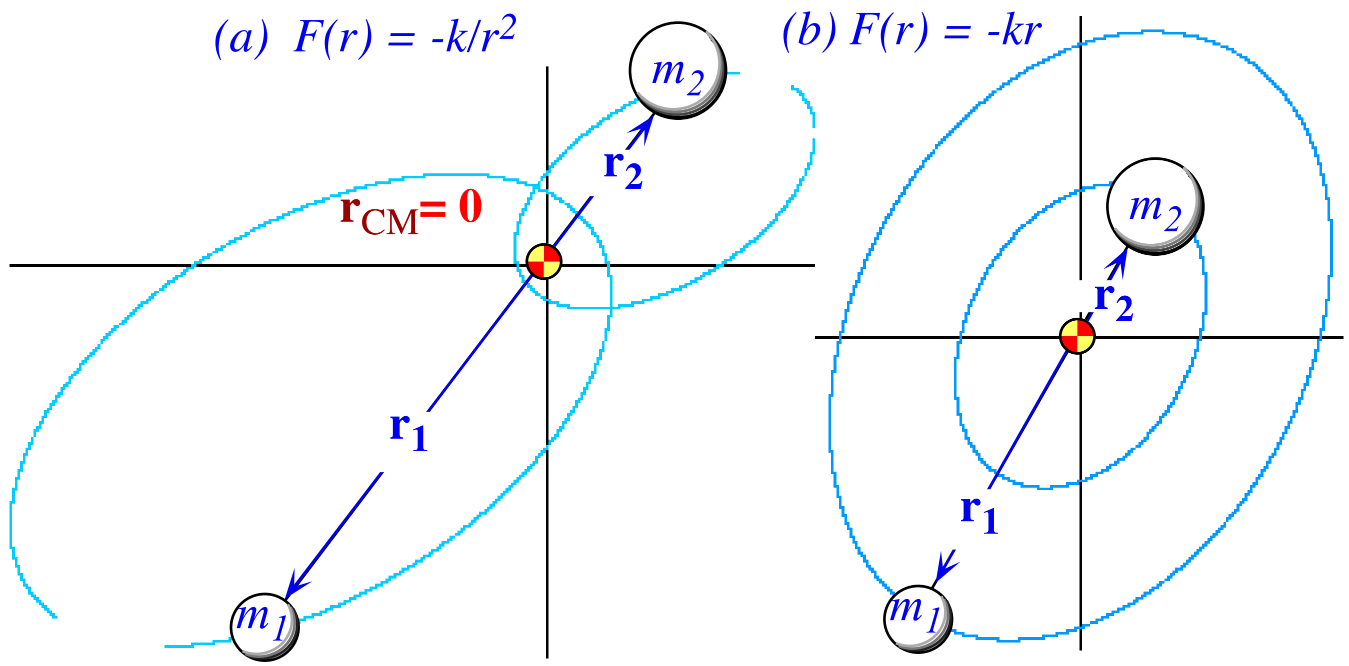

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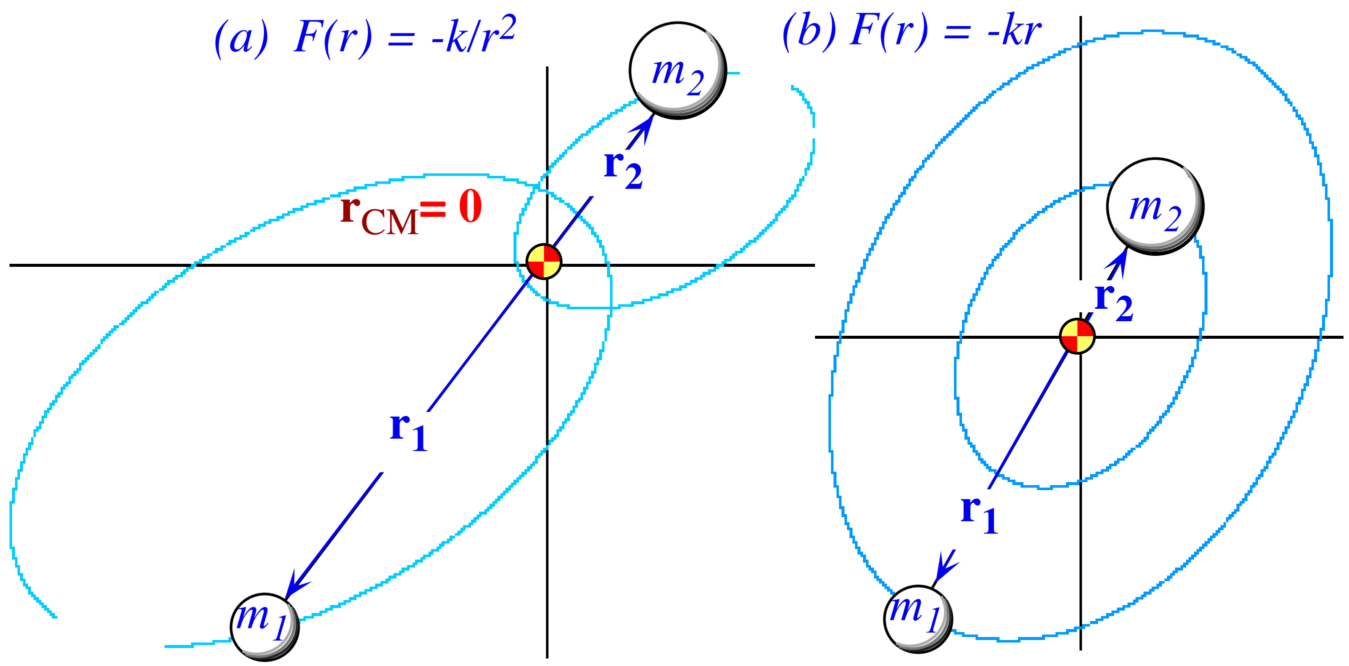
Harmonic oscillator periods and Coulomb orbit periods and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}} \qquad T_{Coul} = 2\pi\sqrt{\frac{\mu a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1{}^3}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2{}^3}{k_2}} \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon$$

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.



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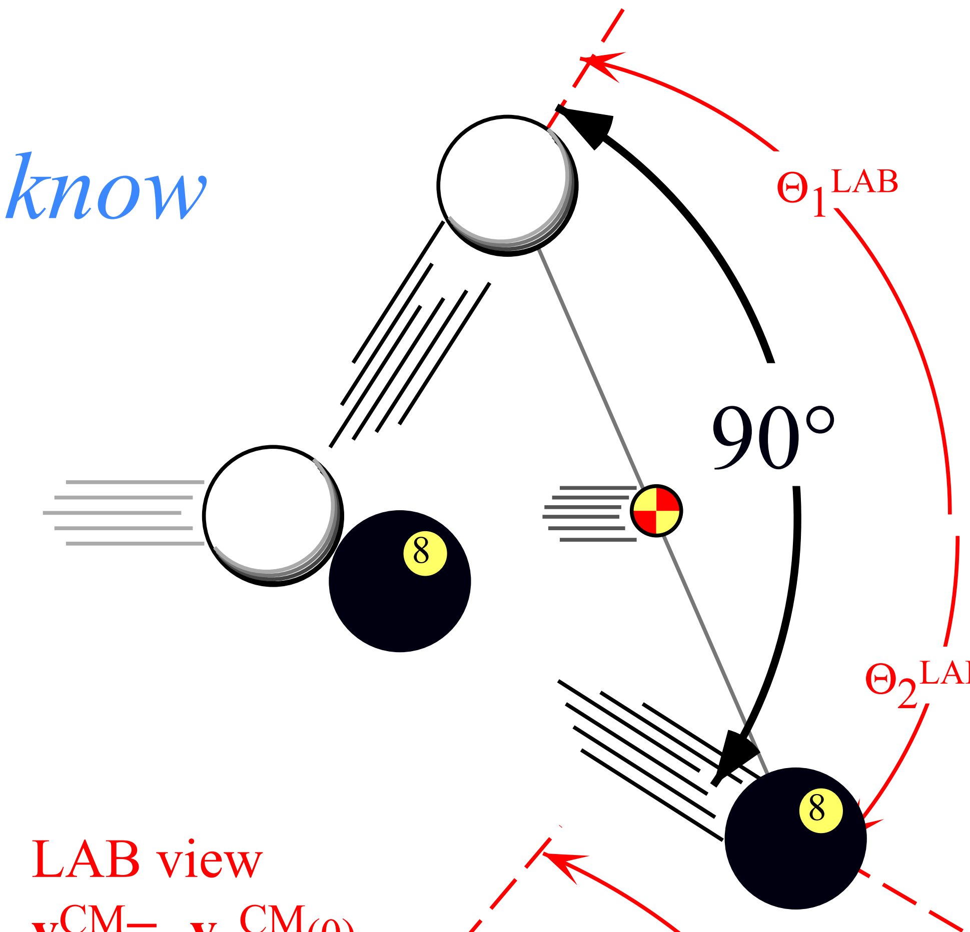
Three Coulomb orbit energy values satisfy the same proportion relation as their axes

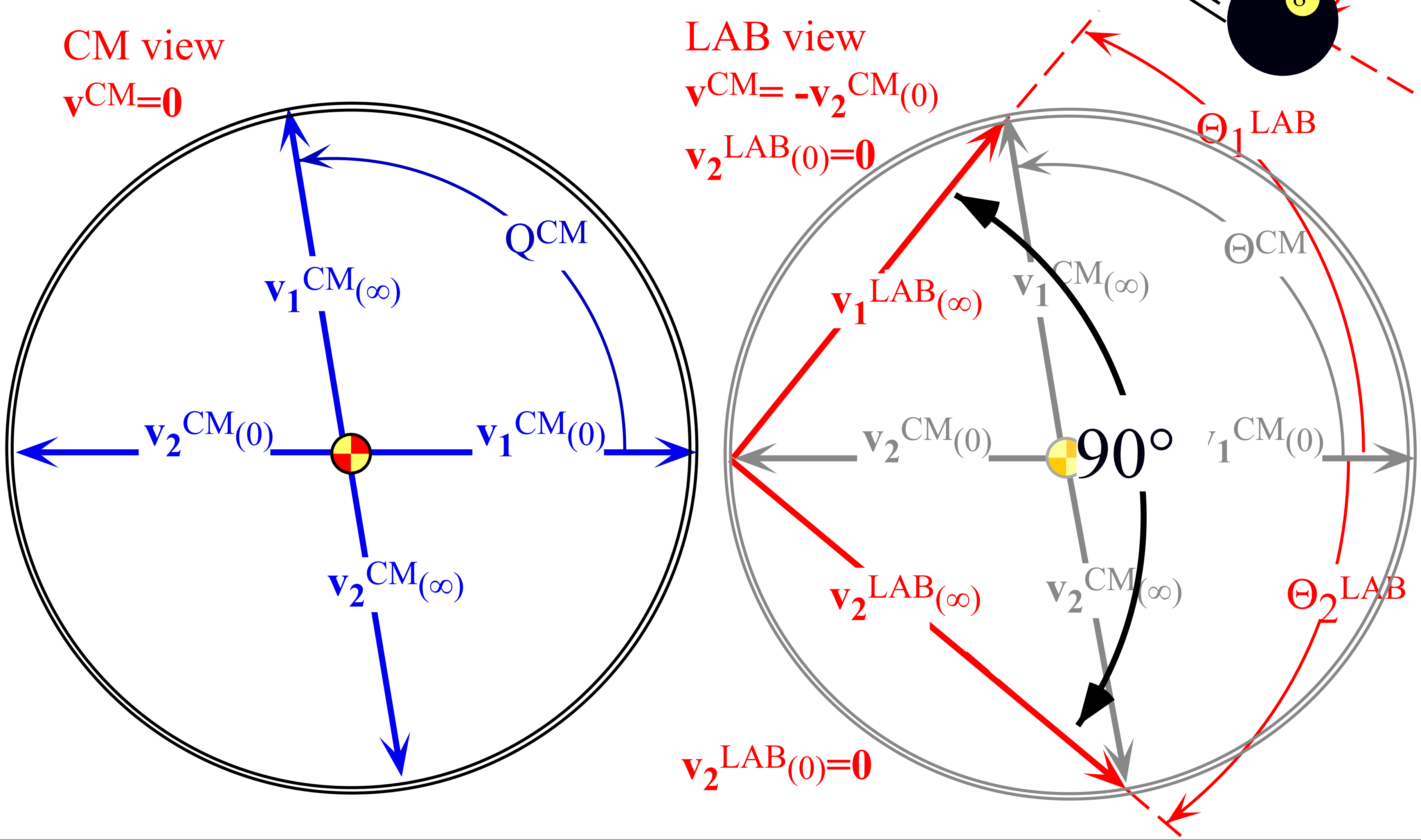
$$E_1 m_1 = E_2 m_2 = E\mu \; , \quad \text{where: } |E_1| = \frac{|k_1|}{2a_1} \; , \quad |E_2| = \frac{|k_2|}{2a_2} \; , \quad |E| = \frac{|k|}{2a} \; .$$

Energy values and axes satisfy similar sum relations

$$E_1 + E_2 = \frac{m_1}{\mu}E + \frac{m_2}{\mu}E = E \; , \quad \text{and: } a_1 + a_2 = \frac{m_1}{\mu}a + \frac{m_2}{\mu}a = a$$

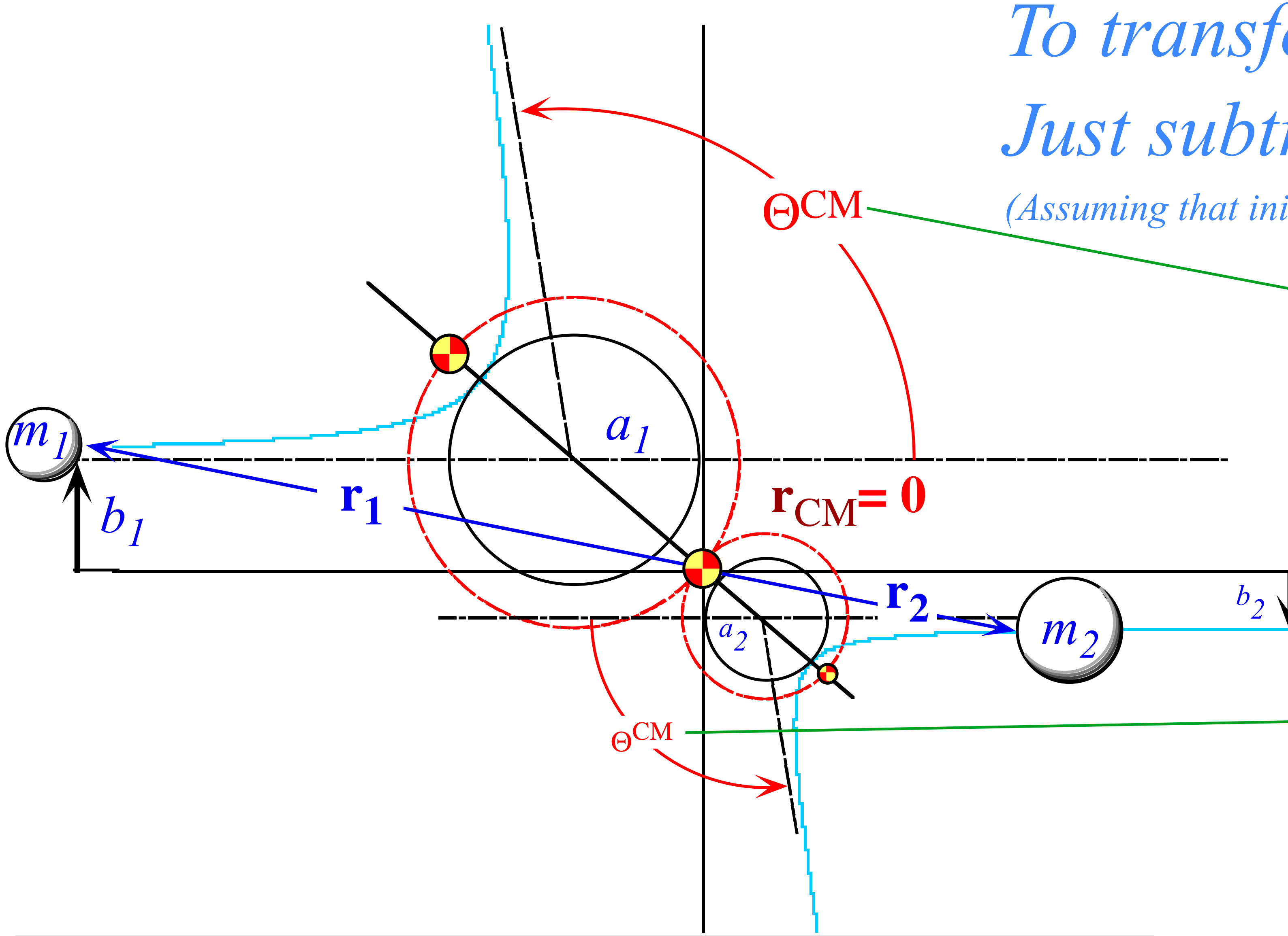
A common type of scattering
(m_1=m_2)
...that every pool shark should know

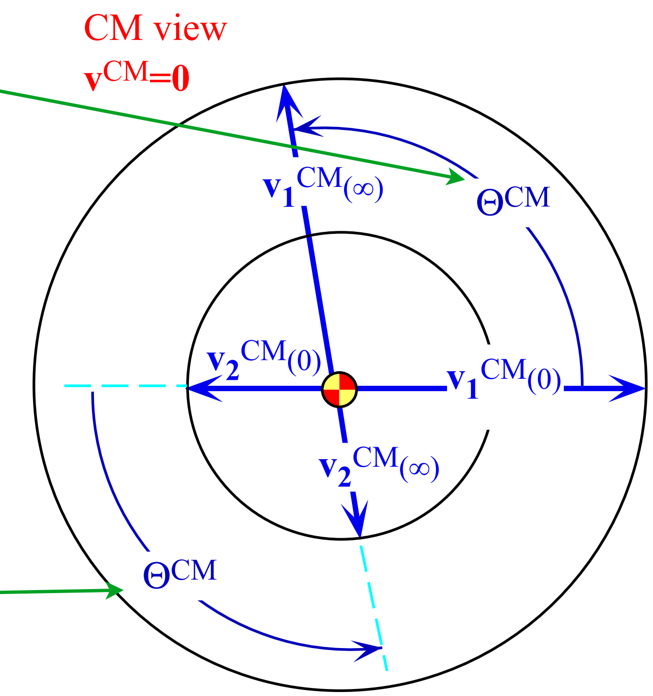


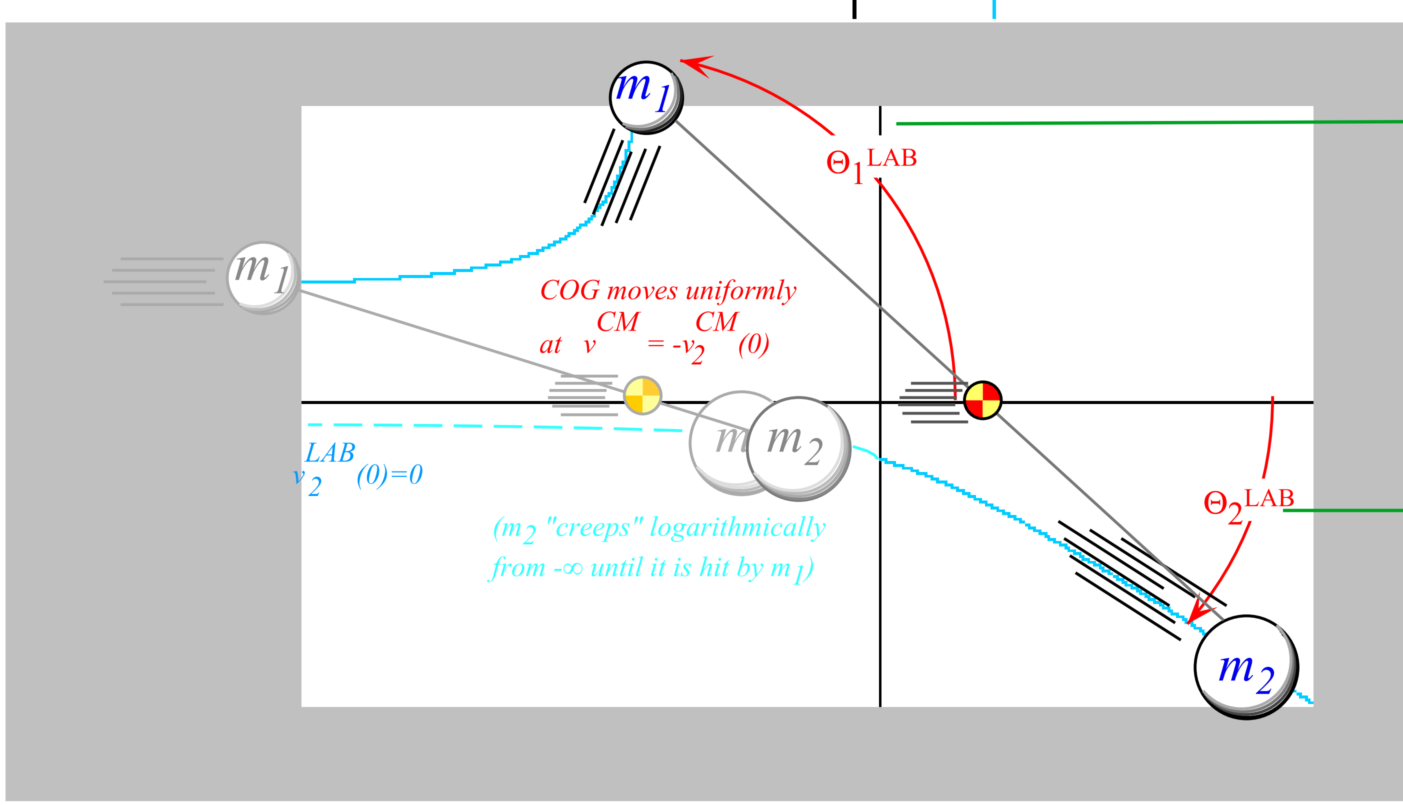


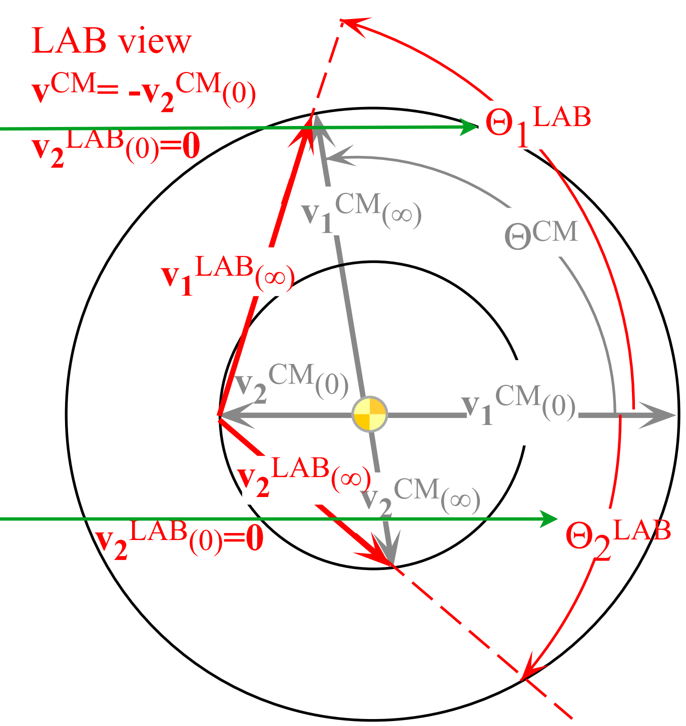
To transform CM to LAB frame
Just subtract $\mathbf{v}_2^{CM}(0)$ from all

(Assuming that initial $\mathbf{v}_2^{LAB}(0)$ is zero so $\mathbf{v}_2^{CM}(0)$ is CM velocity in LAB)









Geometrical Aspects of Classical Coulomb Scattering

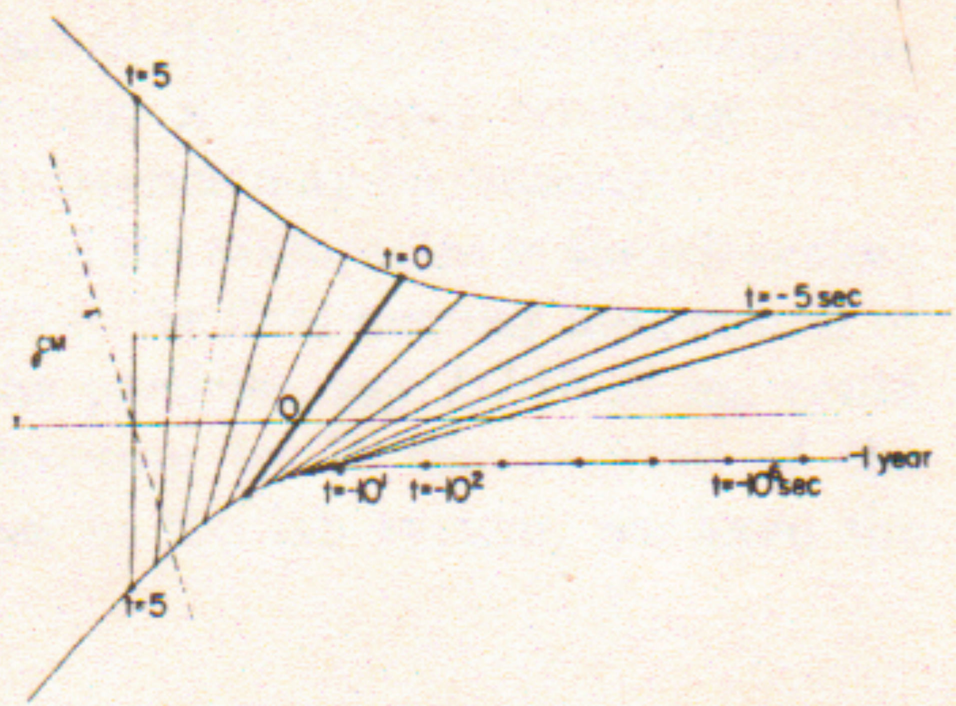


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

The trouble with the Coulomb field is...

$$\int t^{-1} dt = \ln t + C$$

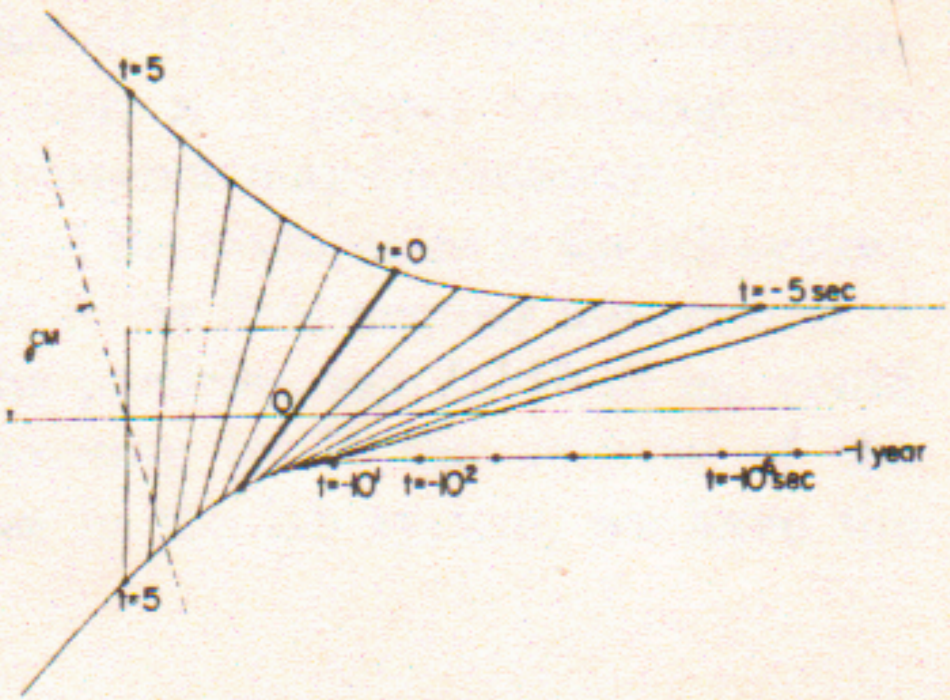
$$v_2^{\text{LAB}}(t) = \int (|F|/m_2)dt$$

$$\cong \int k dt / m_2 [v_1^{\text{CM}}(\text{initial}) t]^2$$

$$\cong [-k/m_2 v_1^{\text{CM}}(\text{initial})^2] t^{-1}$$

1856 / December 1972

From: Geometric aspects of classical Coulomb scattering
American Journal of Physics 40,1852-1856 (1972)
Class project when I taught Jr. CM at Georgia Tech
(Just 5 students)



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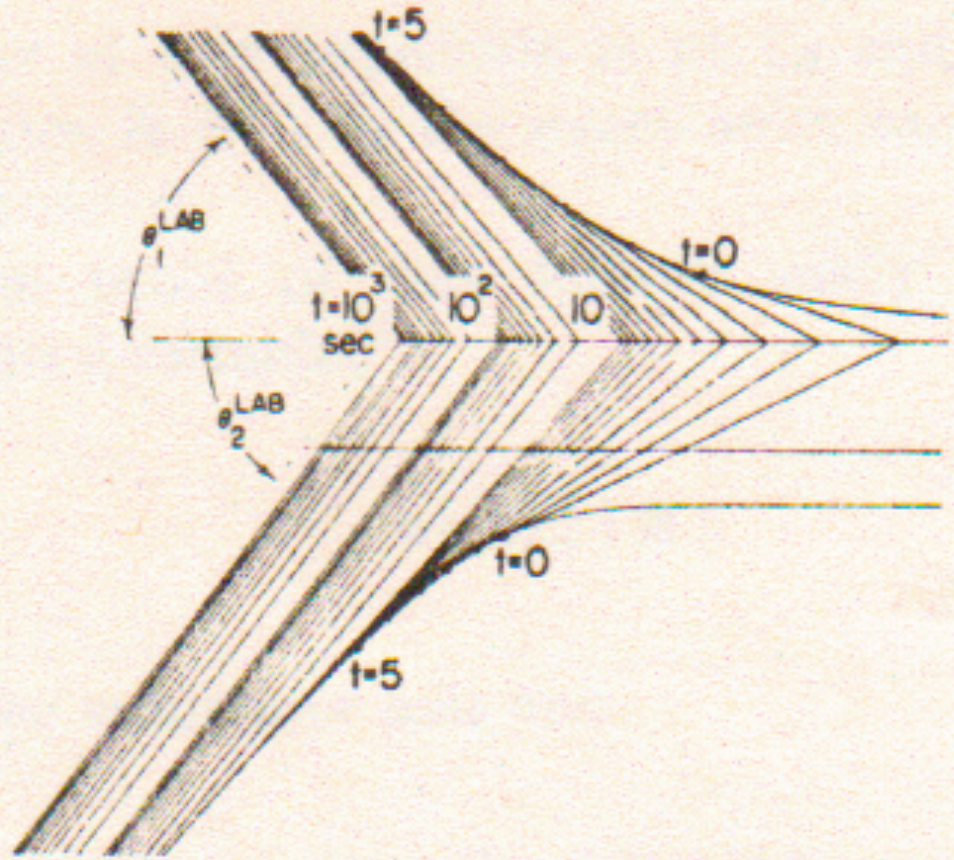

FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t=10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04°, respectively.

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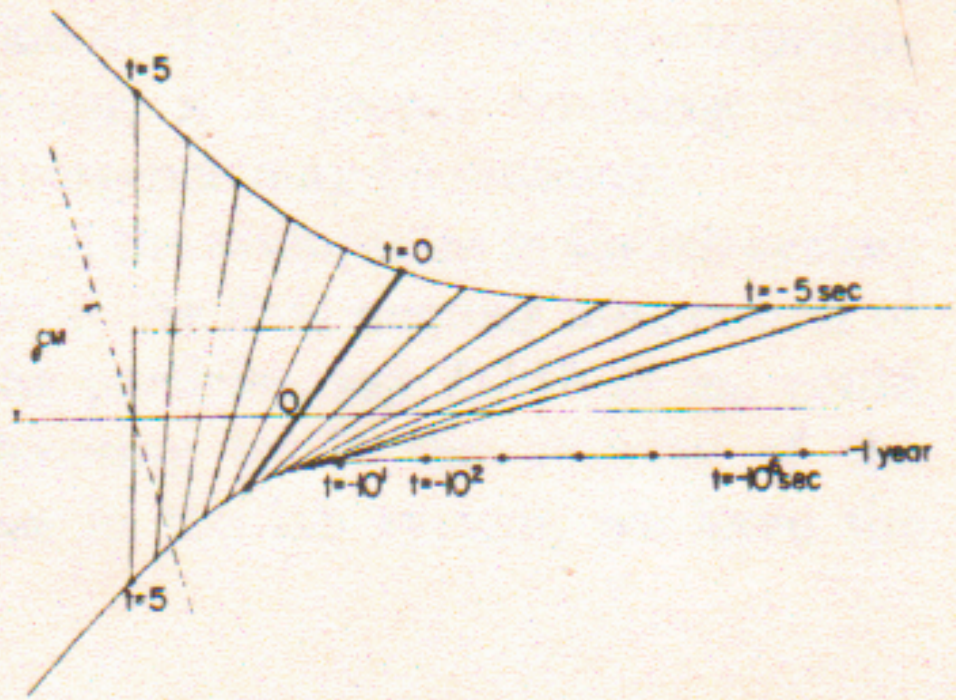

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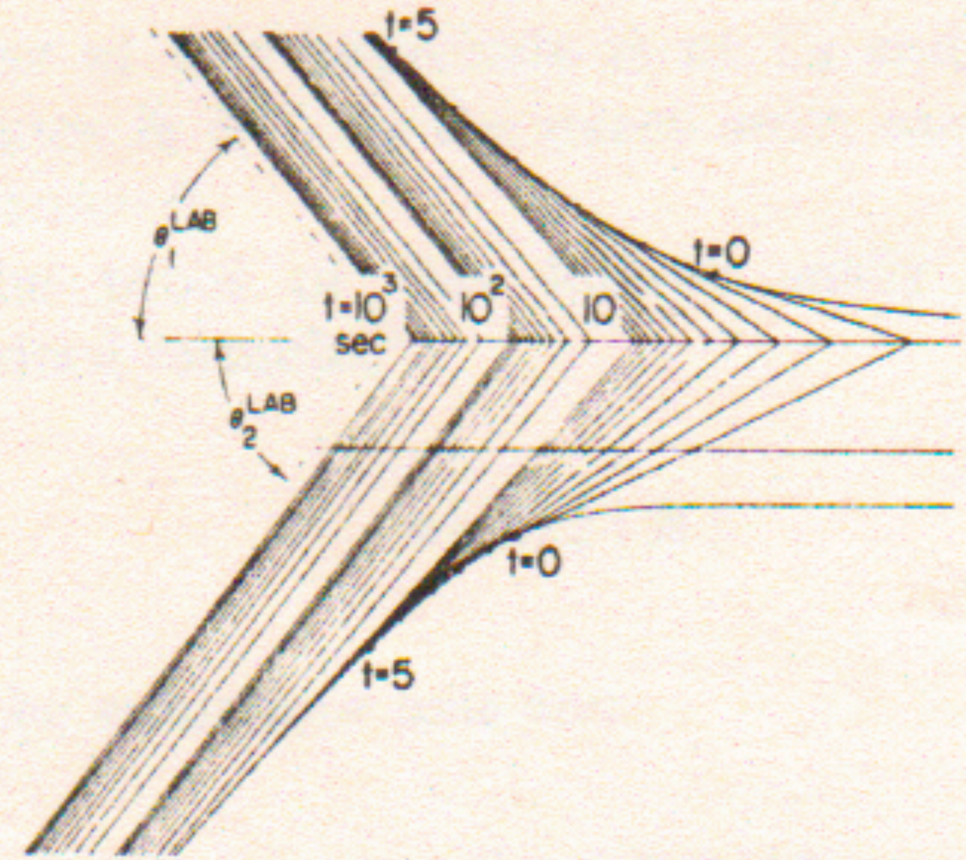

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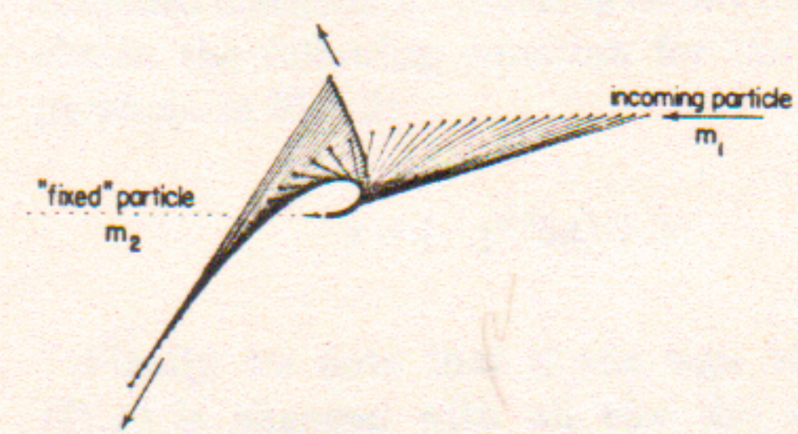

FIG. 7. Attractive Coulomb scattering in laboratory system. This has the same "anomalies" as the repulsive case.

From: Geometric aspects of classical Coulomb scattering
American Journal of Physics 40,1852-1856 (1972)
Class project when I taught Jr. CM at Georgia Tech
(Just 5 students)

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j=\mathbf{r}_j\times m_j\dot{\mathbf{r}}_j\equiv\mathbf{r}_j\times\mathbf{p}_j$

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d**L** /dt gives a rotor Newton equation relating rotor momentum **r**×**p** to rotor force or *torque* **r**×**F**.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k \neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

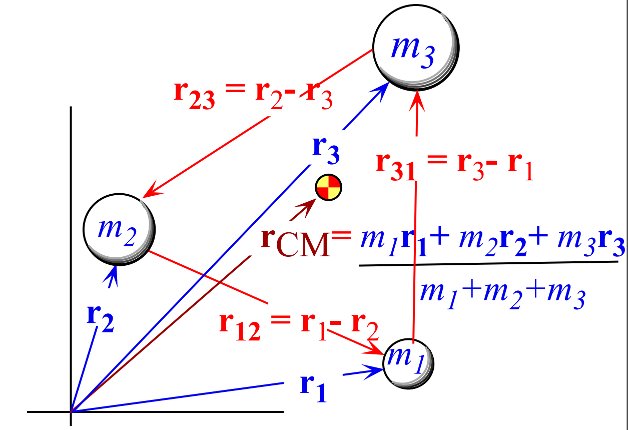

Fig. 6.4.1 Three-particle coordinate vectors

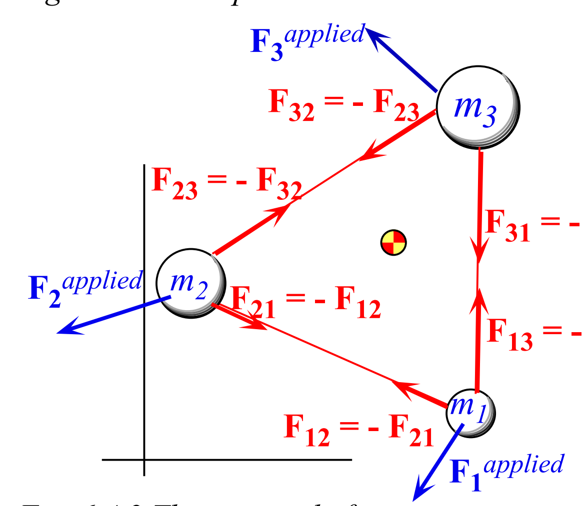

Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*

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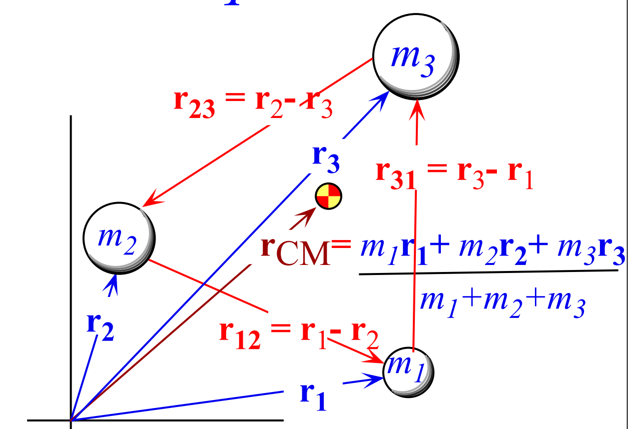
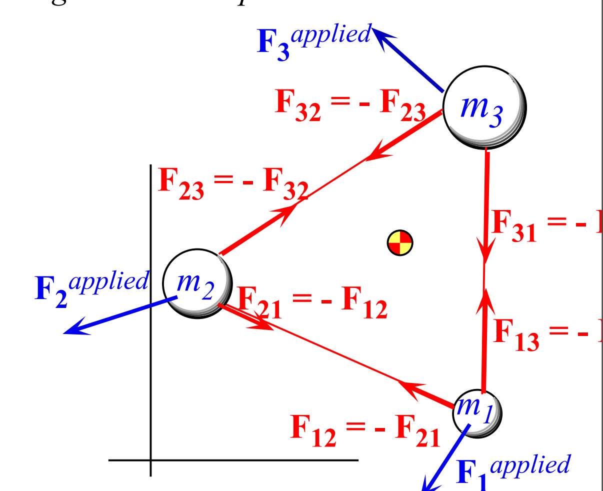


Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint} \right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint} \right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint} \right)$$

$$= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12}^{constraint} + (\mathbf{r}_1 - \mathbf{r}_3) \times \mathbf{F}_{13}^{constraint} + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$



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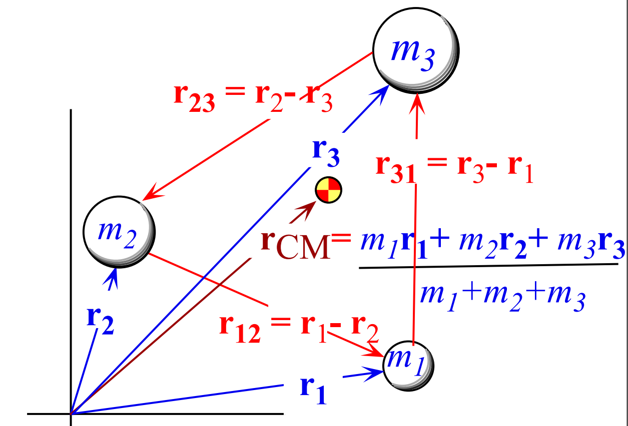
$$= \left(\mathbf{r}_1 - \mathbf{r}_2 \right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3 \right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3 \right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$

However, they vanish if coupling forces act along lines connecting the masses.



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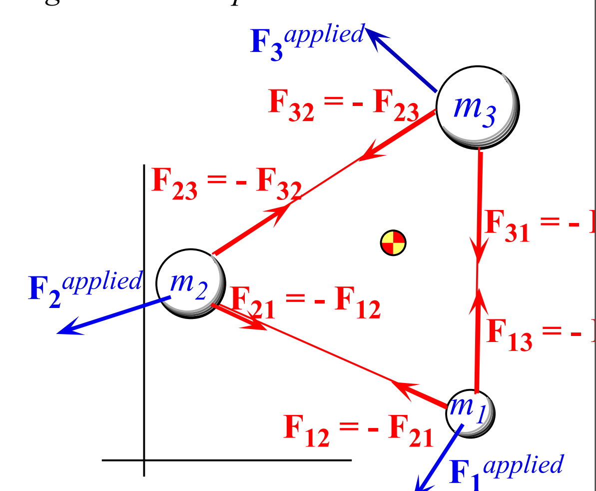

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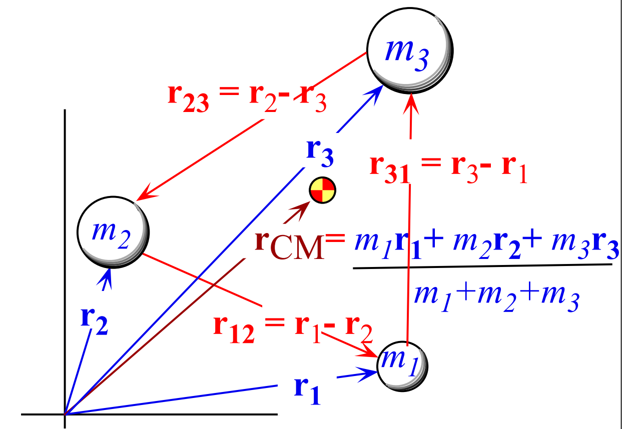

Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3}\sum_{k=1(k\neq j)}^{3}\mathbf{r}_j\times\mathbf{F}_{jk}^{constraint}=\mathbf{r}_1\times\left(\mathbf{F}_{12}+\mathbf{F}_{13}^{constraint}\right)+\mathbf{r}_2\times\left(\mathbf{F}_{21}+\mathbf{F}_{23}^{constraint}\right)+\mathbf{r}_3\times\left(\mathbf{F}_{31}+\mathbf{F}_{32}^{constraint}\right)$$

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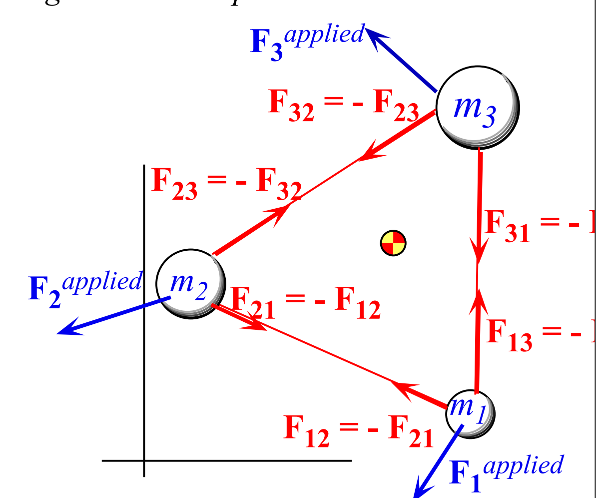

Fig. 6.4.2 Three-particle force vectors

However, they vanish if coupling forces act along lines connecting the masses.

The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt}=\mathbf{N}\text{ , where: }\mathbf{N}=\sum_{j=1}^{3}\mathbf{N}_j\quad\text{and:}\quad\mathbf{N}_j=\sum_{j=1}^{3}\mathbf{r}_j\times\mathbf{F}_j^{applied}$$

Rotational equivalent of Newton's **F**=d**p**/dt *equations:* **N**=d**L**/dt

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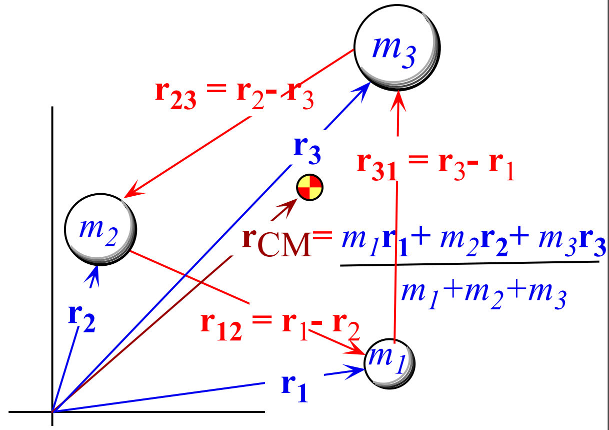

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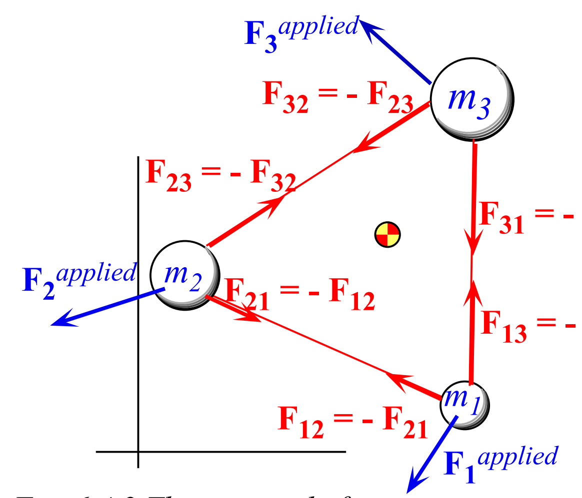

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Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt}=\mathbf{F}\text{ , where: }\mathbf{F}=\sum_{j=1}^{3}\mathbf{F}_j^{applied}$$

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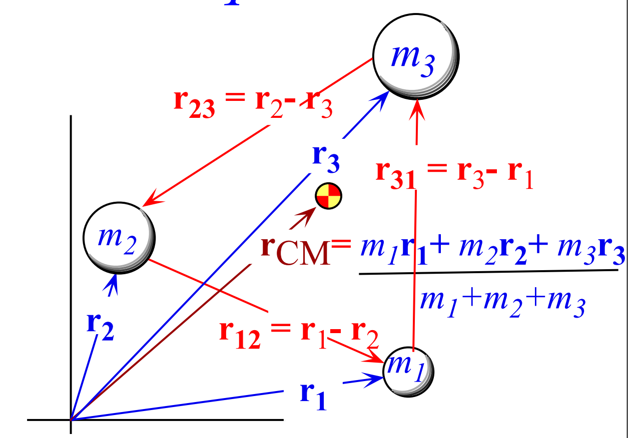

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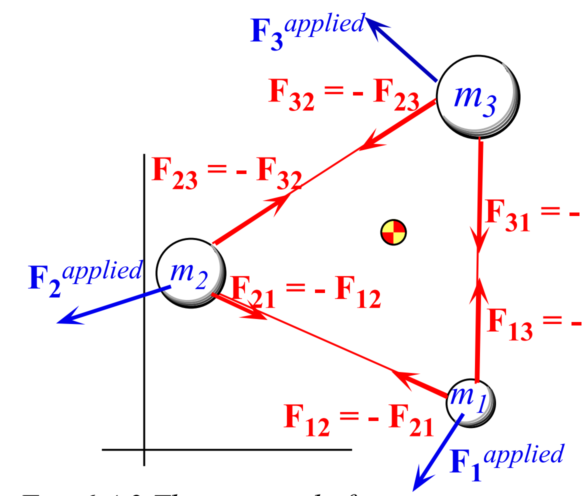

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Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt}=\mathbf{F}\text{ , where: }\mathbf{F}=\sum_{j=1}^{3}\mathbf{F}_j^{applied}$$

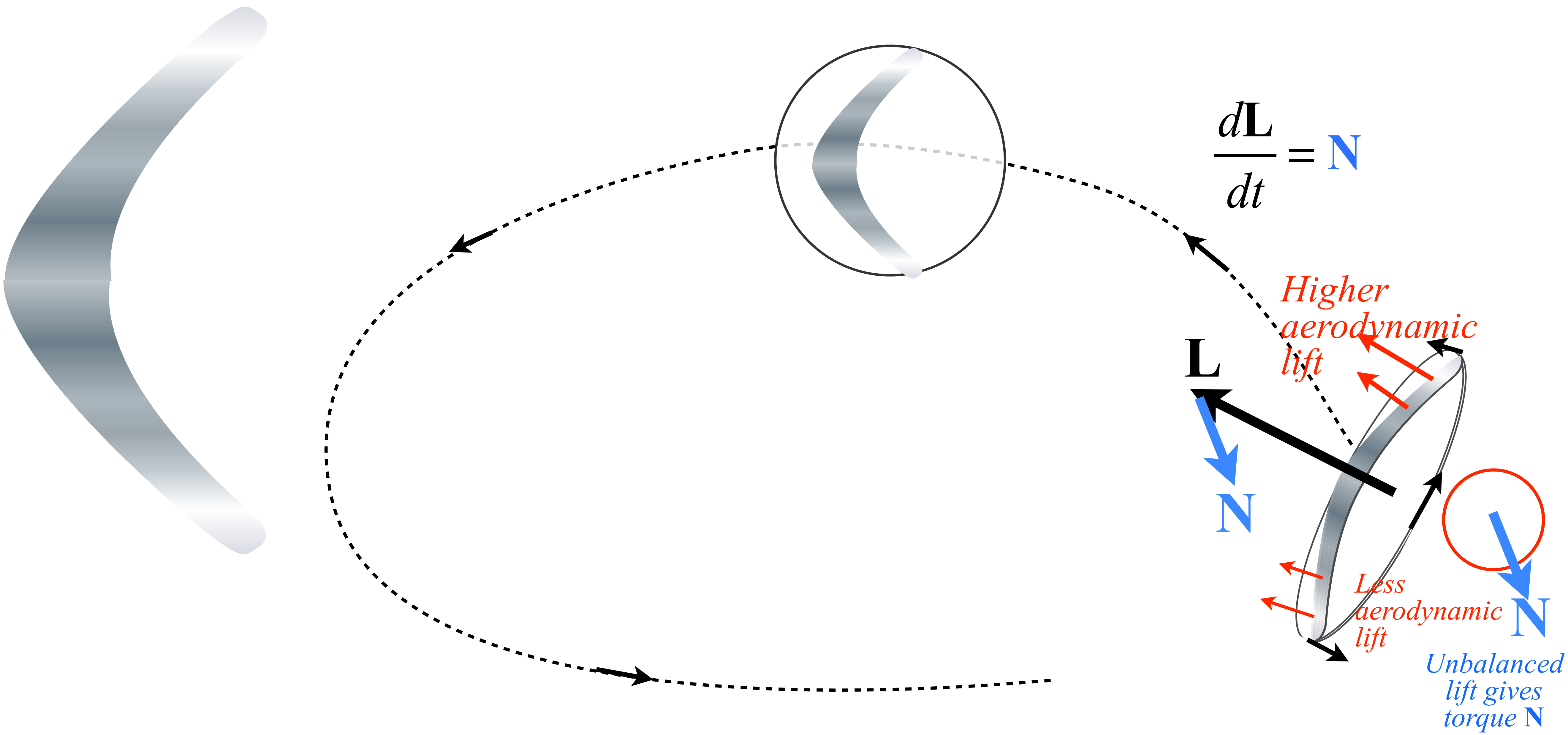
Remaining 3N-6 equations consist of normal mode or GCC equations of some kind.

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

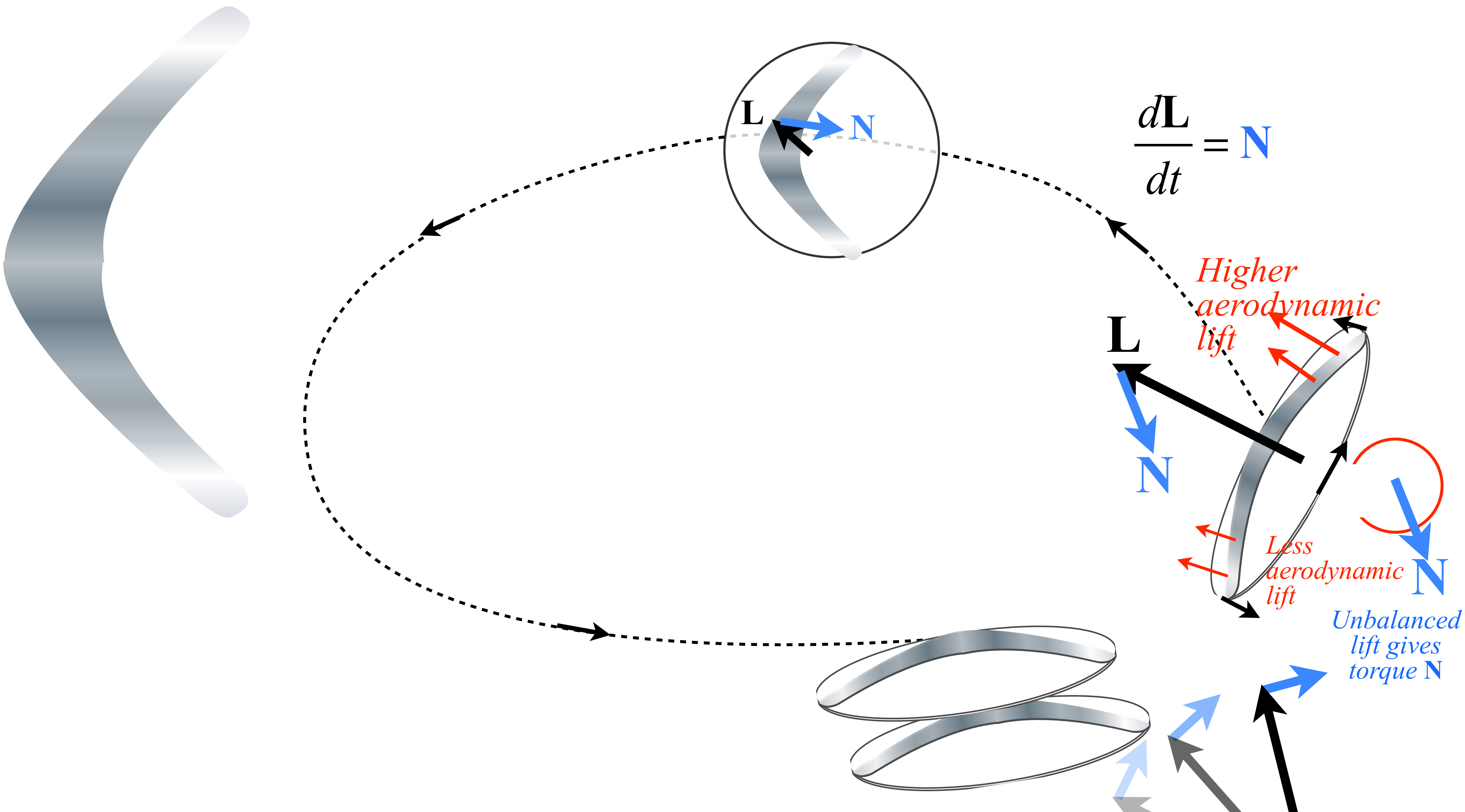
➡ *How to make my boomerang come back*

The gyrocompass and mechanical spin analogy

The Australian Boomerang (that comes back!)



The Australian Boomerang (that comes back and hovers down!)



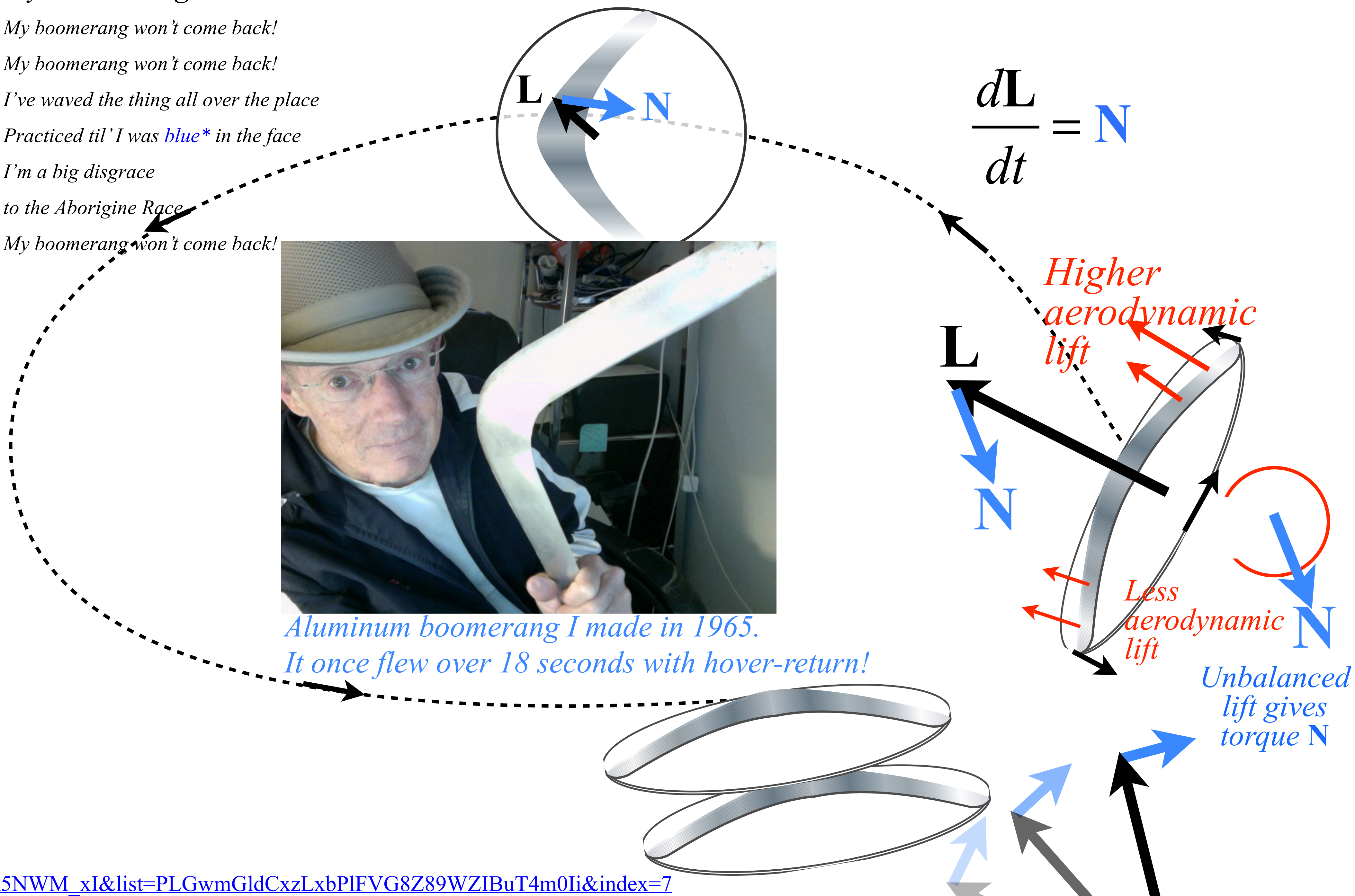
The Australian Boomerang (that comes back and hovers down!)

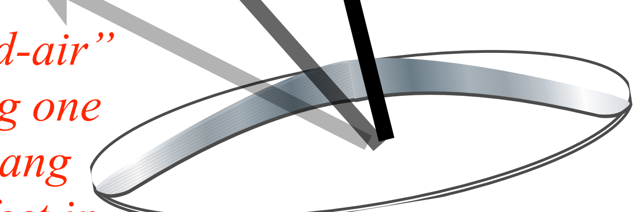
Charlie Drake's famous 1961 song:

My boomerang won't come back!

My boomerang won't come back!
My boomerang won't come back!
I've waved the thing all over the place
Practiced til' I was blue in the face*
I'm a big disgrace
to the Aborigine Race
My boomerang won't come back!

**blue later replaced black*





Aluminum boomerang I made in 1965.
It once flew over 18 seconds with hover-return!

Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.

https://www.youtube.com/watch?v=EXJR5NWM_xI&list=PLGwmGldCxzLxbPlFVG8Z89WZIBuT4m0Ii&index=7

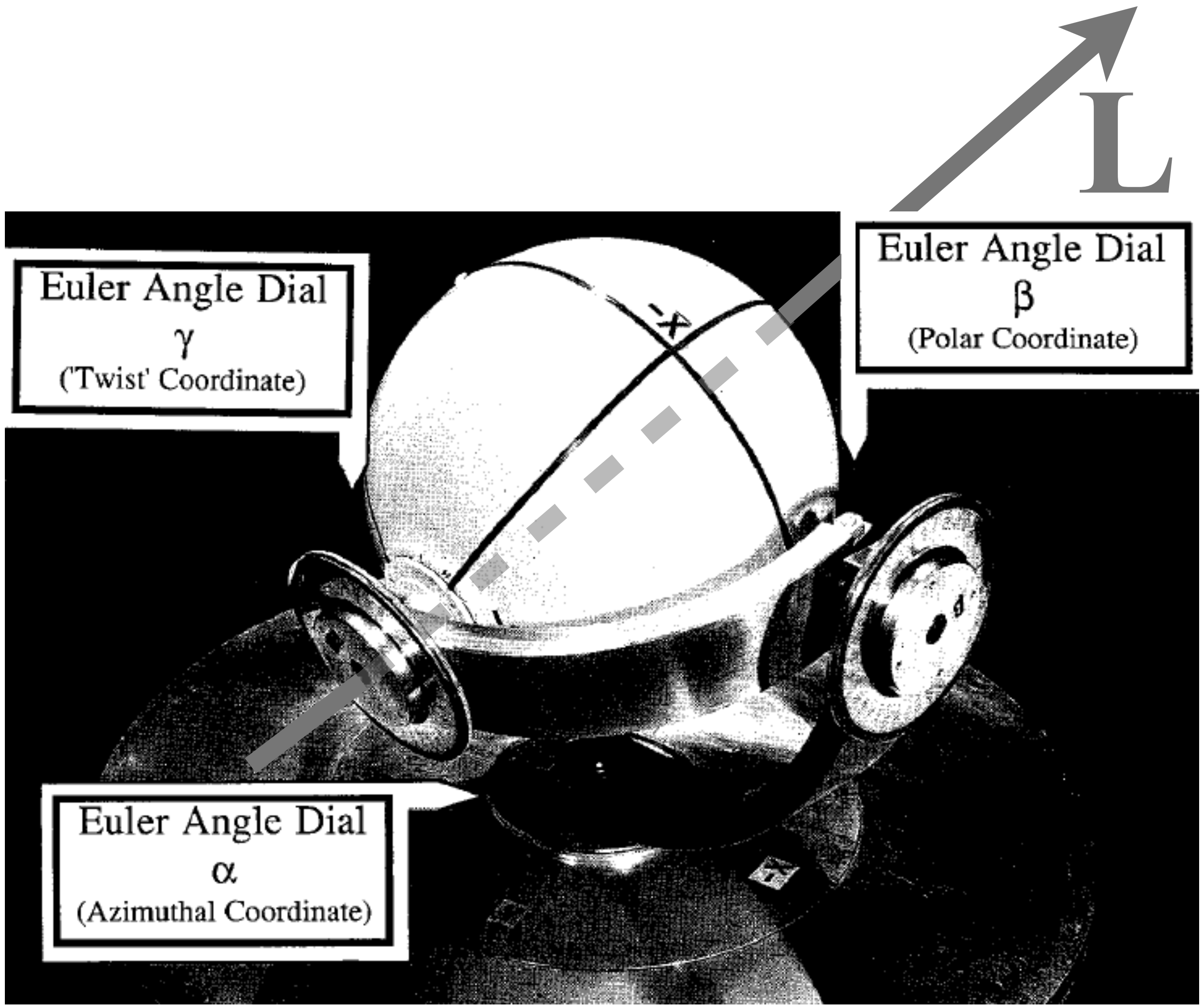
Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

How to make my boomerang come back

→ *The gyrocompass and mechanical spin analogy*

The gyrocompass and mechanical spin analogy

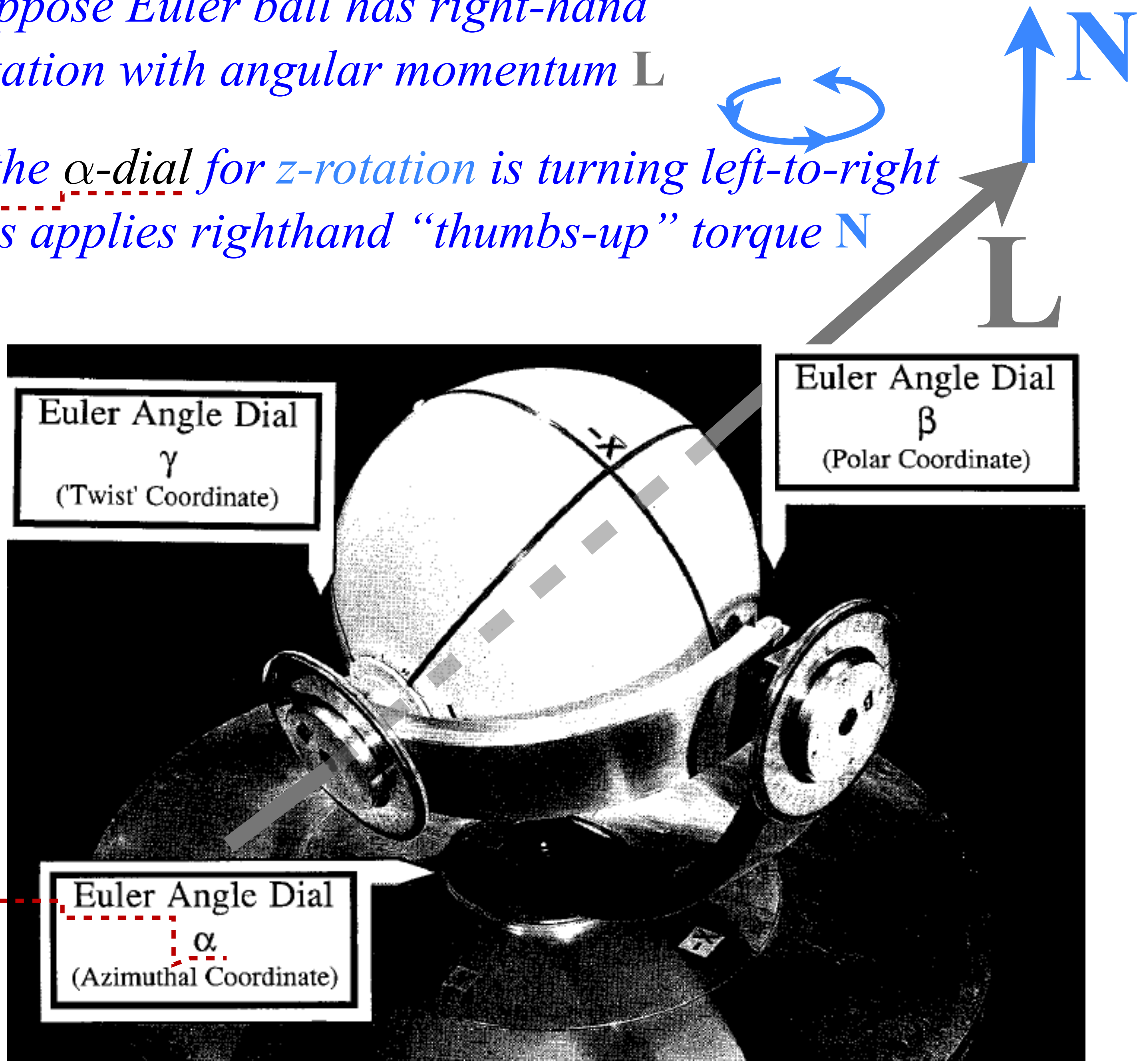
Suppose Euler ball has right-hand rotation with angular momentum **L**



The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque **N**



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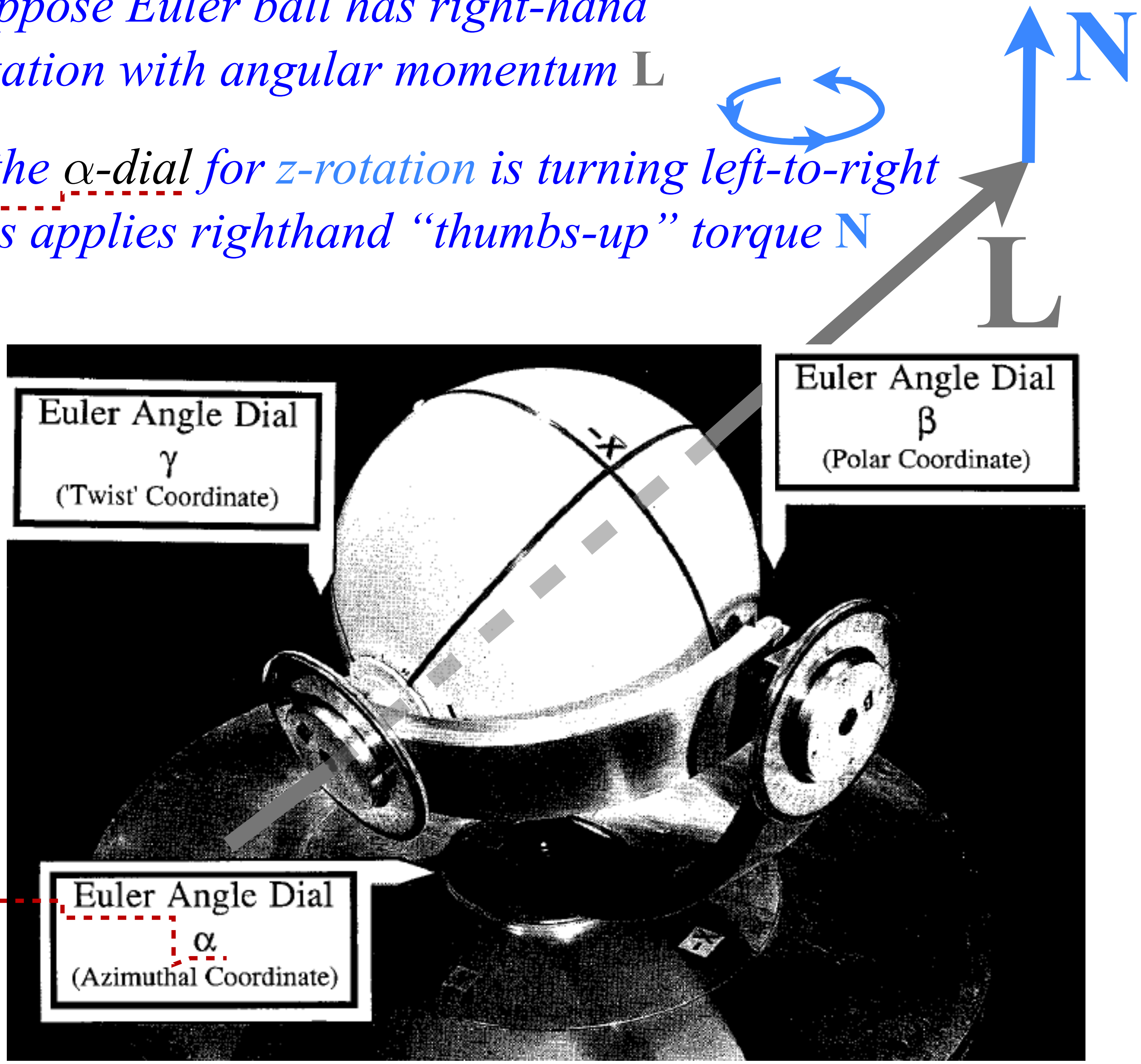

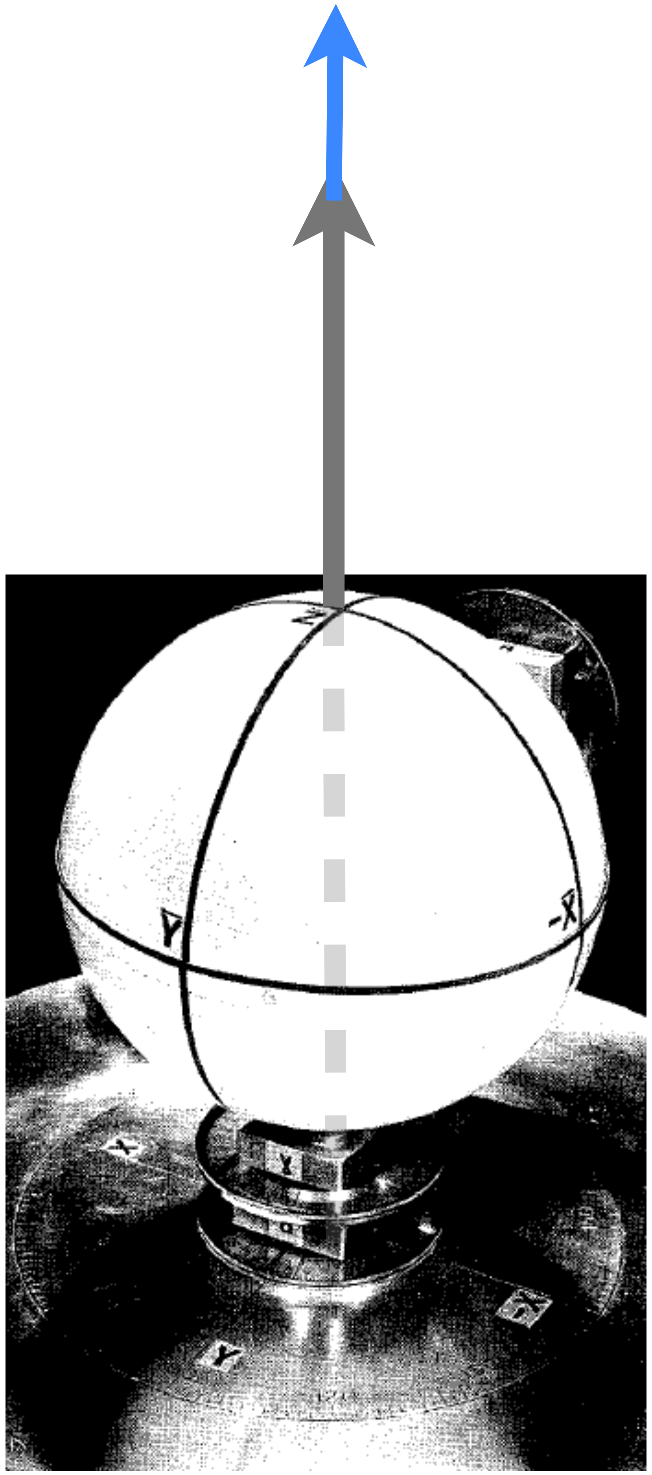

Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

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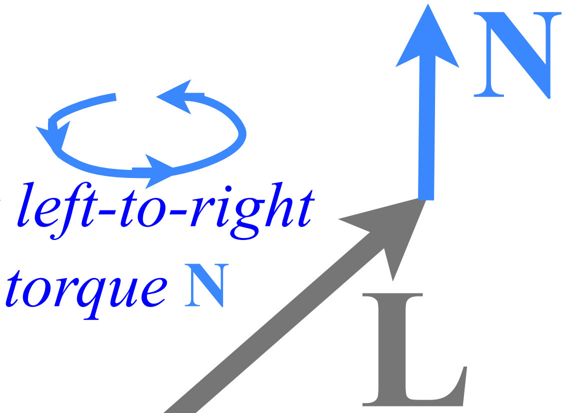

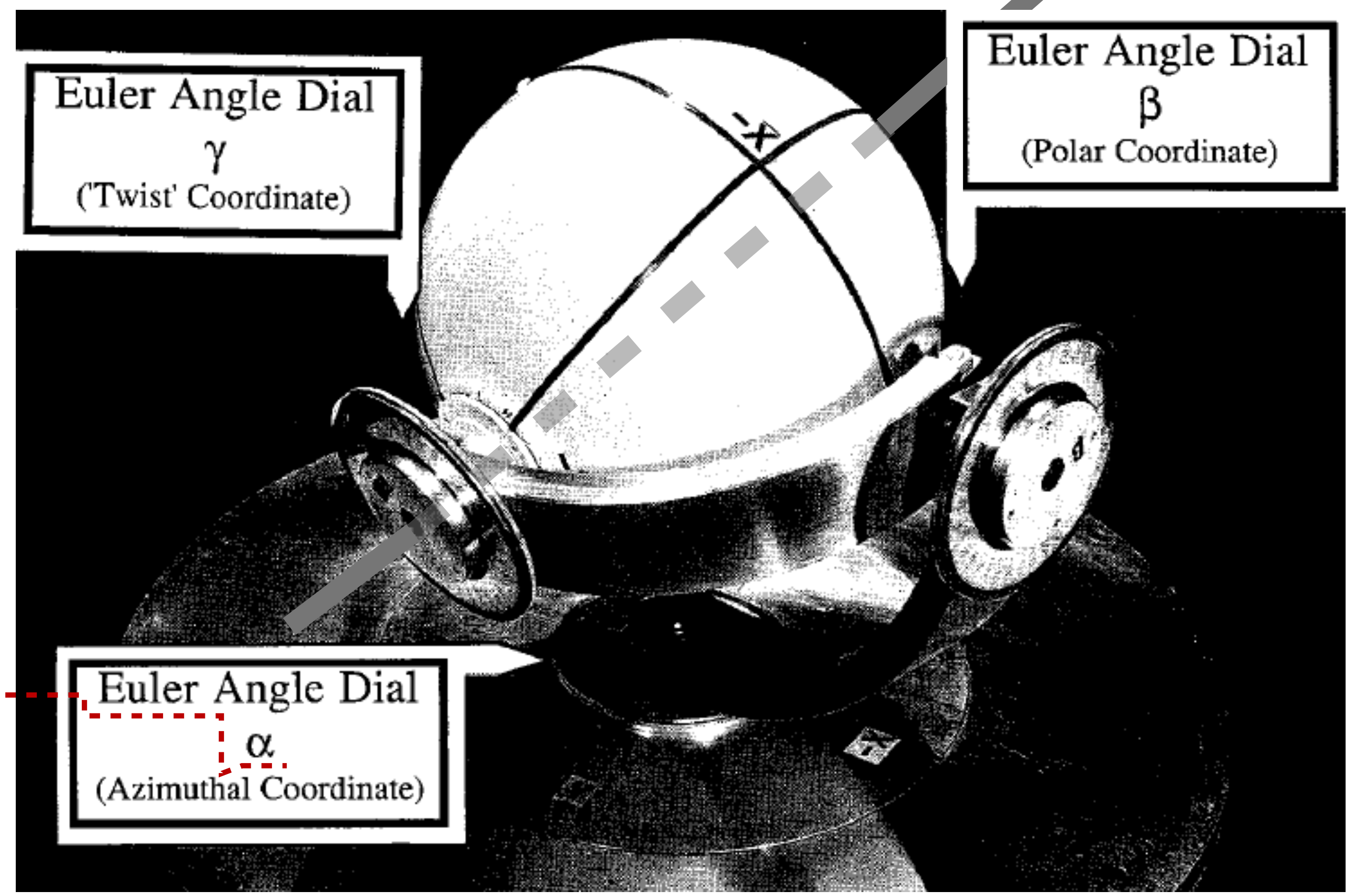

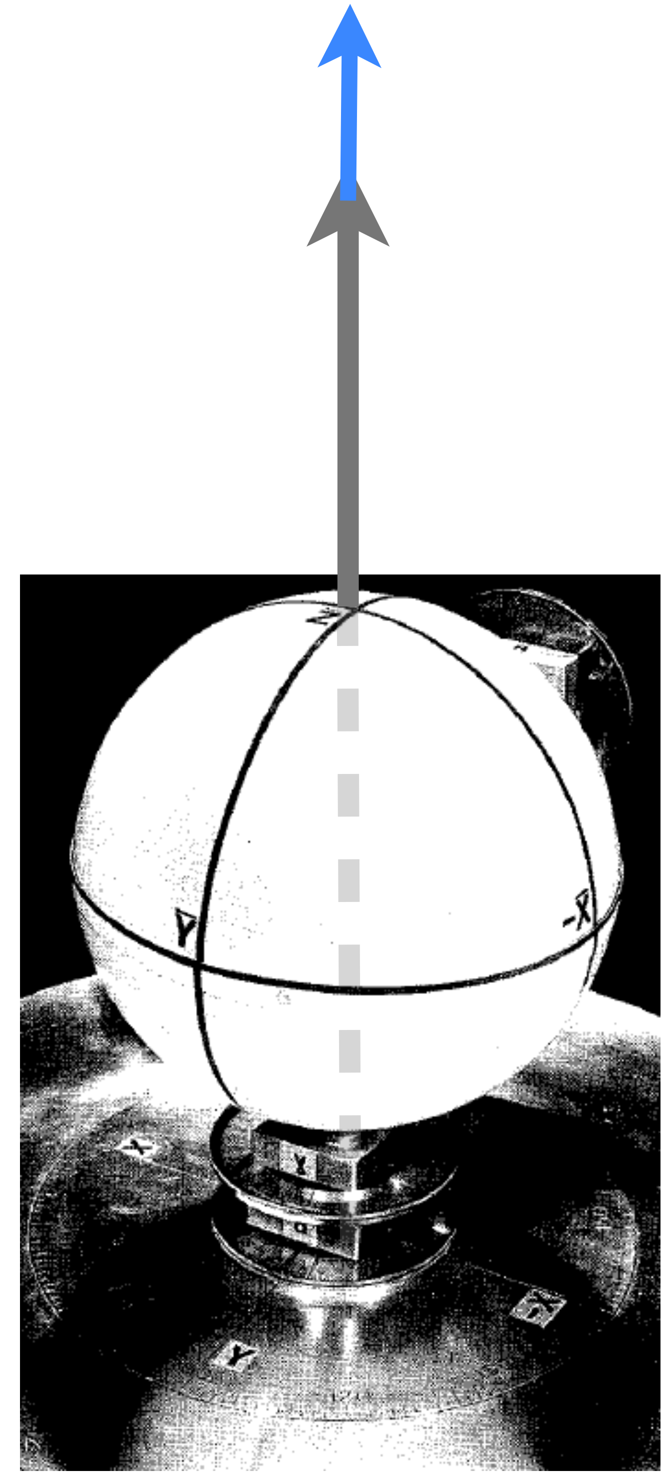

A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.

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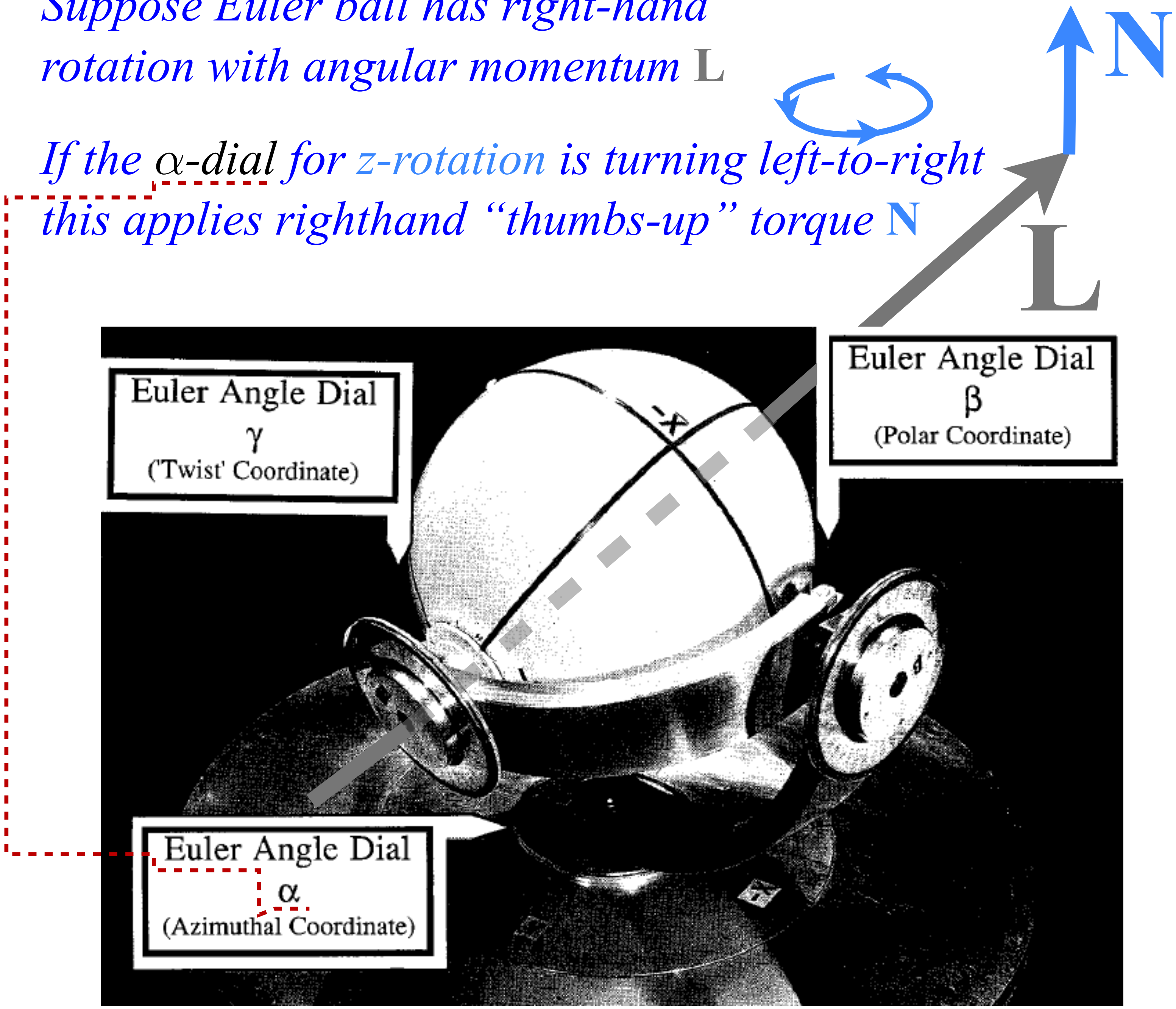

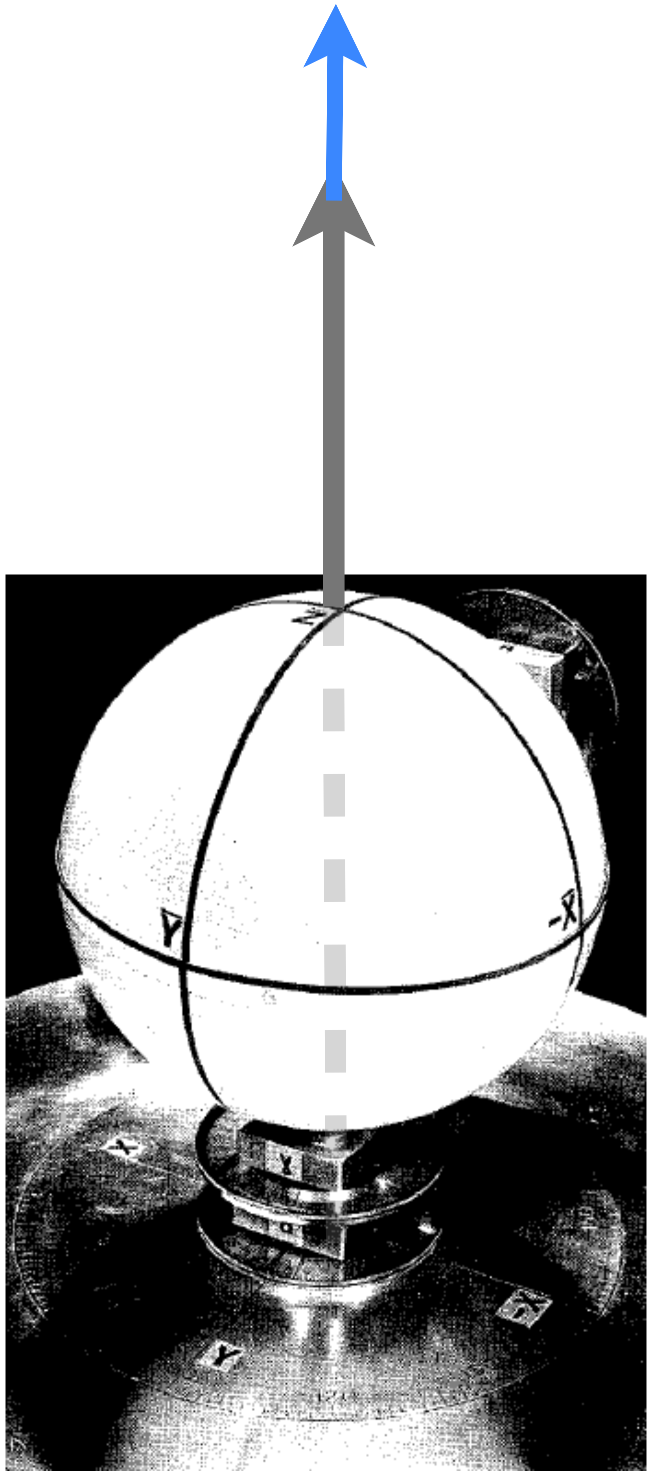

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This is analogous to the tendency for spin magnetic moments to allign (or precess about) the B-direction of a magnetic field

Recall **S***-precession discussion in CMwB Unit 4 Ch.4 and Lect.26*

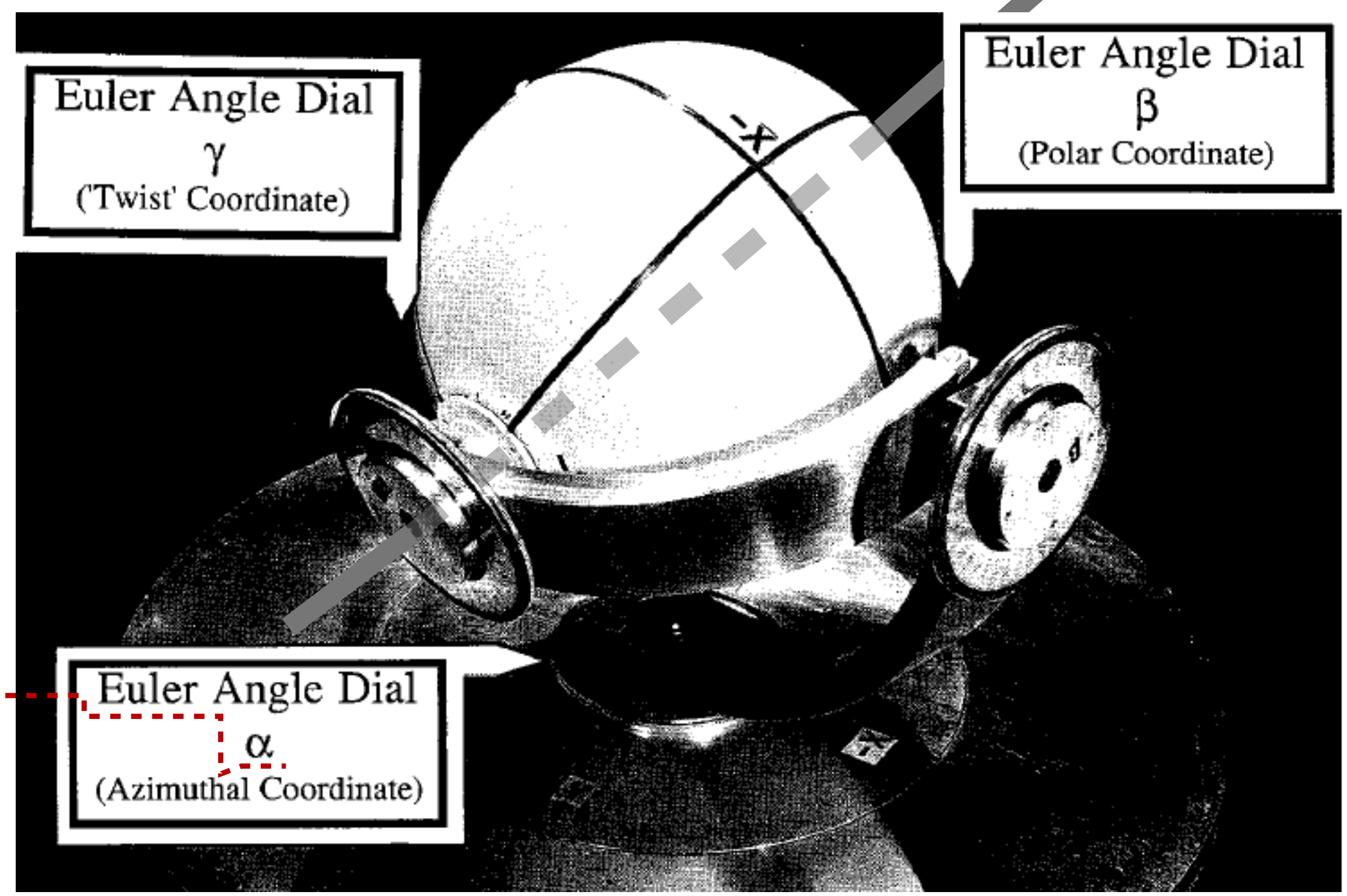
The gyrocompass and mechanical spin analogy

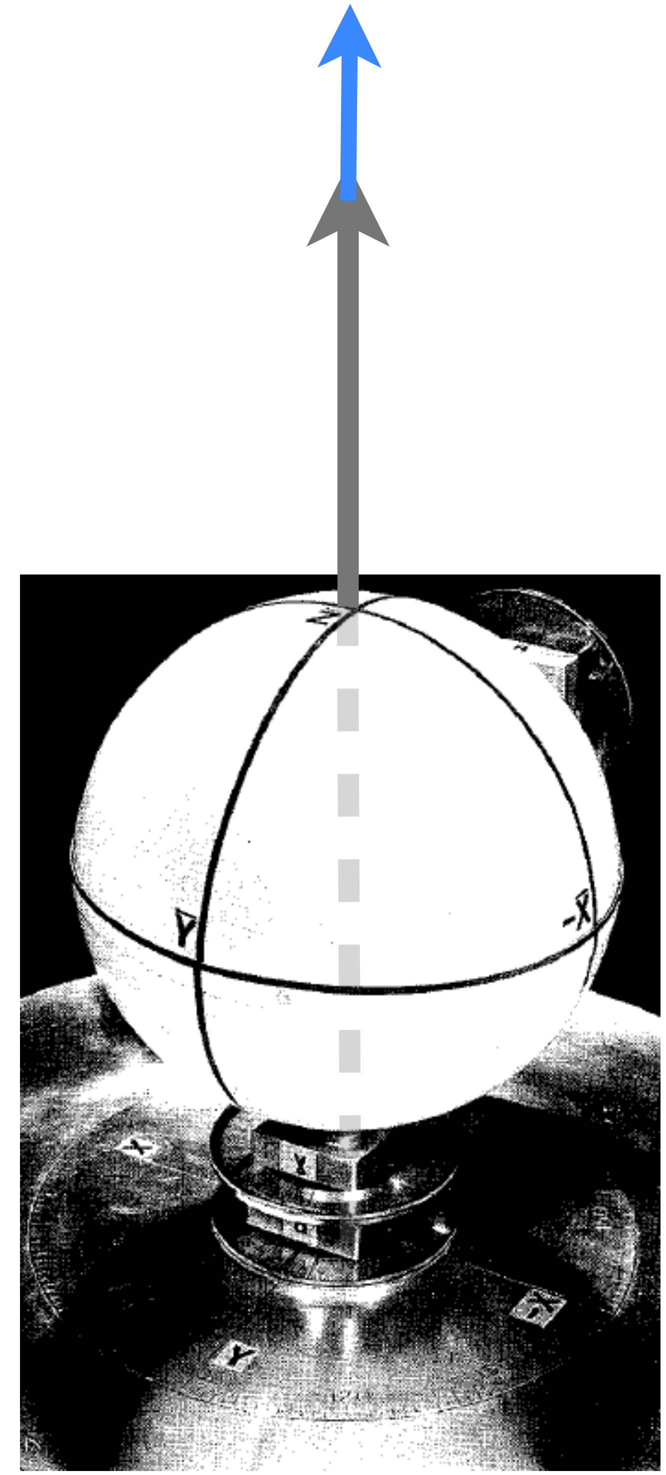
Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α*-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque* **N**

N

L





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Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

General Rule: Gyros tend to "line-up" so they are rotating with whatever is most closely coupled to them.

This is analogous to the tendency for spin magnetic moments to allign (or precess about) the B-direction of a magnetic field

Recall **S***-precession discussion in CMwB Unit 4 Ch.4 and Lect.26*

Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity $\boldsymbol{\omega}$*-ellipsoid vs. angular momentum* $\mathbf{L}$*-ellipsoid*

Lagrangian $\boldsymbol{\omega}$*-equations vs. Hamiltonian momentum* $\mathbf{L}$*-equation*

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j\dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j\mathbf{r}_j \times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \quad \text{with} \quad \mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B}-\left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$$

Consider mass m instantaneously at $\mathbf{r}_m=(x_m,y_m,z_m)=r(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ on a bent axle rotating in a fixed bearing:

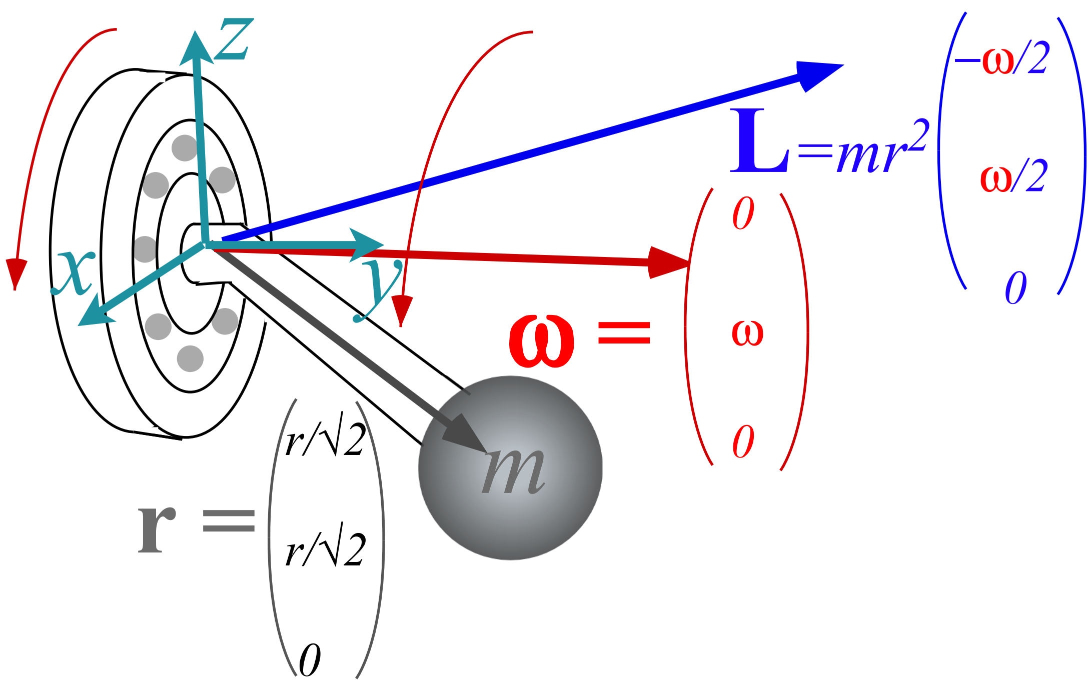

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

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This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{N} \overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{N} m_j \left[(\mathbf{r}_j \bullet \mathbf{r}_j)\mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right]$$

in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j \left[(\mathbf{r}_j \bullet \mathbf{r}_j)\boldsymbol{\omega} - (\mathbf{r}_j \bullet \boldsymbol{\omega})\mathbf{r}_j \right] = \sum_{j=1}^{N} m_j \left[(\mathbf{r}_j \bullet \mathbf{r}_j)\mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right] \bullet \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$$

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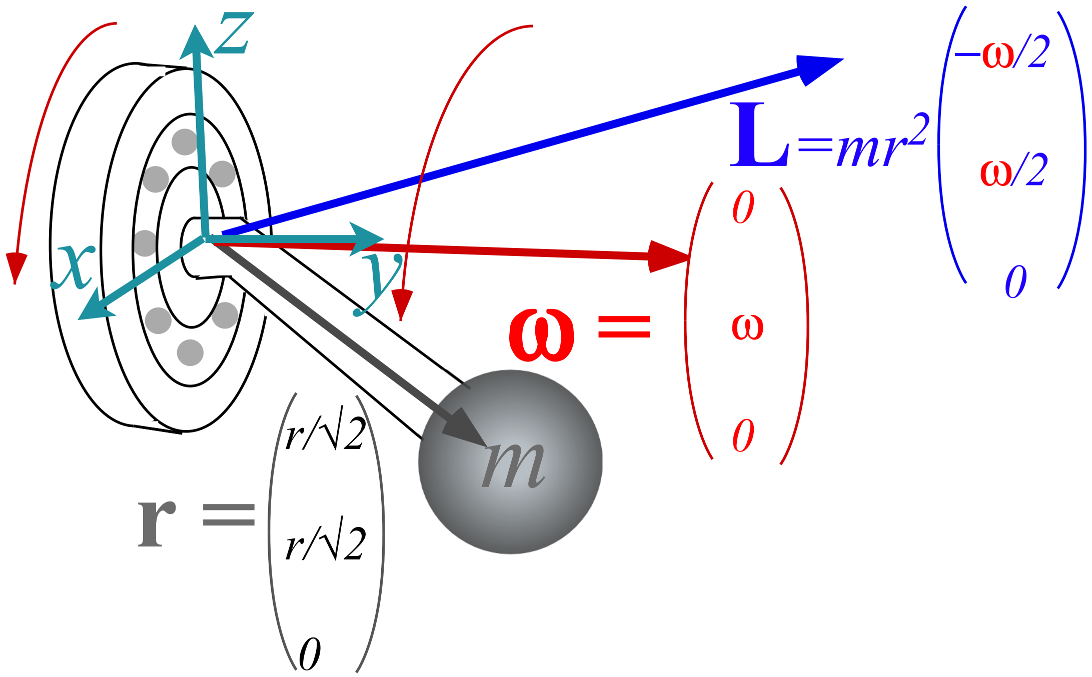

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Matrix form of the ω-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = \sum_{j=1}^{N}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}$$

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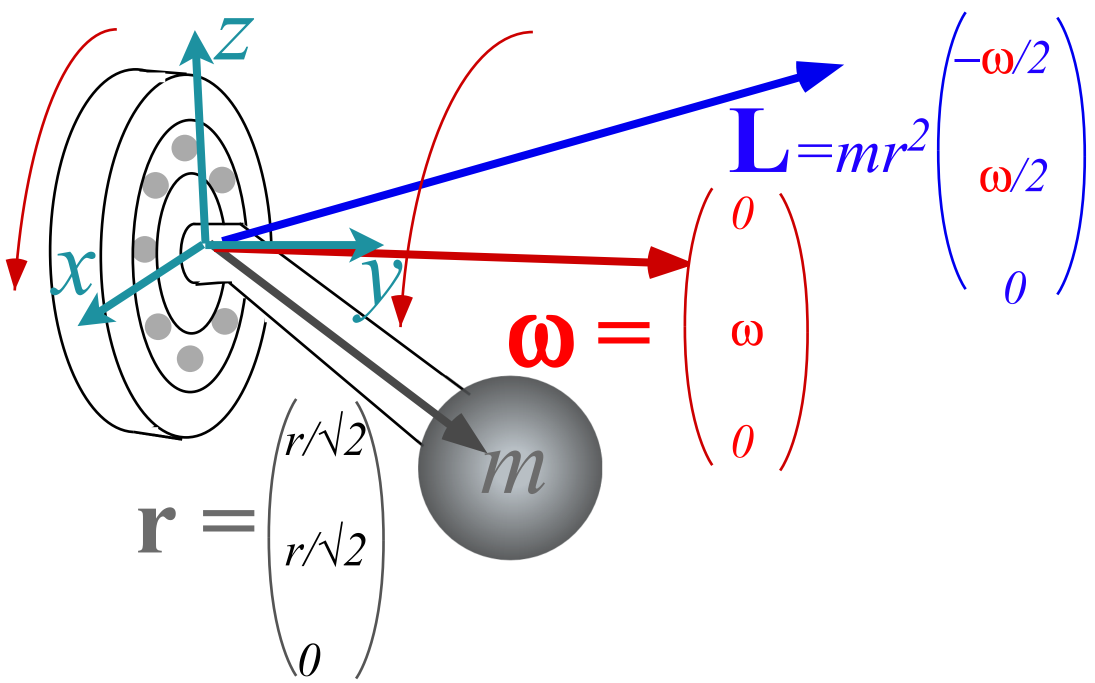

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in the ω-to-$\mathbf{L}$ relation:

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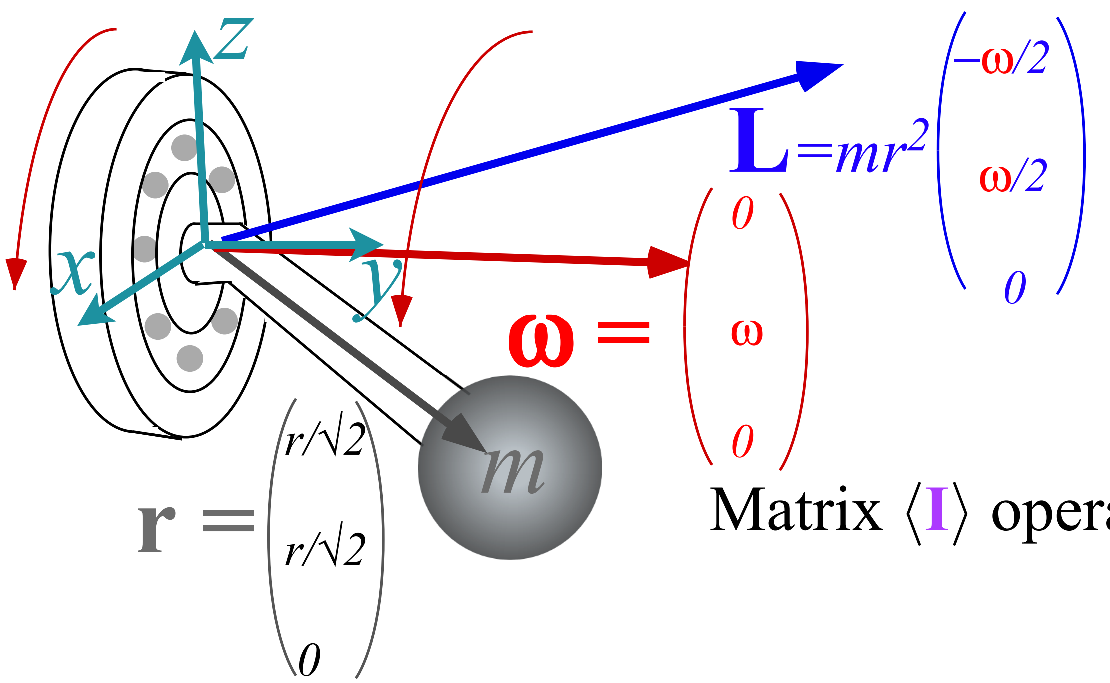
Matrix form of the ω-to-$\mathbf{L}$ relation

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix}$$

using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\langle\overleftrightarrow{\mathbf{I}}\rangle=\sum_{j=1}^{N}\langle\overleftrightarrow{\mathbf{I}}_j\rangle=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m=(x_m,y_m,z_m)=r(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$ on a bent axle rotating in a fixed bearing:



Instantaneous matrix $\langle\mathbf{I}\rangle$ of inertia is:

$$\langle\overleftrightarrow{\mathbf{I}}\rangle=mr^2\begin{pmatrix}(1/\sqrt{2})^2+0 & -(1/\sqrt{2})(1/\sqrt{2}) & -(1/\sqrt{2})0\\ -(1/\sqrt{2})(1/\sqrt{2}) & (1/\sqrt{2})^2+0 & -(1/\sqrt{2})0\\ -0(1/\sqrt{2}) & -0(1/\sqrt{2}) & (1/\sqrt{2})^2+(1/\sqrt{2})^2\end{pmatrix}=mr^2\begin{pmatrix}1/2 & -1/2 & 0\\ -1/2 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \quad \text{and} \quad \mathbf{L}=\sum_{j=1}^{N}\mathbf{r}_j\times m_j\dot{\mathbf{r}}_j=\sum_{j=1}^{N}m_j\mathbf{r}_j\times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \quad \text{with} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\bullet\mathbf{C})\mathbf{B}-(\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}}=\sum_{j=1}^{N}\overleftrightarrow{\mathbf{I}}_j=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]$$

in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

$$\mathbf{L}=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega}-\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right]=\sum_{j=1}^{N}m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega}=\overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix}$$

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle=\sum_{j=1}^{N}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}$$

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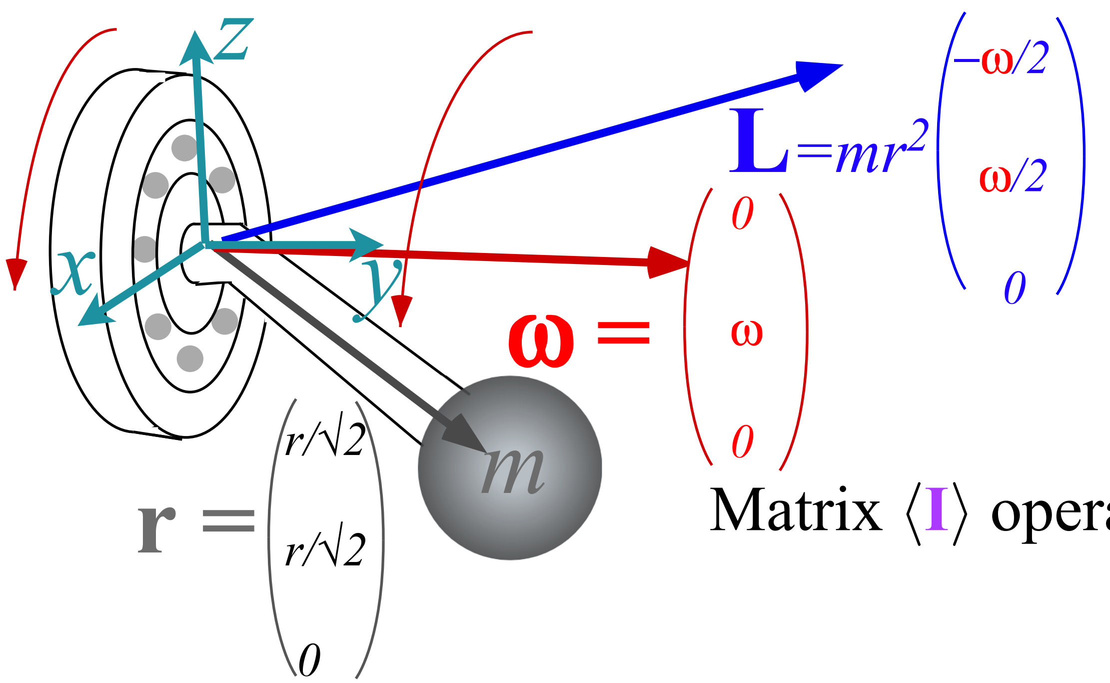

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Instantaneous matrix $\langle\mathbf{I}\rangle$ of inertia is:

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle=mr^2\begin{pmatrix}\left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0\\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0\\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2\end{pmatrix}=mr^2\begin{pmatrix}1/2 & -1/2 & 0\\ -1/2 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=mr^2\begin{pmatrix}1/2 & -1/2 & 0\\ -1/2 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\\omega\\0\end{pmatrix}=mr^2\begin{pmatrix}-1/2\\1/2\\0\end{pmatrix}\omega$$

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

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Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

$$\langle\overleftrightarrow{\mathbf{I}}\rangle=\sum_{j=1}^{N}\langle\overleftrightarrow{\mathbf{I}}_j\rangle=\sum_{j=1}^{N}m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}$$

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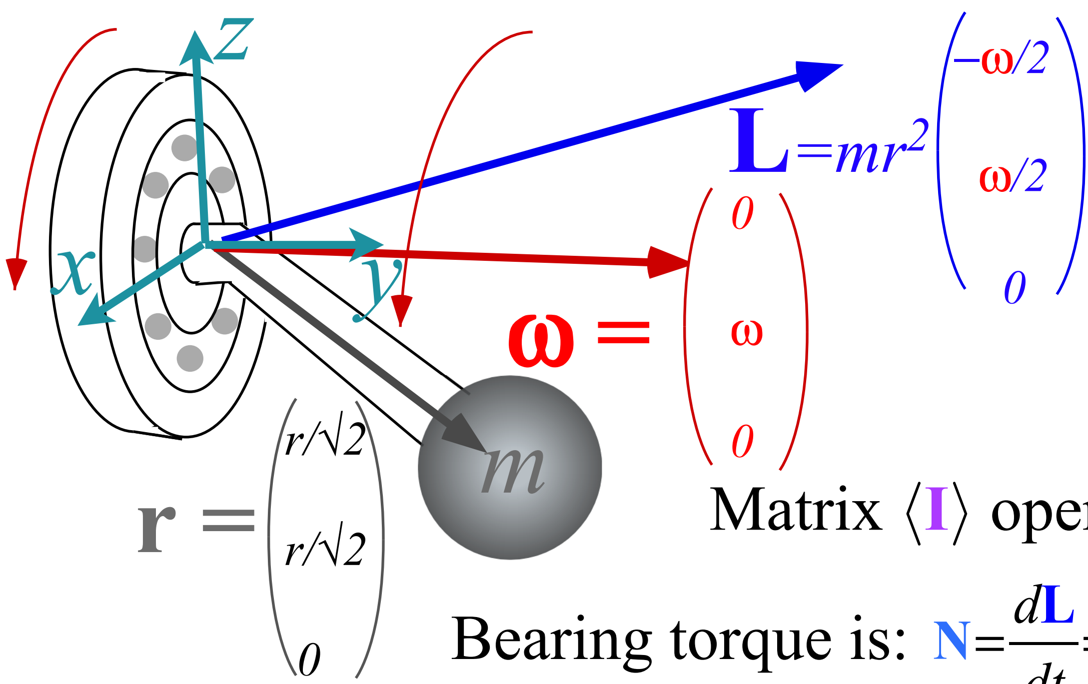

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Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

Bearing torque is: $\mathbf{N}=\dfrac{d\mathbf{L}}{dt}=\boldsymbol{\omega}\times\mathbf{L}$

$$\begin{pmatrix}L_x\\L_y\\L_z\end{pmatrix}=mr^2\begin{pmatrix}1/2 & -1/2 & 0\\ -1/2 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \omega\\ 0\end{pmatrix}=mr^2\begin{pmatrix}-1/2\\ 1/2\\ 0\end{pmatrix}\omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Kinetic energy in terms of velocity ω and rotational Lagrangian

Kinetic energy T of a rotating rigid body can be expressed in terms of the inertia matrix $\mathbf{I}$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j \dot{\mathbf{r}}_j \bullet \dot{\mathbf{r}}_j = \frac{1}{2}\sum_{j=1}^{3} m_j (\boldsymbol{\omega}\times\mathbf{r}_j)\bullet(\boldsymbol{\omega}\times\mathbf{r}_j)$$

Levi-Civita identity

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\bullet\mathbf{C})(\mathbf{B}\bullet\mathbf{D})-(\mathbf{A}\bullet\mathbf{D})(\mathbf{B}\bullet\mathbf{C})$$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j \left[(\boldsymbol{\omega}\bullet\boldsymbol{\omega})(\mathbf{r}_j\bullet\mathbf{r}_j)-(\boldsymbol{\omega}\bullet\mathbf{r}_j)(\mathbf{r}_j\bullet\boldsymbol{\omega})\right]$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\sum_{j=1}^{3} m_j \left[(\mathbf{r}_j\bullet\mathbf{r}_j)\mathbf{1}-(\mathbf{r}_j)(\mathbf{r}_j)\right]\bullet\boldsymbol{\omega}$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\vec{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Kinetic energy is a *quadratic form*

$$T=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\langle\omega|x\rangle & \langle\omega|y\rangle & \langle\omega|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\mathbf{I}|x\rangle & \langle x|\mathbf{I}|y\rangle & \langle x|\mathbf{I}|z\rangle\\ \langle y|\mathbf{I}|x\rangle & \langle y|\mathbf{I}|y\rangle & \langle y|\mathbf{I}|z\rangle\\ \langle z|\mathbf{I}|x\rangle & \langle z|\mathbf{I}|y\rangle & \langle z|\mathbf{I}|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\omega\rangle\\ \langle y|\omega\rangle\\ \langle z|\omega\rangle\end{pmatrix}$$ (Dirac notation)

$$=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\sum_{j=1}^{3} m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

Simplifies in *principle inertial axes {X,Y,Z}* or *body eigen-axes*

$$T=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & I_{XY} & I_{XZ}\\ I_{YX} & I_{YY} & I_{YZ}\\ I_{ZX} & I_{ZY} & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & 0 & 0\\ 0 & I_{YY} & 0\\ 0 & 0 & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}=\frac{I_{XX}\omega_X^2}{2}+\frac{I_{YY}\omega_Y^2}{2}+\frac{I_{ZZ}\omega_Z^2}{2}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$,   generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity $\boldsymbol{\omega}$, momentum $\mathbf{L}$, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$

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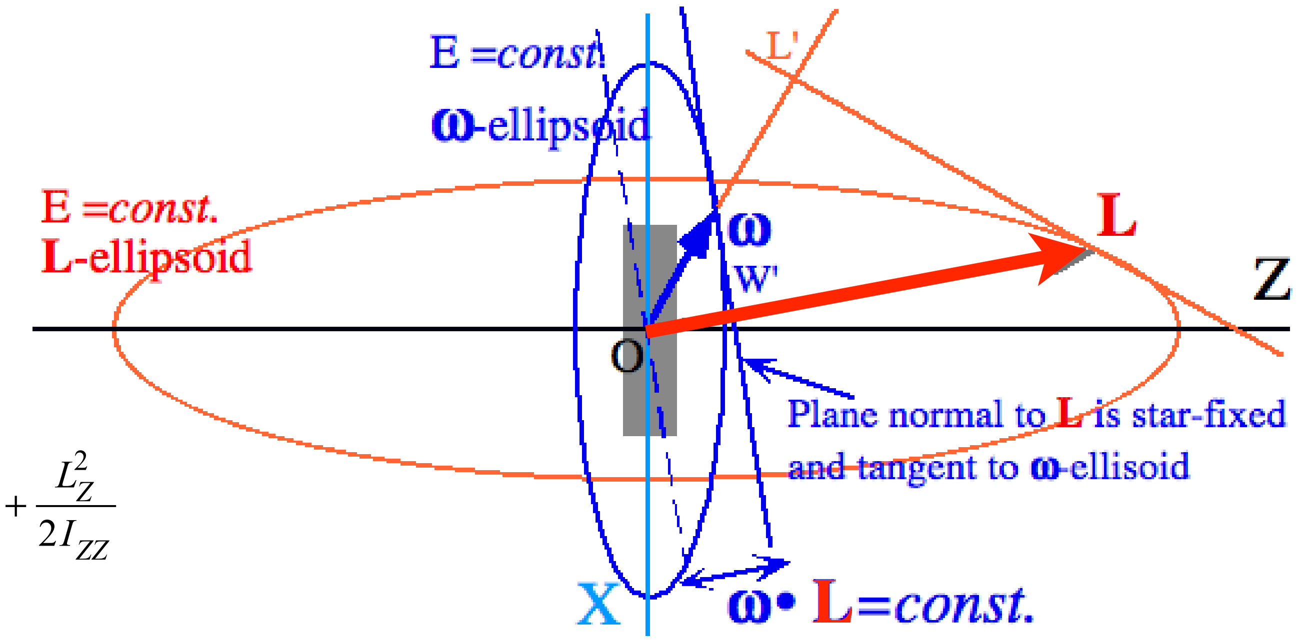

Hamiltonian form is the equation of the *angular momentum or* ***L**-ellipsoid*

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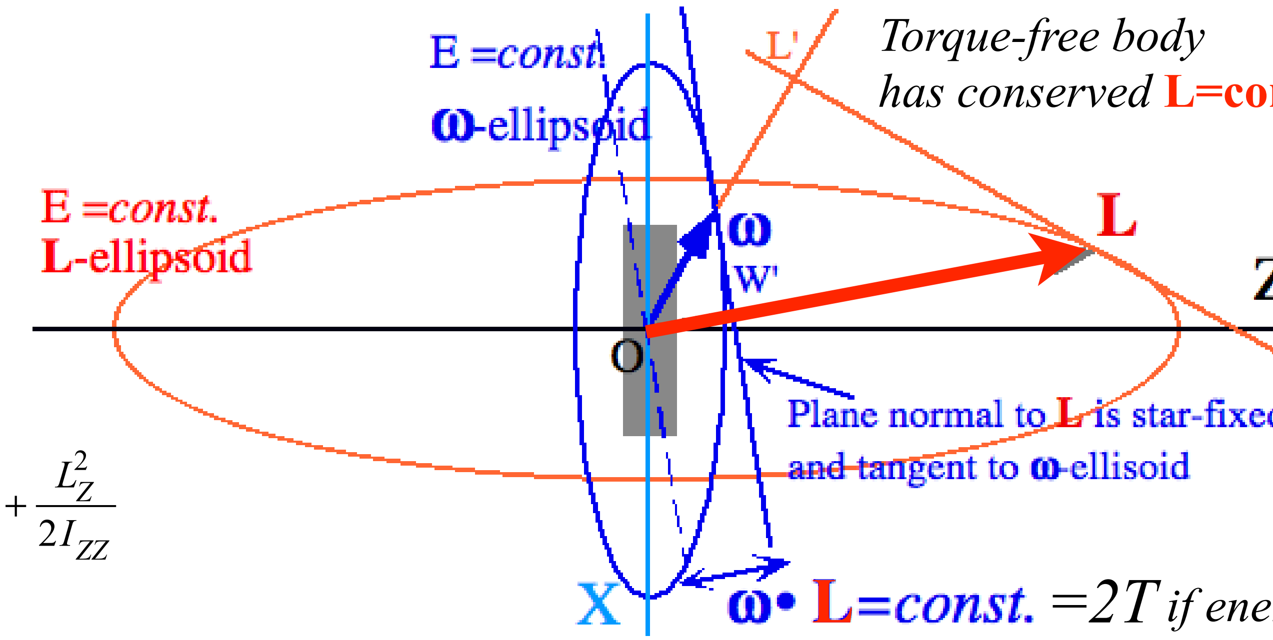
Torque-free body has conserved **L=const**.



ω• L=const. =2T if energy is not dissipated internally

Hamiltonian form is the equation of the *angular momentum or **L**-ellipsoid*

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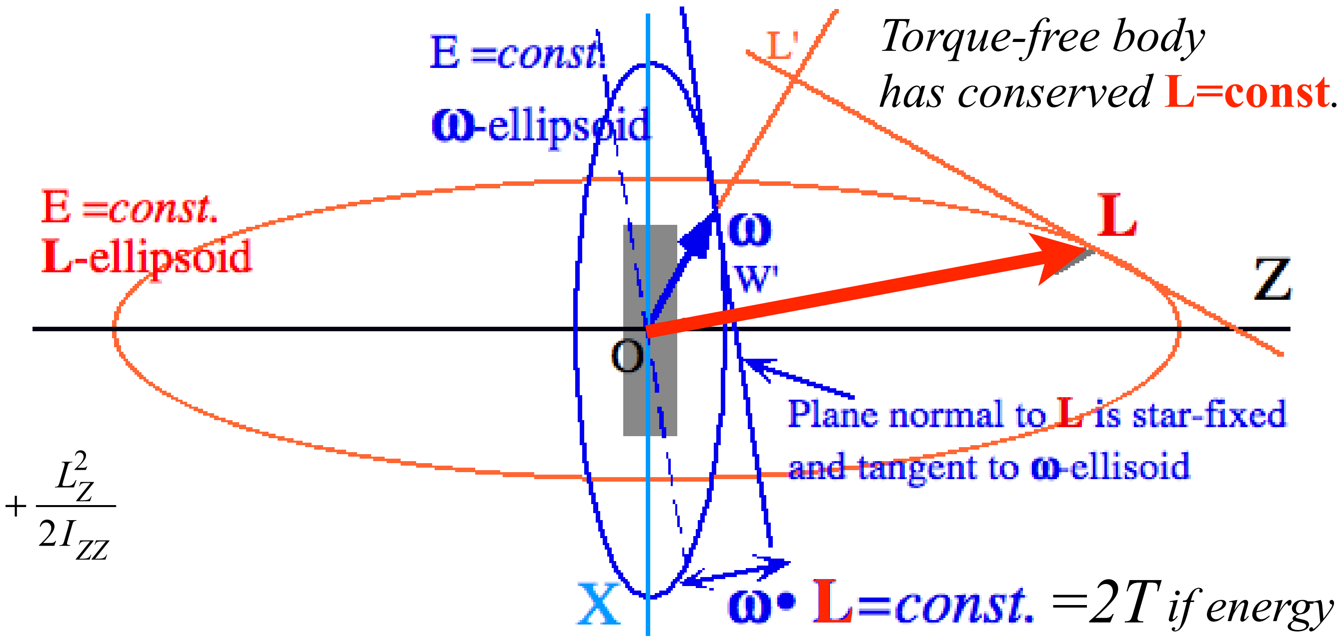

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Canonical momentum: $p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\omega} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

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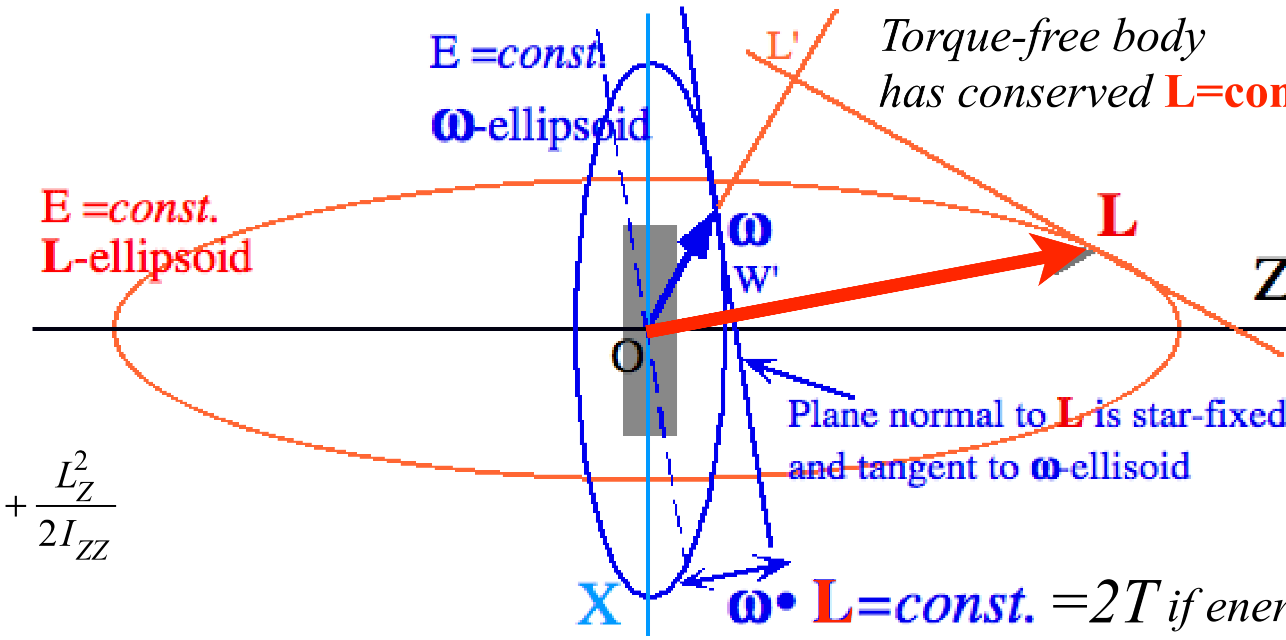

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Hamilton's 1st equations : $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \bullet \mathbf{I}^{-1} \bullet \mathbf{L}}{2} = \mathbf{I}^{-1} \bullet \mathbf{L}$$

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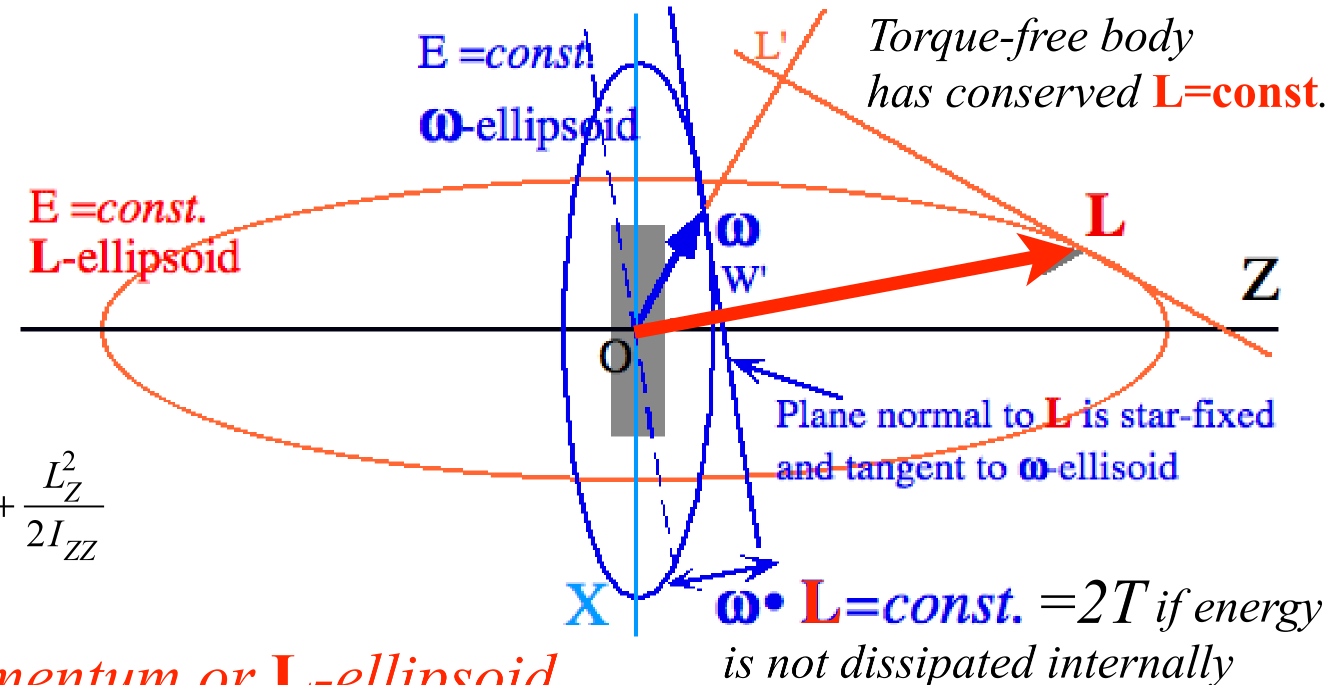
$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

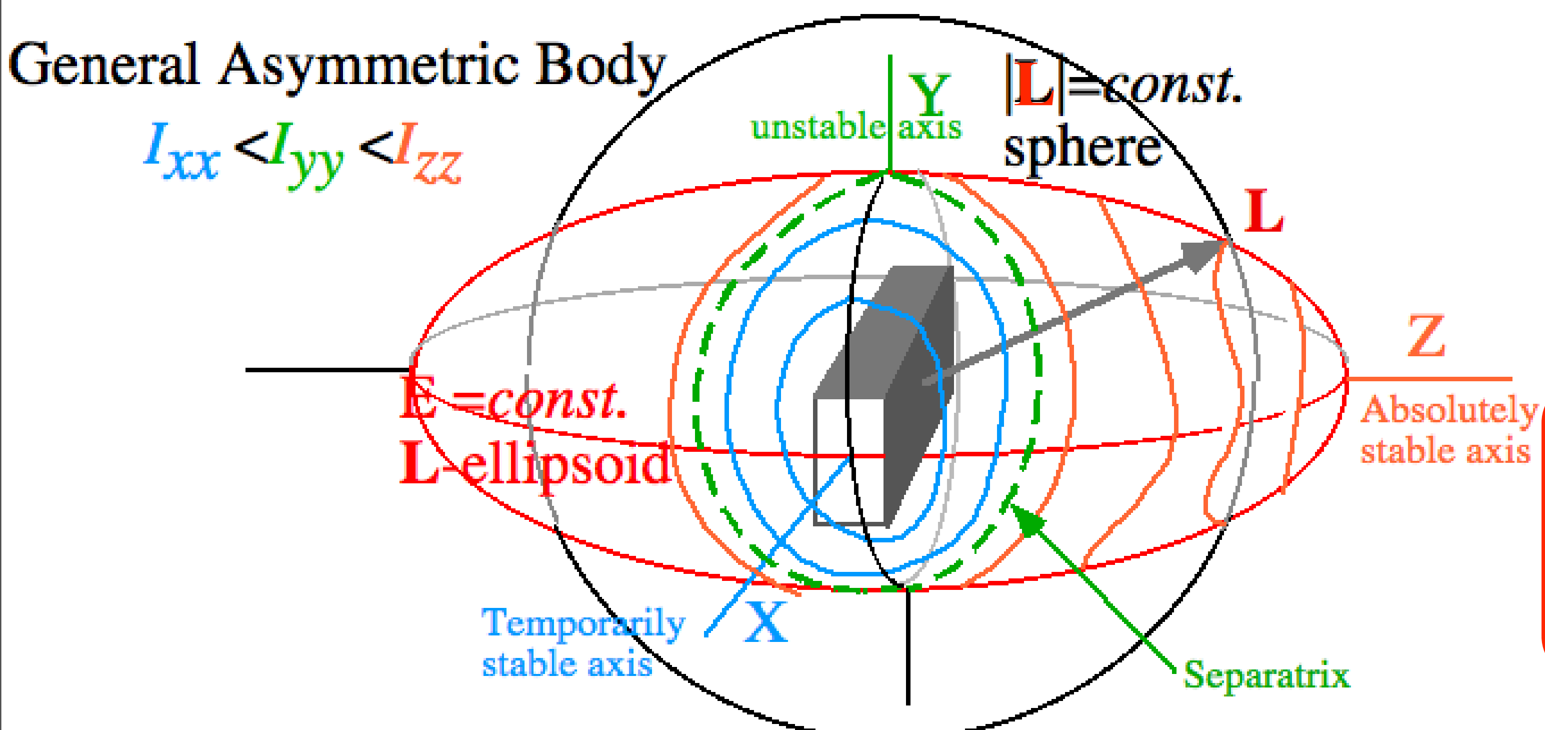
$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*



Canonical momentum: $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Hamilton's 1st equations : $\dot{q}^\mu = \dfrac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \bullet \mathbf{I}^{-1} \bullet \mathbf{L}}{2} = \mathbf{I}^{-1} \bullet \mathbf{L}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

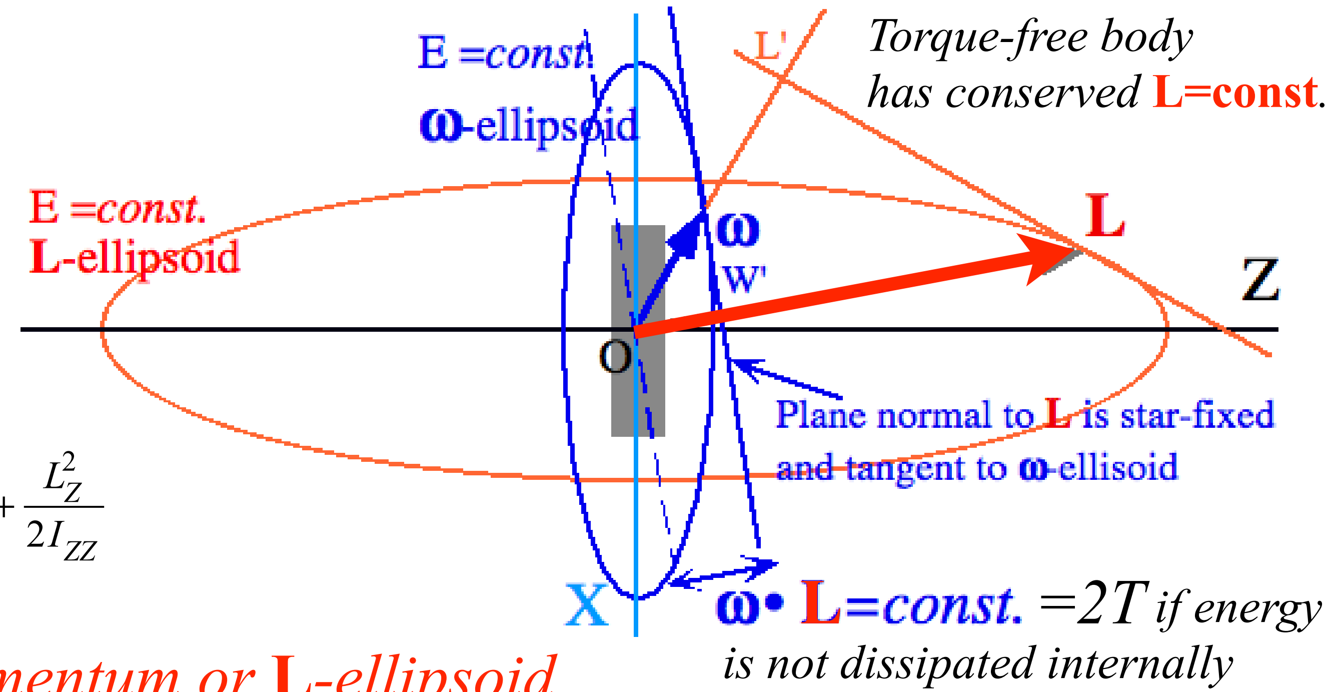
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$

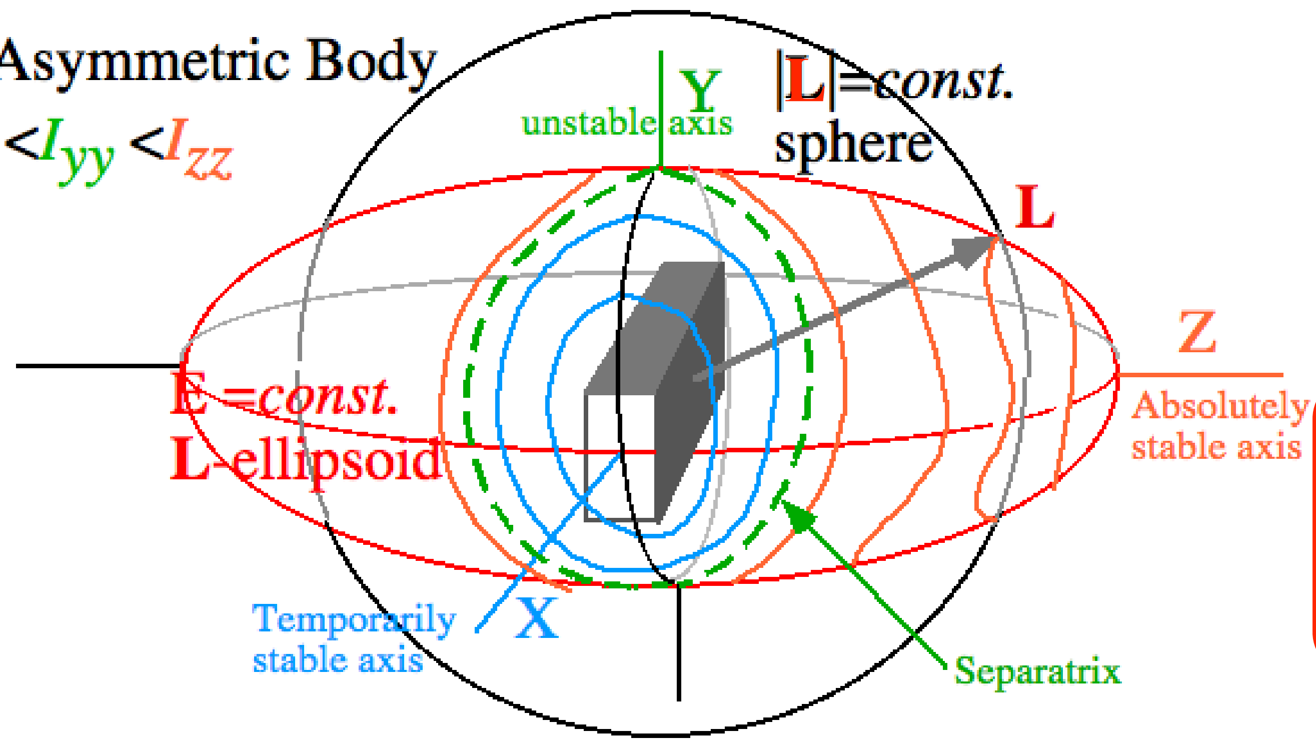


Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid* *ω is generally not conserved unless it is aligned to* ***L*** *or body has symmetry*

General Asymmetric Body

$I_{xx} < I_{yy} < I_{zz}$



Canonical momentum: $p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\omega} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Hamilton's 1st equations : $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \bullet \mathbf{I}^{-1} \bullet \mathbf{L}}{2} = \mathbf{I}^{-1} \bullet \mathbf{L}$$

In body frame momentum ***L*** *moves along intersection of* ***L****-ellipsoid and* ***L****-sphere (Length |****L****| is constant in any classical frame.)*

Rotational Energy Surfaces (RES)

Symmetric, asymmetric, and spherical-top dynamics (Constant **L***)*

BOD-frame cone rolling on LAB frame cone

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)

Rotational Energy Surface (RES) is quadratic multipole function plotted radially

$$E = \frac{\mathbf{J}_x^2}{2I_x} + \frac{\mathbf{J}_y^2}{2I_y} + \frac{\mathbf{J}_z^2}{2I_z} \quad \text{with } J = const.$$

$$= J^2\left(\frac{\sin^2\theta\cos^2\phi}{2I_x} + \frac{\sin^2\theta\sin^2\phi}{2I_y} + \frac{\cos^2\theta}{2I_z}\right)$$

Constant Energy Surface (CES) is asymmetric ellipsoid of constant E

$$E = \frac{\mathbf{J}_x^2}{2I_x} + \frac{\mathbf{J}_y^2}{2I_y} + \frac{\mathbf{J}_z^2}{2I_z} = const.$$

$$or: \quad \frac{\mathbf{J}_x^2}{2EI_x} + \frac{\mathbf{J}_y^2}{2EI_y} + \frac{\mathbf{J}_z^2}{2EI_z} = 1$$

Here notation L or **L** *for angular momentum is replaced by* J *or* **J**

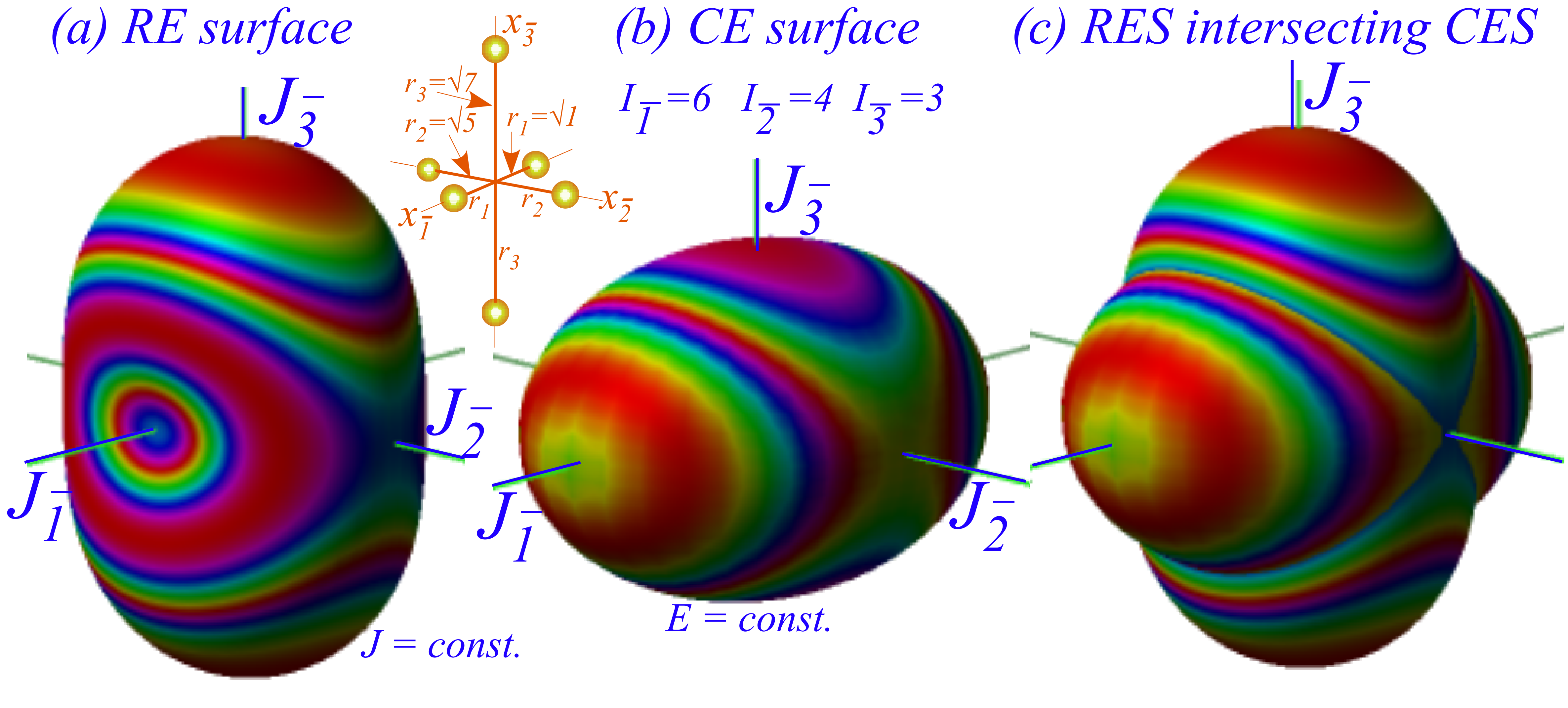


Fig. 6.8.1 Rigid rotor surfaces (a) RES polynomial, (b) CES ellipsoid, and (c) RES and CES intersected.

RES and CES for nearly-symmetric prolate rotors and nearly-symmetric oblate rotors

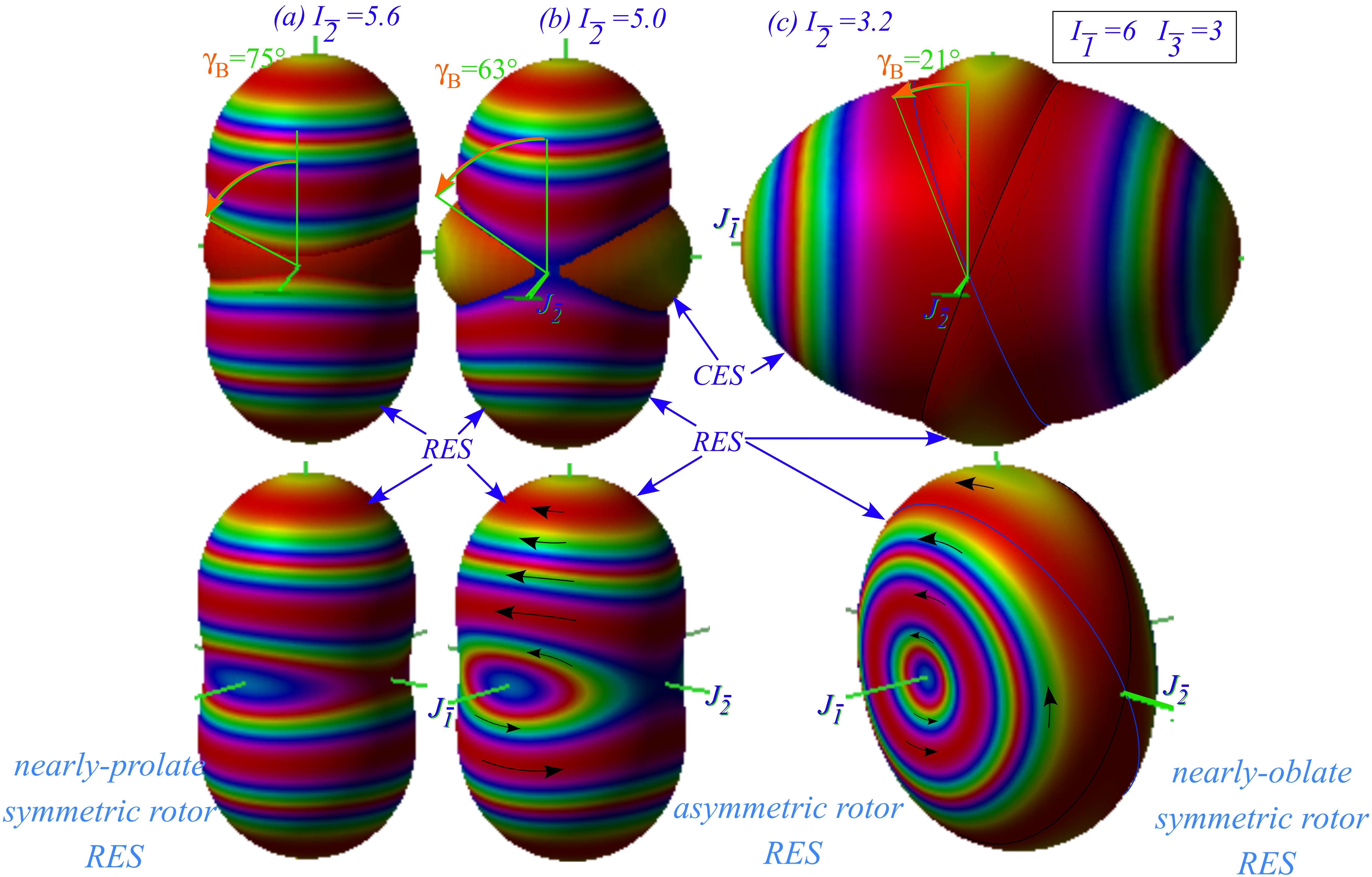
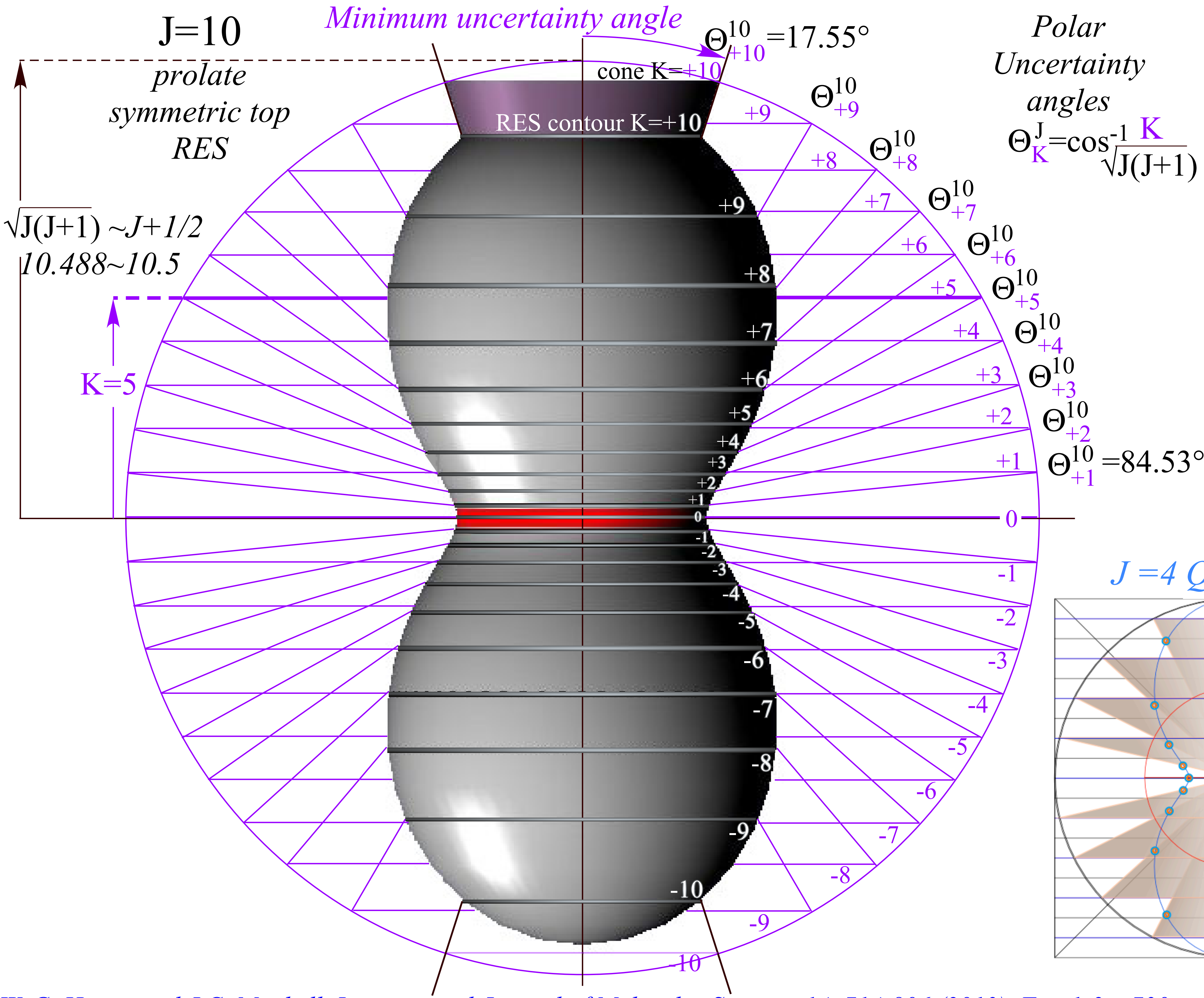
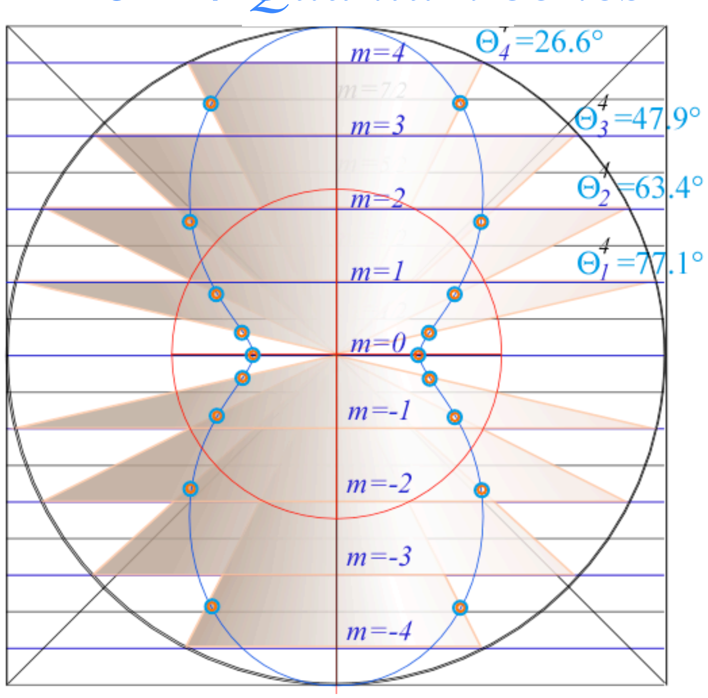


Fig. 6.8.2 Fixed-J- RES with CES at separatrix $E=J^2/2I_{\bar{2}}$ *as* $I_{\bar{2}}$ *varies. (a)* $I_{\bar{2}}$=5.6 and γ_B=75.5° *(Nearly prolate low-E CES), (b)* $I_{\bar{2}}$=5.0 and γ_B= 63.4°, *(c)* $I_{\bar{2}}$=3.2 and γ_B= 20.7° *(Nearly oblate high-E CES).*

RES for symmetric prolate rotor locates J =10 quantum (-J<K<J) levels (at RES-quantum cone intersections)

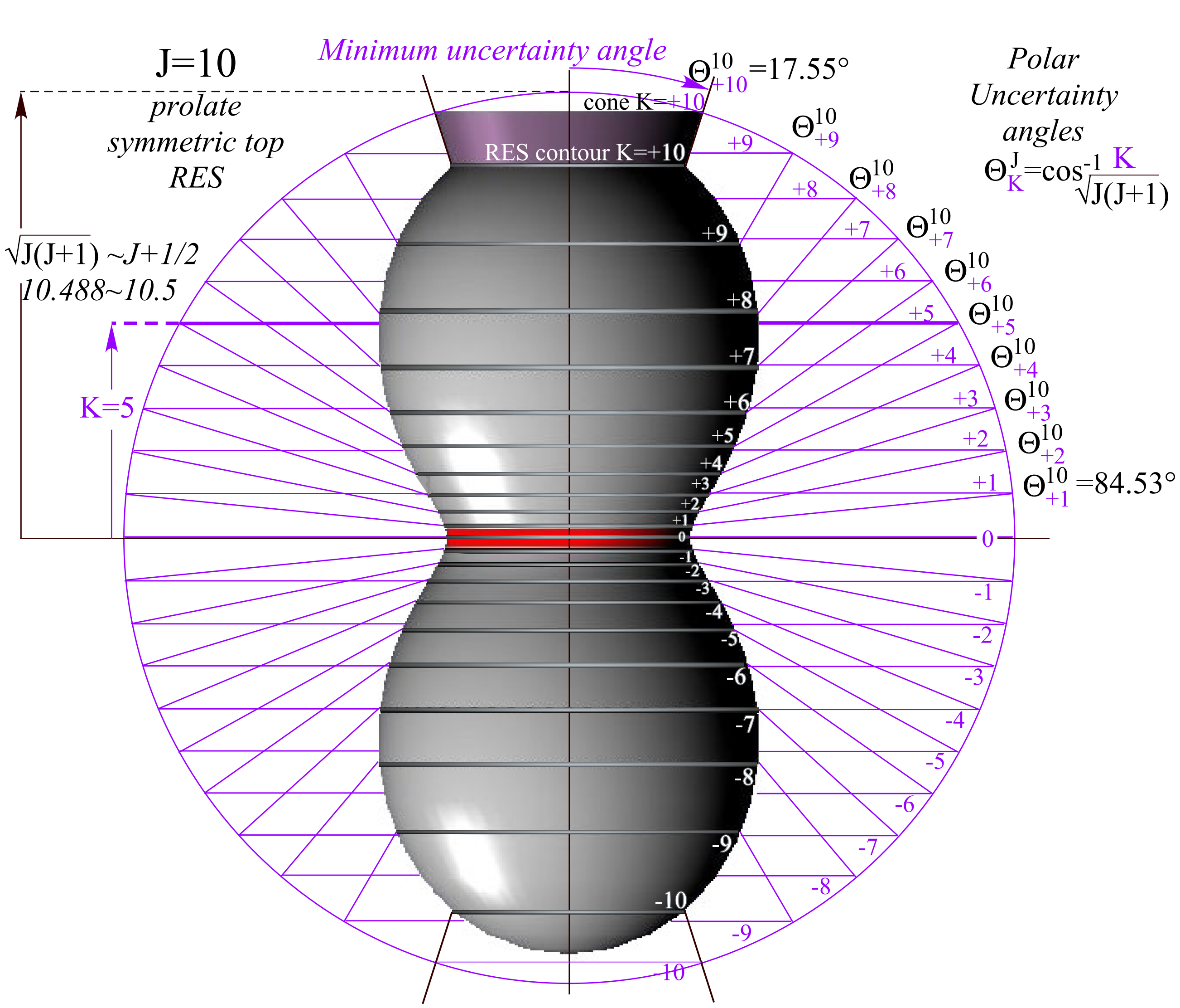


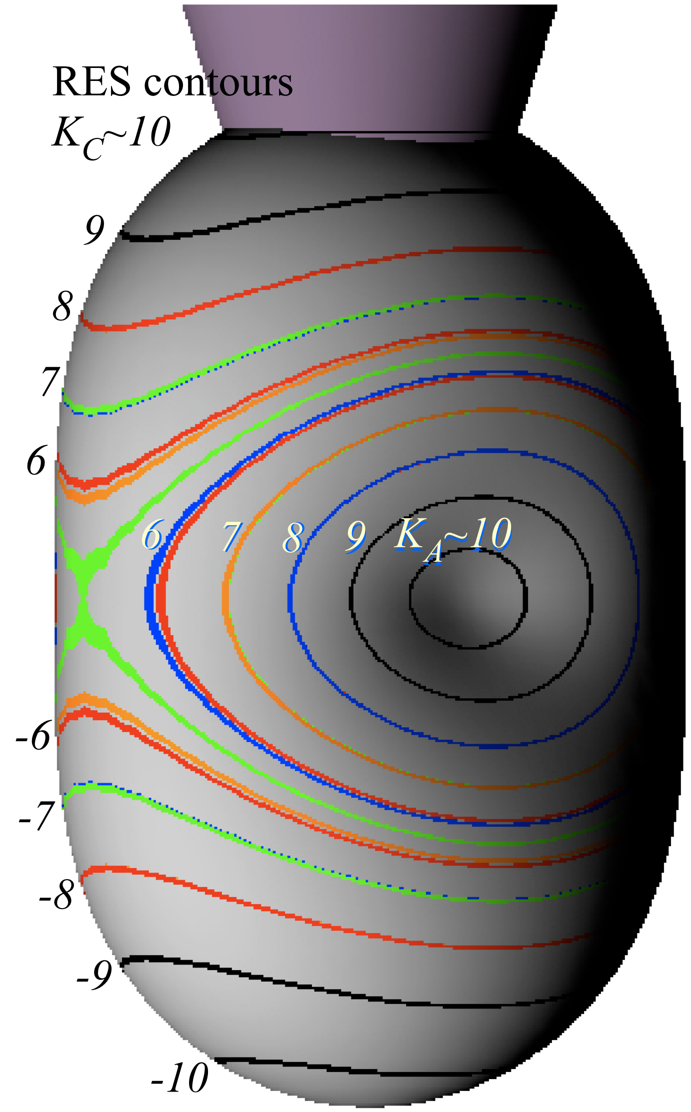
J =4 Quantum cones

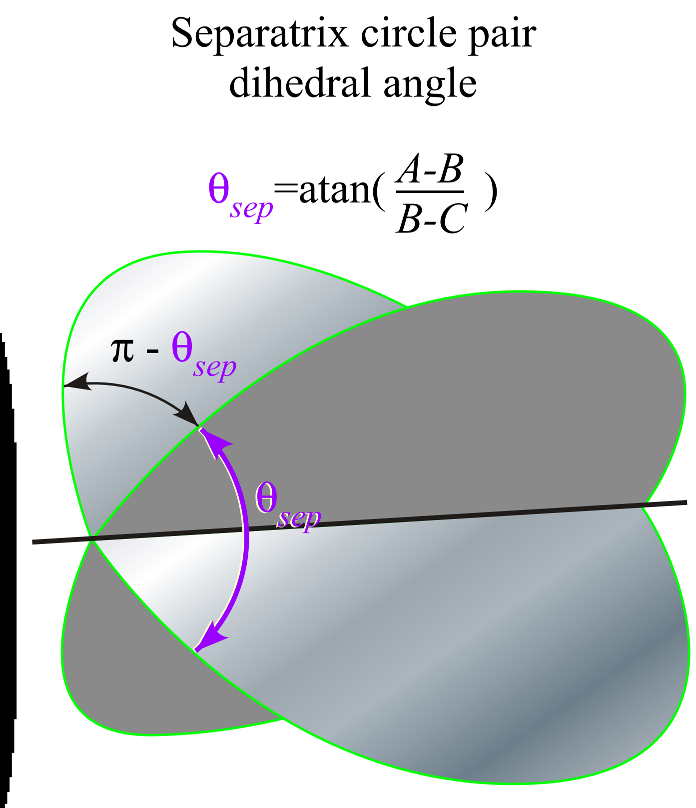


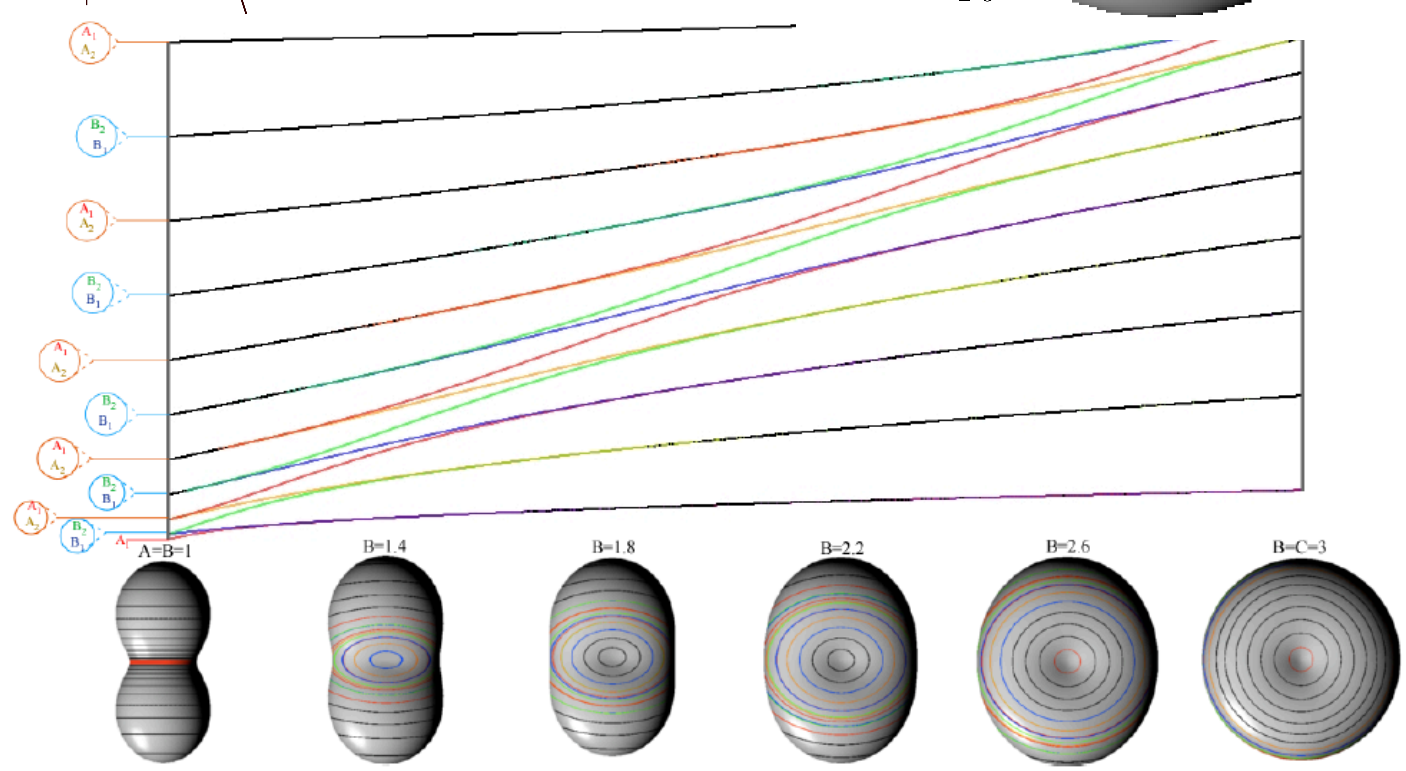
W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-2 p.730

RES for symmetric and asymmetric rotor approximates J =10 (-J<K<J) levels (near RES-quantum cone levels)







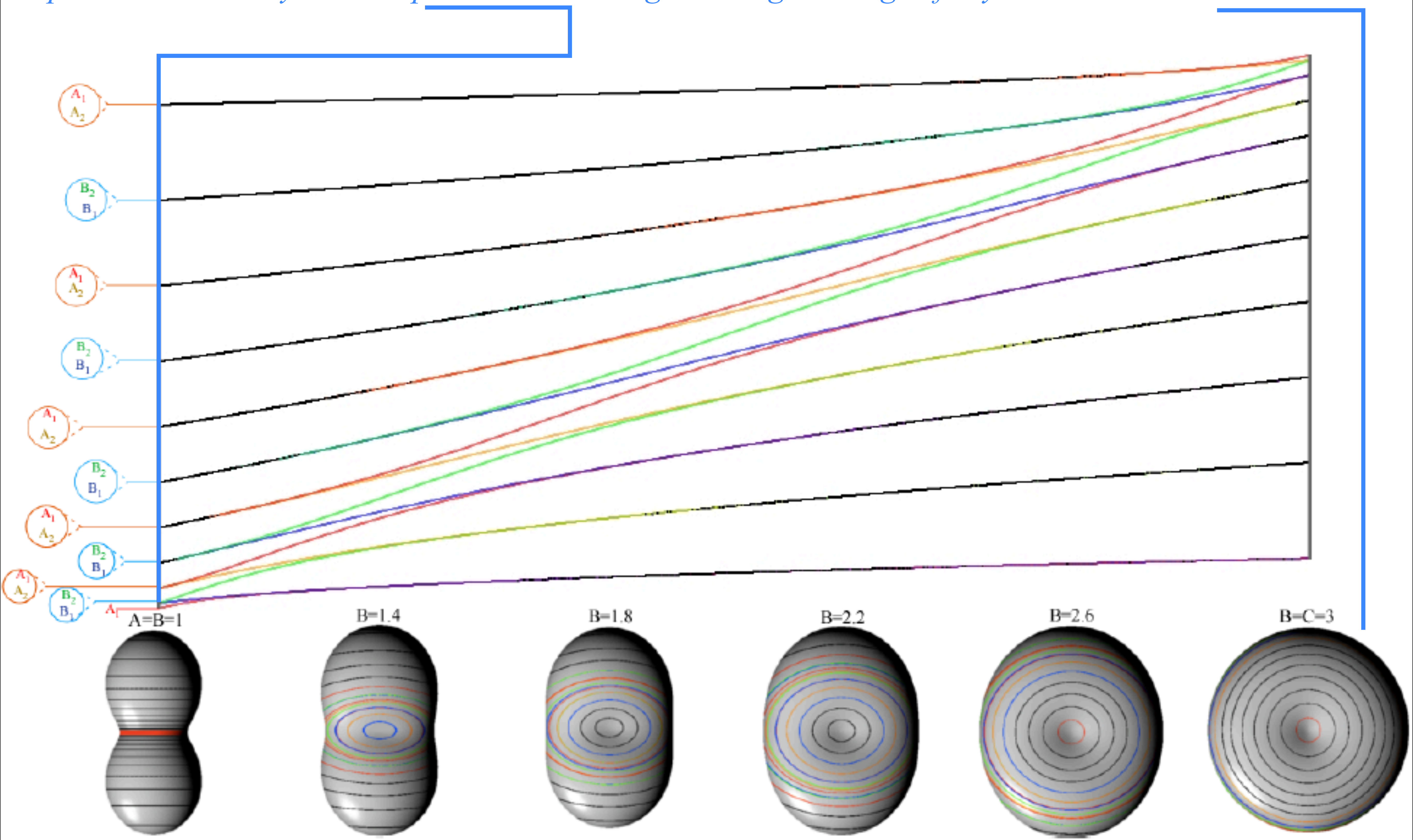


W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730

RES for symmetric prolate rotor locates J =10 quantum (-J<K<J) levels (at RES-quantum cone intersections)

$$E = A\mathbf{J}_x^2 + B\mathbf{J}_y^2 + C\mathbf{J}_z^2 \quad \text{with } J = const.$$

Spectra varies as symmetric prolate RES changes through a range of asymmetric RES to oblate RES



W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 4 p.734

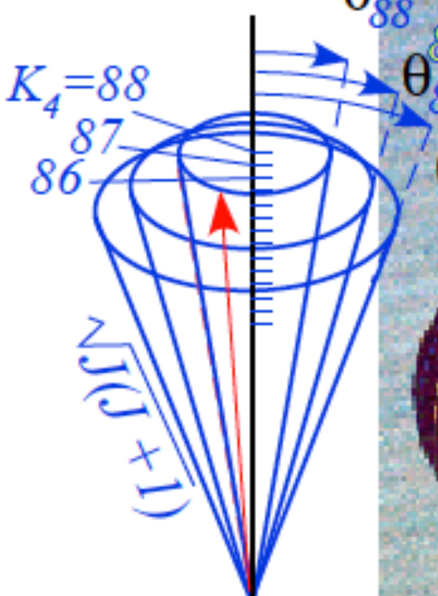
RES for spherical rotor approximates J =88 (-J<K<J) levels of SF_6

$\langle H\rangle \sim \nu_{vib} + BJ(J+1) + \langle H^{Scalar\ Coriolis}\rangle + \langle H^{Tensor\ Centrifugal}\rangle + \langle H^{Tensor\ Coriolis}\rangle + \langle H^{Nuclear\ Spin}\rangle + \ldots$

O_h or T_d Spherical Top: *(Hecht CH_4 Hamiltonian 1960)*

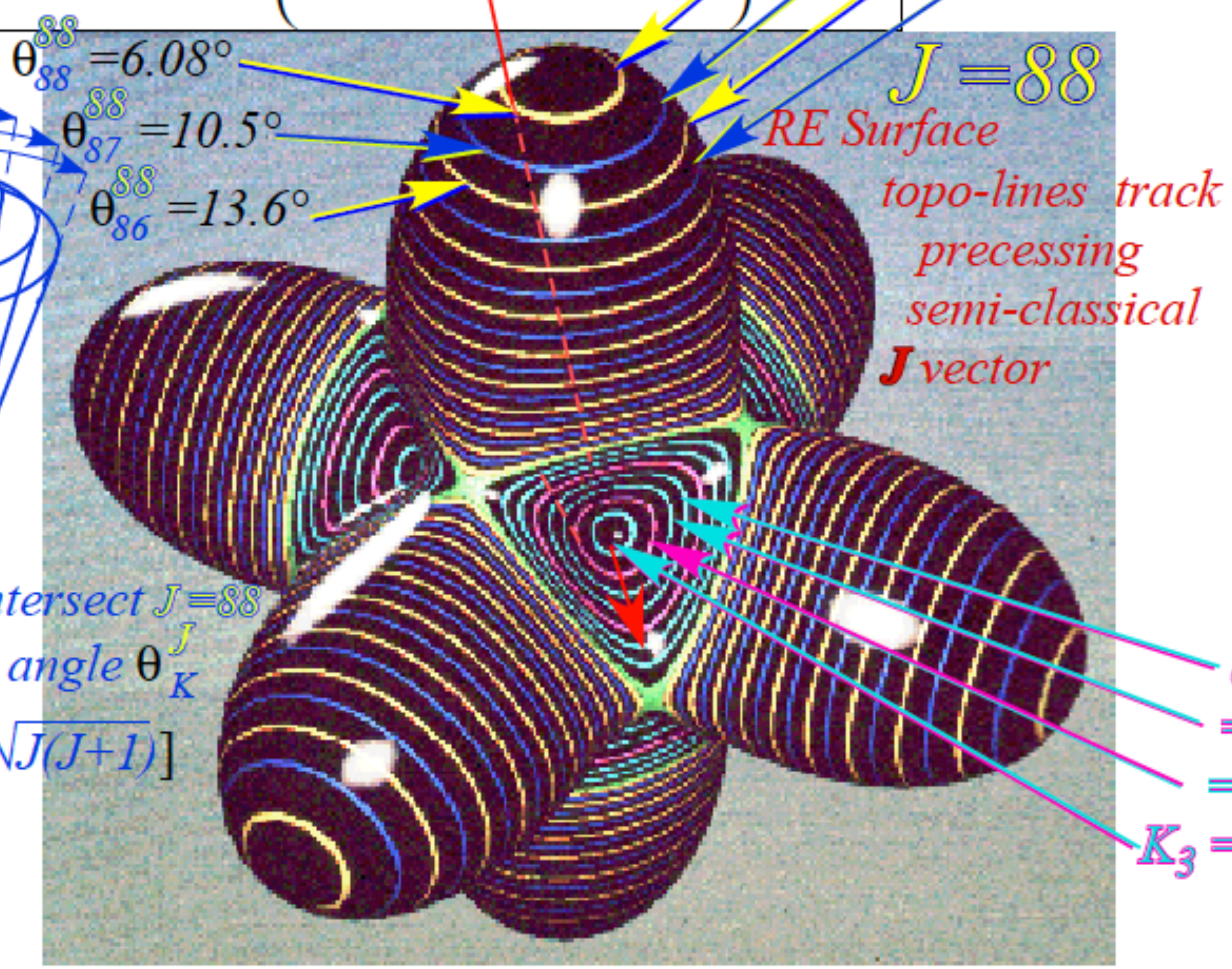
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

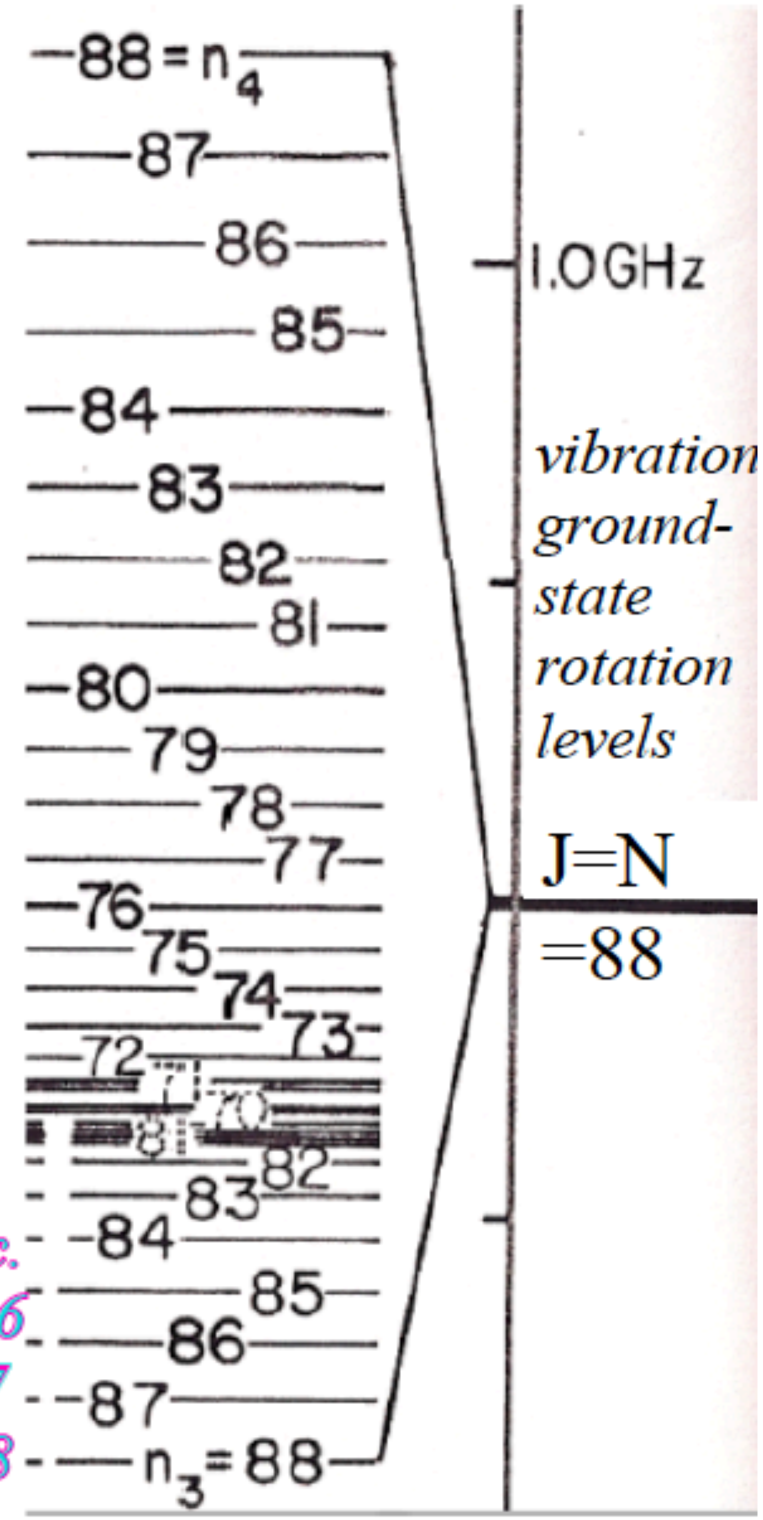
$$= B\mathbf{J}^2 + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



(J,K) cones intersect J=88 RE surface at angle θ_K^J

$\theta_K^J = \text{acos}[K/\sqrt{J(J+1)}]$





(next page shows slice)

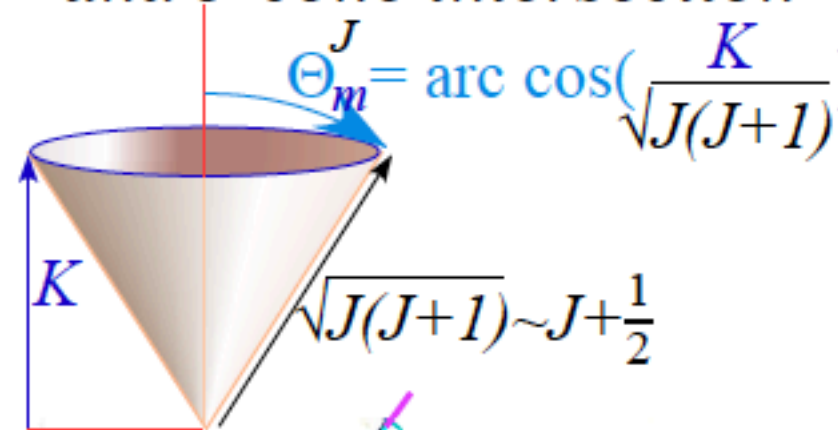
W. G. Harter and J C. Mitchell ,International Symposium on Molecular Spectroscopy, OSU Columbus (2009)

SF_6 Spectra of O_h Ro-vibronic Hamiltonian described by RE Tensor Topography

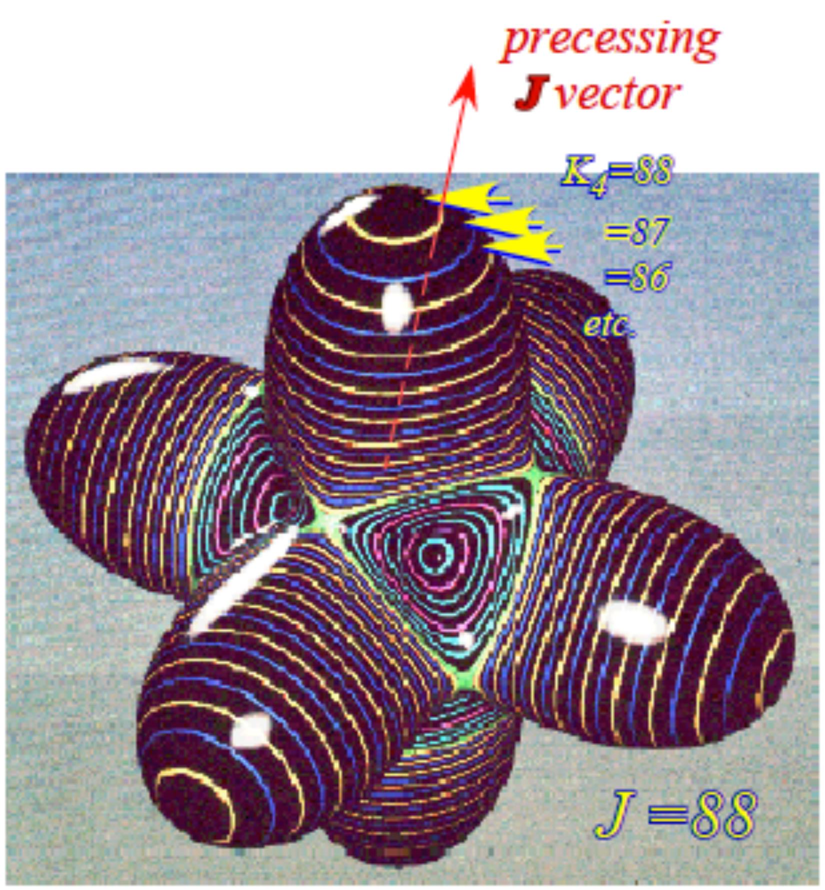
and J-cone intersection

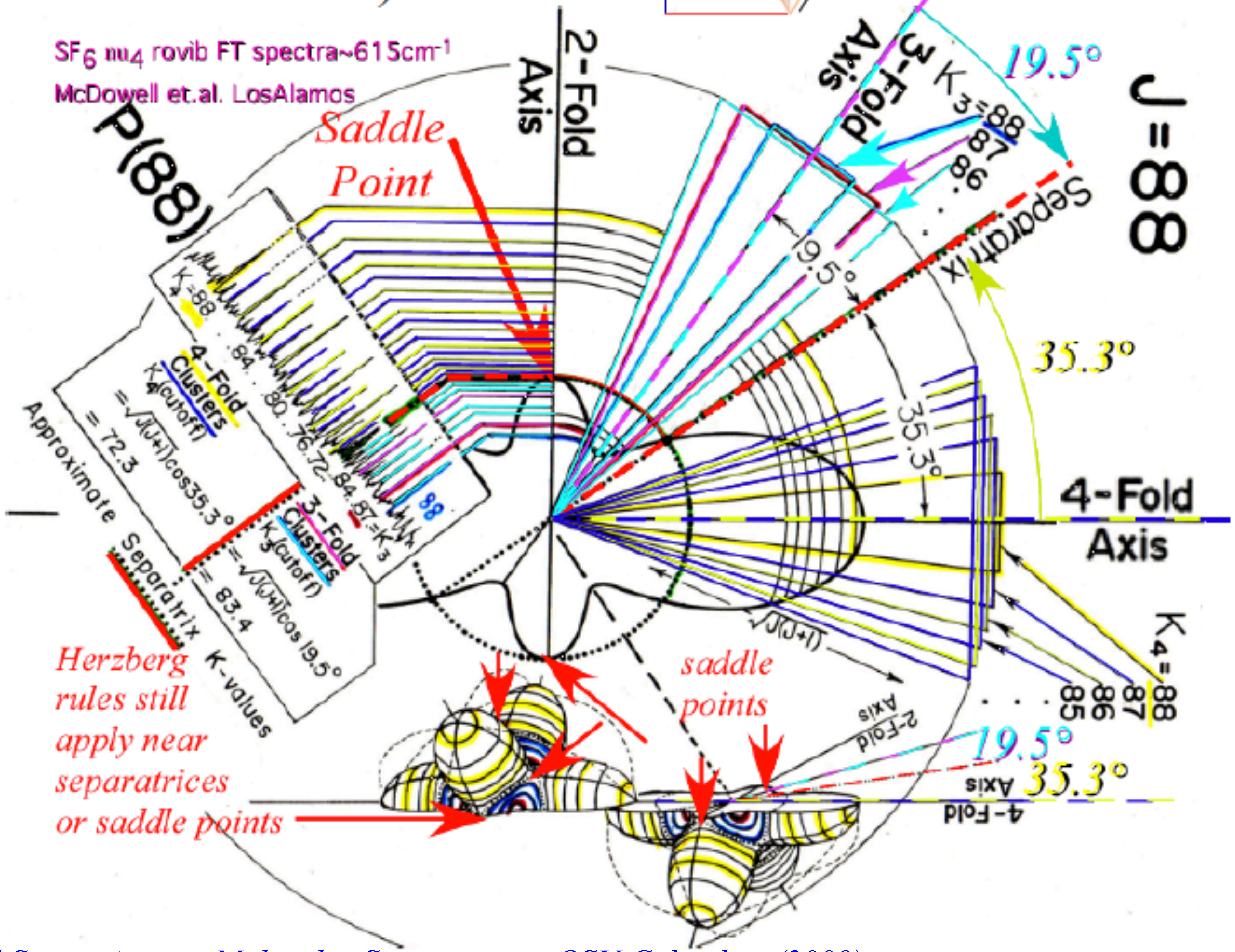
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

$$= \quad B\mathbf{J}^2 \quad + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



Rovibronic Energy (RE) Tensor Surface





W. G. Harter and J C. Mitchell ,International Symposium on Molecular Spectroscopy, OSU Columbus (2009)

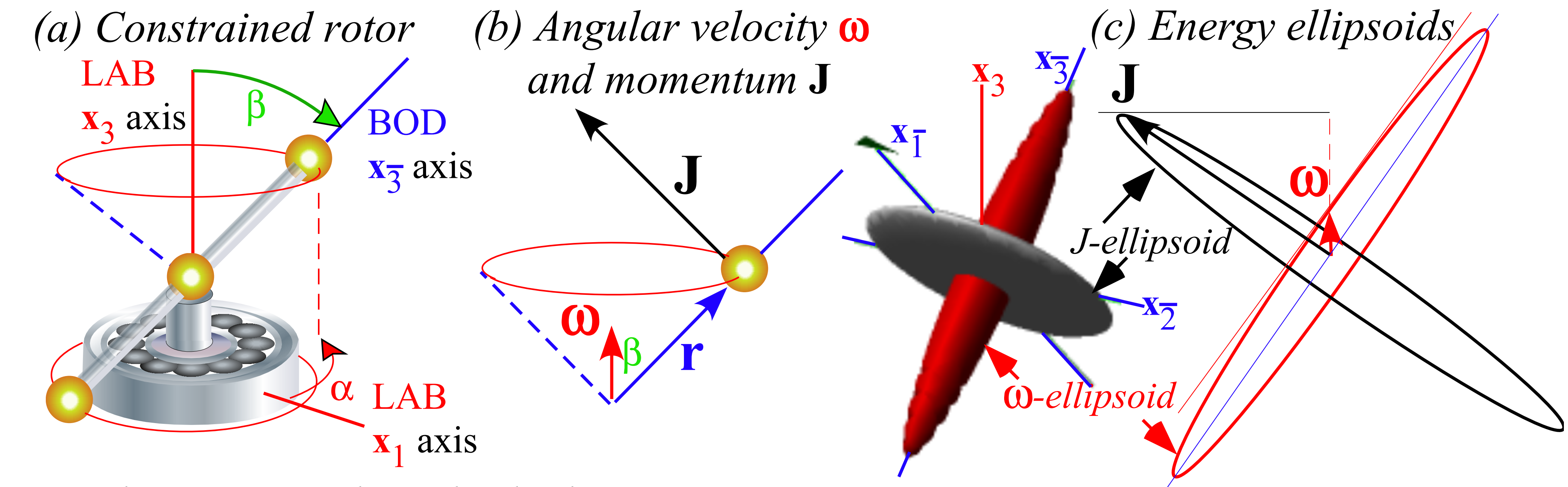


Fig. 6.7.1 Elementary ω-constrained rotor and angular velocity-momentum geometry.

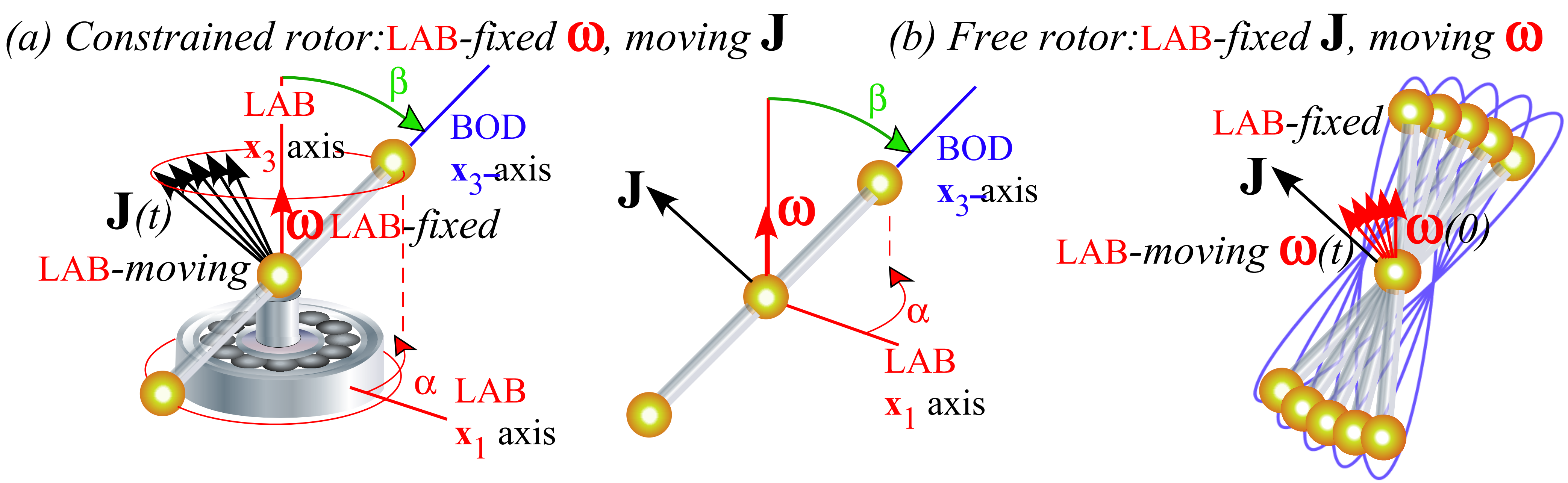
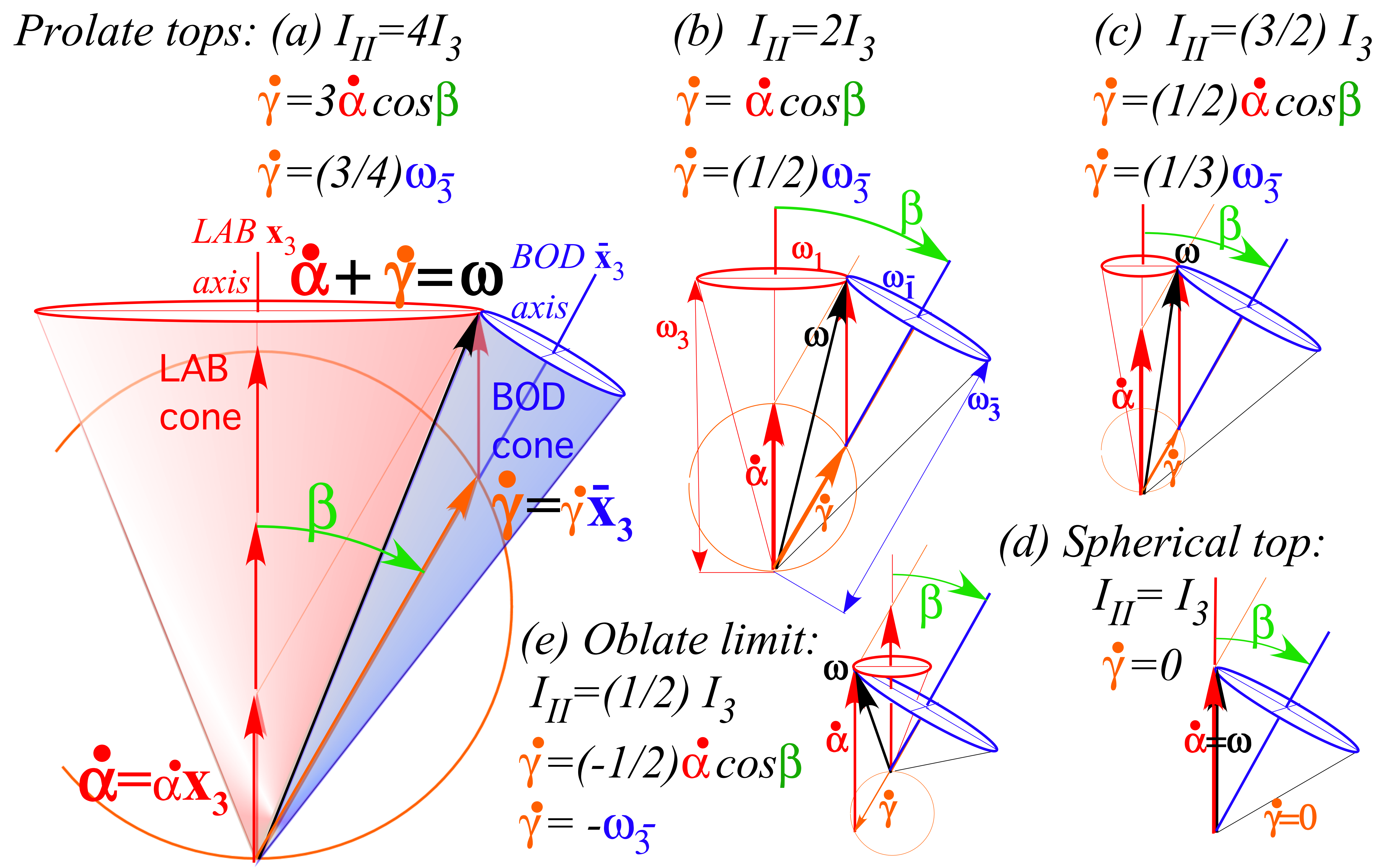


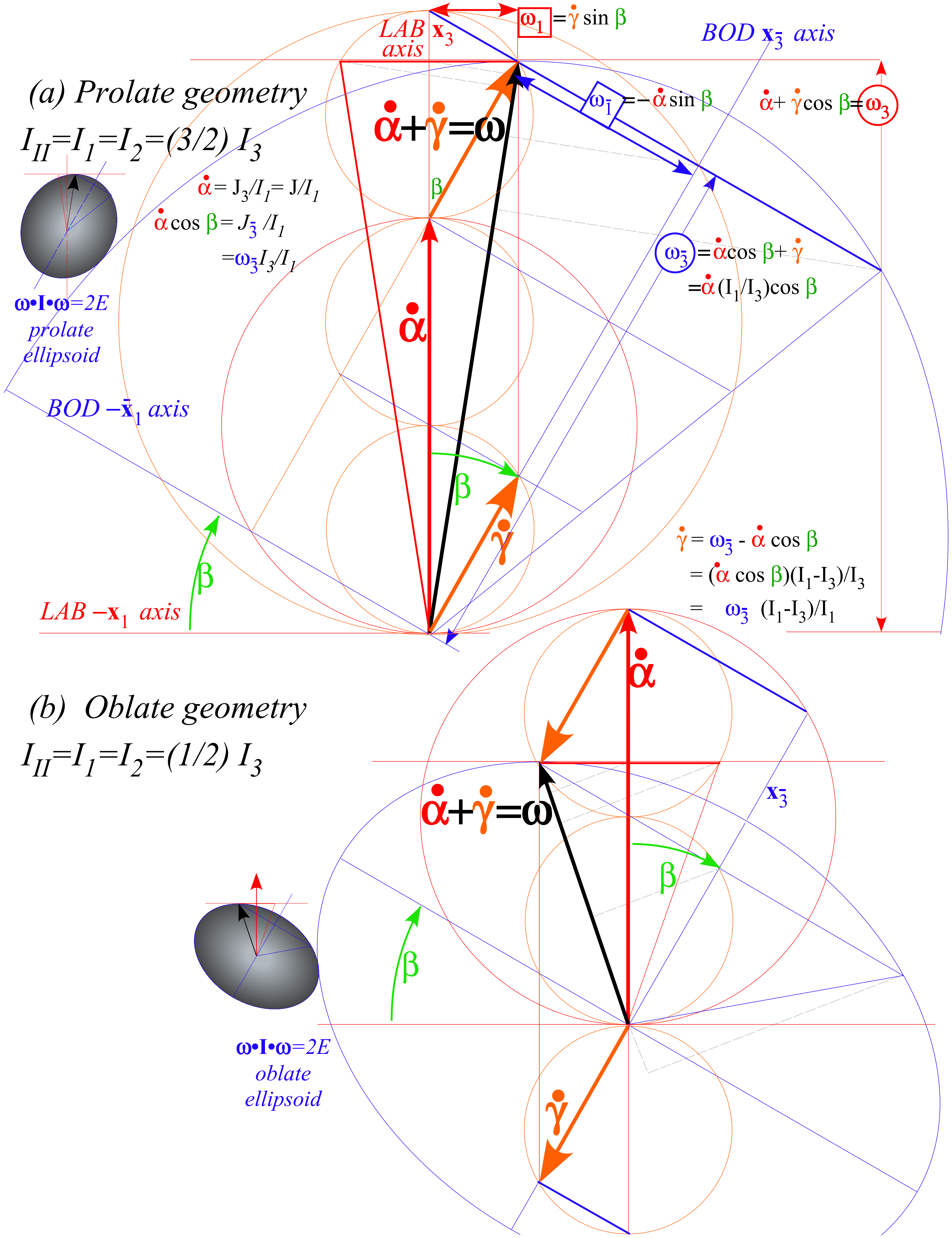
Fig. 6.7.2 Free rotor cut loose from LAB-constraining ω-axis changes dynamics accordingly.

..this was the kind of dynamics that started me dropping superballs...



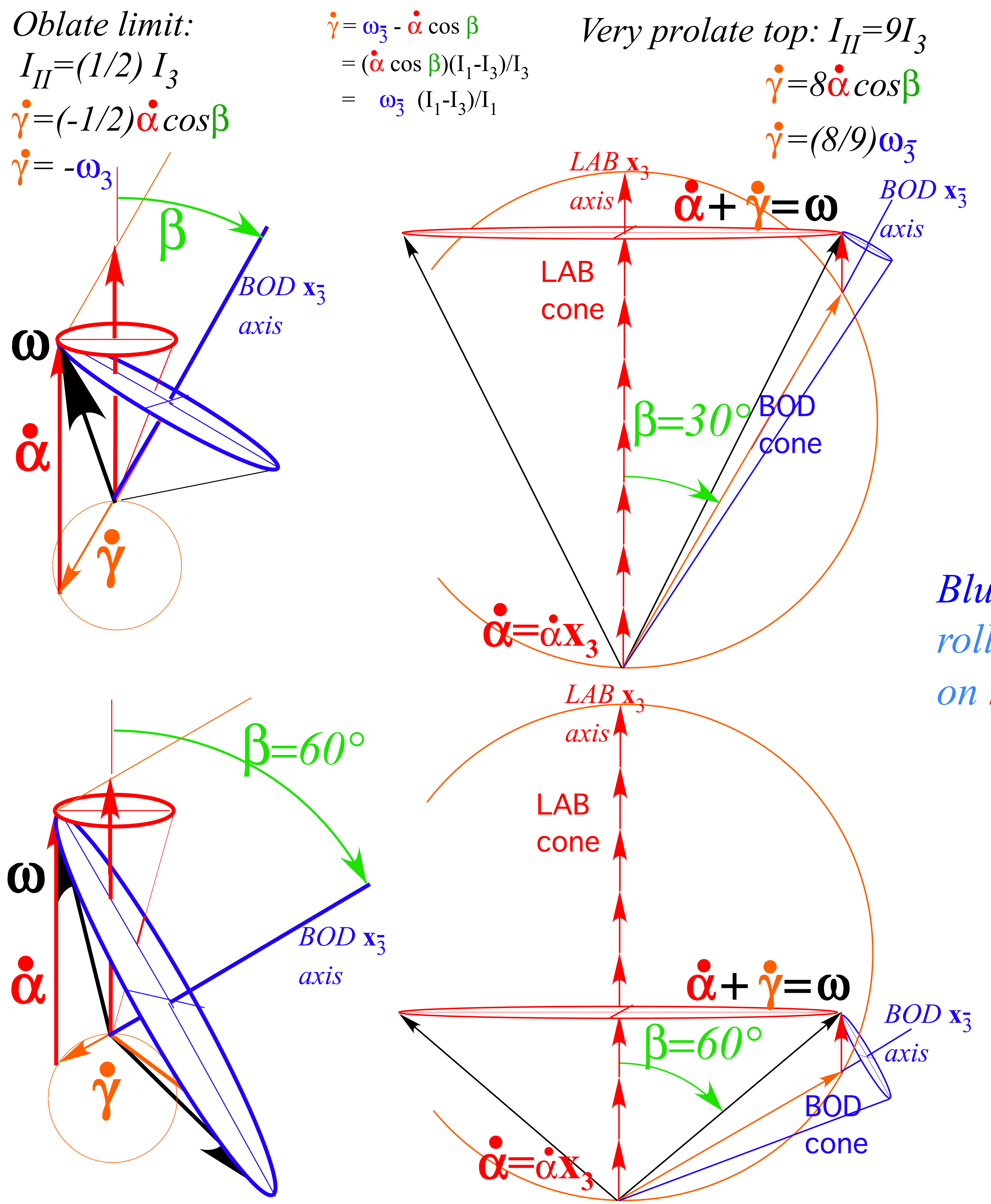
Blue BOD-frame cones roll (around ω-sticking axis) without slipping on red LAB-frame cone

Fig. 6.7.3 Symmetric top ω-cones for β=30° and inertial ratios: (a) $\frac{I_{II}-I_3}{I_3}$ =3, (b) 1, (c) $\frac{1}{2}$, (d) 0, (e) $-\frac{1}{2}$.



Blue BOD-frame cones roll without slipping on red LAB-frame cone

Fig. 6.7.4 Detailed geometry of symmetric top kinetics. (a) Prolate case. (b) Most-oblate case



Blue BOD-frame cones roll without slipping on red LAB-frame cone

Fig. 6.7.5 Extreme cases (Oblate vs. Prolate) of symmetric-top geometry.