Eccentricity vector \( \varepsilon \) and \((\varepsilon, \lambda)\)-geometry of orbital mechanics

- \( \varepsilon \)-vector and Coulomb \( r \)-orbit geometry
- Review and connection to standard development
- \( \varepsilon \)-vector and Coulomb \( p = mv \) geometry
- Example with elliptical orbit

Analytic geometry derivation of \( \varepsilon \)-construction

Algebra of \( \varepsilon \)-construction geometry

Connection formulas for \((a, b)\) and \((\varepsilon, \lambda)\) with \((\gamma, R)\)

Ruler & compass construction of \( \varepsilon \)-vector and orbits

- \((R = -0.375 \text{ elliptic orbit})\)
- \((R = +0.5 \text{ hyperbolic orbit})\)

Properties of Coulomb trajectory families and envelopes

Graphical \( \varepsilon \)-development of orbits

- Launch angle fixed-Varied launch energy
- Launch energy fixed-Varied launch angle
- Launch optimization and orbit family envelopes
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\varepsilon$-vector and Coulomb $p=mv$ geometry

$\varepsilon$-vector and Coulomb $p=mv$ algebra

Example with elliptical orbit

Analytic geometry derivation of $\varepsilon$-construction

Algebra of $\varepsilon$-construction geometry

Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, \rho)$

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
Eccentricity vector $\epsilon$ and $(\epsilon, \lambda)$ geometry of orbital mechanics

Isotropic field $V = V(r)$ guarantees conservation angular momentum vector $L$

$$L = r \times p = m r \times \dot{r}$$

(Review of Lect. 28-29)

Coulomb $V = -k/r$ also conserves eccentricity vector $\epsilon$

$$\epsilon = \hat{r} - \frac{p \times L}{km} = \frac{r}{r} - \frac{p \times (r \times p)}{km}$$

$A = km \cdot \epsilon$ is known as the Laplace-Hamilton-Gibbs-Runge-Lenz vector.

Consider dot product of $\epsilon$ with a radial vector $r$:

$$\epsilon \cdot r = \frac{r \cdot r - r \cdot p \times L}{r} = \frac{r - r \times p \cdot L}{km} = r - \frac{L \cdot L}{km}$$

Let angle $\phi$ be angle between $\epsilon$ and radial vector $r$

$$\epsilon \cdot r = \frac{r \cdot r - r \cdot p \times L}{r} = \frac{r - r \times p \cdot L}{km} = r - \frac{L \cdot L}{km}$$

For $\lambda = L^2/\text{km}$ that matches: $\lambda = \frac{L^2}{1 - \epsilon \cos \phi}$

(a) Attractive ($k > 0$)
- Elliptic ($E < 0$)

(b) Attractive ($k > 0$)
- Hyperbolic ($E > 0$)

(c) Repulsive ($k < 0$)
- Hyperbolic ($E > 0$)

(Rotational momentum $L = r \times p$ is normal to the orbit plane.)

$$\epsilon = \hat{r} - \frac{p \times L}{km}$$

$$\epsilon \cdot p = \frac{p \cdot r - p \cdot p \times L}{km} = p \cdot \dot{r} = pr$$

$$\begin{align*}
\lambda & \text{ if: } \phi = 0 \text{ apogee} \\
\frac{\lambda}{1 - \epsilon} & \text{ if: } \phi = \frac{\pi}{2} \text{ zenith} \\
\frac{\lambda}{1 + \epsilon} & \text{ if: } \phi = \pi \text{ perigee}
\end{align*}$$

Generate symmetry groups: $U(2) \subset U(2)$ or: $R(3) \subset R(3) \times R(3) \subset O(4)$

IHO $V = (k/2)r^2$ also conserves Stokes vector $S$

$$S_A = \frac{1}{2} (x_1^2 + p_1^2 - x_2^2 - p_2^2)$$

$$S_B = x_1 p_1 + x_2 p_2$$

$$S_C = x_1 p_2 - x_2 p_1$$

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Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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  - $\varepsilon$-vector and Coulomb $p=mv$ algebra
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Analytic geometry derivation of $\varepsilon$-construction

Algebra of $\varepsilon$-construction geometry

Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

- $(R=-0.375\text{\ ellipsic orbit})$
- $(R=+0.5\text{\ hyperbolic orbit})$
Geometry of Coulomb orbits (Let: $r = \rho$ here)

\[
\begin{align*}
\frac{r}{\varepsilon} &= \frac{\lambda}{\varepsilon} + r \cos \phi \\
r &= \lambda + r \varepsilon \cos \phi \\
r &= \frac{\lambda}{1 - \varepsilon \cos \phi}
\end{align*}
\]

(Review of Lect. 28-29)

\[
\frac{1}{r} = \frac{1 - \varepsilon \cos \phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon \cos \phi}{\lambda}
\]

All conics defined by:

**Defining eccentricity** $\varepsilon$

Distance to Focal-point $= \varepsilon \cdot$ Distance to Directrix-line

\[
\begin{align*}
\rho &= \frac{\lambda}{(1 + \varepsilon)} \text{ perhelion} \\
\rho &= \frac{\lambda}{1 - \varepsilon} \text{ aphelion}
\end{align*}
\]

**Major axis:** $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

**Focal axis:** $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\varepsilon\lambda/|1-\varepsilon^2|$

**Minor radius:** $b = \sqrt{a^2 - a^2\varepsilon^2} = \sqrt{a\lambda}$ (ellipse: $\varepsilon < 1$)

$\epsilon^2 = 1 - \frac{b^2}{a^2}$ (ellipse: $\varepsilon < 1$) \quad \frac{b^2}{a^2} = \sqrt{1 - \varepsilon^2}$

$\epsilon^2 = 1 + \frac{b^2}{a^2}$ (hyperbola: $\varepsilon > 1$) \quad \frac{b^2}{a^2} = \sqrt{\varepsilon^2 - 1}$

**Parameters**

\[
\begin{align*}
(x, y) &\quad \text{physical parameters} \\
(r, \phi) &\quad \text{parameters}
\end{align*}
\]

\[
\begin{align*}
a &= \frac{k}{2E} \\
E &= \frac{k}{2a} \\
b &= \frac{L}{\sqrt{2m \cdot |E|}} \\
L &= \sqrt{km\lambda}
\end{align*}
\]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\varepsilon$-vector and Coulomb $p = mv$ geometry

$\varepsilon$-vector and Coulomb $p = mv$ algebra

Example with elliptical orbit

Analytic geometry derivation of $\varepsilon$-construction

Algebra of $\varepsilon$-construction geometry

Connection formulas for $(a, b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R = -0.375 \text{ elliptic orbit})$

$(R = +0.5 \text{ hyperbolic orbit})$
Dot product of $\epsilon$ with momentum vector $p$:
\[
\epsilon \cdot p = \frac{p \cdot r - p \cdot p \times L}{r \text{ km}} = p \cdot \hat{r} = p_r = \epsilon p_x
\]

This says:
"Projection of $p$ onto $r$ is eccentricity $\epsilon$ times projection of $p$ onto $\hat{x}$-axis" \\
($\hat{x} = \hat{\epsilon}$)
Dot product of $\epsilon$ with momentum vector $p$:

$$\epsilon \cdot p = \frac{p \cdot r}{r} \cdot \frac{p \cdot p \times L}{km}$$

$$= p \cdot \hat{r} = p_r = \epsilon p_x$$

This says:

"Projection of $p$ onto $r$ is eccentricity $\epsilon$ times projection of $p$ onto $\hat{x}$-axis"

( $\hat{x} = \hat{\epsilon}$ )

Dual radii $r$ and $r'$ locate Thales rectangles in circles with diameters that are tangent vectors $p$ and $-p$

Bisector of angle between dual radii $r$ and $r'$ is normal to tangent vectors $p$ and $-p$

(Review of Lect. 29)
Dot product of $\varepsilon$ with momentum vector $p$:

$$\varepsilon \cdot p = \frac{p \cdot r - p \cdot p \times L}{r} \text{ km}$$

$$= p \cdot \hat{r} = p_r = \varepsilon p_x$$

This says:
"Projection of $p$ onto $r$ is eccentricity $\varepsilon$ times projection of $p$ onto $\hat{x}$-axis"  
($\hat{x} = \hat{\varepsilon}$)

Hyperbola has eccentricity $\varepsilon > 1$

(Here: $\varepsilon = 5/4 = 1.25$)
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\varepsilon$-vector and Coulomb $p = mv$ geometry

$\varepsilon$-vector and Coulomb $p = mv$ algebra

Example with elliptical orbit

Analytic geometry derivation of $\varepsilon$-construction

Algebra of $\varepsilon$-construction geometry

Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R = -0.375$ elliptic orbit$)$

$(R = +0.5$ hyperbolic orbit$)$
\( \varepsilon \)-vector and Coulomb \( \mathbf{p} = m\mathbf{v} \) geometry (Review of Lect. 29 p.50-62)

Finding time derivatives of orbital coordinates \( r, \phi, x, y, \) and eventually velocity \( \mathbf{v} \) or momentum \( \mathbf{p} = m\mathbf{v} \)

Radius \( r \):
\[
\lambda = \frac{L^2}{1 - \varepsilon \cos \phi} \quad \frac{L^2}{km}
\]
\[
r = \frac{dr}{dt} = \frac{L^2}{km} \frac{d}{dt}(-\varepsilon \cos \phi) \quad \frac{L^2}{km} \quad \frac{d}{dt}(1 - \varepsilon \cos \phi)^2
\]
\[
\dot{r} = \frac{L^2}{km} \frac{-\varepsilon \sin \phi \dot{\phi}}{(1 - \varepsilon \cos \phi)^2}
\]
\[
\dot{r} = -\frac{L^2}{km} \left( \frac{km}{L^2} \right)^2 r^2 \dot{\phi} \varepsilon \sin \phi
\]
\[
\dot{r} = -\frac{k}{L^2} m r^2 \dot{\phi} \varepsilon \sin \phi = -\frac{k}{L} \varepsilon \sin \phi
\]

Cartesian \( x = r \cos \phi \):
\[
\dot{x} = \frac{dx}{dt} = \dot{r} \cos \phi - \sin \phi \ r \dot{\phi}
\]
\[
= -\frac{k}{L} \sin \phi
\]

Velocity:
\[
p_x = m \dot{x} = -\frac{mk}{L} \sin \phi
\]

Momentum:
\[
p_y = m \dot{y} = \frac{mk}{L} (\cos \phi - \varepsilon)
\]

Polar angle \( \phi \) using:
\[
L = m r^2 \frac{d \phi}{dt} = m r^2 \dot{\phi}
\]
\[
\dot{\phi} = \frac{L}{m r^2} = \frac{L}{m} \frac{1}{r^2} = \frac{L}{m} \left( \frac{km}{L^2} \right)^2 (1 - \varepsilon \cos \phi)^2
\]
\[
r \dot{\phi} = \frac{L}{mr} = \frac{L}{m} \frac{1}{r} = \frac{L}{m} \left( \frac{km}{L^2} \right) (1 - \varepsilon \cos \phi) = \frac{k}{L} (1 - \varepsilon \cos \phi)
\]
\[\text{using:} \quad \frac{1}{r^2} = \left( \frac{km}{L^2} \right)^2 (1 - \varepsilon \cos \phi)^2 \]
\[
\text{using:} \quad \frac{1}{(1 - \varepsilon \cos \phi)^2} = \left( \frac{km}{L^2} \right)^2 r^2
\]
\[\text{again using:} \quad L = m r^2 \dot{\phi} \]

Cartesian \( y = r \sin \phi \):
\[
\dot{y} = \frac{dy}{dt} = \dot{r} \sin \phi + \cos \phi \ r \dot{\phi}
\]
\[
= \frac{k}{L} (\cos \phi - \varepsilon)
\]

\[\text{Velocity:} \]
\[\text{Momentum:} \]

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Eccentricity vector $\mathbf{e}$ and $(\mathbf{e}, \lambda)$-geometry of orbital mechanics

$\mathbf{e}$-vector and Coulomb $\mathbf{r}$-orbit geometry

Review and connection to standard development

$\mathbf{e}$-vector and Coulomb $\mathbf{p}=m\mathbf{v}$ geometry

$\mathbf{e}$-vector and Coulomb $\mathbf{p}=m\mathbf{v}$ algebra

Example with elliptical orbit

Analytic geometry derivation of $\mathbf{e}$-construction

Algebra of $\mathbf{e}$-construction geometry

Connection formulas for $(a,b)$ and $(\mathbf{e}, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\mathbf{e}$-vector and orbits

$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
\( \boldsymbol{p} = \mathbf{m} \mathbf{v} \) geometry \( (\phi=0) \)

\( p \) is smallest at apogee

\begin{align*}
\lambda &= 1/2 \\
\varepsilon &= \sqrt{3}/2 = 0.866 \\
a &= 2 \\
b &= 1 \\
r &= \frac{\lambda}{1-\varepsilon} \\
r' &= \frac{\lambda}{1+\varepsilon} \\
r &= \frac{\sqrt{3}}{2} \\
R &= \text{Kinetic Energy} \\
\phi &= 0 \\
p &= \text{Coulomb Potential Energy}
\end{align*}
\[ r = \frac{KE}{PE} \]

scale line
bisects
angle \( \hat{\gamma} \) = \( \pi - 2\gamma \)
\( \hat{R} \) is \( \perp \) to \( p \)

\[ p \] momentum line
bisects
angle \( \hat{\gamma} \) = \( 2\gamma \)

Example of geometry for momentum functions:

\[ P_x = m\dot{x} = -\frac{mk}{L} \sin \phi \]

and

\[ P_y = m\dot{y} = \frac{k}{L} (\cos \phi - \varepsilon) \]

Note similarity of \((R, r)\)-triangle in \( r \)-circle of radius \( r \) to that in \( p \)-circle of diameter \( p \) above.

0 > \( R = \frac{KE}{PE} > -1 \) scale subtends angle \( 2\gamma \) with length \( 2r \sin \gamma \) as is derived several pages ahead.
Example of geometry for momentum functions:

\[ P_x = m \dot{x} = -\frac{mk}{L} \sin \phi \]

and

\[ P_y = m \dot{y} = \frac{k}{L} (\cos \phi - \varepsilon) \]
Example of geometry for momentum functions:

\[ P_x = m\dot{x} = -\frac{mk}{L} \sin \phi \]

and

\[ P_y = m\dot{y} = \frac{k}{L} (\cos \phi - \varepsilon) \]
Coulomb $\mathbf{p} = m\mathbf{v}$ geometry ($\phi > 0$)

$\mathbf{p}$ grows as $r$ falls from aphelion

$p$-circle grows as $r$-circle shrinks

$R = \frac{KE}{PE}$ scale line

bisects angle $\gamma + \dot{r} = \pi - 2\gamma$

$\dot{R}$ is $\perp$ to $\mathbf{p}$

$p$ momentum line

bisects angle $\gamma + \dot{r} = 2\gamma$

$p = 2$

$b = 1$

$\varepsilon = \sqrt{3}/2$

$= 0.866$

$\lambda = 1/2$

Example of geometry for momentum functions:

$P_x = m\dot{x} = -\frac{mk}{L} \sin \phi$

and

$P_y = m\dot{y} = \frac{k}{L} (\cos \phi - \varepsilon)$

$0 > R = KE/PE > -1$ scale subtends angle $2\gamma$ with length $2r \sin \gamma$ as is derived several pages ahead.

Note similarity of $(R, r)$-triangle in $r$-circle of radius $r$ to that in $p$-circle of diameter $p$ above.
Example of geometry for momentum functions:

\[ P_x = m \dot{x} = -\frac{mk}{L} \sin \phi \]

and

\[ P_y = m \dot{y} = \frac{k}{L} (\cos \phi - \varepsilon) \]

Note similarity of \((\mathbf{R}, r)\)-triangle in \(r\)-circle of radius \(r\) to that in \(p\)-circle of diameter \(p\) above.
\[ a = 2 \]

\[ b = 1 \]

\[ \epsilon = \sqrt{3}/2 = 0.866 \]

\[ \lambda = 1/2 \]

\[ y = a(\cos \phi - \epsilon) = -a \epsilon \phi \]

\[ x = -a \sin \phi = -a \]

\[ \dot{r}'' = a \epsilon \]

\[ \gamma = \theta + \phi \]

\[ \text{perhelion} \]

\[ \lambda/(1+\epsilon) \]

\[ \text{focal radius} \]

\[ a \epsilon \]

\[ \text{aphelion} \]

\[ \lambda/(1-\epsilon) \]

\[ R = \frac{KE}{PE} \]

\[ \text{scale line} \]

\[ \text{bisects} \]

\[ \text{angle} \quad \alpha_{+}^{r} = \pi - 2 \gamma \]

\[ \hat{R} \text{ is } \perp \text{ to } p \]

\[ \text{p momentum line} \]

\[ \text{bisects} \]

\[ \text{angle} \quad \alpha_{-}^{r} = 2 \gamma \]

\[ 0 > R = KE/PE > -1 \]

scale subtends angle \( 2\gamma \) with length \( 2r \sin \gamma \) as is derived several pages ahead.

Note similarity of \((R, r)\)-triangle in \(r\)-circle of radius \(r\) to that in \(p\)-circle of diameter \(p\) above.

Example of geometry for momentum functions:

\[ P_x = m\dot{x} = -\frac{mk}{L} \sin \phi \]

and

\[ P_y = m\dot{y} = \frac{k}{L} (\cos \phi - \epsilon) \]
Example of geometry for momentum functions:

\[ P_x = m \ddot{x} = -\frac{mk}{L} \sin \phi \]

\[ P_y = m \ddot{y} = \frac{k}{L} (\cos \phi - \varepsilon) \]

Note similarity of \((R, r)\)-triangle in \(r\)-circle of radius \(r\) to that in \(p\)-circle of diameter \(p\) above.
Coulomb $p = mv$ geometry ($\phi > 0$)

$p$ grows as $r$ falls from apogee

$p$-circle grows as $r$-circle shrinks

$R = \frac{KE}{PE}$ scale line

bisects

angle $\phi_+ \phi_-' = \pi - 2\gamma$

$\vec{R}$ is $\perp$ to $p$

Example of geometry

for momentum functions:

$P_x = m\dot{x} = \frac{-mk}{L} \sin \phi$

and

$P_y = m\dot{y} = \frac{k}{L} (\cos \phi - \varepsilon)$

Note similarity of $(R, r)$-triangle in $r$-circle of radius $r$ to that in $p$-circle of diameter $p$ above.
**Example of geometry for momentum functions:**

\[ P_x = m\dot{x} = -\frac{mk}{L} \sin \phi \]

and

\[ P_y = m\dot{y} = \frac{k}{L} (\cos \phi - \varepsilon) \]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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- Ruler & compass construction of $\varepsilon$-vector and orbits

- $(R=-0.375$ elliptic orbit$)$
- $(R=+0.5$ hyperbolic orbit$)$
**ε-vector** and Coulomb orbit construction steps

Pick launch point \( P \) (radius vector \( r \ )) and elevation angle \( γ \) from radius (momentum initial \( p \) direction )

Copy F-center circle around launch point \( P \)
Copy elevation angle \( γ (\angle FP′) \) onto \( \angle P′PQ \)
Extend resulting line \( QP′Q' \) to make focus locus

Reason for focus locus:
Line \( r \) from 1st focus \( F \) “reflects” off line \( p \) (or \( P′P \)) toward 2nd focus \( F′ \) somewhere so incident-angle \( γ \) equals reflected-angle \( γ \)

Pick initial \( R = KE/PE \) value (here \( R = -3/8 \)) Draw \( ε \)-vector from focus \( F \) to \( R \)-point and beyond to 2nd focus \( F′ \)

Copy double angle \( 2γ (\angle FPQ) \) onto \( \angle PFT \)
Extend \( \angle PFT \) chord \( PT \) to make \( R \)-ratio scale line
Label chord \( PT \) with \( R = 0 \) at \( P \) and \( R = -1.0 \) at \( T \).
Mark \( R \)-line fractions \( R = 0, +1/4, +1/2, \ldots \) above \( P \) and \( R = 0, -1/8, -1/4, -1/2, \ldots, -3/4 \) below \( P \) and \( -5/4, -3/2, \ldots \) below \( T \).

Mark \( R \)-line fractions \( R = 0, +1/4, +1/2, \ldots \) above \( P \) and \( R = 0, -1/8, -1/4, -1/2, \ldots, -3/4 \) below \( P \) and \( -5/4, -3/2, \ldots \) below \( T \).

\[
R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} - \frac{k}{r(0)}
\]

\[
= \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}
\]

Focus \( F \) and 2nd focus \( F′ \) allow final construction of orbital trajectory.
Here it is an \( R = -3/8 \) ellipse.
(Detailed Analytic geometry of \( ε \)-vector follows.)
Fig. 5.4.2 Construction of eccentricity vector $\epsilon$ and orbit from initial $\mathbf{r}$, $\mathbf{p}$ with $KE/PE=-3/8$. Next several pages give step-by-step constructions of $\epsilon$-vector and Coulomb orbit and trajectory physics.
**ε-vector and Coulomb orbit construction steps**

Pick launch point $\mathbf{P}$

(radius vector $\mathbf{r}$)

and elevation angle $\gamma$ from radius

(momentum initial $\mathbf{p}$ direction)

Next several pages give step-by-step constructions of $\varepsilon$-vector and Coulomb orbit and trajectory physics
**ε-vector and Coulomb orbit construction steps**

Pick launch point \( P \)
(radius vector \( r \))
and elevation angle \( \gamma \) from radius
(momentum initial \( p \) direction)

Copy \( F \)-center circle around launch point \( P \)
Copy elevation angle \( \gamma (\angle FPP') \) onto \( \angle P'PQ \)
Extend resulting line \( PQP' \) to make focus locus

Reason for focus locus:
Line \( r \) from 1st focus \( F \) "reflects" off
line \( p \) (or \( P'P \)) toward 2nd focus \( F' \) somewhere
so incident-angle \( \gamma \) equals reflected-angle \( \gamma \)

Next several pages give step-by-step constructions of **ε-vector** and Coulomb orbit and trajectory physics
**ε-vector** and Coulomb orbit construction steps

Pick launch point $P$ (radius vector $r$) and elevation angle $\gamma$ from radius (momentum initial $p$ direction)

Copy F-center circle around launch point $P$

Copy elevation angle $\gamma (\angle FP'P)$ onto $\angle P'PQ$

Extend resulting line $QPQ'$ to make **focus locus**

Reason for focus locus: Line $r$ from 1st focus $F$ "reflects" off line $p$ (or $P'P$) toward 2nd focus $F'$ somewhere so incident-angle $\gamma$ equals reflected-angle $\gamma$

Copy double angle $2\gamma (\angle FPQ)$ onto $\angle PFT$

Extend $\angle PFT$ chord $PT$ to make $R$-ratio scale line

Label chord $PT$ with $R=0$ at $P$ and $R=-1.0$ at $T$.

Mark $R$-line fractions $R=0, +1/4, +1/2,...$ above $P$ and $R=0, -1/8,-1/4,-1/2,...,-3/4$ below $P$ and $-5/4,-3/2,...$ below $T$.

Copy $F$-center circle around launch point $P$

Copy elevation angle $\gamma (\angle FPP')$ onto $\angle P'PQ$

Extend resulting line $QPQ'$ to make **focus locus**

Copy $F$-center circle around launch point $P$

Copy elevation angle $\gamma (\angle FP'P)$ onto $\angle P'PQ$

Extend resulting line $QPQ'$ to make **focus locus**

Line $r$ from 1st focus $F$ "reflects" off line $p$ (or $P'P$) toward 2nd focus $F'$ somewhere so incident-angle $\gamma$ equals reflected-angle $\gamma$
**ε-vector** and Coulomb orbit construction steps

Pick launch point \( P \) (radius vector \( r \ ))
and elevation angle \( \gamma \) from radius (momentum initial \( p \) direction )

Copy F-center circle around launch point \( P \)
Copy elevation angle \( \gamma (\angle FP'P') \) onto \( \angle P'PQ \)
Extend resulting line \( QPQ' \) to make **focus locus**

Copy double angle \( 2\gamma (\angle FPQ) \) onto \( \angle PFT \)
Extend \( \angle PFT \) chord \( PT \) to make \( R \)-**ratio scale line**
Label chord \( PT \) with \( R=0 \) at \( P \) and \( R=-1.0 \) at \( T \).
Mark \( R \)-line fractions \( R=0, +1/4, +1/2,... \) above \( P \) and \( R=0, -1/8,-1/4,-1/2,...,-3/4 \) below \( P \) and \( -5/4,-3/2,... \) below \( T \).

Reason for **focus locus**:
Line \( r \) from 1st focus \( F \) “reflects” off line \( p \) (or \( P'P \)) toward 2nd focus \( F' \) somewhere so incident-angle \( \gamma \) equals reflected-angle \( \gamma \)

**ε-vector** and Coulomb orbit construction steps

Copy F-center circle around launch point \( P \)
Copy elevation angle \( \gamma (\angle FP'P') \) onto \( \angle P'PQ \)
Extend resulting line \( QPQ' \) to make **focus locus**

Copy double angle \( 2\gamma (\angle FPQ) \) onto \( \angle PFT \)
Extend \( \angle PFT \) chord \( PT \) to make \( R \)-**ratio scale line**
Label chord \( PT \) with \( R=0 \) at \( P \) and \( R=-1.0 \) at \( T \).
Mark \( R \)-line fractions \( R=0, +1/4, +1/2,... \) above \( P \) and \( R=0, -1/8,-1/4,-1/2,...,-3/4 \) below \( P \) and \( -5/4,-3/2,... \) below \( T \).

\[
R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} \frac{-k}{r(0)}
\]

\[
R = \frac{\text{Initial KE}}{\text{Initial PE}} = \pm \frac{v^2(0)}{v^2(\infty)}
\]
**ε-vector** and Coulomb orbit construction steps

Pick launch point $P$ (radius vector $r$)

and elevation angle $\gamma$ from radius (momentum initial $p$ direction)

Copy F-center circle around launch point $P$

Copy elevation angle $\gamma(\angle FP')$ onto $\angle P'PQ$

Extend resulting line $QPQ'$ to make focus locus

**Reason for focus locus:** Line $r$ from 1st focus $F$ “reflects” off line $p$ (or $P'P$) toward 2nd focus $F'$ somewhere so incident-angle $\gamma$ equals reflected-angle $\gamma$

Copy double angle $2\gamma(\angle FPQ)$ onto $\angle PFT$

Extend $\angle PFT$ chord $PT$ to make $R$-ratio scale line

Label chord $PT$ with $R=0$ at $P$ and $R=-1.0$ at $T$.

Mark $R$-line fractions $R=0,\ +1/4,\ +1/2,\ ...$ above $P$ and $R=0,\ -1/8,\ -1/4,\ -1/2,\ ...,\ -3/4$ below $P$ and $-5/4,\ -3/2,\ ...$ below $T$.

Pick initial $R=KE/PE$ value (here $R=-3/8$)

Draw $\epsilon$-vector from focus $F$ to R-point and beyond to 2nd focus $F'$

**Initial KE**

$\text{Initial PE} = \frac{m v^2(0)}{2} - k / r(0)$

$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0)}{2} / \frac{-k}{r(0)}$

$= \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$

Focus $F$ and 2nd focus $F'$ allow final construction of orbital trajectory.

Here it is an $R=-3/8$ ellipse.

(Detailed Analytic geometry of $\epsilon$-vector follows.)
**ε-vector and Coulomb orbit construction steps**

Pick launch point \( P \) (radius vector \( r \)) and elevation angle \( \gamma \) from radius (momentum initial \( p \) direction)

Copy F-center circle around launch point \( P \)
Copy elevation angle \( \gamma (\angle FPP') \) onto \( \angle P'PQ \)
Extend resulting line \( QPQ' \) to make **focus locus**

Reason for **focus locus**:
Line \( r \) from 1st focus \( F \) “reflects” off line \( p \) (or \( P'P \)) toward 2nd focus \( F' \) somewhere so incident-angle \( \gamma \) equals reflected-angle \( \gamma \)

Copy double angle \( 2\gamma (\angle FPQ) \) onto \( \angle PFT \)
Extend \( \angle PFT \) chord \( PT \) to make **\( R \)-ratio scale line**
Label chord \( PT \) with \( R=0 \) at \( P \) and \( R=-1.0 \) at \( T \).
Mark **\( R \)-line** fractions \( R=0, +1/4, +1/2,... \) above \( P \) and \( R=0, -1/8, -1/4, -1/2,..., -3/4 \) below \( P \) and \( -5/4, -3/2,... \) below \( T \).

**Pick initial \( R=KE/PE \) value**
(here \( R=+1/2 \)) Draw **ε-vector** from focus \( F \) to \( R \)-point
(Here it intersects 2nd focus \( F' \)

**Focus F and 2nd focus F' allow final construction of orbital trajectory.**
Here it is an \( R=+1/2 \) hyperbola.

(Detailed Analytic geometry of \( ε \)-vector follows.)

\[
R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} \quad \frac{-k}{r(0)}
\]

\[
= \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}
\]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

$\varepsilon$-vector and Coulomb $r$-orbit geometry

Review and connection to standard development

$\varepsilon$-vector and Coulomb $p=mv$ geometry

$\varepsilon$-vector and Coulomb $p=mv$ algebra

Example with elliptical orbit

Analytic geometry derivation of $\varepsilon$-construction

$\blacktriangleright$ Algebra of $\varepsilon$-construction geometry

Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R = -0.375$ elliptic orbit$)$

$(R = +0.5$ hyperbolic orbit$)$
Analytic geometry derivation of \( \varepsilon \)-constructions

\[
\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \frac{(mv_0)(mv_0 r_0)\sin \gamma}{km} \hat{L}_p
\]

where: \( \hat{L}_p \equiv p \times L \)

Fig. 5.4.3
Construction of eccentricity vector \( \varepsilon \) and orbit from initial \( r, p \) with \( KE/PE=+1/2 \).

\[
R = \frac{Initial \ KE}{Initial \ PE} = \frac{mv^2(0)}{2} - k / r(0)
\]

\[
= \pm \left( \frac{Initial \ velocity \ \text{Escape velocity}}{v^2(\infty)} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}
\]
Analytic geometry derivation of $\varepsilon$-constructions

Attracted bound ellipses

Attracted unbound hyperbolas

Repelled unbound hyperbolas

Repelled bound ellipses

Fig. 5.4.3

Construction of eccentricity vector $\varepsilon$ and orbit from initial $r, p$ with $KE/PE=+1/2$.

$$\varepsilon = \hat{r} - \frac{p \times L}{km} = \hat{r} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km} \hat{L}_{p\times}$$

where: $L_{p\times} \equiv p \times L$

$$\varepsilon = \hat{r} + 2\sin\gamma \frac{mv_0^2/2}{-k/r_0} \hat{L}_{p\times} = \hat{r} + 2\sin\gamma \frac{KE}{PE} \hat{L}_{p\times}$$

$$R = \frac{Initial KE}{Initial PE} = \frac{mv^2(0)/2}{-k/r(0)}$$

$$= \pm \left( \frac{Initial\ velocity}{Escape\ velocity} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$
The eccentricity vector is:
\[ \mathbf{\varepsilon} = \hat{r} + 2 \sin \gamma \frac{m v_0^2}{k r_0} \hat{\mathbf{L}}_p = \hat{r} + 2 \sin \gamma \frac{KE}{PE} \hat{\mathbf{L}}_p \]

where: \( \mathbf{L}_p \times \equiv \mathbf{p} \times \mathbf{L} \)

The eccentricity vector is:
\[ \mathbf{\varepsilon} = \left( \begin{array}{c} \cos \gamma \\ \sin \gamma \end{array} \right) + 2 \sin \gamma \left( \begin{array}{c} 0 \\ 1 \end{array} \right) R = \left( \begin{array}{c} \cos \gamma \\ (2R+1) \sin \gamma \end{array} \right) \]

Analytic geometry derivation of \( \varepsilon \)-constructions

Attracted bound ellipses
Attracted unbound hyperbolas
Repelled unbound hyperbolas
Repelled bound ellipses

\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0) / 2}{-k / r(0)} = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]
Analytic geometry derivation of $\varepsilon$-constructions

The eccentricity vector is:

$$\varepsilon = \hat{r} + 2 \sin \gamma \frac{mv_0^2/2}{-k/r_0} \hat{L}_{pX} = \hat{r} + 2 \sin \gamma \frac{KE}{PE} \hat{L}_{pX}$$

For: $\gamma = 45^\circ$ and: $R = +\frac{1}{2}$

$$\varepsilon = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}(2R+1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 2/\sqrt{2} \end{pmatrix}$$

$$R = \frac{Initial \ KE}{Initial \ PE} = \frac{mv^2(0)/2}{-k/r(0)}$$

$$= \pm \left( \frac{Initial \ velocity}{Escape \ velocity} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$
Analytic geometry derivation of $\varepsilon$-constructions

\[ \varepsilon = \hat{r} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \hat{r} - \left( \frac{mv_0}{mv_0 r_0} \right) \sin \gamma \hat{L}_{p^x} \]

where: $\mathbf{L}_{p^x} \equiv \mathbf{p} \times \mathbf{L}$

The eccentricity vector is:

\[ \varepsilon = \hat{r} + 2 \sin \gamma \frac{mv_0^2}{-k/r_0} \hat{L}_{p^x} = \hat{r} + 2 \sin \gamma \frac{KE}{PE} \hat{L}_{p^x} \]

For: $\gamma = 45^\circ$ and: $R = + \frac{1}{2}$

\[ \varepsilon = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + 2 \sin \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} R = \begin{pmatrix} \cos \gamma \\ (2R+1) \sin \gamma \end{pmatrix} \]

The eccentricity parameter defined by:

\[ \varepsilon^2 = \cos^2 \gamma + (2R+1)^2 \sin^2 \gamma = 1 + \frac{a^2}{b^2} = 1 + 4(R+1)\sin^2 \gamma = \frac{5}{2} \]

\[ R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{mv^2(0)}{2} \left( -k / r(0) \right) = \pm \left( \frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)} \]
Eccentricity vector $\varepsilon$ and $(\varepsilon, \lambda)$-geometry of orbital mechanics

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Algebra of $\varepsilon$-construction geometry

Connection formulas for $(a, b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

$(R=-0.375 \text{ elliptic orbit})$

$(R=+0.5 \text{ hyperbolic orbit})$
Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1)$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon > 1)$$

Three pairs of parameters for Coulomb orbits:
1. Cartesian $(a,b)$, 2. Physics $(E,L)$, 3. Polar $(\varepsilon, \lambda)$

Now we relate a 4th pair: 4. Initial $(\gamma, R)$
Algebra of $\varepsilon$-construction geometry

The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$

= $1 - \frac{b^2}{a^2}$ for ellipse ($\varepsilon < 1$) where: $4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$

= $1 + \frac{b^2}{a^2}$ for hyperbola ($\varepsilon > 1$) where: $4R(R+1)\sin^2\gamma = \frac{b^2}{a^2} = \varepsilon^2 - 1$

Three pairs of parameters for Coulomb orbits:
1. Cartesian $(a, b)$, 2. Physics $(E, L)$, 3. Polar $(\varepsilon, \lambda)$

Now we relate a 4th pair: 4. Initial $(\gamma, R)$
The eccentricity parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$.

$\epsilon^2 = 1 + 4R(R+1)\sin^2 \gamma$

$= 1 - \frac{b^2}{a^2}$ for ellipse $(\epsilon < 1)$ where: $4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \epsilon^2 - 1$ implying: $R(R+1) < 0$

$= 1 + \frac{b^2}{a^2}$ for hyperbola $(\epsilon > 1)$ where: $4R(R+1)\sin^2 \gamma = +\frac{b^2}{a^2} = \epsilon^2 - 1$ implying: $R(R+1) > 0$
Algebra of ε-construction geometry

The **eccentricity** parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

Three pairs of parameters for Coulomb orbits:
1. Cartesian $(a,b)$, 2. Physics $(E,L)$, 3. Polar $(\varepsilon, \lambda)$

Now we relate a 4th pair: 4. Initial $(\gamma, R)$

\[ \varepsilon^2 = 1 + 4R(R+1)\sin^2 \gamma \]

- For ellipse $(\varepsilon < 1)$ where: $4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $-R^2 > R$)  
  (or: $0 > R > -1$)

- For hyperbola $(\varepsilon > 1)$ where: $4R(R+1)\sin^2 \gamma = \frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $-R^2 < R$)  
  (or: $0 < R < -1$)
The eccentricity parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

\[ \varepsilon^2 = 1 + 4R(R+1)\sin^2 \gamma \]

\[ = 1 - \frac{b^2}{a^2} \text{ for ellipse } (\varepsilon < 1) \text{ where: } 4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ implying: } R(R+1) < 0 \] (or: $R^2 > R$)

\[ = 1 + \frac{b^2}{a^2} \text{ for hyperbola } (\varepsilon > 1) \text{ where: } 4R(R+1)\sin^2 \gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1 \text{ implying: } R(R+1) > 0 \] (or: $R^2 < R$)

Total $-\frac{k}{2a} = E = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

\[ -\frac{k}{2a} = E = KE + PE = R \cdot PE + PE = (R+1)PE = (R+1) \frac{-k}{r} \text{ or: } \frac{1}{2a} = \frac{1}{r} = R+1 \]
The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$ to individual radii $a$, $b$, and $\lambda$.

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

- For ellipse ($\varepsilon < 1$) where: $4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $-R^2 > R$)
- For hyperbola ($\varepsilon > 1$) where: $4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $-R^2 < R$)

Total $\frac{-k}{2a} = E = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

$$\frac{-k}{2a} = E = KE + PE = R \cdot PE + PE = (R + 1)PE = \frac{-(R+1)}{r} a$$

or: $$\frac{1}{2a} = \frac{1}{r} = \frac{(R+1)}{r}$$

$$a = \frac{r}{2(R+1)} = \left(\frac{1}{2(R+1)}\right)$$ assuming *unit* initial radius ($r = 1$).
The *eccentricity* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

\[ \epsilon^2 = 1 + 4R(R+1)\sin^2 \gamma \]

- for ellipse ($\epsilon < 1$) where: \(4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \epsilon^2 - 1\) implying: $R(R+1) < 0$ (or: $R^2 > R$)
- for hyperbola ($\epsilon > 1$) where: \(4R(R+1)\sin^2 \gamma = \frac{b^2}{a^2} = \epsilon^2 - 1\) implying: $R(R+1) > 0$ (or: $-R^2 < R$)

Total \(\frac{-k}{2a} = E = KE + PE\) relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

\(\frac{-k}{2a} = E = KE + PE = R \cdot PE + PE = (R + 1)PE = \frac{R}{r} - k\) or: \(1 = \frac{R}{2a} \frac{1}{r} = (R + 1)\)

\[ a = \frac{r}{2(R+1)} = \frac{1}{2(R+1)} \text{ assuming unit initial radius } (r\equiv 1). \]

\(4R(R+1)\sin^2 \gamma = \pm \frac{b^2}{a^2}\) implies: \(2\sqrt{R(R+1)}\sin \gamma = \frac{b}{a}\) or: \(b = 2a\sqrt{R(R+1)}\sin \gamma\)

\[ b = r\sqrt{\frac{\pm R}{R+1}}\sin \gamma = \sqrt{\frac{\pm R}{R+1}}\sin \gamma \text{ assuming unit initial radius } (r\equiv 1) \]
The **eccentricity** parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

\[ \varepsilon^2 = 1 + 4R(R+1)\sin^2 \gamma \]

- for ellipse ($\varepsilon < 1$) where: $4R(R+1)\sin^2 \gamma = -\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) < 0$ (or: $-R^2 > R$)
- for hyperbola ($\varepsilon > 1$) where: $4R(R+1)\sin^2 \gamma = +\frac{b^2}{a^2} = \varepsilon^2 - 1$ implying: $R(R+1) > 0$ (or: $-R^2 < R$)

Total $\frac{-k}{2a} = E = KE + PE$ relates ratio $R = \frac{KE}{PE}$ to individual radii $a$, $b$, and $\lambda$.

\[ \frac{-k}{2a} = E = KE + PE = R \cdot PE + PE = (R + 1)PE = (R + 1) \frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a} = (R + 1) \frac{1}{r} \]

\[ a = \frac{r}{2(R+1)} = \left( \frac{1}{2(R+1)} \right) \quad \text{assuming unit initial radius ($r \equiv 1$).} \]

\[ 4R(R+1)\sin^2 \gamma = \mp \frac{b^2}{a^2} \quad \text{implies:} \quad 2\sqrt{\mp R(R+1)} \sin \gamma = \frac{b}{a} \quad \text{or:} \quad b = 2a\sqrt{\mp R(R+1)} \sin \gamma \]

\[ b = r \sqrt{\mp R \over R+1} \sin \gamma = \sqrt{\mp R \over R+1} \sin \gamma \quad \text{assuming unit initial radius ($r \equiv 1$).} \]

**Latus radius** is similarly related:

\[ \lambda = \frac{b^2}{a} = \mp 2r R \sin^2 \gamma \]
Algebra of $\varepsilon$-construction geometry

The 

\[
\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma
\]

\[
= 1 - \frac{b^2}{a^2} \text{ ellipse (}\varepsilon < 1) \quad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2}
\]

\[
= 1 + \frac{b^2}{a^2} \text{ hyperbola (}\varepsilon > 1) \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2}
\]

Now we relate a 4th pair: 4. Initial $(\gamma, R)$

\[
a = \frac{r}{2(R+1)} \left( \frac{1}{2(R+1)} \right) \quad \text{assuming } \textbf{unit} \text{ initial radius } (r \equiv 1).
\]

\[
b = r \sqrt{\frac{\mp R}{R+1} \sin \gamma} \left( \frac{\sqrt{\mp R}}{R+1} \sin \gamma \right) \quad \text{assuming } \textbf{unit} \text{ initial radius } (r \equiv 1)
\]

Latus radius is similarly related:

\[
\lambda = \frac{b^2}{a} = \mp 2rR\sin^2\gamma
\]

From $\varepsilon^2$ result (at top):

\[
\frac{b}{a} = 2\sqrt{\mp R(R+1)\sin \gamma} = \sqrt{\pm(1 - \varepsilon^2)}
\]
Eccentricity vector $\varepsilon$ and $(\varepsilon,\lambda)$-geometry of orbital mechanics

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Connection formulas for $(a,b)$ and $(\varepsilon,\lambda)$ with $(\gamma,R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

$\Rightarrow$  
$(R=-0.375$  elliptic orbit)  
$(R=+0.5$  hyperbolic orbit)
Extend FP to make major axis sum \( FPP' : (r + r' = 2a) \) at \( P' \) intersect of \( r' \)-arc \( F'P' \).

\( R = -3/8 \) elliptic orbit construction

\( \gamma = 45^\circ \)
Strike radius-\( r \) arc about point \( P' \) to intersect original radius-\( r \) circle about focus \( F \) at ends of bisection line \( BB' \). Draw radius-\( a \) circle at \( F \) tangent to bisection line \( BB' \).

\[ R = -\frac{3}{8} \text{ elliptic orbit construction} \]

\[ \gamma = 45^\circ \]
Strike radius-\( r \) arc about point \( P' \) to intersect original radius-\( r \) circle about focus \( F \) at ends of bisection line \( BB' \). Draw radius-\( a \) circle at tangent to bisection line \( BB' \).

Draw radius-\( a \) circle at \( F' \)

Draw radius-\( a \) and radius-\( b \) circles at \( O \) (Center of bisection line (\( \pm b \)).

\[ R = \frac{-3}{8} \] elliptic orbit construction

\[ R = \frac{-3}{8} \]

\[ \gamma = 45^\circ \]
strike radius-r arc about point P to intersect original radius-r circle about focus F at ends of bisection line BB'.

Draw radius-a circle at F'.

Draw radius-a and radius-b circles at O (Center of bisection line (±b)).

R=-3/8 elliptic orbit construction

γ=45°
Draw radius-\(a\) circle at \(F'\)

Draw radius-\(a\) and radius-\(b\) circles at \(O\)  
(Center of bisection line (\(\pm b\)).) Do (\(a, b\))-ellipse construction.

\[
\varepsilon = \sqrt{1 + 4R(R+1)\sin^2\gamma} = \frac{\sqrt{34}}{8} = .73
\]

\[
a = \frac{1}{2(R+1)} = \frac{4}{5}
\]

\[
b = \sqrt{\frac{R}{R+1}} \sin\gamma = \frac{3}{\sqrt{10}} = .54
\]

\[
\lambda = \frac{b^2}{a} = 2R \sin^2\gamma = \frac{3}{8} = .375
\]

\[
\frac{b}{a} = 2\sqrt{R(R+1)\sin^2\gamma} = \tan 34^\circ
\]
Eccentricity vector $\varepsilon$ and ($\varepsilon, \lambda$)-geometry of orbital mechanics

- $\varepsilon$-vector and Coulomb $r$-orbit geometry
- Review and connection to standard development
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- $\varepsilon$-vector and Coulomb $p=mv$ algebra
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Connection formulas for $(a,b)$ and $(\varepsilon, \lambda)$ with $(\gamma, R)$

Ruler & compass construction of $\varepsilon$-vector and orbits

- $(R=-0.375 \text{ elliptic orbit})$
- $(R=+0.5 \text{ hyperbolic orbit})$
Major diameter $2a$ is difference $(r-r' = 2a)$.
Major radius $a$ is half of difference $(r-r')/2 = a$
Major diameter $2a$ needs to be centered on $F-F'$ focal axis

$R = +1/2$ hyperbolic orbit construction

$\gamma = 45^\circ$
Major diameter $2a$ is difference $(r-r'=2a)$.
Major radius $a$ is half of difference $(r-r')/2=a$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F$-$P$ radius $r$ using $F$-$P$ circle intersections to define $r/2$ sections.

$$R = \frac{1}{2} \text{ hyperbolic orbit construction}$$

$$\gamma = 45^\circ$$
Major diameter $2a$ is difference $(r-r') = 2a$.
Major radius $a$ is half of difference $(r-r')/2 = a$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$.

\[ R=+\frac{1}{2} \text{ hyperbolic orbit construction} \]
\[ \gamma=45^\circ \]
Major diameter $2a$ is difference $(r-r' = 2a)$.
Major radius $a$ is half of difference $(r-r')/2 = a$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.
1. Bisect $F$-$P$ radius $r$ using $F$-$P$ circle intersections to define $r/2$ sections.
2. Bisect $F$-$F'$ focal axis using $F$-$F'$ circle intersections to locate orbit center $C$.
3. Bisect $F'$-$P$ radius $r'$ using $F'$-$P$ circle intersections.

$R = +1/2$ hyperbolic orbit construction

$\gamma = 45^\circ$
Major diameter $2a$ is difference $(r-r'=2a)$.
Major radius $a$ is half of difference $(r-r')/2 = a$.
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F$-$P$ radius $r$ using $F$-$P$ circle intersections to define $r/2$ sections.
2. Bisect $F$-$F'$ focal axis using $F$-$F'$ circle intersections to locate orbit center $C$.
3. Bisect $F'$-$P$ radius $r'$ using $F'$-$P$ circle intersections.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$. 

$C \frac{r}{2} a = \frac{(r-r')}{2}$

$KE_{PE} + 0.50$

$\gamma = 45^\circ$

Wednesday, December 24, 2014
Major diameter $2a$ is difference $(r-r'=2a)$.

Major radius $a$ is half of difference $(r-r')/2=a$.

Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$.
5. Copy circle of major radius $a=(r-r')/2$ about orbit center $C$.

$\varepsilon$
Major diameter \(2a\) is difference \((r-r')=2a\).
Major radius \(a\) is half of difference \((r-r')/2=a\)
Major diameter \(2a\) needs to be centered on \(F-F'\) focal axis

1. Bisect \(F-P\) radius \(r\) using \(F-P\) circle intersections to define \(r/2\) sections.
2. Bisect \(F-F'\) focal axis using \(F-F'\) circle intersections to locate orbit center \(C\).
3. Bisect \(F'-P\) radius \(r'\) using \(F'-P\) circle intersections.
4. Swing radius \(r'/2\) onto \(r/2\) section to make major radius \(a=(r-r')/2\).
5. Copy circle of major radius \(a=(r-r')/2\) about orbit center \(C\).
6. Draw focal circle of diameter \(2a\) about orbit center \(C\).
Major diameter $2a$ is difference $(r-r'=2a)$.
Major radius $a$ is half of difference $(r-r')/2=a$
Major diameter $2a$ needs to be centered on $F-F'$ focal axis.

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
2. Bisect $F-F'$ focal axis using $F-F'$ circle intersections to locate orbit center $C$.
4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a=(r-r')/2$.
5. Copy circle of major radius $a=(r-r')/2$ about orbit center $C$.
6. Draw focal circle of diameter $2a\varepsilon$ about orbit center $C$.
7. Erect minor radius $b$ tangent to $a$-circle from point $a$ on $C\varepsilon$-axis to point $b$ on focal circle.
Major diameter $2a$ is difference $(r-r' = 2a)$.  
Major radius $a$ is half of difference $(r-r')/2 = a$.

Major diameter $2a$ needs to be centered on $F-F'$ focal axis:

1. Bisect $F-P$ radius $r$ using $F-P$ circle intersections to define $r/2$ sections.
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4. Swing radius $r'/2$ onto $r/2$ section to make major radius $a = (r-r')/2$.
5. Copy circle of major radius $a = (r-r')/2$ about orbit center $C$.
6. Draw focal circle of diameter $2a$ about orbit center $C$.
7. Erect minor radius $b$ tangent to $a$-circle from point $a$ on $C\varepsilon$-axis to point $b$ on focal circle.
8. Complete orbit $a$-$\mathbf{x}$-$b$ box between focal circle and $a$-circle and its diagonal asymptotes.

$R = +1/2$ hyperbolic orbit construction

$R = +1/2$

$\gamma = 45^\circ$

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9. Draw section of hyperbolic orbit.)
Construction based on: $r-r'=2a$ or: $r'=r-2a$

$1^{st}$ draw an $r$-arc about focus $F$. 

$9.~\text{Draw section of hyperbolic orbit.}$
9. Draw section of hyperbolic orbit.

Construction based on: $r-r' = 2a$ or $r' = r-2a$

1. Draw an r-arc about focus F.
2. Set compass to $(r-2a)$ using r-arc-minus-2a on Cε-line.

$R = +1/2$ hyperbolic orbit construction

$R = +1/2$

$γ = 45°$
9. Draw section of hyperbolic orbit.

Construction based on: $r-r'=2a$ or: $r'=r-2a$

1st draw an $r$-arc about focus $F$.

2nd set compass to $(r-2a)$ using $r$-arc-minus-$2a$ on $C\varepsilon$-line.

3rd draw $(r-2a)$-arc about focus $F'$. 

$R=+1/2$ hyperbolic orbit construction

$\gamma=45^\circ$
9. Draw section of hyperbolic orbit.

Construction based on: \( r - r' = 2a \) or: \( r' = r - 2a \)

1st draw an \( r \)-arc about focus \( F \).

2nd set compass to \( (r - 2a) \) using \( r \)-arc-minus-\( 2a \) on \( C\epsilon \)-line.

3rd draw \( (r - 2a) \)-arc about focus \( F' \). Orbit points at intersections.
9. Draw section of hyperbolic orbit.

R = +1/2 hyperbolic orbit construction

\( \gamma = 45^\circ \)
9. Draw section of hyperbolic orbit.

\[ \gamma = 45^\circ \]

\[ R = +1/2 \]

**R** positive half hyperbolic orbit construction
9. Draw section of hyperbolic orbit.

\[ KE_{PE} + 0.50 \]

\[ +0.25 \]

\[ 0.0 \]

\[ -0.50 \]

\[ -1.0 \]

\[ R = +1/2 \]

\[ \gamma = 45^\circ \]

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9. Draw section of hyperbolic orbit.

\[ \varepsilon = \sqrt{1 + 4R(R+1)\sin^2\gamma} = \sqrt{\frac{3}{2}} = 1.58 \]

\[ a = \frac{1}{2(R+1)} = \frac{1}{3} = .33 \]

\[ b = \sqrt{\frac{R}{R+1}} \sin\gamma = \frac{1}{\sqrt{6}} = .408 \]

\[ \lambda = \frac{b^2}{a} = 2R\sin^2\gamma = \frac{1}{2} = .5 \]

\[ \frac{b}{a} = 2\sqrt{R(R+1)}\sin\gamma = \tan 50.7^\circ \]

\[ R=+1/2 \text{ hyperbolic orbit construction} \]

\[ \gamma = 45^\circ \]
Properties of Coulomb trajectory families and envelopes

Graphical $\epsilon$-development of orbits

- Launch angle fixed - Varied launch energy
- Launch energy fixed - Varied launch angle
- Launch optimization and orbit family envelopes
Graphs and protractors make Coulomb trajectory analysis easier.
Start with initial angle
\( \alpha = 20^\circ \)
(horz. elev.)

or
\( \gamma = 70^\circ \)
(rad. elev.)

for velocity
\( v(0) \) or \( -v(0) \)

Label Main Focus \( F \)
Construct focus locus for 2nd foci \( F' \)

Launch Elevation Angle

\( \alpha - \gamma = 50^\circ \)

\( 2(\alpha - \gamma) = 100^\circ \)
Label Main Focus $F$

Construct focus locus for 2nd foci $F'$

Construct R-line normal to initial velocity $\pm v(0)$ line

Start with initial angle

$\alpha = 20^\circ$

(horiz. elev.)

or

$\gamma = 70^\circ$

(rad. elev.)

for velocity $v(0)$ or $-v(0)$

$2(\alpha - \gamma) = 100^\circ$
Start with initial angle \( \alpha = 20° \) (horiz. elev.) or \( \gamma = 70° \) (rad. elev.) for velocity \( v(0) \) or \(-v(0)\).

Label Main Focus F

Construct focus locus for 2\(^{nd}\) foci \( F' \)

Construct R-line normal to initial velocity ±\( v(0) \) line
Start with initial angle \( \alpha = 20° \)

(horiz. elev.)

or

\( \gamma = 70° \)

(rad. elev.)

for velocity \( v(0) \) or \(-v(0)\)

Label Main Focus \( F \)

Construct focus locus for 2nd foci \( F' \)

Construct \( R\)-line normal to initial velocity \( \pm v(0) \) line

\[ 2(\alpha - \gamma) = 100° \]
Start with initial angle $\alpha = 20^\circ$ (horiz. elev.) or $\gamma = 70^\circ$ (rad. elev.) for velocity $v(0)$ or $-v(0)$.

Label Main Focus $F$. Construct focus locus for 2nd foci $F'$.

Construct R-line normal to initial velocity $\pm v(0)$ line.

$2(\alpha - \gamma) = 100^\circ$
Start with initial angle

\[ \alpha = 20^\circ \]

(horiz. elev.)

or

\[ \gamma = 70^\circ \]

(rad. elev.)

for velocity

\[ \mathbf{v}(0) \text{ or } -\mathbf{v}(0) \]

or \[ -\mathbf{v}(0) \]

Label Main Focus \( F \)

Construct \textit{R-line normal} to initial velocity \( \mathbf{v}(0) \) line

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)-\text{sect R-line normal} \) to mark \( R = KE/PE = 0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \( \textit{R-line} \)-point

Start with initial angle \( \alpha = 20^\circ \) (horiz. elev.)

or \( \gamma = 70^\circ \) (rad. elev.)

for velocity

\[ \mathbf{v}(0) \text{ or } -\mathbf{v}(0) \]

or \[ -\mathbf{v}(0) \]
Start with initial angle

\( \alpha = 20^\circ \) (horiz. elev.)
or
\( \gamma = 70^\circ \) (rad. elev.)

for velocity

\( v(0) \) or \( -v(0) \)

Label Main Focus \( F \)

Construct \textit{R-line normal} to initial velocity \( v(0) \) line.

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)\)-sect \textit{R-line normal} to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots \) for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \textit{R-line}-point and beyond to prime foci \( F' \).
This \( R = -9/8 \) \( \varepsilon \)-line hits focus-locus far away.

This \( R = -1 \) \( \varepsilon \)-line intersects focus-locus at \( \pm \infty \).

Label Main Focus \( F \)

Construct \( R \)-line normal to initial velocity \( v(0) \) line

Construct focus locus for prime foci \( F' \)

Start with initial angle \( \alpha = 20^\circ \) (horiz. elev.)

or \( \gamma = 70^\circ \) (rad. elev.) for velocity

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \( R \)-line-point and beyond to prime foci \( F' \)

This \( (R = \pm \infty) \varepsilon \)-line intersects focus-locus on unit circle. \([R = \pm \infty] \varepsilon \)-line parallel to \( R \)-scale line.

This \( (R = -1) \varepsilon \)-line parallel to focus-locus

\[ v(0) \]
Properties of Coulomb trajectory families and envelopes

Graphical $\epsilon$-development of orbits

Launch angle fixed-Varied launch energy

$\Rightarrow$ Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes
Start with initial velocity $v(0)$ or $-v(0)$

Label Main Focus $F$

Construct $R$-line normal to initial velocity $v(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, ...$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$ to each $R$-line-point and beyond to prime foci $F'$

Range bisection circles (these are not orbits) indicate reentry ranges
Start with initial velocity $v(0)$ or $-v(0)$

Label Main Focus $F$

Construct $R$-line normal to initial velocity $v(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$

to each $R$-line-point and beyond to prime foci $F'$

$2a = r' + r = r' + 1$

Range bisection circles (these are not orbits) indicate reentry ranges

$\v(0)$ Same arc centered on unit circle measures “string length” $2a = r' + r = r' + 1$
Start with initial velocity \( \mathbf{v}(0) \) or \(-\mathbf{v}(0)\)

Label Main Focus \( F \)

Construct \textit{R-line normal} to initial velocity \( \mathbf{v}(0) \) line

Construct \textit{focus locus} for prime foci \( F' \)

\((N=8)\)-sect \textit{R-line normal} to mark \( R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots \)

for eccentricity \( \varepsilon \)-vector scale

Extend eccentricity \( \varepsilon \)-vectors from the main Focus \( F \) to each \textit{R-line}-point and beyond to prime foci \( F' \)

\textit{Range bisection circles} (these are not orbits) indicate reentry ranges

\textit{Construct ellipse point by point}

\( 2a=r'+r=r'+1 \)

Same arc centered on unit circle measures “string length”
Label Main Focus $F$

Construct \textit{R-line normal} to initial velocity $\mathbf{v}(0)$ line

Construct \textit{focus locus} for prime foci $F'$

$(N=8)$-sect \textit{R-line normal} to mark $R = KE/PE = 0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$ to each \textit{R-line}-point and beyond to prime foci $F'$

Range bisection indicates re-entry ranges

Maximum range limit for this elevation angle $\alpha = 20^\circ$ has $R = -1$ and $\varepsilon = 1$ (parabola)

$\varepsilon$-line parallel to focus-locus

This puts 2nd focus at $\infty$. 
Label Main Focus $F$

Construct R-line normal to initial velocity $\mathbf{v}(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect R-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$

to each R-line-point and beyond to prime foci $F'$

Maximum range limit for this elevation angle $\alpha=20^\circ$ is range $\phi=280^\circ$

Maximum range limit for this elevation angle $\alpha=20^\circ$ has $R=-1$ and $\varepsilon=1$ (parabola)

This puts 2nd focus at $\infty$. 

Range bisection indicates re-entry ranges

\[ (R=-1) \varepsilon = 1 \text{-line parallel to focus-locus} \]
Label Main Focus $F$

Construct *R-line normal* to initial velocity $v(0)$ line

Construct *focus locus* for prime foci $F'$

$(N=8)$-sect *R-line normal* to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, ...$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$ to each *R-line*-point and beyond to prime foci $F'$

*Maximum* range limit for this elevation angle $\alpha=20^\circ$ is range $\phi=280^\circ$

Range bisection indicates re-entry ranges

Maximum range limit for this elevation angle $\alpha=20^\circ$ has $R=-1$ and $\varepsilon=1$ (*parabola*)

This puts 2nd focus at $\infty$. 

\[ (R=-1) \quad \varepsilon = 1 \text{-line parallel to focus-locus} \]
Maximum range limit for this elevation angle $\alpha = 20^\circ$ has $R = -1$ and $\varepsilon = 1$ (parabola).

This puts 2nd focus at $\infty$. 

Revu: geometry of parabola "kites"
Properties of Coulomb trajectory families and envelopes
Graphical $\epsilon$-development of orbits
Launch angle fixed-Varied launch energy
$\Rightarrow$ Launch energy fixed-Varied launch angle
Launch optimization and orbit family envelopes
Start with initial velocity $v(0)$ or $-v(0)$

Label Main Focus $F$

Construct $R$-line normal to initial velocity $v(0)$ line

Construct focus locus for prime foci $F'$

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, ...$

for eccentricity $\varepsilon$-vector scale

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$ to each $R$-line point and beyond to prime foci $F'$

Start with initial velocity $v(0)$ or $-v(0)$ or $-v(0)$
Label Main Focus $F$

Construct $R$-line normal to initial velocity $\mathbf{v}(0)$ line.

Construct focus locus for prime foci $F'$.

$(N=8)$-sect $R$-line normal to mark $R=KE/PE=0, \pm 1/8, \pm 2/8, \pm 3/8, \ldots$ for eccentricity $\varepsilon$-vector scale.

Extend eccentricity $\varepsilon$-vectors from the main Focus $F$ to each $R$-line-point and beyond to prime foci $F'$

focus locus for fixed Energy or fixed $R=KE/PE=-5/8$

focus locus for fixed launch angle $\alpha = 20^\circ$

Range Longitude
Properties of Coulomb trajectory families and envelopes

Graphical $\varepsilon$-development of orbits

Launch angle fixed-Varied launch energy
Launch energy fixed-Varied launch angle

$\Rightarrow$ Launch optimization and orbit family envelopes
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

Problem:

Find trajectory angle of minimum energy to fly 90° of arc (1/4 around planet)
Problem:

Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus $F'$ lies on radial line that bisects longitude angle
Problem:
Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus F' lies on radial line that bisects longitude angle

Optimal prime focus F' lies on line connecting START and FINISH at tangent point of minimal energy circle SF'.

Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

**Problem:**
Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

**Solution:** Prime focus $F'$ lies on radial line that bisects longitude angle

Optimal prime focus $F'$ lies on line connecting $\text{START}$ and $\text{FINISH}$ at tangent point of minimal energy circle $SF'$.

$R$-line normal must bisect angle $FSF'$ connecting foci $F$ and $F'$ and is normal to initial launch vector $v_0$.
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes.

**Problem:**
Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

**Solution:** Prime focus \( F' \) lies on radial line that bisects longitude angle

Optimal prime focus \( F' \) lies on line connecting **START** and **FINISH**
at tangent point of minimal energy circle \( SF' \).

R-line normal must bisect angle \( FSF' \) connecting foci \( F \) and \( F' \) and is normal to initial launch vector \( v_0 \) with launch angle \( \alpha = 22.5° \)

The \( \varepsilon \)-vector and \( R \)-value:

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Wednesday, December 24, 2014
Problem: 
Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus $F'$ lies on radial line that bisects longitude angle

Optimal prime focus $F'$ lies on line connecting START and FINISH at tangent point of minimal energy circle $SF'$.

R-line normal must bisect angle $FSF'$ connecting foci $F$ and $F'$ and is normal to initial launch vector $v_0$ with launch angle $\alpha = 22.5°$

The $\varepsilon$-vector and $R$-value:

Maximum range 269.999°.
Coulomb envelope geometry

(a) Focus locus for KE/PE = R = -3/8

(b) Caustic for KE/PE = R = -3/8

(c) Diving orbit

Ideal comet “heads” or “tails” in solar wind
Optimum energy angle relations

\[ \Theta = \frac{\pi}{2} - \theta \]

\[ = 2\theta - \frac{\pi}{2} \]

\[ \Theta = \frac{\pi}{2} - \theta \]

\[ \rho/2 \]

\[ \rho/2 \]

\[ \theta = \frac{(\pi - \rho)}{4} \]

\[ \rho = \pi - 4\theta \]

Achieve range \( \rho \)

Optimum (smallest) focus-focus circle to

These angles are equal because of equal-focal reflection angles