Lecture 29
Tue. 12.10.2014

Geometry and Symmetry of Coulomb Orbital Dynamics I.

(Ch. 2-4 of Unit 5 12.11.14)

Rutherford scattering and differential scattering cross-sections

Parabolic "kite" and envelope geometry

Eccentricity vector $\boldsymbol{\varepsilon}$ and (ε,λ)-geometry of orbital mechanics

$\boldsymbol{\varepsilon}$-vector and Coulomb $\mathbf{r}$-orbit geometry

Review and connection to standard development

$\boldsymbol{\varepsilon}$-vector and Coulomb $\mathbf{p}=m\mathbf{v}$ geometry

$\boldsymbol{\varepsilon}$-vector and Coulomb $\mathbf{p}=m\mathbf{v}$ algebra

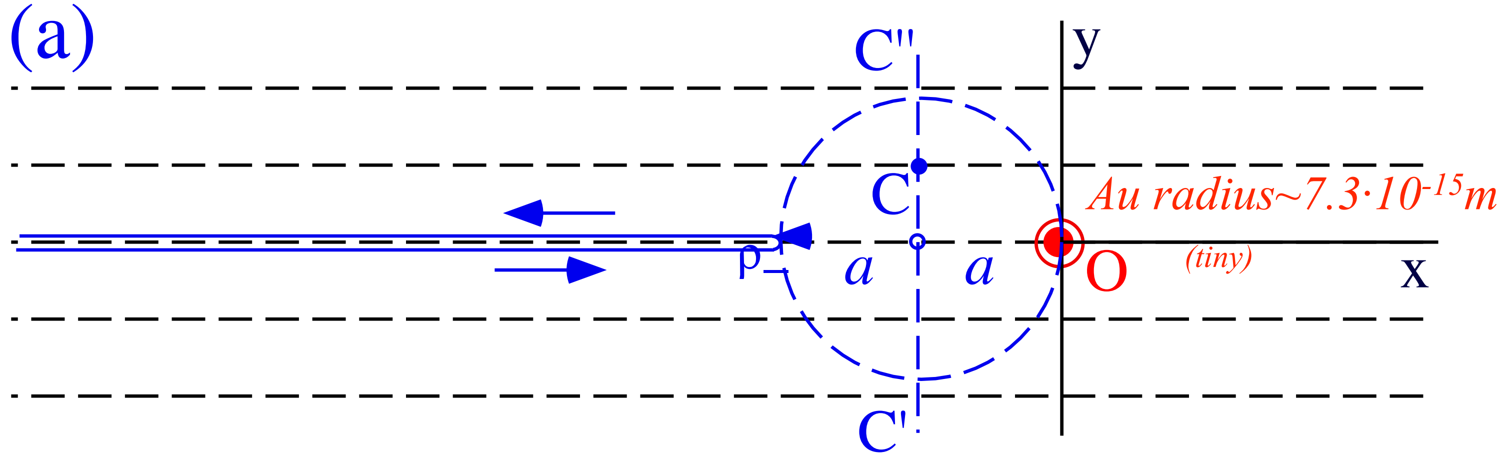
Example with elliptical orbit

Analytic geometry derivation of $\boldsymbol{\varepsilon}$-construction

Algebra of $\boldsymbol{\varepsilon}$-construction geometry

Connection formulas for (a,b) and (ε,λ) with (γ,R)

➤ *Review and added: Rutherford scattering and differential scattering cross-sections*
Parabolic "kite" and envelope geometry

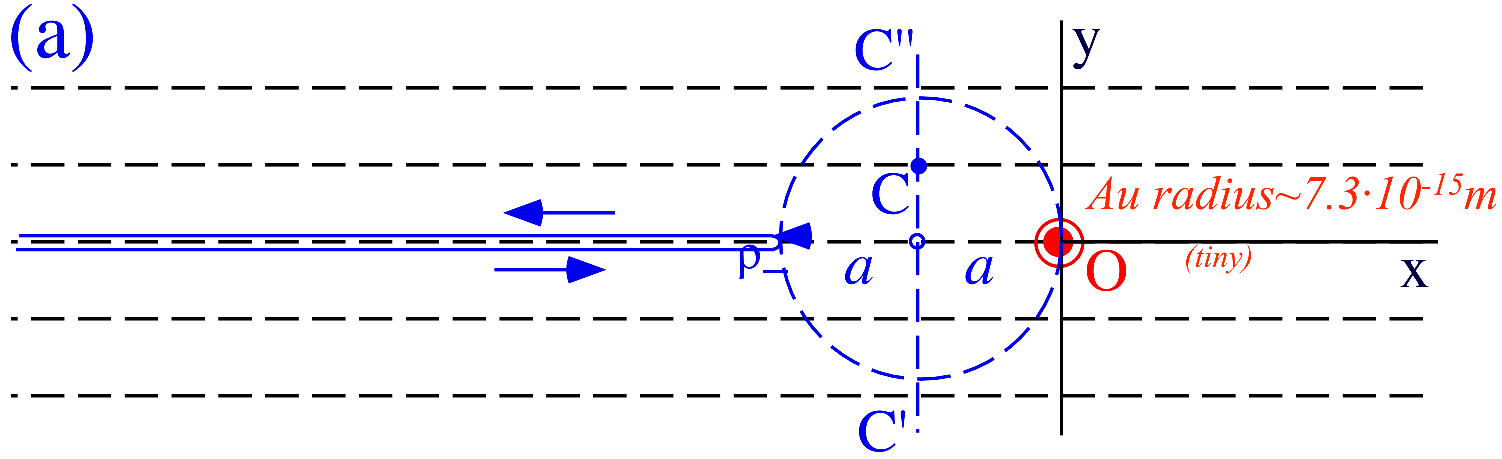


Rutherford scattering of α^{+2}

particles from Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach 2a.

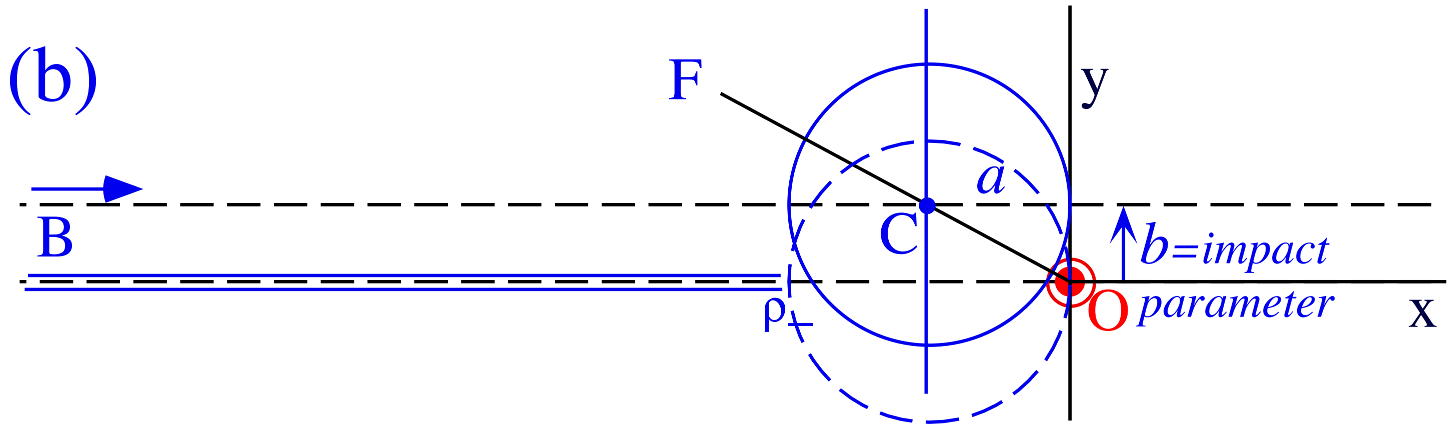
(E=k/2a) $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

Assume "Dead-On" closest approach $2a$.

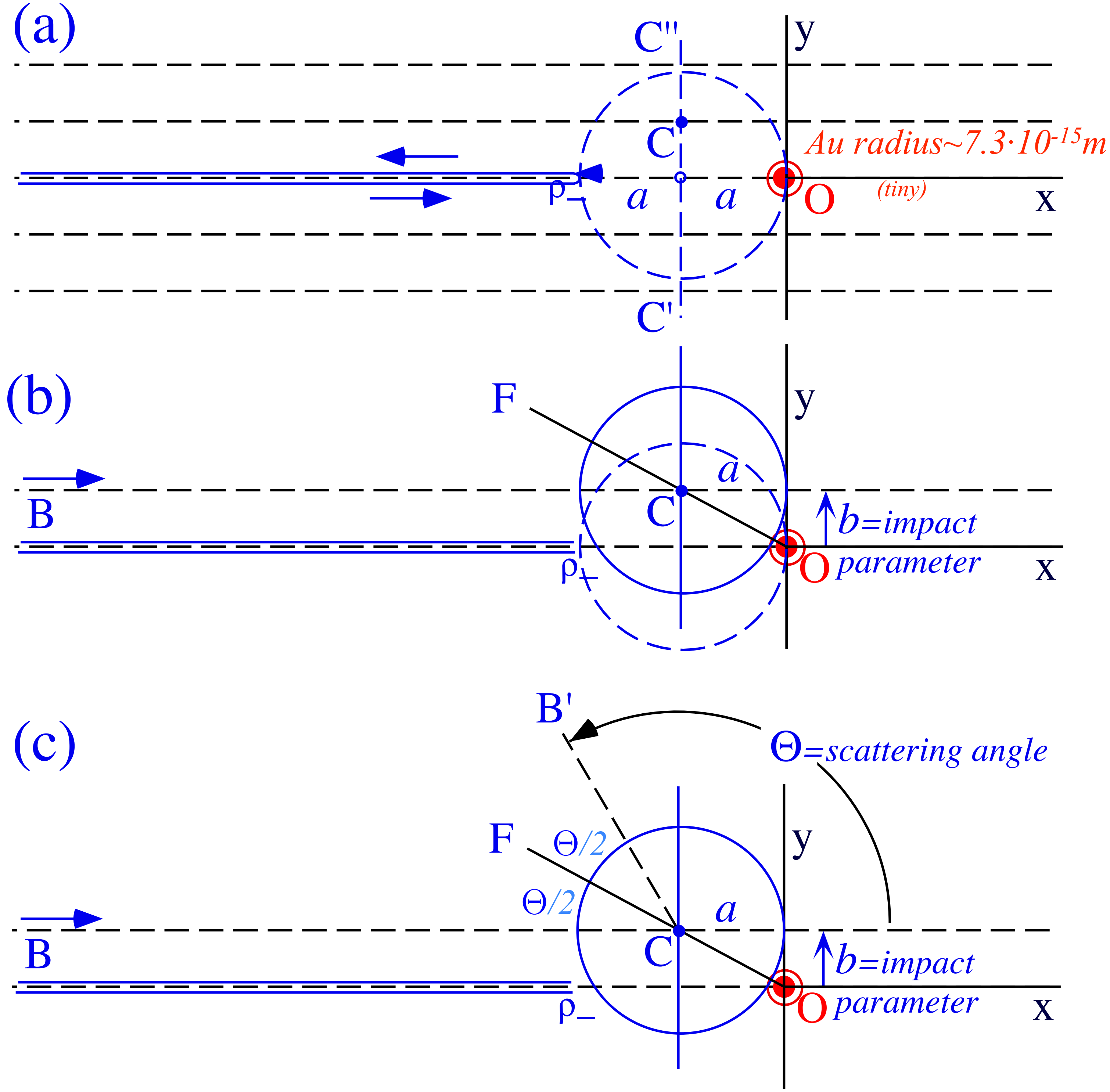
$(E=k/2a)$ $a\sim 10^{-11}m >> 7.3\cdot 10^{-15}m$



Pick an "impact parameter" line $y = b$.

Draw circle of radius a *around center point* $C=(-a,b)$ *tangent to* y*-axis.*

Draw "focus-locus" line OCF.



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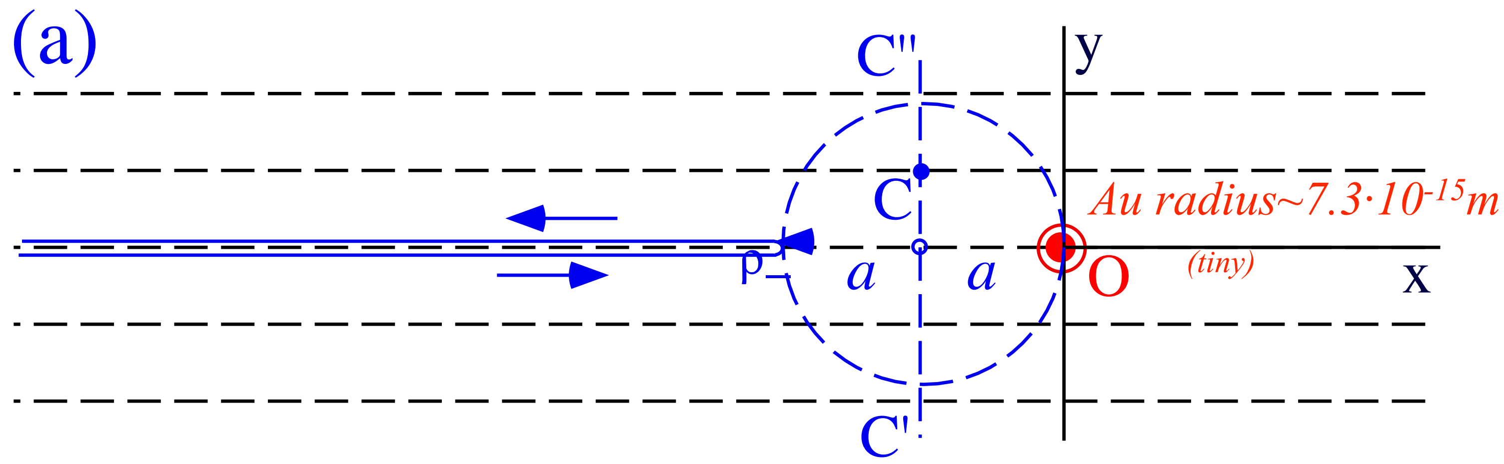
Draw "focus-locus" line OCF.

Copy angle ∠BCF *(equal to* $\Theta/2$*)*

to make angle ∠FCB′ *(also equal to* $\Theta/2$*)*

Resulting line CB′ *is outgoing asymptote at scattering angle* Θ *.*



Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O

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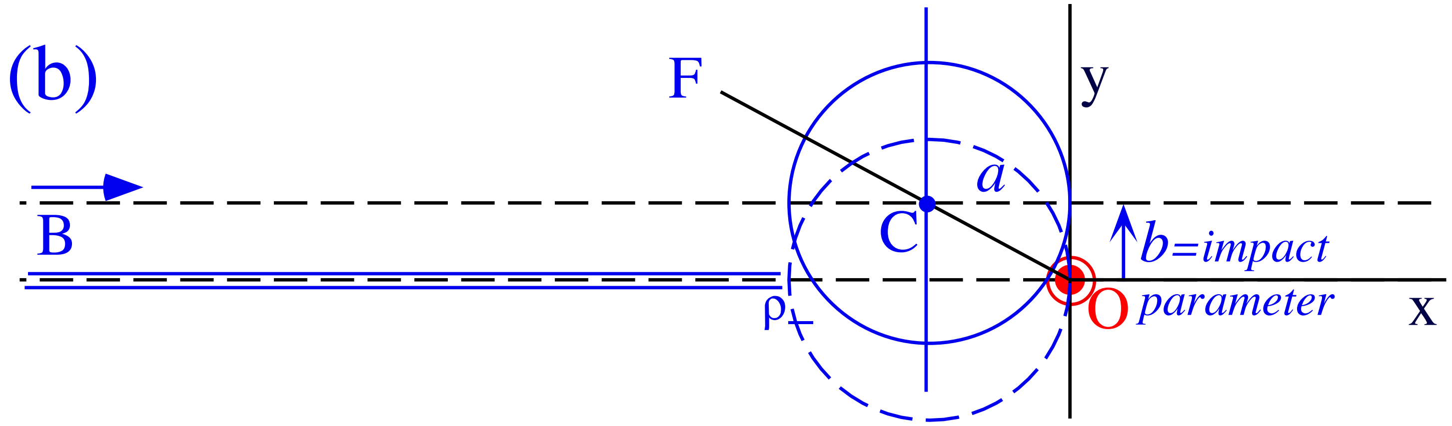
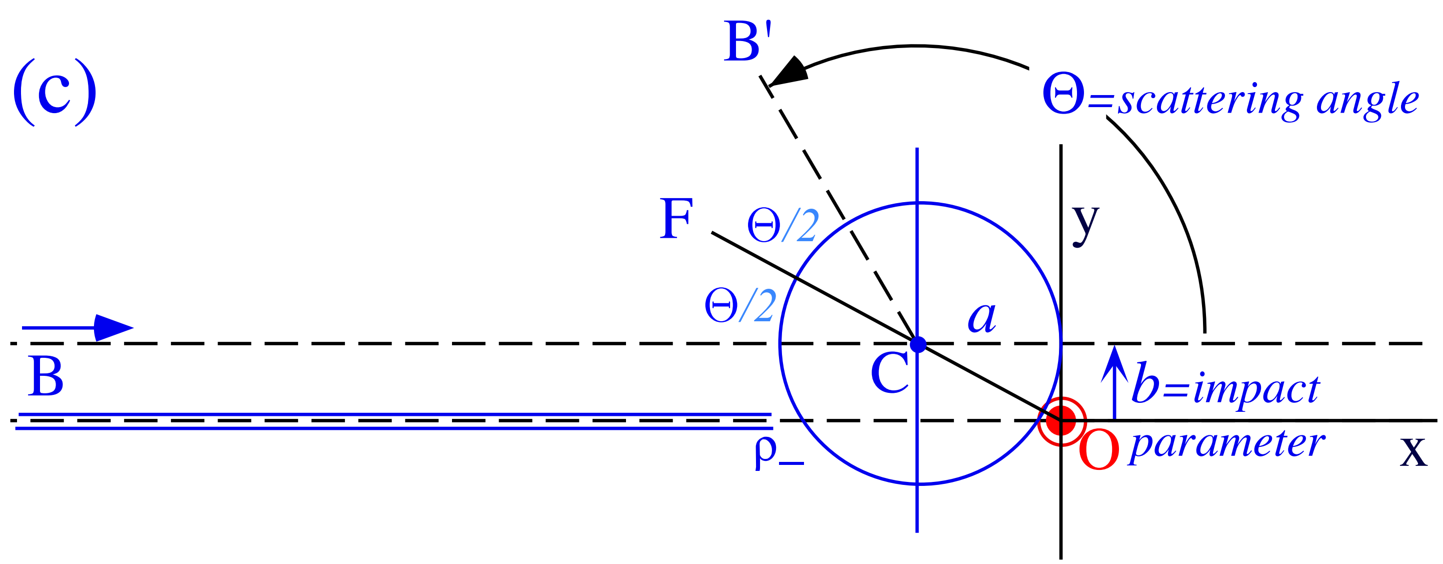
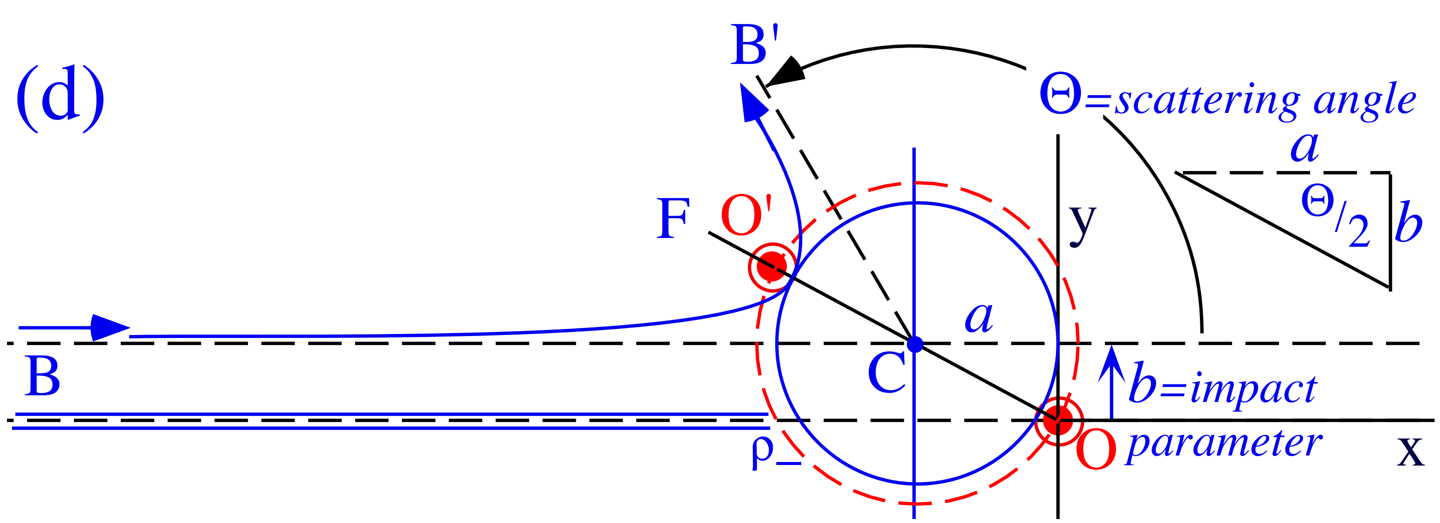


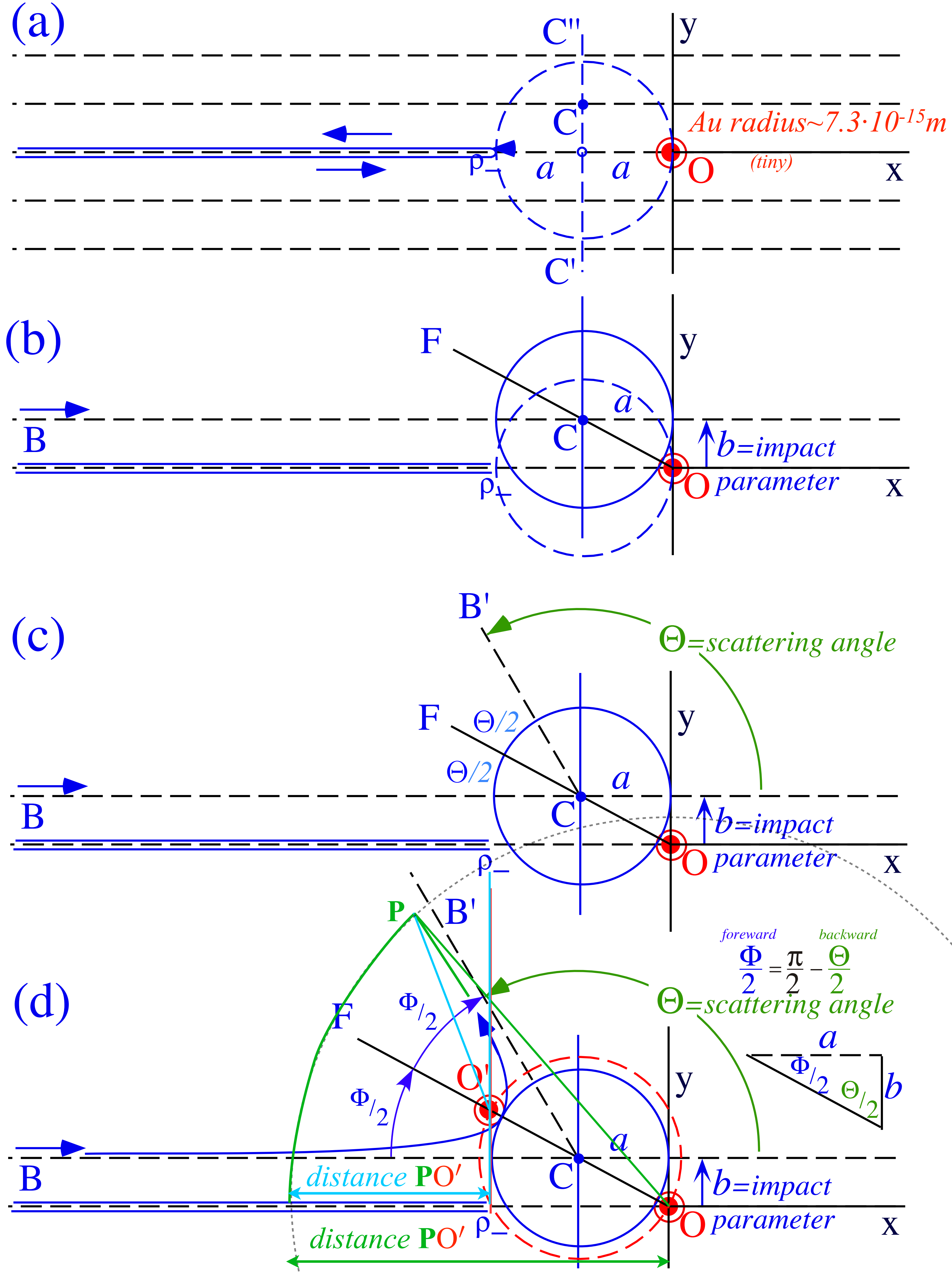
Pick an "impact parameter" line y =b.
*Draw circle of radius **a** around*
center point C=(-*a*,*b*) *tangent to* *y-axis.*
Draw "focus-locus" line OCF.



Copy angle ∠BCF *(equal to Θ/2)*
to make angle ∠FCB′ *(also equal to Θ/2)*
Resulting line CB′ *is outgoing asymptote*
at scattering angle Θ .



Locate secondary focus O′ *by drawing*
circle around point C *of diameter* CO
thru point O. *Diameter* O′CO *is 2aε.*
Hyperbolic orbit points **P** *now found*
*using constant 2a=*PO-PO′



Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O

Assume "Dead-On" closest approach $2a$.

($E=k/2a$) $\quad$ *$a\sim 10^{-11}m >> 7.3\cdot 10^{-15}m$*

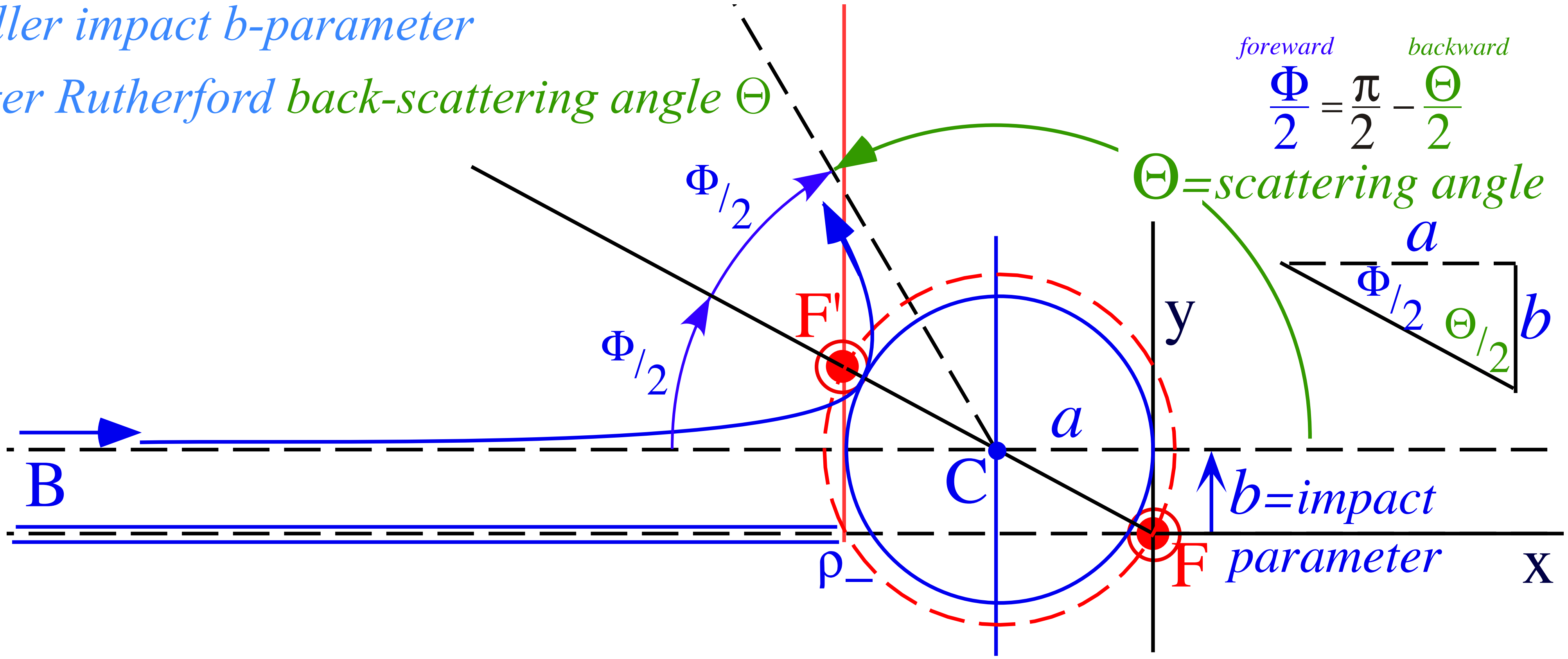
Pick an "impact parameter" line $y=b$.
Draw circle of radius a around center point $C=(-a,b)$ tangent to y-axis.
Draw "focus-locus" line OCF.

Copy angle ∠BCF (equal to $\Theta/2$) to make angle ∠FCB′ (also equal to $\Theta/2$)
Resulting line CB′ is outgoing asymptote at scattering angle Θ.

Locate secondary focus O′ by drawing circle around point C of diameter CO thru point O. Diameter O′CO is $2a\varepsilon$.
*Hyperbolic orbit points **P** now found using constant $2a$=**P**O-**P**O′*

Smaller impact b-parameter

Larger Rutherford back-scattering angle Θ



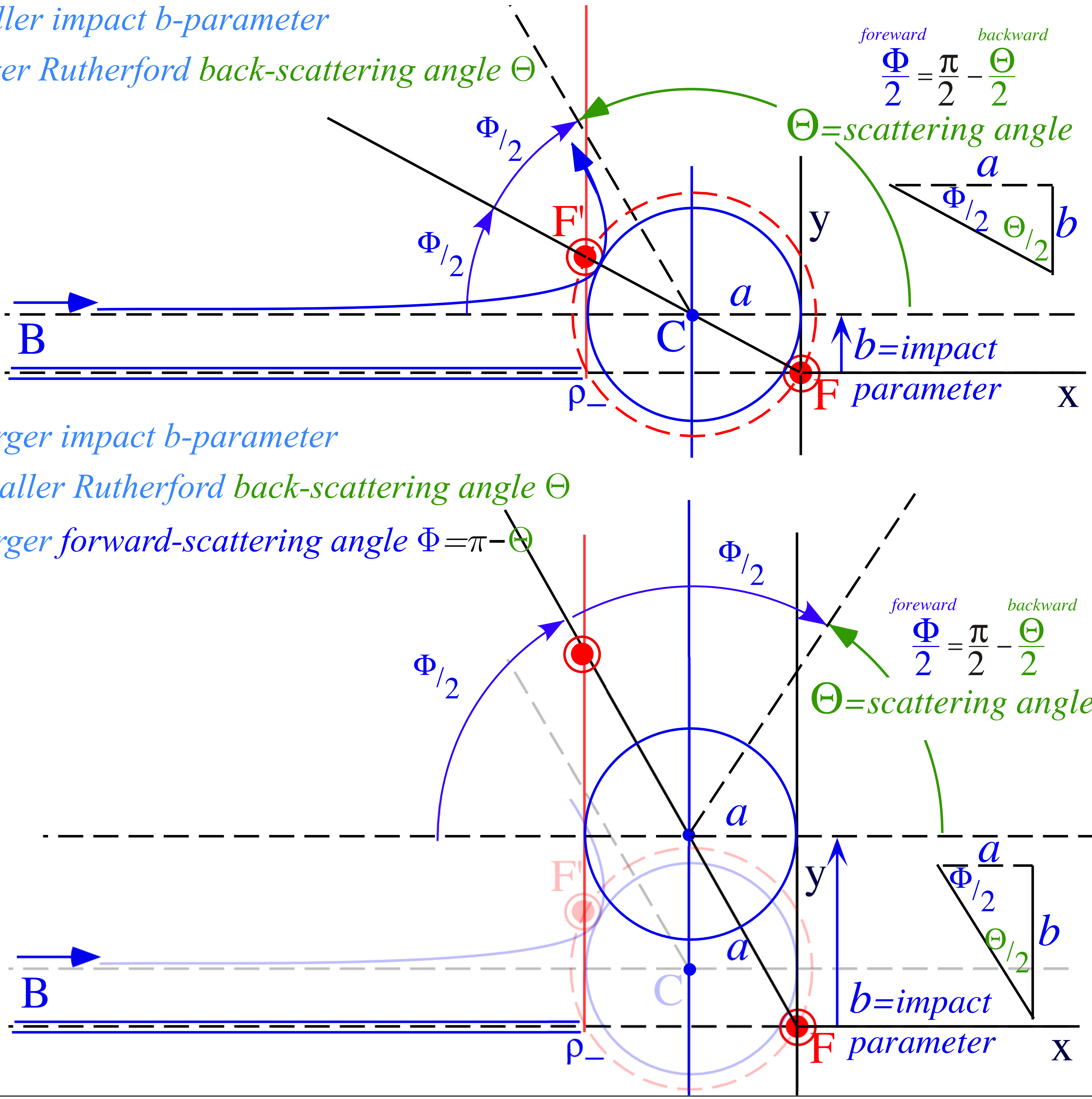
Smaller impact b-parameter

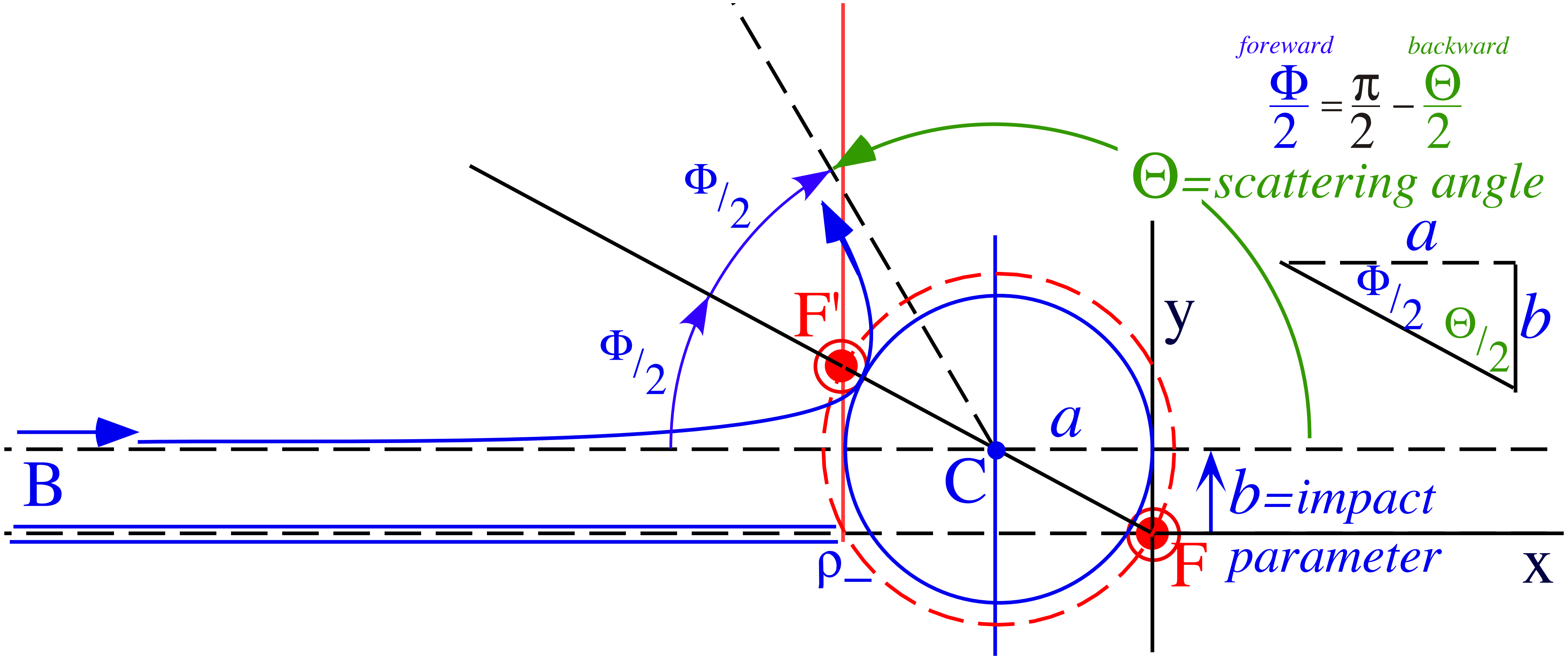
Larger Rutherford back-scattering angle Θ

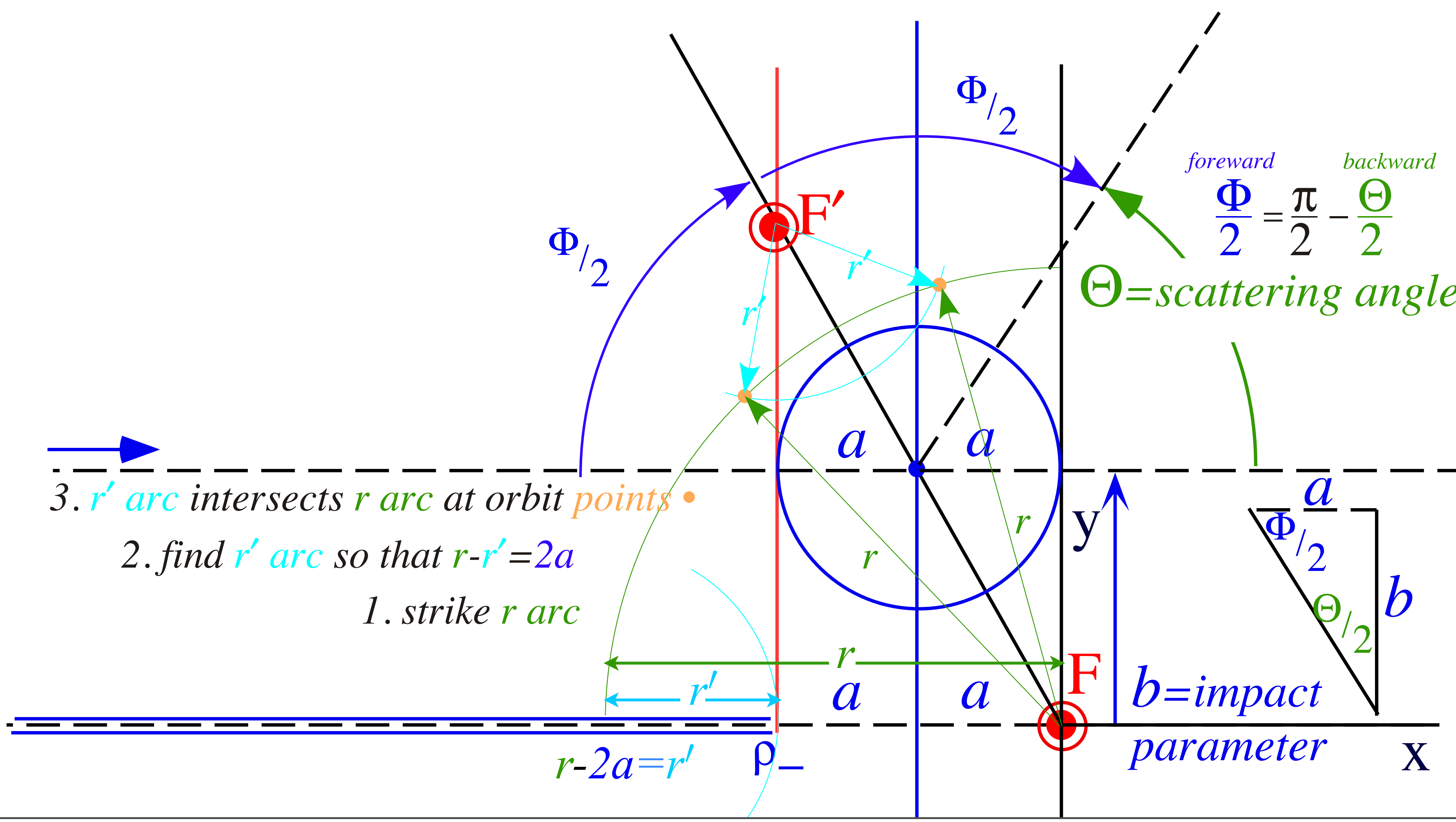
Larger impact b-parameter

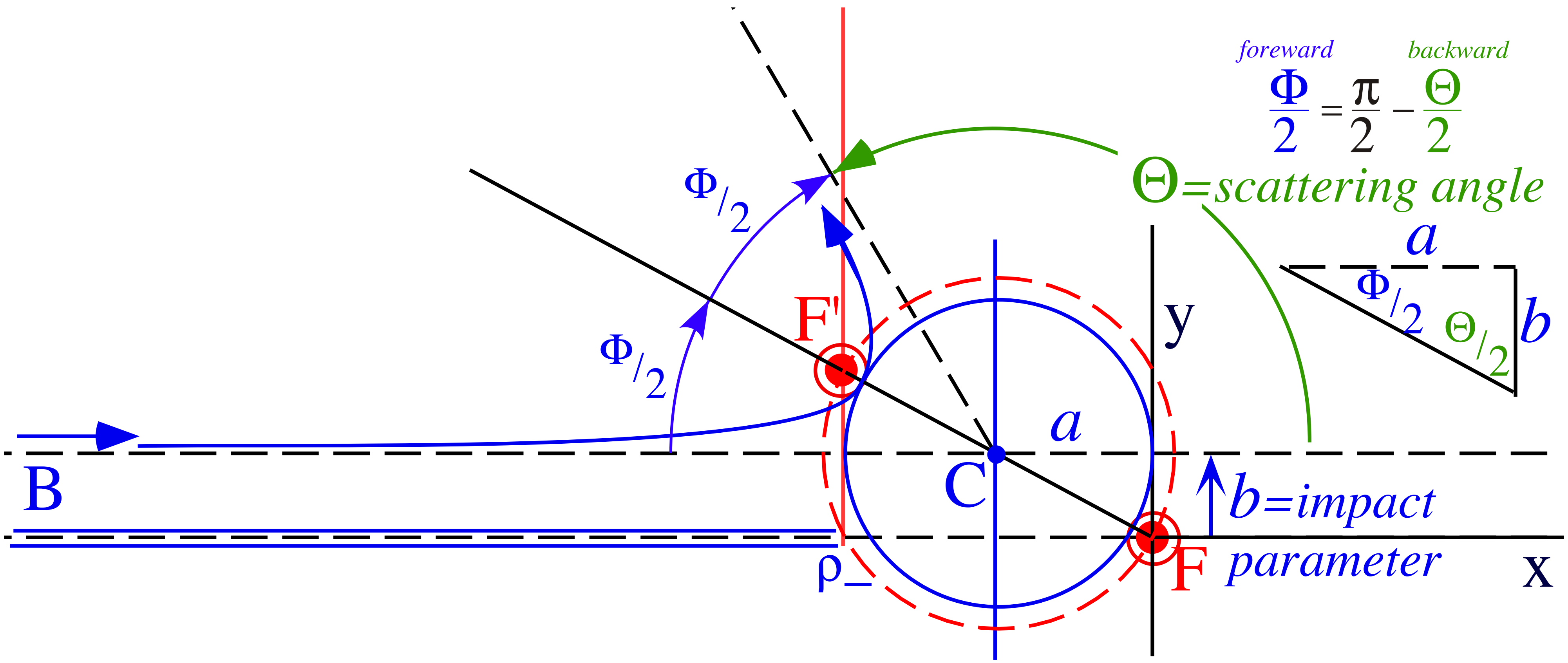
Smaller Rutherford back-scattering angle Θ

Larger forward-scattering angle $\Phi = \pi - \Theta$




foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
F'
y
a
Φ/2
Θ/2
b
a
C
B
b=impact parameter
F
x
ρ_


Φ/2
foreward
backward
Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
F′
r′
r′
a
a
3. r′ arc intersects r arc at orbit points
2. find r′ arc so that r-r′=2a
1. strike r arc
r
r
y
a
Φ/2
Θ/2
b
r
r′
a
a
F
b=impact parameter
x
r-2a=r′
ρ_



foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
Φ/2
F'
a
b
Θ/2
y
a
B
C
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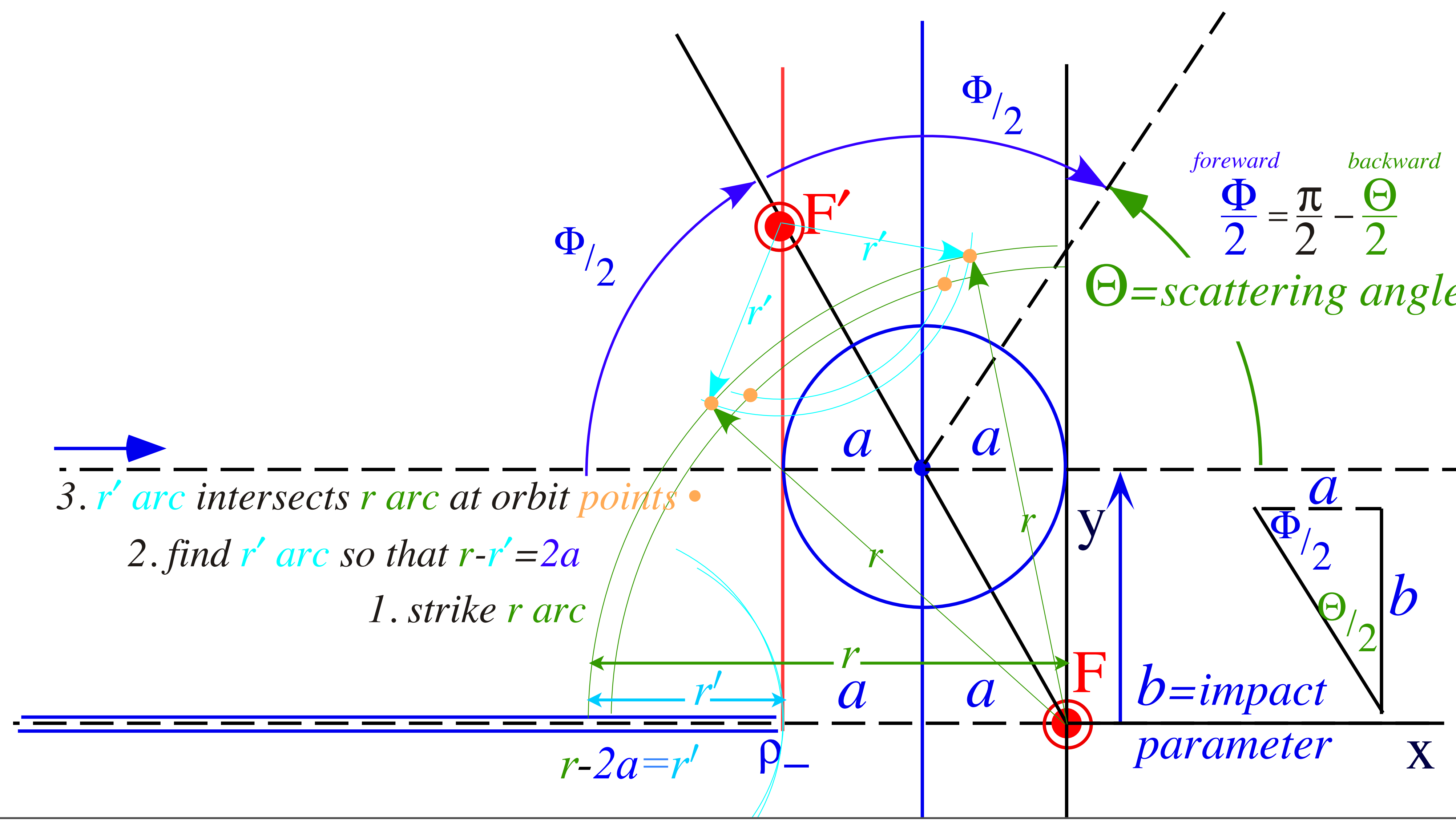
x
ρ_



Φ/2
foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Φ/2
F′
r′
r′
Θ=scattering angle
a
a
3. r′ arc intersects r arc at orbit points •
2. find r′ arc so that r-r′=2a
1. strike r arc
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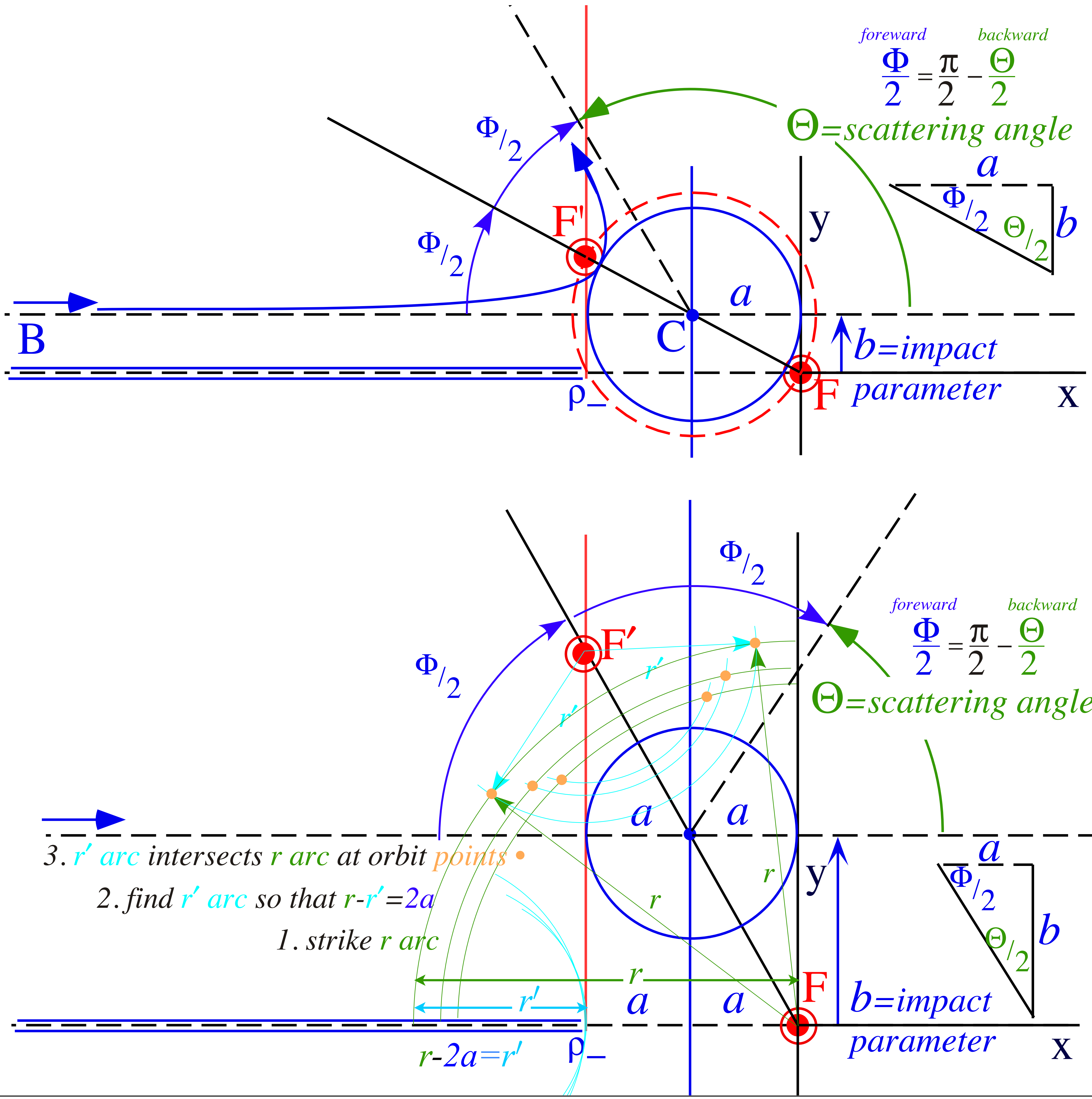
Φ/2
b
Θ/2
r
r′
a
a
F
b=impact parameter
r-2a=r′
ρ_
x


foreward
backward
$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
Θ=scattering angle
Φ/2
F'
a
b
Θ/2
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B
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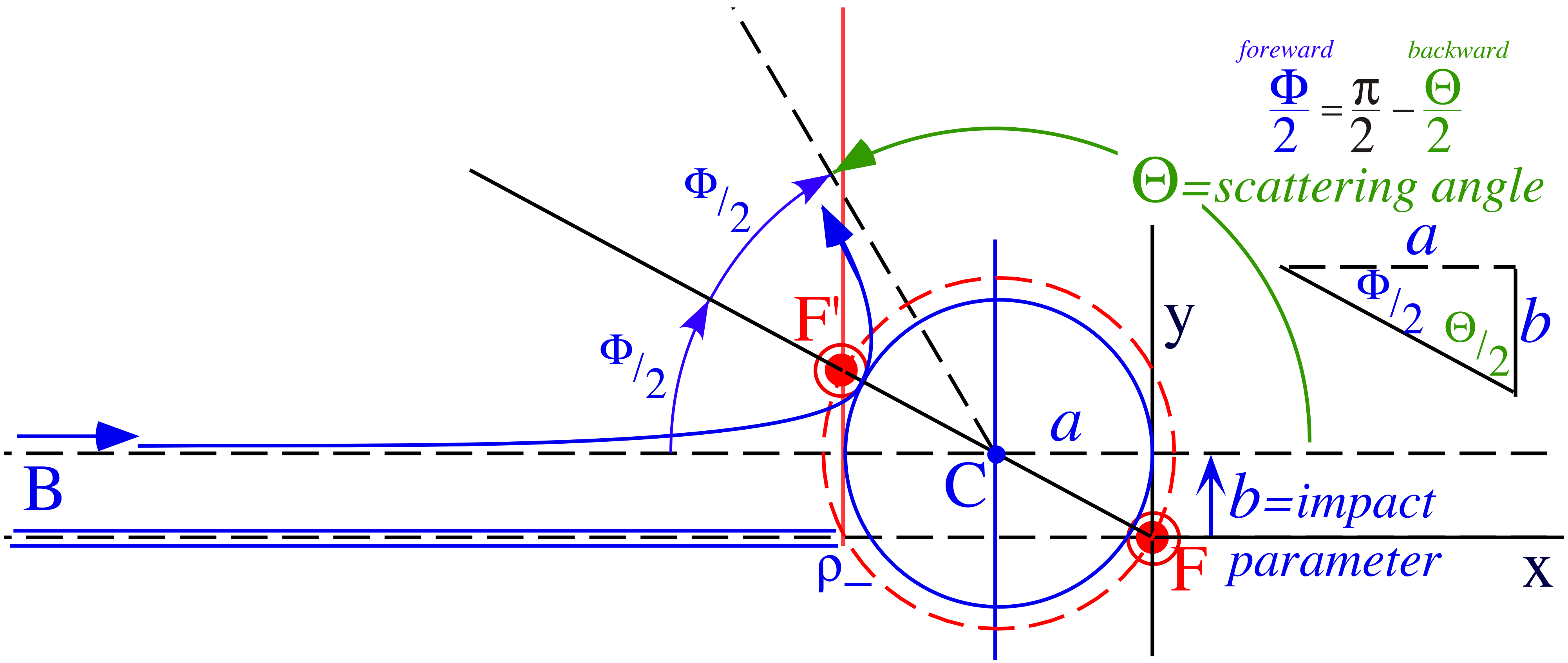
F
ρ_
x
Φ/2
F′
r′
a
r
3. r′ arc intersects r arc at orbit points •
2. find r′ arc so that r-r′=2a
1. strike r arc
r-2a=r′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Θ=scattering angle
Φ/2
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ρ_
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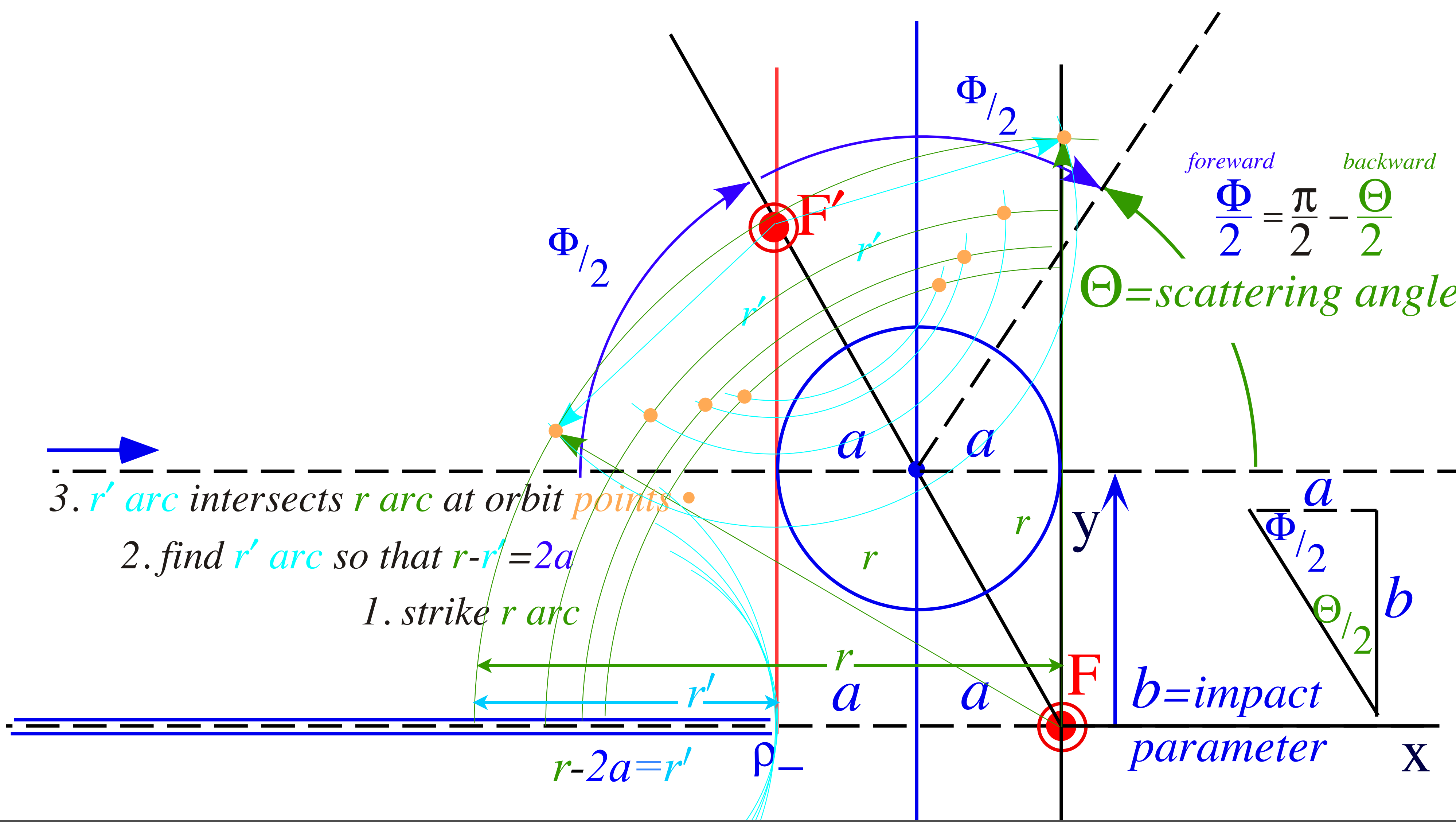
Φ/2
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F′
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r-2a=r′
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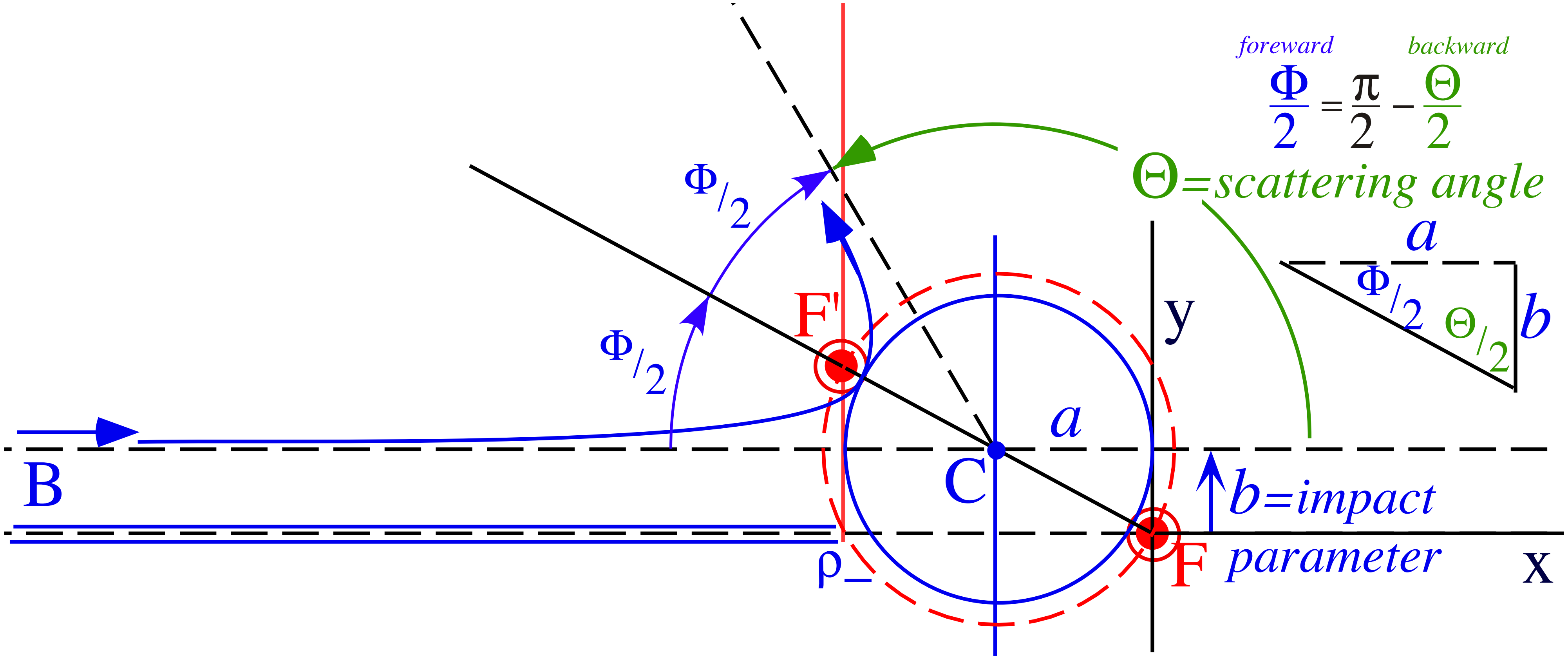
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F'
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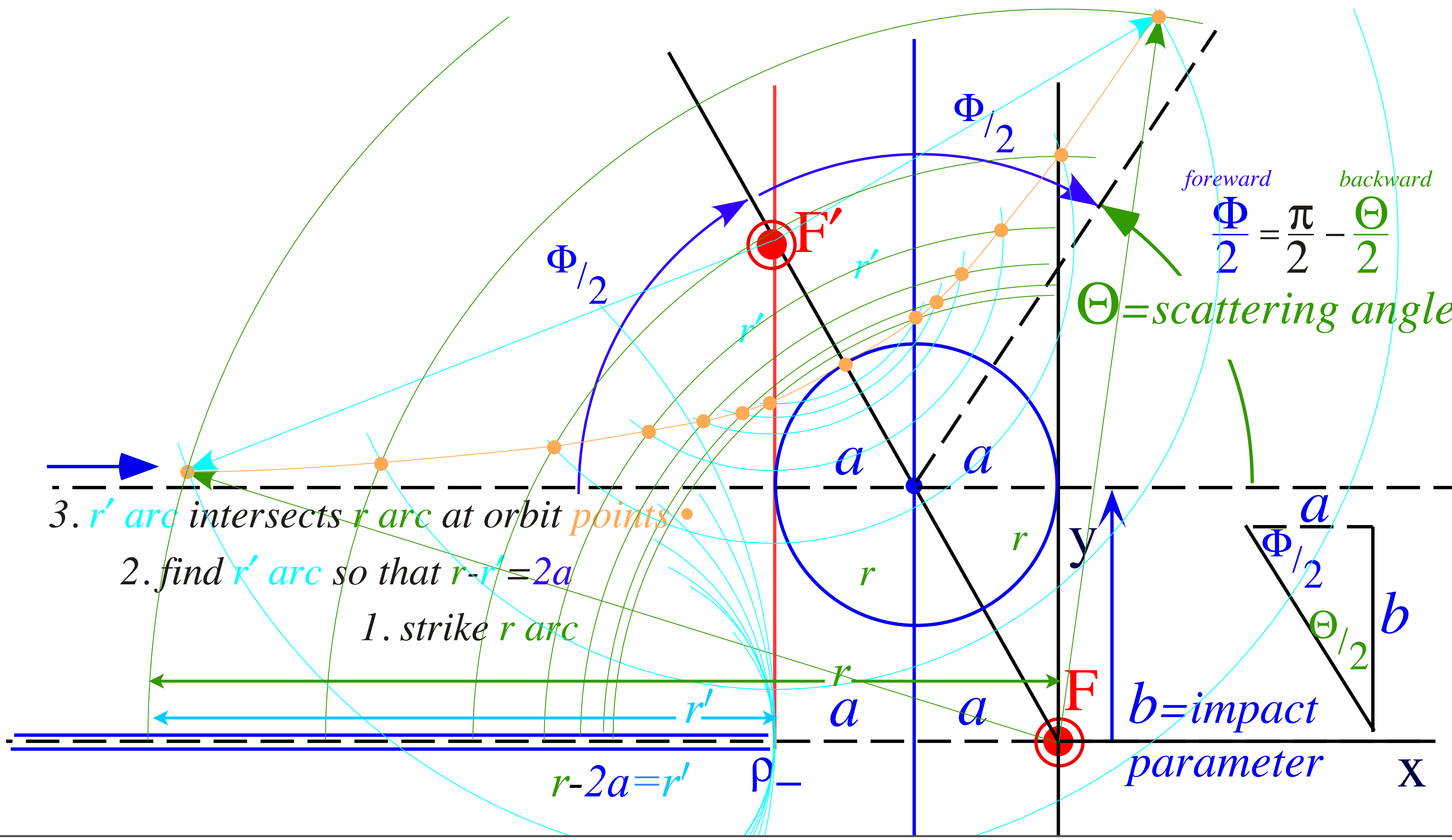
ρ_
F
x


Φ/2
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$\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$
F′
Φ/2
Θ=scattering angle
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a
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2. find r′ arc so that r-r′=2a
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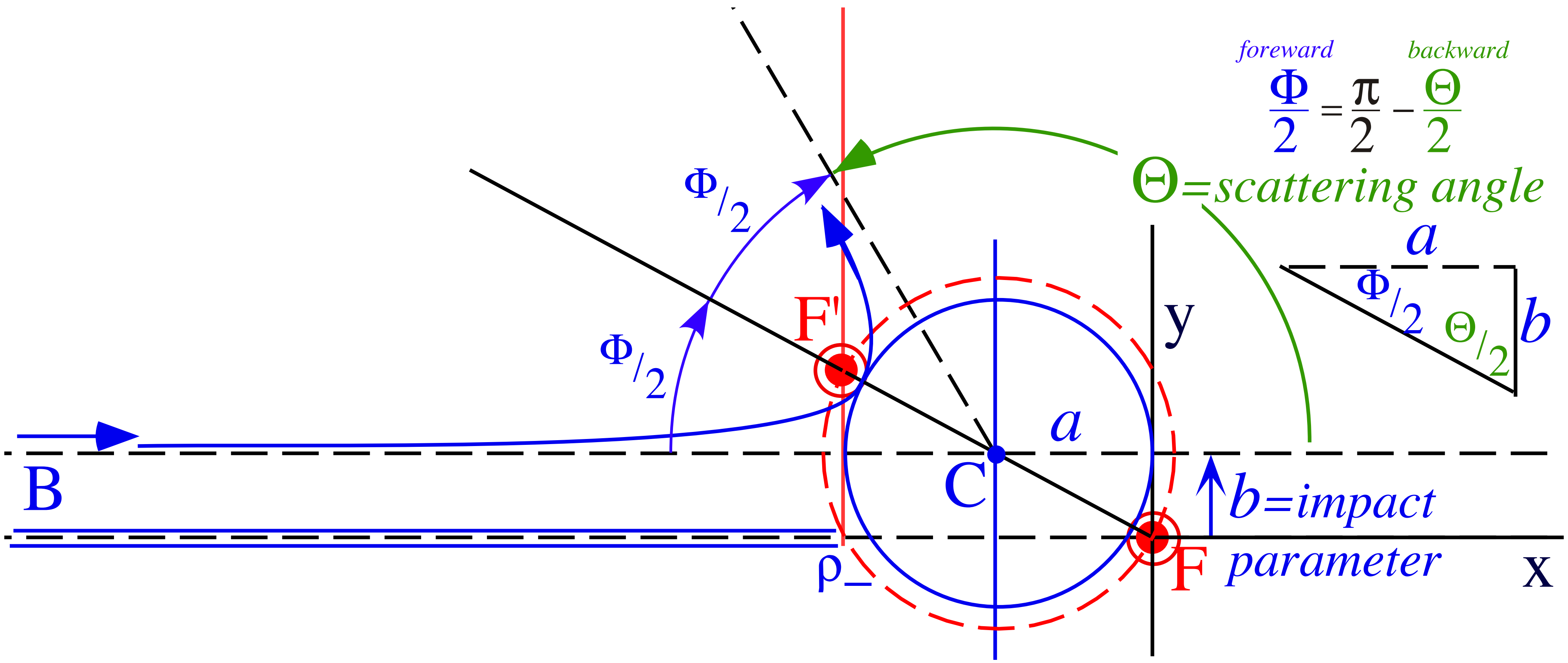
a
Φ/2
Θ/2
b
y
F
b=impact parameter
x
r-2a=r′
ρ_



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Φ/2 = π/2 − Θ/2
Θ=scattering angle
Φ/2
Φ/2
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Φ/2
Θ/2
F'
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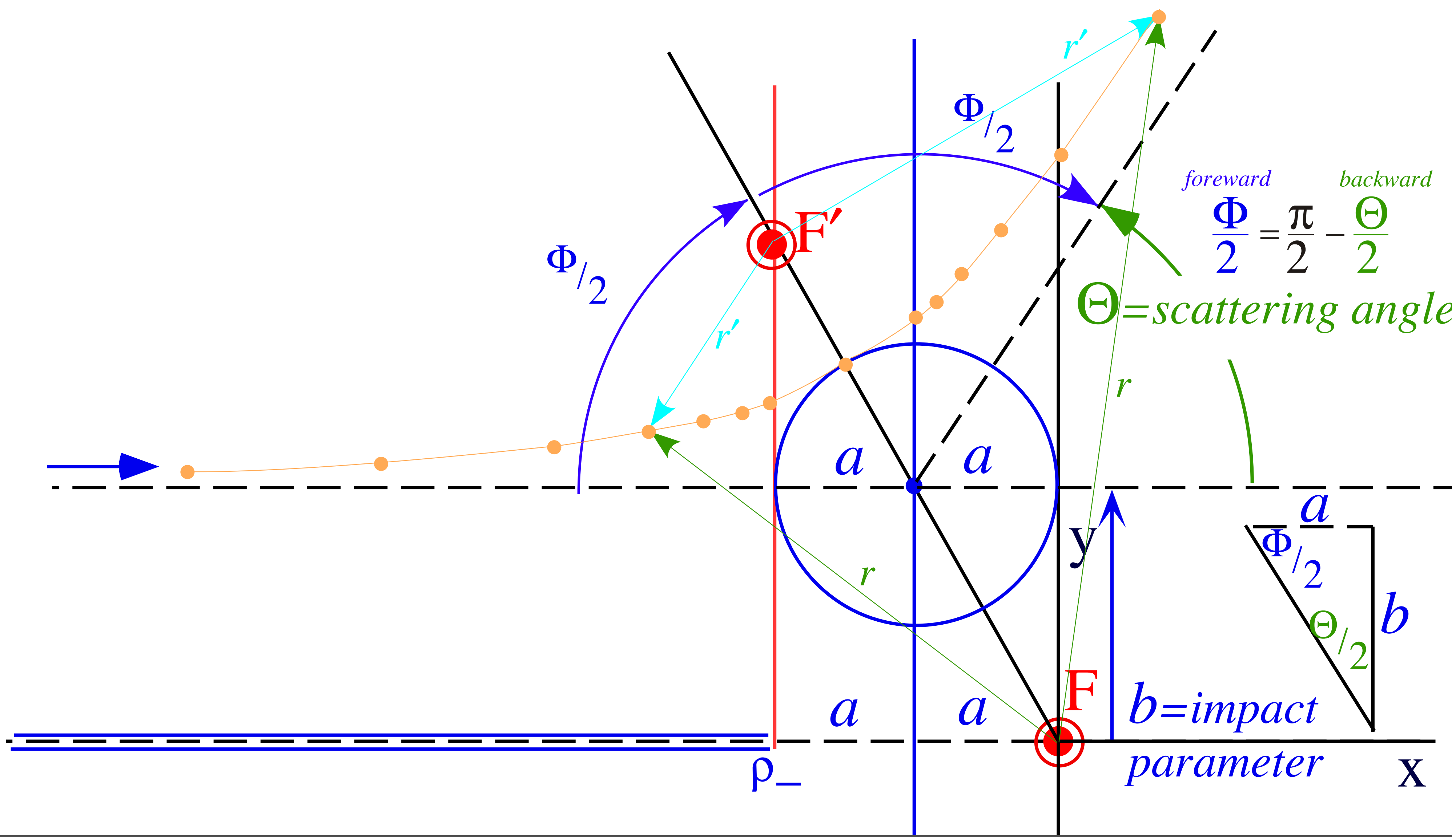
B
C
b=impact
parameter
F
ρ_
x



r′
Φ/2
foreward
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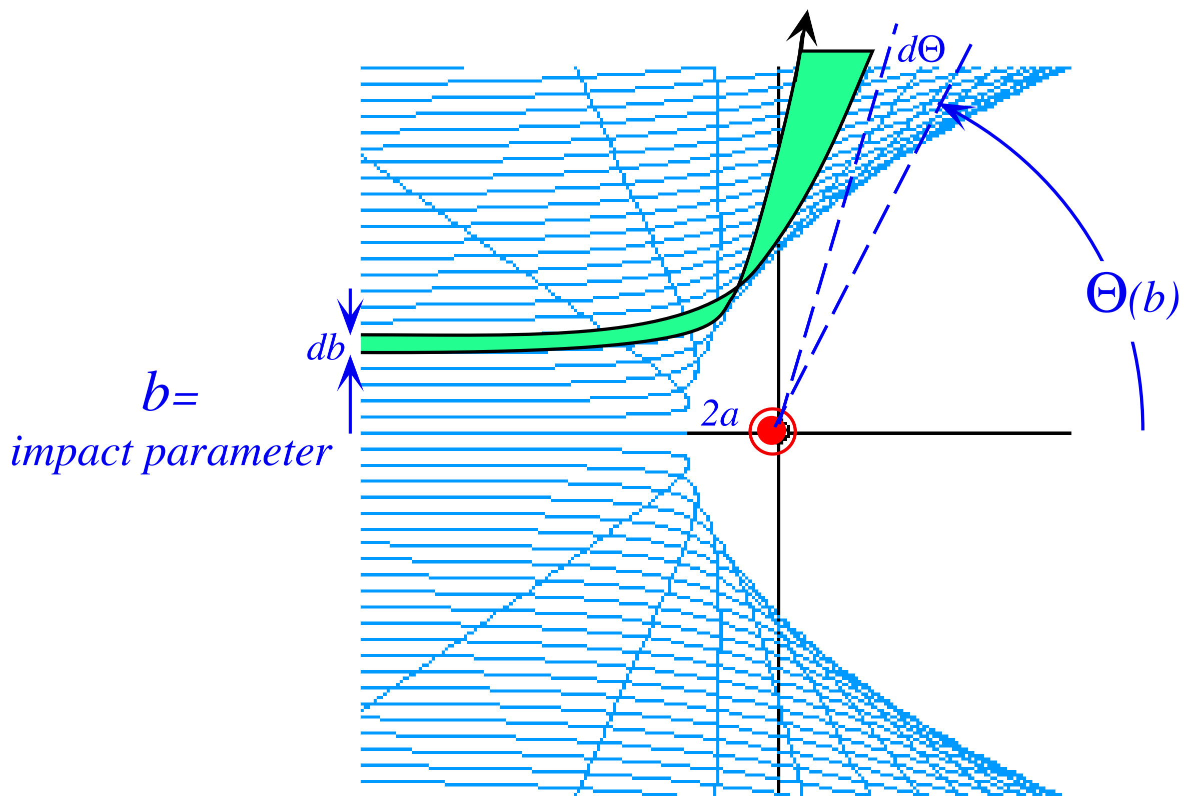
Φ/2 = π/2 − Θ/2
F′
Φ/2
Θ=scattering angle
r′
r
a
a
a
y
Φ/2
b
Θ/2
r
F
a
a
b=impact
parameter
ρ_
x

Review: Coulomb scattering geometry

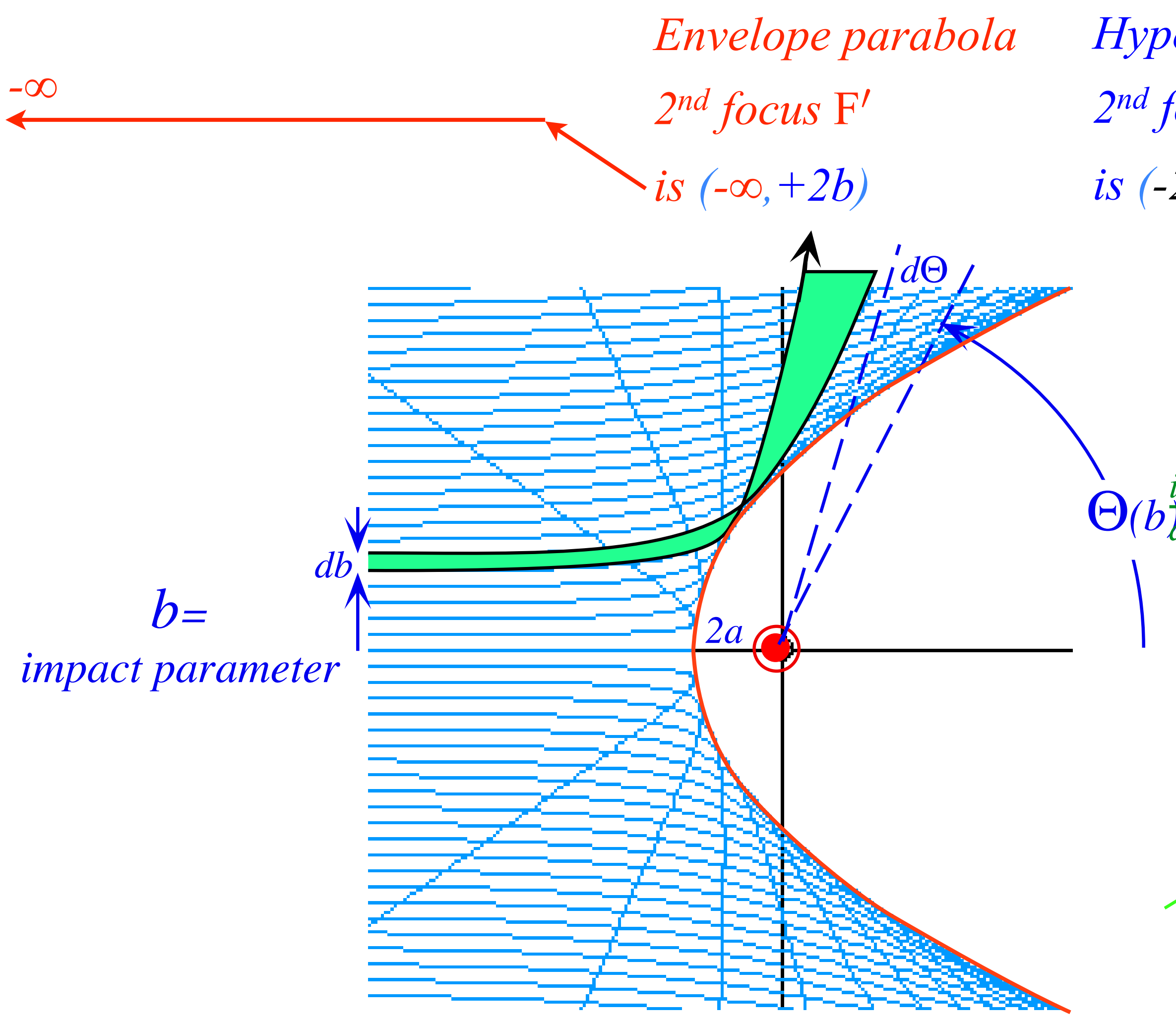
Review and added: Rutherford scattering and differential scattering cross-sections

➔ *Parabolic "kite" and envelope geometry*

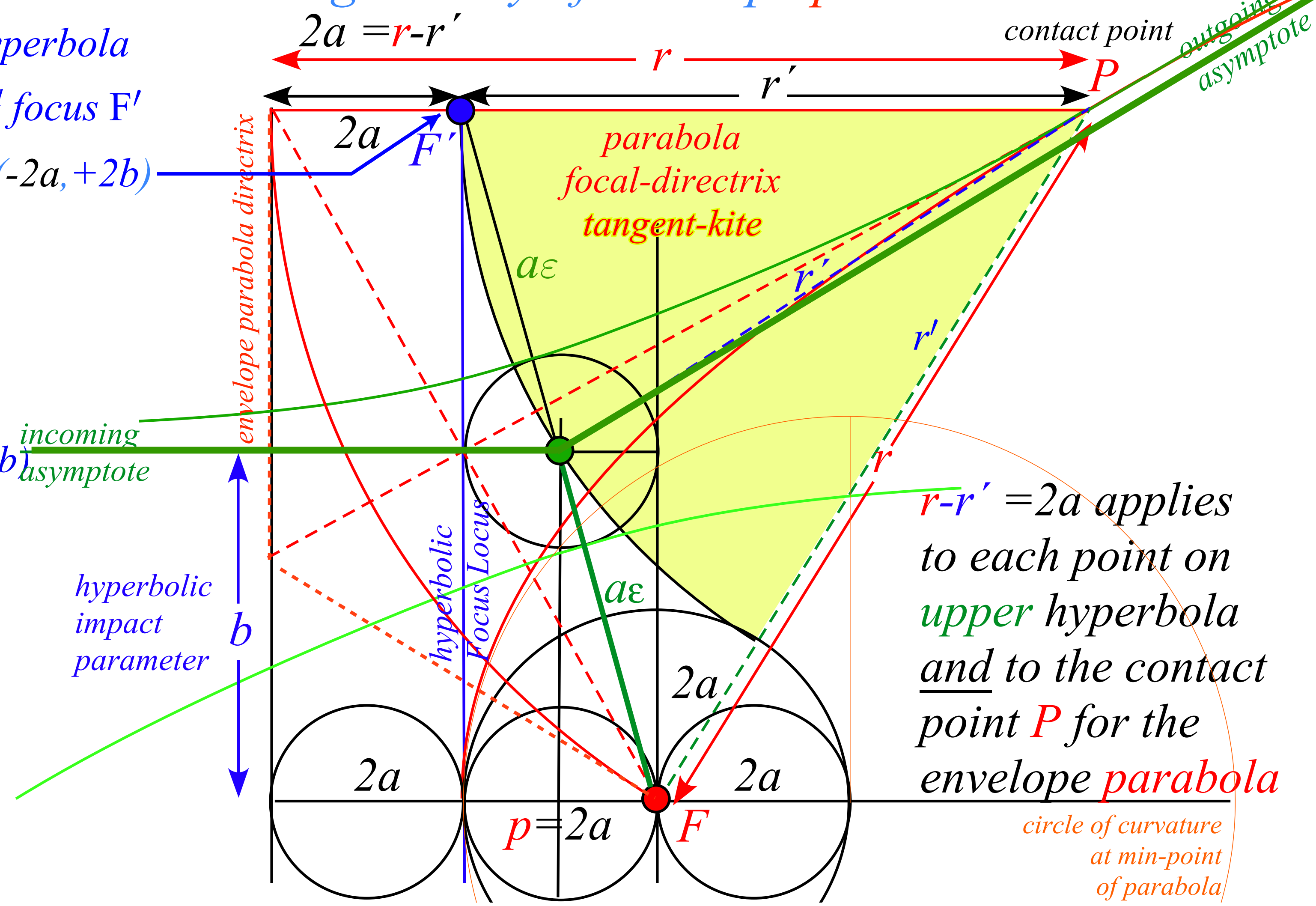
Rutherford scattering geometry



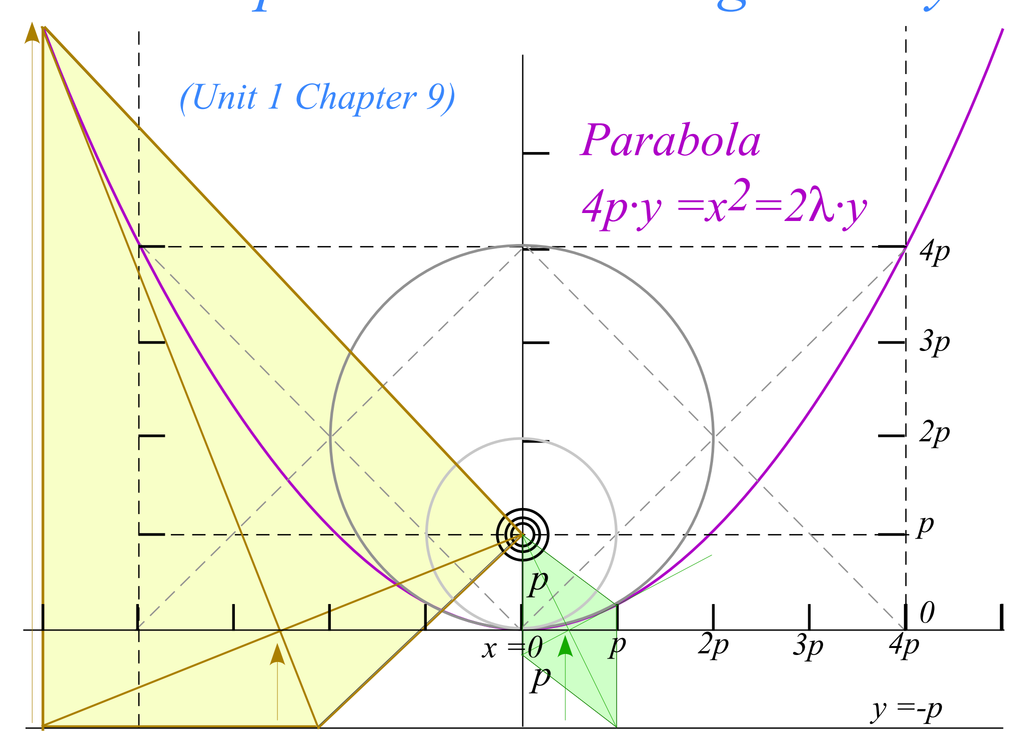
Rutherford scattering geometry

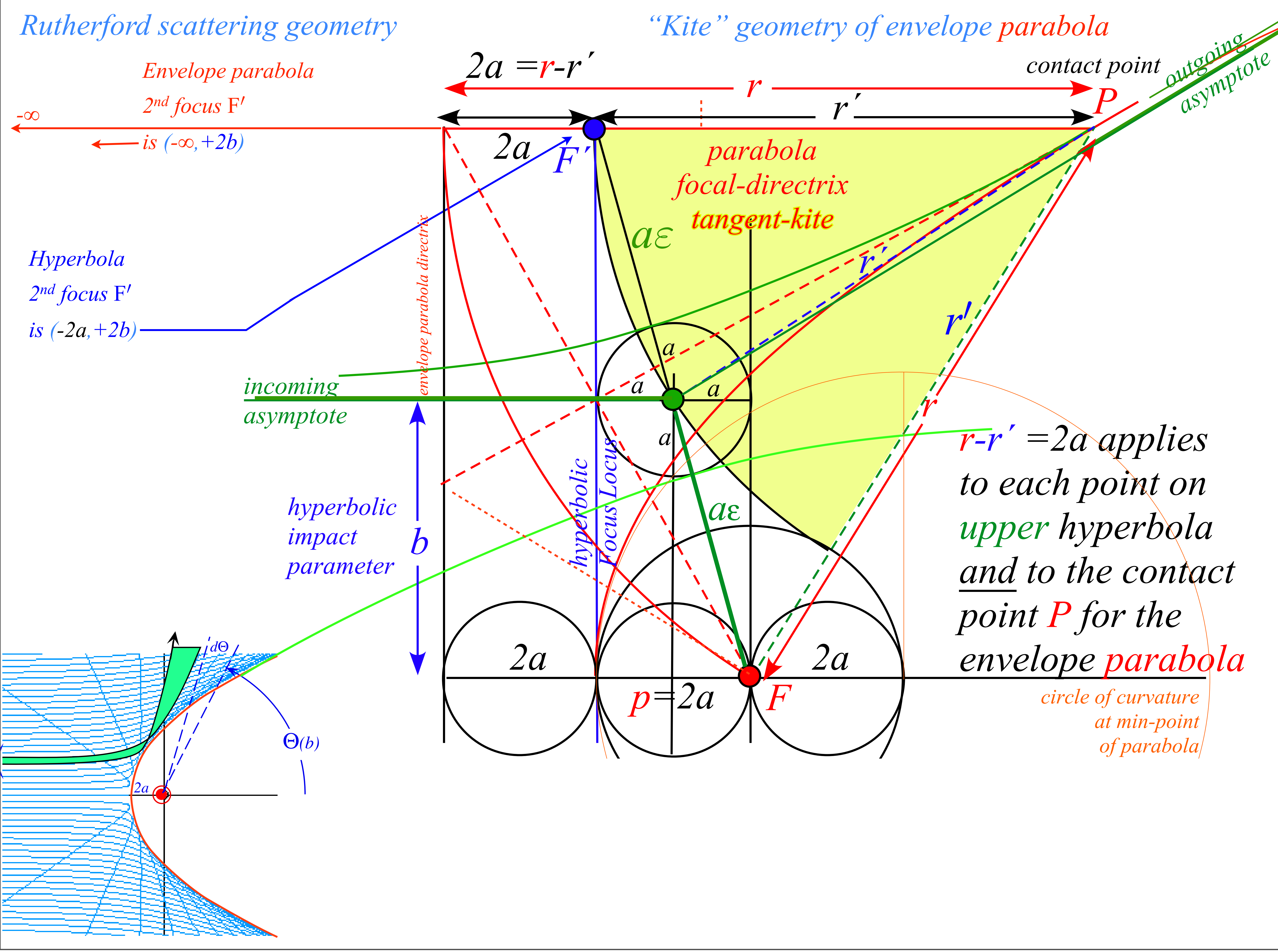


"Kite" geometry of envelope parabola



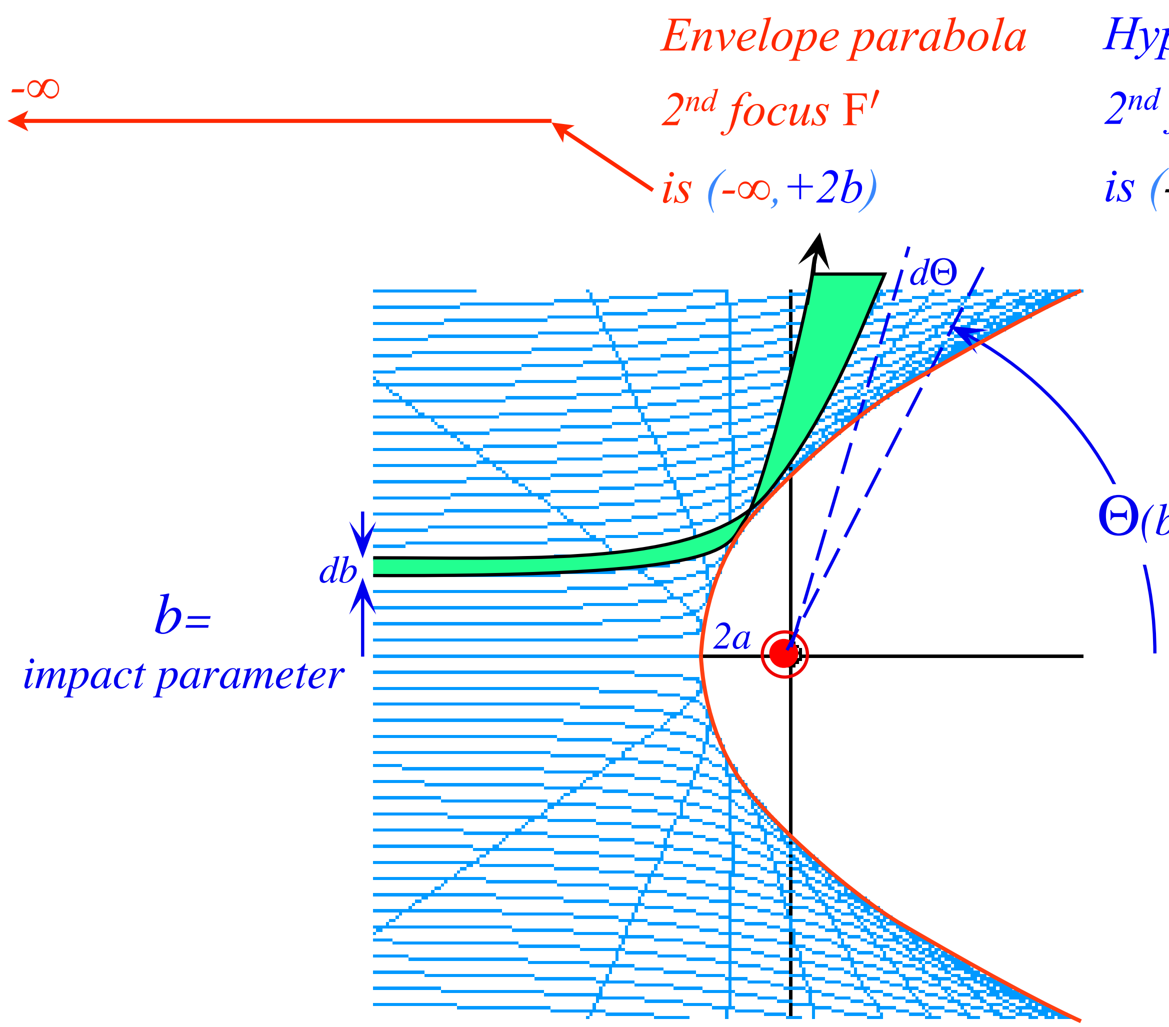
Recall parabolic " kite" geometry



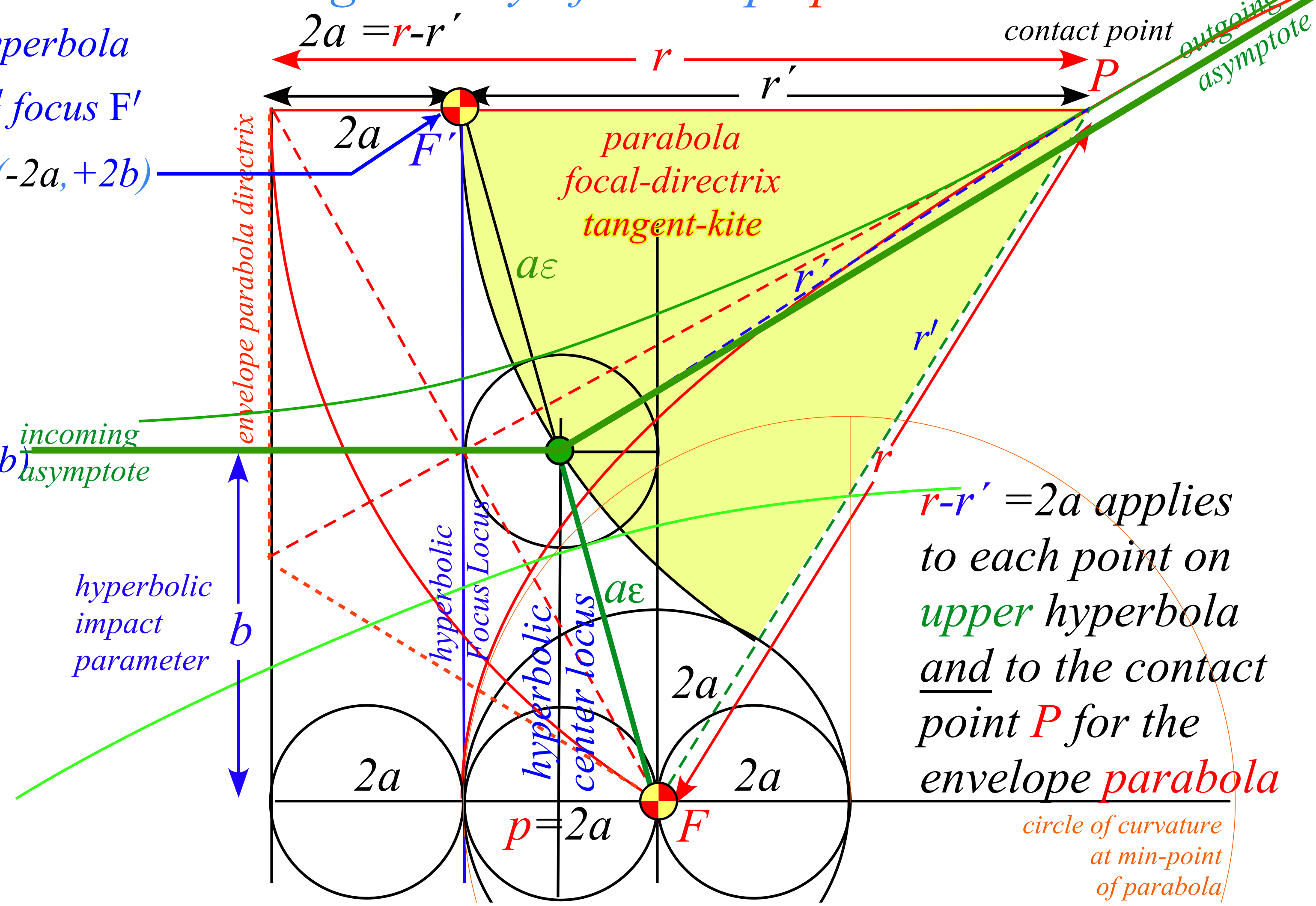


Rutherford scattering geometry
"Kite" geometry of envelope parabola
Envelope parabola
2nd focus F′
is (-∞,+2b)
-∞
2a =r-r′
r
r′
2a
F′
contact point
P
outgoing asymptote
parabola
focal-directrix
tangent-kite
aε
Hyperbola
2nd focus F′
is (-2a,+2b)
envelope parabola directrix
incoming asymptote
a
b
hyperbolic impact parameter
hyperbolic Focus Locus
r-r′ =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola
p=2a
F
circle of curvature at min-point of parabola
dΘ
Θ(b)

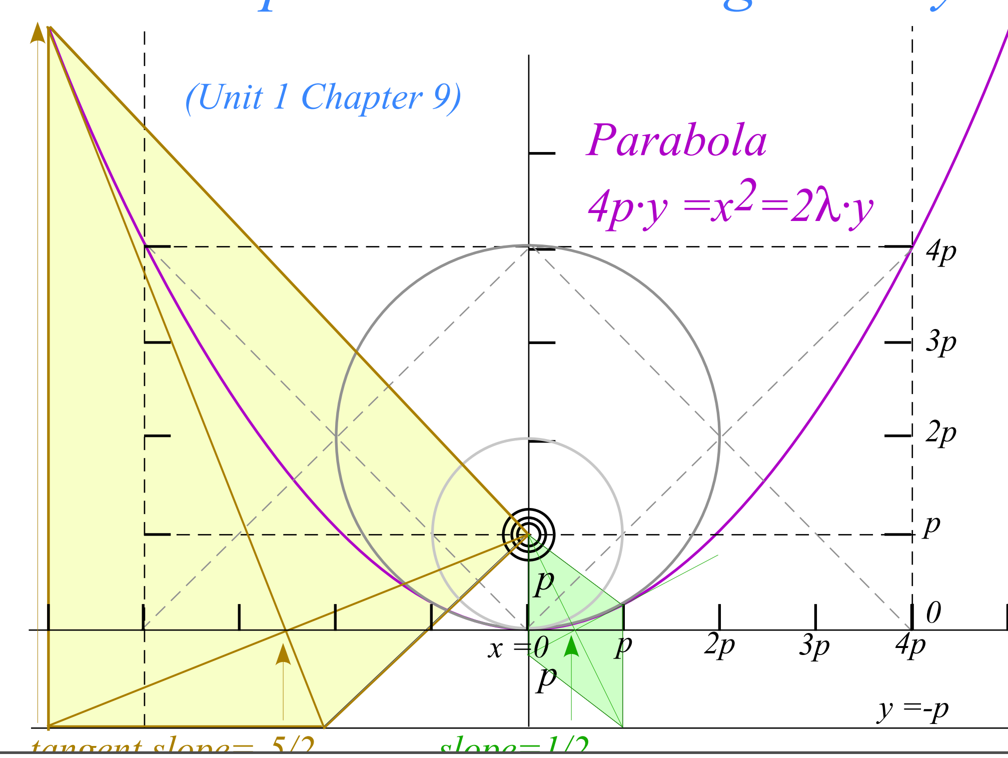
Rutherford scattering geometry



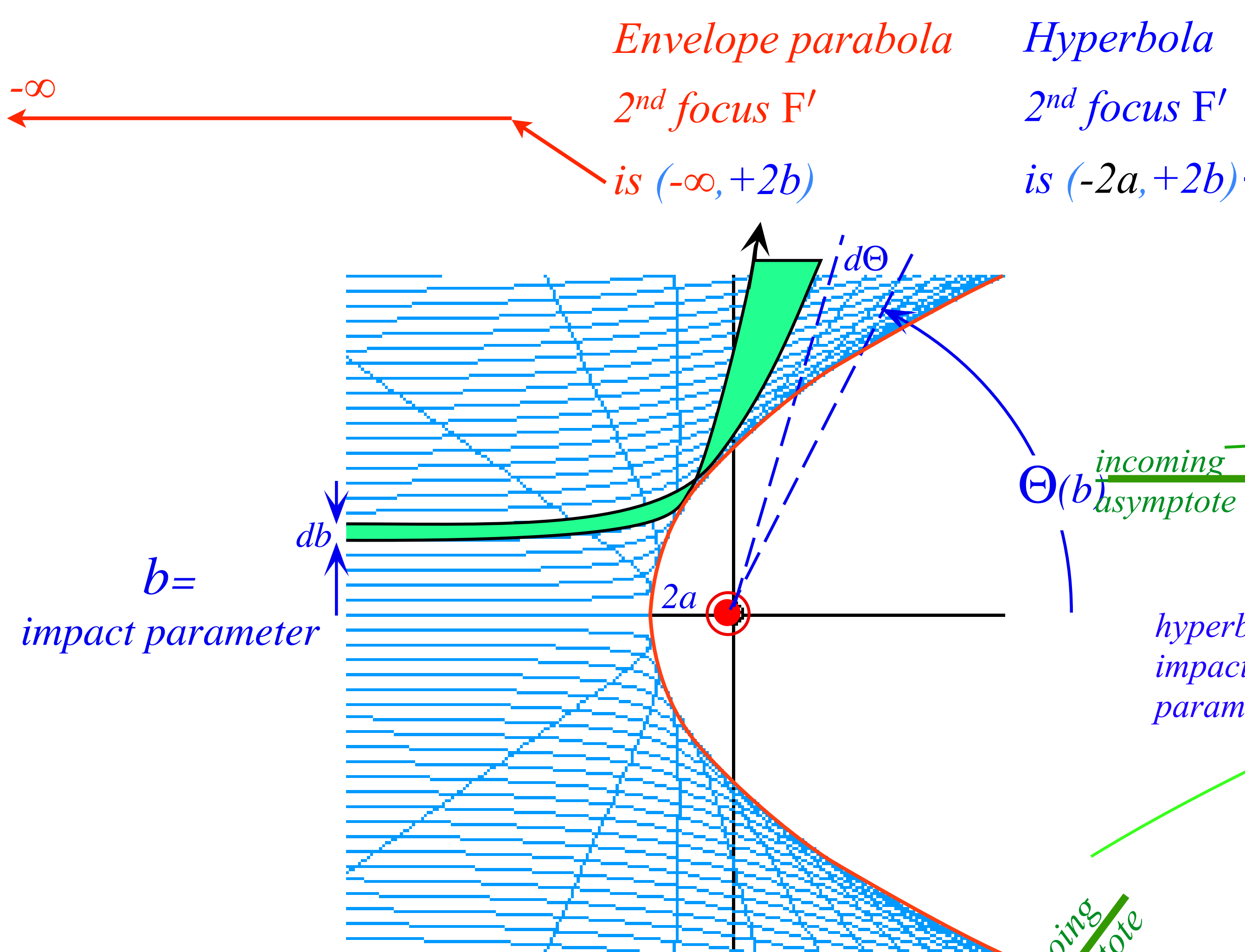
"Kite" geometry of envelope parabola



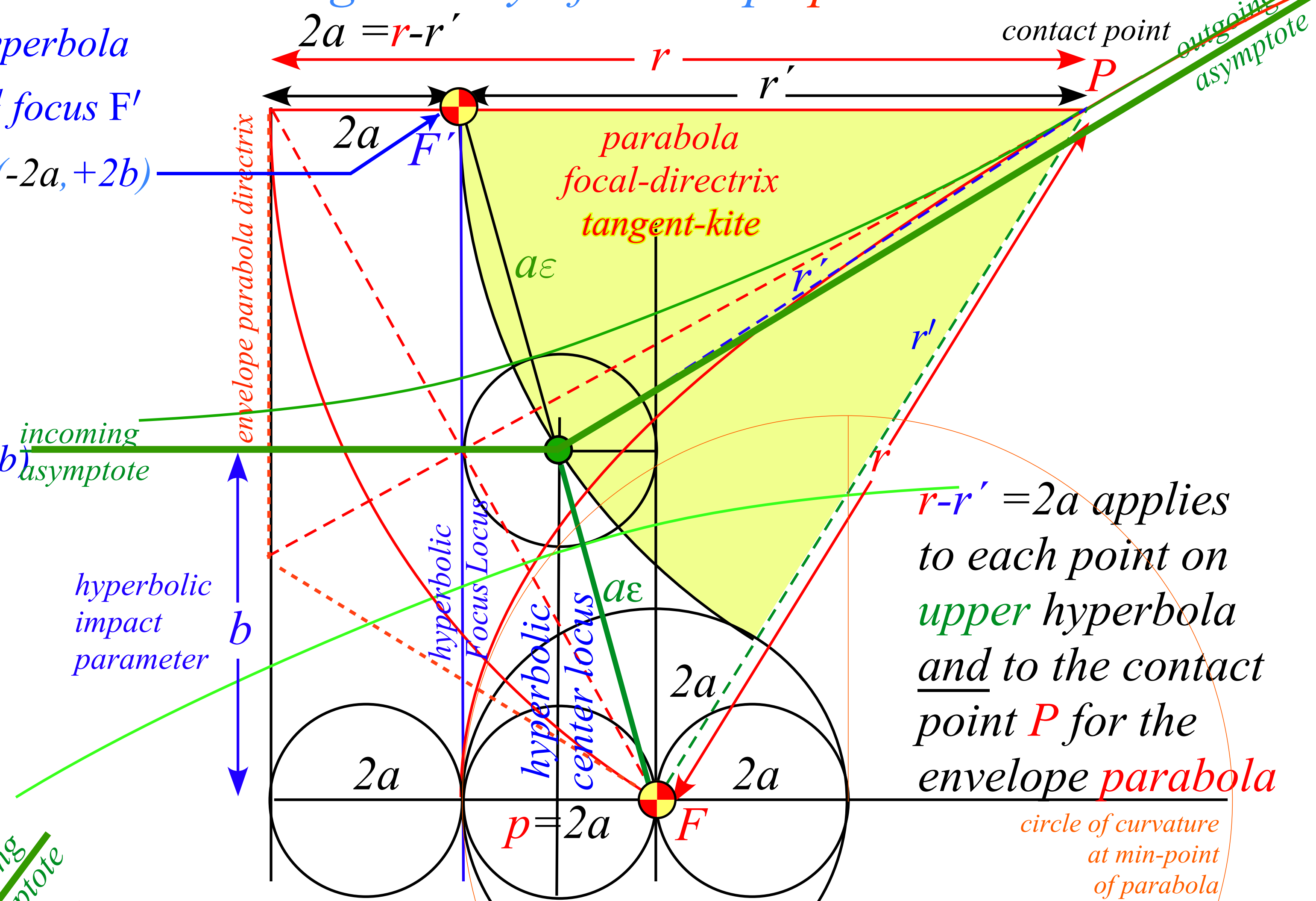
Recall parabolic " kite" geometry



Rutherford scattering geometry



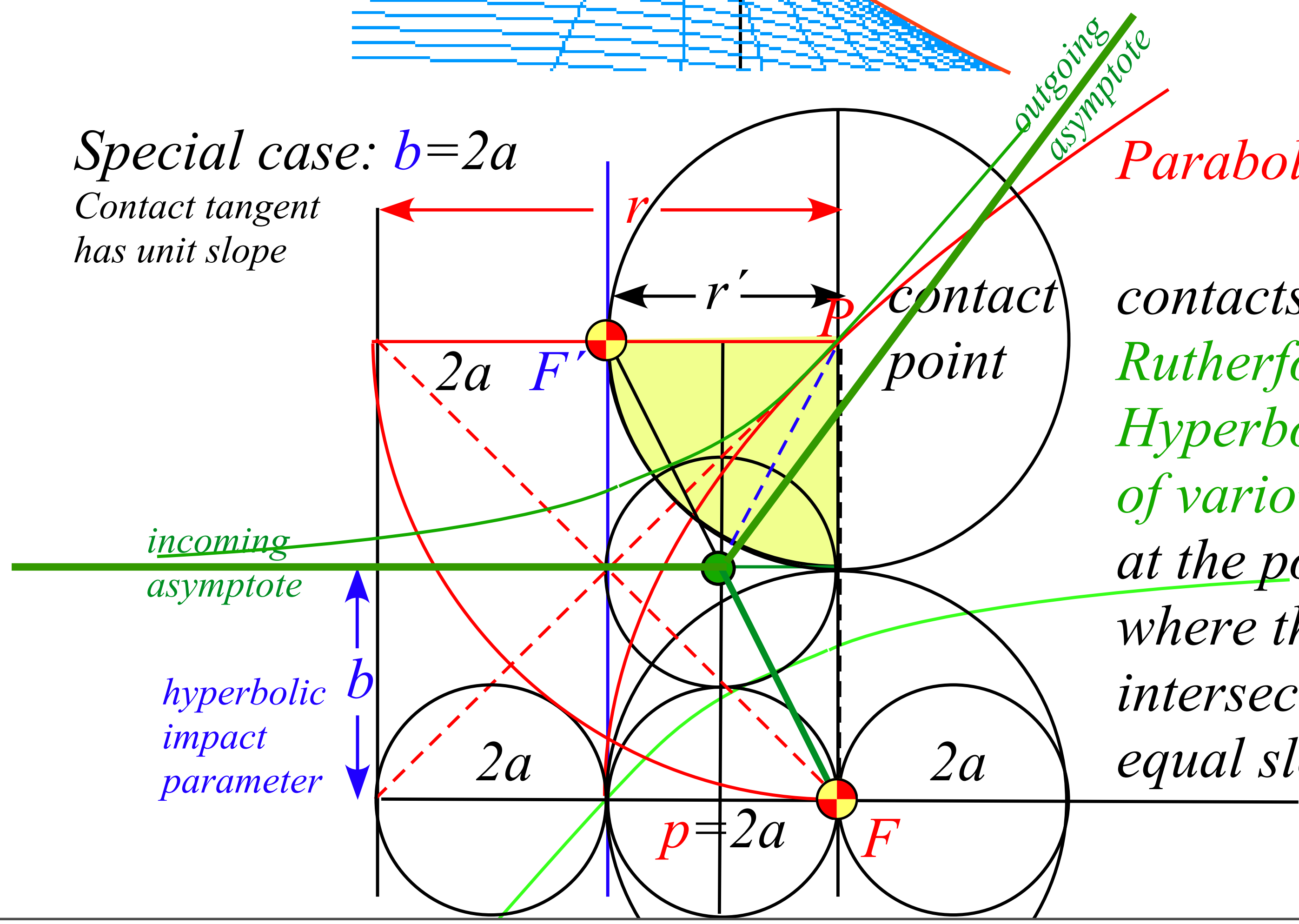
"Kite" geometry of envelope parabola



r-r' =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola

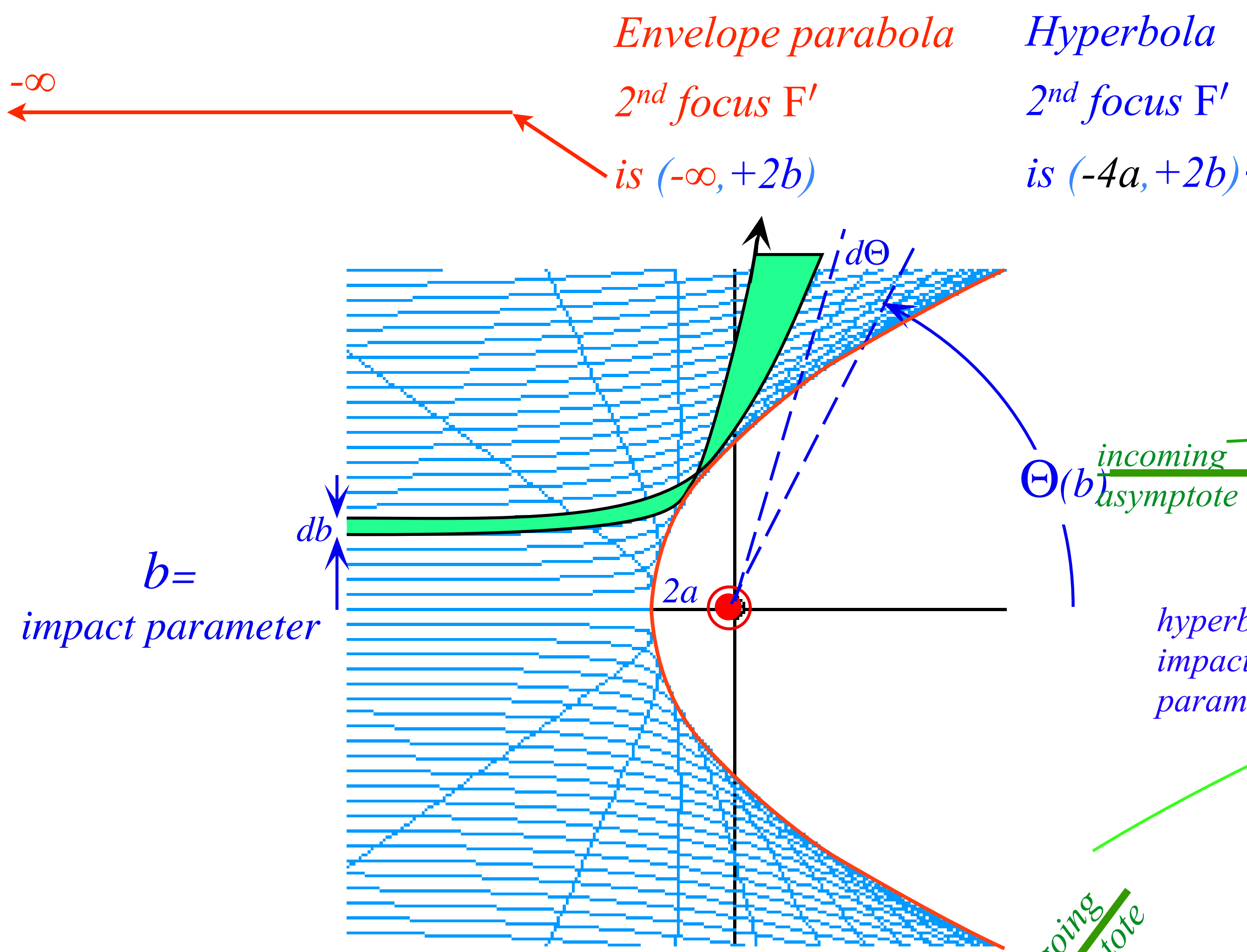
Special case: b=2a

Contact tangent has unit slope

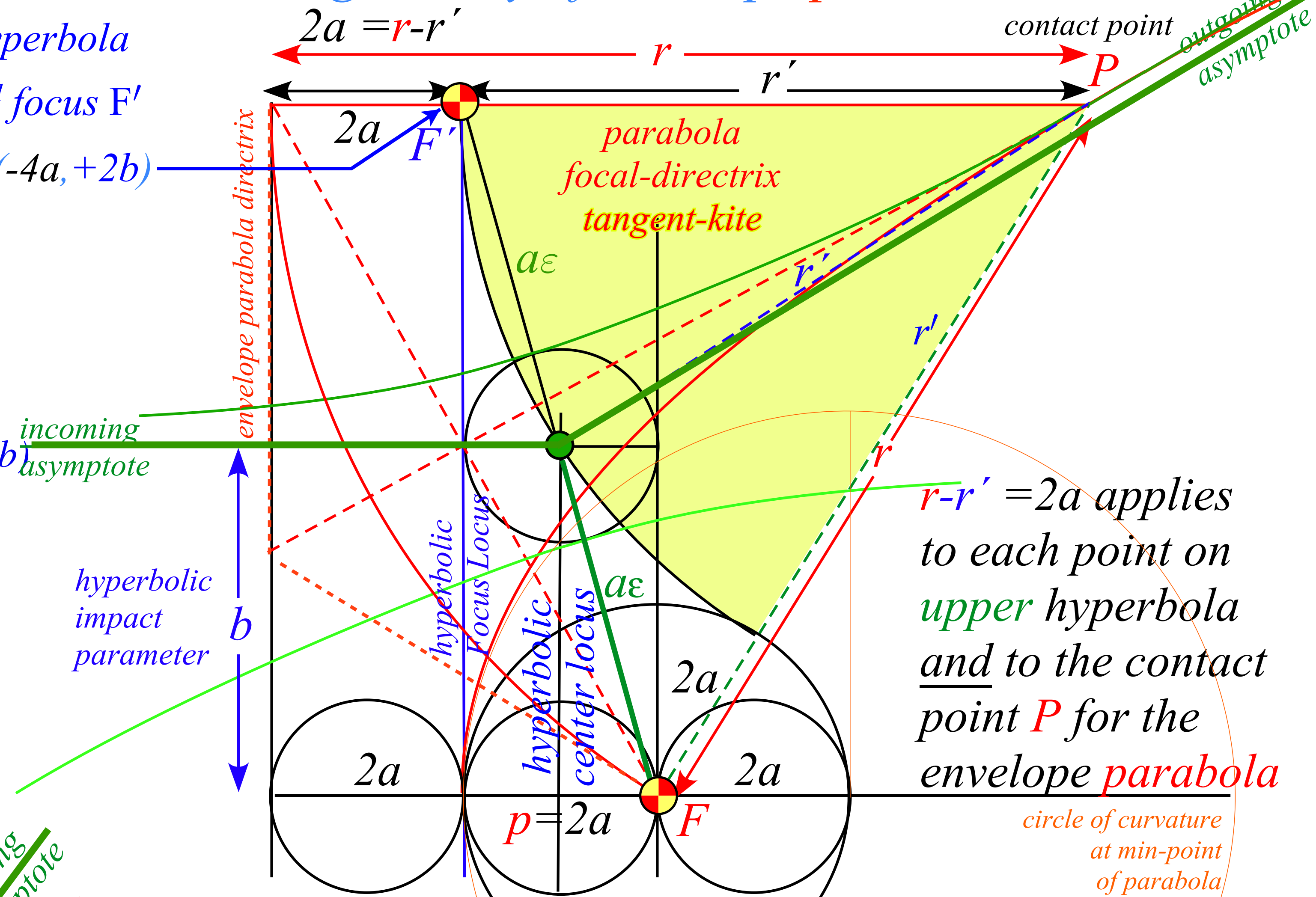


contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Rutherford scattering geometry

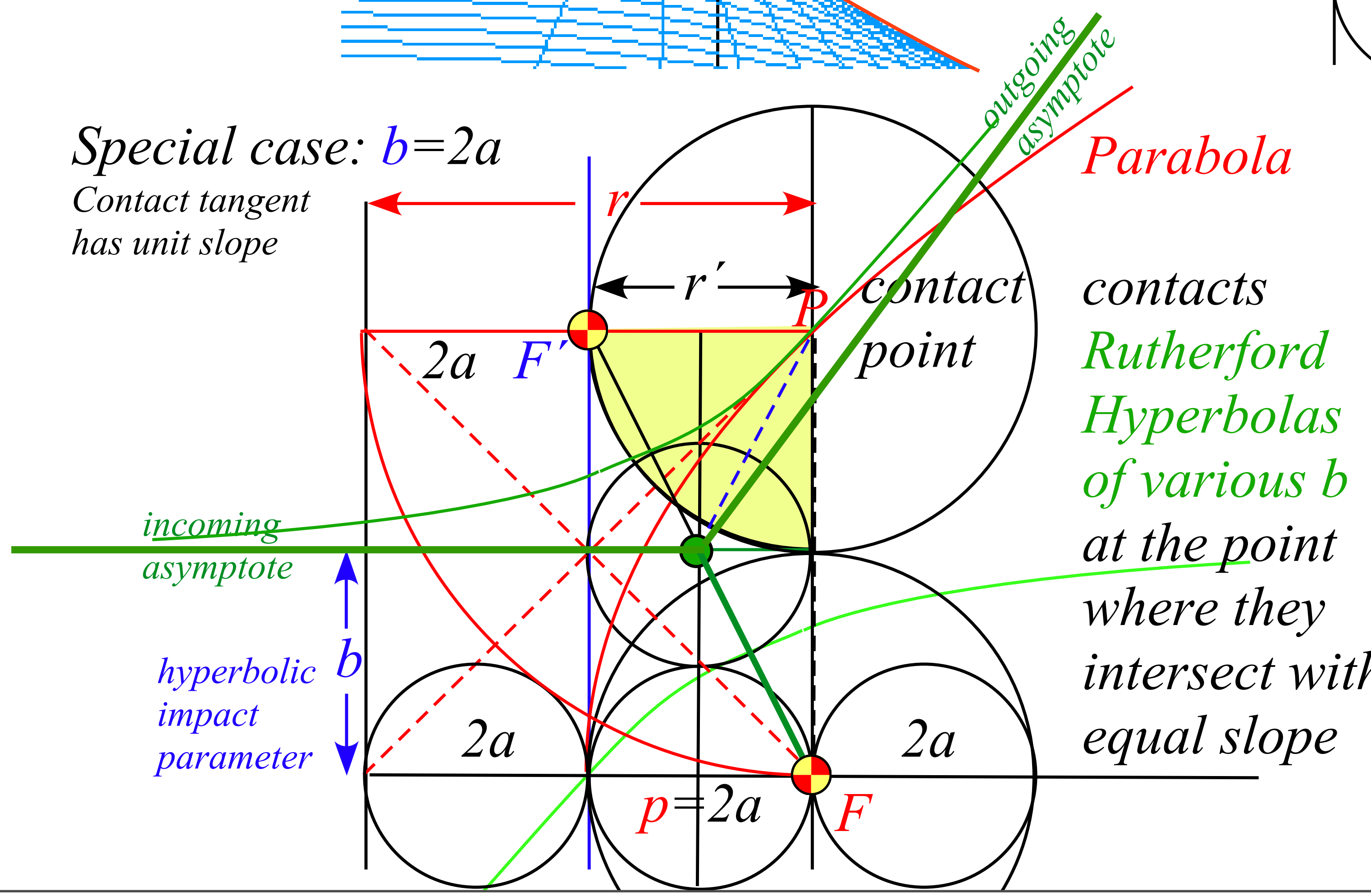


"Kite" geometry of envelope parabola



Special case: b=2a

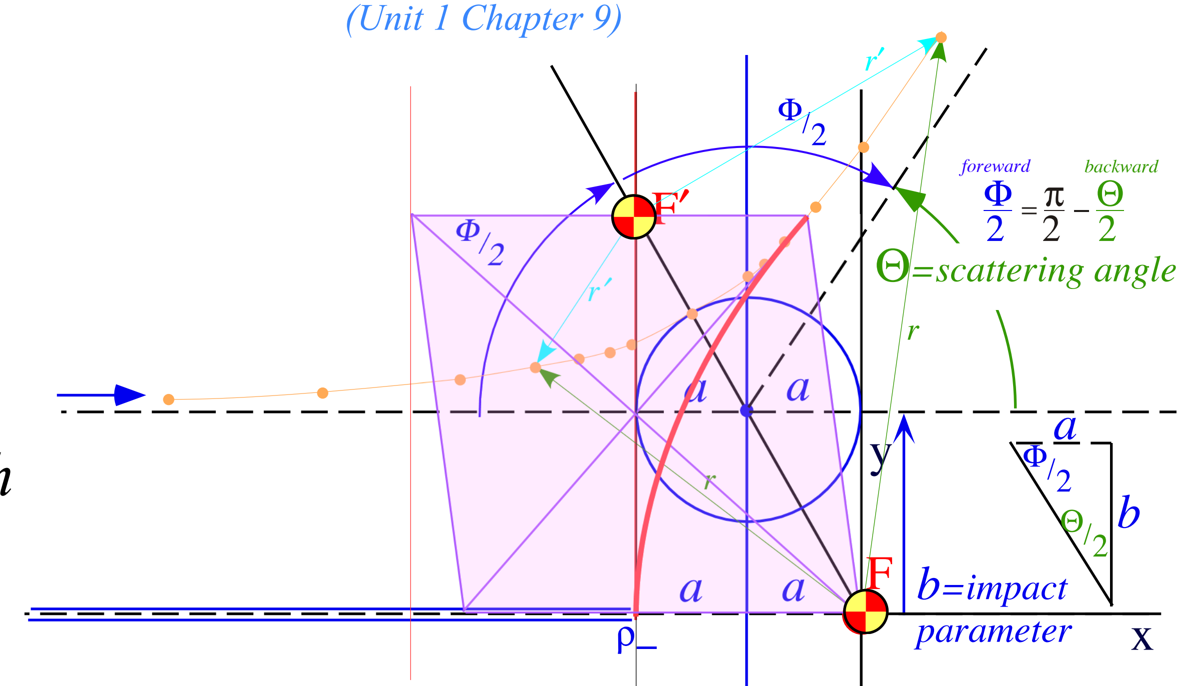
Contact tangent
has unit slope



contacts
Rutherford
Hyperbolas
of various b
at the point
where they
intersect with
equal slope

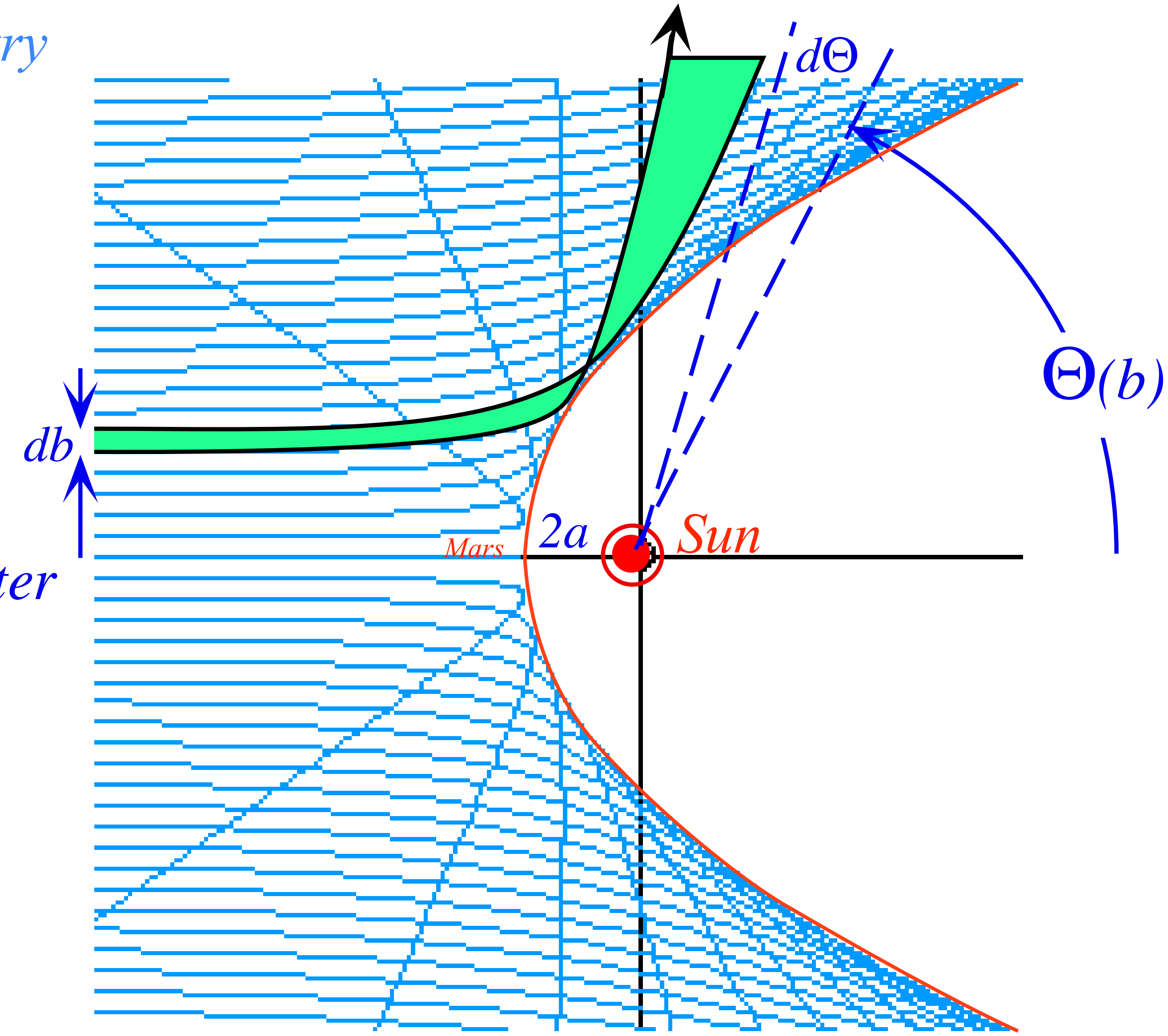
Recall parabolic " kite" geometry

(Unit 1 Chapter 9)



Rutherford scattering geometry

b= impact parameter



Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

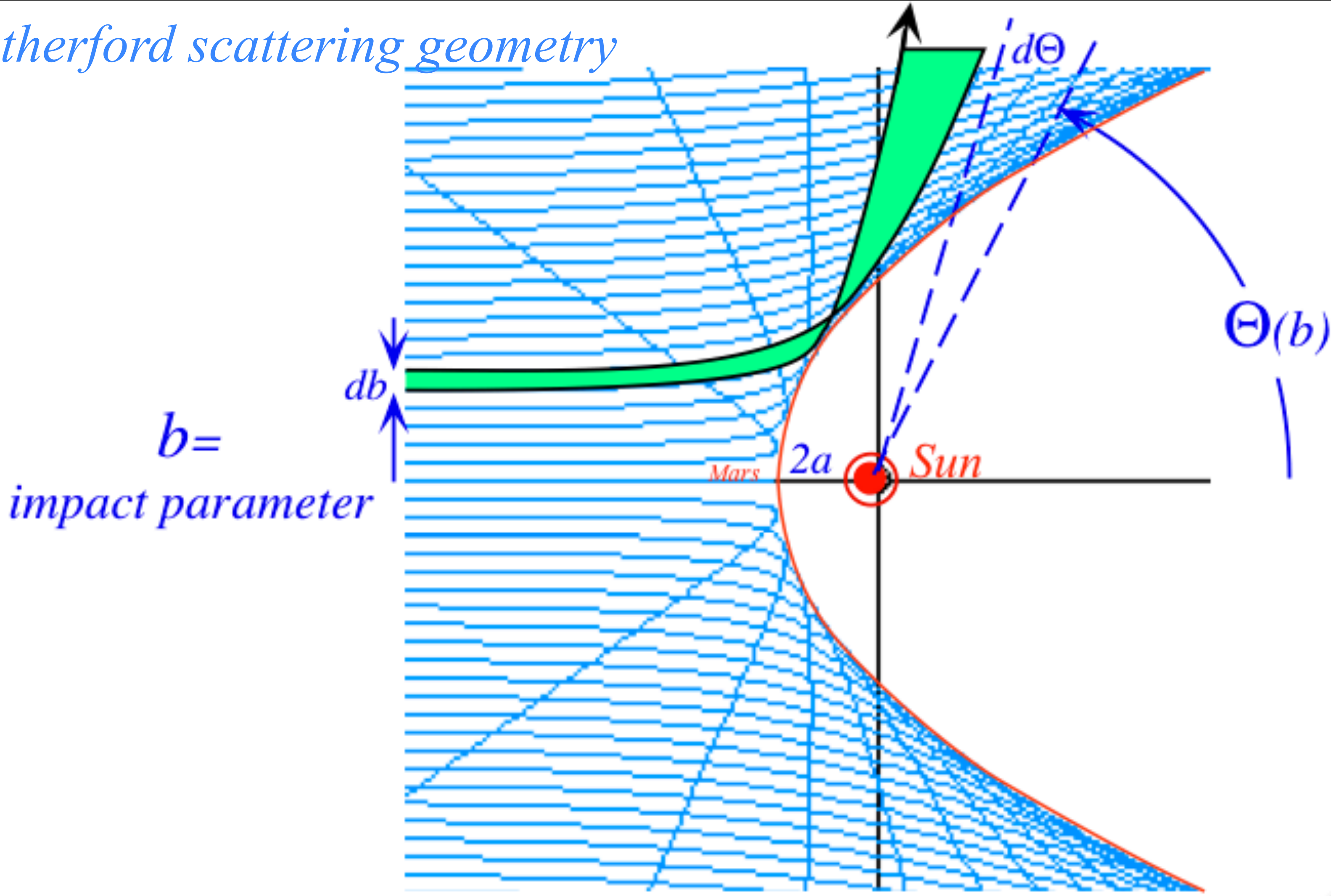
Incremental window $d\sigma = b \cdot db$ normal to beam axis at $x=-\infty$ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at $R=+\infty$ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Ratio $\dfrac{d\sigma}{d\Omega} = \dfrac{b\,db\,d\varphi}{\sin\Theta d\Theta d\varphi} = \dfrac{b}{\sin\Theta}\dfrac{db}{d\Theta}$ is called the *differential scattering crossection (DSC)*

Geometry $b = a\ \cot\dfrac{\Theta}{2} = \dfrac{k}{2E}\cot\dfrac{\Theta}{2}$ gives the *Rutherford DSC*. $\dfrac{d\sigma}{d\Omega} = \dfrac{k^4}{16E^2}\sin^{-4}\dfrac{\Theta}{2}$

Agrees exactly with 1st Born approximation to *quantum* Coulomb DSC!

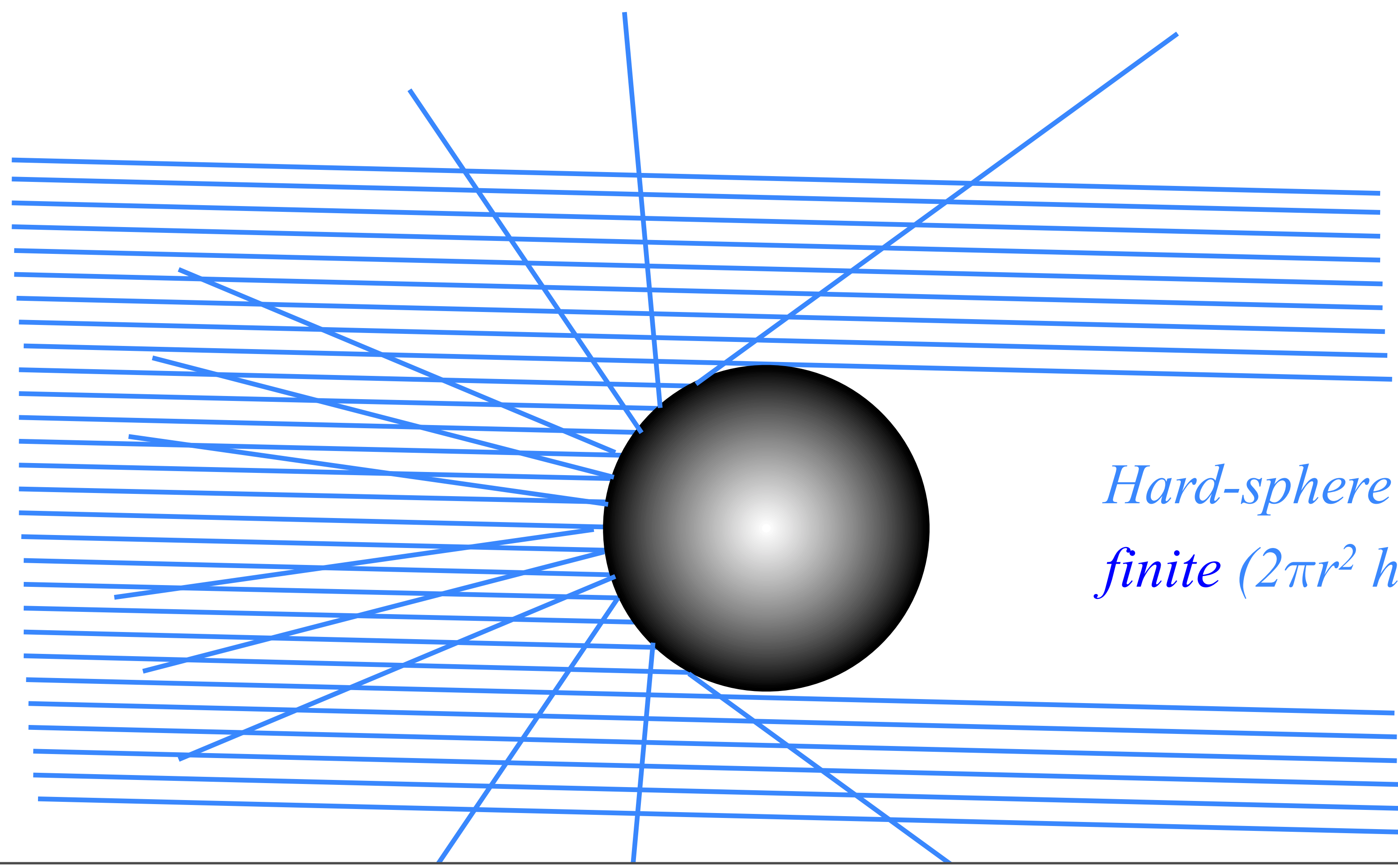
Rutherford scattering geometry



Two Extremes:

*Rutherford (Coulomb) scattering has **infinite** (∞) total cross section*

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \frac{k^4}{16E^2} \sin^{-4}\frac{\Theta}{2} = \infty$$



*Hard-sphere scattering has **finite** ($2\pi r^2$ here) total cross section*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

➔ $\boldsymbol{\varepsilon}$*-vector and Coulomb* **r***-orbit geometry*

Review and connection to standard development

$\boldsymbol{\varepsilon}$*-vector and Coulomb* **p**$=m$**v** *geometry*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* **p**$=m$**v** *algebra*

Example with elliptical orbit

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Algebra of $\boldsymbol{\varepsilon}$*-construction geometry*

Connection formulas for (a,b) and (ε,λ) *with* (γ,R)

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Eccentricity vector **ε** *and (ε,λ) geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* **ε**

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

Eccentricity vector ε *and (ε,λ) geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* ε

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p} \times \mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A = \tfrac{1}{2}(x_1^2 + p_1^2 - x_2^2 - p_2^2)$$
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Generate symmetry groups: $U(2) \subset U(2)$

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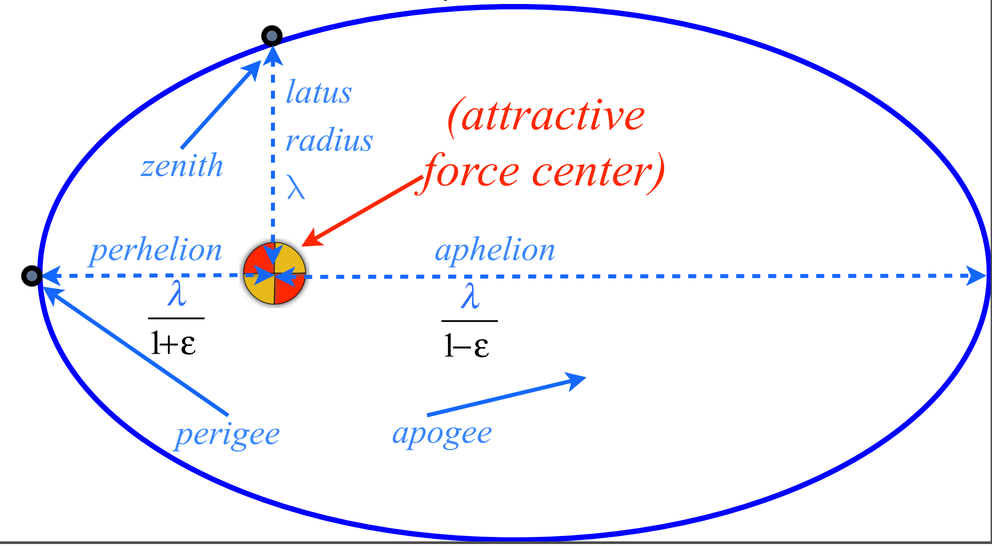

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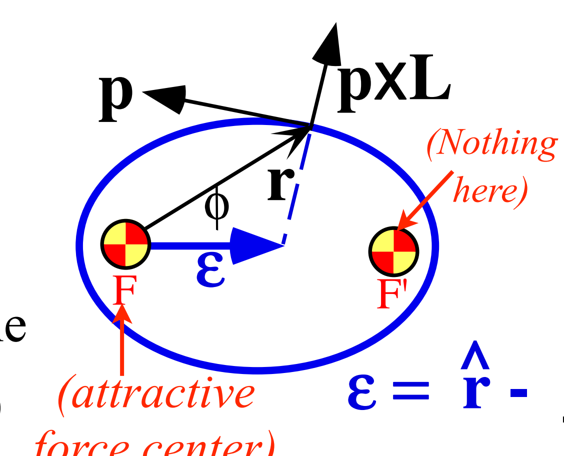

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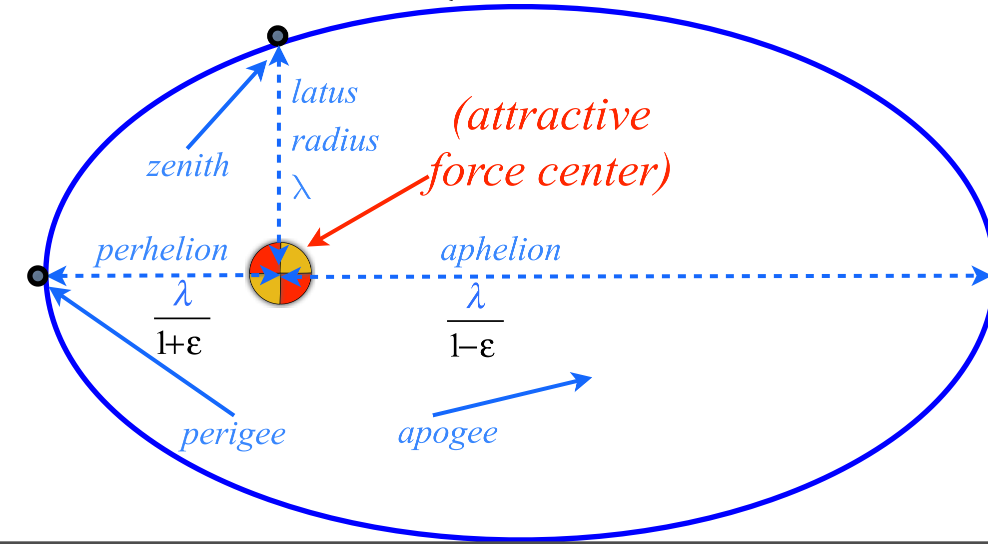

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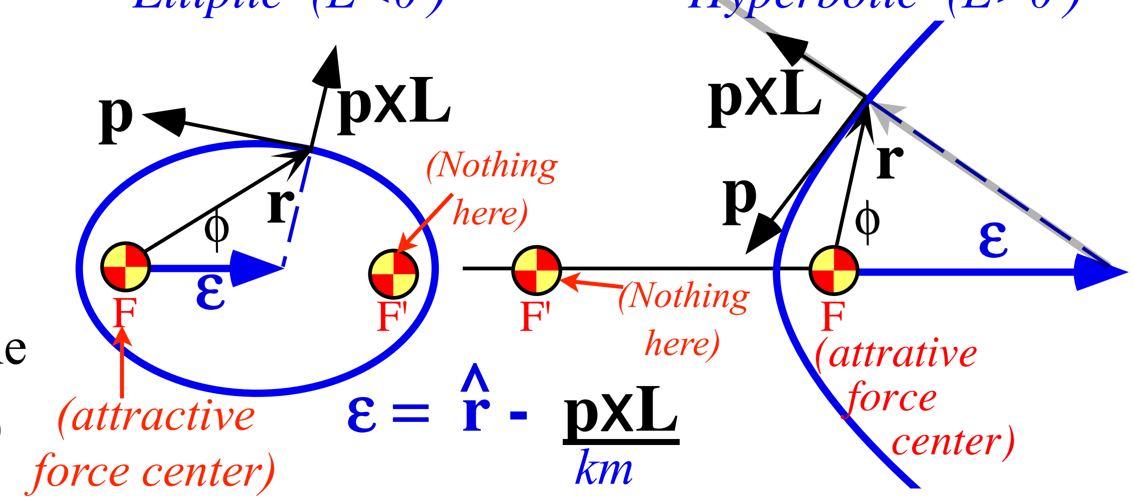

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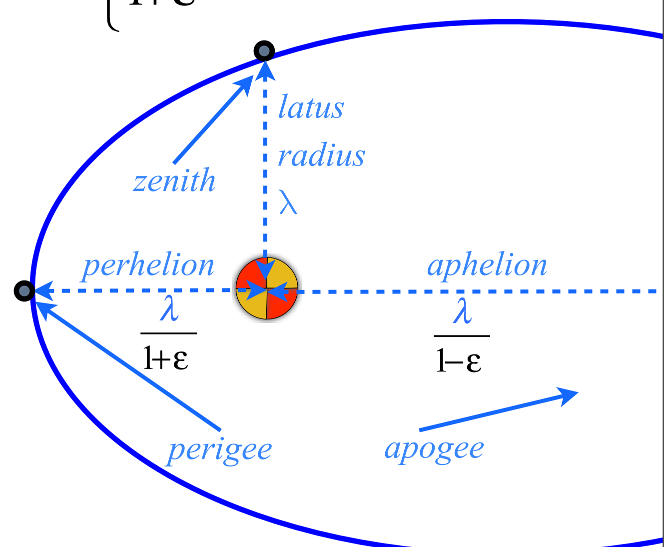

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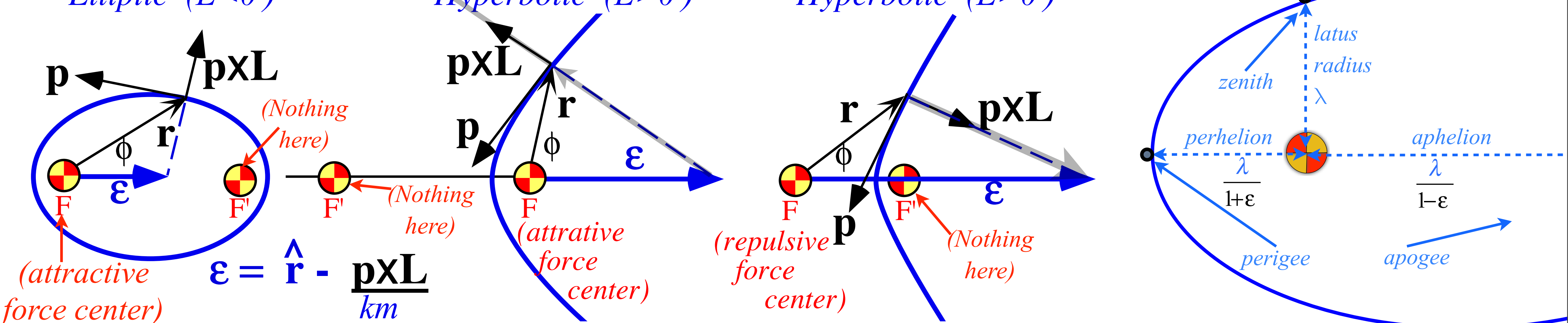


Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* **r***-orbit geometry*

➔ *Review and connection to standard development*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *geometry*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *algebra*

Example with elliptical orbit

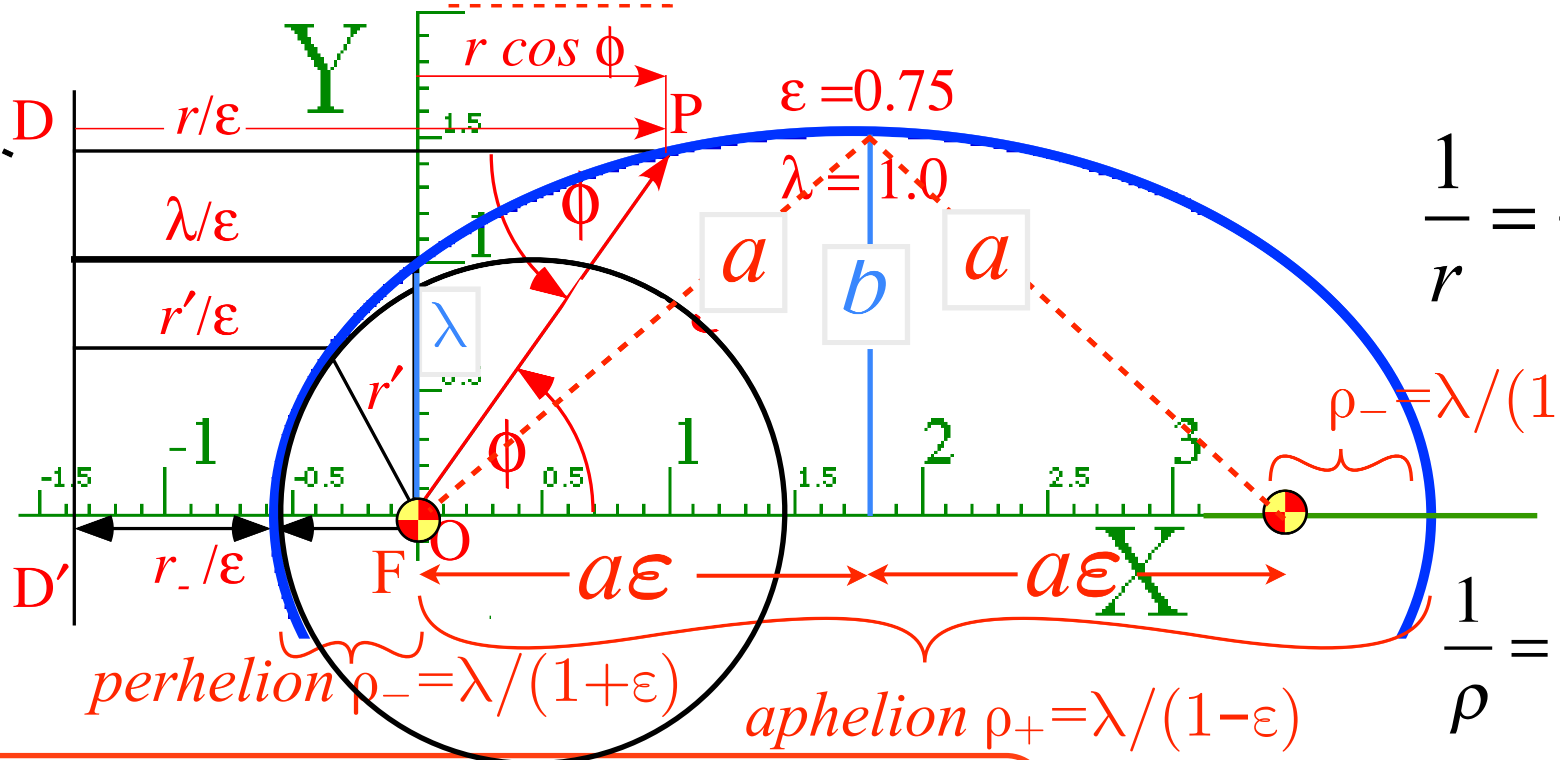
Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Algebra of $\boldsymbol{\varepsilon}$*-construction geometry*

Connection formulas for (a,b) and (ε,λ) *with* (γ,R)

(From Lecture 28 p. 64-74) *Geometry of Coulomb orbits (Let: r = ρ here)*

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by:

*Defining **eccentricity** ε*

Distance to F~ocal~-point $= \varepsilon \cdot$ Distance to D~irectrix~-line

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon<1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon>1$)*

(x,y) *parameters*	*physical constants*	(r,ϕ) *parameters*
$a = \dfrac{k}{2E}$	$E = \dfrac{k}{2a}$	$\varepsilon = \sqrt{\dfrac{k^2m + 2L^2E}{k^2m}} = \sqrt{1 \pm \dfrac{b^2}{a^2}}$
$b = \dfrac{L}{\sqrt{2m\lvert E\rvert}}$	$L = \sqrt{km\lambda}$	$\lambda = \dfrac{L^2}{km} = \dfrac{b^2}{a}$

$\varepsilon^2 = 1 - \dfrac{b^2}{a^2}$ (ellipse: $\varepsilon<1$) $\dfrac{b^2}{a^2} = \sqrt{1-\varepsilon^2}$

$\varepsilon^2 = 1 + \dfrac{b^2}{a^2}$ (hyperbola: $\varepsilon>1$) $\dfrac{b^2}{a^2} = \sqrt{\varepsilon^2 - 1}$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon<1$)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb: $\varepsilon>1$)*

Eccentricity vector **ε** *and* (ε,λ)*-geometry of orbital mechanics*

ε-*vector and Coulomb* **r**-*orbit geometry*

Review and connection to standard development

➤ **ε**-*vector and Coulomb* **p**=*m***v** *geometry*

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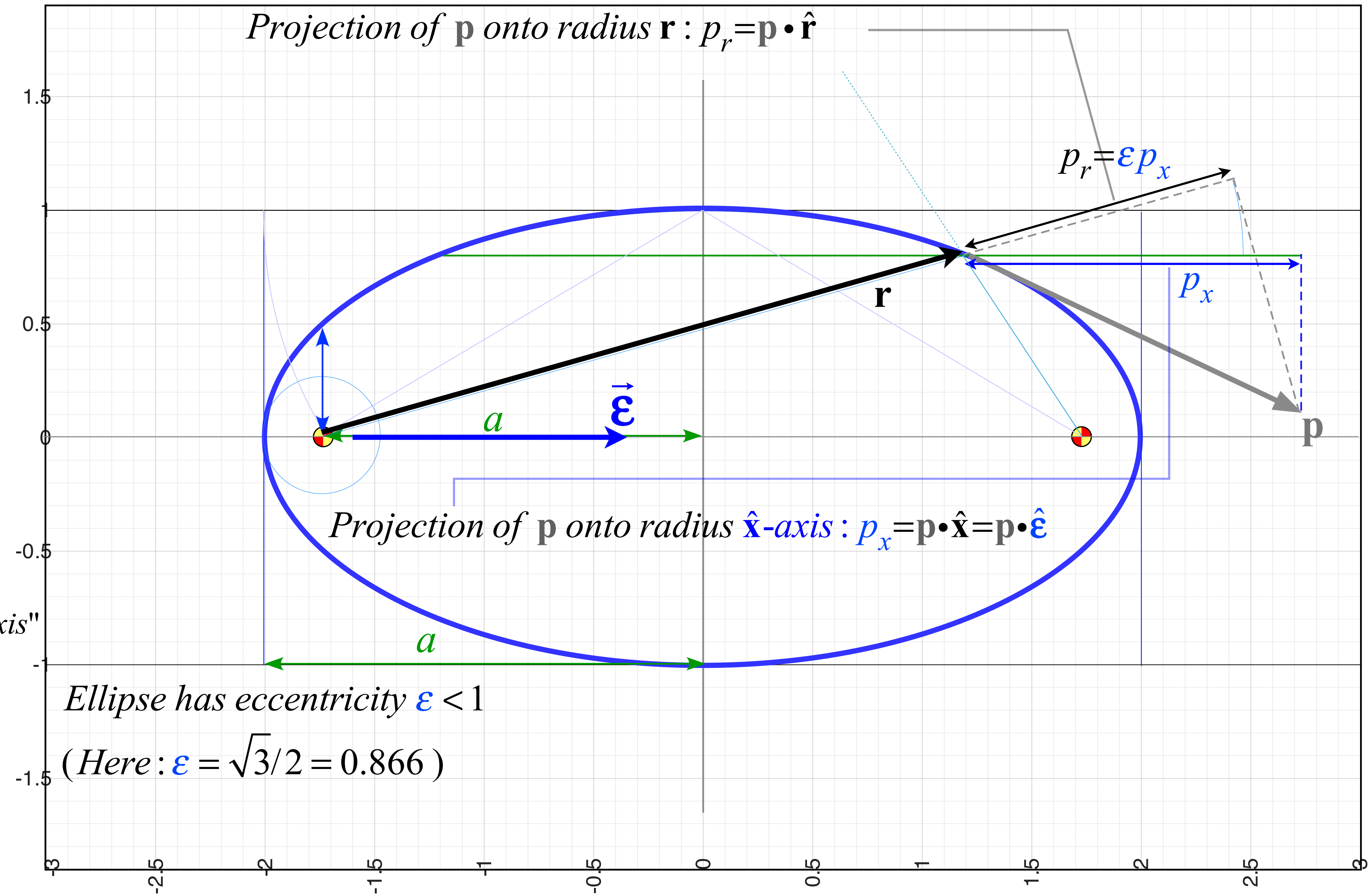
$$= \mathbf{p}\bullet\hat{\mathbf{r}} = p_r = \varepsilon p_x$$

This says:

"*Projection of* $\mathbf{p}$ *onto* $\mathbf{r}$ *is eccentricity* ε *times projection of* $\mathbf{p}$ *onto* $\hat{\mathbf{x}}$*-axis*"

($\hat{\mathbf{x}} = \hat{\boldsymbol{\varepsilon}}$)



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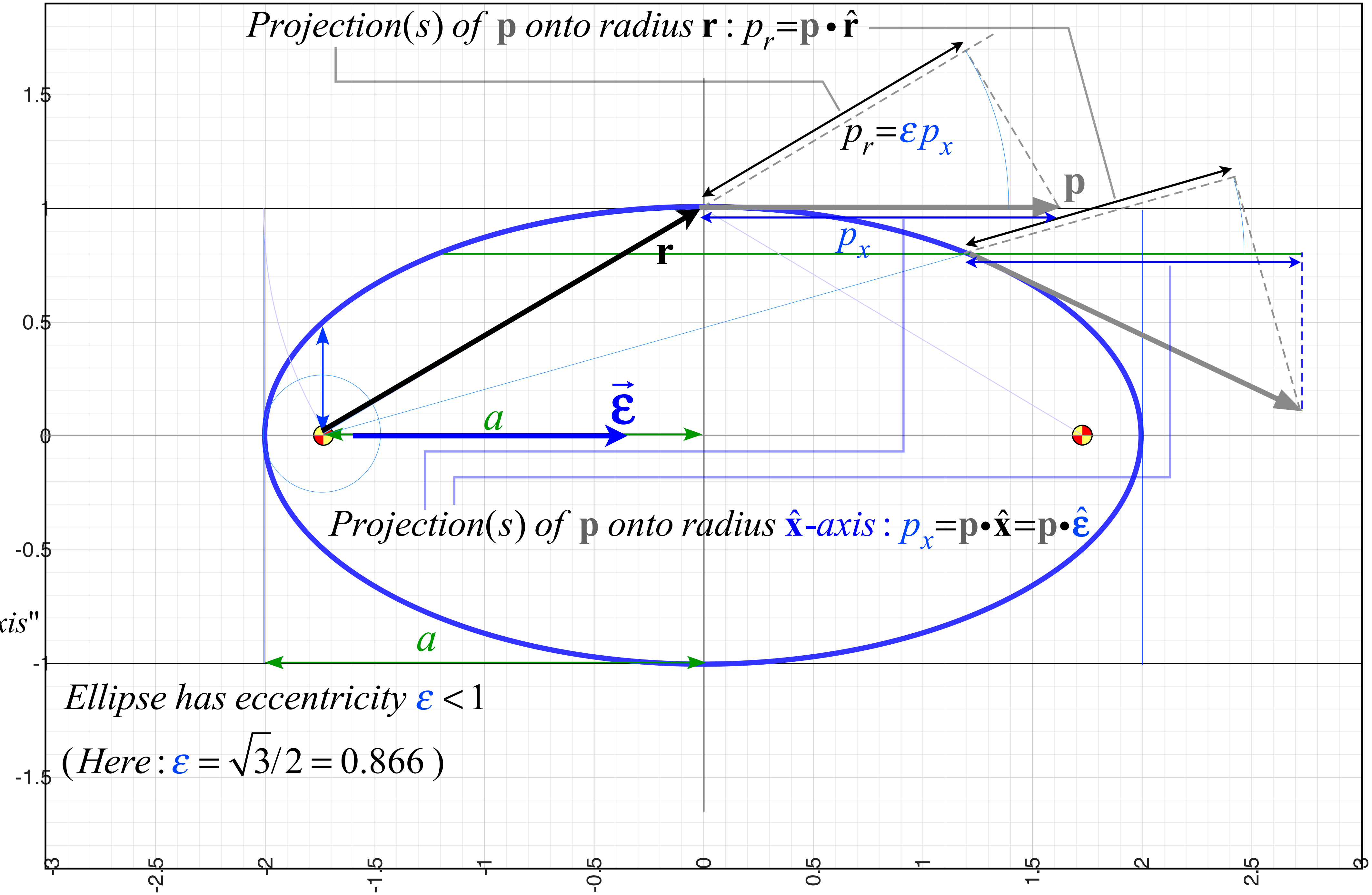

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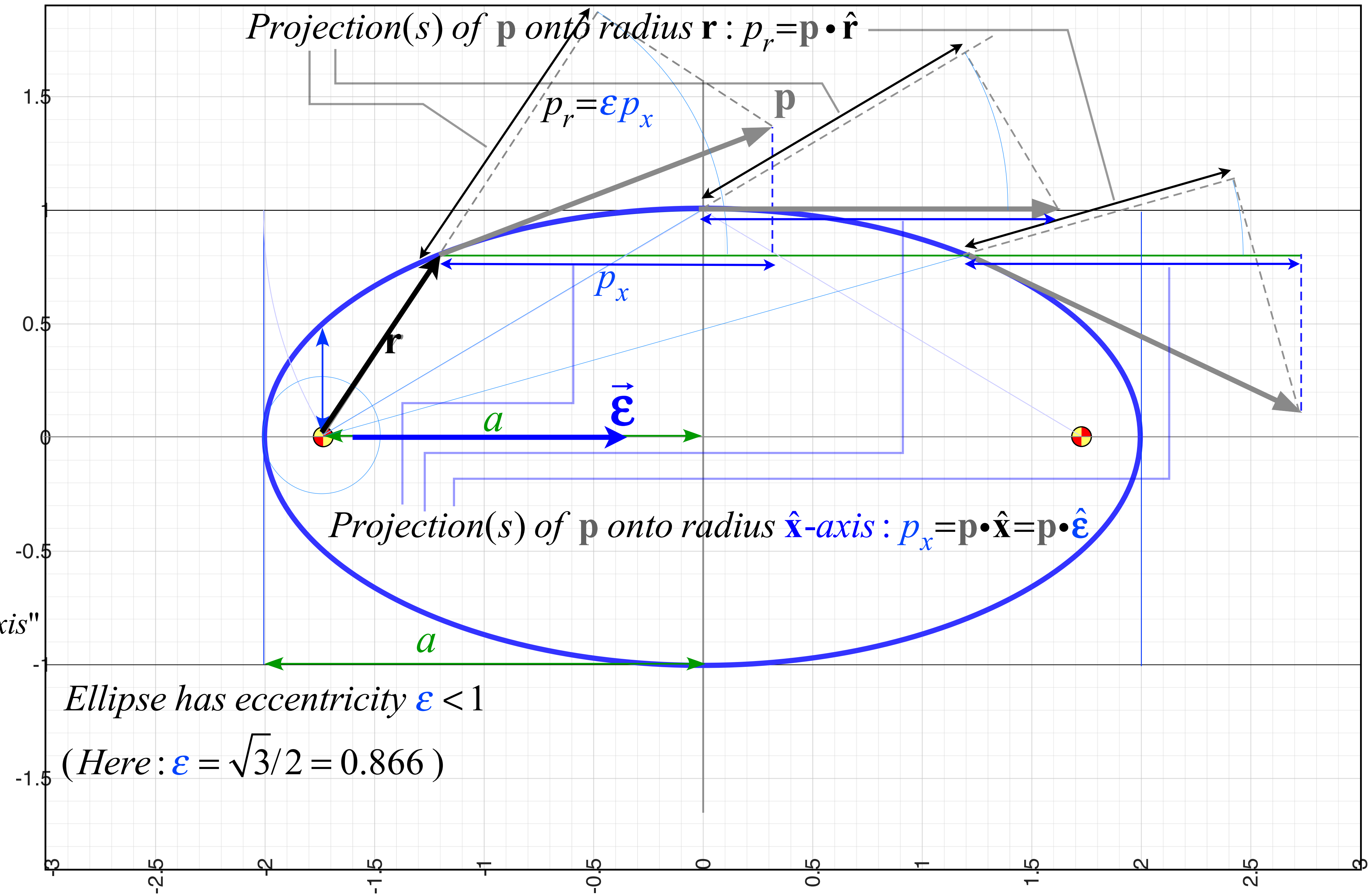

Dot product of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon} \bullet \mathbf{p} = \frac{\mathbf{p} \bullet \mathbf{r}}{r} - \frac{\mathbf{p} \bullet \mathbf{p} \times \mathbf{L}}{km}$$

$$= \mathbf{p} \bullet \hat{\mathbf{r}} = p_r = \varepsilon p_x$$

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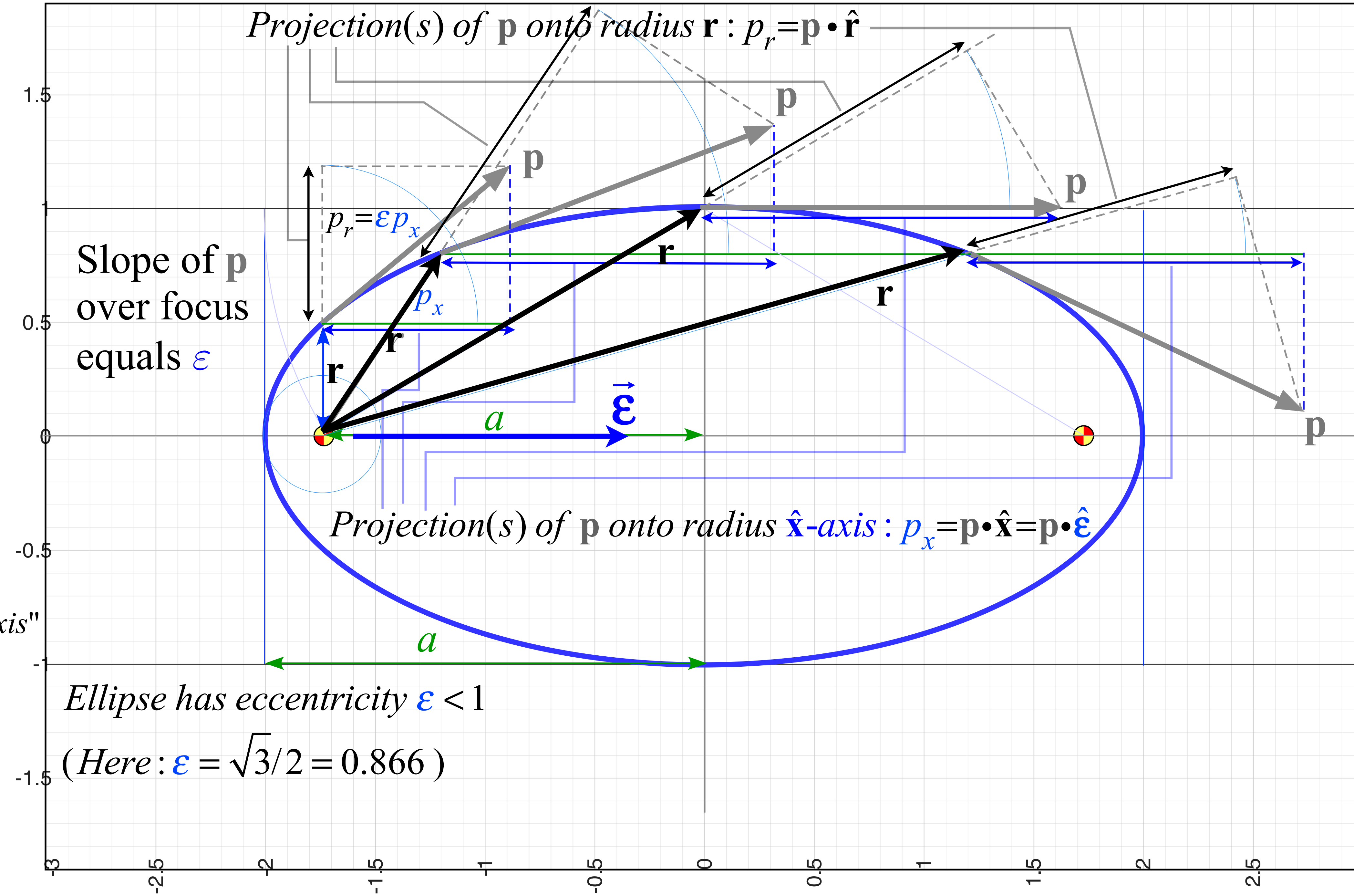

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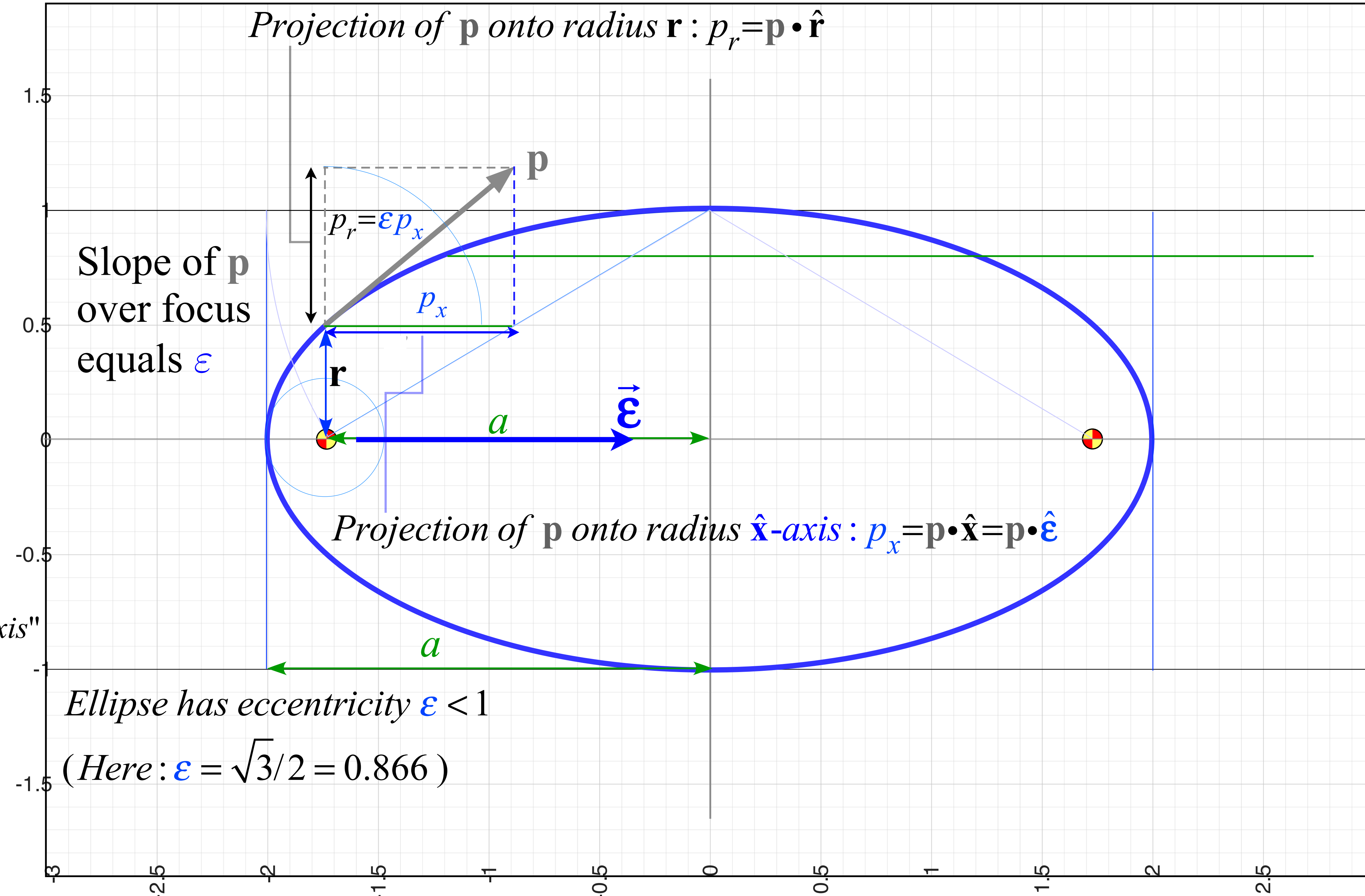

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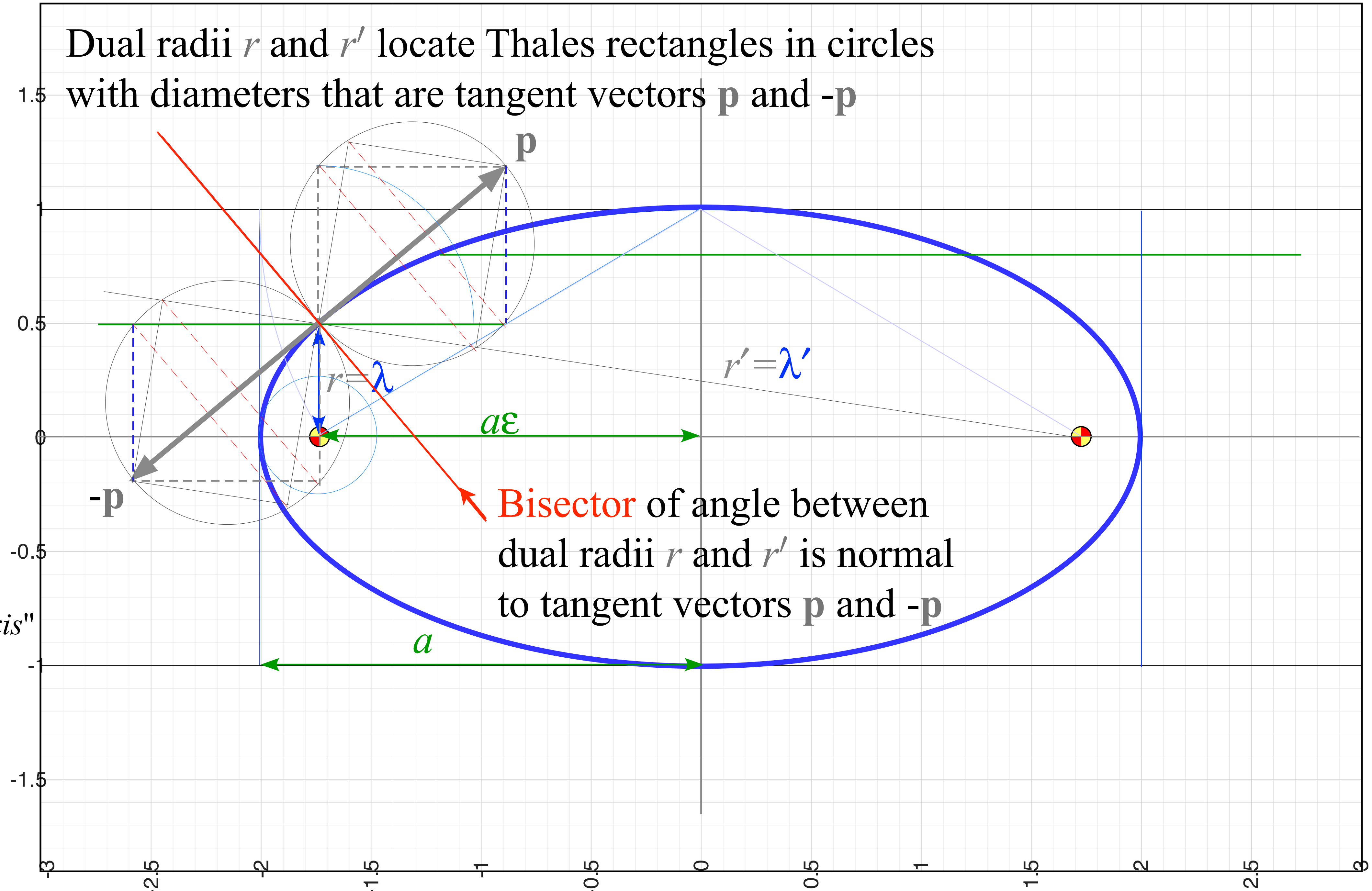

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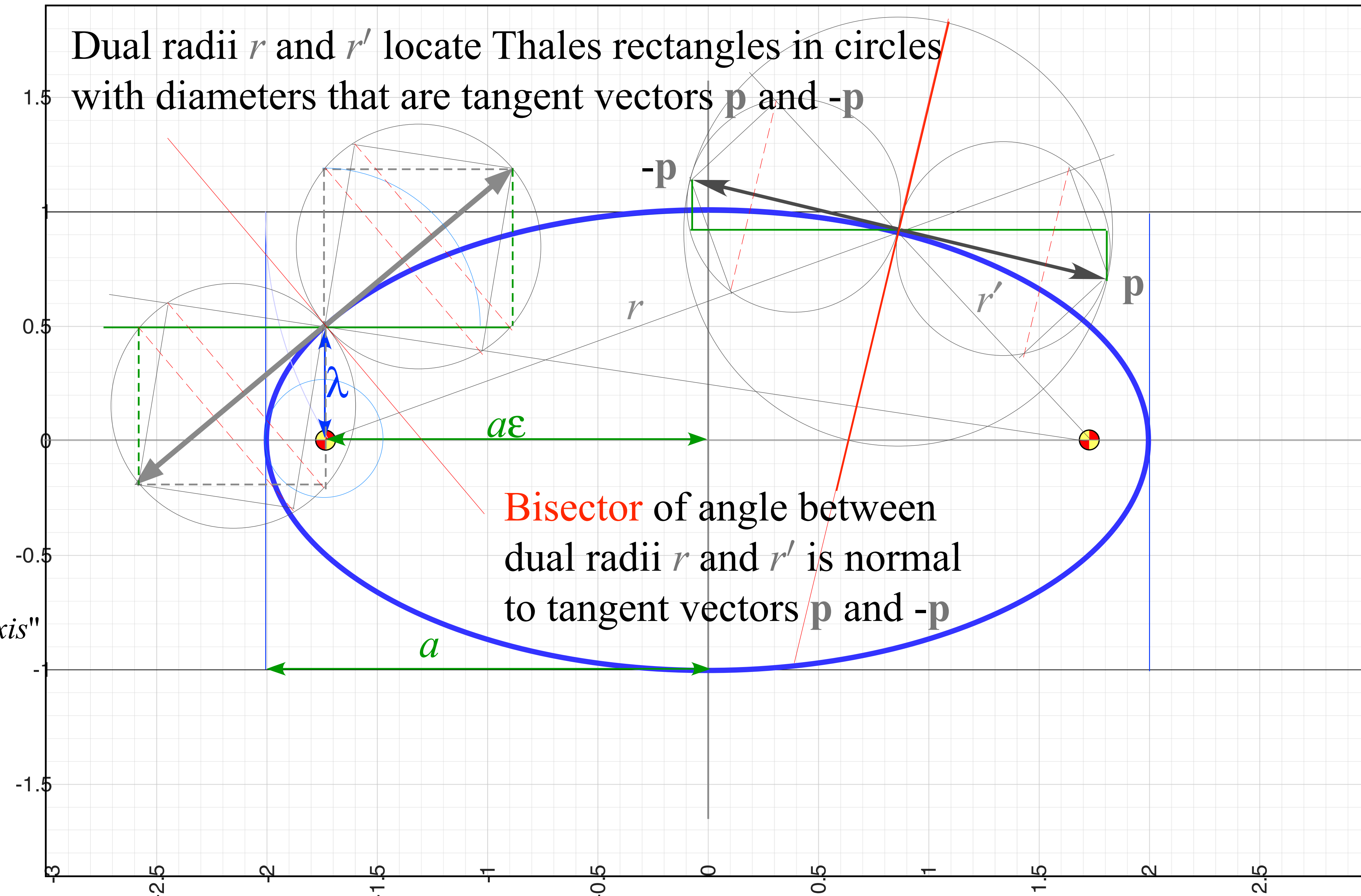

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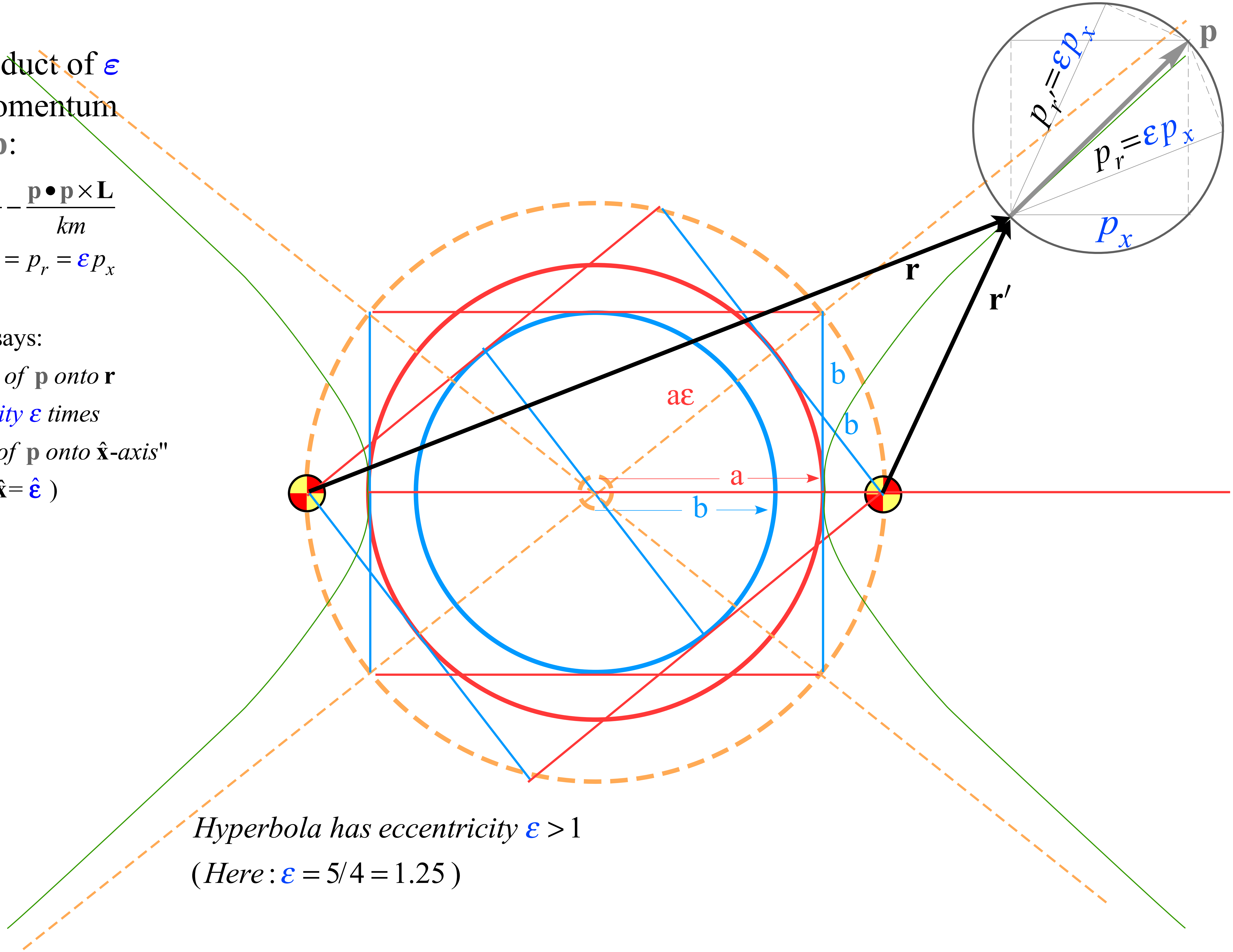

Hyperbola has eccentricity $\varepsilon > 1$

$(Here: \varepsilon = 5/4 = 1.25)$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* **r***-orbit geometry*

Review and connection to standard development

$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *geometry*

➔ $\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *algebra*

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ε-vector and Coulomb p=mv algebra

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Velocity:

Momentum:

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Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

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➔ *Example with elliptical orbit*

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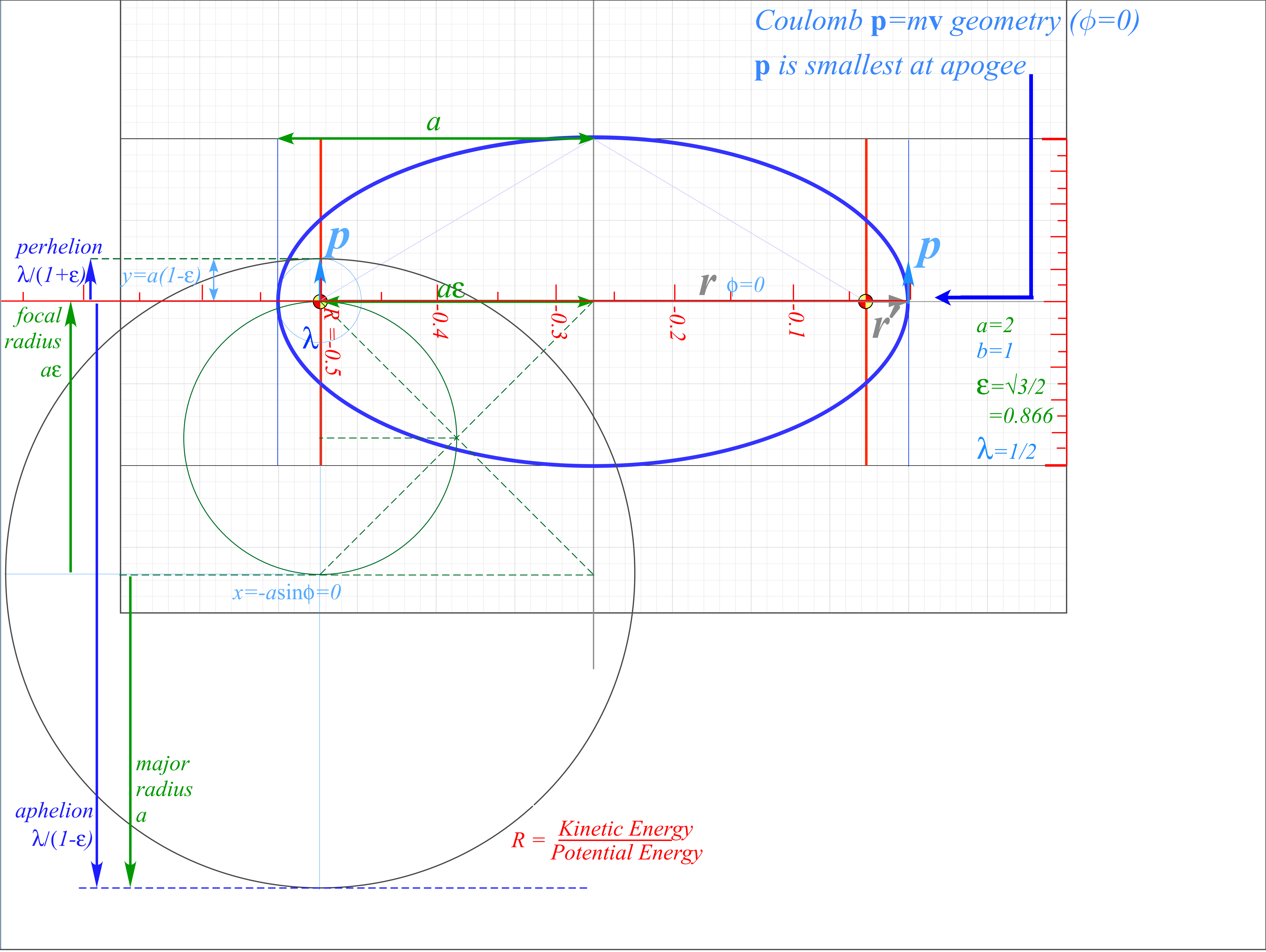
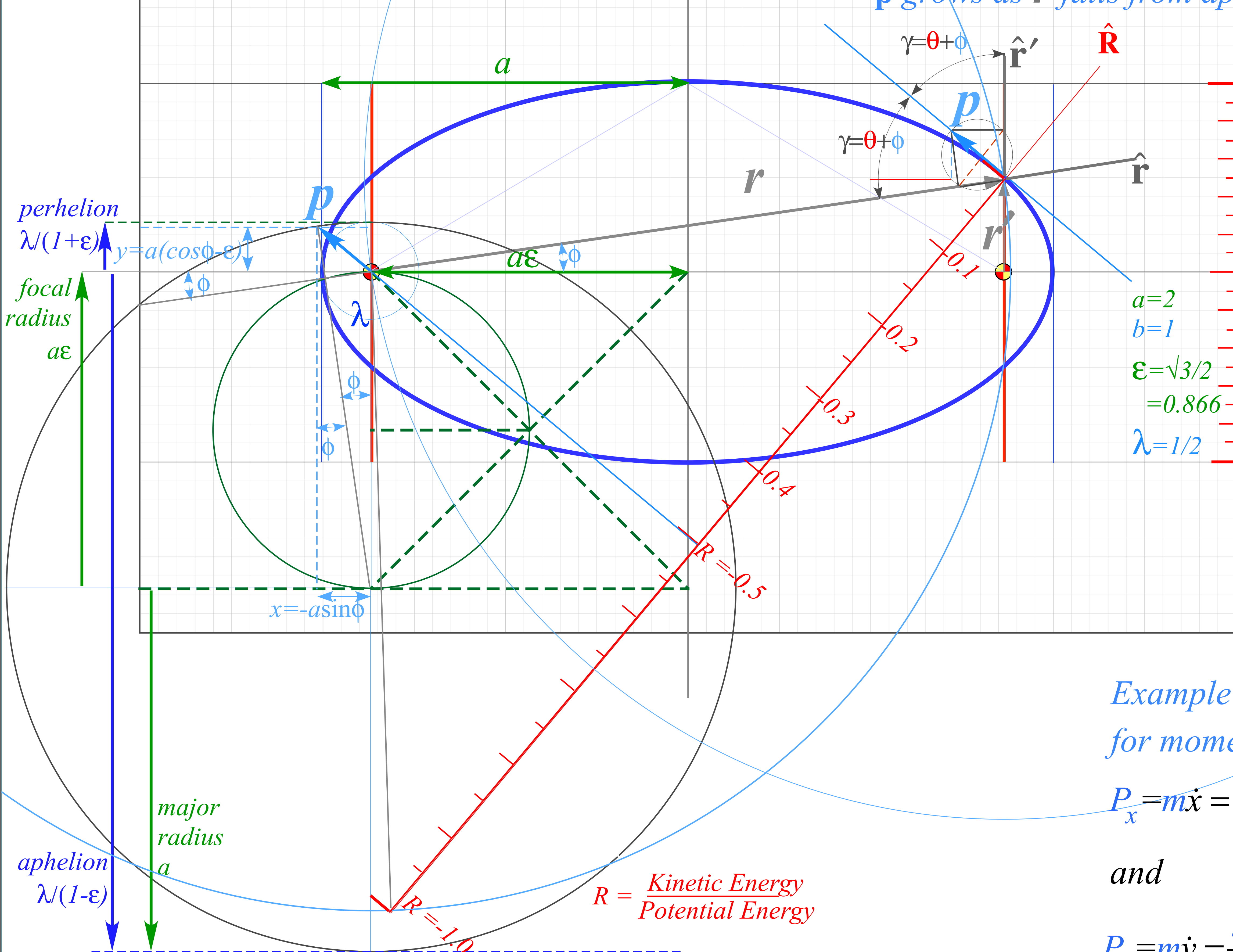


Coulomb **p**=m**v** geometry (φ=0)
p is smallest at apogee
a
p
p
perhelion
λ/(1+ε)
y=a(1-ε)
aε
r φ=0
r'
focal
radius
aε
λ
R=-0.5
-0.4
-0.3
-0.2
-0.1
a=2
b=1
ε=√3/2
=0.866
λ=1/2
x=-asinφ=0
major
radius
a
aphelion
λ/(1-ε)
R = Kinetic Energy / Potential Energy

Coulomb **p**=*m***v** *geometry (ϕ >0)*

p *grows as* ***r*** *falls from apogee*



$R=\frac{KE}{PE}$ *scale line* bisects angle$\sphericalangle^{+\hat{r}'}_{+\hat{r}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is ⊥ to **p**

p *momentum line* bisects angle$\sphericalangle^{+\hat{r}'}_{-\hat{r}}=2\gamma$

Example of geometry for momentum functions:

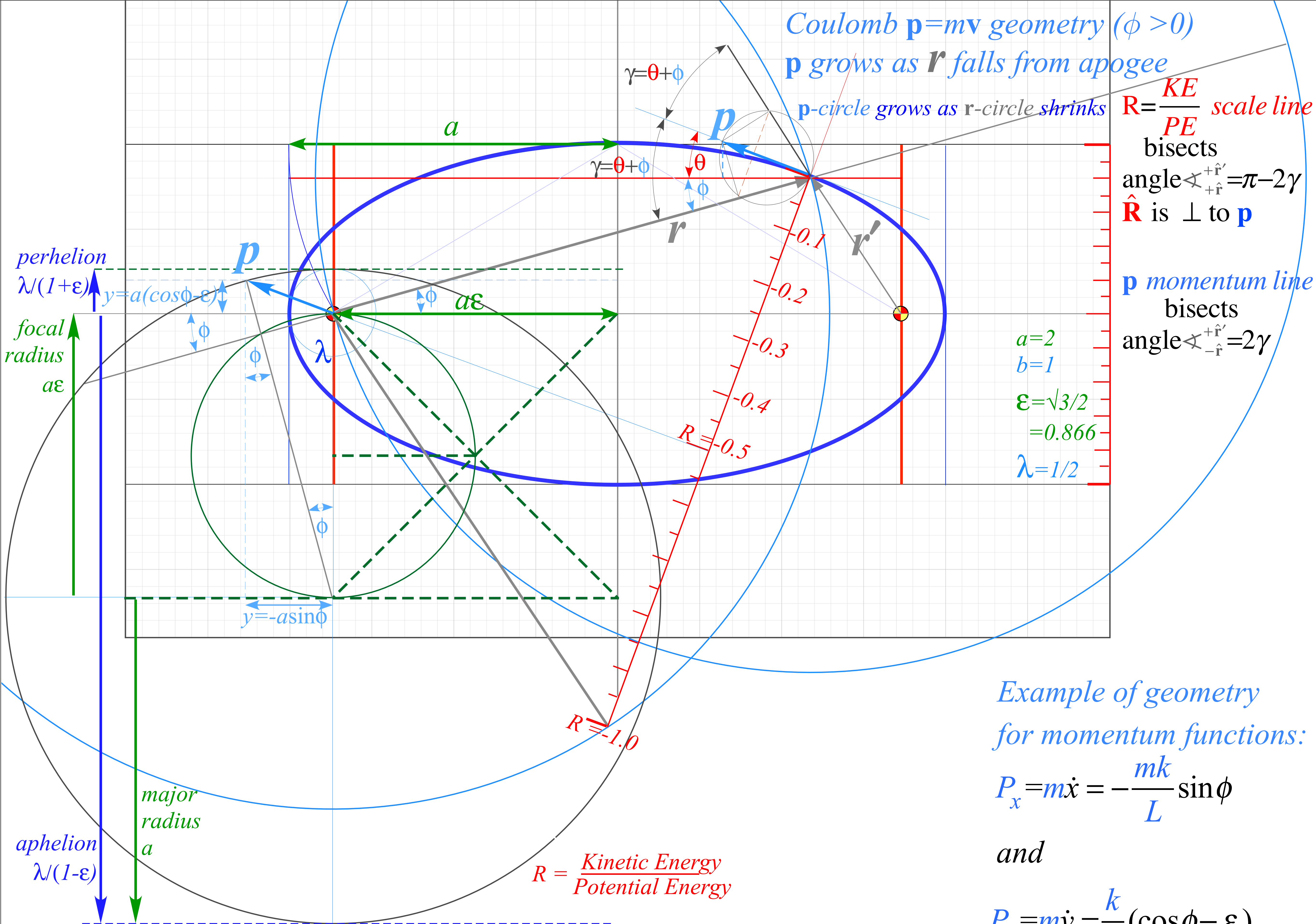
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{k}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived several pages ahead.*

*Note similarity of (***R***,***r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*



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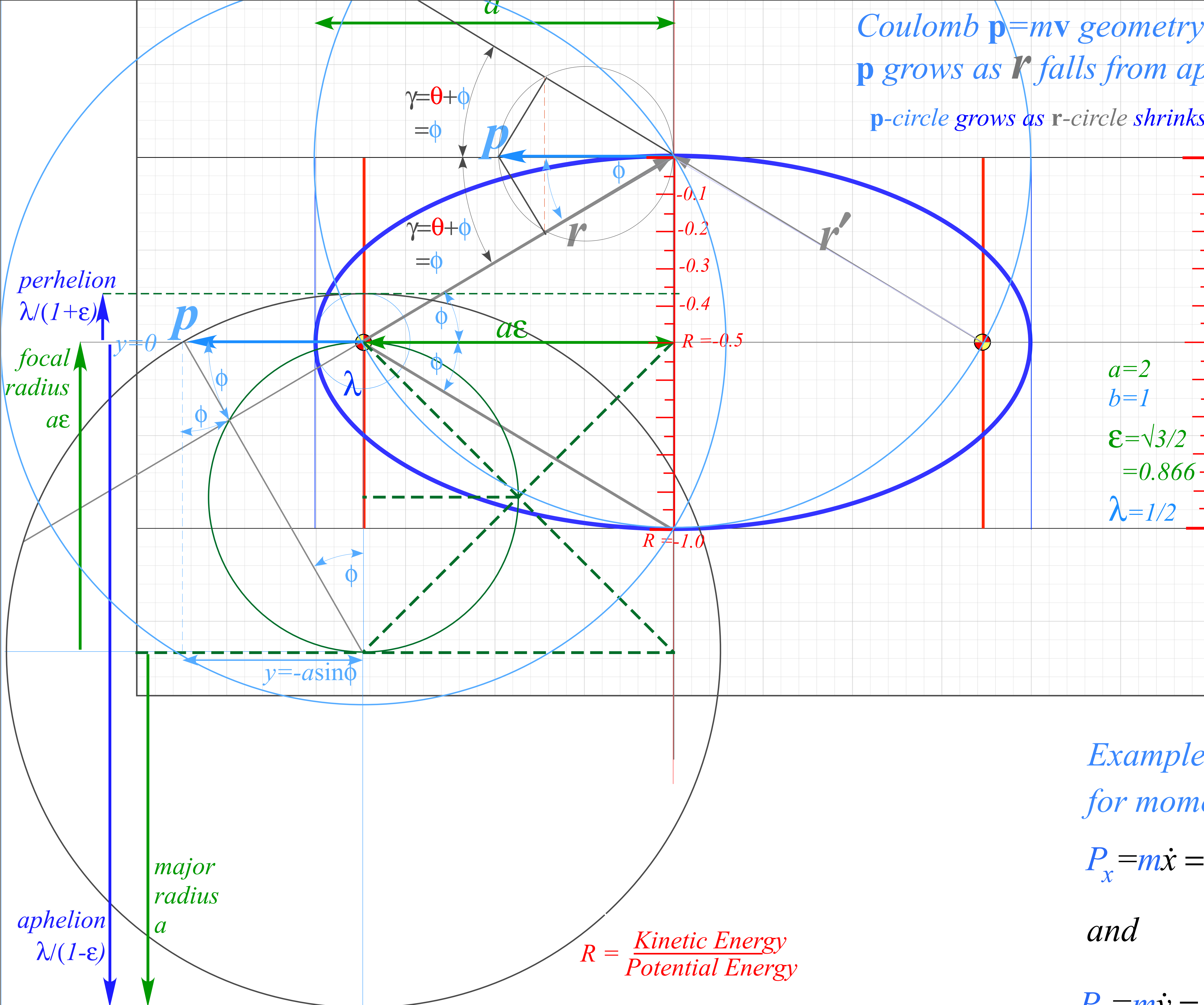
*Note similarity of (***R**,**r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Coulomb **p**=m**v** geometry (ϕ >0)
p grows as **r** falls from apogee



Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{k}{L}(\cos\phi - \varepsilon)$$

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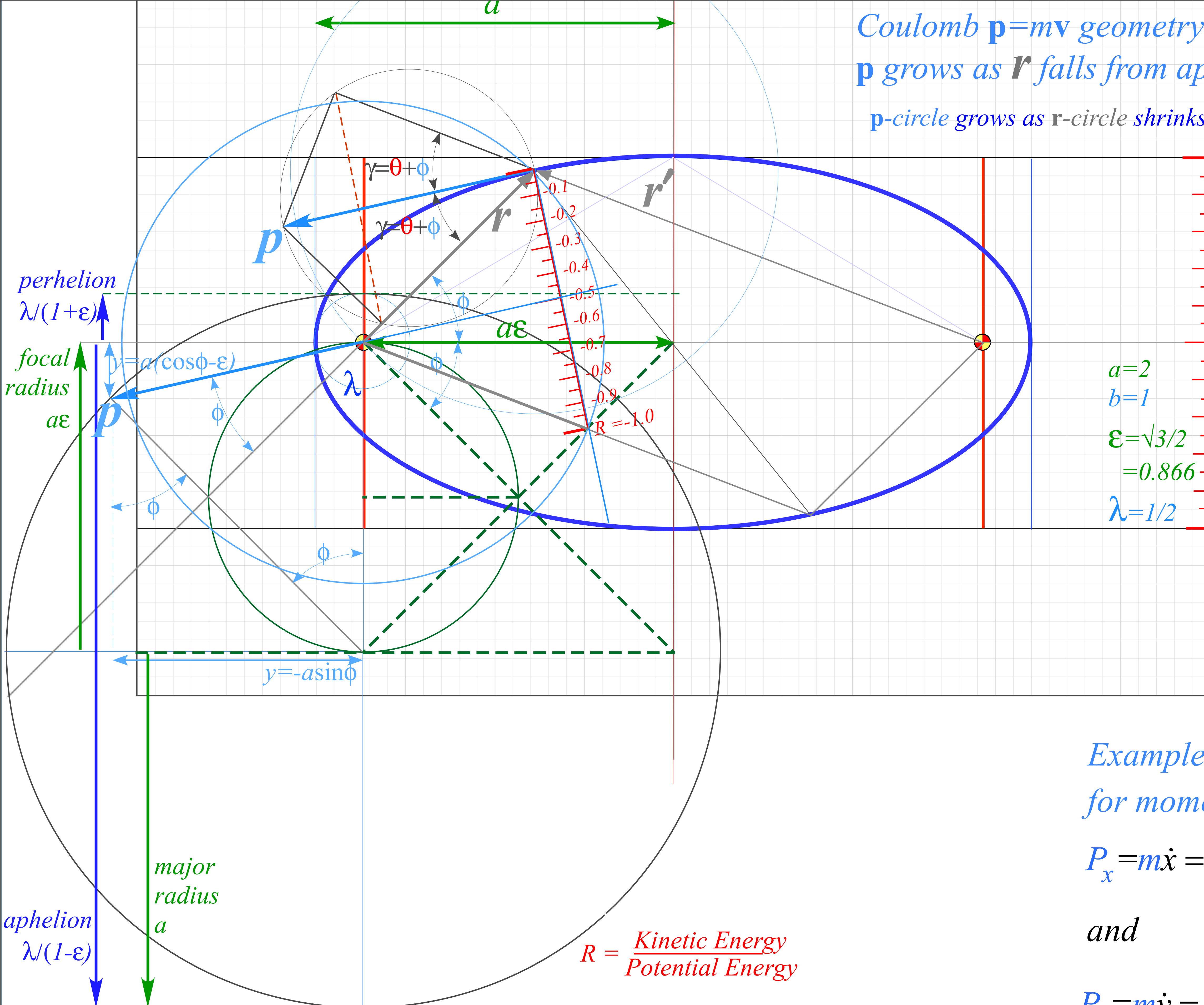

0>R=KE/PE>-1 *scale subtends angle* 2γ *with length* $2r\sin\gamma$ *as is derived several pages ahead.*

Note similarity of (**R**,**r**)-*triangle in* **r**-*circle of radius* r *to that in* **p**-*circle of diameter* p *above.*

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and

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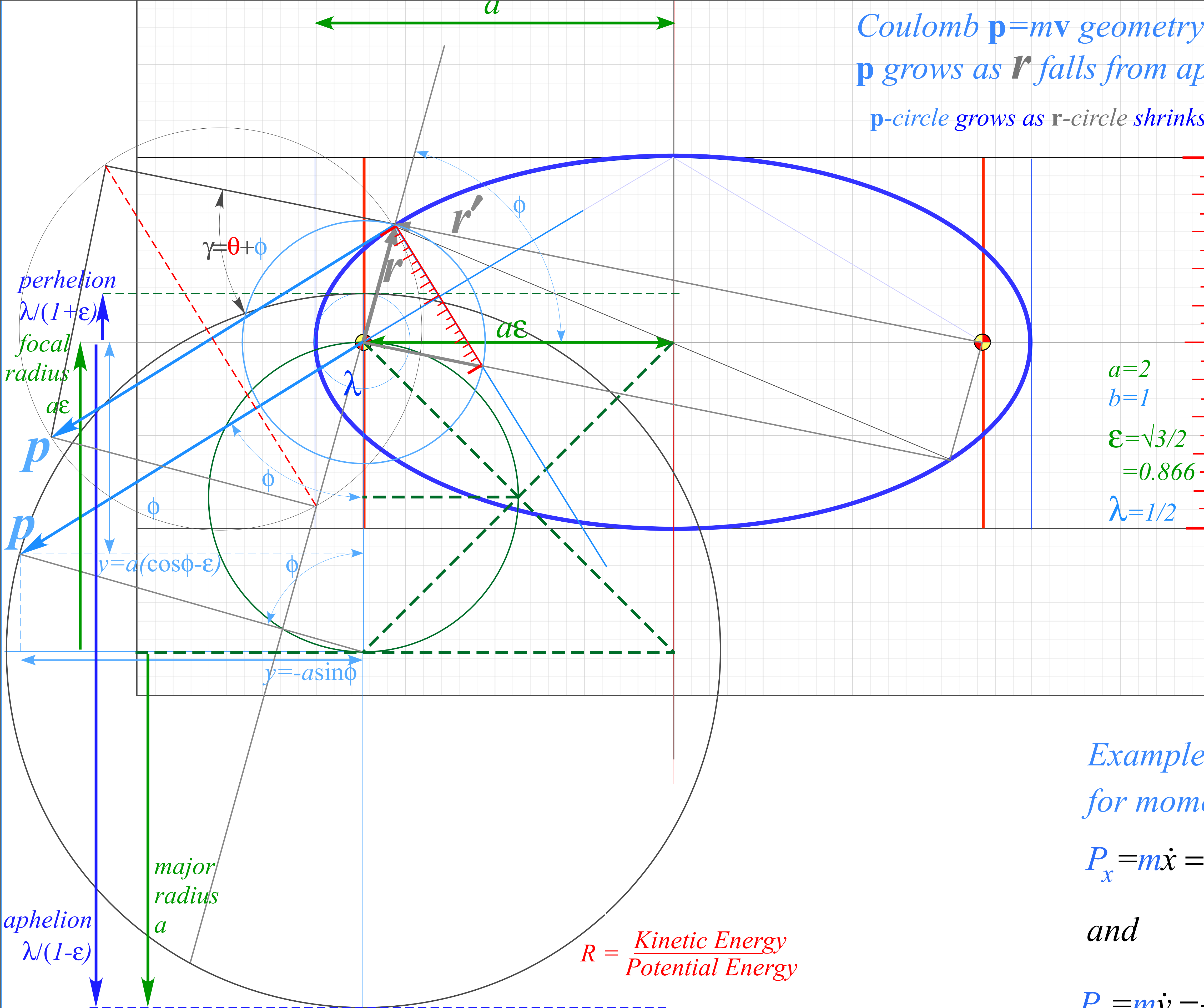
Coulomb **p**=*m***v** geometry ($\phi > 0$)

p grows as ***r*** falls from apogee

p-circle grows as **r**-circle shrinks



$\mathbf{R}=\frac{KE}{PE}$ *scale line* bisects angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{+\hat{\mathbf{r}}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is $\perp$ to **p**

p *momentum line* bisects angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{-\hat{\mathbf{r}}}=2\gamma$

Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{k}{L}(\cos\phi - \varepsilon)$$

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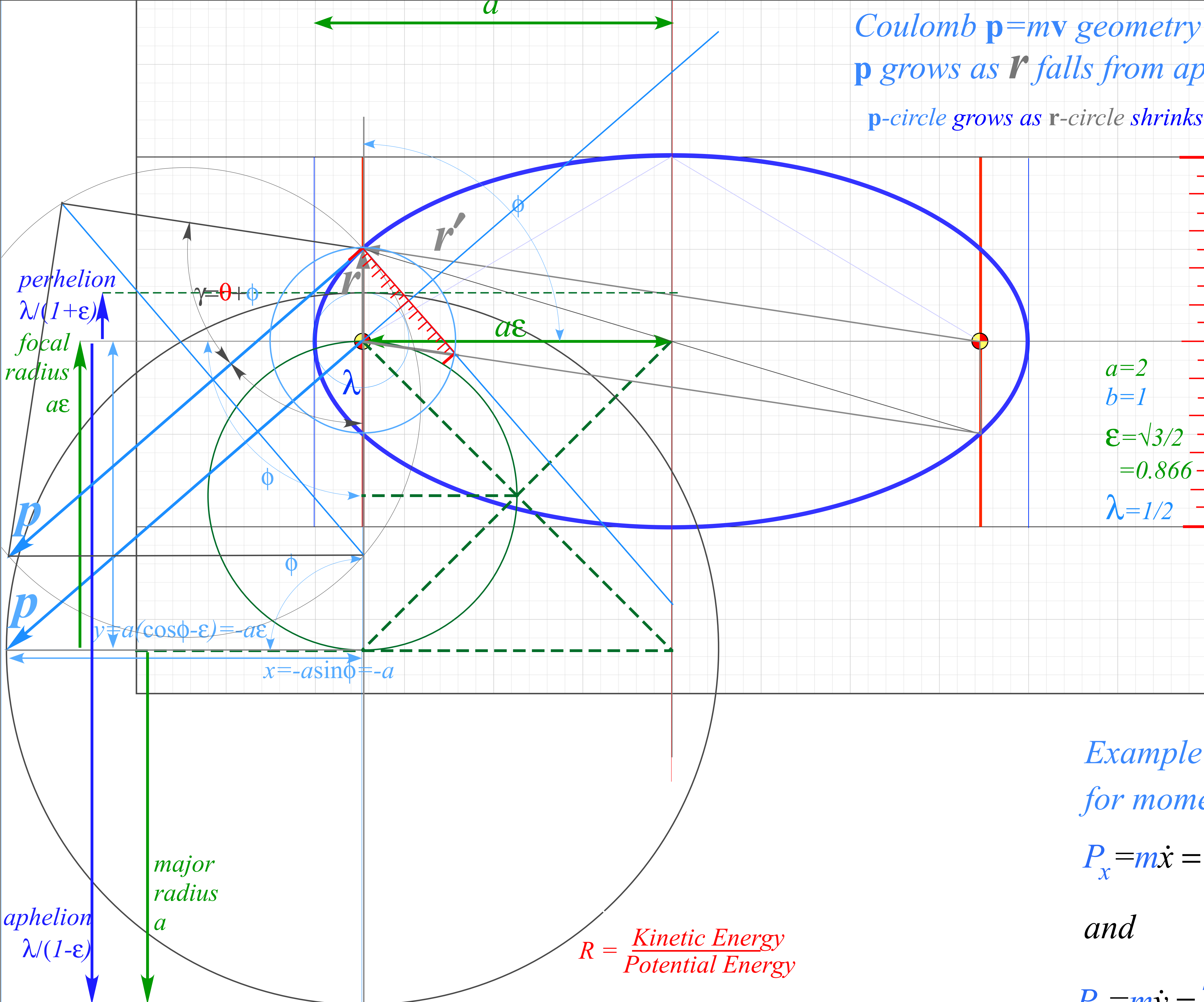
*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*

Coulomb **p**=m**v** geometry ($\phi > 0$)

p grows as *r* falls from apogee

p-circle grows as **r**-circle shrinks



$R = \frac{KE}{PE}$ scale line bisects angle $\sphericalangle^{+\hat{r}'}_{+\hat{r}} = \pi - 2\gamma$

$\hat{\mathbf{R}}$ is $\perp$ to **p**

p momentum line bisects angle $\sphericalangle^{+\hat{r}'}_{-\hat{r}} = 2\gamma$

Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

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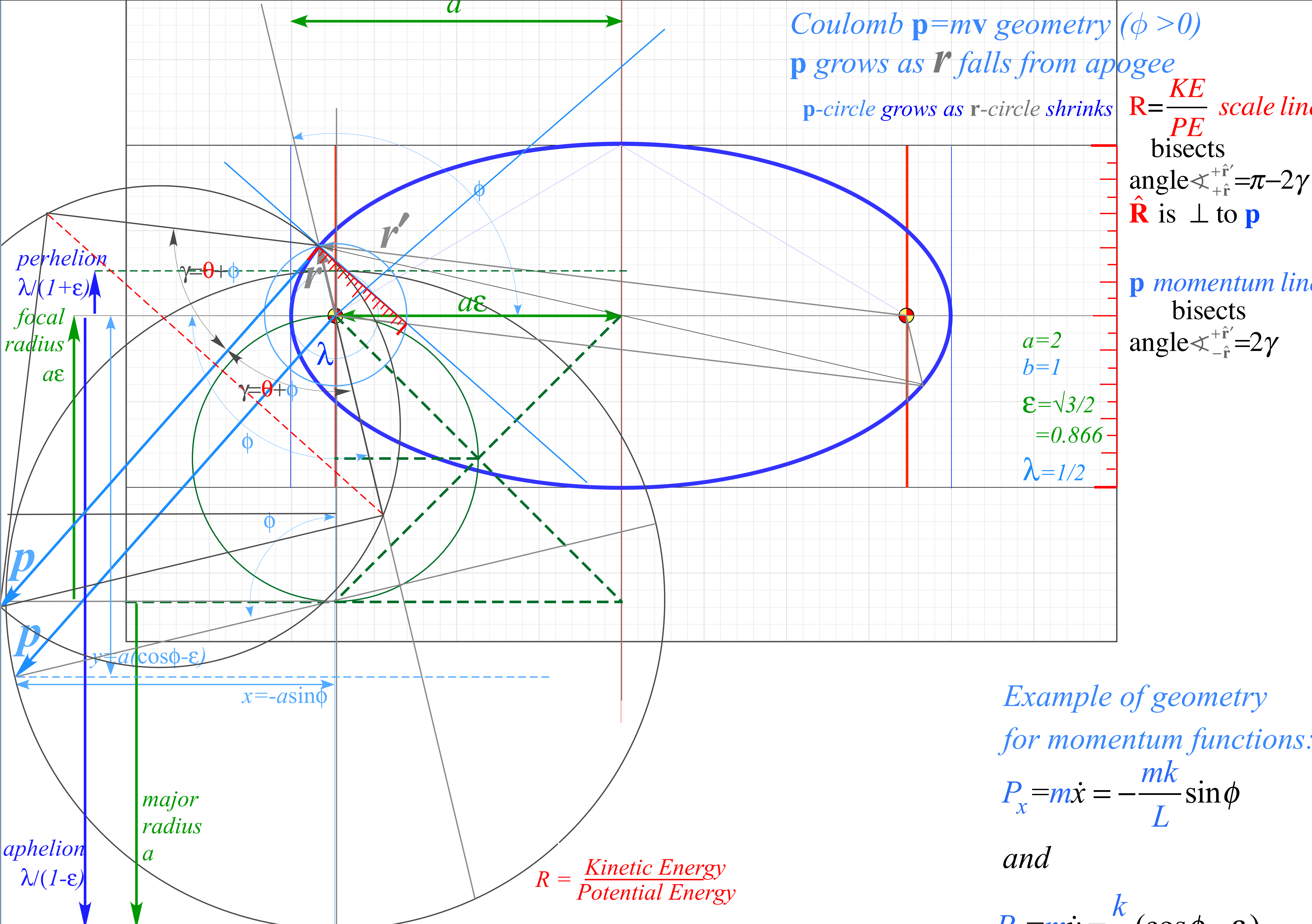
Coulomb $\mathbf{p}=m\mathbf{v}$ geometry ($\phi > 0$)

p grows as **r** falls from apogee

p-circle grows as **r**-circle shrinks



$0>R=KE/PE>-1$ *scale subtends angle* 2γ *with length* $2r\sin\gamma$ *as is derived several pages ahead.*

*Note similarity of (***R***,***r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Example of geometry for momentum functions:

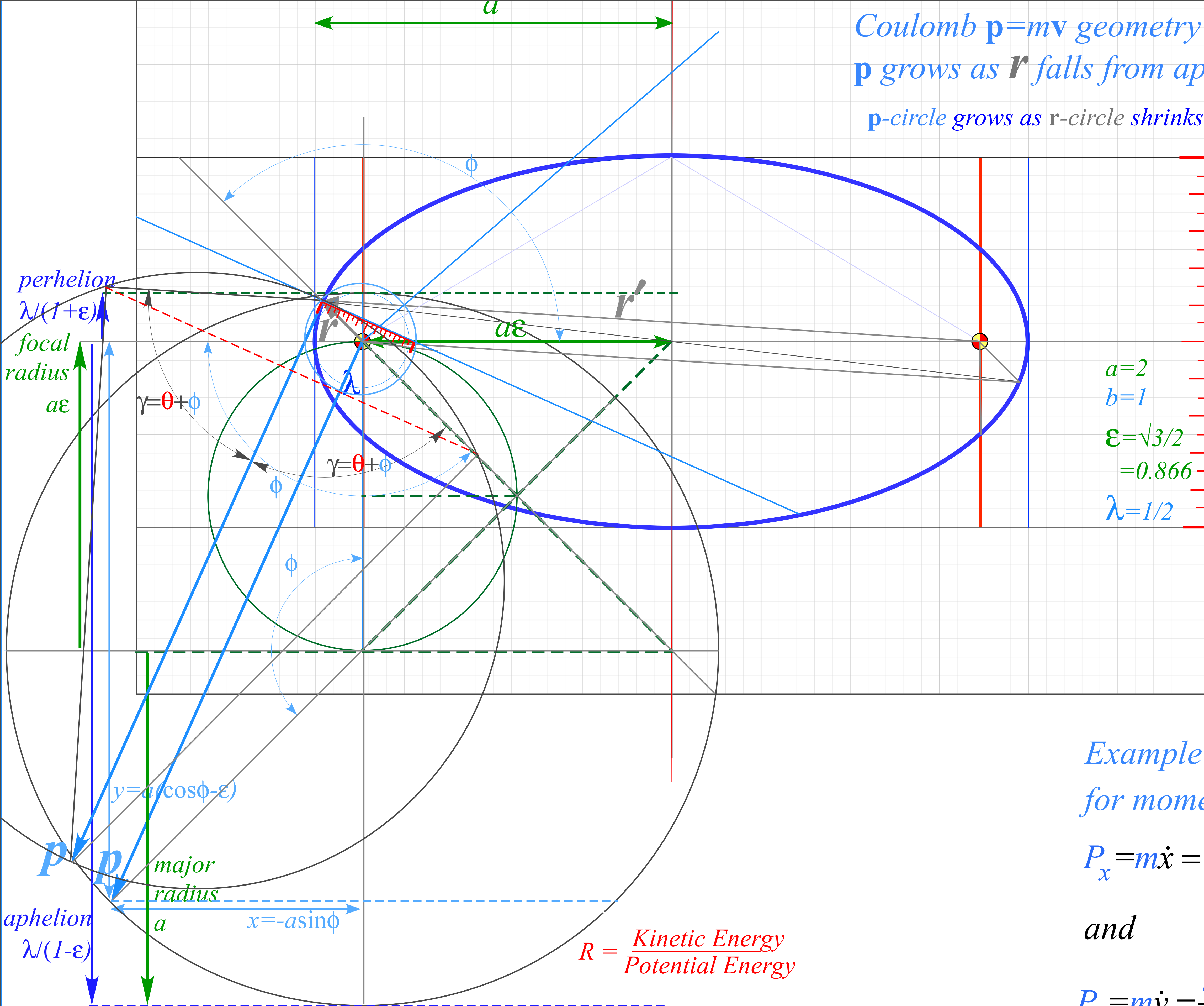
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{k}{L}(\cos\phi - \varepsilon)$$

Coulomb **p**=*m***v** geometry (ϕ >0)
p grows as ***r*** falls from apogee

p-circle grows as **r**-circle shrinks



$R = \dfrac{KE}{PE}$ *scale line*

bisects

angle $\measuredangle^{+\hat{\mathbf{r}}'}_{+\hat{\mathbf{r}}} = \pi - 2\gamma$

$\hat{\mathbf{R}}$ is $\perp$ to **p**

p *momentum line*

bisects

angle $\measuredangle^{+\hat{\mathbf{r}}'}_{-\hat{\mathbf{r}}} = 2\gamma$

R = *Kinetic Energy* / *Potential Energy*

Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

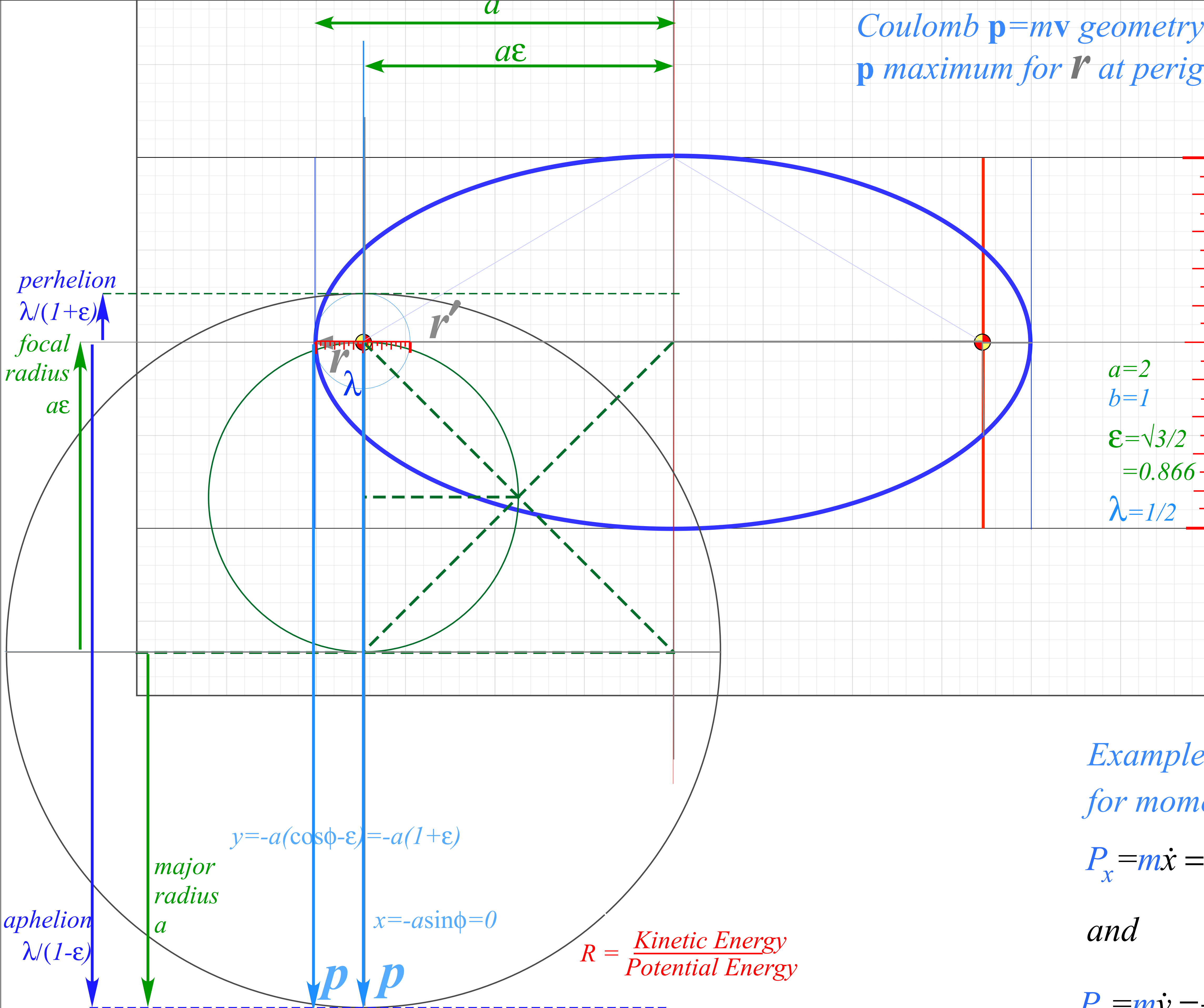
and

$$P_y = m\dot{y} = \frac{k}{L}(\cos\phi - \varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sin*γ as is derived several pages ahead.*

*Note similarity of (***R***,***r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Coulomb **p**=*m***v** *geometry* ($\phi > 0$)
p *maximum for* ***r*** *at perigee*



Example of geometry for momentum functions:

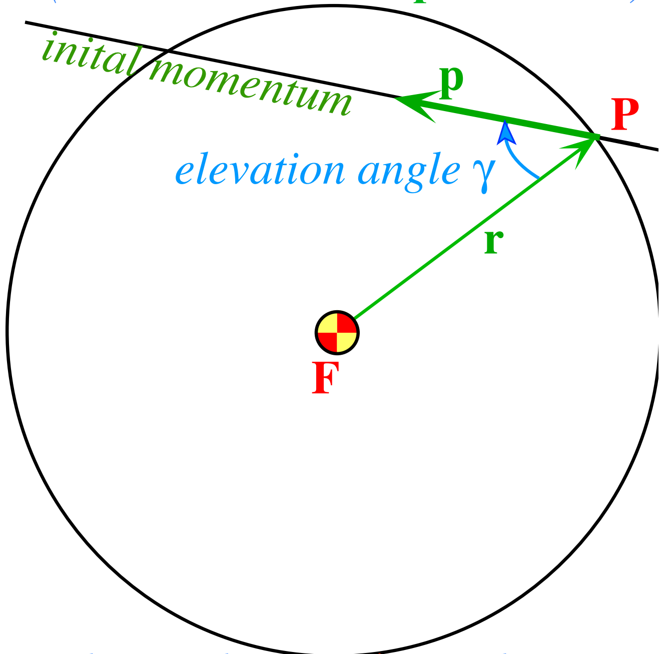
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

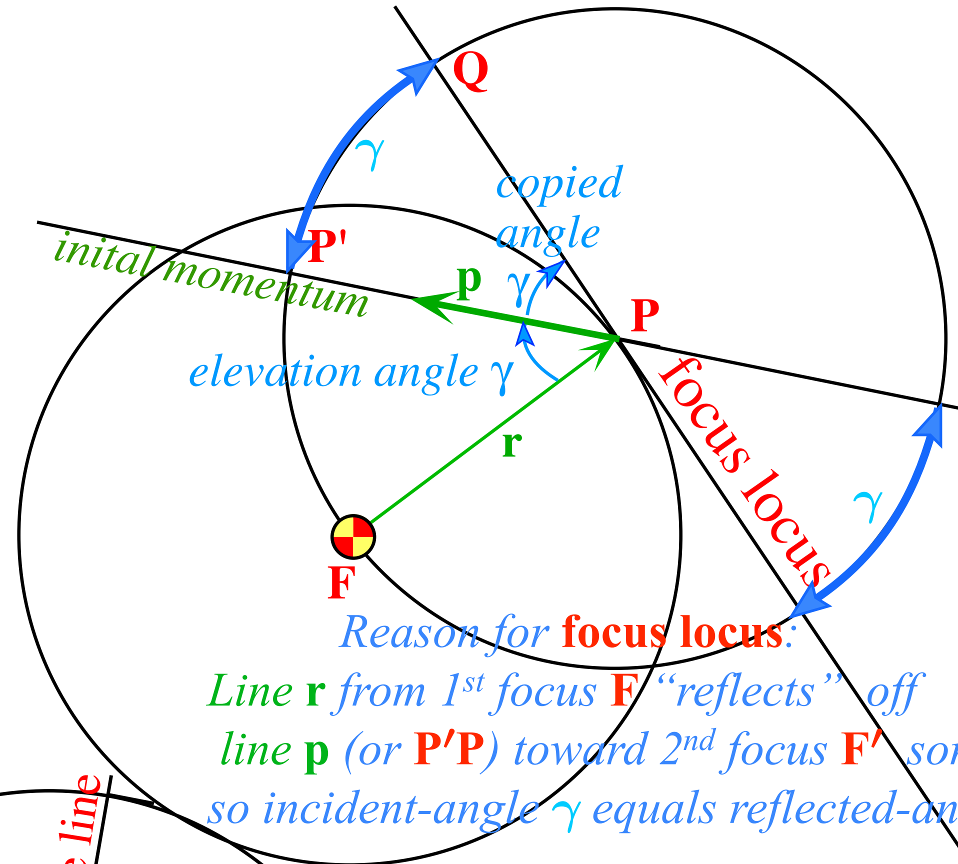
$$P_y = m\dot{y} = \frac{k}{L}(\cos\phi - \varepsilon)$$

ε-vector and Coulomb orbit construction steps

Pick launch point P
*(radius vector **r**)*
and elevation angle γ from radius
*(momentum initial **p** direction)*

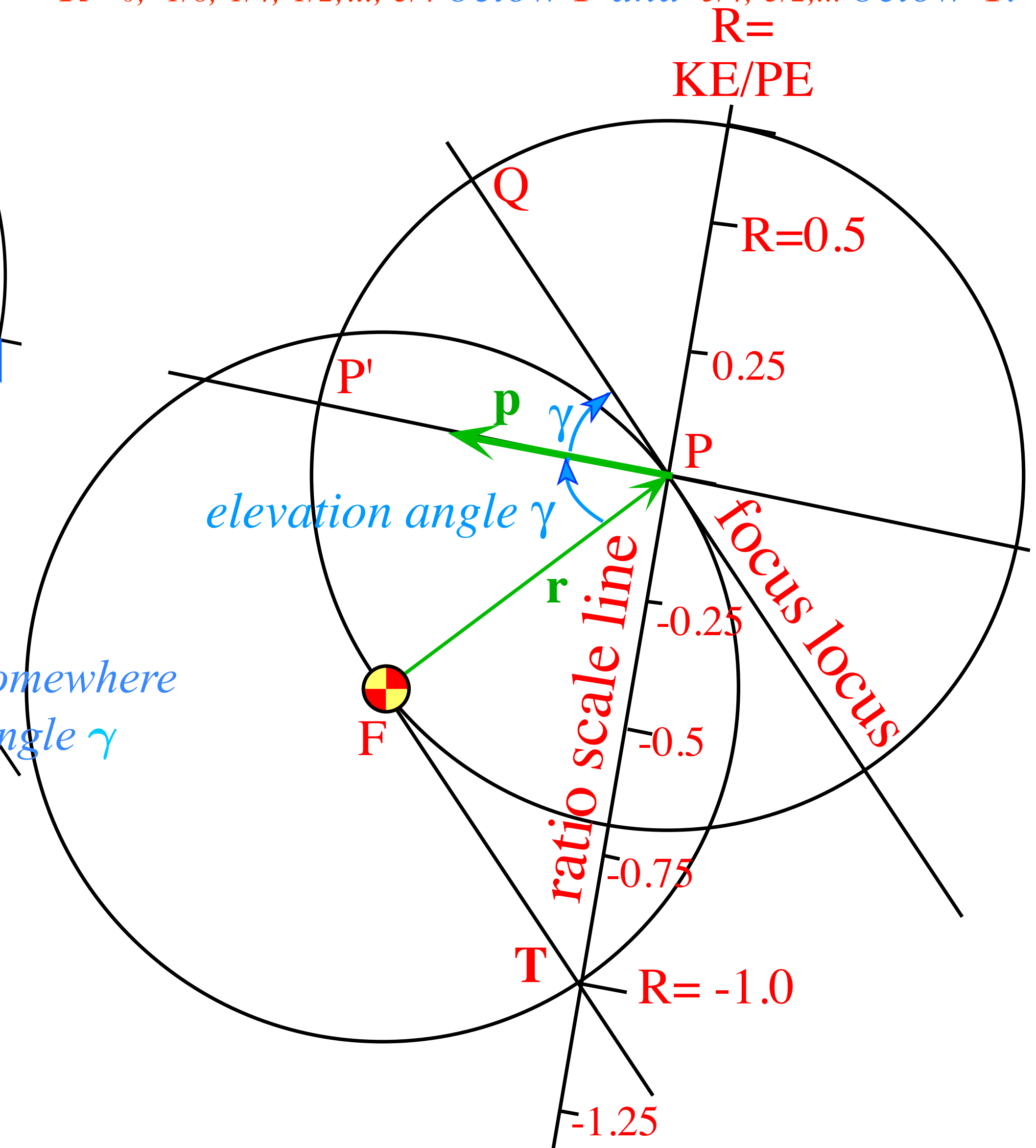


Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ to make **focus locus**

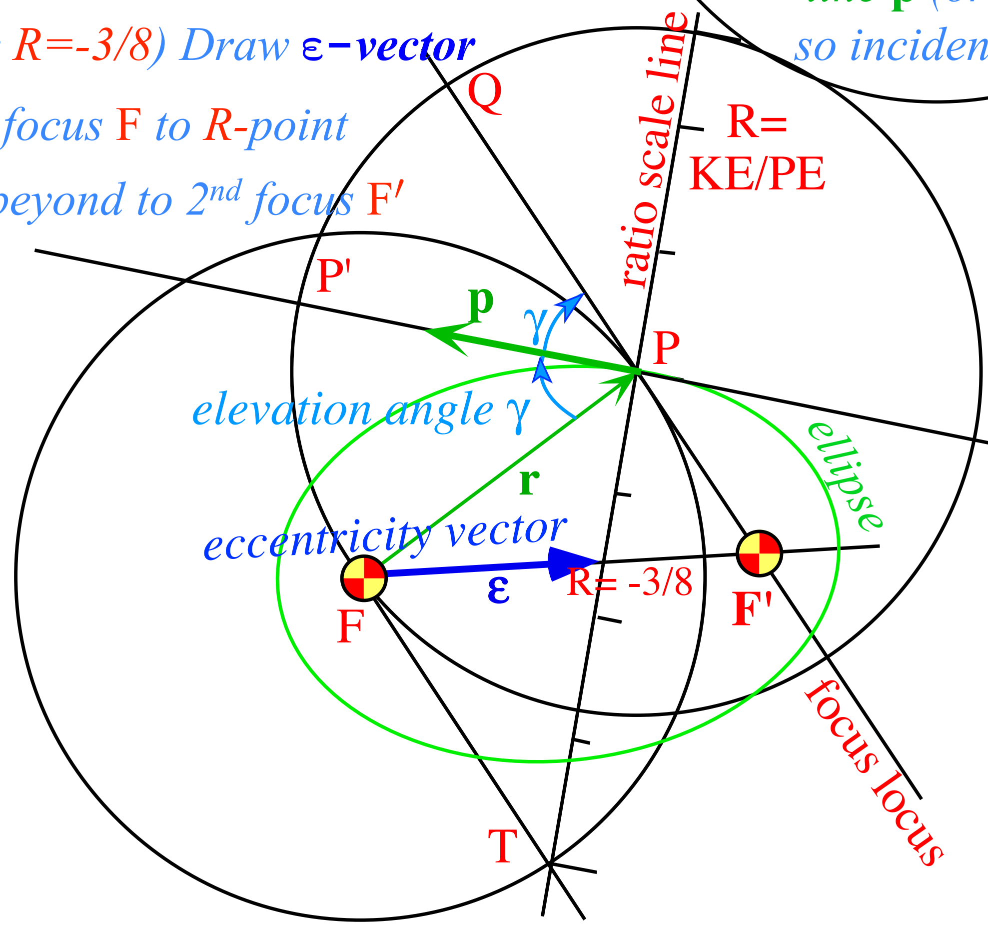


Reason for **focus locus***:*
*Line **r** from 1st focus **F** "reflects" off*
*line **p** (or **P′P**) toward 2nd focus **F′** somewhere*
so incident-angle γ equals reflected-angle γ

Copy double angle 2γ (∠FPQ) onto ∠PFT
*Extend ∠PFT chord PT to make R-***ratio scale line**
Label chord PT with R=0 at P and R=-1.0 at T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above P and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P and -5/4,-3/2,... below T.



Pick initial R=KE/PE value
(here R=-3/8) Draw ***ε−vector***
from focus F to R-point
and beyond to 2nd focus F′



$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm \left(\frac{\text{Initial velocity}}{\text{Escape velocity}} \right)^2 = \pm \frac{v^2(0)}{v^2(\infty)}$$

focus F and 2nd focus F′ allow final
construction of orbital trajectory.
Here it is an R=-3/8 ellipse.

*(Detailed Analytic geometry of **ε-vector** follows.)*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* **r**-*orbit geometry*

Review and connection to standard development

$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *geometry*

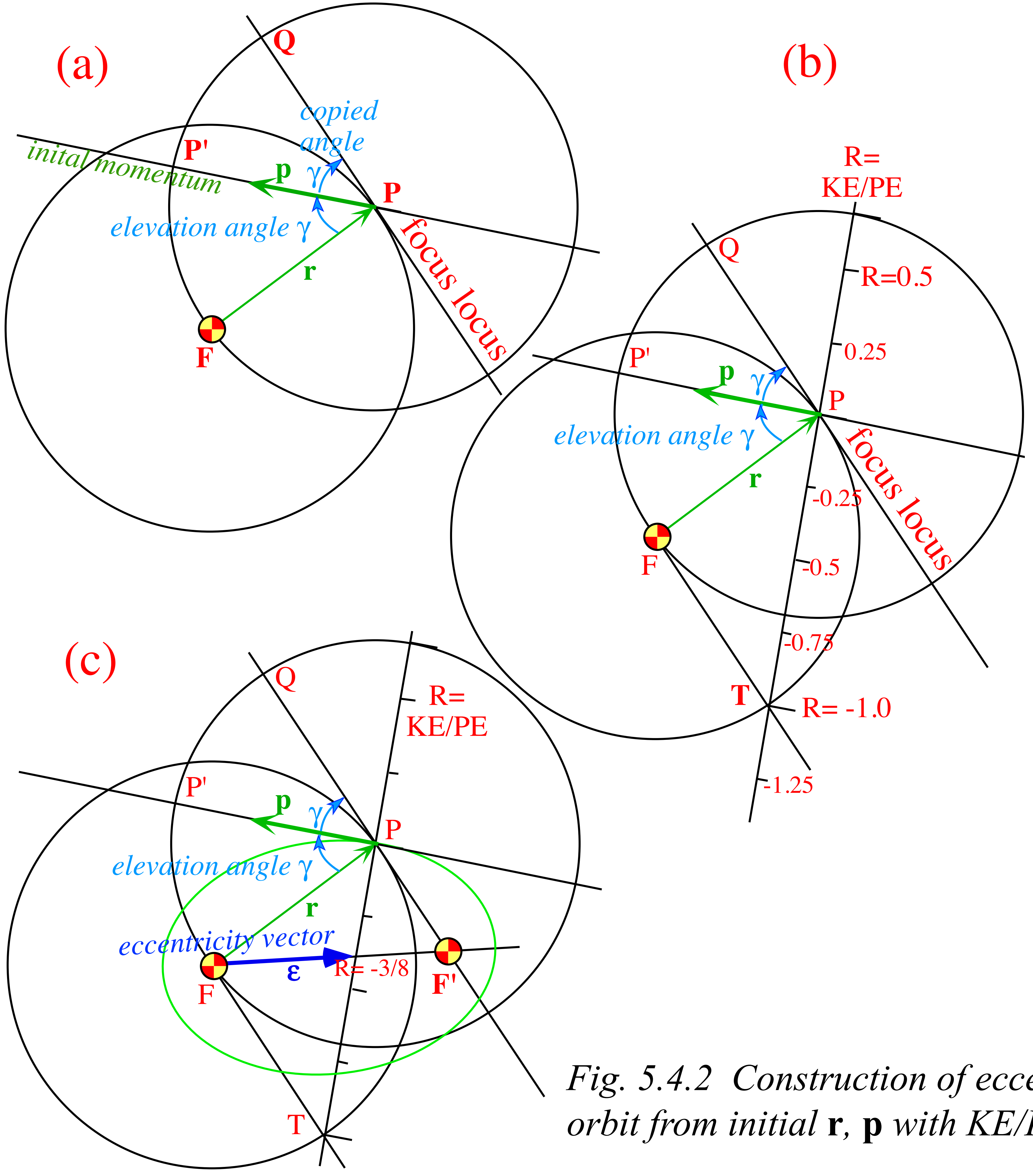
$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *algebra*

Example with elliptical orbit

➔ *Analytic geometry derivation of* $\boldsymbol{\varepsilon}$*-construction*

Algebra of $\boldsymbol{\varepsilon}$*-construction geometry*

Connection formulas for (a,b) and (ε,λ) *with* (γ,R)

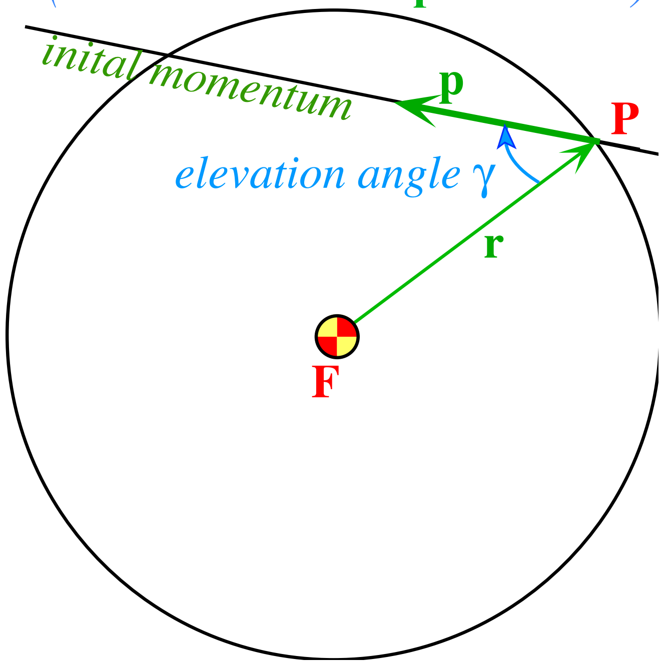


Next several pages give step-by-step constructions of **ε***-vector and Coulomb orbit and trajectory physics*

Fig. 5.4.2 Construction of eccentricity vector ε *and orbit from initial* **r**, **p** *with KE/PE=-3/8.*

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*



Next several pages give step-by-step constructions of ε-vector and Coulomb orbit and trajectory physics

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
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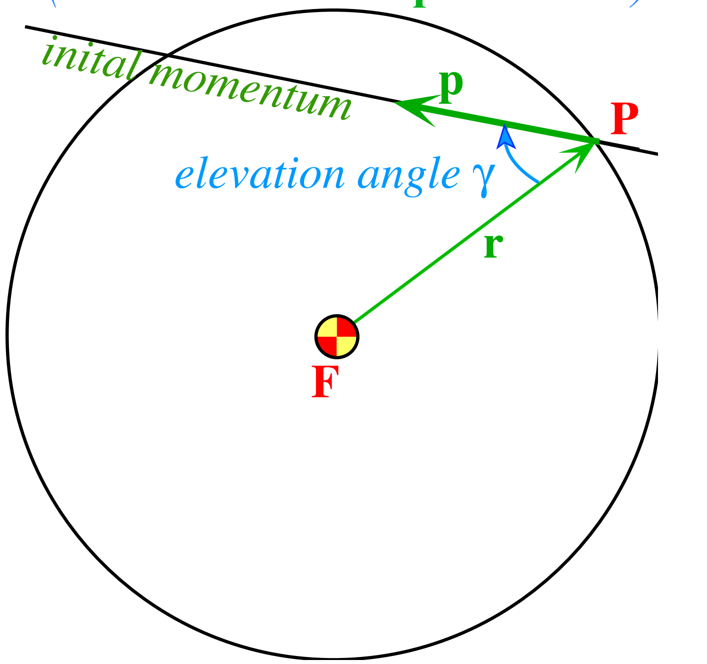
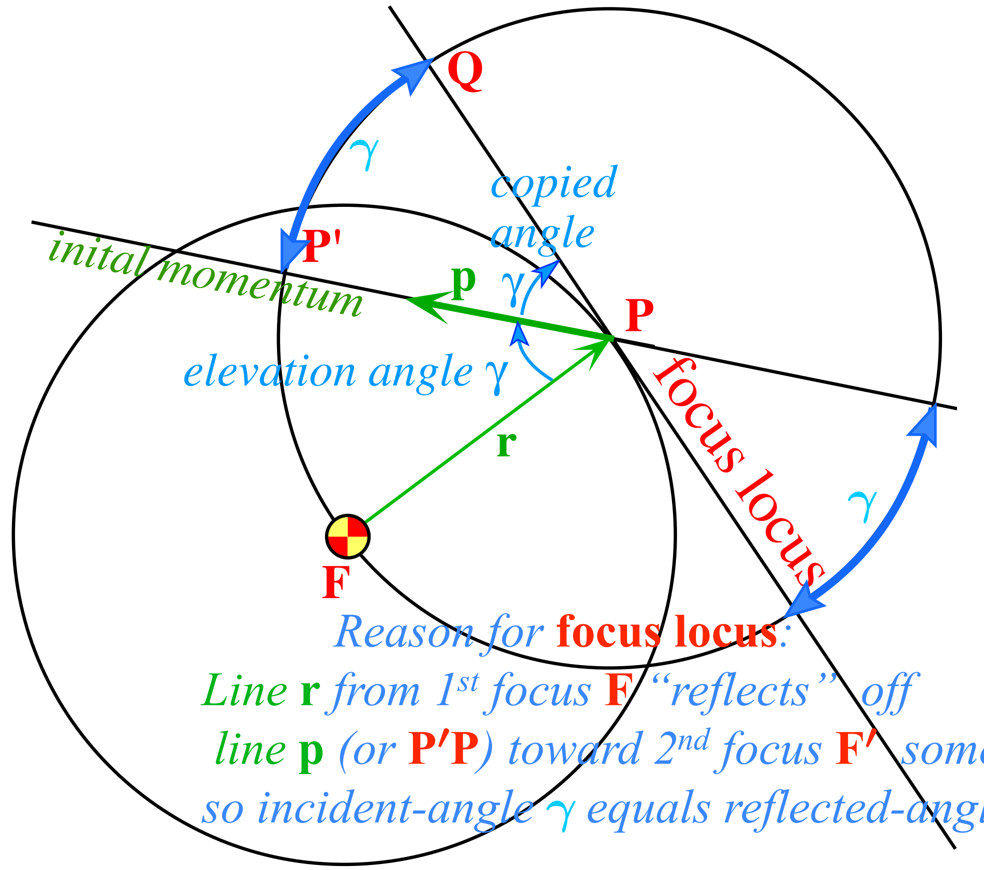


Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus**:
Line **r** *from 1st focus* **F** *"reflects" off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ *equals reflected-angle* γ

Next several pages give step-by-step constructions of ε-vector and Coulomb orbit and trajectory physics

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
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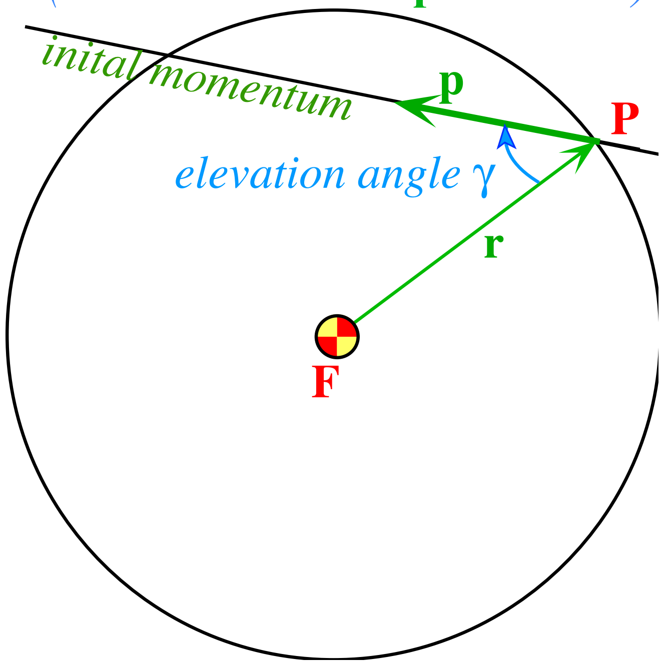

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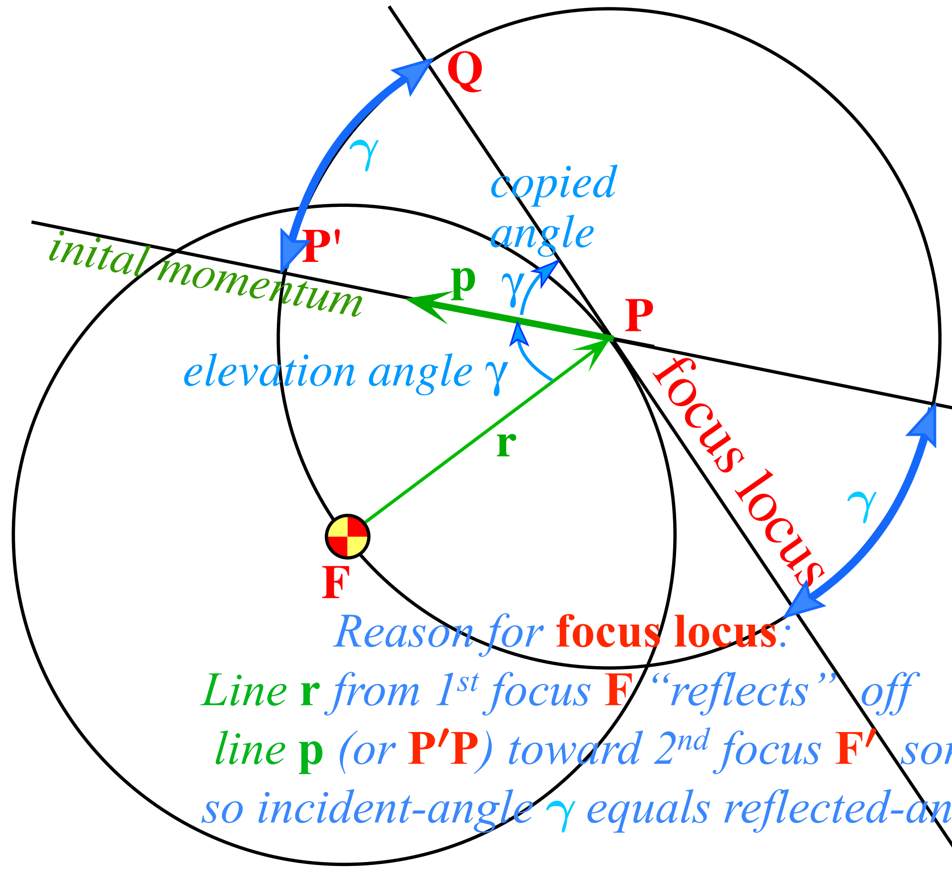

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Extend ∠PFT chord PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
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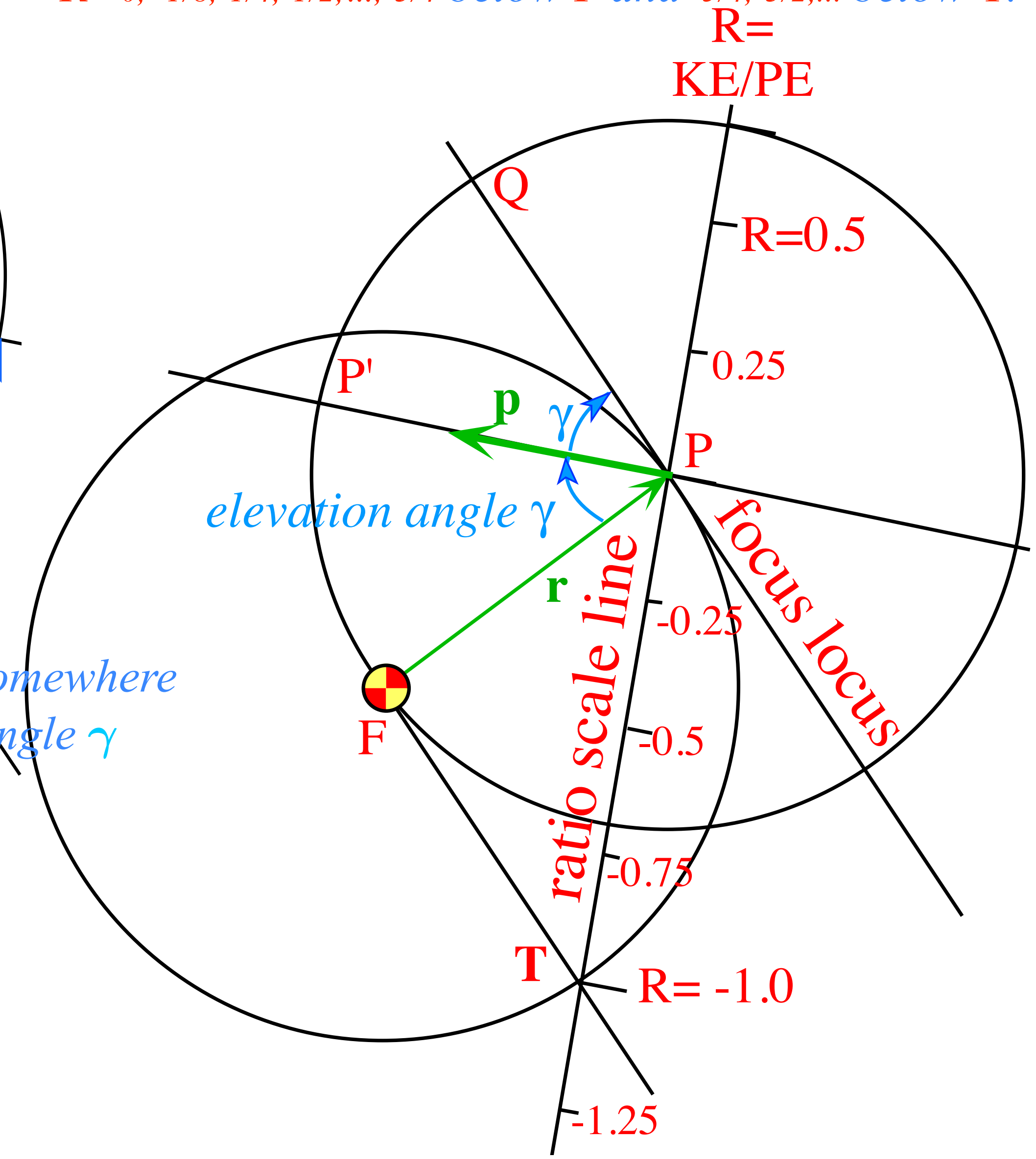
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.



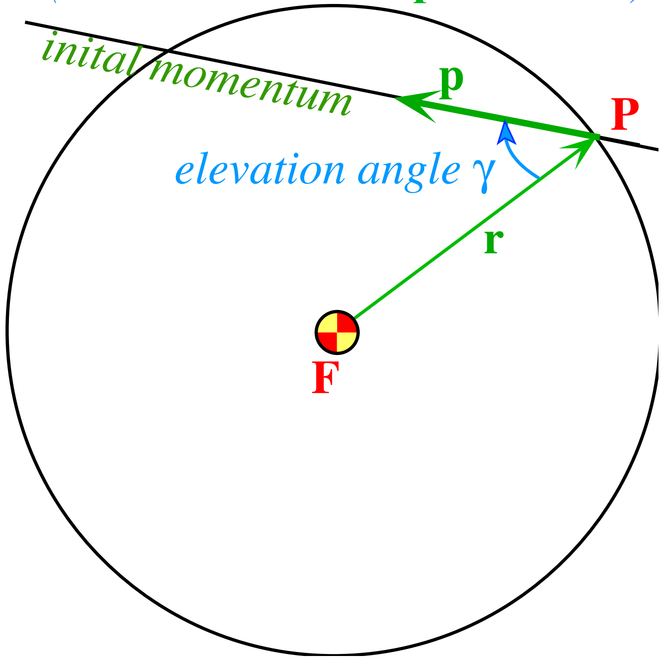
ε-vector and Coulomb orbit construction steps

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

(momentum initial **p** *direction)*



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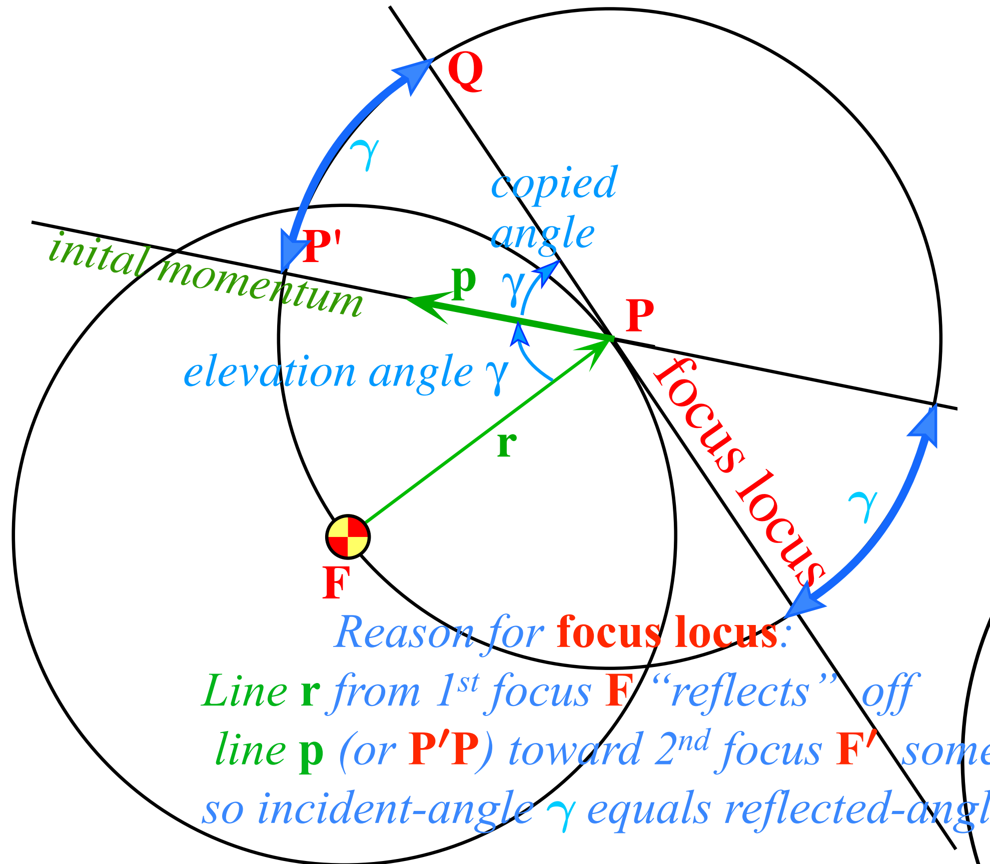
Extend resulting line QPQ′ *to make* **focus locus**



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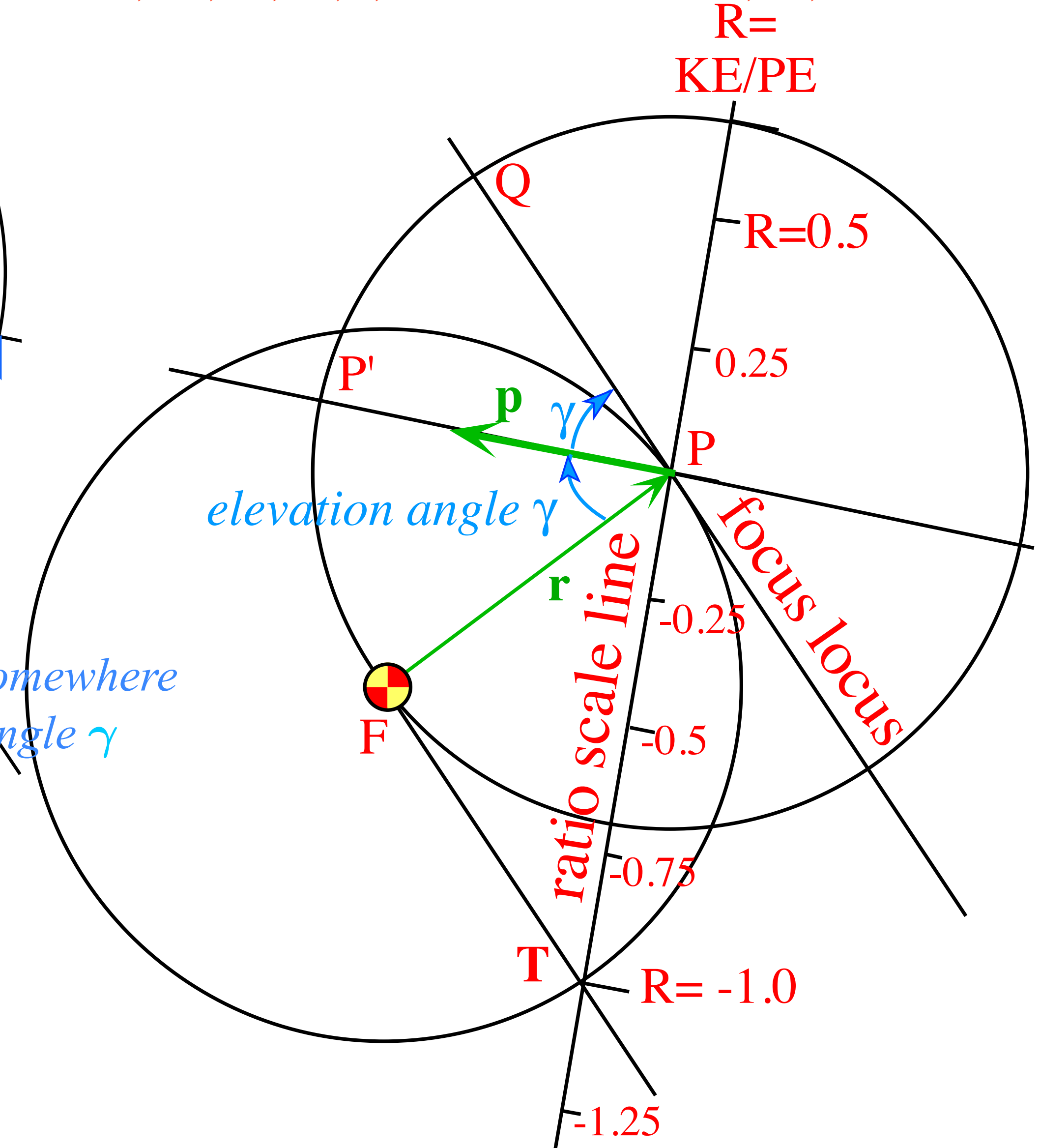
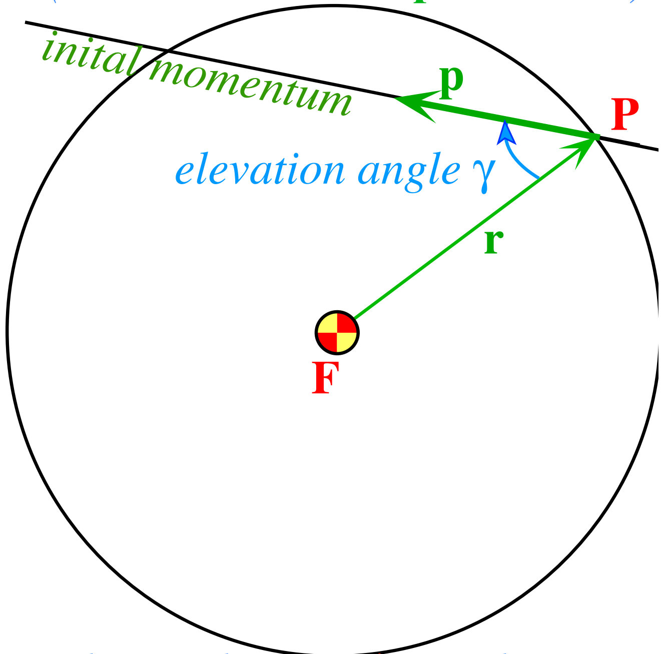


$$R = \frac{\text{Initial } KE}{\text{Initial } PE} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

ε-vector and Coulomb orbit construction steps

Pick launch point P
*(radius vector **r**)*
and elevation angle γ from radius
*(momentum initial **p** direction)*



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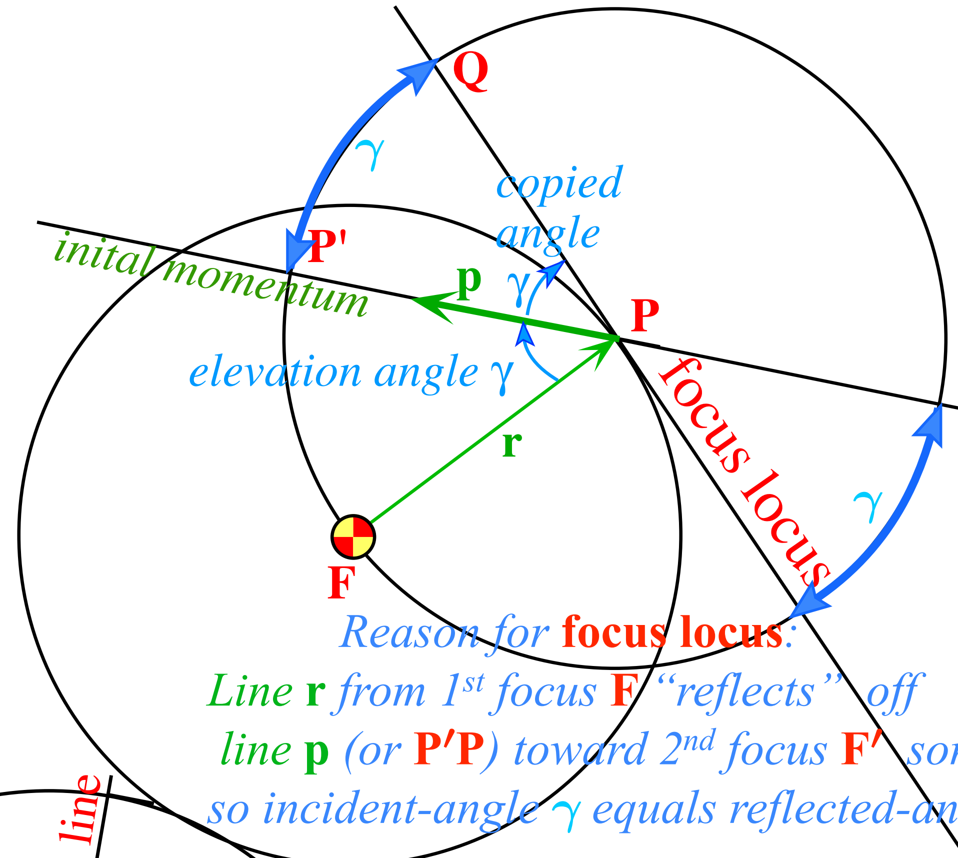
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
*Extend resulting line QPQ′ to make **focus locus***



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*Extend ∠PFT chord PT to make R-**ratio scale line***
Label chord PT with R=0 at P and R=-1.0 at T.
*Mark R-**line** fractions R=0, +1/4, +1/2,... above P and*
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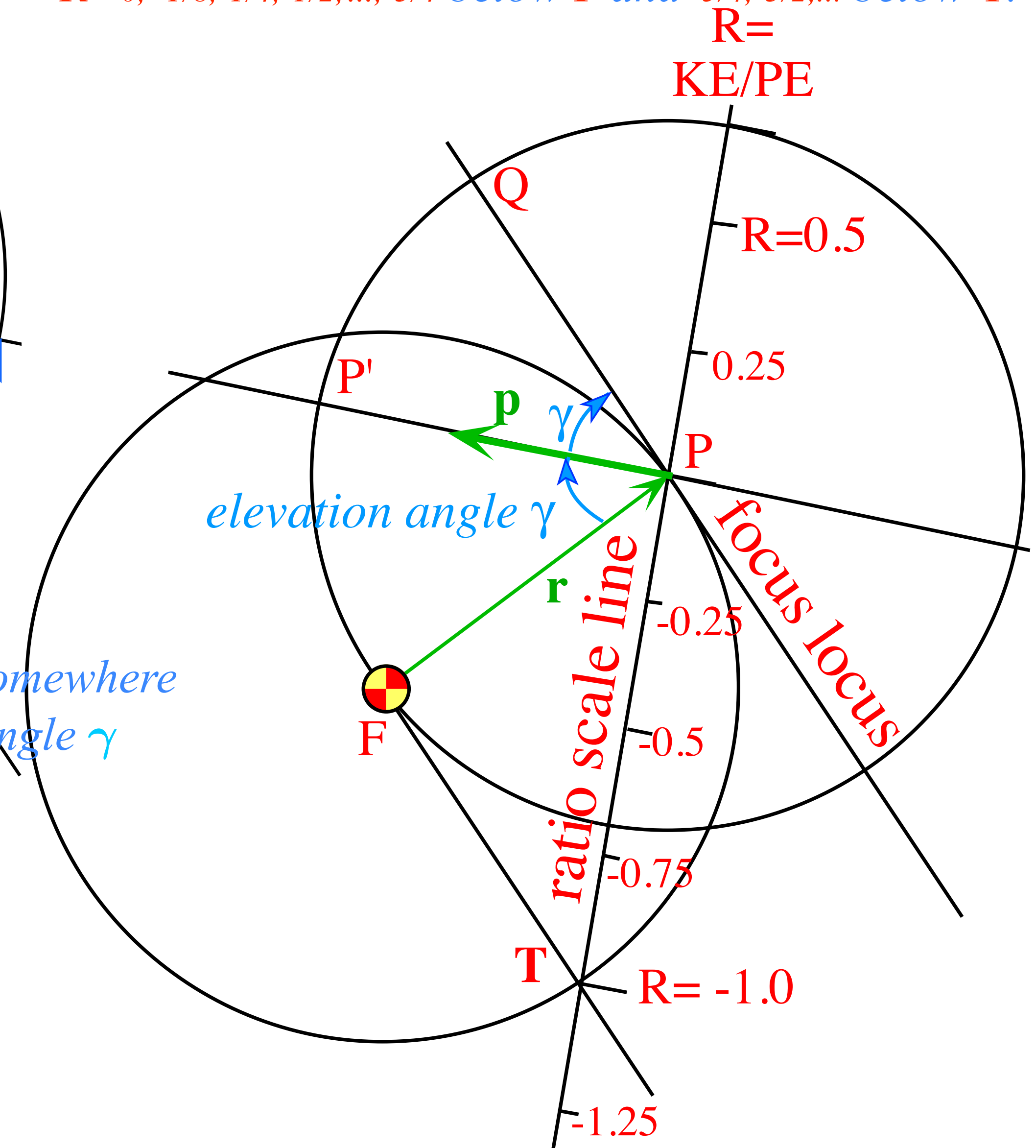
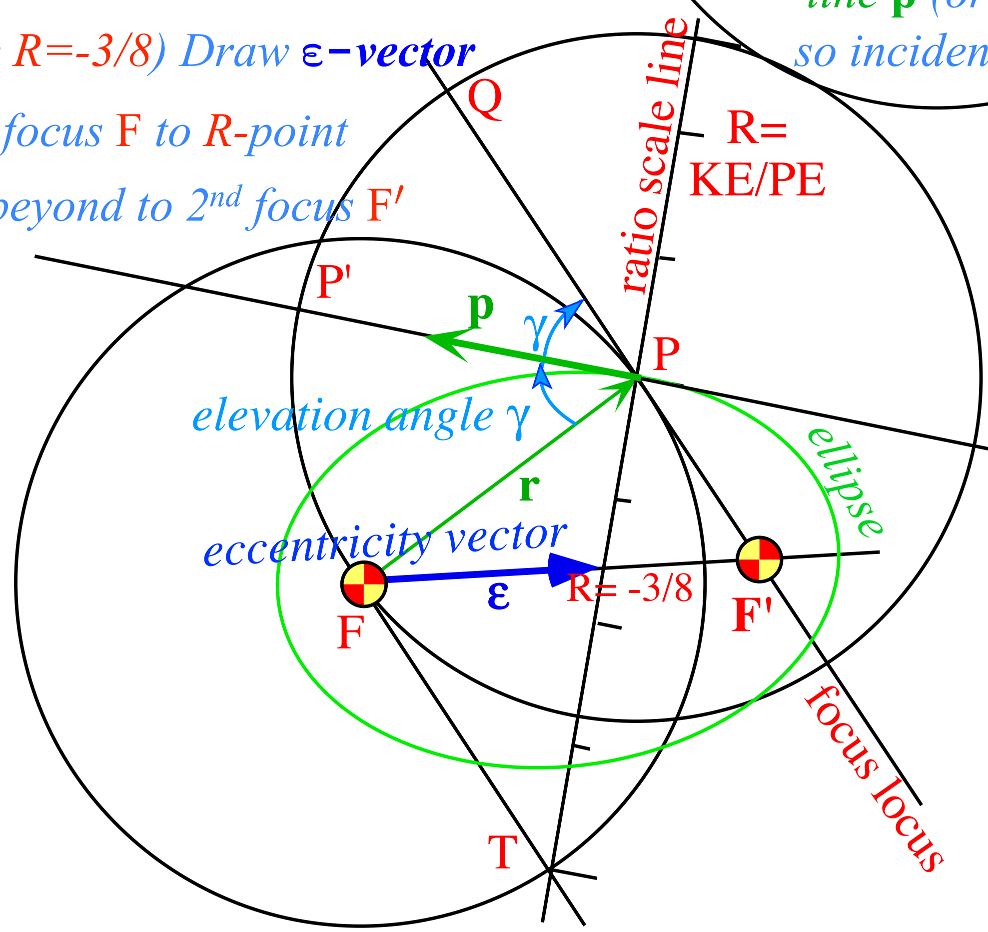


Pick initial R=KE/PE value
*(here R=-3/8) Draw **ε−vector***
from focus F to R-point
and beyond to 2nd focus F′



$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

focus F and 2nd focus F′ allow final
construction of orbital trajectory.
Here it is an R=-3/8 ellipse.

*(Detailed Analytic geometry of **ε-vector** follows.)*

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

Copy F-center circle around launch point P
Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*
Extend resulting line QPQ′ *to make* **focus locus**

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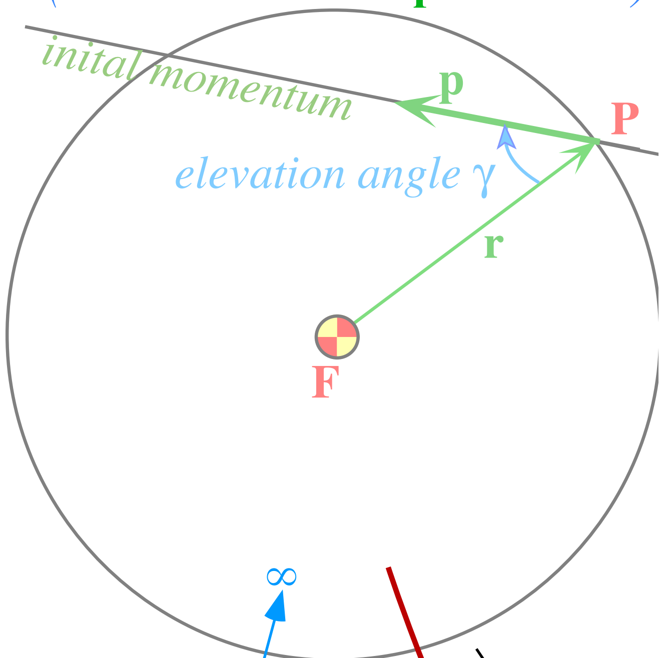
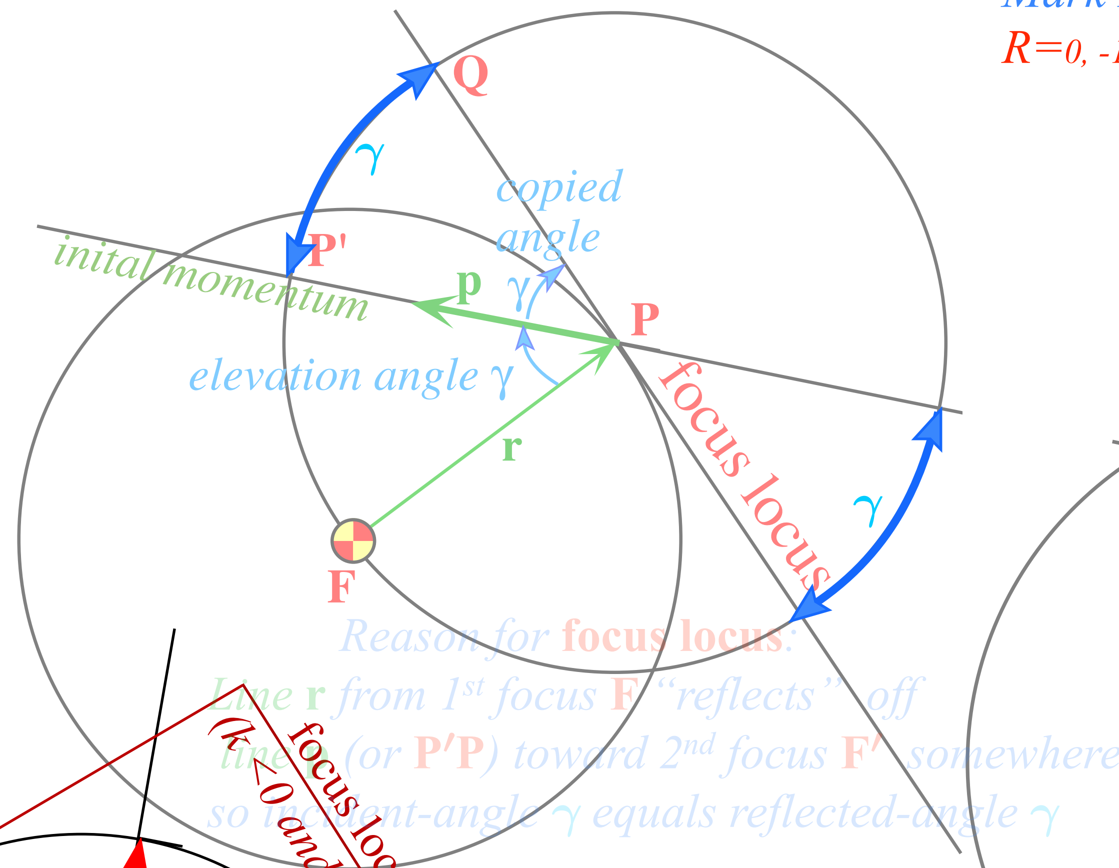
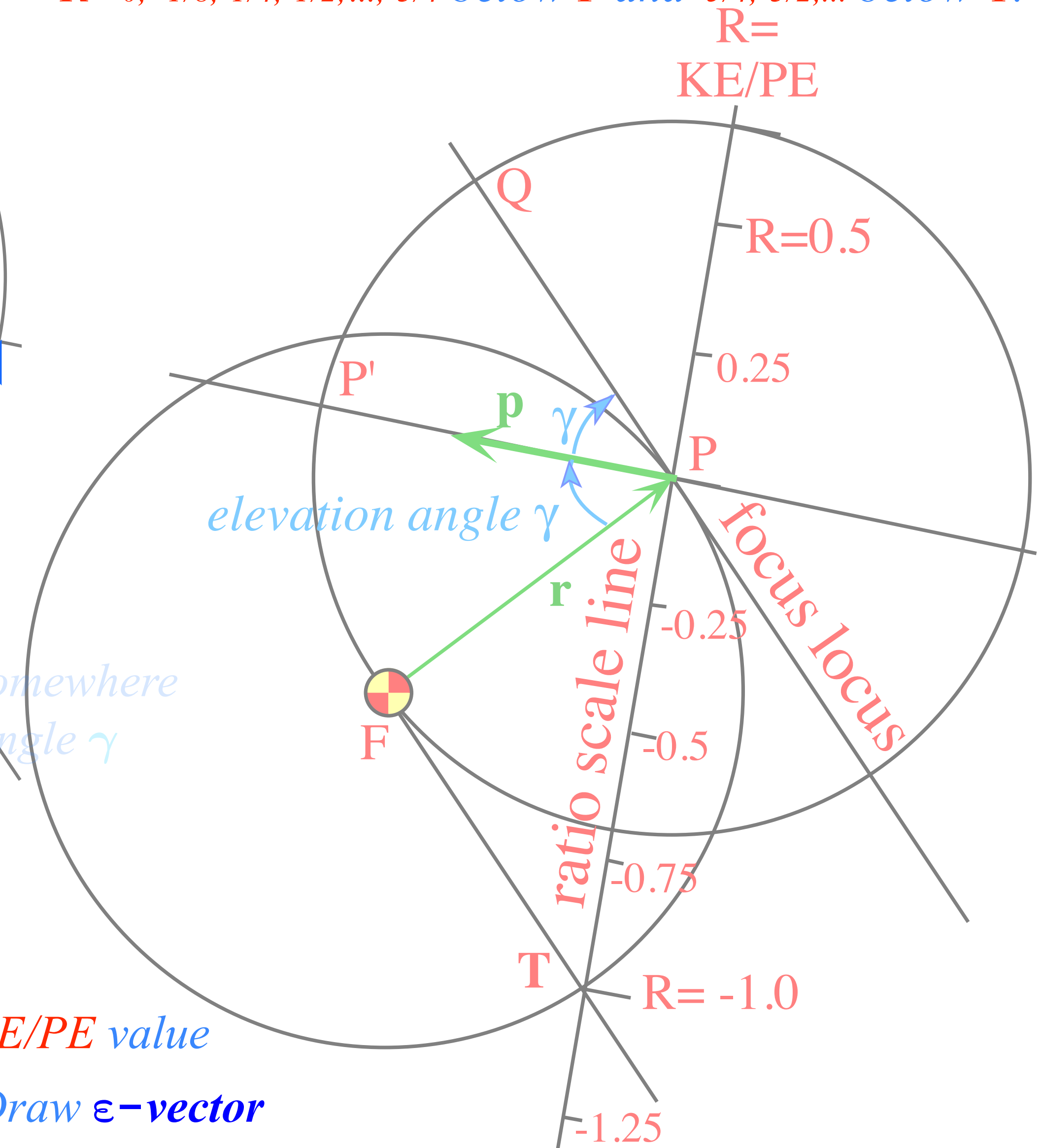
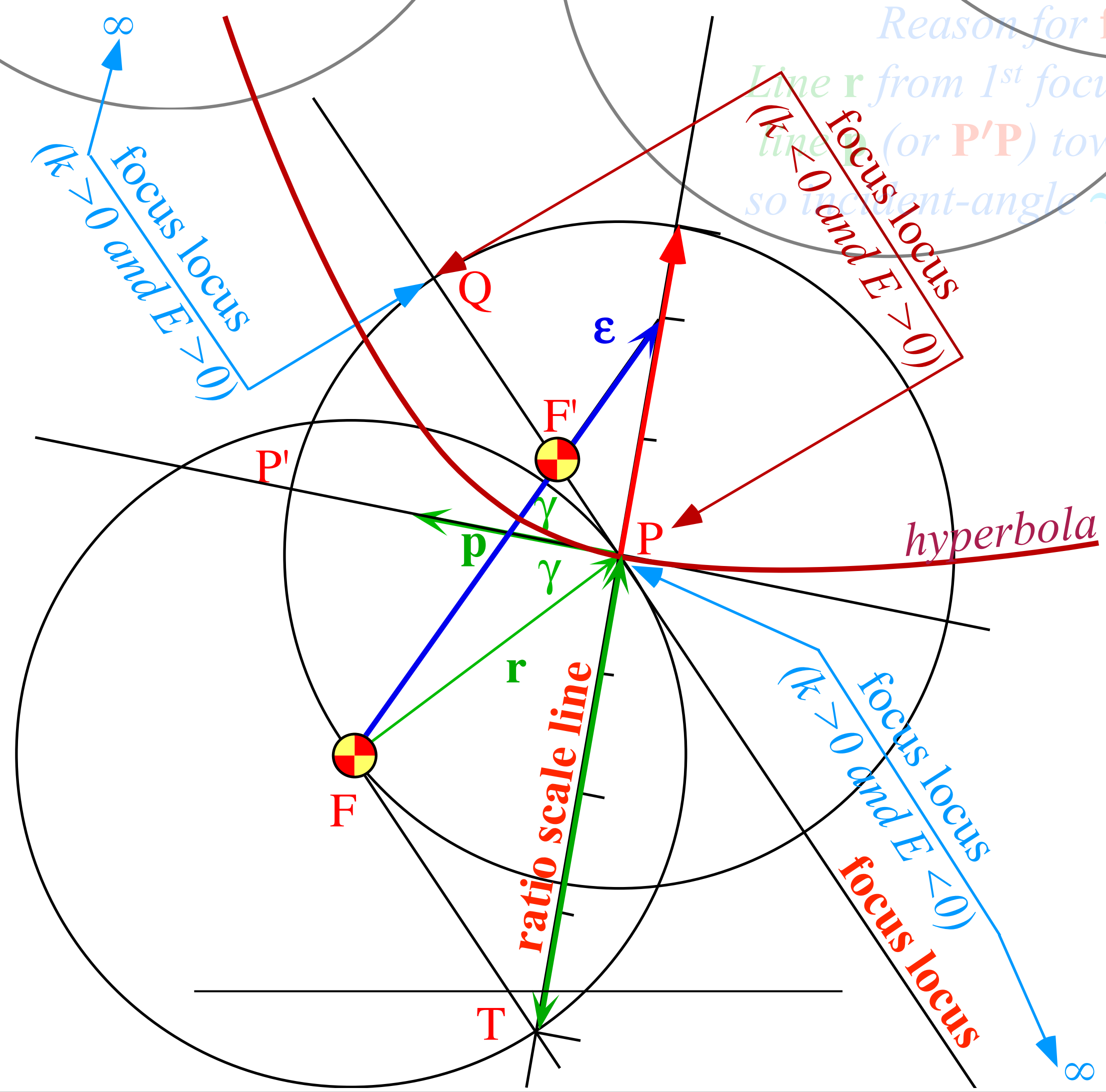








Pick initial R=KE/PE value
(here R=+1/2) Draw **ε−*vector***
from focus F *to R-point*
(Here it intersects 2nd focus F′

> *focus* F *and 2nd focus* F′ *allow final construction of orbital trajectory.*
> *Here it is an R=+1/2 hyperbola.*

$$R=\frac{\text{Initial } KE}{\text{Initial } PE}=\frac{m v^2(0)/2}{-k/r(0)}$$

$$=\pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2=\pm\frac{v^2(0)}{v^2(\infty)}$$

(Detailed Analytic geometry of ***ε-vector*** *follows.)*

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

ε-*vector and Coulomb* **r**-*orbit geometry*

Review and connection to standard development

ε-*vector and Coulomb* **p**=*m***v** *geometry*

ε-*vector and Coulomb* **p**=*m***v** *algebra*

Example with elliptical orbit

Analytic geometry derivation of **ε**-*construction*

➔ *Algebra of* **ε**-*construction geometry*

Connection formulas for (a,b) and (ε,λ) with (γ,R)

Analytic geometry derivation of ε-constructions

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

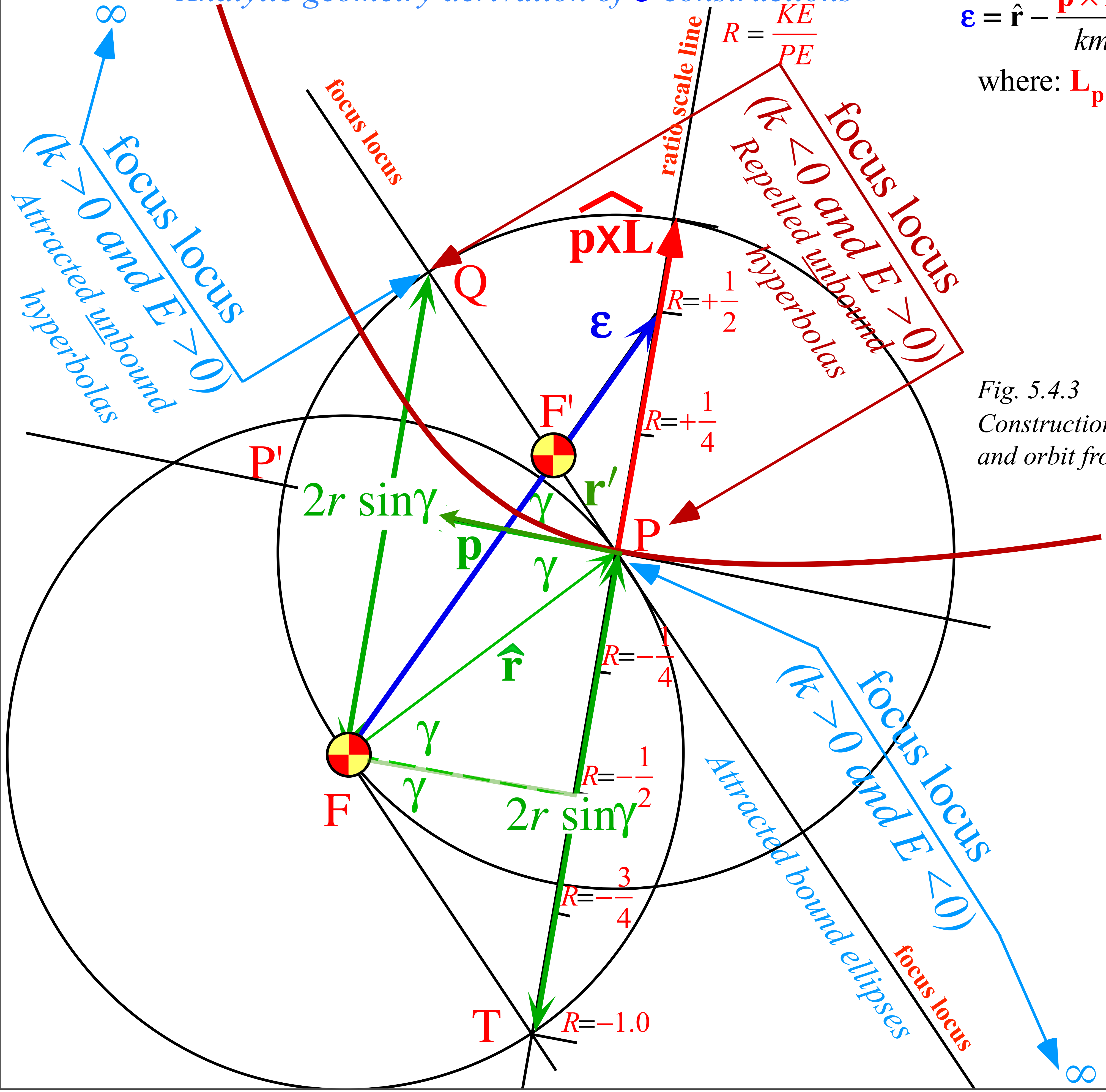


Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Analytic geometry derivation of ε-constructions

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

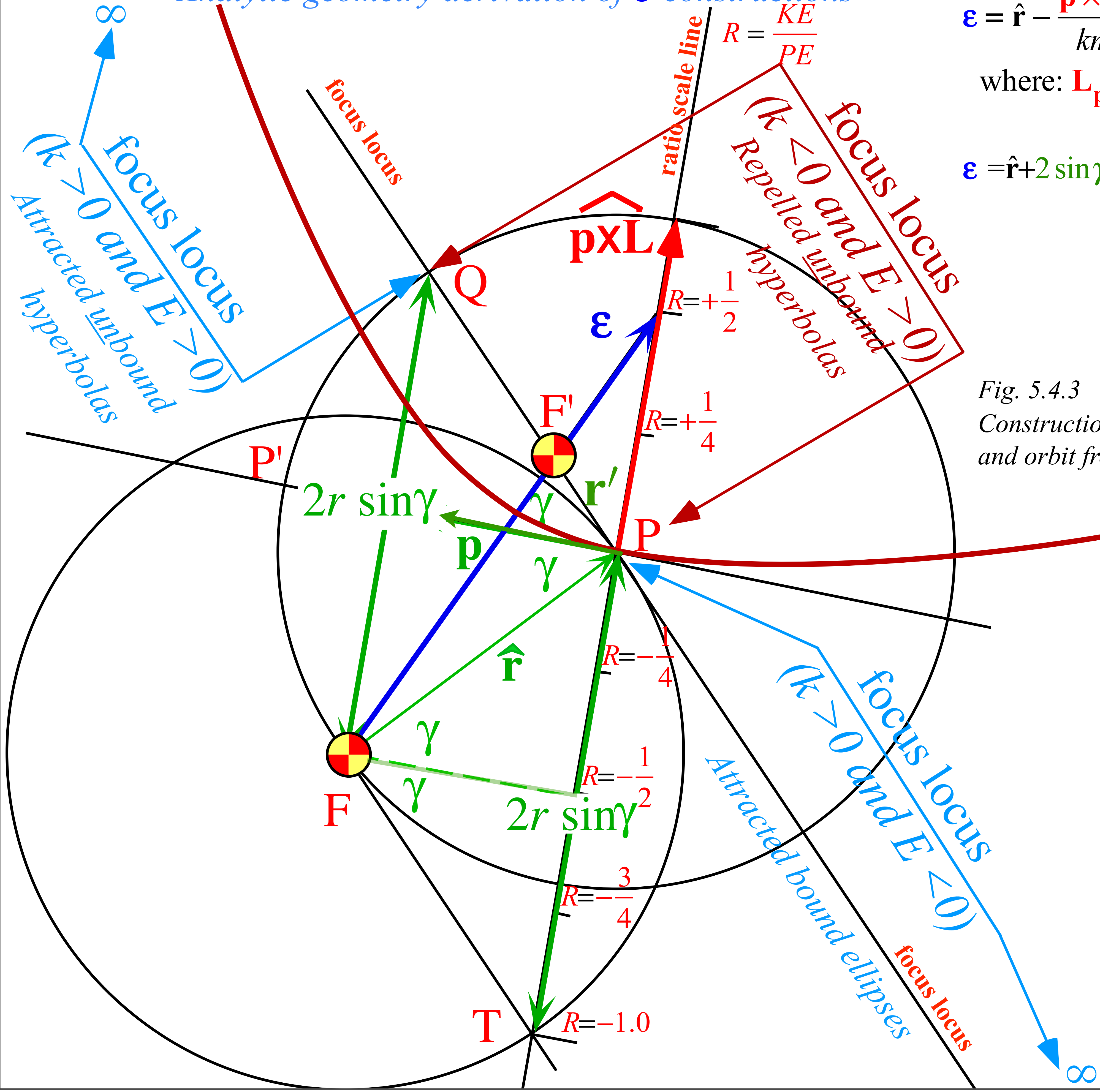
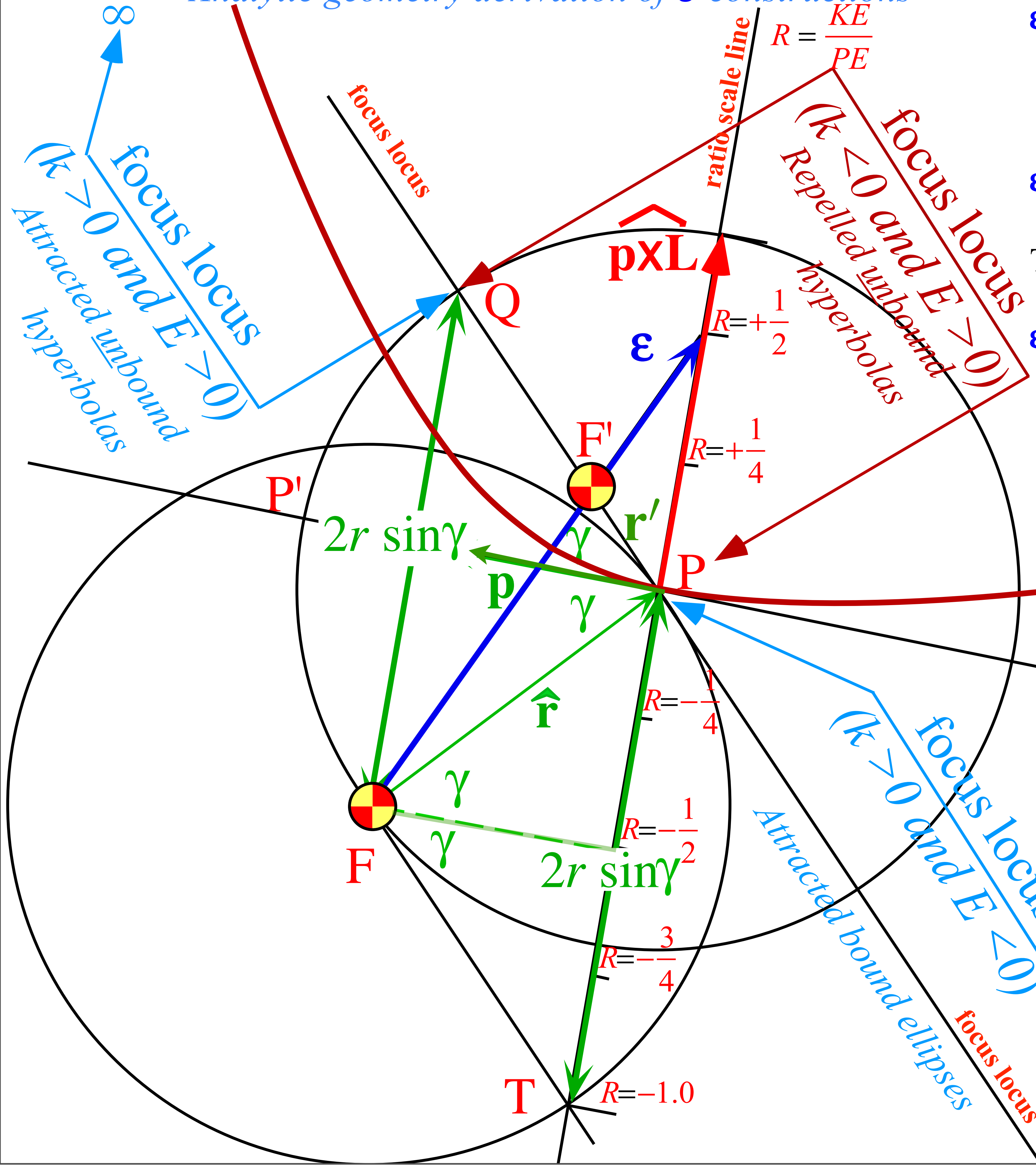


Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Analytic geometry derivation of ε-constructions



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

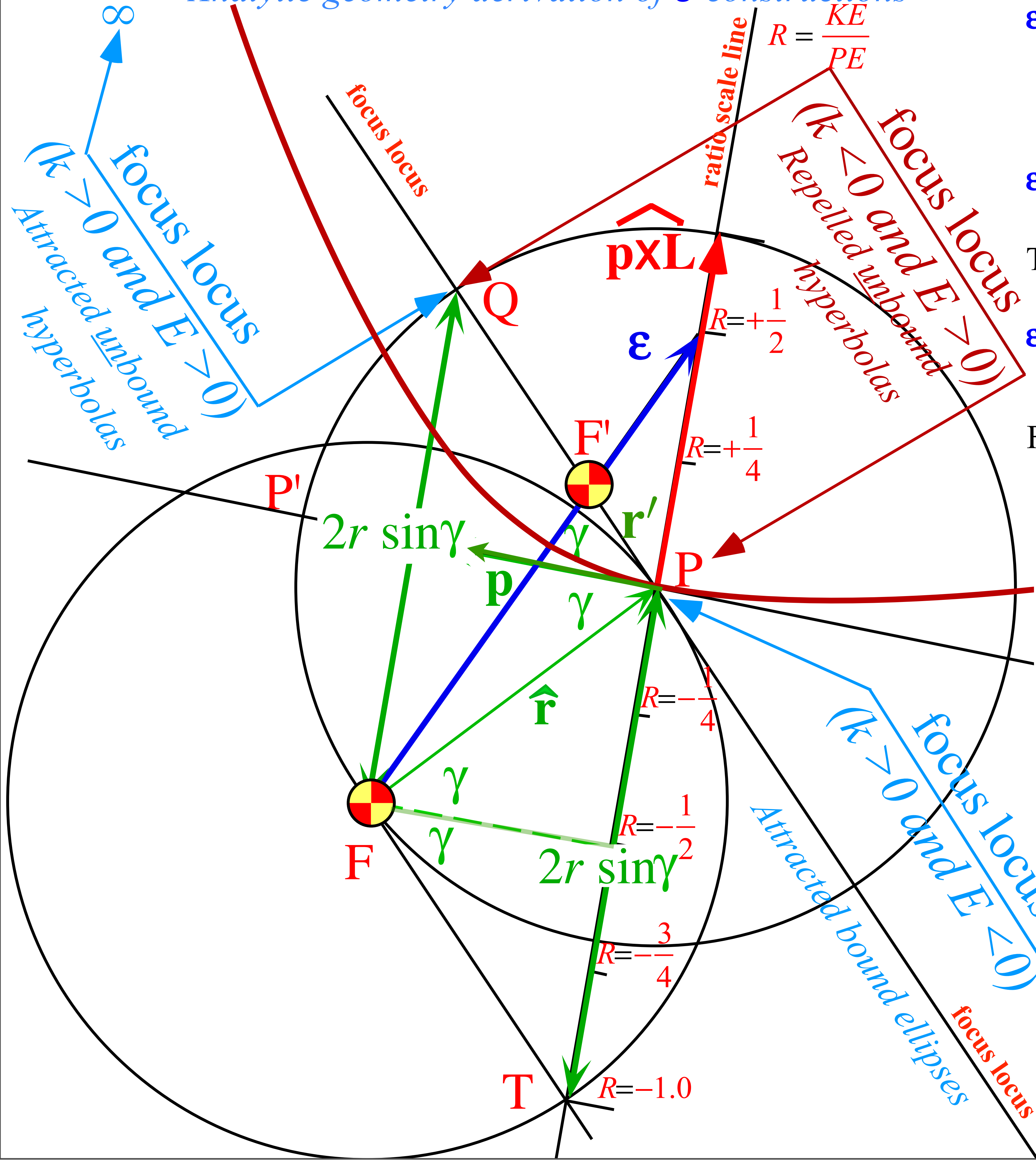
The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Analytic geometry derivation of ε-constructions



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For: $\gamma=45^\circ$ and: $R=+\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

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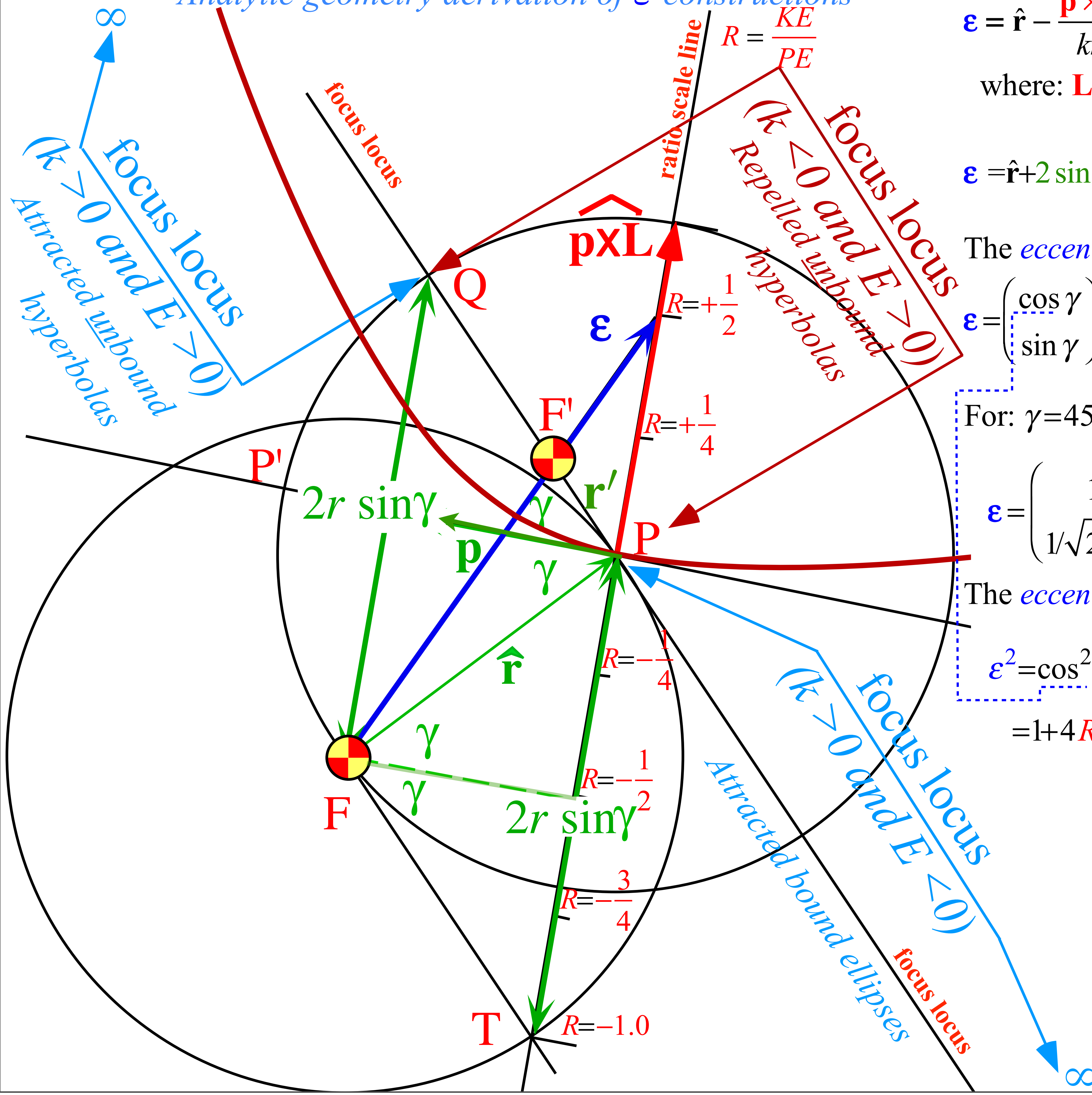

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The *eccentricity* parameter defined by:

$$\varepsilon^2 = \cos^2\gamma + (2R+1)^2\sin^2\gamma = 1 \pm \frac{a^2}{b^2}$$
$$= 1 + 4R(R+1)\sin^2\gamma = \frac{5}{2}$$

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Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* **r***-orbit geometry*

Review and connection to standard development

$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *geometry*

$\boldsymbol{\varepsilon}$*-vector and Coulomb* $\mathbf{p}=m\mathbf{v}$ *algebra*

Example with elliptical orbit

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Algebra of $\boldsymbol{\varepsilon}$*-construction geometry*

➔ *Connection formulas for (a,b) and* (ε,λ) *with* (γ,R)

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1)$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola} \ (\varepsilon > 1)$$

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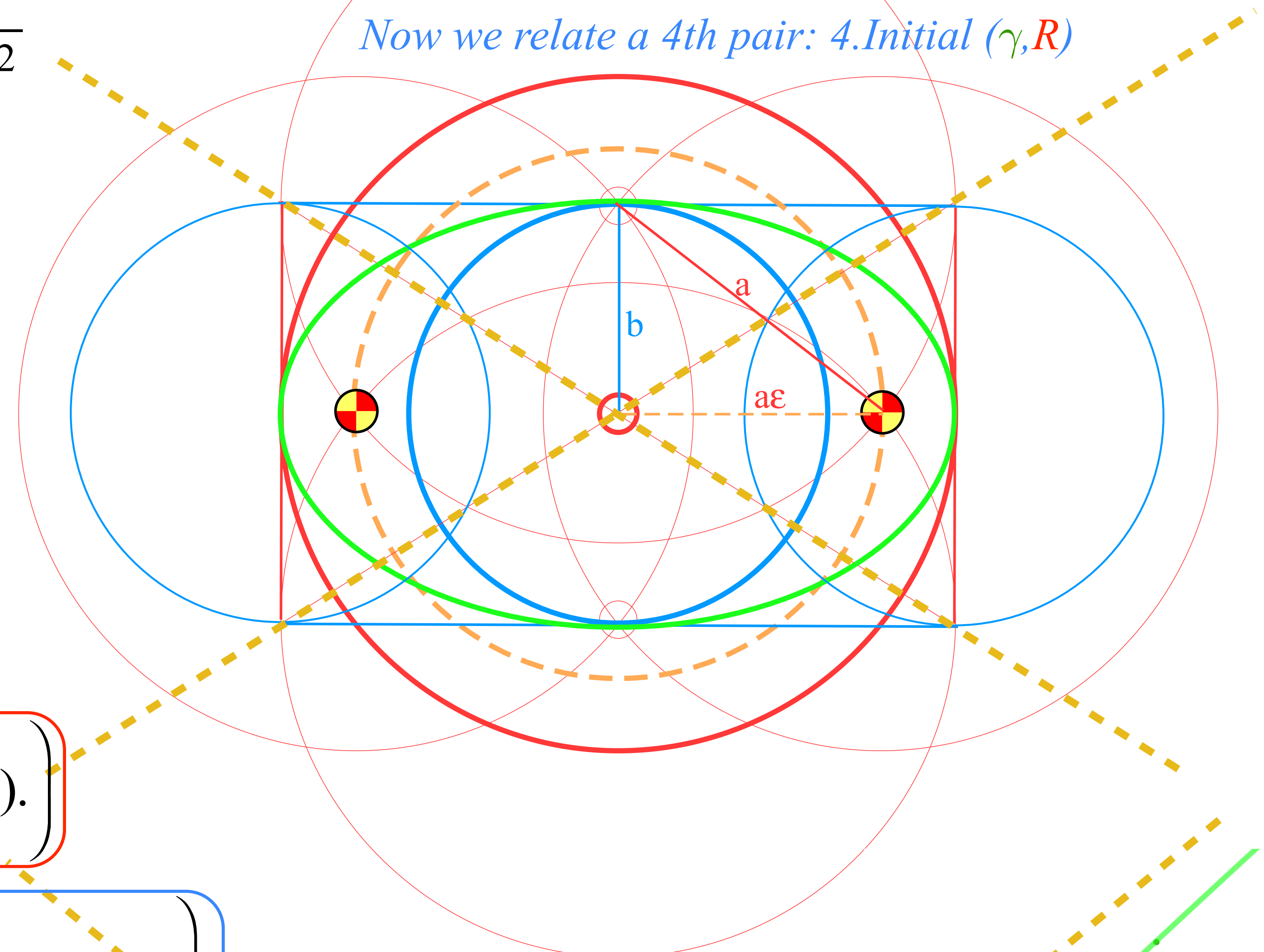

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From ε^2 result (at top):

$$\frac{b}{a} = 2\sqrt{\mp R(R+1)}\sin\gamma = \sqrt{\pm(1-\varepsilon^2)}$$