Lecture 8
Thur. 9.19.2013

Kepler Geometry of IHO (Isotropic Harmonic Oscillator) Elliptical Orbits

(Ch. 9 and Ch. 11 of Unit 1)

Constructing 2D IHO orbits by phasor plots

Review of phasor "clock" geometry (From Lecture 7)
Integrating IHO equations by phasor geometry

Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry
Calculus and vector geometry of IHO orbits
A confusing introduction to Coriolis-centrifugal force geometry

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived rigorously)
Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived later)
Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ (Derived rigorously)
Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)

Brief introduction to matrix quadratic form geometry

BoxIt simulation of U(2) orbits
http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

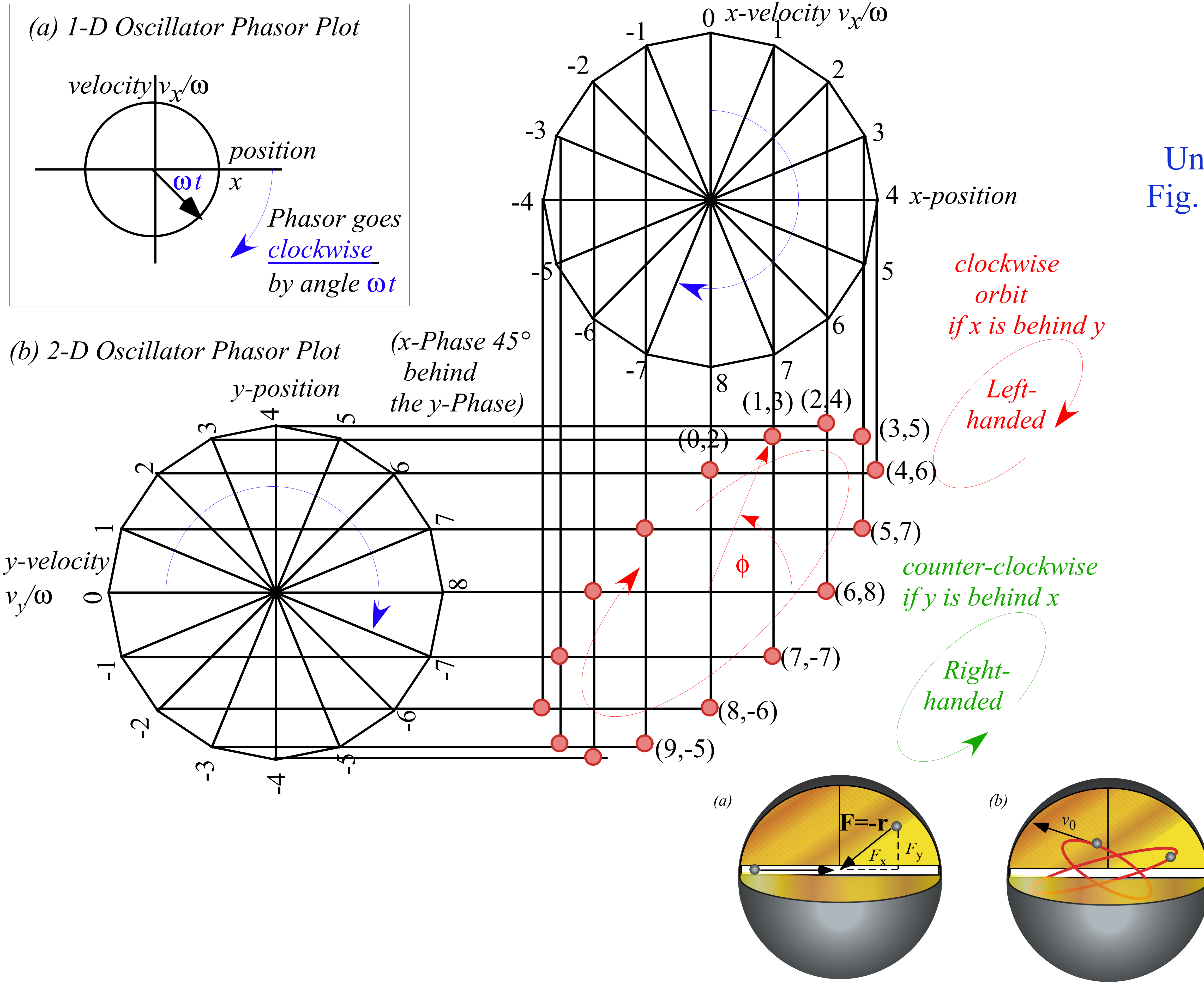
Constructing 2D IHO orbits by phasor plots

→ *Review of phasor "clock" geometry (From Lecture 7)*

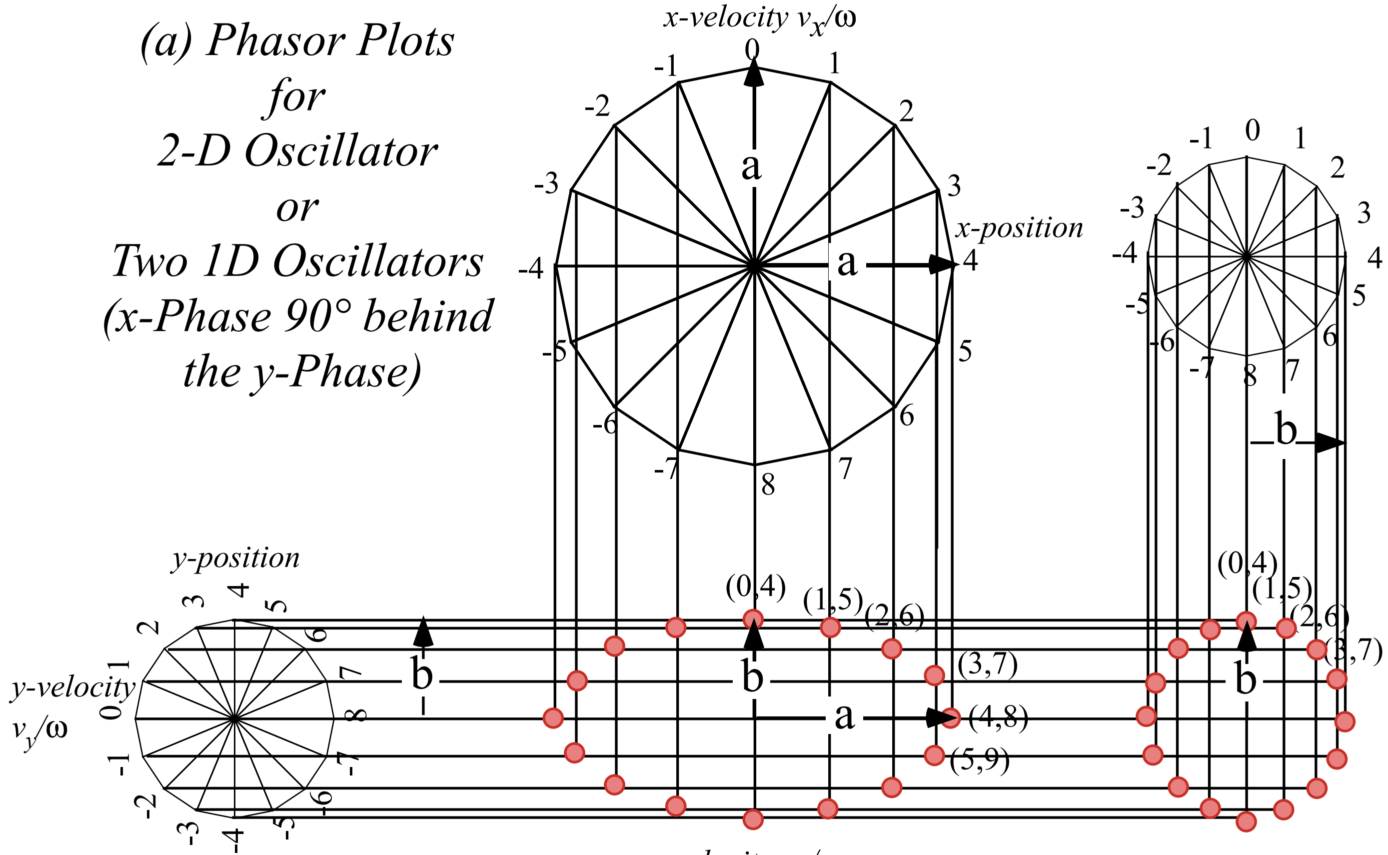
Integrating IHO equations by phasor geometry

Isotropic Harmonic Oscillator phase dynamics in uniform-body

Unit 1
Fig. 9.10

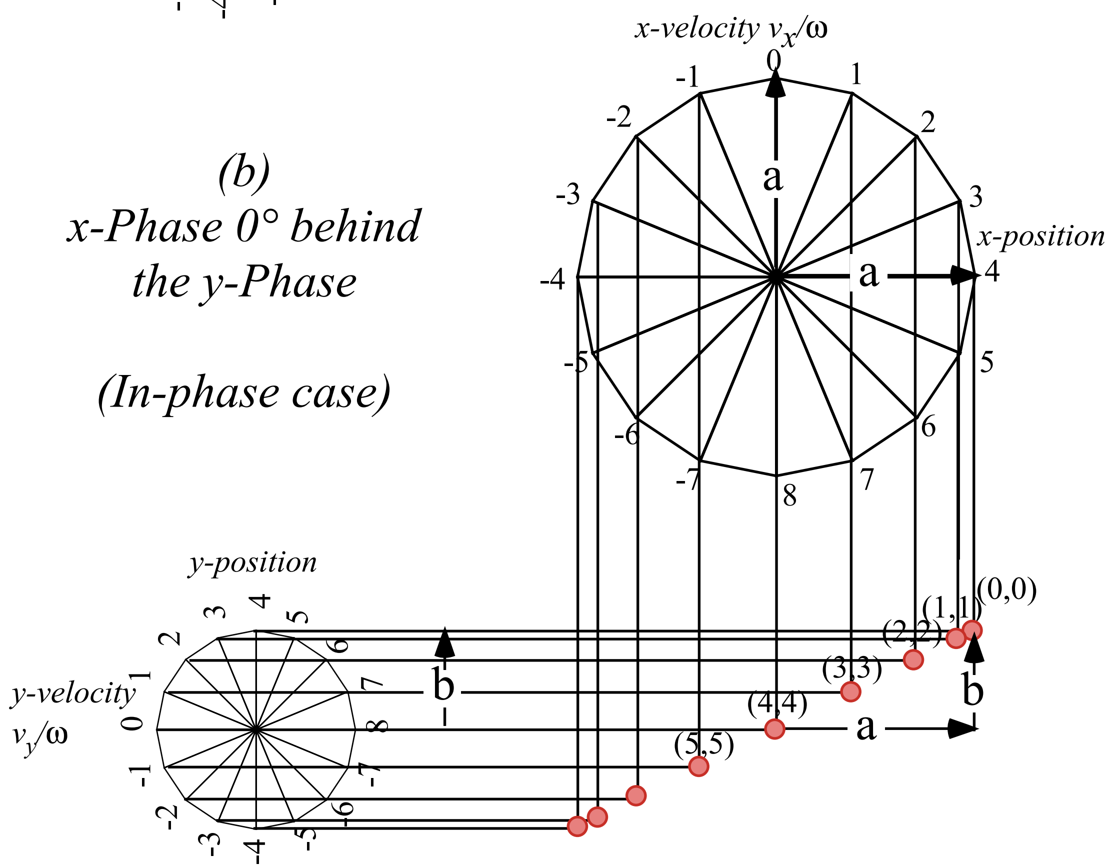


(a) Phasor Plots for 2-D Oscillator or Two 1D Oscillators (x-Phase 90° behind the y-Phase)

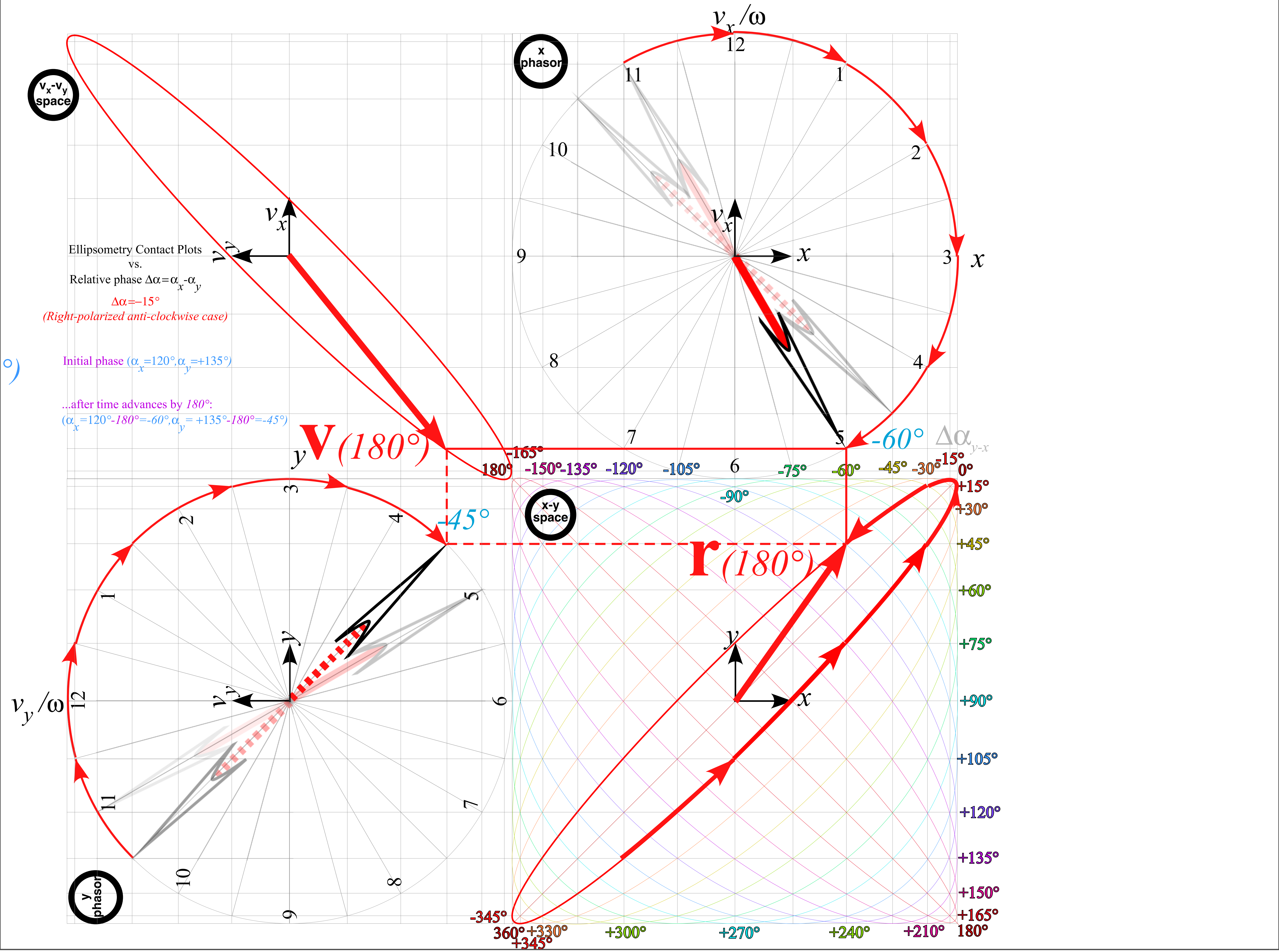


(b) x-Phase 0° behind the y-Phase

(In-phase case)



These are more generic examples with radius of x-phasor differing from that of the y-phasor.

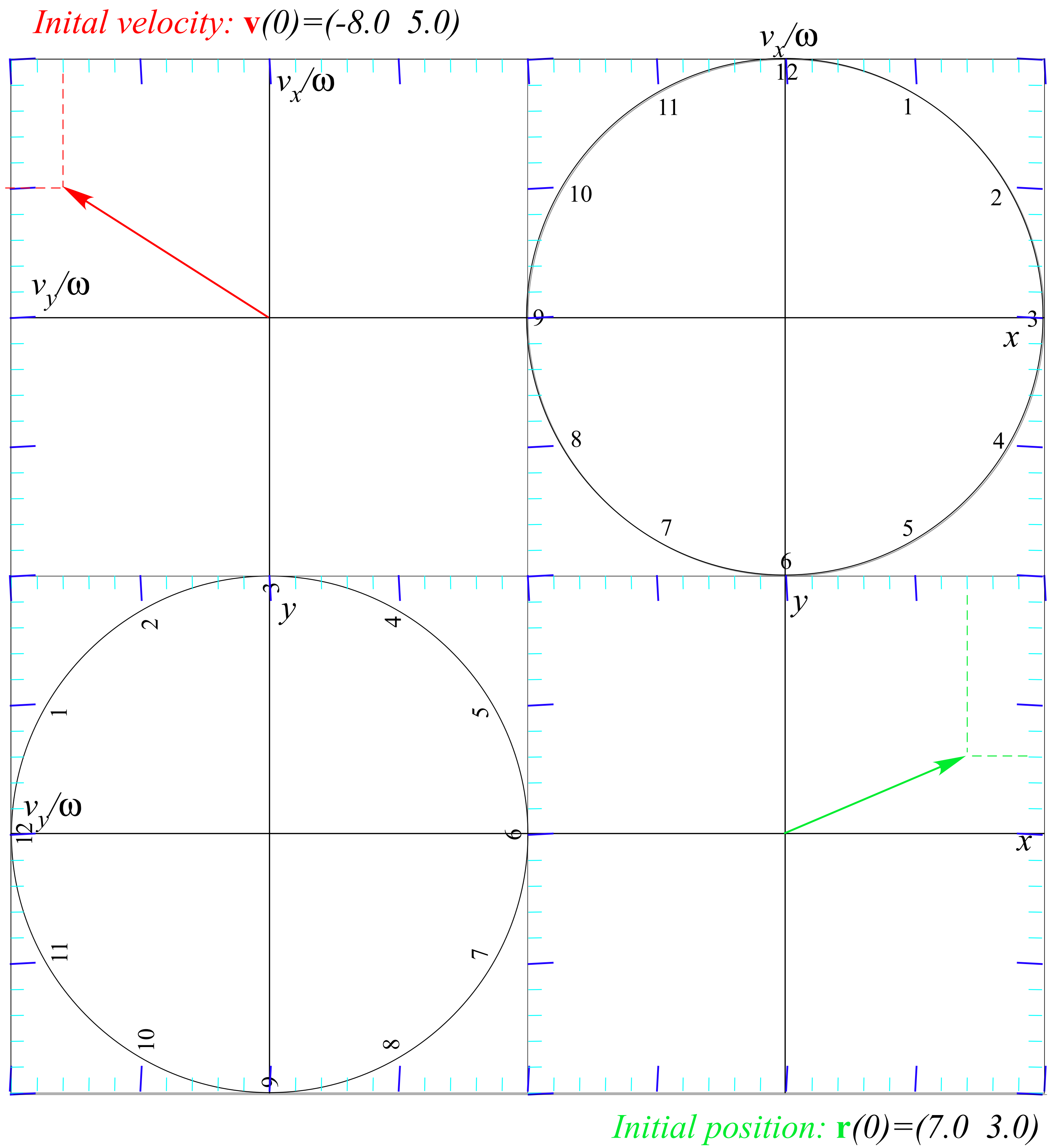


Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=–15°
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 180°:
(α_x=120°-180°=-60°,α_y= +135°-180°=-45°)
v_x-v_y space
x phasor
y phasor
x-y space
v_x/ω
v_y/ω
v_x
v_y
x
y
V(180°)
r(180°)
-60°
-45°
Δα_y-x
180°
-165°
-150°
-135°
-120°
-105°
-90°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°
+210°
+240°
+270°
+300°
+330°
+345°
360°
-345°

Constructing 2D IHO orbits by phasor plots

Review of phasor "clock" geometry (From Lecture 7)

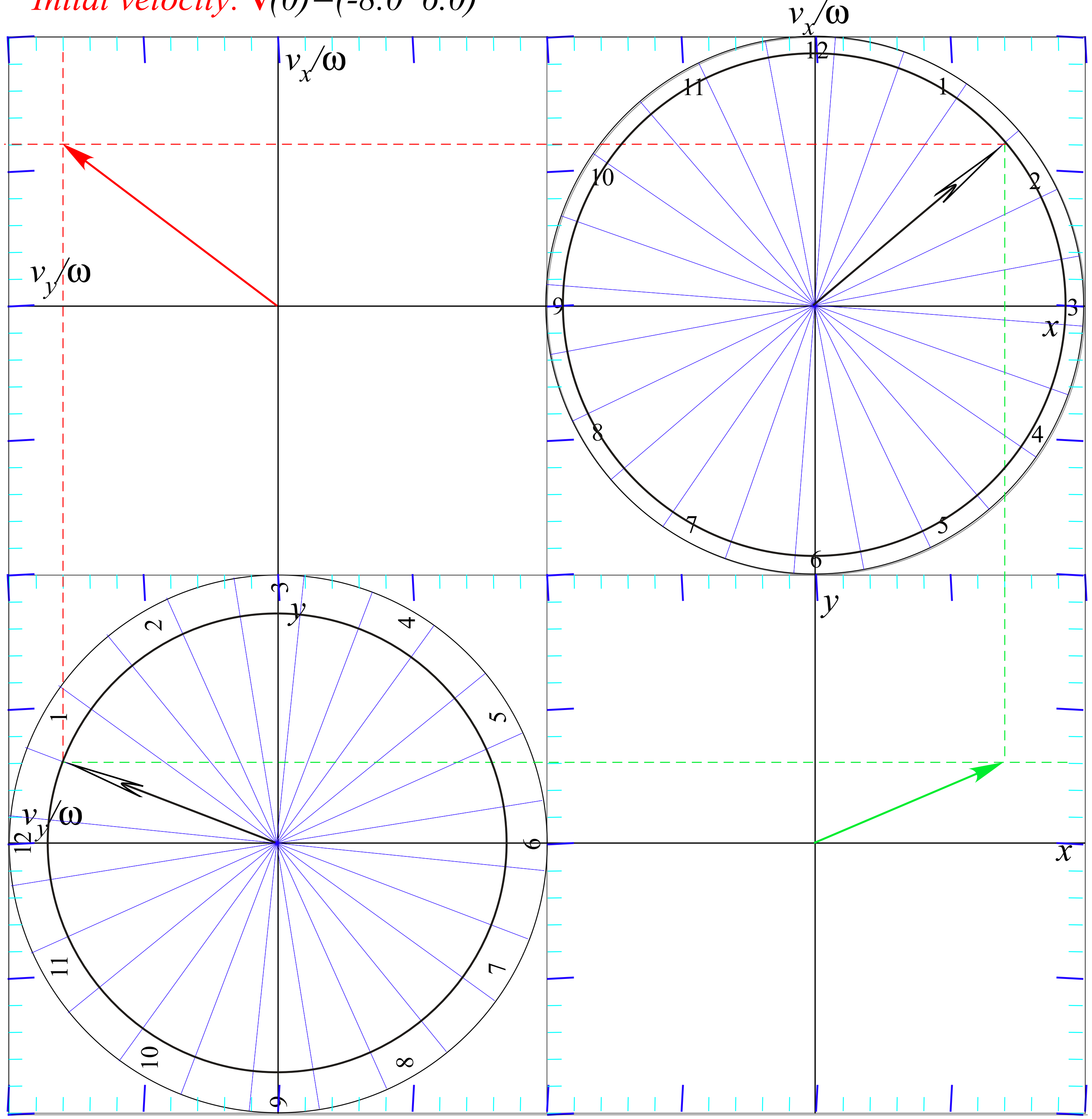
→ *Integrating IHO equations by phasor geometry (case of unequal x and y phasor area)*



[BoxIt simulation of U(2) orbits](http://www.uark.edu/ua/modphys/markup/BoxItWeb.html)

http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

Inital velocity: **v**(0)=(-8.0 6.0)



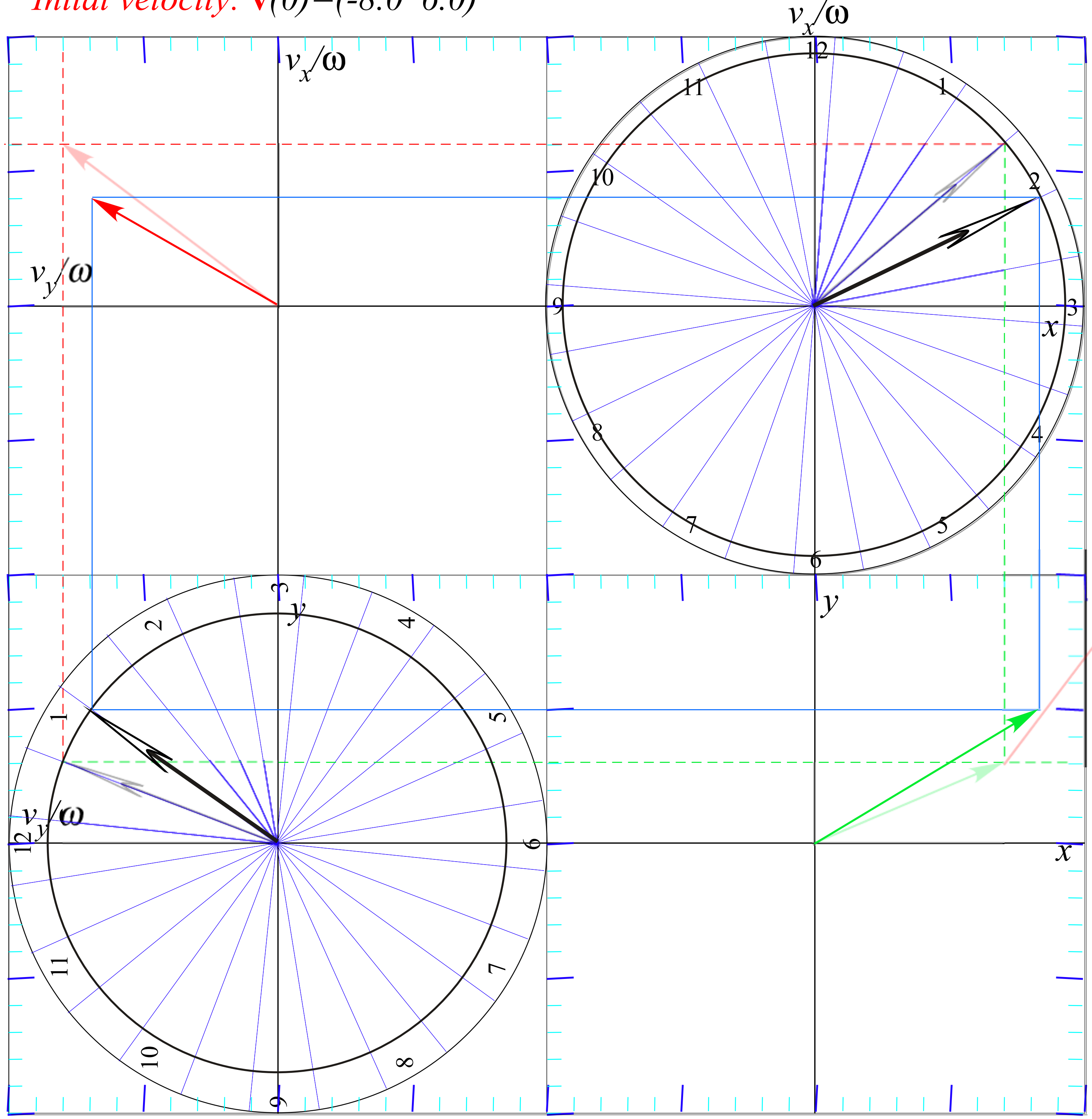
Arbitrary initial position
r(0)=(x(0),y(0))

and initial velocity
v(0)=(v_x(0),v_y(0))

Usually have x and y
phasor circles of unequal size

Initial position: **r**(0)=(7.0 3.0)

Inital velocity: **v***(0)=(-8.0 6.0)*

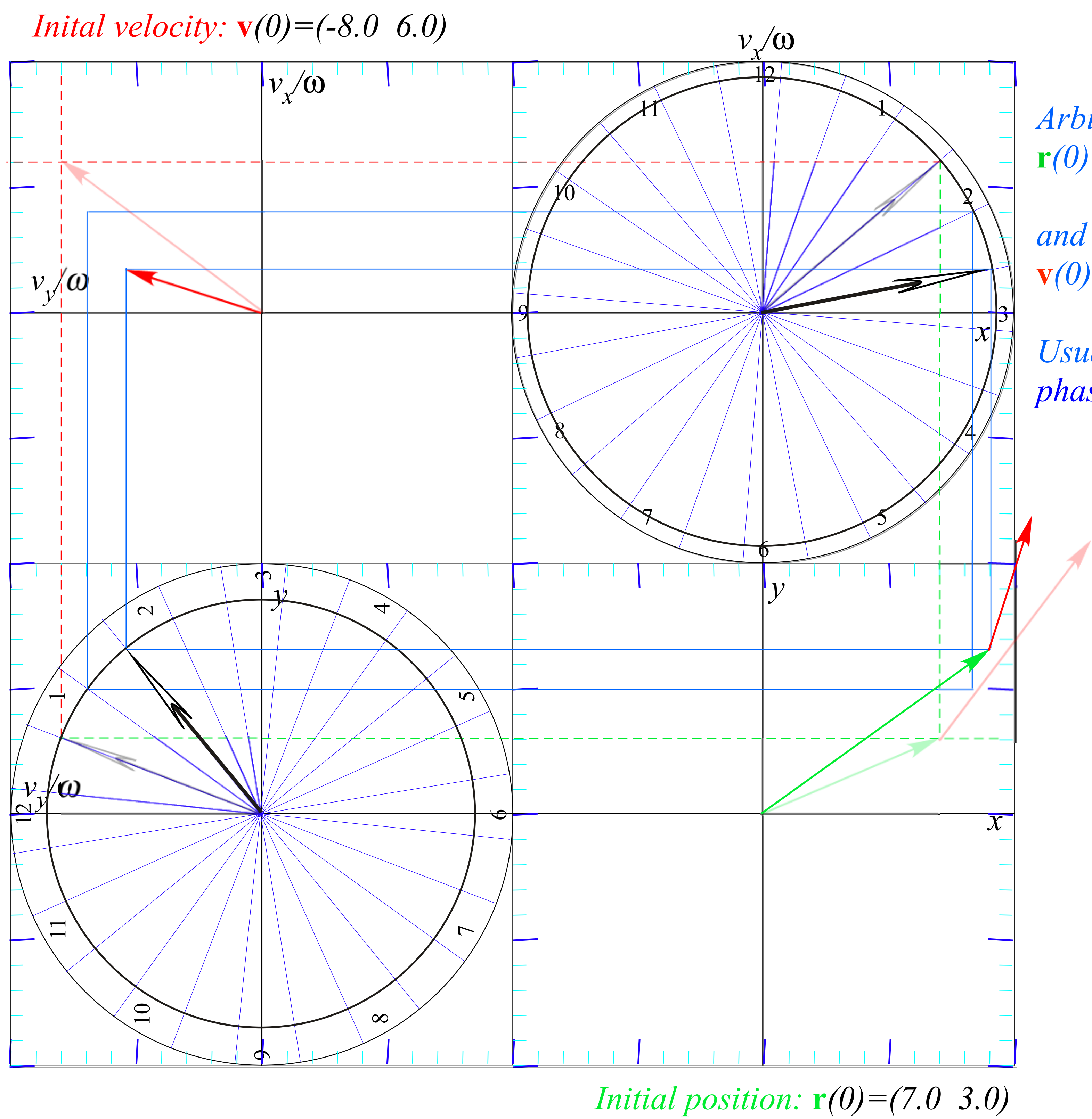


Arbitrary initial position
r*(0)=(x(0),y(0)*

and initial velocity
v*(0)=*($v_x(0)$,$v_y(0)$

Usually have x and y
phasor circles of unequal size

Initial position: **r***(0)=(7.0 3.0)*

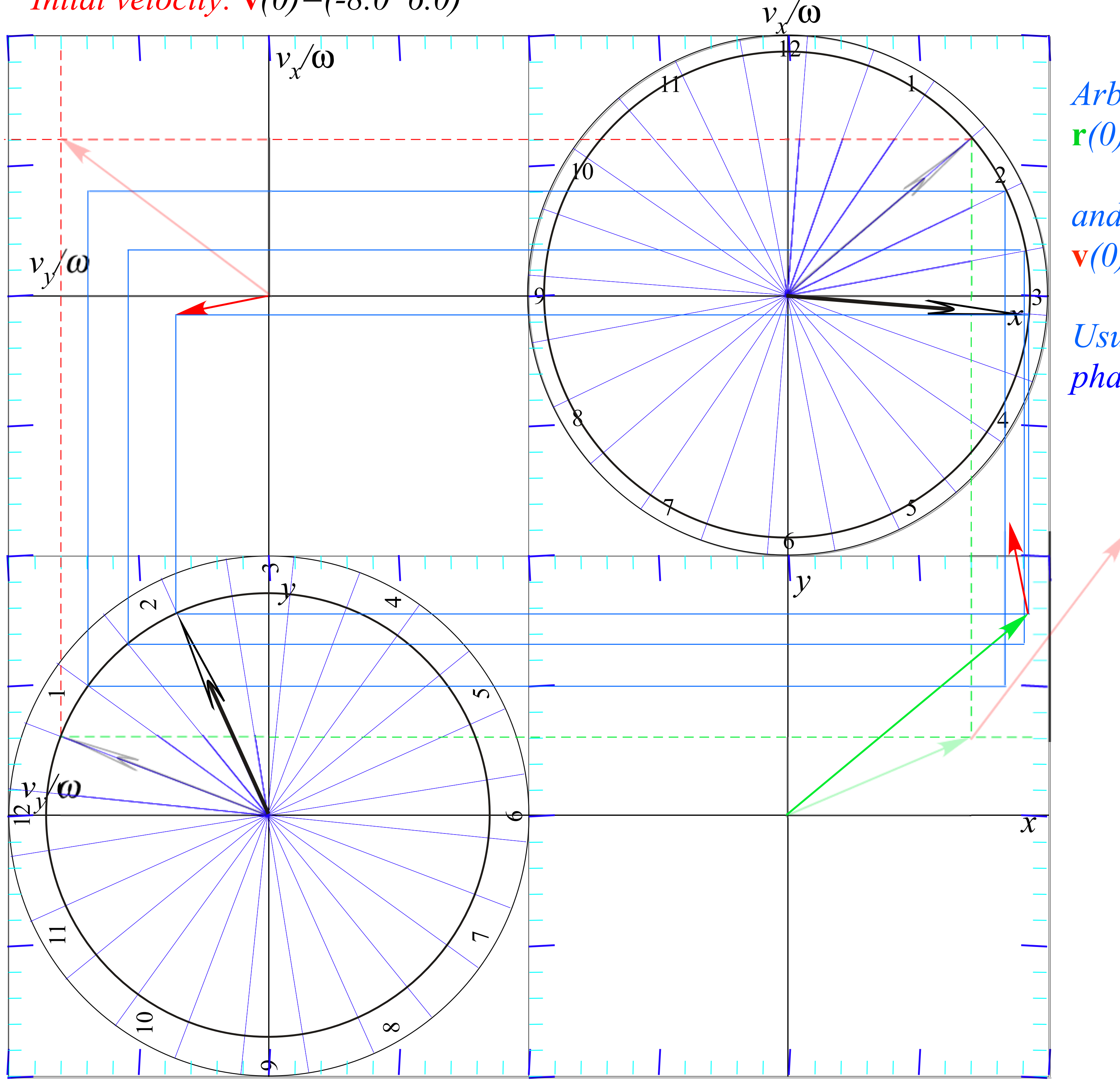


Arbitrary initial position
r*(0)=(x(0),y(0))*

and initial velocity
v*(0)=(*v_x*(0),*v_y*(0))*

Usually have x and y
phasor circles of unequal size

Inital velocity: **v**(0)=(-8.0 6.0)



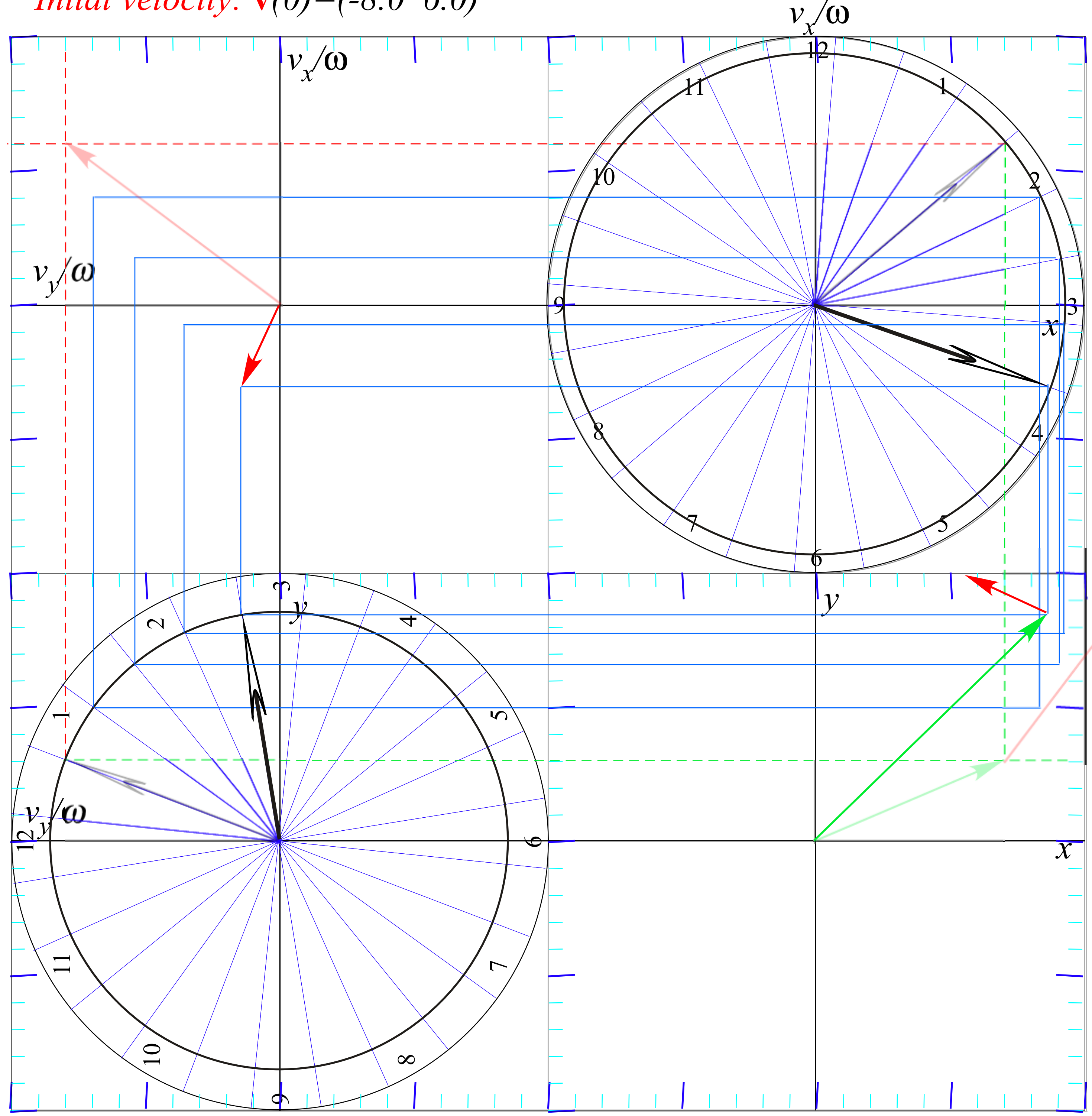
Arbitrary initial position
r(0)=(x(0),y(0)

and initial velocity
v(0)=(v_x(0),v_y(0)

Usually have x and y
phasor circles of unequal size

Initial position: **r**(0)=(7.0 3.0)

Inital velocity: **v***(0)=(-8.0 6.0)*



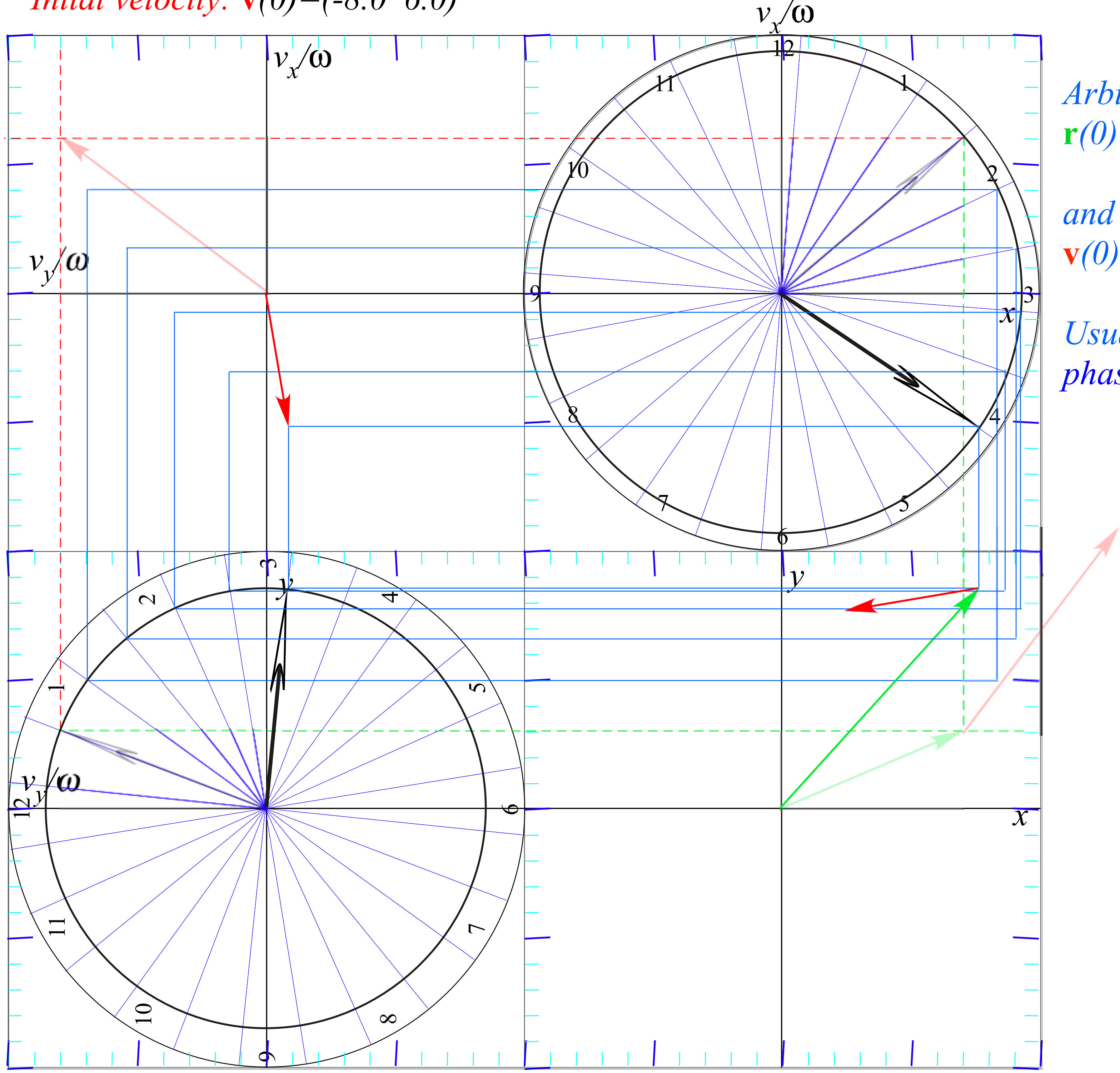
Arbitrary initial position
r*(0)=(x(0),y(0))*

and initial velocity
v*(0)=(*v_x*(0),*v_y*(0))*

Usually have x and y
phasor circles of unequal size

Initial position: **r***(0)=(7.0 3.0)*

Inital velocity: $\mathbf{v}(0)=(-8.0\ \ 6.0)$



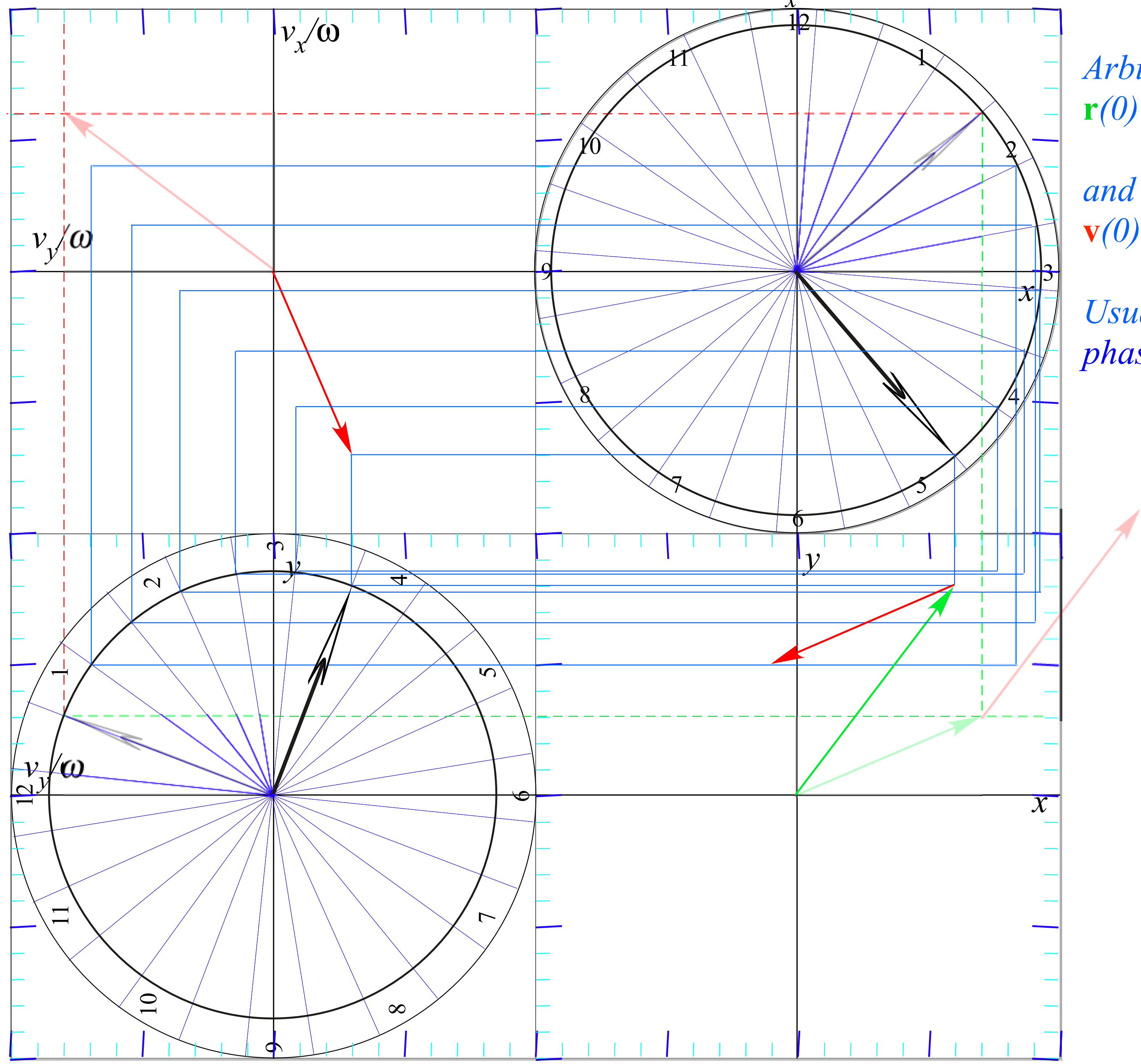
Arbitrary initial position
$\mathbf{r}(0)=(x(0),y(0))$

and initial velocity
$\mathbf{v}(0)=(v_x(0),v_y(0))$

Usually have x and y phasor circles of unequal size

Initial position: $\mathbf{r}(0)=(7.0\ \ 3.0)$

Inital velocity: **v***(0)=(-8.0 6.0)*



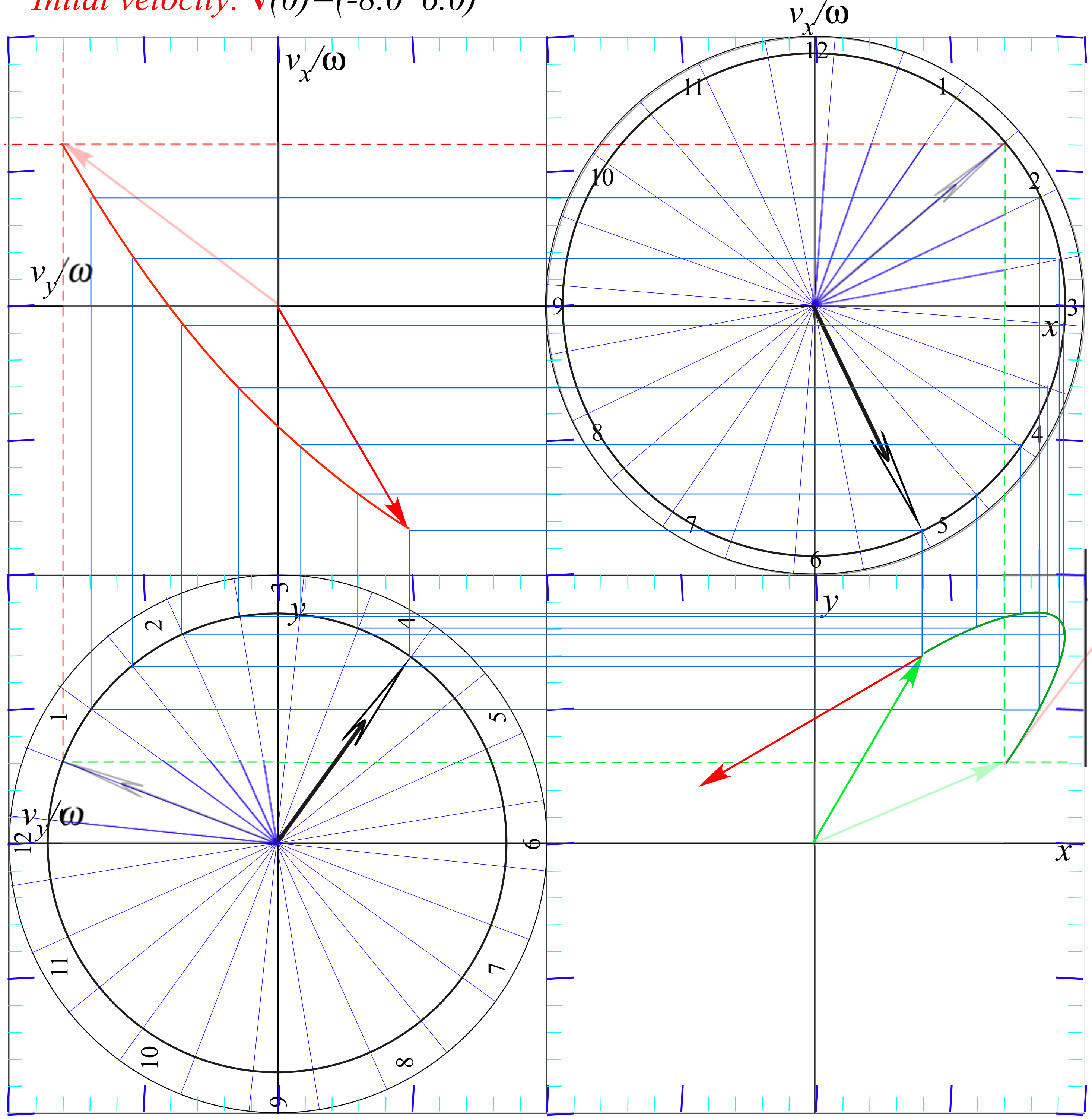
Arbitrary initial position
r*(0)=(x(0),y(0))*

and initial velocity
v*(0)=($v_x(0)$,$v_y(0)$)*

Usually have x and y
phasor circles of unequal size

Initial position: **r***(0)=(7.0 3.0)*

Inital velocity: **v***(0)=(-8.0 6.0)*



Initial position: **r***(0)=(7.0 3.0)*

Arbitrary initial position
r*(0)=(x(0),y(0)*

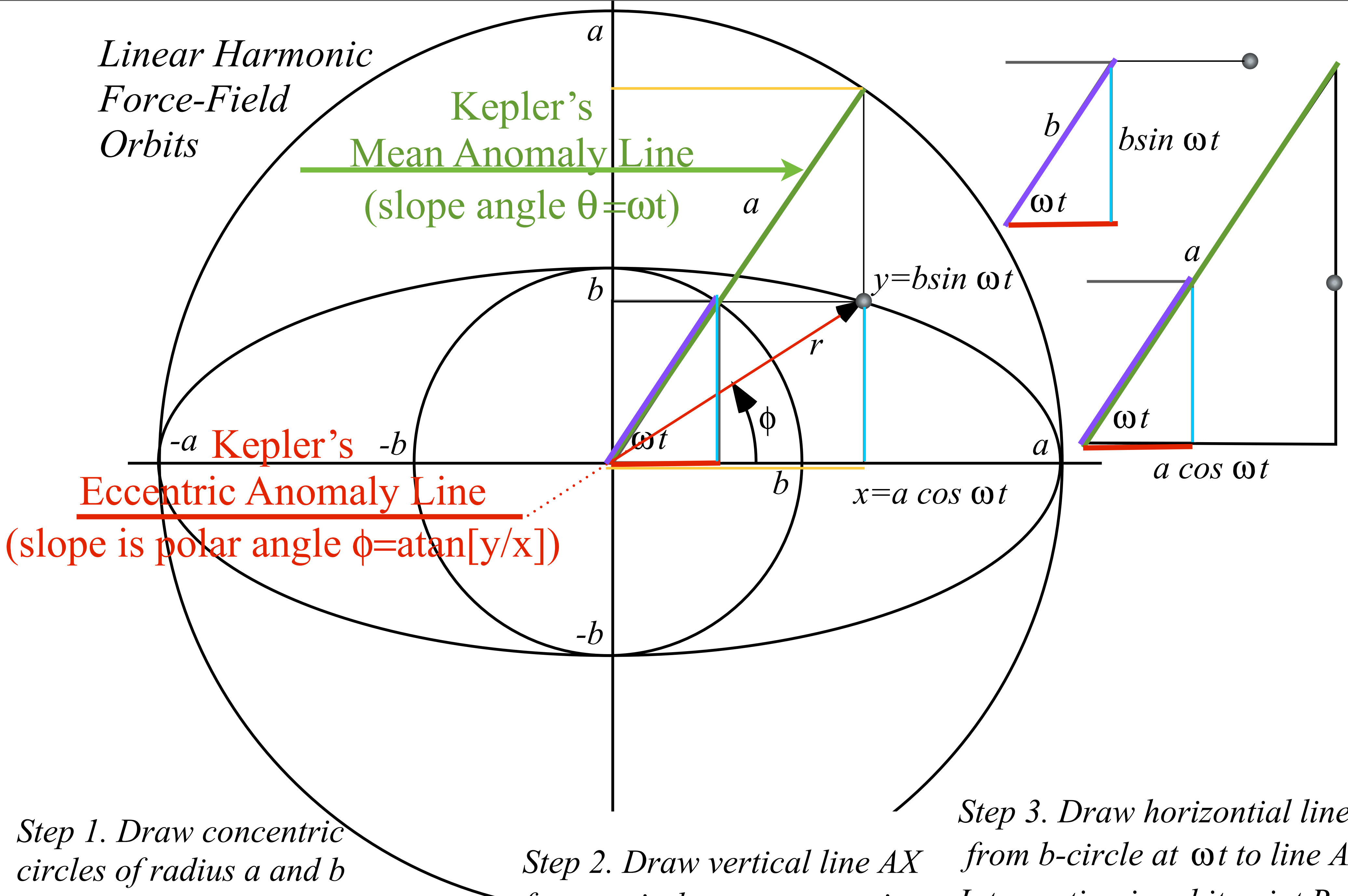
and initial velocity
v*(0)=(v_x(0),v_y(0)*

Usually have x and y
phasor circles of unequal size

Constructing 2D IHO orbits using Kepler anomaly plots

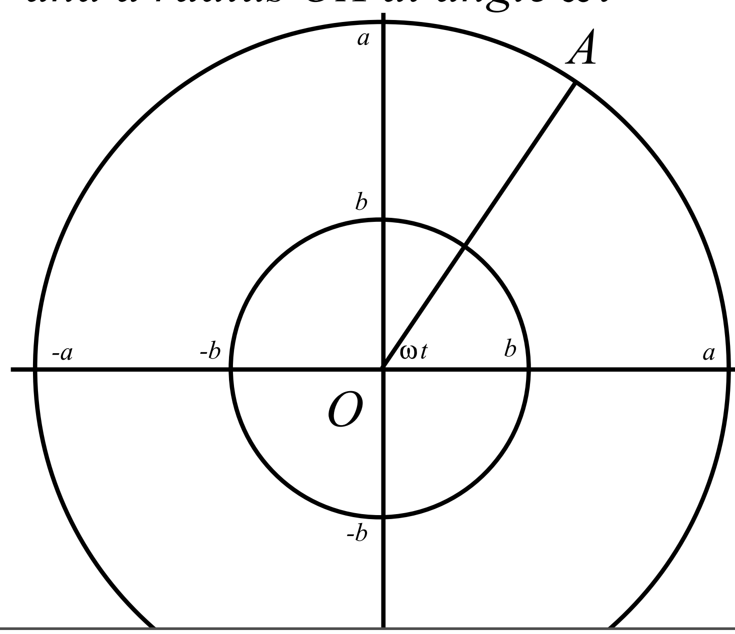
→ *Mean-anomaly and eccentric-anomaly geometry*
Calculus and vector geometry of IHO orbits
A confusing introduction to Coriolis-centrifugal force geometry

Linear Harmonic Force-Field Orbits

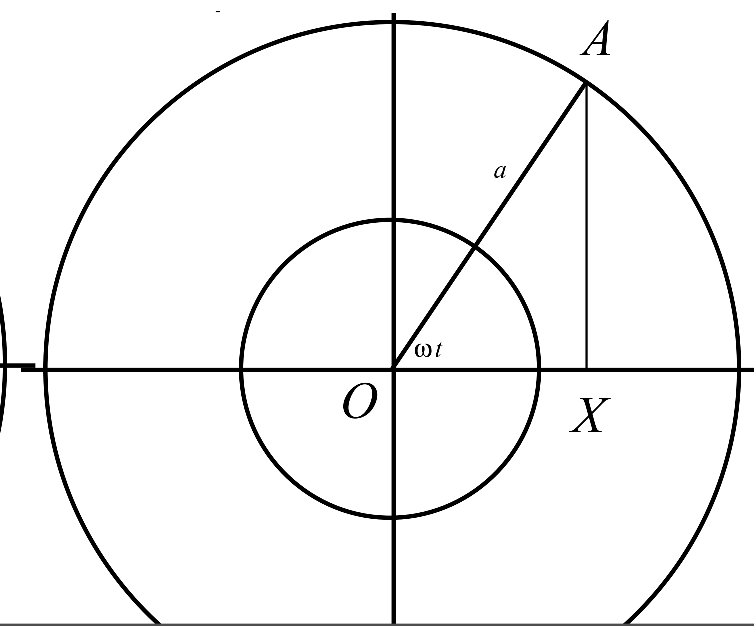


Unit 1
Fig. 11.1
(top 2/3's)

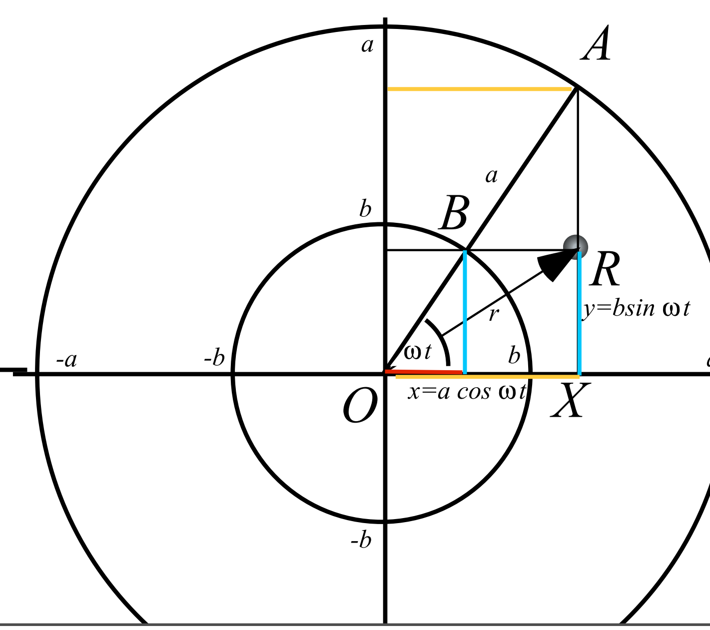
Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt

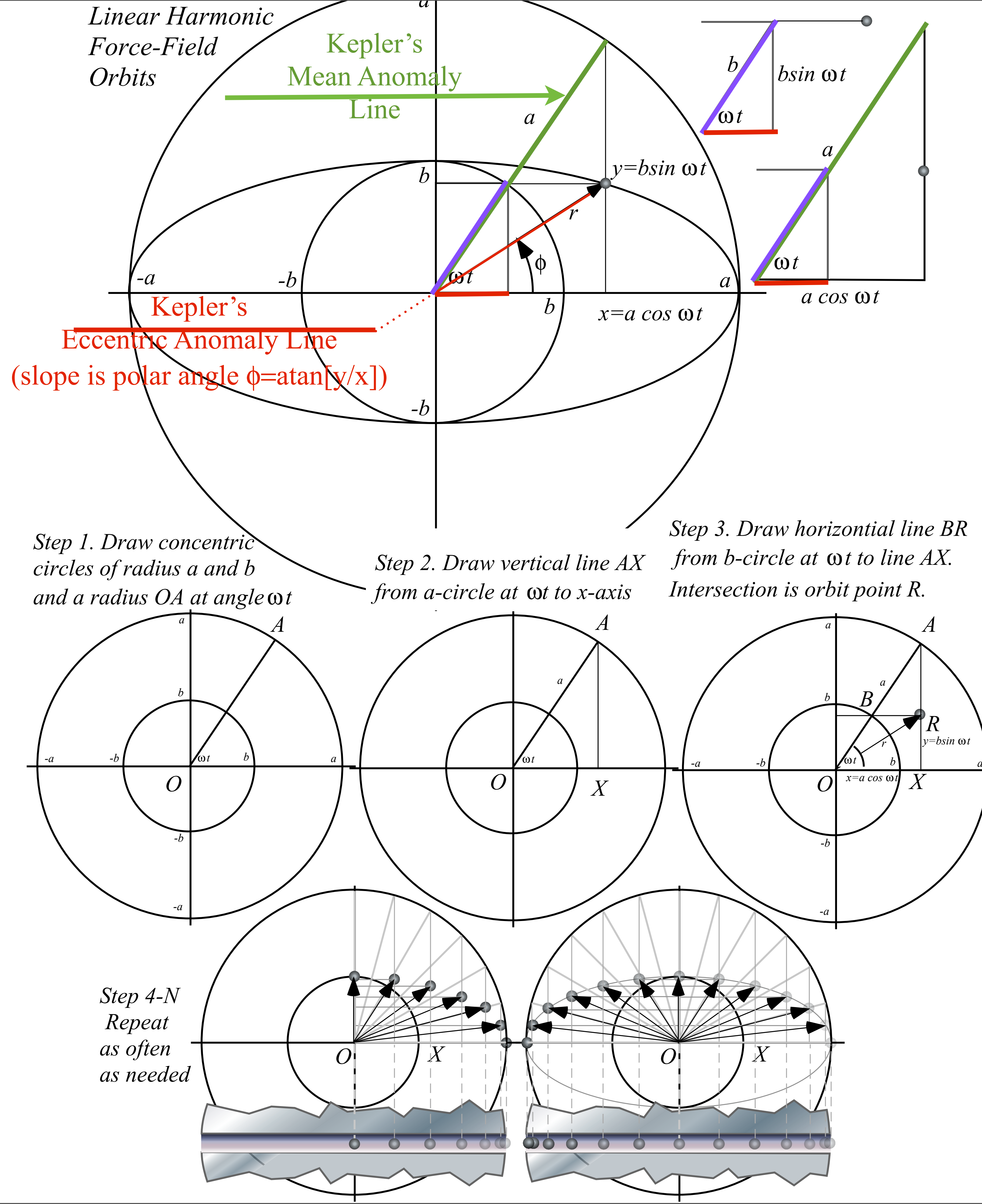


Step 2. Draw vertical line AX from a-circle at ωt to x-axis



Step 3. Draw horizontial line BR from b-circle at ωt to line AX. Intersection is orbit point R.





Unit 1
Fig. 11.1

Constructing 2D IHO orbits using Kepler anomaly plots

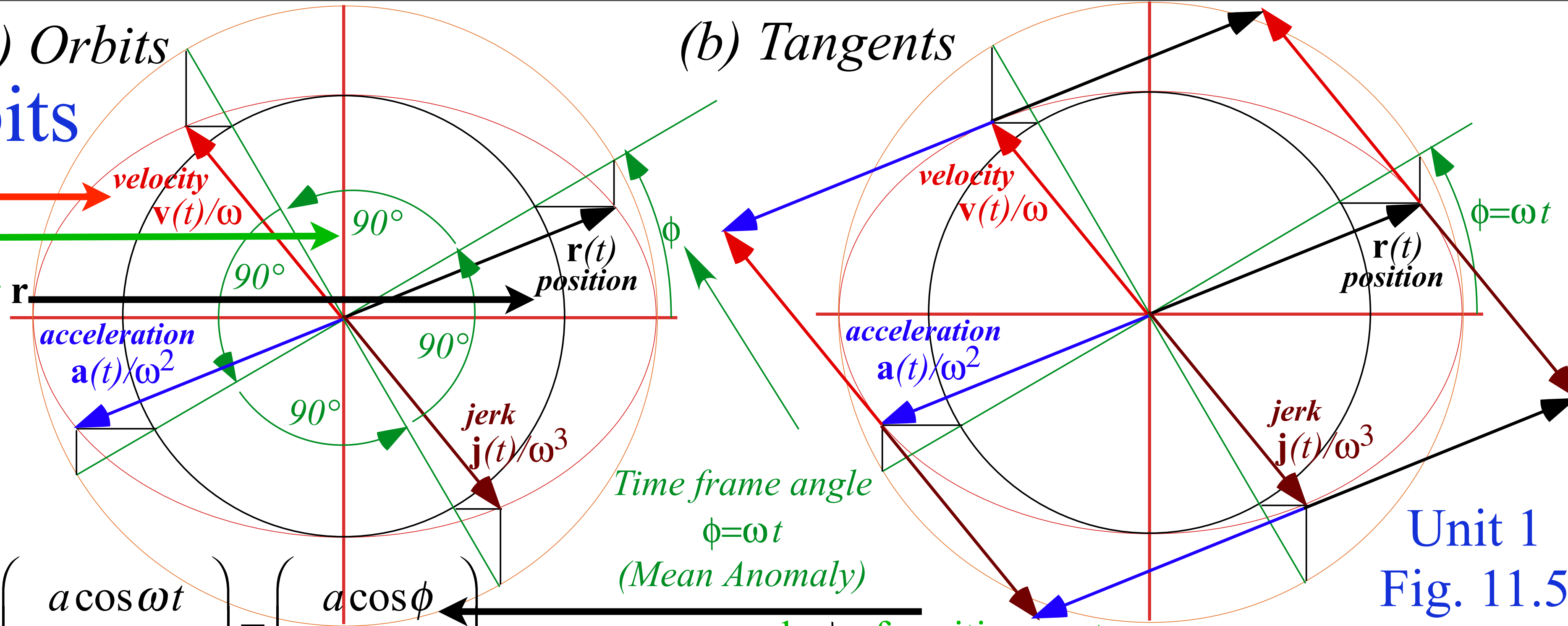
Mean-anomaly and eccentric-anomaly geometry

→ *Calculus and vector geometry of IHO orbits*

A confusing introduction to Coriolis-centrifugal force geometry

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**



Unit 1
Fig. 11.5

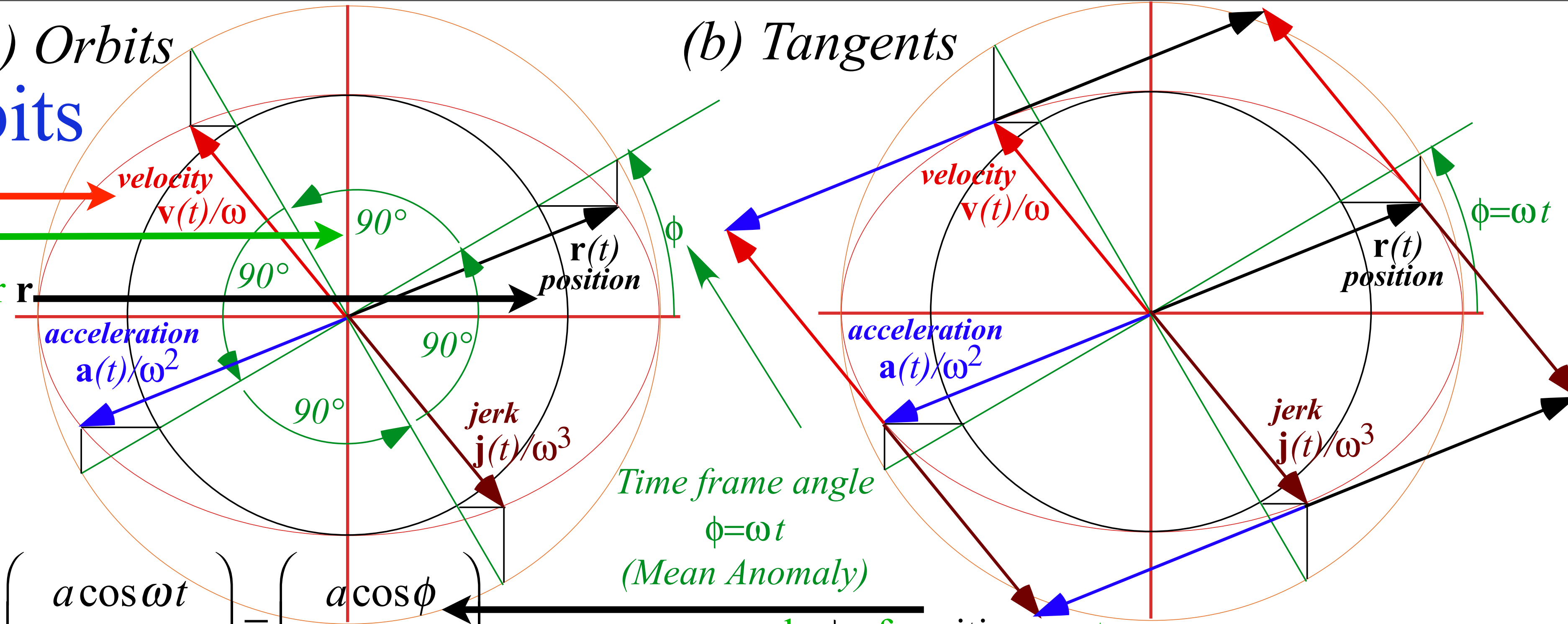
$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly ϕ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**



$$\text{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly φ of position vector **r** rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\text{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\text{for } \omega = 1)$$

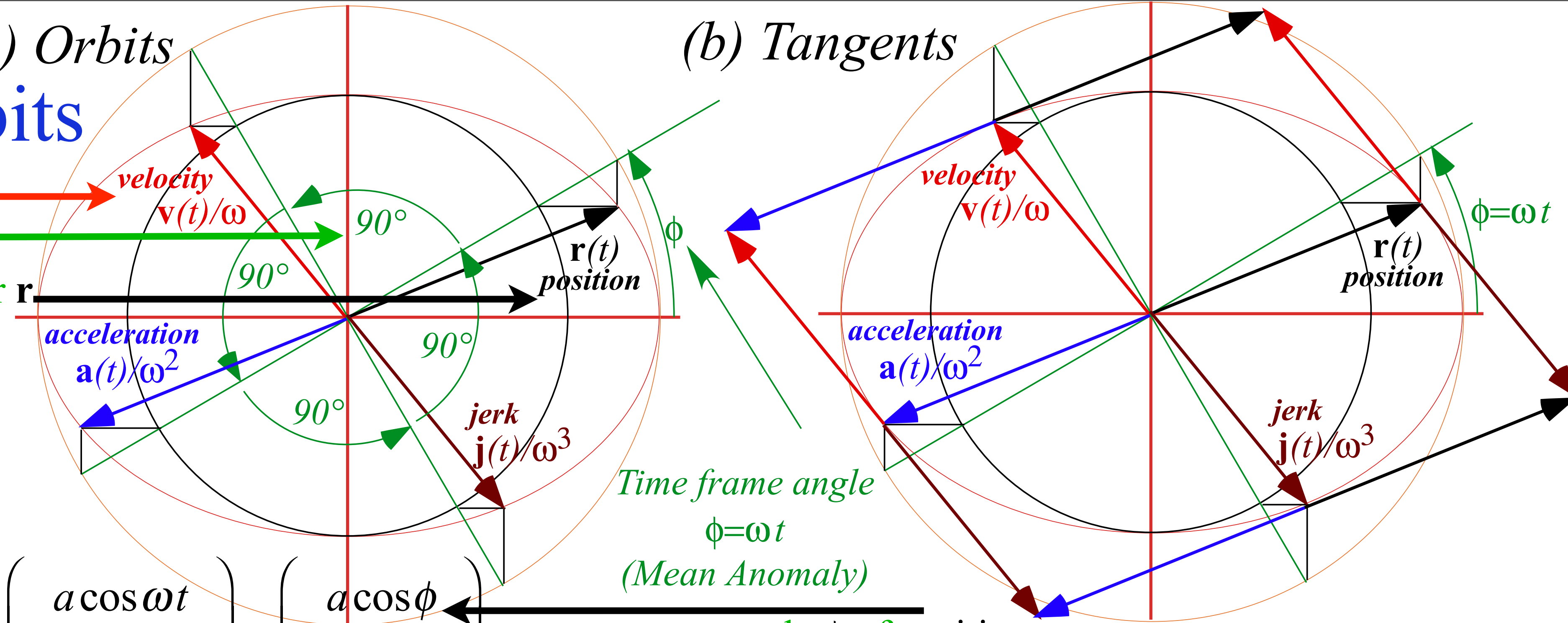
m.a. φ+π/2 of vector **v** rotated by another π/2 is *m.a.* of vector **a**

$$\text{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**



mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

m.a. φ+π/2 of vector **v** rotated by
another π/2 is *m.a.* of vector **a**

$$\textit{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

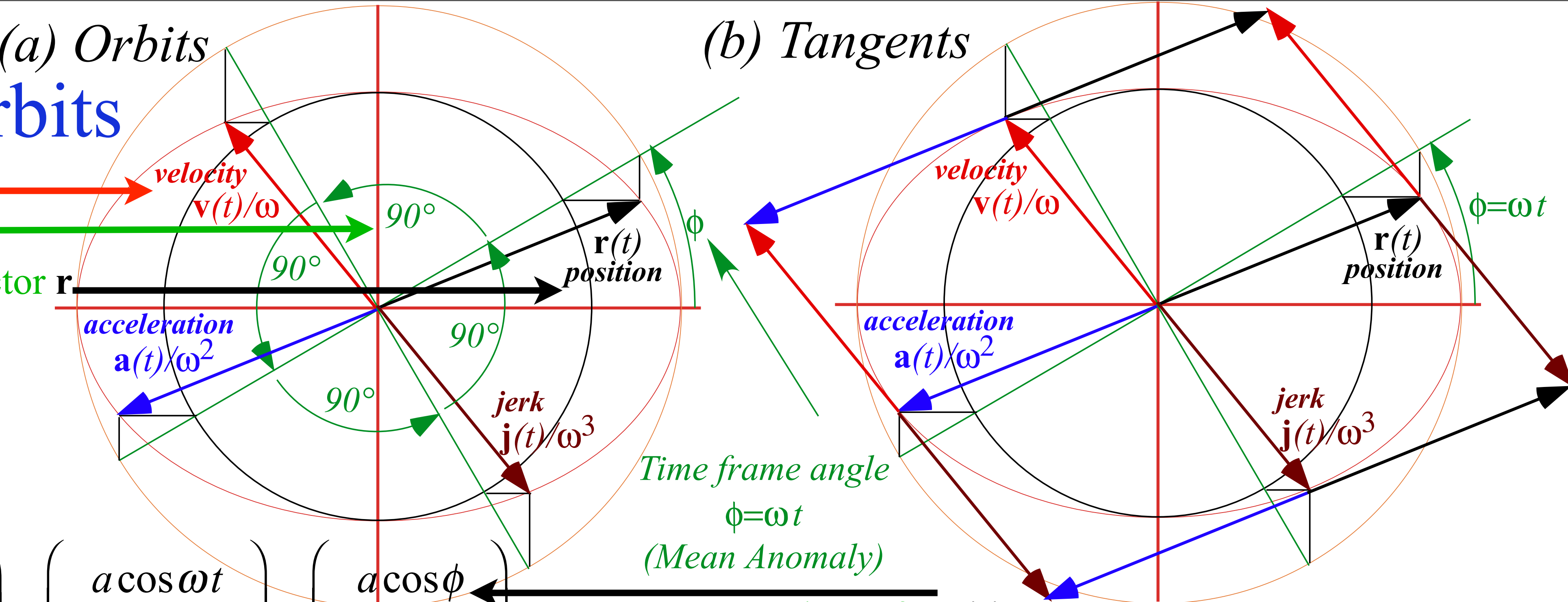
or change of velocity

$$\textit{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi + \frac{3\pi}{2}\right) \\ b\sin\left(\phi + \frac{3\pi}{2}\right) \end{pmatrix}$$

...and so forth...

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**



mean-anomaly φ of position vector **r** rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

m.a. φ+π/2 of vector **v** rotated by another π/2 is *m.a.* of vector **a**

$$\textit{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

...and so forth...

$$\textit{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi + \frac{3\pi}{2}\right) \\ b\sin\left(\phi + \frac{3\pi}{2}\right) \end{pmatrix}$$

...and so on...

$$\textit{inauguration or change of jerk}: \mathbf{i} = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix} = \frac{d\mathbf{j}}{dt} = \dot{\mathbf{j}} = \ddot{\mathbf{a}} = \dddot{\mathbf{v}} = \ddddot{\mathbf{r}} = \frac{d^4\mathbf{r}}{dt^4} = \begin{pmatrix} a\cos\left(\phi + \frac{4\pi}{2}\right) \\ b\sin\left(\phi + \frac{4\pi}{2}\right) \end{pmatrix}$$

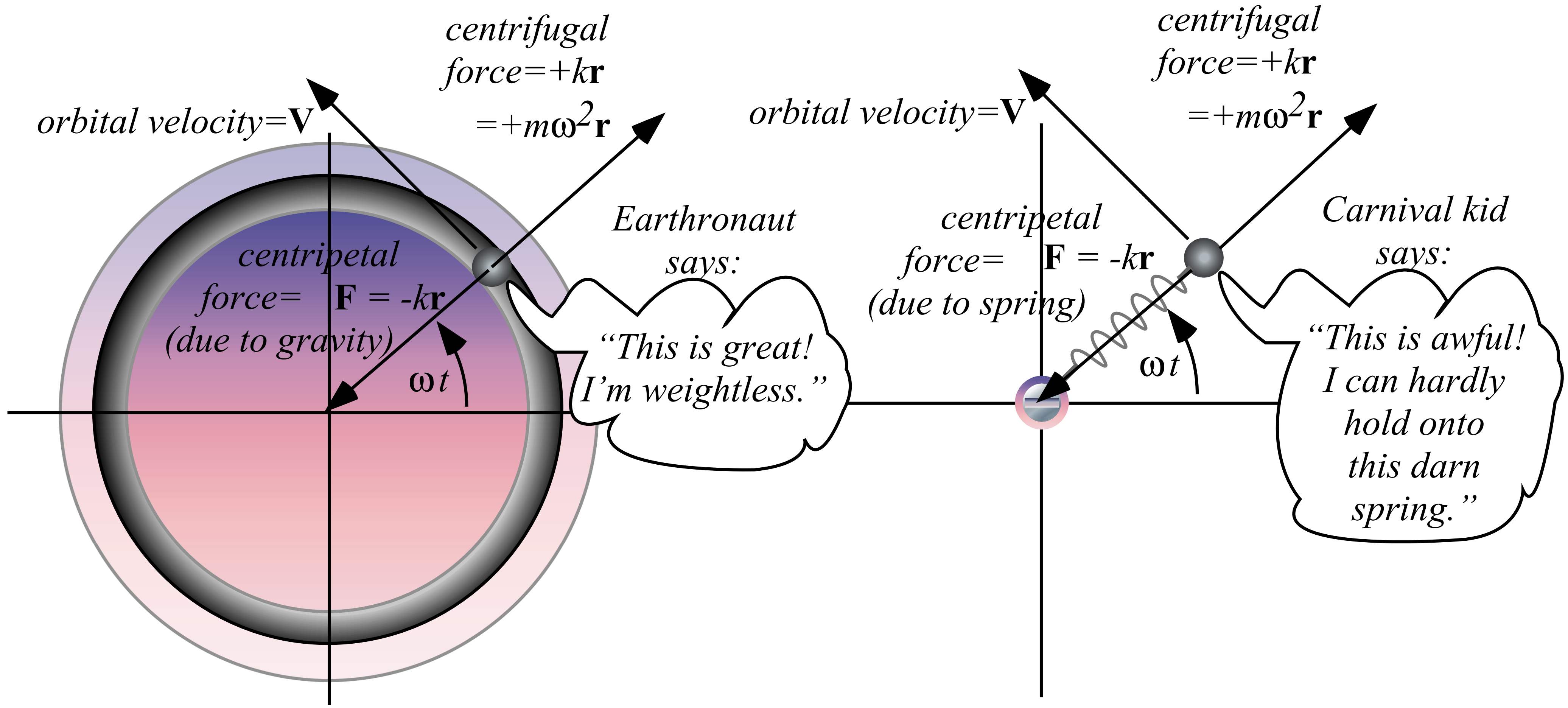
Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry

Calculus and vector geometry of IHO orbits

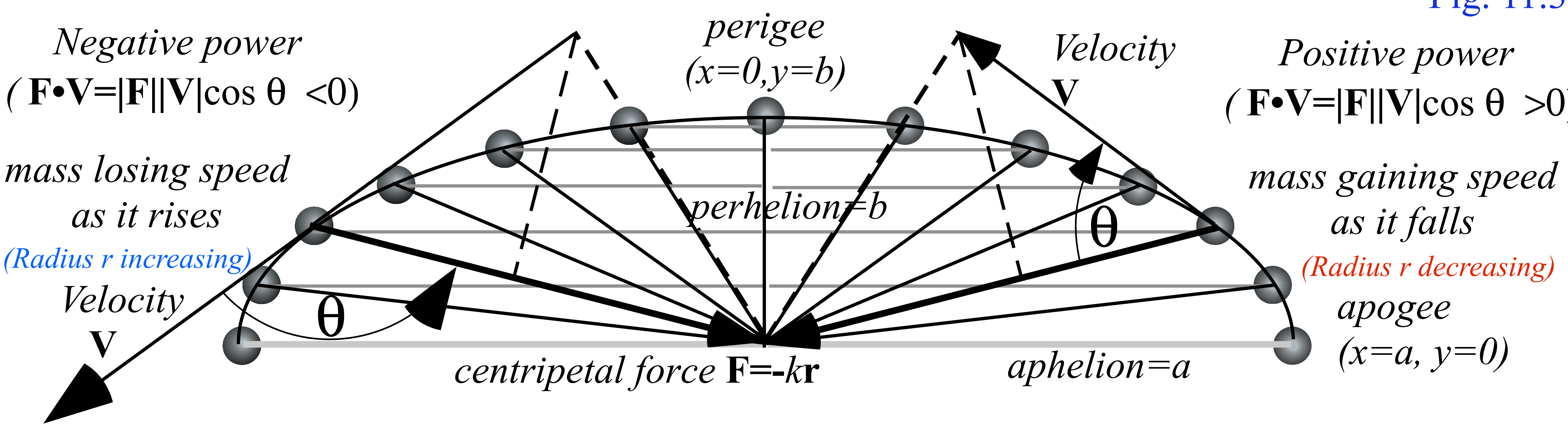
→ *A confusing introduction to Coriolis-centrifugal force geometry*

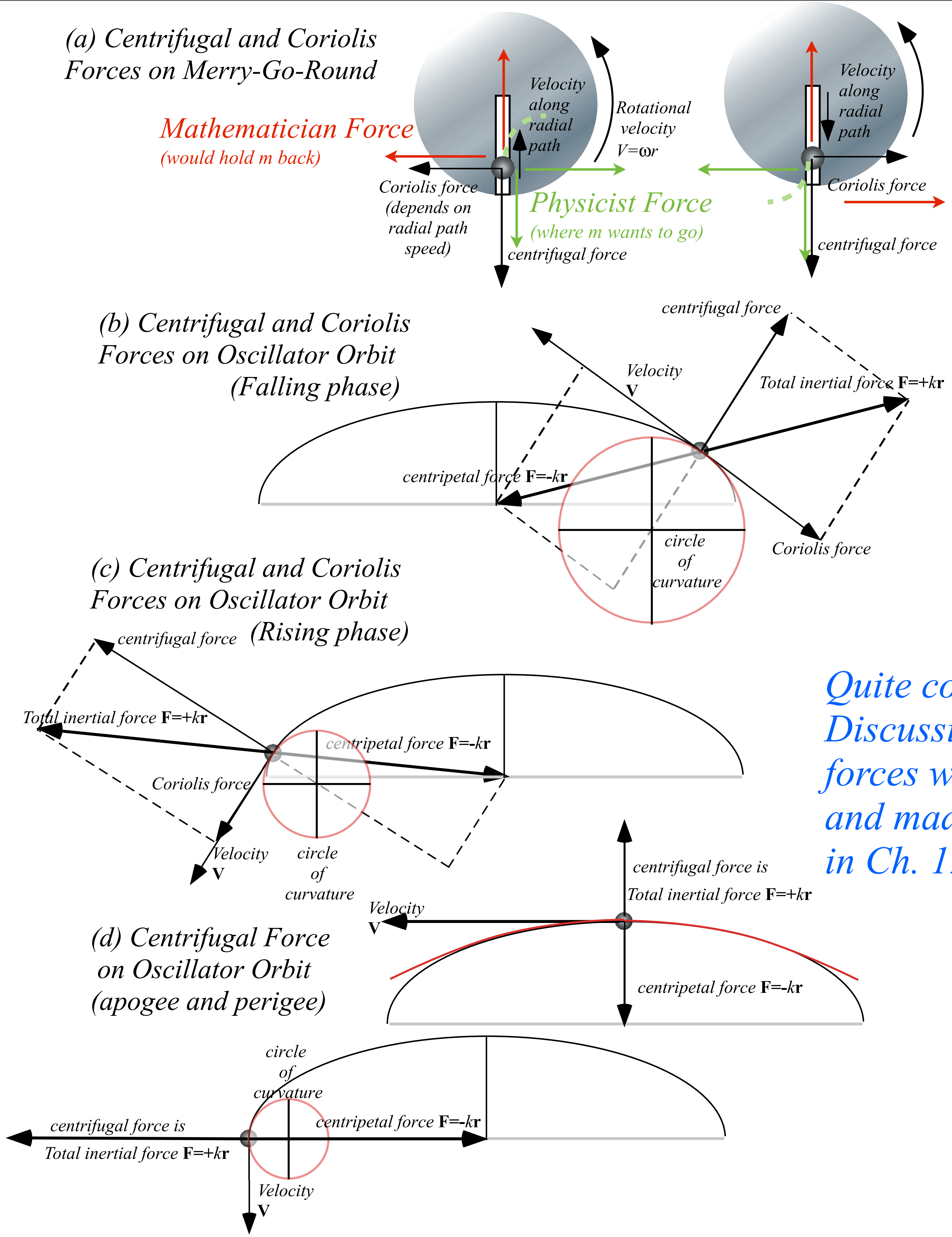
(a) "Earthronaut" orbiting tunnel inside Earth (b) "Carnival kid" orbiting in space attached to a spring



Unit 1 Fig. 11.2

Unit 1 Fig. 11.3





Unit 1
Fig. 11.4
a-d

Quite confusing?
Discussion of Coriolis forces will be done more elegantly and made more physically intuitive in Ch. 12 of Unit1 and in Unit 6.

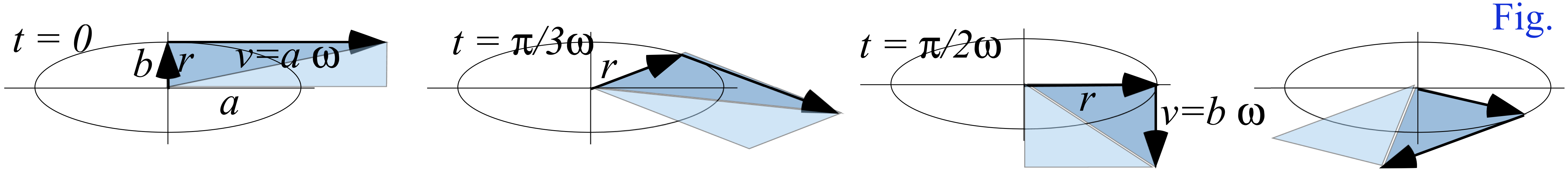
Some Kepler's "laws" for central (isotropic) force F(r)

→ *Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived rigorously)*
Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived later)
Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ (Derived rigorously)
Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)

Some Kepler's "laws" for central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - b\sin\omega t\cdot(-a\omega\sin\omega t) = ab\cdot\omega$$

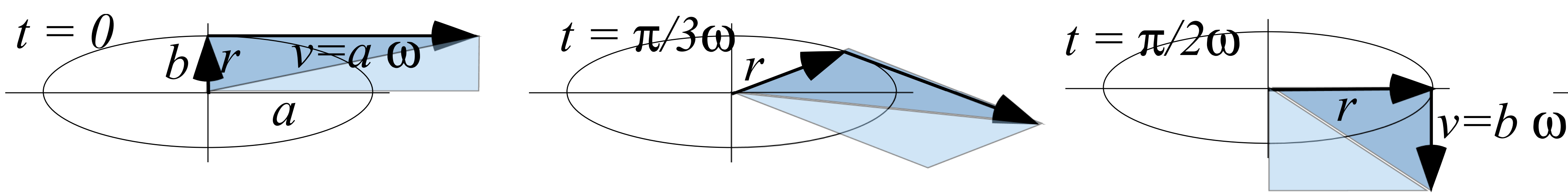
✓ *for IHO*

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} \qquad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - a\sin\omega t\cdot(-b\omega\sin\omega t) = ab\cdot\omega$$

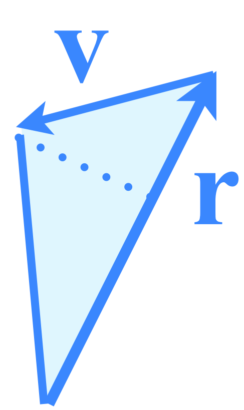
✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

✓ *for IHO*

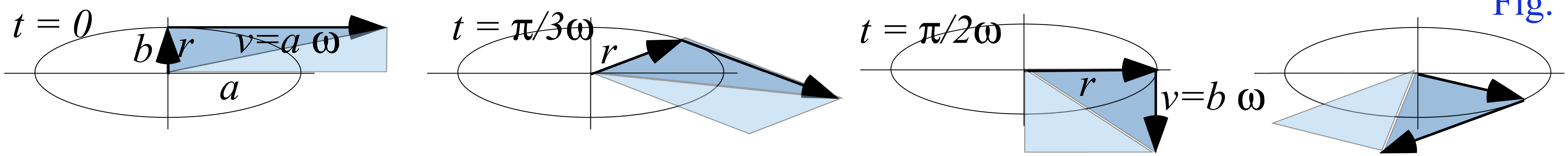
$$|\mathbf{r}\times\mathbf{v}| = r\cdot v\cdot \sin\measuredangle_r^v$$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - a\sin\omega t\cdot(-b\omega\sin\omega t) = ab\cdot\omega$$

✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

✓ *for IHO*

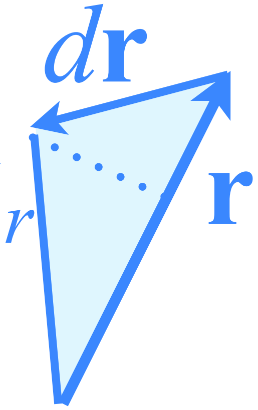
3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T$$

by 2.

✓ *for IHO*

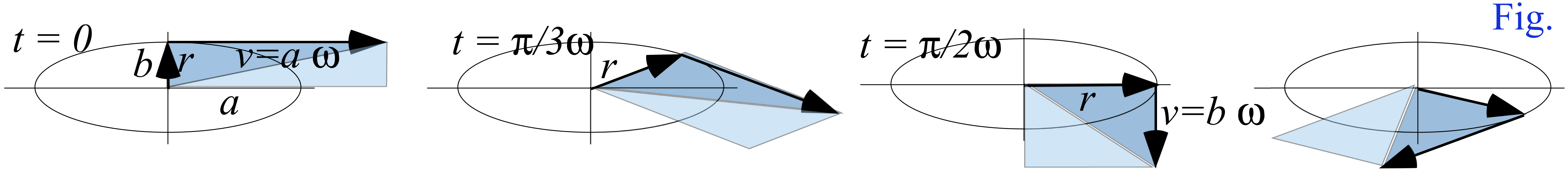
$$|\mathbf{r}\times d\mathbf{r}| = r\cdot dr\cdot \sin\measuredangle_r^{dr}$$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - a\sin\omega t\cdot(-b\omega\sin\omega t) = ab\cdot\omega$$

✓ *for IHO*

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega = m\cdot ab\cdot\frac{2\pi}{\tau}$$

✓ *for IHO*

3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T$$

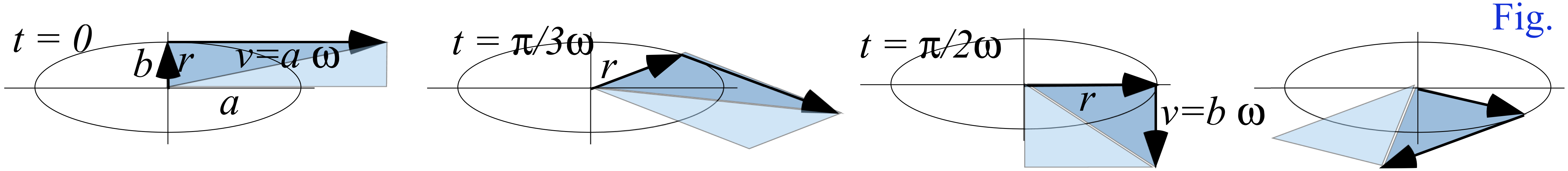
✓ *for IHO*

In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - a\sin\omega t\cdot(-b\omega\sin\omega t) = ab\cdot\omega$$

✓ *for IHO*

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,\mathbf{r}\times\mathbf{v} = m(r_x v_y - r_y v_x) = m\cdot ab\cdot\omega = m\cdot ab\cdot\frac{2\pi}{\tau}$$

✓ *for IHO*

3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T$$

✓ *for IHO*

In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit)

(Recall from Lecture 7: $\omega = \sqrt{k/m} = \sqrt{G\rho_\oplus 4\pi/3}$)

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived rigorously)

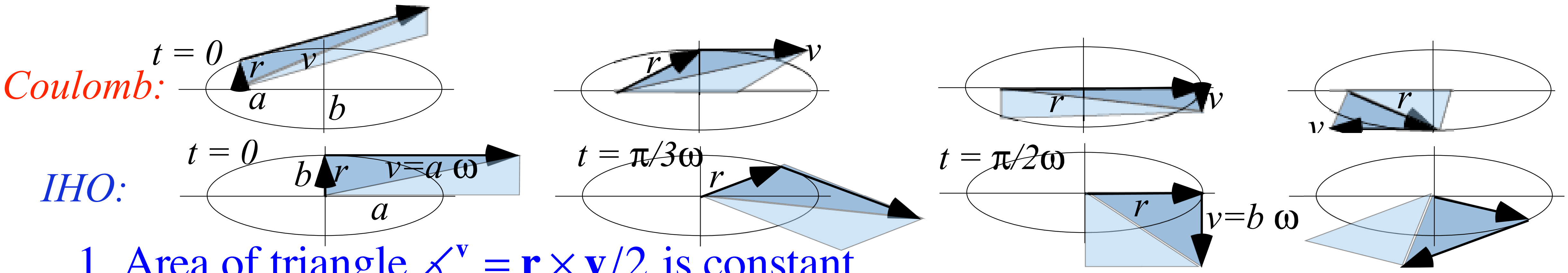
→ *Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived later)*

Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ (Derived rigorously)

Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

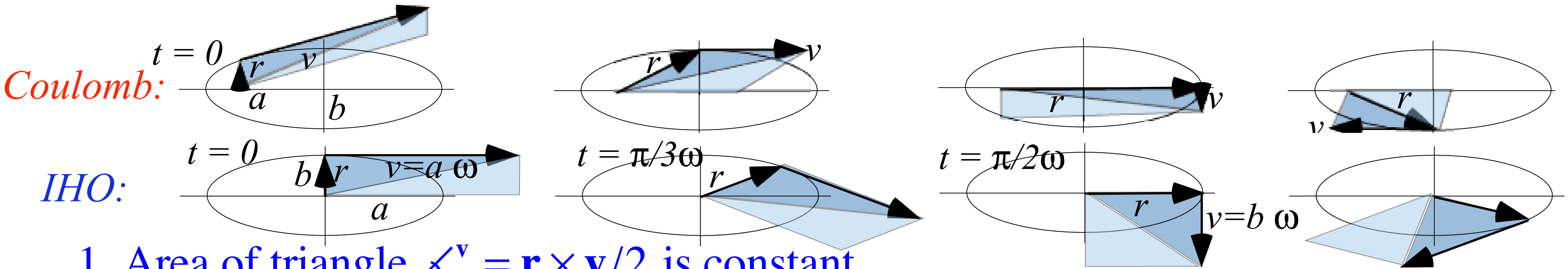
$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = \begin{cases} ab\cdot\sqrt{G\rho_\oplus 4\pi/3} & \text{for IHO} \\ a^{-1/2}b\sqrt{GM_\oplus} & \text{for Coul.} \end{cases}$$

✓ for IHO

✓ for Coul.

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = \begin{cases} ab\cdot\sqrt{G\rho_\oplus 4\pi/3} & \text{for IHO} \\ a^{-1/2}b\sqrt{GM_\oplus} & \text{for Coul.} \end{cases}$$

✓ for IHO

✓ for Coul.

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

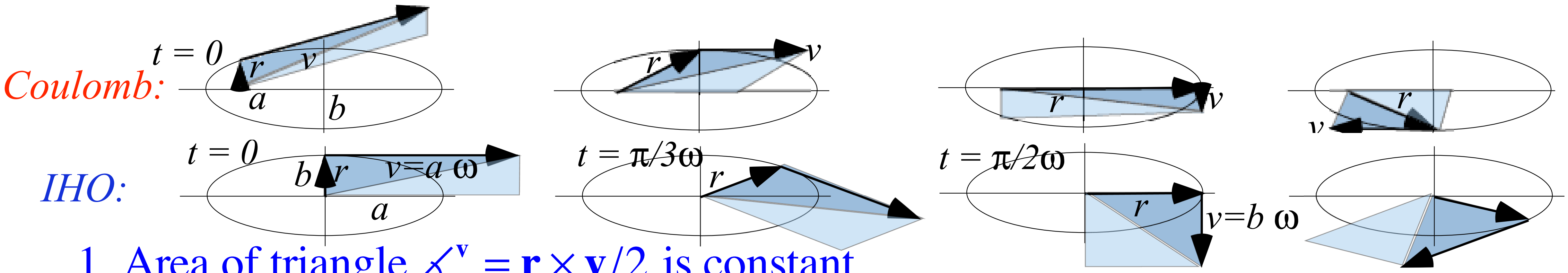
$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = \begin{cases} m\cdot ab\cdot\sqrt{G\rho_\oplus 4\pi/3} & \text{for IHO} \\ m\cdot a^{-1/2}b\sqrt{GM_\oplus} & \text{for Coul.} \end{cases}$$

✓ for IHO

✓ for Coul.

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = \begin{cases} ab\cdot\sqrt{G\rho_{\oplus}4\pi/3} & \text{for IHO} \\ a^{-1/2}b\sqrt{GM_{\oplus}} & \text{for Coul.} \end{cases}$$

✓ for IHO

✓ for Coul.

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = \begin{cases} m\cdot ab\cdot\sqrt{G\rho_{\oplus}4\pi/3} & \text{for IHO} \\ m\cdot a^{-1/2}b\sqrt{GM_{\oplus}} & \text{for Coul.} \end{cases}$$

✓ for IHO

✓ for Coul.

3. Equal area is swept by radius vector in each equal time interval T

In one period:

$$\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_{\tau}}{L} = \frac{2m\cdot ab\cdot\pi}{L} = \begin{cases} \dfrac{2m\cdot ab\cdot\pi}{m\cdot ab\cdot\sqrt{G\rho_{\oplus}4\pi/3}} = \dfrac{2\pi}{\sqrt{G\rho_{\oplus}4\pi/3}} & \text{for IHO} \\ \dfrac{2m\cdot ab\cdot\pi}{m\cdot a^{-1/2}b\sqrt{GM_{\oplus}}} = \dfrac{2\pi}{a^{-3/2}\sqrt{GM_{\oplus}}} & \text{for Coul.} \end{cases}$$

that is ω_{IHO}

that is ω_{Coul}

Applies to any central F(r)

Applies to IHO and Coulomb

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived rigorously)

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived later)

→ *Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ (Derived rigorously)*

Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}=\begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2=k$

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived rigorously)

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived later)

Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ (Derived rigorously)

→ *Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)*

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2=k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

$$E=KE+PE=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2-\frac{k}{r}=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2-\frac{GM_\oplus m}{r}=-\frac{GM_\oplus m}{a}$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite

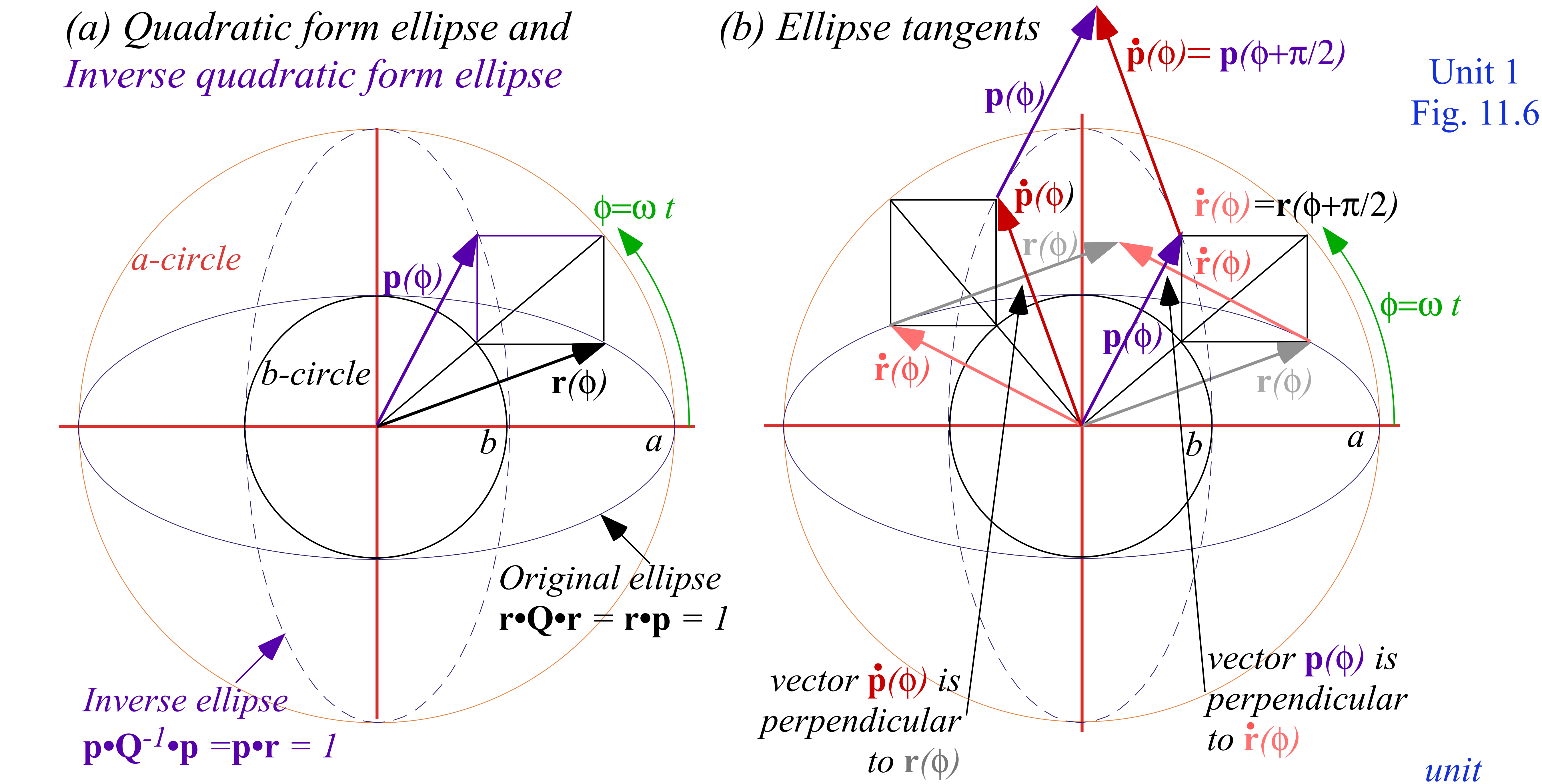
$$\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$$

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = 1 = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

A inverse matrix Q^{-1} *generates an ellipse by* **p**•*Q*$^{-1}$•**p**=*1 called* inverse *or* dual *ellipse:*

$$\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1$$

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix} = 1 = \begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix} = a^2 p_x^2 + b^2 y^2$$



Note some quadratic form mutual duality relations:

unit mutual projection

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $x = r_x = a\cos\phi = a\cos\omega t$, $y = r_y = b\sin\phi = b\sin\omega t$ so: $\boxed{\mathbf{p}\bullet\mathbf{r}=1}$

***p** is perpendicular to velocity **v**=**ṙ** , a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $\dot{r}_x = -a\sin\phi$, $\dot{r}_y = b\cos\phi$ and: $p_x = (1/a)\cos\phi$, $p_y = (1/b)\sin\phi$