Lecture 21
Tue. 11.06.2012

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

Some Ways to do constraint analysis

- *Way 1. Simple constraint insertion*
- *Way 2. GCC constraint webs*
    - *Find covariant force equations*
    - *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

- *Way 3. OCC constraint webs*
    - *Preview of atomic-Stark orbits*
        - *Classical Hamiltonian separability*
- *Way 4. Lagrange multipliers*
    - *Lagrange multiplier as eigenvalues*
    - *Multiple multipliers*
    - *"Non-Holonomic" multipliers*

Some Ways to do constraint analysis

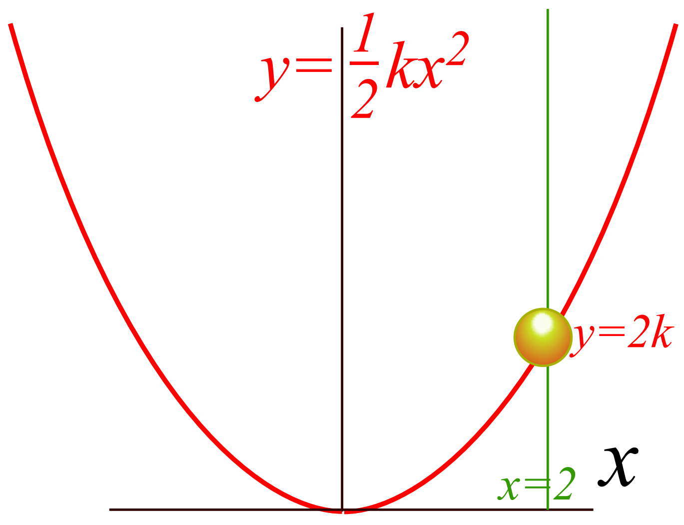
→ *Way 1. Simple constraint insertion*

Way 2. GCC constraint webs

  - *Find covariant force equations*
  - *Compare covariant vs. contravariant forces*

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}\overbrace{(m\dot{x}^2+m\dot{y}^2)}-mgy$$

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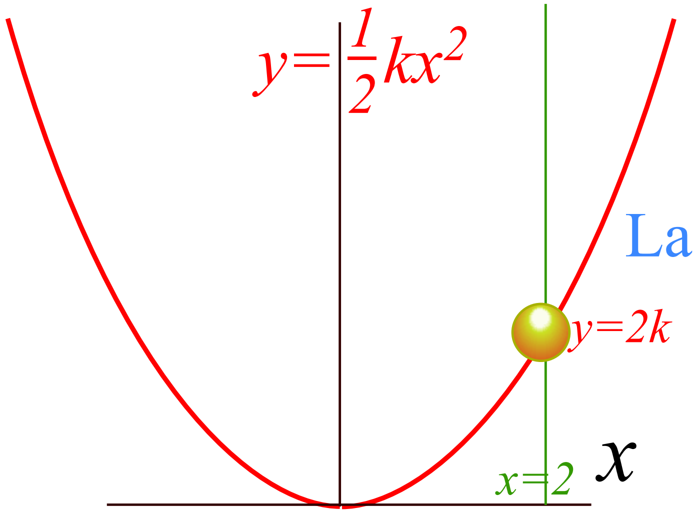

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Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial \dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

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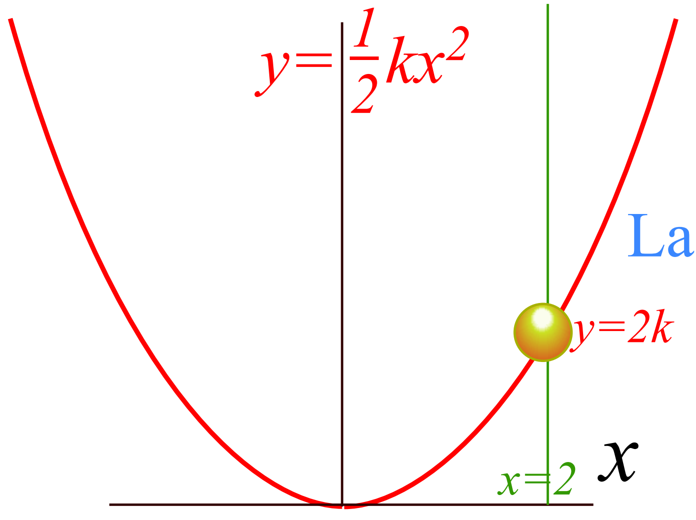

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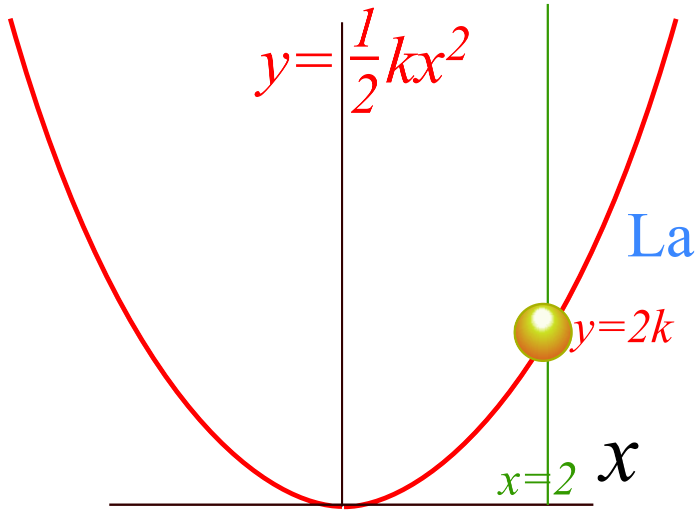

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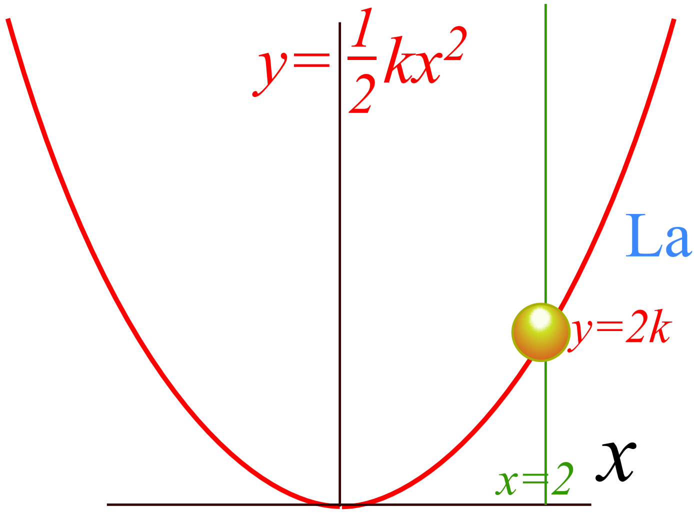

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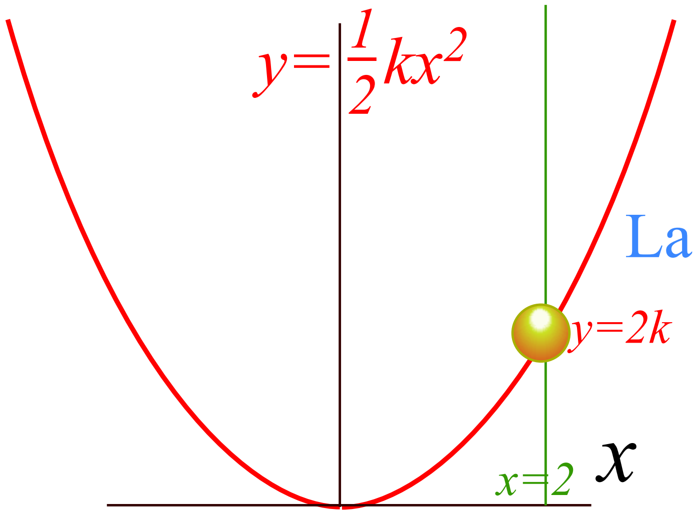

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}$$

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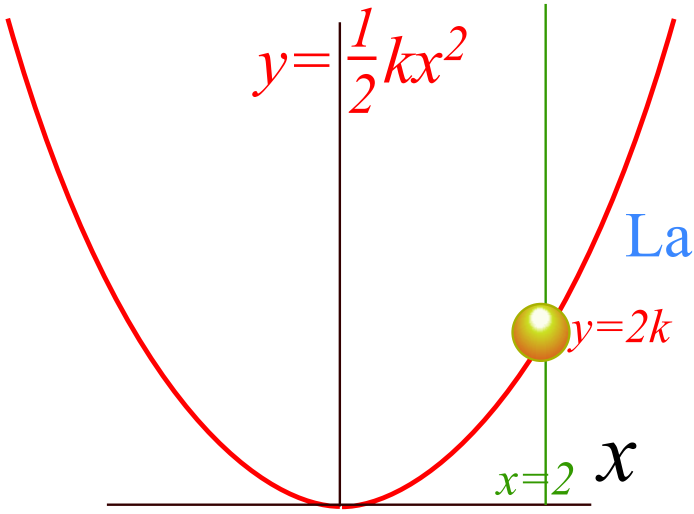

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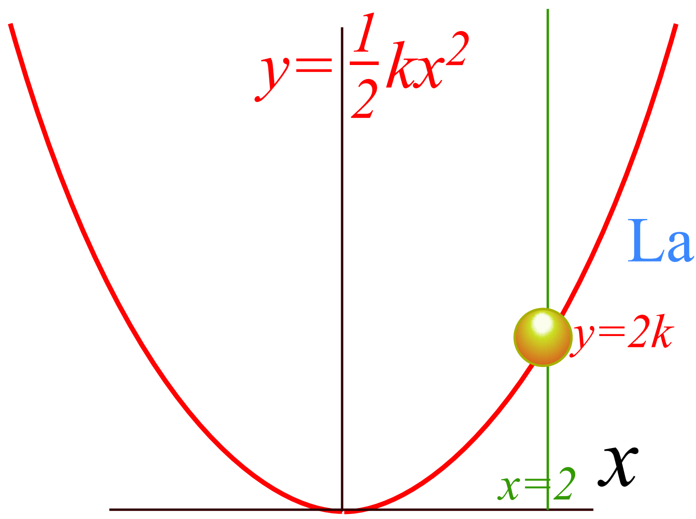

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$$L = \frac{1}{2}\left(m\dot{x}^2 + m\dot{y}^2\right) - mgy$$

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$$L = \frac{1}{2}\left(m\dot{x}^2 + m(kx\dot{x})^2\right) - m\frac{1}{2}gkx^2$$

$$= \frac{m}{2}\left(\dot{x}^2 + k^2x^2\dot{x}^2 - gkx^2\right)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + k^2x^2\dot{x})$$

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Lagrange equation $\dot{p}_x = f_x = \frac{\partial L}{\partial x}$

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$$\dot{p}_x = m(1 + k^2x^2)\ddot{x} \qquad = -mk^2x\dot{x}^2 - mgkx$$

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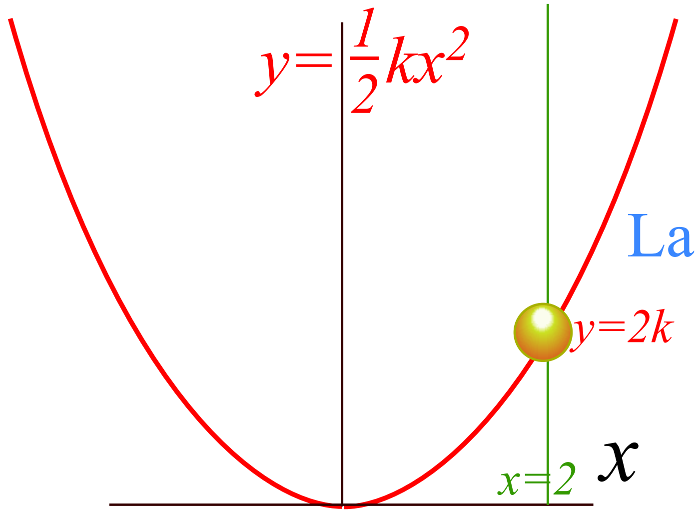

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$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$\dot{p}_x=m(1+k^2x^2)\ddot{x}\qquad=-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

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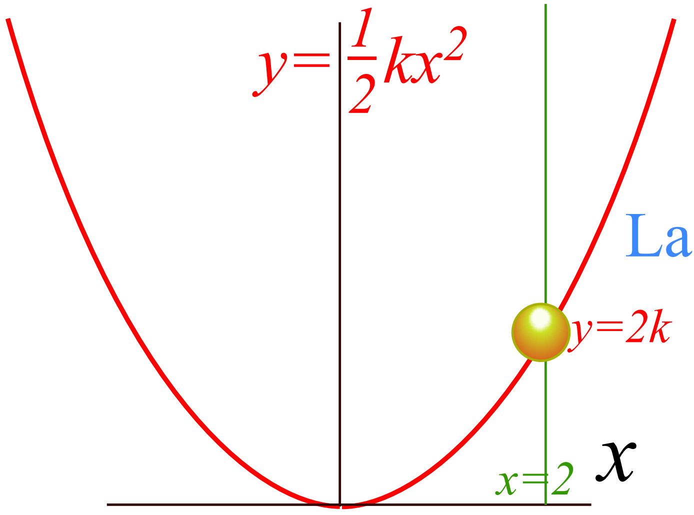

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$\dot{p}_x=m(1+k^2x^2)\ddot{x} \qquad =-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

$$\boxed{\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx}$$

Some Ways to do constraint analysis

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- → Way 2. GCC constraint webs
    - Find covariant force equations
    - Compare covariant vs. contravariant forces

Way 2. *GCC constraint webs.*

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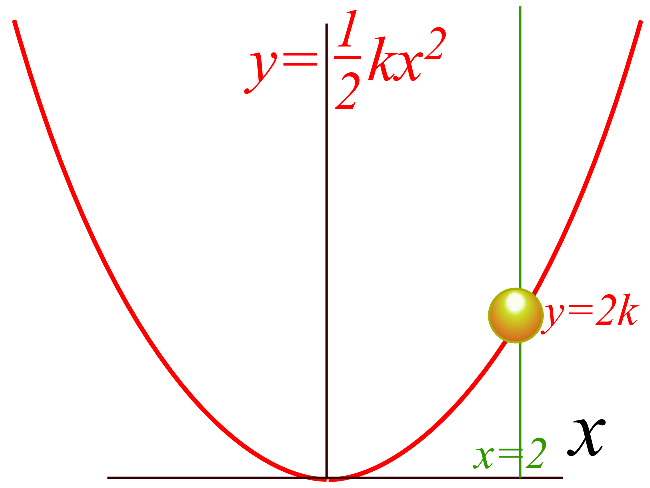
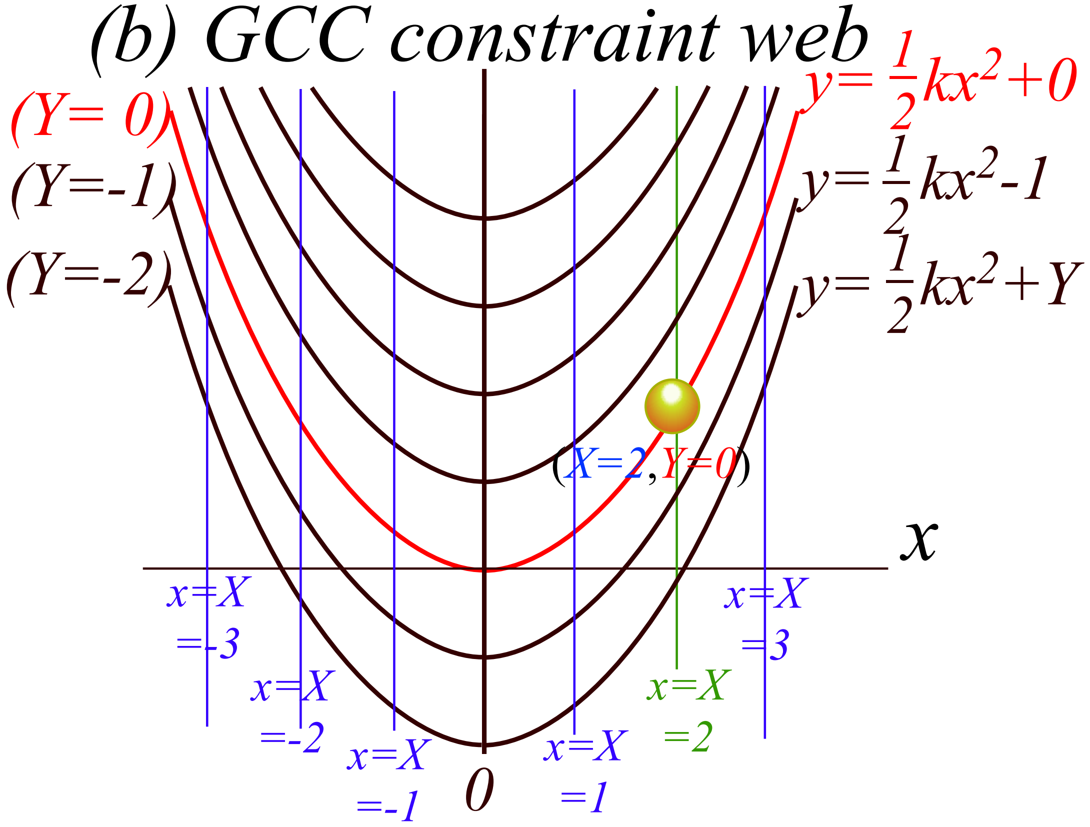


$$x = X$$
$$y=\frac{1}{2}kx^2+Y$$

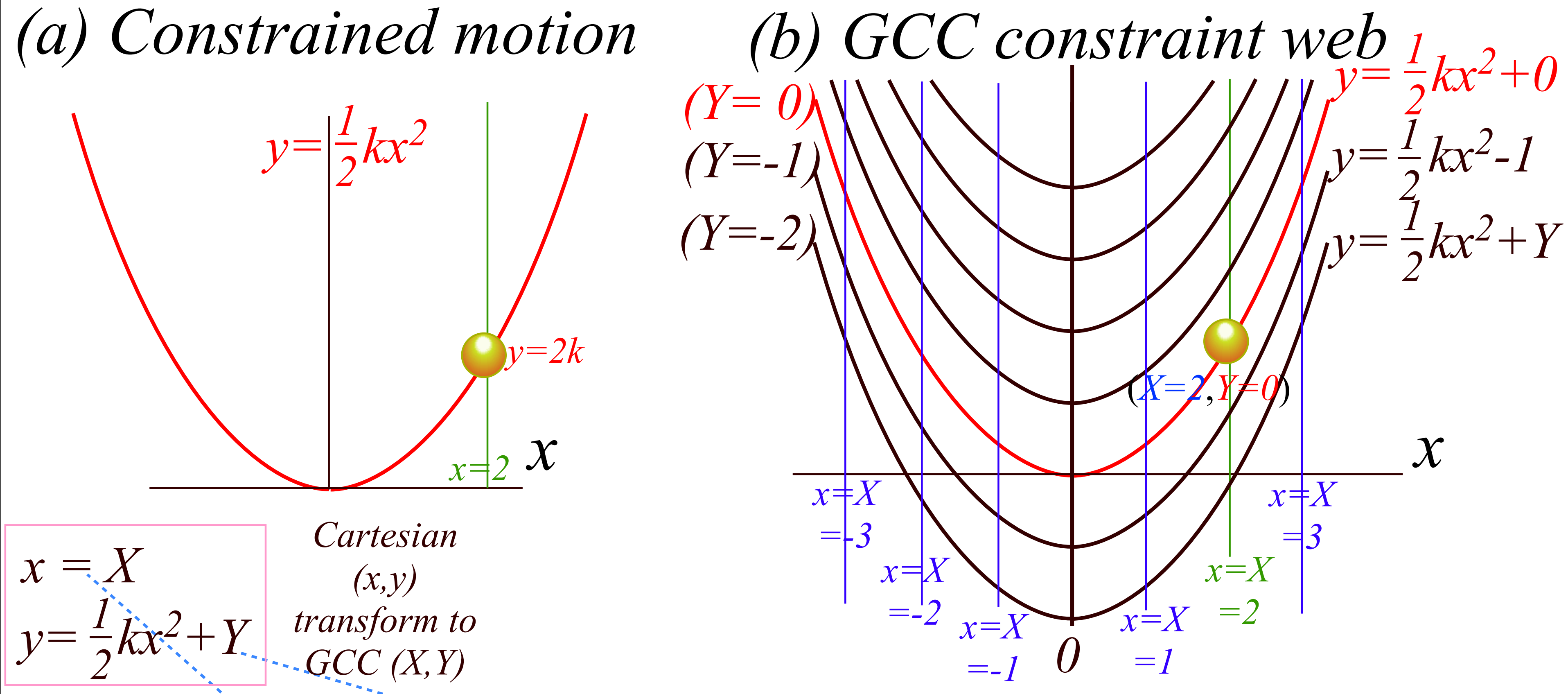
Cartesian (x,y) transform to GCC (X,Y)

(b) GCC constraint web



Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

Way 2. GCC constraint webs.



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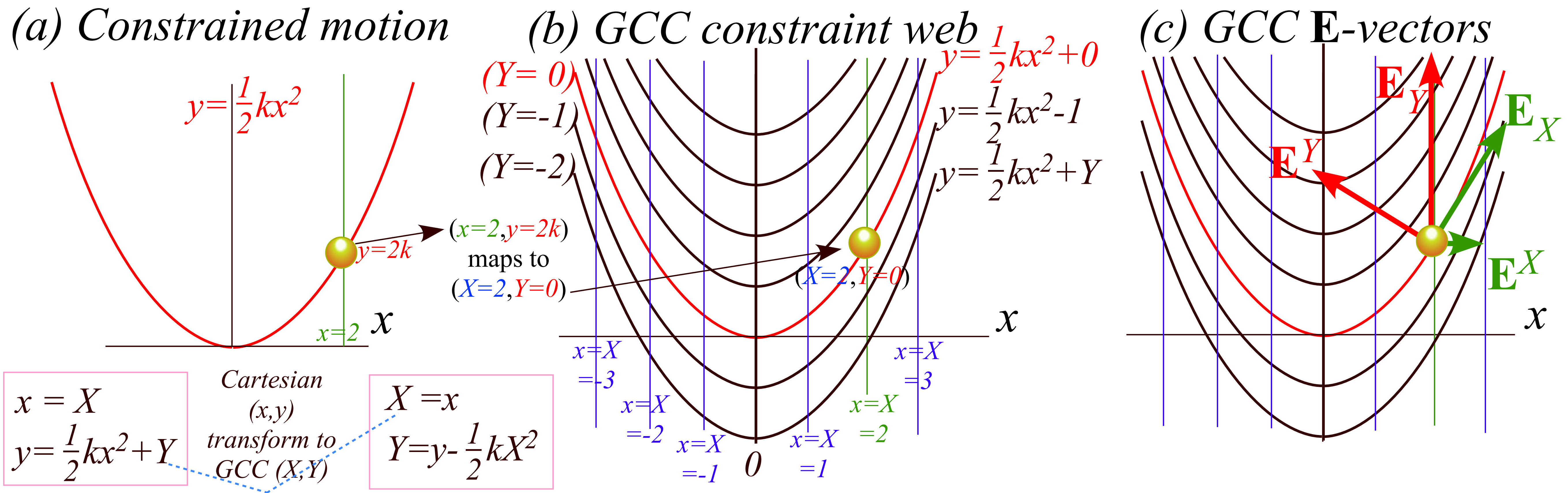
(a) Constrained motion

(b) GCC constraint web

(c) GCC **E**-vectors

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Cartesian (x,y) transform to GCC (X,Y)

$X=x$

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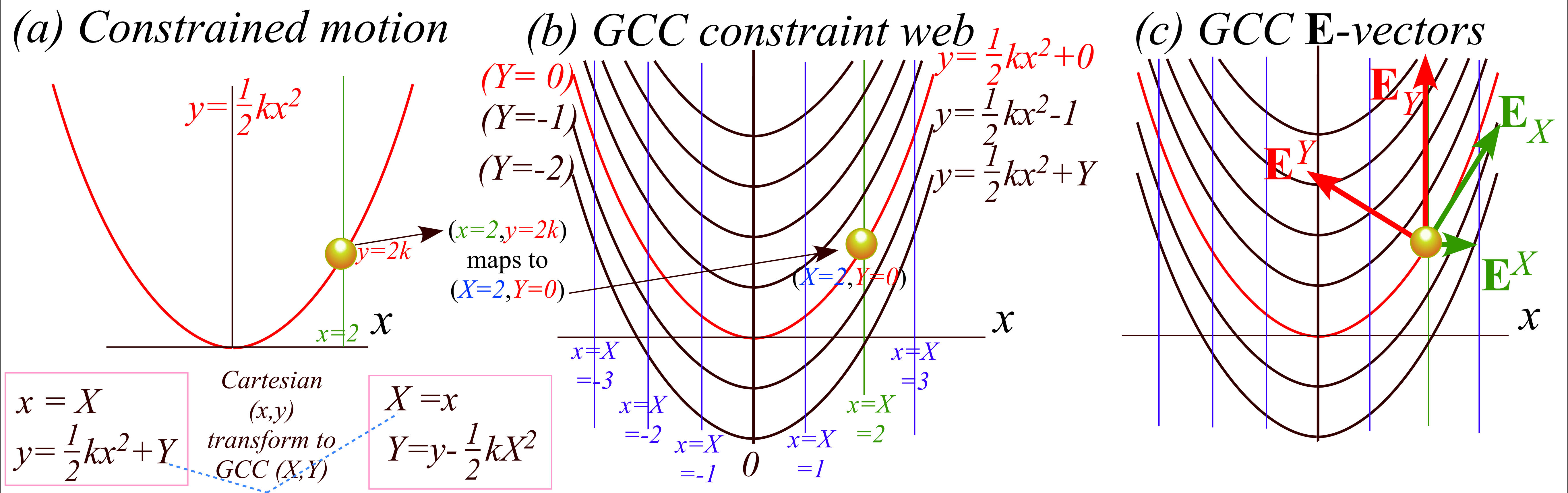
Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$x=q^1=X$ $\quad y=1/2kx^2+q^2=kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix}1\\kx\end{pmatrix},\ \mathbf{E}_Y=\begin{pmatrix}0\\1\end{pmatrix}$$

Way 2. GCC constraint webs.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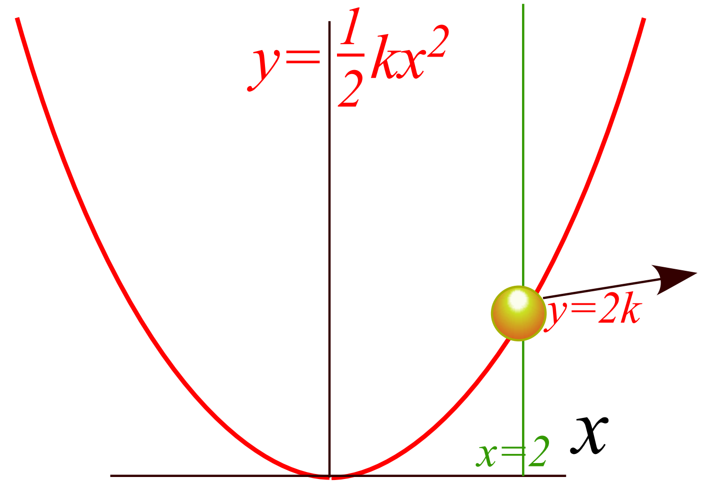
Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \begin{array}{l} \mathbf{E}^X=\begin{pmatrix} 1 & 0 \end{pmatrix} \\ \mathbf{E}^Y=\begin{pmatrix} -kx & 1 \end{pmatrix} \end{array}$$

Way 2. GCC constraint webs.



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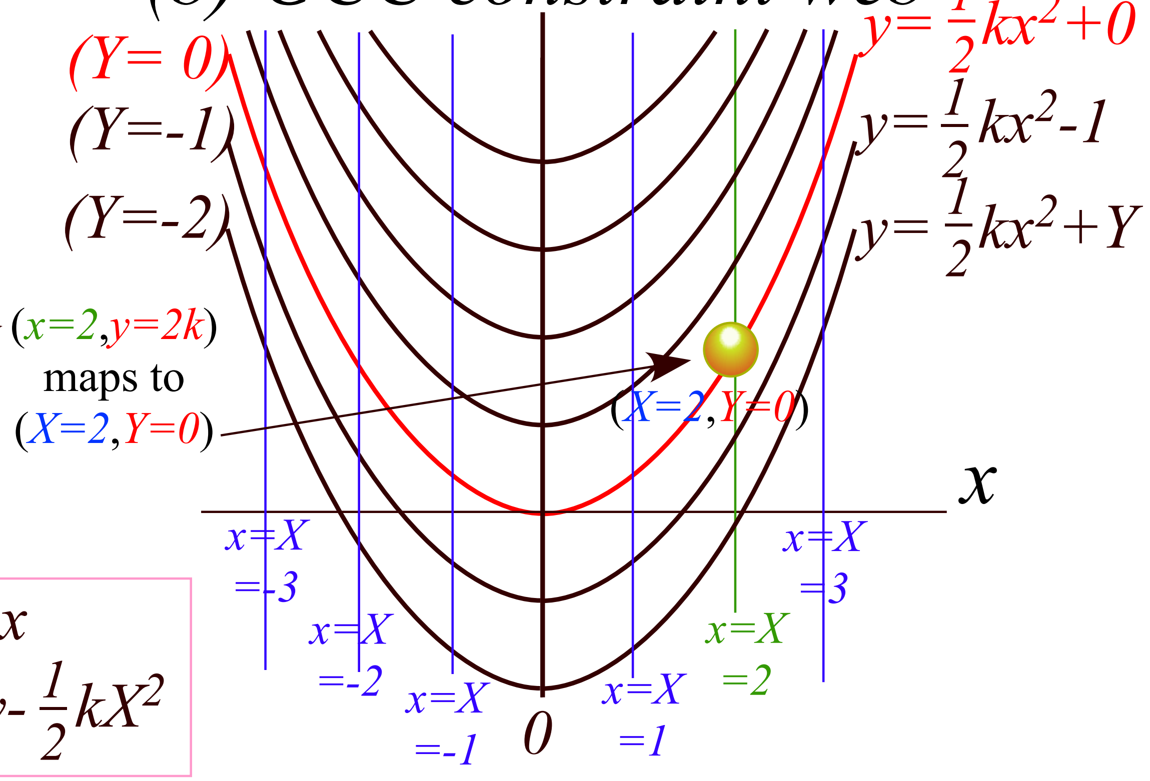

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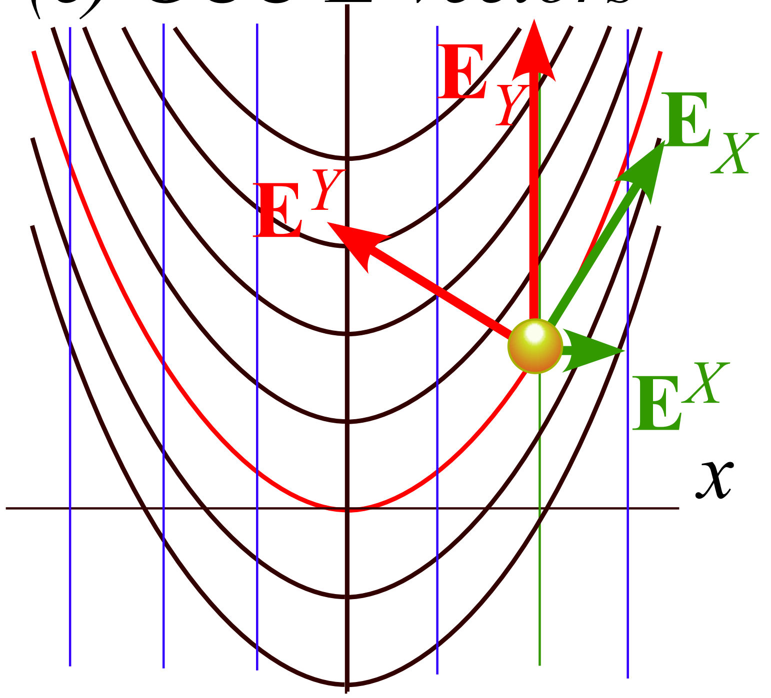

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Cartesian (x,y) transform to GCC (X,Y)

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Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

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Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

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$$\mathbf{E}^X=\begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\mathbf{E}^Y=\begin{pmatrix} -kx & 1 \end{pmatrix}$$

Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}$$

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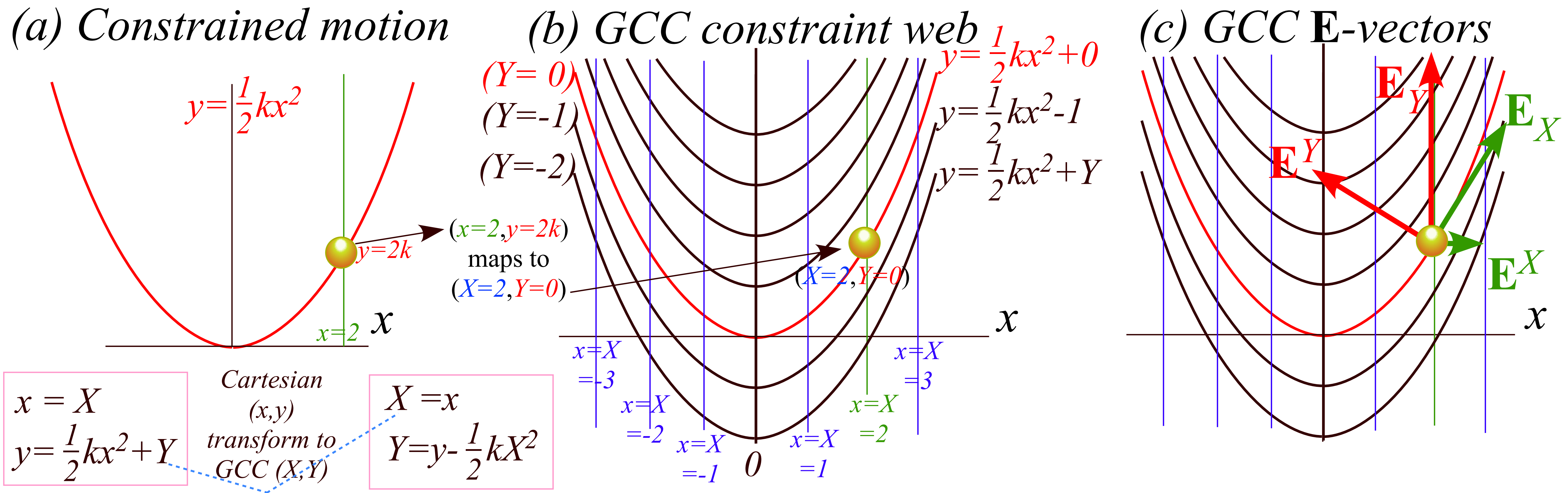
Way 2. GCC constraint webs.



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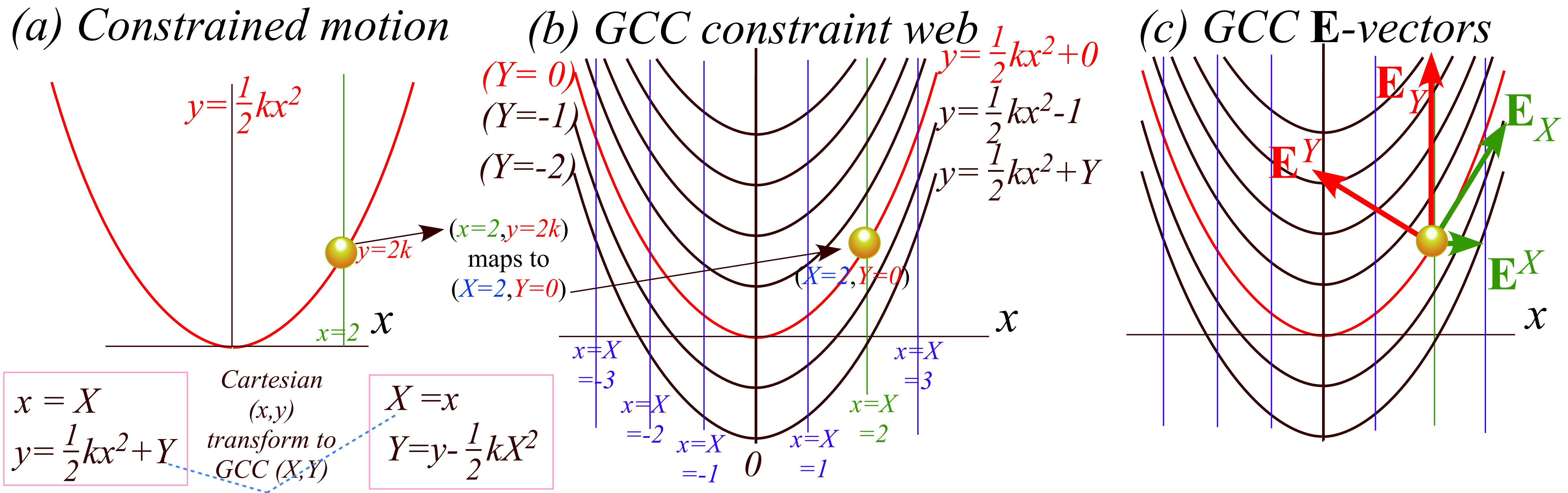
Find: 1st coordinate differentials and velocity relations:
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} \qquad \begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB}=mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB}=J_{AC}J_{BC}=(JJ^\dagger)_{AB}$

$$m\begin{pmatrix}\mathbf{E}_X\bullet\mathbf{E}_X & \mathbf{E}_X\bullet\mathbf{E}_Y\\ \mathbf{E}_Y\bullet\mathbf{E}_X & \mathbf{E}_Y\bullet\mathbf{E}_Y\end{pmatrix}=\begin{pmatrix}\gamma_{XX} & \gamma_{XY}\\ \gamma_{YX} & \gamma_{YY}\end{pmatrix}=m\begin{pmatrix}1+k^2x^2 & kx\\ kx & 1\end{pmatrix}$$

Way 2. GCC constraint webs.



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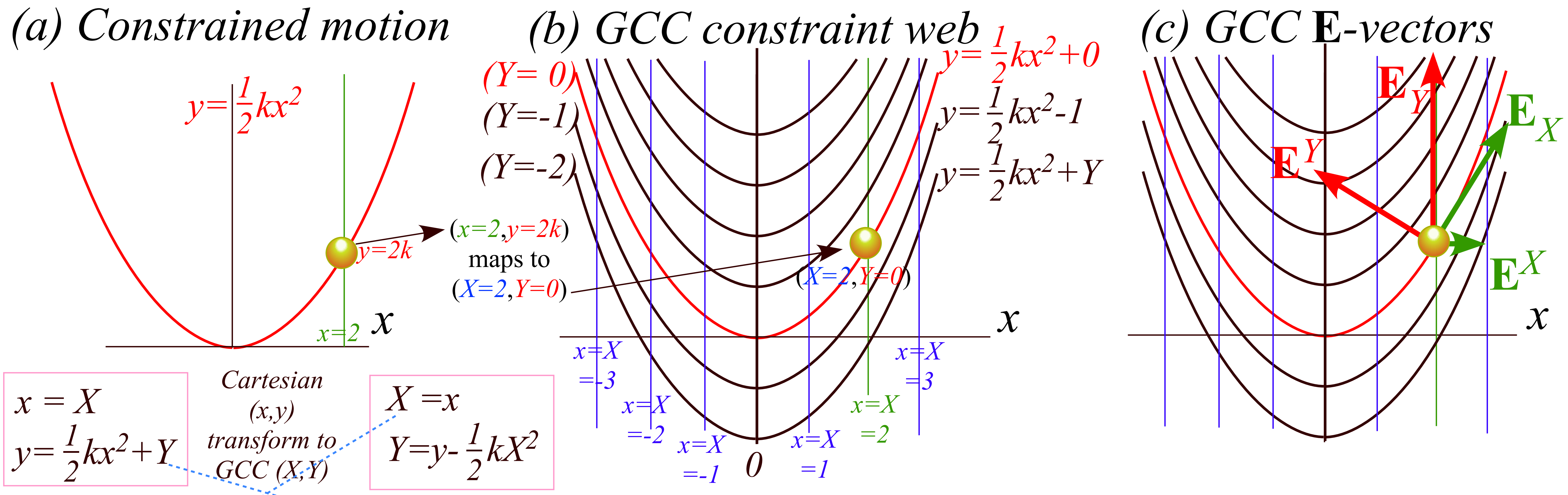

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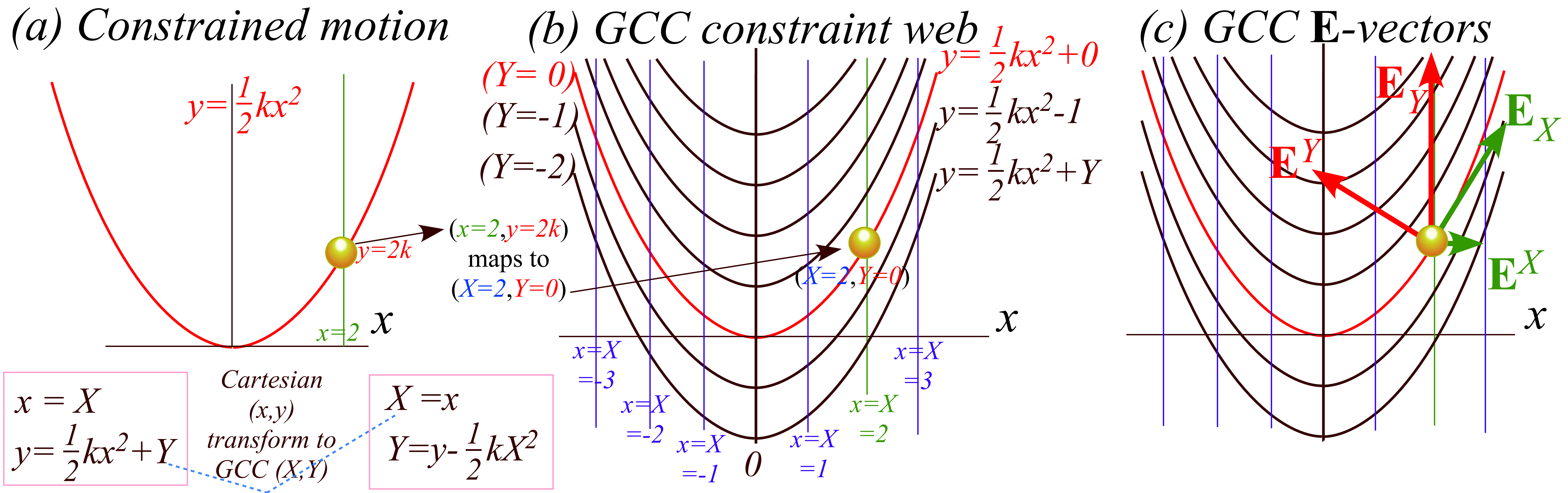

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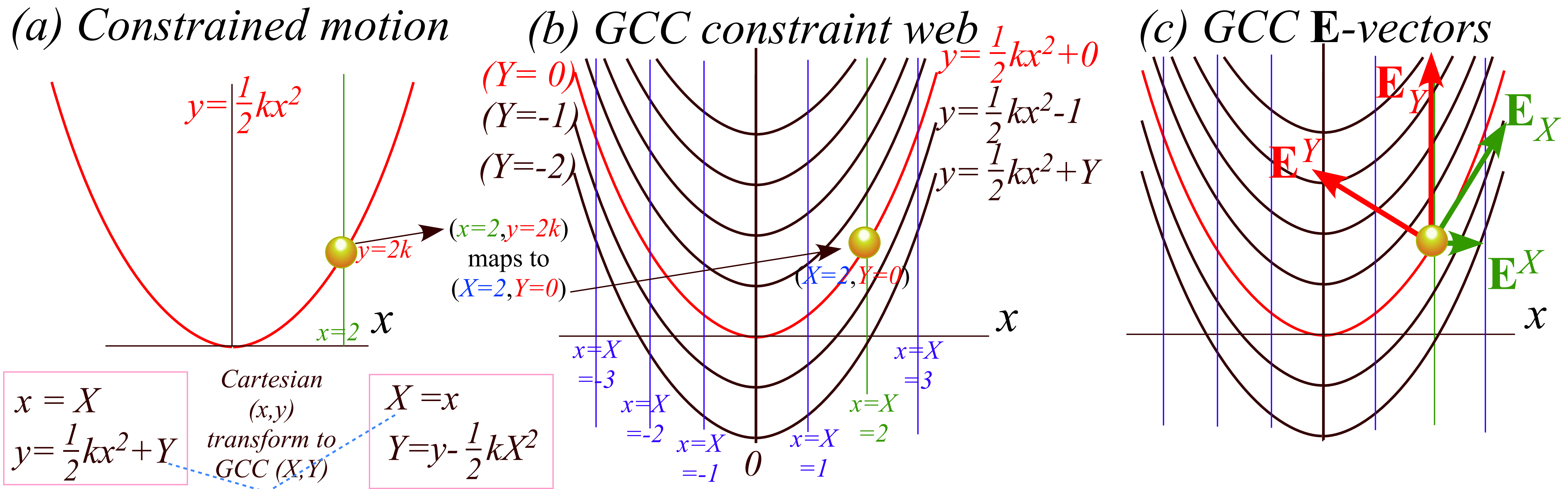

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Some Ways to do constraint analysis

- *Way 1. Simple constraint insertion*
- *Way 2. GCC constraint webs*
  - → *Find covariant force equations*
  - *Compare covariant vs. contravariant forces*

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero.($F_X^{cov}=0=F_Y^{cov}$)

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$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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$\ddot{x} = 0 = \ddot{X}$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

- Find covariant force equations
- → Compare covariant vs. contravariant forces

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

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Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

$$= 0 \cdot \nabla X + F_Y^{cov}\nabla Y$$

$$= 0 \cdot \mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

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$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$

$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

So constraint requirements in covariant equations are $F_X^{cov}=0$ and $F_Y^{cov}\neq 0$. (with: $\dot{Y}=0=\ddot{Y}$).

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Frictionless constraint forces have *covariant* components F_B^{cov}

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Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\begin{aligned}\mathbf{F}(Y=const.) &= F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y \\ &= 0\cdot\nabla X + F_Y^{cov}\nabla Y \\ &= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y\end{aligned}$$

So constraint requirements in covariant equations are $F_X^{cov}=0$ and $F_Y^{cov}\neq 0$. (with: $\dot Y = 0 = \ddot Y$).

$$\dot Y = 0 = \ddot Y$$

$$m\begin{pmatrix}(1+k^2X^2)\ddot X + 0 + k^2X\dot X^2 + gkX \\ kX\ddot X + 0 + k\dot X^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix} \qquad \ddot X = -\frac{k^2X\dot X^2 + gkX}{1+k^2X^2} = -\frac{k\dot X^2 + g}{1+k^2X^2}kX$$

FINALLY ! We get the Way 1. solution

$$\boxed{\ddot x = \frac{-k\dot x^2 - g}{1+k^2x^2}kx}$$

Constraint force components are covariant

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$$\begin{aligned}\mathbf{F}(Y=const.) &= F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y\\ &= 0\cdot\nabla X + F_Y^{cov}\nabla Y\\ &= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y\end{aligned}$$

So constraint requirements in covariant equations are $F_X^{cov}=0$ and $F_Y^{cov}\neq 0$. (with: $\dot Y = 0 = \ddot Y$).

$$\dot Y = 0 = \ddot Y$$

$$m\begin{pmatrix}(1+k^2X^2)\ddot X + 0 + k^2X\dot X^2 + gkX\\ kX\ddot X + 0 + k\dot X^2 + g\end{pmatrix} = \begin{pmatrix}0\\ F_Y^{cov}\end{pmatrix} \quad \ddot X = -\frac{k^2X\dot X^2 + gkX}{1+k^2X^2} = -\frac{k\dot X^2+g}{1+k^2X^2}kX$$

$$\mathbf{F} = \quad F_Y^{cov} \quad \mathbf{E}^Y$$

$$= m(kX\ddot X + 0 + k\dot X^2 + g)\begin{pmatrix}-kX\\ 1\end{pmatrix}$$

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Frictionless constraint forces have *covariant* components F_B^{cov}

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Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

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Centripetal
force mkv^2+mg
(what the roller-coaster rider feels)

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Frictional or driving forces have *contravariant* components F^A_{con}

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$$\begin{aligned}\mathbf{F} = \quad & F_Y^{cov} \quad \mathbf{E}^Y\\ = & \; m(kX\ddot{X}+0+k\dot{X}^2+g)\begin{pmatrix}-kX\\1\end{pmatrix}\\ = & \; m\left(\frac{\cancel{k^2X^2(k\dot{X}^2+g)}}{1+k^2X^2}+\frac{(k\dot{X}^2+g)(1+\cancel{k^2X^2})}{1+k^2X^2}\right)\begin{pmatrix}-kX\\1\end{pmatrix}\end{aligned}$$

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Constraint requirements in contravariant equations (with: $\dot{Y} = 0 = \ddot{Y}$).

$$\begin{pmatrix} \ddot{X} + 0 \\ \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

$$\begin{aligned}-g = \ddot{y} &= \frac{d^2}{dt^2}\left(\frac{1}{2}kX^2 + Y\right) \\ &= k\dot{X}^2 + kX\ddot{X} + \ddot{Y}(= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0)\end{aligned}$$

Other Ways to do constraint analysis

→ *Way 3. OCC constraint webs*
    - *Preview of atomic-Stark orbits*
        - *Classical Hamiltonian separability*

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Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

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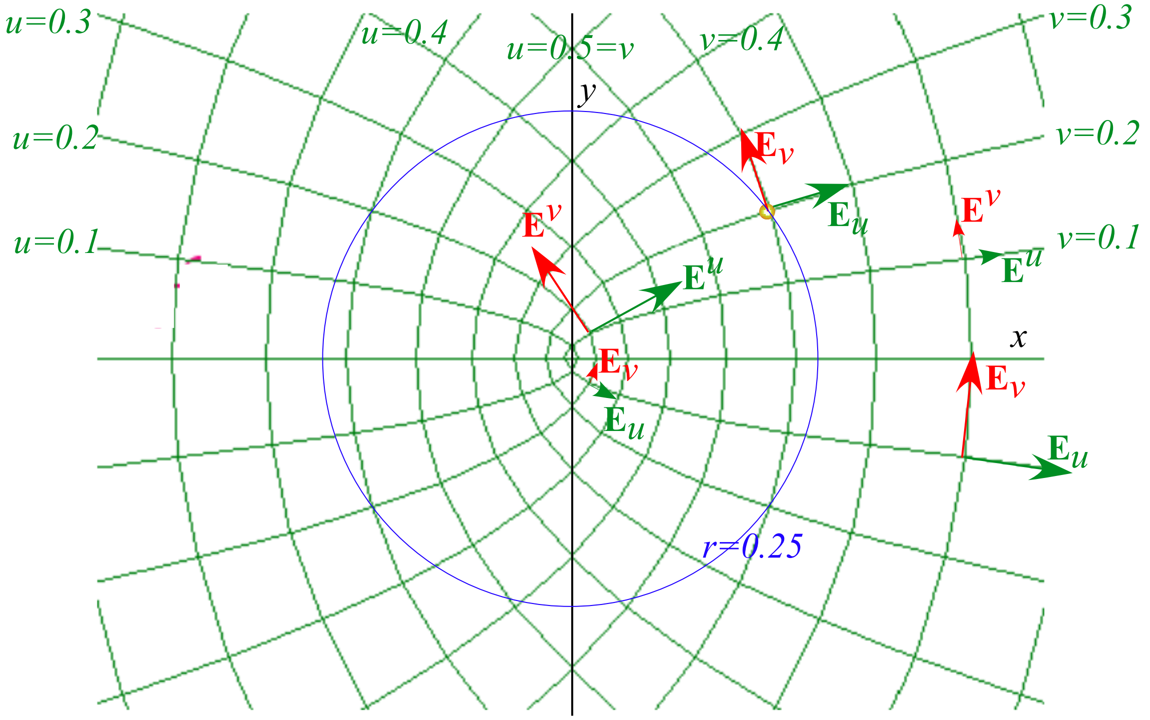

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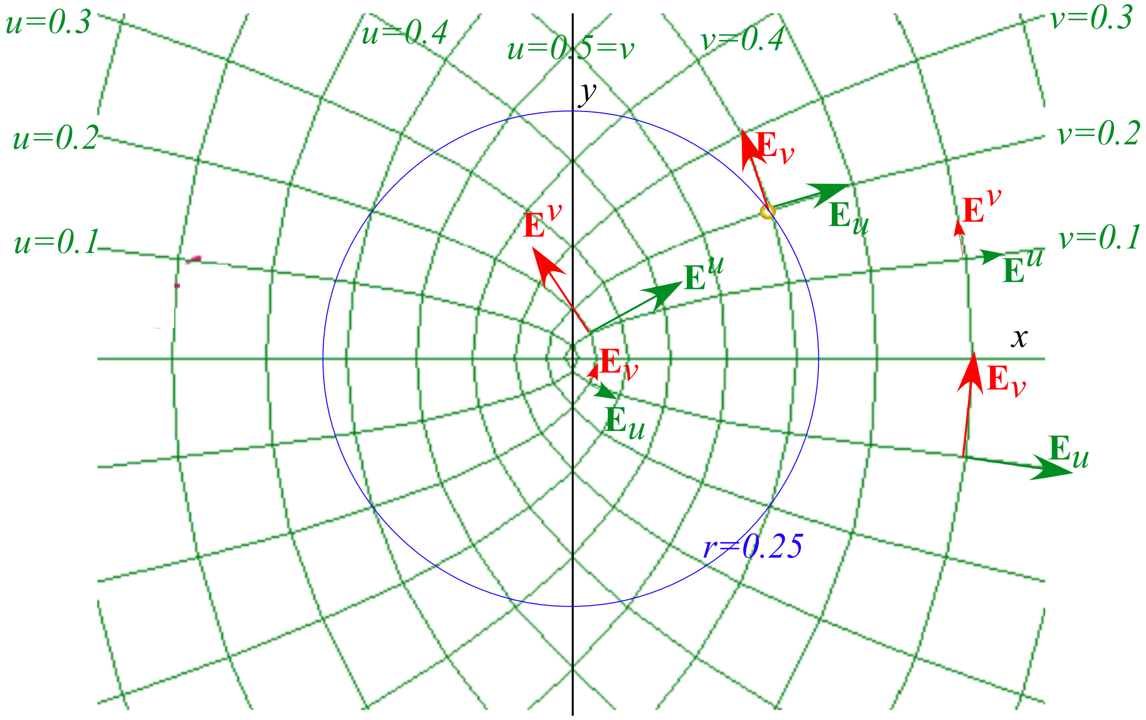

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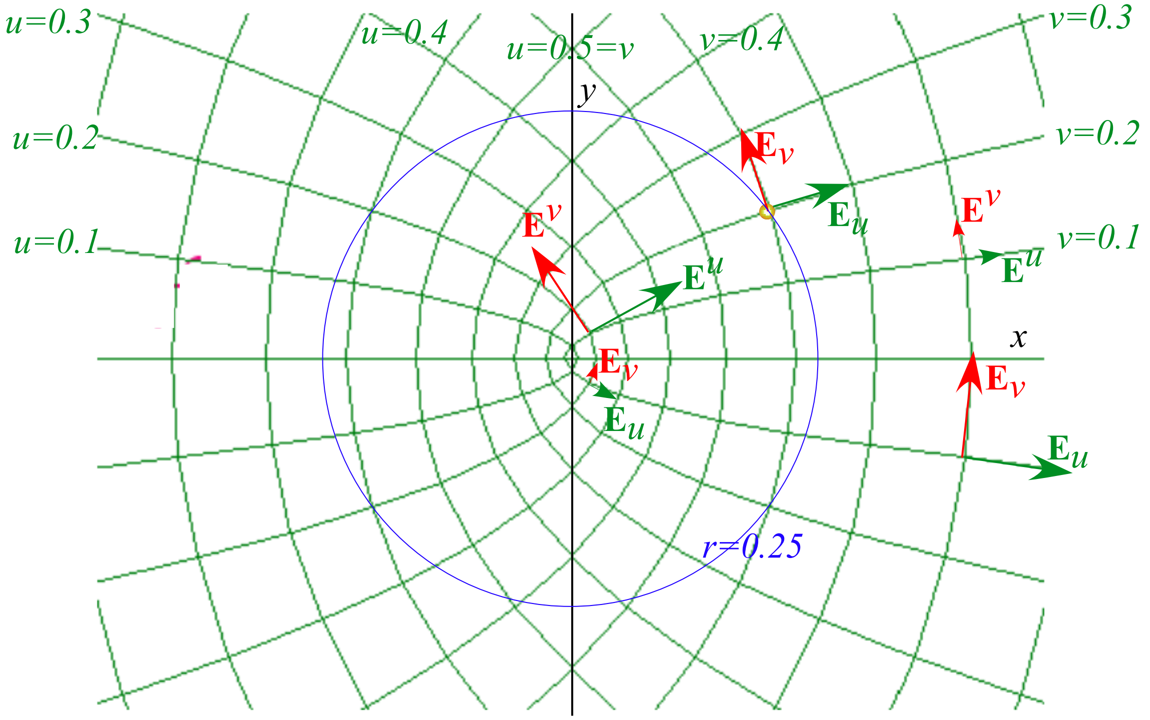

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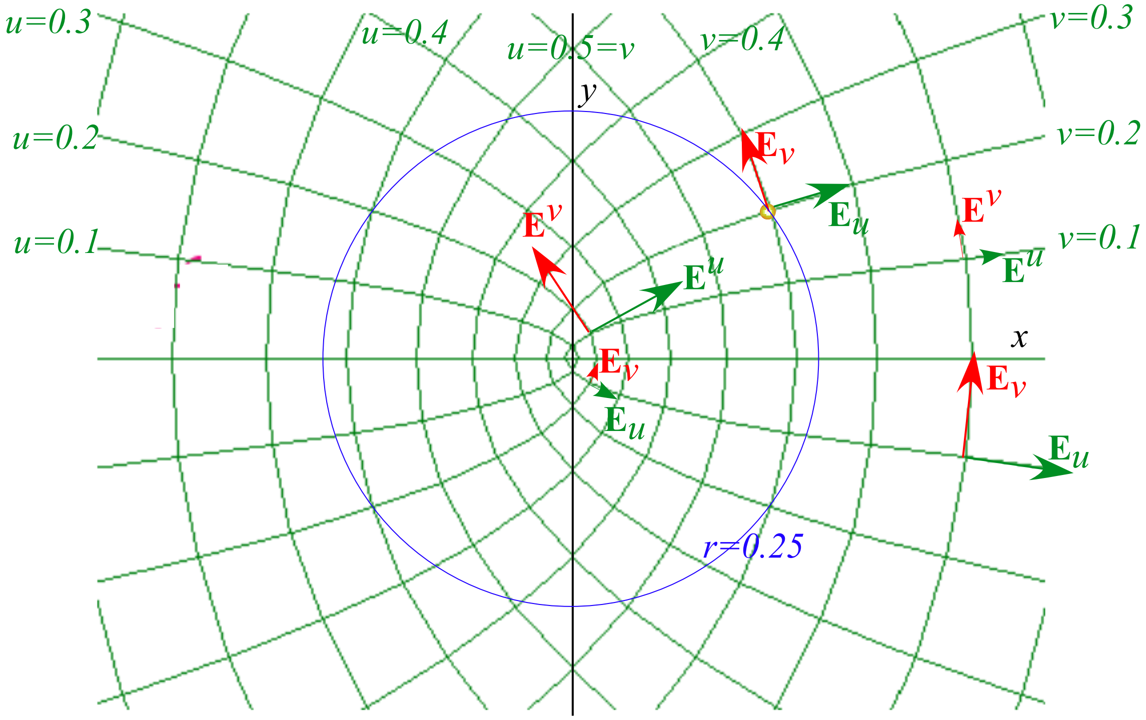
Gives confocal parabolics



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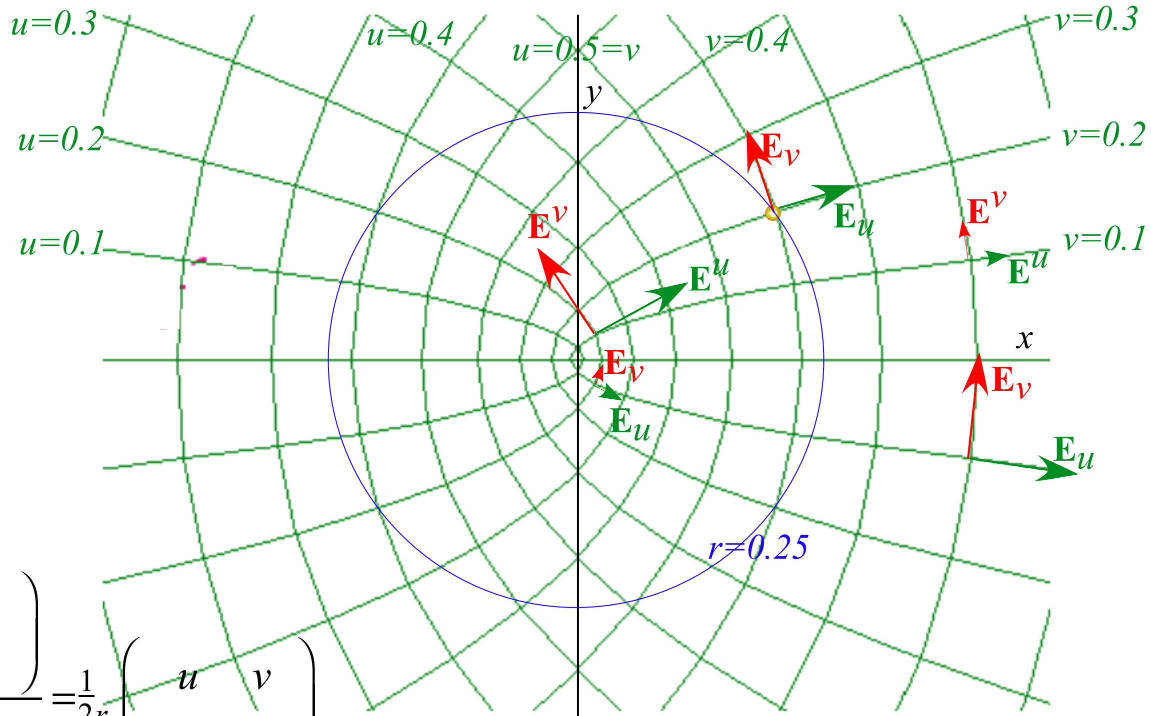

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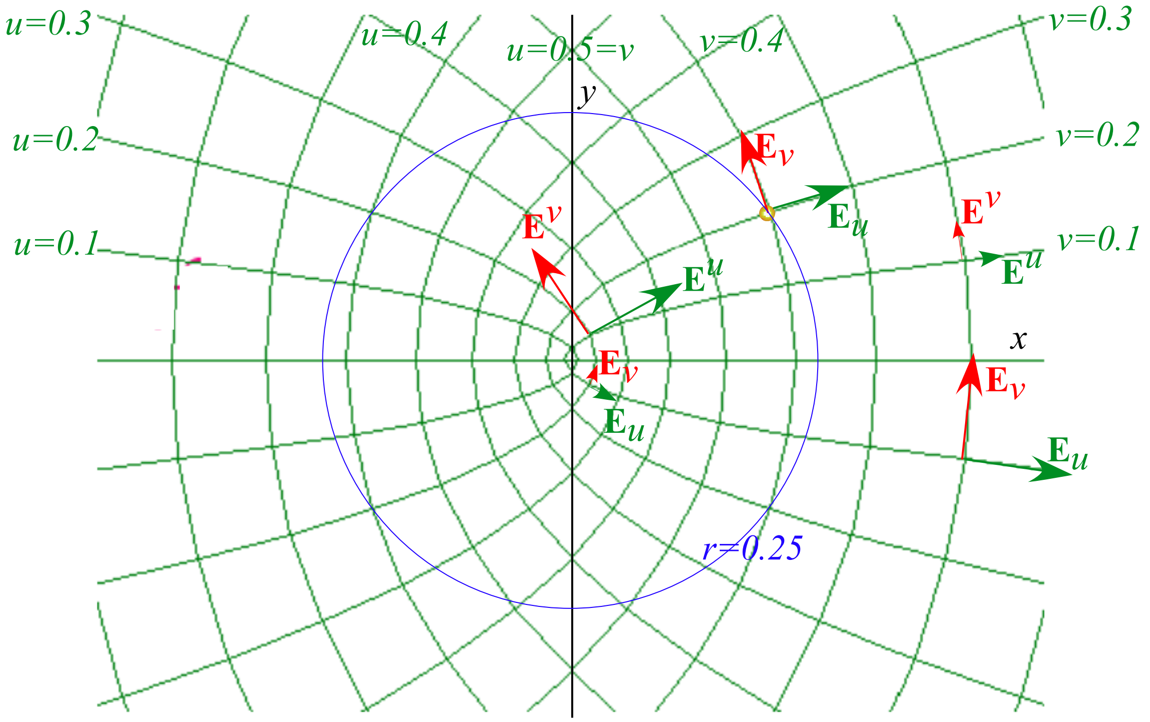

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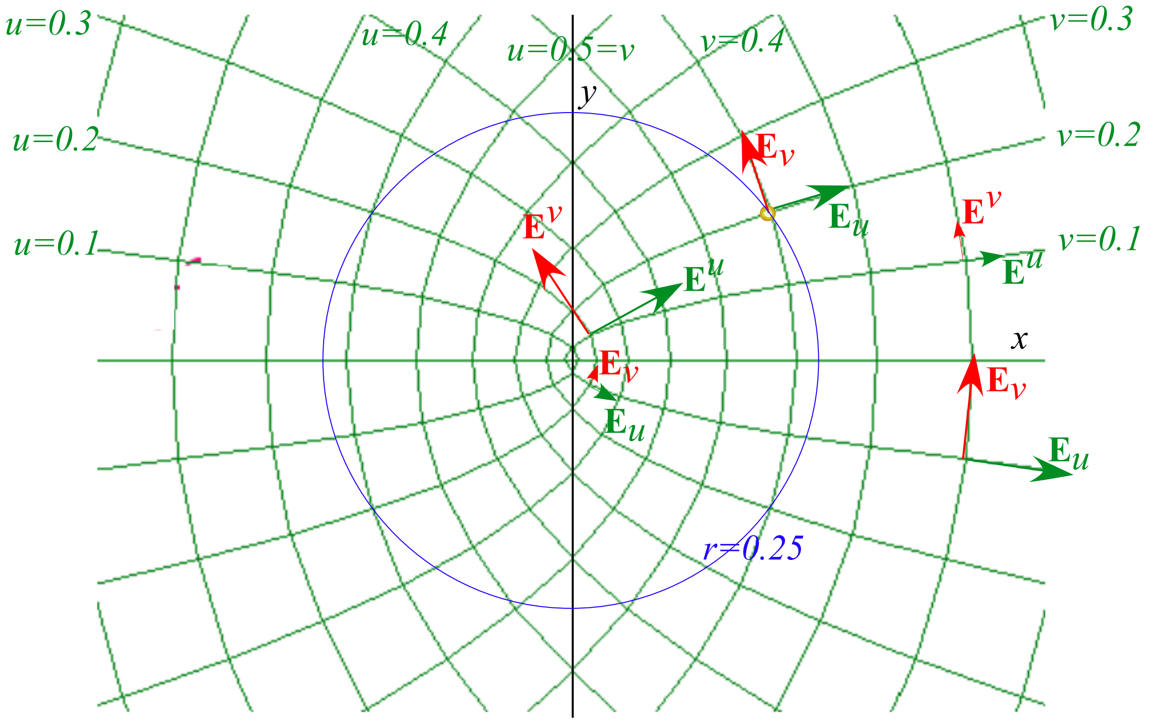

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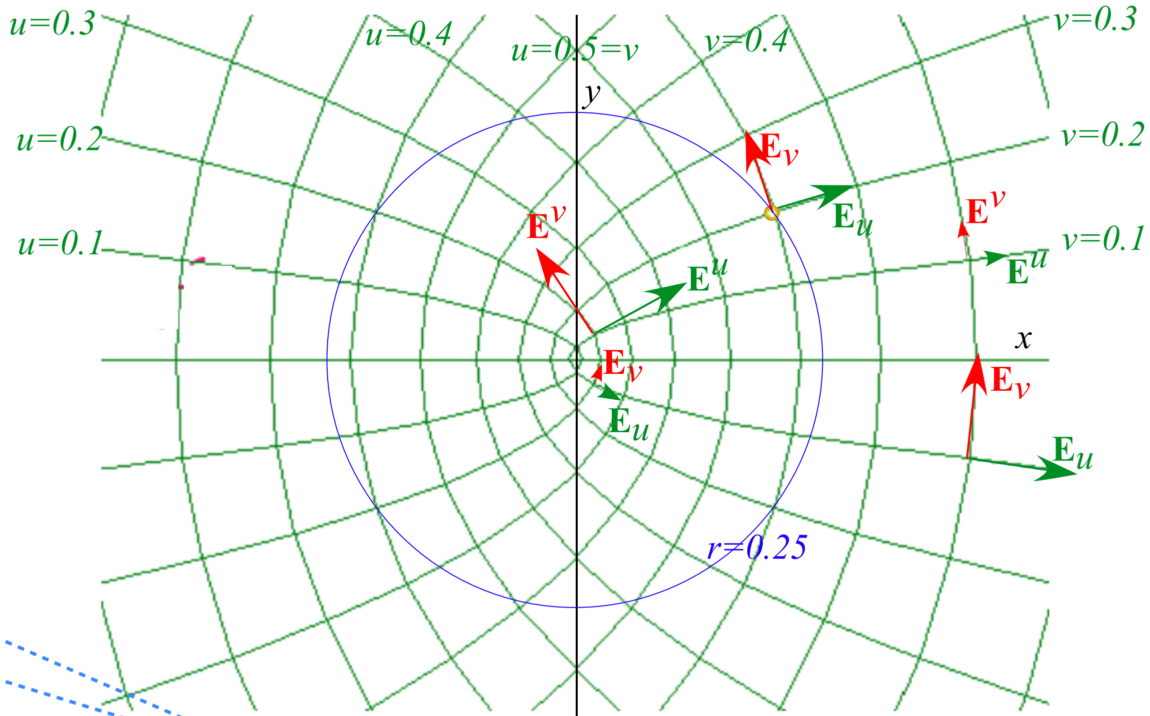

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$$H=\tfrac{1}{2m}(g^{ab}p_ap_b)+V=\tfrac{1}{2m}(g^{uu}p_u^2+g^{vv}p_v^2)+V=\frac{p_u^2+p_v^2}{8m(u^2+v^2)}+V$$

$$V=\varepsilon x+k/r$$

Stark-Coulomb potential

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$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2 \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

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$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

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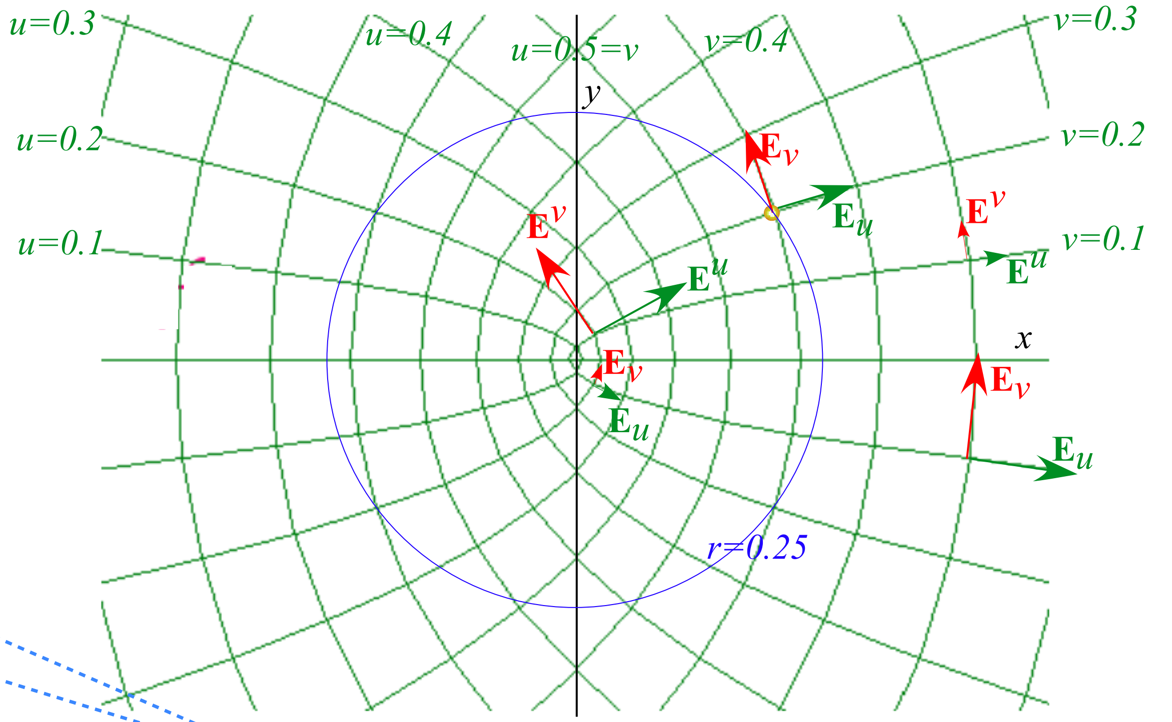
Gives confocal parabolics



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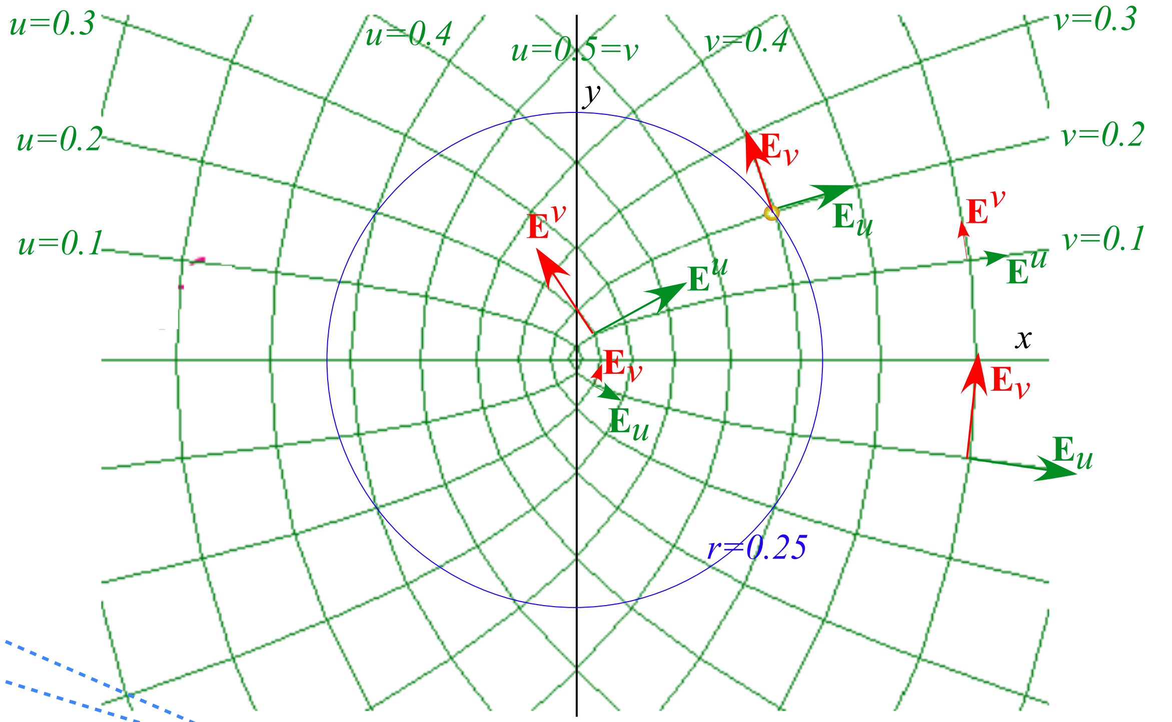

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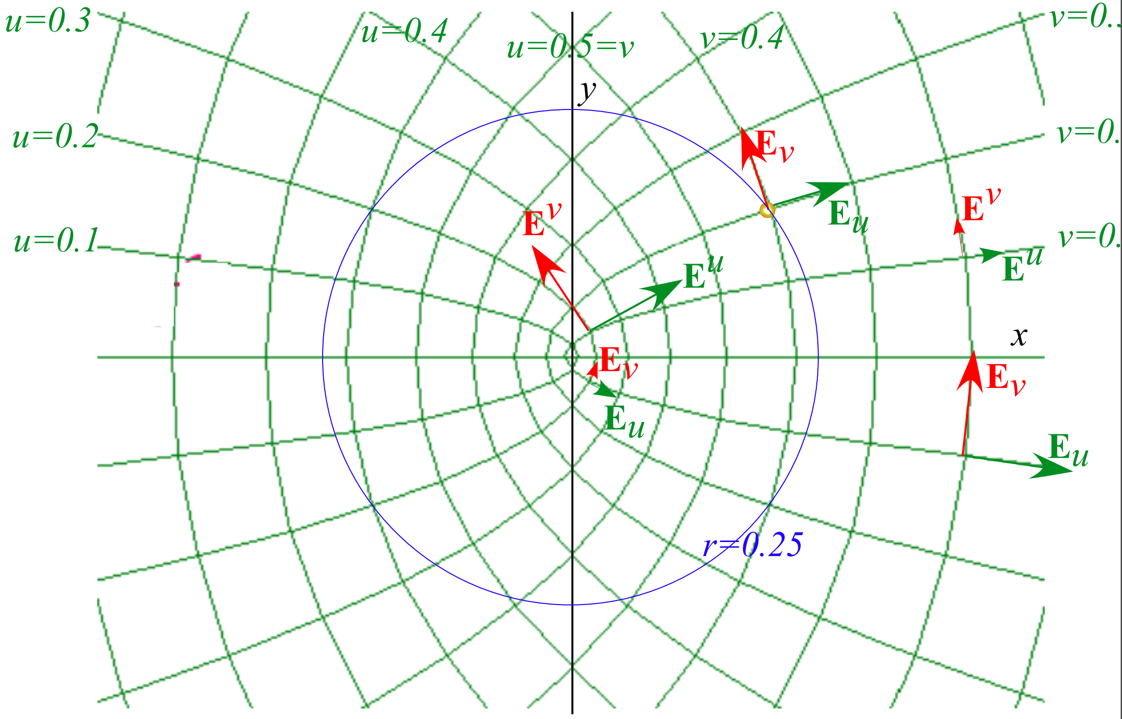

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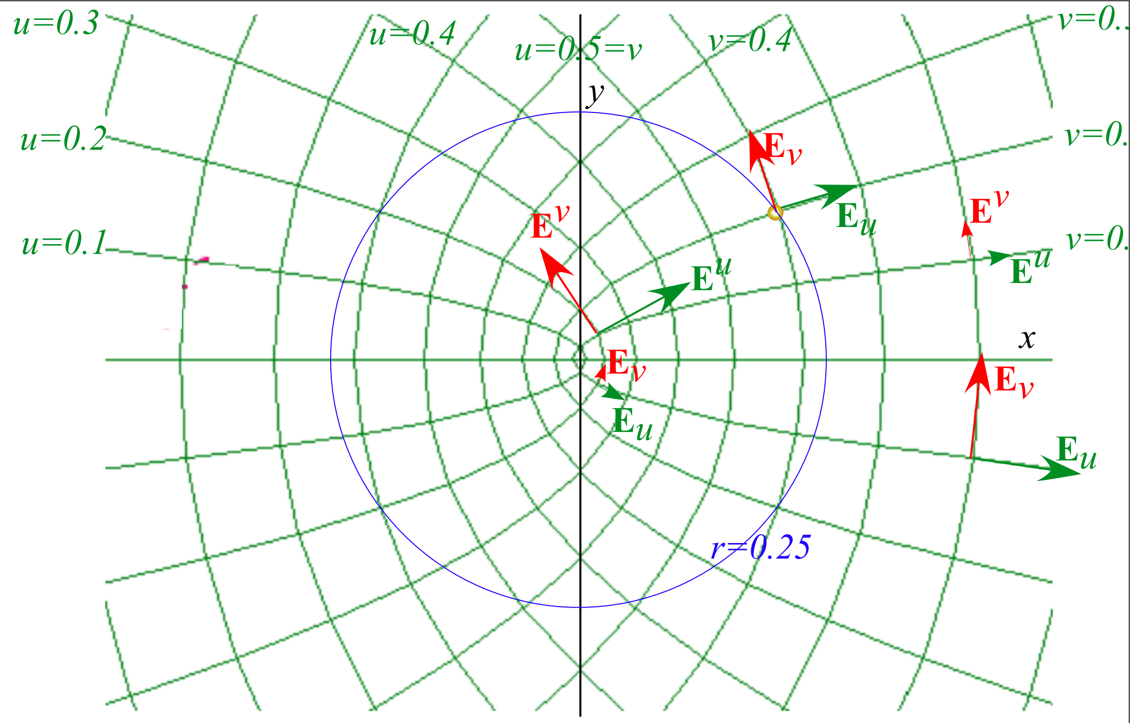

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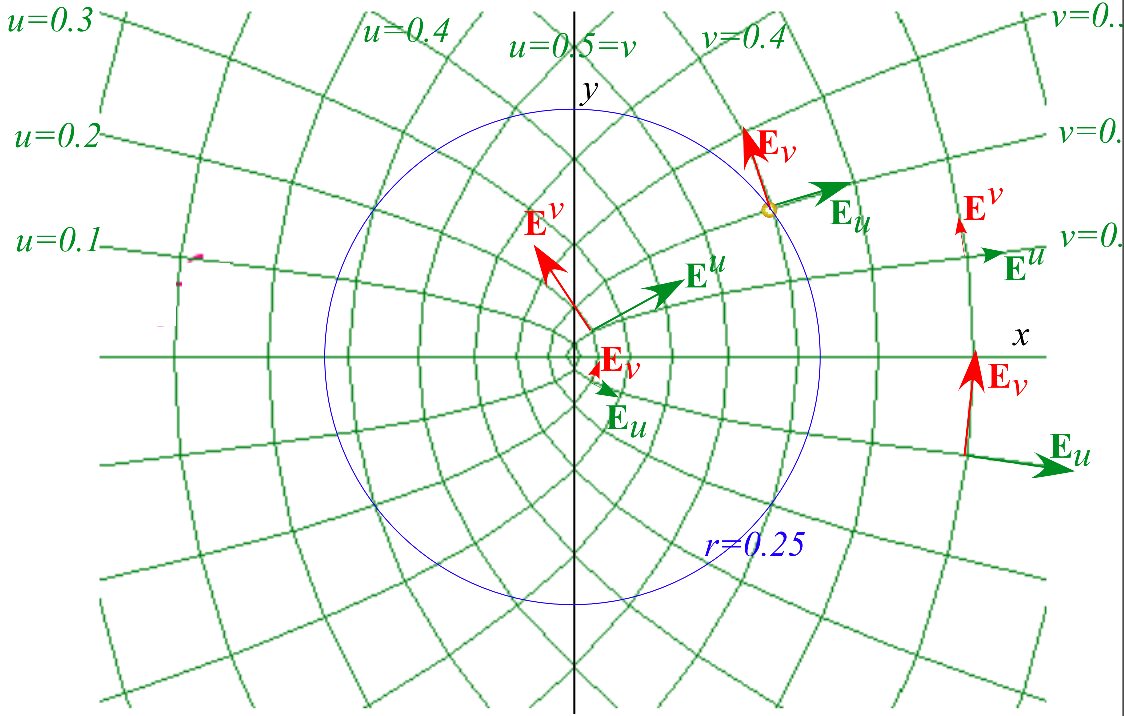

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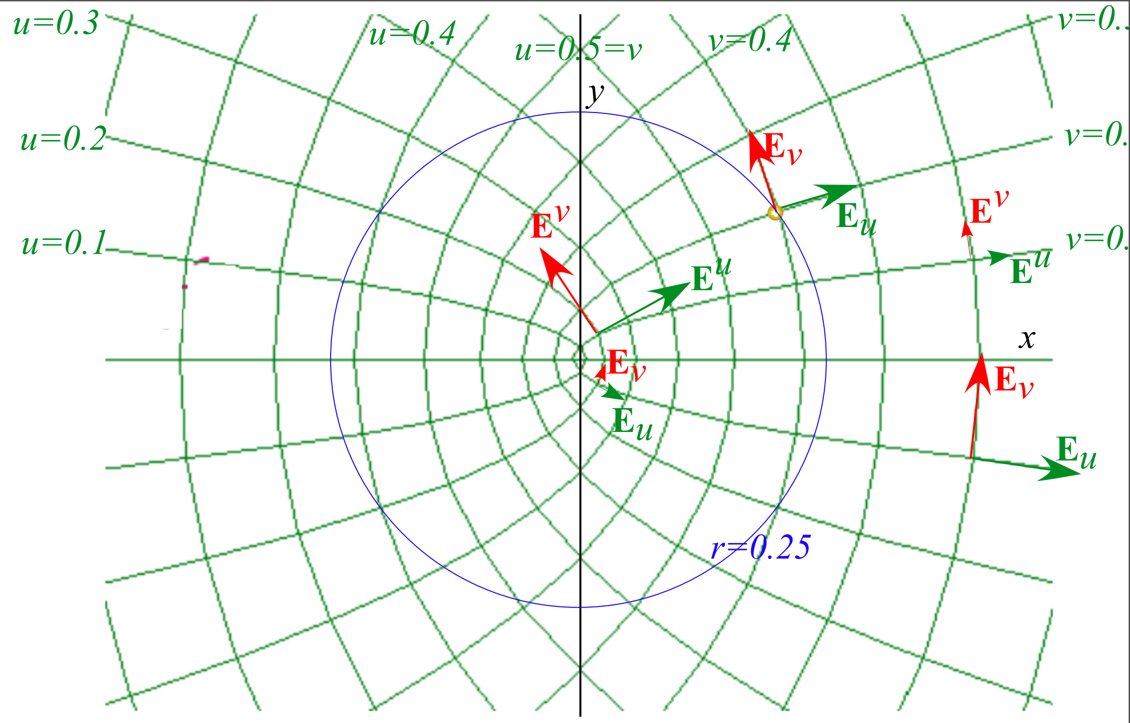

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Zero Stark-field *(ε=0)* gives h_u or h_v harmonic oscillation if $E<0$. It's unstable or anharmonic otherwise.

$$\dot{p}_u=-\frac{\partial h_u}{\partial u}=-8Eu+16\varepsilon u^3 \qquad \dot{u}=\frac{\partial h_u}{\partial p_u}=p_u/m \qquad \dot{p}_v=-\frac{\partial h_v}{\partial v}=-8Ev-16\varepsilon v^3 \qquad \dot{v}=\frac{\partial h_v}{\partial p_v}=p_v/m$$

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Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

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Then the λ function gives the new constrained *x*-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola *y=1/2kx²* is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda\nabla c^1 = \lambda\nabla(\tfrac{1}{2}kx^2 - y) = \lambda\begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda\begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda\begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Constraint function *y=1/2kx²* gives derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$ that give multiplier λ.

$$\lambda = m(-k\dot{x}^2 - kx\ddot{x} - g)$$

Then the λ function gives the new constrained *x*-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

(Same solution as before)

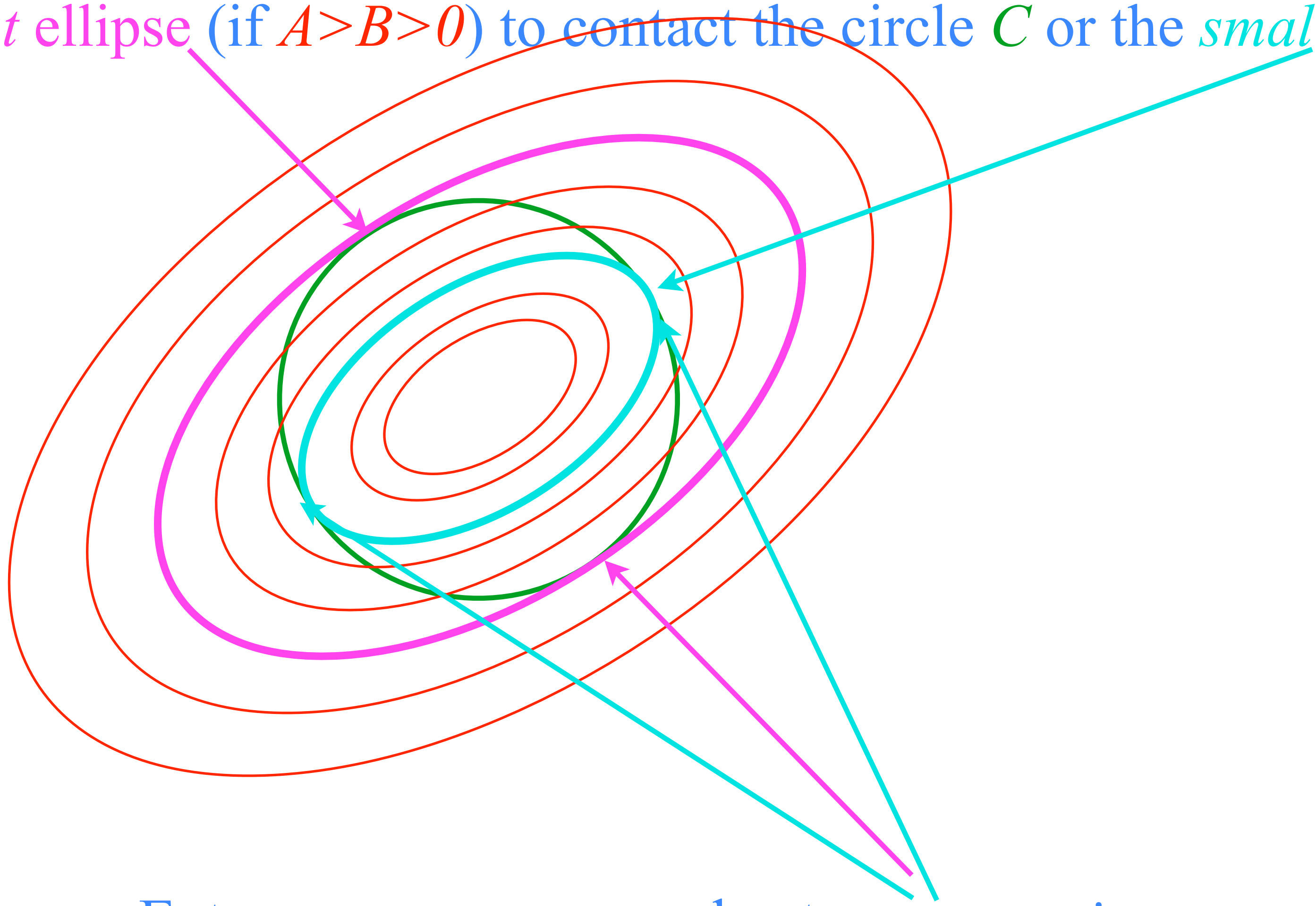
$$\boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2x^2}kx}$$

Other Ways to do constraint analysis

- *Way 3. OCC constraint webs*
  - *Preview of atomic-Stark orbits*
    - *Classical Hamiltonian separability*
- *Way 4. Lagrange multipliers*
  - ➔ *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *“Non-Holonomic” multipliers*

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix}\partial_x H\\ \partial_y H\end{pmatrix} = \lambda \cdot \begin{pmatrix}\partial_x C\\ \partial_y C\end{pmatrix}$$

$$\begin{pmatrix}Ax+By\\ Bx+Dy\end{pmatrix} = \lambda \cdot \begin{pmatrix}x\\ y\end{pmatrix}$$

Extreme cases occur only at *contact points*

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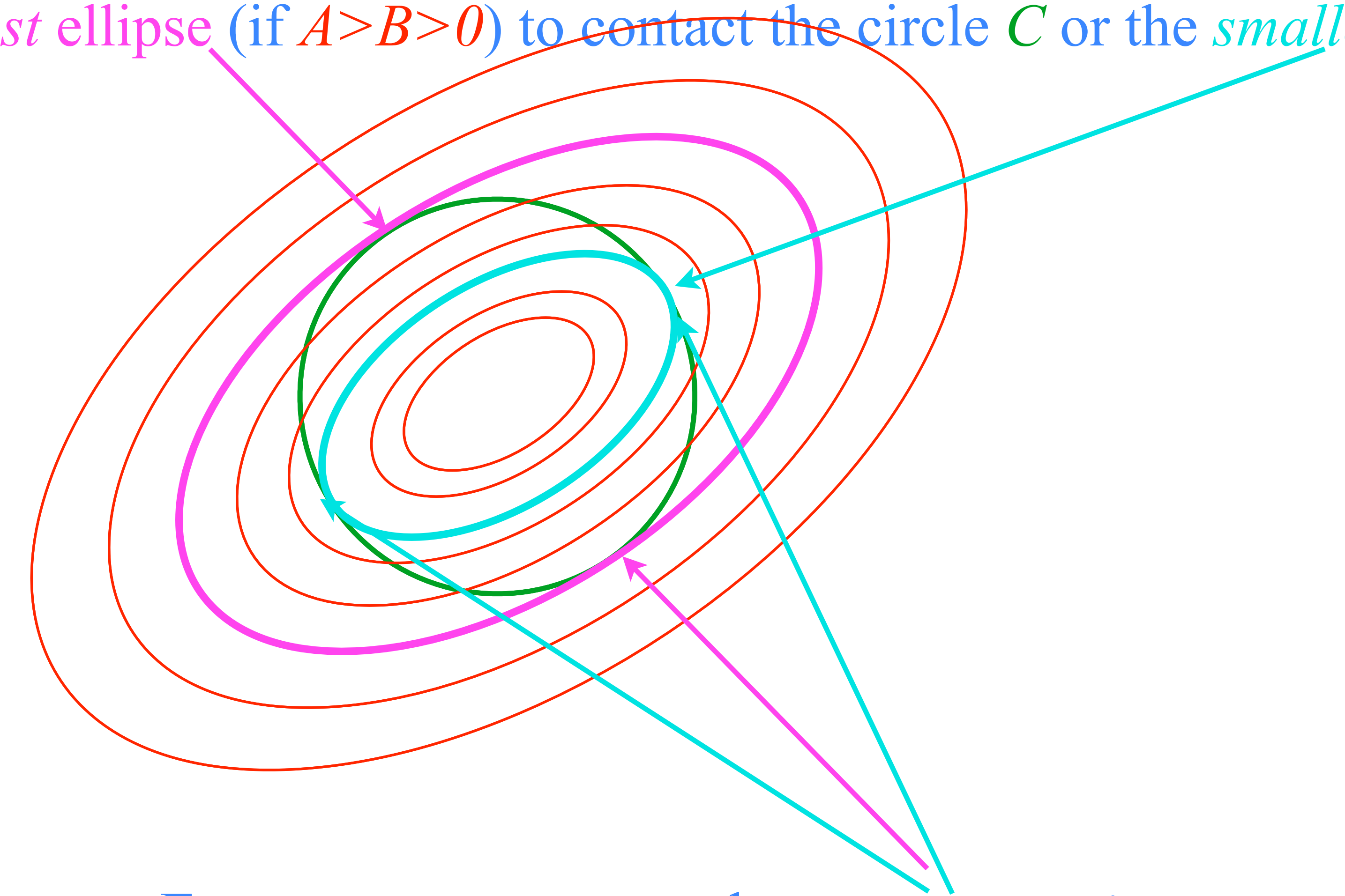

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This amounts to a *λ-eigenvalue-eigenvector* equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we label eigenvalues λ with a Greek "L")

Other Ways to do constraint analysis

Way 3. OCC constraint webs
    Preview of atomic-Stark orbits
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        Lagrange multiplier as eigenvalues
    → *Multiple multipliers*
        "Non-Holonomic" multipliers

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{\qquad \partial c}{\partial q^k} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k} \bullet \frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k \bullet \nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda \frac{\partial c}{\partial q^1} \\ \lambda \frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda \frac{\partial c}{\partial q^k}$$

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Two or more constraints $c^1(q^k)=const., c^2(q^k)=const.,\cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

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- → *"Non-Holonomic" multipliers*

Constraints may be determined by differential relations that are not integrable. Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1}dq^1 + \frac{\partial c^1}{\partial q^2}dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1}dq^1 + \frac{\partial c^2}{\partial q^2}dq^2 + \ldots$$

$$\vdots \qquad\qquad \vdots$$

General differential constraint relations

$$0 = C_1^1 dq^1 + C_2^1 dq^2 + \ldots$$

$$0 = C_1^2 dq^1 + C_2^2 dq^2 + \ldots$$

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Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \ldots$$

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I guess that means that integrable ones are holonomic, but why do we need the bigger words.

A requirement for integrability (or “holonomicty”) is that double differentials are symmetric.

$$\frac{\partial^2 c^{\gamma}}{\partial q^j \partial q^k} = \frac{\partial^2 c^{\gamma}}{\partial q^k \partial q^j}$$

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Force components $F_k^{\gamma} = \frac{\partial c^{\gamma}}{\partial q^k} = C_k^{\gamma}$ must satisfy *reciprocity relations* to be gradients of a c^{γ} function.

Integral constraint differentials

$$\frac{\partial F_k^{\gamma}}{\partial q^j} = \frac{\partial^2 c^{\gamma}}{\partial q^j \partial q^k} = \frac{\partial F_j^{\gamma}}{\partial q^k}$$

General differential constraint relations

$$\frac{\partial C_k^{\gamma}}{\partial q^j} \begin{matrix} \textit{may or} \\ \textit{may not be} \end{matrix} \frac{\partial C_j^{\gamma}}{\partial q^k}$$