Reimann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

Christoffel g-derivative formula
What’s a tensor? What’s not?

Riemann equations of motion (No explicit t-dependence and fixed GCC)

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Separation of GCC Equations: Effective Potentials
Small radial oscillations
Cycloid vs Pendulum
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

Christoffel g-derivative formula

What’s a tensor? What’s not?
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $U$ are due to:

(1) changing $U^m$ components

\[
\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} \left( E_m \right) + U^n \frac{\partial E_n}{\partial q^i}
\]

(2) curving GCC vectors $E_n$. 

Tuesday, October 30, 2012
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

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\]

2. curving GCC vectors $\mathbf{E}_n$

Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}^\ell$ or else $\mathbf{E}_m$

\[
\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $\mathbf{U}$ are due to:

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\]

2. curving GCC vectors $\mathbf{E}_n$

\[
\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{i n;\ell} \mathbf{E}_\ell
\]

**Christoffel coefficients $\Gamma_{ij;k}$ of the first kind**

defined by:

\[
\Gamma_{i n;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_\ell = \Gamma_{ni;\ell}
\]
**Covariant derivative and Christoffel Coefficients** $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. **Changing $U^m$ components**

\[
\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^m \frac{\partial E_n}{\partial q^i}
\]

2. **Curving GCC vectors $E_n$**

\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;}^\ell E^\ell = \Gamma_{in;}^m E^m
\]

**Derivative of $E_n$** is expressed using $E^\ell$ or else $E^m$

**Christoffel coefficients $\Gamma_{ij;}^k$ of the first kind** defined by:

\[
\Gamma_{in;}^\ell = \frac{\partial E_n}{\partial q^i} \cdot E^\ell = \Gamma_{ni;}^\ell
\]

**Christoffel coefficients $\Gamma_{ij;}^k$ the second kind** defined by:

\[
\Gamma_{in;}^m = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni;}^m
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. changing $U^m$ components
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\]

2. curving GCC vectors $E_n$

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$
\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell = \Gamma_{in;m} E_m
\]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:
\[
\Gamma_{in;\ell} = \frac{\partial E^n_i}{\partial q^i} \cdot E_\ell = \Gamma_{ni;\ell}
\]

Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:
\[
\Gamma_{in;m} = \frac{\partial E^n_i}{\partial q^i} \cdot E^m = \Gamma_{ni;m}
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Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

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\[ \frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n \frac{\partial E_n}{\partial q^i} \]

2. curving GCC vectors $E_n$

\[ \frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E_\ell = \Gamma_{in;\ell}^m E_m \]

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

\[ \frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E_\ell = \Gamma_{in;\ell}^m E_m \]

**Christoffel coefficients $\Gamma_{ij;k}$ of the first kind**

defined by:

\[ \Gamma_{in;\ell} = \frac{\partial E_n^i}{\partial q^i} \cdot E_\ell = \Gamma_{ni;\ell} \]

$i, n$ to $n, i$ symmetry guaranteed here

**Christoffel coefficients $\Gamma_{ij;k}$ the second kind**

defined by:

\[ \Gamma_{in;\ell}^m = \frac{\partial E_n^i}{\partial q^i} \cdot E_m^\ell = \Gamma_{ni}^m \]

$i, n$ to $n, i$ symmetry guaranteed here

Q: Do we need a third kind of $\Gamma$-coefficient or a $\Lambda$-coefficient? (to differentiate \textit{contravariant} $E^n$ or \textit{covariant} $U_n$)

\[ \frac{\partial E^n}{\partial q^i} = \Lambda_{im} E^m \]

where: $\Lambda_{im} = \frac{\partial E^n}{\partial q^i} \cdot E_m$
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. changing $U^m$ components

\[
\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i}(E_m) + U^n \frac{\partial E^n}{\partial q^i}
\]

2. curving GCC vectors $E_n$

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell}^m E_m = \Gamma_{ni;}^m E_m
\]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:

\[
\Gamma_{in;\ell}^m = \frac{\partial E_n^\ell}{\partial q^i} \cdot E_m = \Gamma_{ni;}^m E_m
\]

$i,n$ to $n,i$ symmetry guaranteed here

Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:

\[
\Gamma_{in;}^m = \frac{\partial E^m}{\partial q^i} = \Gamma_{ni;}
\]

$i,n$ to $n,i$ symmetry guaranteed here

Q: Do we need a third kind of $\Gamma$-coefficient or a $\Lambda$-coefficient? (to differentiate contravariant $E^n$ or covariant $U_n$)

A: NO! That $\Lambda$-coefficient is just a $\Gamma$-coefficient with a (-).

\[
0 = \frac{\partial (E^n \cdot E_m)}{\partial q^i} = \frac{\partial E^n}{\partial q^i} \cdot E_m + E^n \cdot \frac{\partial E_m}{\partial q^i}
\]

So: $\Lambda_{im}^n = -\Gamma_{im}^n$
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. **Changing $U^m$ components**
\[
\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n \frac{\partial E^n}{\partial q^i}
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2. **Curving GCC vectors $E_n$**
\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;}^\ell E_\ell = \Gamma_{in;}^m E_m
\]

Derivative of $E_n$ is expressed using $E_\ell$ or else $E_m$

\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{in;}^\ell E_\ell = \Gamma_{in;}^m E_m
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**Christoffel coefficients $\Gamma_{ij;k}$ of the first kind**
defined by:
\[
\Gamma_{in;}^\ell = \frac{\partial E_n^i}{\partial q^i} \bullet E_\ell = \Gamma_{ni;}^\ell
\]

**Christoffel coefficients $\Gamma_{ij;}^k$ the second kind**
defined by:
\[
\Gamma_{in;}^m = \frac{\partial E_n^i}{\partial q^i} \bullet E^m = \Gamma_{ni;}^m
\]

Any vector derivative can be expressed using $\Gamma_{ij;}^k$ in terms of $E_m$
\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) E_m
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

(1) changing $U^m$ components
\[ \frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n \frac{\partial E^n}{\partial q^i} \]

(2) curving GCC vectors $E_n$
\[ \frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell = \Gamma_{in;}^m E_m \]

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$
\[ \frac{\partial E_n}{\partial q^i} \cdot E^\ell = \Gamma_{in;}^m \]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:
\[ \Gamma_{in;\ell} = \frac{\partial E_n}{\partial q^i} \cdot E^\ell = \Gamma_{ni;\ell} \]

Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:
\[ \Gamma_{in;}^m = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni;}^m \]

Any vector derivative can be expressed using $\Gamma_{ij;k}$ in terms of $E_m$ or $E^m$
\[ \frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im;}^n \right) E^m \]
\[ \frac{\partial E^n}{\partial q^i} \cdot E_m = -E^n \cdot \frac{\partial E_m}{\partial q^i} \]
So:
\[ \Lambda_{im}^n = -\Gamma_{im}^n \]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. changing $U^m$ components
   \[
   \frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n \frac{\partial E_n}{\partial q^i}
   \]

2. curving GCC vectors $E_n$
   \[
   \frac{\partial E_n}{\partial q^i} = \Gamma_{in;l} \ell = \Gamma_{i;n;l} E_m
   \]

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

\[
\frac{\partial E_n}{\partial q^i} = \Gamma_{i;n;l} \ell = \Gamma_{i;n;l} E_m
\]

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:
\[
\Gamma_{in;l} = \frac{\partial E_n}{\partial q^i} \cdot E_\ell = \Gamma_{ni;l}
\]

Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:
\[
\Gamma_{i;n;l} = \frac{\partial E_m}{\partial q^i} \cdot E_n = \Gamma_{ni;m}
\]

Any vector derivative can be expressed using $\Gamma_{ij;k}$ in terms of $E_m$ or $E^n$

\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{i;n;m} \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{n;m} \right) E^m
\]

So:
\[
\Lambda_{i;m}^n = -\Gamma_{i;n;m}
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^k$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. changing $U^m_i$ components

\[
\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j_j E_j) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n_i \frac{\partial E^n}{\partial q^i}
\]

2. curving GCC vectors $E_n$

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

\[
\frac{\partial E^n}{\partial q^i} = \Gamma_{in;}^\ell \ E^\ell = \Gamma_{in;}^m \ E^m
\]

Christoffel coefficients $\Gamma_{ij;}^k$ of the first kind defined by:

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\]

Christoffel coefficients $\Gamma_{ij;}^k$ the second kind defined by:

\[
\Gamma_{in;}^m = \frac{\partial E^n}{\partial q^i} \cdot E^m = \Gamma_{ni;}^m
\]

Any vector derivative can be expressed using $\Gamma_{ij;}^k$ in terms of $E^m$ or $E^m$

\[
\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im;}^n \right) E^m
\]

Defining covariant derivative $U^m_{;i}$ of a contravariant component $U^m_i$

\[
U^m_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m
\]

So:

\[
\Lambda^m_{im} = -\Gamma^m_{im}
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC $q^m$ derivatives of vectors $U$ are due to:

1. changing $U^m$ components

$$\frac{\partial U}{\partial q^i} = \frac{\partial}{\partial q^i} \left( U^j E_j \right) = \frac{\partial U^m}{\partial q^i} (E_m) + U^n \frac{\partial E_n}{\partial q^i}$$

2. curving GCC vectors $E_n$

Derivative of $E_n$ is expressed using $E^\ell$ or else $E_m$

$$\frac{\partial E_n}{\partial q^i} = \Gamma_{in;\ell} E^\ell = \Gamma_{in}^m E_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind defined by:

$$\Gamma_{in;\ell} = \frac{\partial E_n}{\partial q^i} \cdot E^\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij;k}$ the second kind defined by:

$$\Gamma_{in}^m = \frac{\partial E_n}{\partial q^i} \cdot E^m = \Gamma_{ni}^m$$

Any vector derivative can be expressed using $\Gamma_{ij;k}$ in terms of $E_m$ or $E^m$

$$\frac{\partial U}{\partial q^i} = \left( \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in}^m \right) E_m = \left( \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im}^n \right) E^m$$

$$= \quad U^m_{;i} \quad E_m = \quad U_{m;i} \quad E^m$$

Defining covariant derivative $U^m_{;i}$ of a contravariant component $U^m$

$$U^m_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in}^m$$

...and covariant derivative $U_{m;i}$ of a covariant component $U_m$

$$U_{m;i} = \frac{\partial U^m}{\partial q^i} - U^n \Gamma_{im}^n$$
Intrinsic derivatives:
(Mathematicians being cute)
Defining intrinsic derivative of contravariant vector components.

\[
\frac{\delta V^k}{\delta t} = \frac{dV^k}{dt} + \Gamma^k_{mn} V^m \dot{q}^n = \frac{\partial V^k}{\partial q^n} \dot{q}^n + \Gamma^k_{mn} V^m \dot{q}^n = V^k_{;n} \dot{q}^n
\]

\[
F_k = \frac{\delta p_k}{\delta t}
\]

Tensor chain rules.

\[
\frac{\delta V^k}{\delta t} = V^k_{;n} \dot{q}^n, \quad \text{replaces:} \quad \frac{dV^k}{dt} = \frac{\partial V^k}{\partial q^n} \dot{q}^n \quad \text{where:} \quad V^k_{;n} = \frac{\partial V^k}{\partial q^n} + \Gamma^k_{mn} V^m
\]

Defining intrinsic derivative of covariant vector components.

\[
\frac{\delta V_k}{\delta t} = \frac{dV_k}{dt} - \Gamma^m_{kn} V^m \dot{q}^n = \frac{\partial V_k}{\partial q^n} \dot{q}^n - \Gamma^m_{kn} V^m \dot{q}^n = V_k_{;n} \dot{q}^n
\]

\[
F^k = \frac{\delta p^k}{\delta t}
\]

\[
\frac{\delta V_k}{\delta t} = V_k_{;n} \dot{q}^n, \quad \text{replaces:} \quad \frac{dV_k}{dt} = \frac{\partial V_k}{\partial q^n} \dot{q}^n \quad \text{where:} \quad V_k_{;n} = \frac{\partial V_k}{\partial q^n} - \Gamma^m_{kn} V^m
\]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

Christoffel g-derivative formula
What’s a tensor? What’s not?
Christoffel g-derivative formula

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]
Christoffel g-derivative formula

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = \Gamma_{nm;i} + \Gamma_{in;m}
\]  \quad \text{(switched } i \leftrightarrow n)\

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]  \quad \text{(switched } i \leftrightarrow m)
Christoffel g-derivative formula

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

(switched i ↔ n)

(switched i ↔ m)
Christoffel g-derivative formula

\[
\frac{\partial (\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} 
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i ↔ n)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i ↔ m)}
\]
Christoffel g-derivative formula

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i }\leftrightarrow\text{ n)} \]

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i }\leftrightarrow\text{ m)} \]

Gives the Christoffel formula

\[ \Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \]
Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}^k$

Christoffel g-derivative formula

What’s a tensor? What’s not?
\[
\begin{align*}
\frac{\partial (E_m \cdot E_n)}{\partial q^i} &= \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \\
\frac{\partial g_{mn}}{\partial q^i} &= \Gamma_{im;n} + \Gamma_{in;m} \\
\frac{\partial g_{mi}}{\partial q^n} &= -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched } i \leftrightarrow n) \\
\frac{\partial g_{in}}{\partial q^m} &= \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched } i \leftrightarrow m) \\
\end{align*}
\]

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

What's a tensor? What's not?

Chain-saw-sums transform a "bar-frame" view

\[
\begin{align*}
\bar{U}^m_{;n} &= \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m \\
\end{align*}
\]

of covariant derivative

\[
\begin{align*}
U^m_{;n} &= \frac{\partial U}{\partial q^n} \cdot E_m
\end{align*}
\]
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m)}
\]

Chain-saw-sums transform a "bar-frame" view

\[
\bar{U}^m_{\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m
\]

of covariant derivative

\[
U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m
\]

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im; n} + \Gamma_{in; m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm; i} - \Gamma_{in; m} \quad \text{(switched } i \leftrightarrow n) \]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im; n} + \Gamma_{mn; i} \quad \text{(switched } i \leftrightarrow m) \]

Gives the Christoffel formula

\[
\Gamma_{im; n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

Chain-saw-sums transform a "bar-frame" view \( \bar{U}^{\bar{m}}; \bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{E}^{\bar{m}} \) of covariant derivative \( U_m; n = \frac{\partial U}{\partial q^n} \cdot E_m \)

\[
\bar{U}^{\bar{m}}; \bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{E}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial U}{\partial q^n} \cdot \bar{E}^{\bar{m}}
\]

The transformation of \( U_m; n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{\ell; n}^m \) is that of general 2nd-rank tensor \( T^m_n \)

\[
T^{\bar{m}}; \bar{n} = \frac{\partial q^m}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T^m_n
\]
\[
\frac{\partial \left( \mathbf{E}_m \cdot \mathbf{E}_n \right)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \\
- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \\
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \\
\]
(switched i \leftrightarrow n)

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

Chain-saw-sums transform a "bar-frame" view

\[
\tilde{U}^\bar{m} : \bar{n} = \frac{\partial \bar{U}}{\partial q^{\bar{n}}} \cdot \bar{E}^m
\]

of covariant derivative

\[
U^m : n = \frac{\partial U}{\partial q^n} \cdot \mathbf{E}_m
\]

The transformation of

\[
U^m : n = \frac{\partial U^m}{\partial q^n} + \mathcal{U}^{\ell} \Gamma_{n;\ell}^m
\]

is that of general 2nd-rank tensor

\[
T^\bar{m} : \bar{n} = \frac{\partial \bar{q}^\bar{m}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^\bar{n}} T^m_n
\]

The transformation of

\[
U^m : n = \frac{\partial U^m}{\partial q^n}
\]

is NOT that simple.
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{im;n}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

Gives the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

Chain-saw-sums transform a "bar-frame" view \(\bar{U}^m_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^n} \cdot \bar{E}^m\) of covariant derivative \(U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m\)

\[
\frac{\partial \bar{U}^m}{\partial \bar{q}^n} \cdot \bar{E}^m = \frac{\partial \bar{U}^n}{\partial \bar{q}^m} \cdot \frac{\partial \bar{U}^m}{\partial \bar{q}^n} \cdot \bar{E}^m = \frac{\partial q^n}{\partial \bar{q}^m} \cdot \frac{\partial U^m}{\partial q^n} \cdot E_m
\]

The transformation of \(U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{n;\ell}^m\) is that of general 2nd-rank tensor \(T_{n;\bar{m}}^m\)

The transformation of \(U^m_{;n} = \frac{\partial U^m}{\partial q^n}\) is NOT that simple. At first it looks possible.

\[
\frac{\partial \bar{U}^m}{\partial \bar{q}^n} = \frac{\partial q^n}{\partial \bar{q}^m} \cdot \frac{\partial U^m}{\partial q^n}
\]
\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}
\]

\[
\frac{\partial g_{im}}{\partial q^n} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m)}
\]

Chain-saw-sums transform a "bar-frame" view
\[
\bar{U}^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E^m
\]

of covariant derivative
\[
U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m
\]

Gives the Christoffel formula
\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

The transformation of \( U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{n;\ell}^m \) is that of general 2nd-rank tensor \( T^m_n \)

The transformation of \( U^m_{;n} = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.

But, still need to write \( \frac{\partial U^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).
What’s a tensor? What’s not?

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = \Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i ↔ n)} \]

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i ↔ m)} \]

Chain-saw-sums transform a "bar-frame" view \[ \bar{ U}_m = \frac{\partial \bar{U}_m}{\partial q^\bar{n}} \cdot \bar{E}_m \] of covariant derivative \[ U_m = \frac{\partial U}{\partial q^n} \cdot E_m \]

\[ \bar{ U}_m = \frac{\partial \bar{U}_m}{\partial q^\bar{n}} \cdot \bar{E}_m = \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial \bar{U}_m}{\partial \bar{q}^n} \cdot \bar{E}_m = \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial \bar{U}_m}{\partial \bar{q}^n} \cdot \frac{\partial q^m}{\partial q^\bar{m}} E_m \]

The transformation of \[ U_m = \frac{\partial U^m}{\partial q^n} + U^t \Gamma_{nt}^m \] is that of general 2nd-rank tensor \[ T_{mn} \]

The transformation of \[ U^m = \frac{\partial U^m}{\partial q^n} \] is NOT that simple. At first it looks possible.

The transformation of \[ U^m, n = \frac{\partial U^m}{\partial q^n} \] is NOT that simple. At first it looks possible. standard contra-tran: \[ \bar{ U}^m \]

But, still need to write \[ \frac{\partial \bar{U}^m}{\partial q^n} \] in terms of \[ \frac{\partial U^m}{\partial q^n} \].

\[ \frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial q^m}{\partial q^n} \left( \frac{\partial \bar{q}^m}{\partial q^n} U^m \right) \]
\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}
\]

(switched i \leftrightarrow n)

Given the Christoffel formula

\[
\Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

Chain-saw-sums transform a "bar-frame" view of covariant derivative

\[
\bar{U}^m;\bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^\bar{n}} \cdot \bar{E}^m
\]

The transformation of \(U^m;\bar{n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma_{n\ell;}^m\) is that of general 2nd-rank tensor \(T^n_m\)

The transformation of \(U^m,n = \frac{\partial U^m}{\partial q^n}\) is NOT that simple. At first it looks possible.

\[
\frac{\partial \bar{U}^m}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{q}^\bar{m}}{\partial q^n} \frac{\partial \bar{U}^m}{\partial \bar{q}^{\bar{n}}} + \frac{\partial \bar{q}^\bar{n}}{\partial q^n} \frac{\partial \bar{U}^m}{\partial \bar{q}^{\bar{m}}} 
\]

But, still need to write \(\frac{\partial \bar{U}^m}{\partial \bar{q}^\bar{n}}\) in terms of \(\frac{\partial U^m}{\partial q^n}\).

\[
\frac{\partial U^m}{\partial q^n} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}^m}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{m}}} \frac{\partial \bar{U}^m}{\partial \bar{q}^{\bar{m}}} + U^m \frac{\partial \bar{q}^\bar{m}}{\partial q^n} \frac{\partial \bar{U}^m}{\partial \bar{q}^{\bar{m}}}
\]
What’s a tensor? What’s not?

\[
\frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im; n} + \Gamma_{in; m}
\]

\[
\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm; i} - \Gamma_{in; m} \quad \text{(switched i } \leftrightarrow \text{ n)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im; n} + \Gamma_{mn; i} \quad \text{(switched i } \leftrightarrow \text{ m)}
\]

Chain-saw-sums transform a "bar-frame" view \( \bar{U}^m_{; n} = \frac{\partial \bar{U}}{\partial q^m} \cdot \bar{E}^m \)

of covariant derivative \( U^m_{; n} = \frac{\partial U}{\partial q^m} \cdot E_m \)

\( \bar{U}^m_{; n} = \frac{\partial \bar{U}}{\partial q^m} \cdot \bar{E}^m = \frac{\partial q^n}{\partial q^m} \cdot \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{E}^m = \frac{\partial q^n}{\partial q^m} \cdot \frac{\partial \bar{U}}{\partial q^n} \cdot \frac{\partial q^n}{\partial q^m} E_m \)

Gives the Christoffel formula

\[
\Gamma_{im; n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)
\]

The transformation of \( U^m_{; n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma^m_{\ell; n} \)

is that of general 2nd-rank tensor \( T^m_{; n} \)

The transformation of \( U^m_{; n} \) is NOT that simple. At first it looks possible. \( \frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^n}{\partial q^m} \frac{\partial \bar{U}^m}{\partial q^n} \)

But, still need to write \( \frac{\partial \bar{U}^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[
\frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^n}{\partial q^m} \frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^n}{\partial q^m} \frac{\partial q^n}{\partial q^m} U^m = \frac{\partial q^n}{\partial q^m} \frac{\partial q^n}{\partial q^m} \bar{E}^m + U^m \frac{\partial q^n}{\partial q^m} \frac{\partial q^n}{\partial q^m}
\]

1\(^{\text{st}}\) term is OK, but 2\(^{\text{nd}}\) term is zero only if Jacobian is constant matrix!
What’s a tensor? What’s not?

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m} \]

\[ \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \] (switched i \leftrightarrow n)

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \] (switched i \leftrightarrow m)

Chain-saw-sums transform a "bar-frame" view \[ \bar{U}^m;\bar{n} = \frac{\partial \bar{U}}{\partial \bar{q}^m} \cdot \bar{E}^m \] of covariant derivative \[ U^m; n = \frac{\partial U}{\partial q^n} \cdot E_m \]

Gives the Christoffel formula

\[ \Gamma_{im;n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \]

The transformation of \( U^m; n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma^m_{n \ell} \) is that of general 2nd-rank tensor \( T^m_n \)

The transformation of \( U^m, n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible.

But, still need to write \( \frac{\partial \bar{U}^m}{\partial \bar{q}^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[ \frac{\partial \bar{U}^m}{\partial \bar{q}^n} = \frac{\partial \bar{q}^m}{\partial q^m} \frac{\partial q^n}{\partial q^n} \frac{\partial U^m}{\partial q^n} \] holds if and only if \( \frac{\partial}{\partial q^n} \left( \frac{\partial \bar{q}^m}{\partial q^m} \right) = 0 \)

1\textsuperscript{st} term is OK, but 2\textsuperscript{nd} term is zero only if Jacobian is constant matrix!
\[
\frac{\partial \left( E_m \cdot E_n \right)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i}
\]

\[
\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}
\]

\[
- \frac{\partial g_{mi}}{\partial q^n} = \Gamma_{nm;i} - \Gamma_{in;m} \quad \text{(switched i } \leftrightarrow \text{n)}
\]

\[
\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i } \leftrightarrow \text{m)}
\]

The transformation of \( U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot E_m \) is that of general 2nd-rank tensor \( T^m_{;n} \).

The transformation of \( U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma^m_{n;\ell} \) is NOT that simple. At first it looks possible.

But, still need to write \( \frac{\partial \tilde{U}^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[
\frac{\partial \tilde{U}^m}{\partial q^n} = \frac{\partial q^m}{\partial q^n} \frac{\partial U^m}{\partial q^n} \quad \text{holds if and only if } \frac{\partial}{\partial q^n} \left( \frac{\partial q^m}{\partial q^n} \right) = 0
\]

Otherwise, \( U^m_{;n} \) needs “correction” \( U^\ell \Gamma^m_{n;\ell} \).
What's a tensor? What's not?

\[ \frac{\partial (E_m \cdot E_n)}{\partial q^i} = \frac{\partial E_m}{\partial q^i} \cdot E_n + E_m \cdot \frac{\partial E_n}{\partial q^i} \]

\[ \frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im; n} + \Gamma_{in; m} \]

\[ \frac{\partial g_{ni}}{\partial q^m} = -\Gamma_{n;m;i} - \Gamma_{i;n;m} \quad \text{(switched i} \leftrightarrow \text{n)} \]

\[ \frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mni} \quad \text{(switched i} \leftrightarrow \text{m)} \]

The transformation of \( U^m; n = \frac{\partial U}{\partial q^n} \cdot E_m \)

\[ \bar{U}^m; n = \frac{\partial \bar{U}}{\partial q^\bar{n}} \cdot \bar{E}^m = \frac{\partial \bar{U}}{\partial q^n} \frac{\partial q^n}{\partial q^\bar{n}} \cdot \bar{E}^m = \frac{\partial \bar{q}^\bar{m}}{\partial q^n} \frac{\partial q^n}{\partial q^\bar{m}} \cdot E_m \]

The transformation of \( U^m; n = \frac{\partial U^m}{\partial q^n} + U^\ell \Gamma^m_{n\ell} \) is that of general 2nd-rank tensor \( T^m_{n\bar{n}} \)

\[ \bar{T}^m_{n\bar{n}} = \frac{\partial \bar{q}^\bar{m}}{\partial q^\bar{n}} \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial U^m}{\partial q^n} \]

The transformation of \( U^m; n = \frac{\partial U^m}{\partial q^n} \) is NOT that simple. At first it looks possible. \[ \frac{\partial \bar{U}^m}{\partial q^\bar{n}} = \frac{\partial \bar{U}^m}{\partial q^n} = \frac{\partial q^n}{\partial q^\bar{n}} = \frac{\partial U^m}{\partial q^n} \]

But, still need to write \( \frac{\partial \bar{U}^m}{\partial q^n} \) in terms of \( \frac{\partial U^m}{\partial q^n} \).

\[ \frac{\partial \bar{U}^m}{\partial q^\bar{n}} = \frac{\partial \bar{q}^\bar{m}}{\partial q^\bar{n}} \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial U^m}{\partial q^n} \]

\[ \frac{\partial \bar{U}^m}{\partial q^\bar{n}} = \frac{\partial \bar{q}^\bar{m}}{\partial q^n} \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial U^m}{\partial q^n} + U^m \frac{\partial q^n}{\partial q^\bar{n}} \frac{\partial q^n}{\partial q^\bar{n}} \]

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

Otherwise, \( U^m; n \) needs “correction” \( U^\ell \Gamma^m_{n\ell} \). And, that \( U^\ell \Gamma^m_{n\ell} \) cannot be a \( T^m_{n\bar{n}} \)-tensor either!
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)
Riemann equations of motion (No explicit $t$-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

$$T = \frac{1}{2} M_{jk} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m + \frac{\partial x^j}{\partial t} \right) \left( \frac{\partial x^k}{\partial q^n} \dot{q}^n + \frac{\partial x^k}{\partial t} \right)$$

*All explicit-$t$-dependent terms are zero*
**Riemann equations of motion** (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy

$$T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$$

to GCC

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell}$$

All explicit-t-dependent terms are zero.
Riemann equations of motion (No explicit \( t \)-dependence and fixed GCC)

Kinetic metric \( \gamma_{mn} \) is a covariant tensor transform of an original Cartesian inertia tensor \( M_{ij} \)

\[
\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}
\]

Converts Cartesian kinetic energy \( T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \) to GCC \( T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n \)

Lagrange equations for fixed GCC convert to tensor form

\[
F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)
\]

1\textsuperscript{st} term involves \textit{covariant momentum} \( p_\ell \).

\[
p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n
\]

All explicit-\( t \)-dependent terms are zero.
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^\ell} \right) - \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$

1st term involves covariant momentum $p_\ell$. Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

\[ \gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \]

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Riemann equations of motion

\[ \frac{\partial}{\partial t} \gamma_{mn} \partial_{x^j} \partial_{x^k} \dot{q}^m \dot{q}^n = \Sigma_{\text{all explicit-t-dependent terms are zero}} \]

Lagrange equations for fixed GCC convert to tensor form

\[ F_\ell = \frac{d}{dt} \left( \gamma_{\ell m} \dot{q}^m \dot{q}^n \right) \]

1st term involves \textit{covariant momentum} $p_\ell$:

\[ p_\ell = \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n \]

Inverse \textit{contravariant kinetic metric} $\gamma^{mn}$ gives velocity $\dot{q}^n$

\[ \dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n \]

Canonical Lagrange equations valid for \textit{all} GCC, fixed or explicit in time $t$:

\[ F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} \]

The “4-wheel-drive garbage truck”

Tuesday, October 30, 2012
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_\ell} \right) - \frac{\partial T}{\partial q_\ell} = \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q_\ell}$$

$1^{st}$ term involves covariant momentum $p_\ell$.

Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial \dot{q}^n} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q_\ell} \dot{q}^m \dot{q}^n$$

All explicit-t-dependent terms are zero.
Riemann equations of motion (No explicit \( t \)-dependence and fixed GCC)

Kinetic metric \( \gamma_{mn} \) is a covariant tensor transform of an original Cartesian inertia tensor \( M_{ij} \)

\[
\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}
\]

Converts Cartesian kinetic energy \( T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \) to GCC \( T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n \)

Lagrange equations for fixed GCC convert to tensor form

\[
F_\ell = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^\ell} \right) - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} d \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)
\]

1\(^{st}\) term involves covariant momentum \( p_\ell \).

Inverse contravariant kinetic metric \( \gamma^{mn} \) gives velocity \( \dot{q}^n \)

\[
p_\ell \equiv \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma^{\ell n} \dot{q}^n
\]

Canonical Lagrange equations valid for all GCC, fixed or explicit in time \( t \):

Following is for fixed GCC only:

\[
F_\ell = \frac{d}{dt} \left( \gamma^{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial \dot{q}^\ell} \dot{q}^m \dot{q}^n \gamma^{\ell n} + \dot{q}^n \frac{d\gamma^{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial \dot{q}^\ell} \dot{q}^m \dot{q}^n
\]

Time derivative of kinetic metric is expanded by chain rule.

\[
\frac{d\gamma^{\ell n}}{dt} = \frac{\partial \gamma^{\ell n}}{\partial q^m} \dot{q}^m
\]

All explicit-\( t \)-dependent terms are zero.

The “4-wheel-drive garbage truck”
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$

1st term involves covariant momentum $p_\ell$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial \dot{q}^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n$$

Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d}{dt} \gamma_{\ell n} + \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$

Time derivative of kinetic metric is expanded by chain rule.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{\partial}{\partial q^m} \gamma_{\ell n} \dot{q}^m - \frac{1}{2} \frac{\partial}{\partial q^\ell} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right)$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q_m} \frac{\partial x^k}{\partial q_n}$$

Converts Cartesian kinetic energy

$$T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$$

to GCC

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \gamma_{\ell n} \dot{q}^n - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q_\ell} \dot{q}^m \dot{q}^n$$

1st term involves covariant momentum $p_\ell$.

Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \gamma_{\ell n} \dot{q}^n - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q_\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \frac{d}{dt} \gamma_{\ell n} \dot{q}^n$$

The "4-wheel-drive garbage truck"

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d}{dt} \gamma_{\ell n} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Rearrange to expose Christoffel coefficients:

$$\Gamma_{\ell mn} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^\ell} + \frac{\partial g_{mn}}{\partial q^\ell} - \frac{\partial g_{mn}}{\partial q^\ell} \right)$$

The “4-wheel-drive garbage truck”
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_{\ell} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^\ell} \right) - \frac{1}{2} \frac{\partial T}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

The $1^{st}$ term involves covariant momentum $p_\ell$. Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$.

Following is for fixed GCC only:

$$F_{\ell} = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im:n} = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^l} + \frac{\partial g_{im}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^\ell} \right) - \frac{1}{2} \frac{\partial T}{\partial q^\ell} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

1st term involves covariant momentum $p_\ell$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{d}{dt} \left( \gamma_{mn} \dot{q}^m \dot{q}^n \right) = \gamma_{\ell n} \dot{q}^n$$

Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \frac{d}{dt} \gamma_{\ell n} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im:n}^\ell = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^\ell} + \frac{\partial g_{im}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

This gives covariant Riemann equations

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \Gamma_{im:n}^\ell \dot{q}^m \dot{q}^n$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[ \frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

All explicit-t-dependent terms are zero

The "4-wheel-drive garbage truck"
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric $\gamma_{mn}$ is a covariant tensor transform of an original Cartesian inertia tensor $M_{ij}$

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^n} \dot{q}^n \dot{q}^m$$

1st term involves covariant momentum $p_\ell$. Inverse contravariant kinetic metric $\gamma^{mn}$ gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time $t$:

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} \left( \gamma_{\ell n} \dot{q}^n \right) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \frac{d \gamma_{\ell n}}{dt} + \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^n} \dot{q}^n \dot{q}^m$$

Time derivative of kinetic metric is expanded by chain rule.

$$\gamma_{\ell n} \ddot{q}^n + \frac{d \gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Rearrange to expose Christoffel coefficients:

$$\Gamma_{im;}^n = \frac{1}{2} \left( \frac{\partial g_{mn}}{\partial q^\ell} + \frac{\partial g_{im}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

This gives covariant Riemann equations and contravariant Riemann equations.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;}^n \dot{q}^m \dot{q}^n$$

All explicit-t-dependent terms are zero

The “4-wheel-drive garbage truck”

$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}$

Tuesday, October 30, 2012
Riemann equations of motion (No explicit t-dependence and fixed GCC)

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[ \frac{\partial x}{\partial \rho} = \cos \phi, \quad \frac{\partial y}{\partial \rho} = \sin \phi, \quad \frac{\partial z}{\partial \rho} = 0 \]

\[ \frac{\partial x}{\partial \phi} = -\rho \sin \phi, \quad \frac{\partial y}{\partial \phi} = \rho \cos \phi, \quad \frac{\partial z}{\partial \phi} = 0 \]

\[ \frac{\partial x}{\partial z} = 0, \quad \frac{\partial y}{\partial z} = 0, \quad \frac{\partial z}{\partial z} = 1 \]

$J = \begin{pmatrix} \cos \phi & -\rho \sin \phi & 0 \\ \rho \cos \phi & \sin \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$J^{-1} = \begin{pmatrix} \cos \phi & \rho \cos \phi & 0 \\ -\rho \sin \phi & \sin \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

$\mathbf{E}_\rho$ $\mathbf{E}_\phi$ $\mathbf{E}_z$

$F = F_\rho \mathbf{E}_\rho + F_\phi \mathbf{E}_\phi + F_z \mathbf{E}_z$

$= f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$
\begin{align*}
\langle J \rangle &= \begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix},
\langle K \rangle &= \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \cos \phi & 0 \\
0 & 0 & \frac{\partial \rho}{\partial z} = 1
\end{pmatrix}
\end{align*}
$$

\[\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow\]

\[\mathbf{E}_\rho \quad \mathbf{E}_\phi \quad \mathbf{E}_z \quad = \langle J^{-1} \rangle\]

Covariant forces

\[F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0\]

\[F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0\]

\[F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)


\[
\begin{align*}
\left\langle J \right\rangle &= \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & 0 \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1 
\end{pmatrix},
\left\langle K \right\rangle &= \begin{pmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & 0 \\
-\frac{\sin \phi}{\rho} & \frac{\cos \phi}{\rho} & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1 
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\therefore x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{align*}
\]

**Covariant forces**

\[
\begin{align*}
F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\
F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\end{align*}
\]

**Covariant kinetic metric**

\[
\begin{align*}
\gamma_{\rho\rho} &= m \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^i}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m E_\rho \cdot E_\rho = m (\cos^2 \phi + \sin^2 \phi) = m \\
\gamma_{\rho\phi} &= m \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^i}{\partial \rho} \frac{\partial x^j}{\partial \phi} = m E_\rho \cdot E_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\
\gamma_{\phi\phi} &= m \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^i}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m E_\phi \cdot E_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\
\gamma_{zz} &= m \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^i}{\partial z} \frac{\partial x^j}{\partial z} = m E_z \cdot E_z = m
\end{align*}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\mathbf{J} = \begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial y}{\partial \rho} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \phi} = \sin \phi & \frac{\partial x}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[
\mathbf{\Gamma} = \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[
\mathbf{J}^{-1} = \begin{pmatrix}
\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z
\end{pmatrix}
\]

Covariant forces

\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]

\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]

\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Covariant kinetic metric

\[
\gamma^{\rho\rho} = m \left( \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \rho} \right) = m \cos^2 \phi + \sin^2 \phi = m
\]

\[
\gamma^{\phi\phi} = m \left( \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} \right) = m \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi = m \rho^2
\]

\[
\gamma^{zz} = m \left( \frac{\partial x}{\partial z} \frac{\partial x}{\partial z} \right) = m
\]

Contravariant kinetic metric

\[
\gamma_{\rho\rho} = 1 / m
\]

\[
\gamma_{\phi\phi} = 1 / \left( m \rho^2 \right)
\]

\[
\gamma_{zz} = 1 / m
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}, \quad \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix} \leftarrow E^\rho \\
\begin{pmatrix}
x = \rho \cos \phi \\
y = \rho \sin \phi \\
z = z
\end{pmatrix} \leftarrow E^\phi
\]

\[\mathbf{\nabla} J = \frac{\partial x}{\partial \rho} = \cos \phi \quad \frac{\partial x}{\partial \phi} = -\rho \sin \phi \quad \frac{\partial x}{\partial z} = 0
\]

\[\mathbf{\nabla} y = \frac{\partial y}{\partial \rho} = \sin \phi \quad \frac{\partial y}{\partial \phi} = \rho \cos \phi \quad \frac{\partial y}{\partial z} = 0
\]

\[\mathbf{\nabla} z = \frac{\partial z}{\partial \rho} = 0 \quad \frac{\partial z}{\partial \phi} = \rho \cos \phi \quad \frac{\partial z}{\partial z} = 1
\]

\[J^{-1} = \begin{pmatrix}
\frac{\partial \rho}{\partial \rho} = \rho \\
\frac{\partial \phi}{\partial \rho} = 0 \\
\frac{\partial z}{\partial \rho} = 0
\end{pmatrix}
\]

\[\mathbf{E}^\rho = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z
\]

\[\mathbf{E}^\phi = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z
\]

\[\mathbf{E}^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z
\]

Covariant forces
\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]
\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0
\]
\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Lagrangian
\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\rho}^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

Covariant kinetic metric
\[
\gamma_{\rho\rho} = m  \\
\gamma_{\phi\phi} = \frac{1}{m} \\
\gamma_{zz} = \frac{1}{m}
\]

Contravariant kinetic metric
\[
\gamma^{\rho\rho} = \frac{1}{m} \\
\gamma^{\phi\phi} = \frac{1}{m^2} \\
\gamma^{zz} = \frac{1}{m}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = \frac{1}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\rho}{\sin \phi} & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[\left\langle \mathbf{J} \right\rangle = \begin{pmatrix} \mathbf{E}_\rho \\ \mathbf{E}_\phi \\ \mathbf{E}_z \end{pmatrix} = (\mathbf{J}^{-1}) \]

**Covariant forces**

\[
\begin{align*}
F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\
F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\end{align*}
\]

**Lagrangian**

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

\[
\begin{align*}
\dot{\rho} &= \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \frac{\partial}{\partial \rho} \\
\dot{\phi} &= \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \frac{\partial}{\partial \phi} \\
\dot{z} &= \frac{\partial T}{\partial z} = \gamma_{z\phi} \frac{\partial}{\partial z}
\end{align*}
\]

\[
\begin{align*}
p_\rho &= \dot{\rho} = \gamma_{\rho\rho} p_\rho \\
p_\phi &= \dot{\phi} = \gamma_{\phi\phi} p_\phi \\
p_z &= \dot{z} = \gamma_{z\phi} p_z
\end{align*}
\]

\[
\begin{align*}
p_\rho &= m \dot{\rho} \\
p_\phi &= m \rho^2 \dot{\phi} \\
p_z &= m \dot{z}
\end{align*}
\]

**(Covariant kinetic metric)**

\[
\gamma_{\rho\rho} = \frac{1}{m} \\
\gamma_{\phi\phi} = \frac{1}{(m \rho^2)} \\
\gamma_{zz} = \frac{1}{m}
\]

**(Contravariant kinetic metric)**
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\langle J \rangle = \begin{bmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial y}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial x}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1 \\
\end{bmatrix}, \quad \langle K \rangle = \begin{bmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \rho}{\partial x} = -\sin \phi & \frac{\partial \rho}{\partial y} = \cos \phi & 0 \\
0 & 0 & \frac{\partial \rho}{\partial z} = 1 \\
\end{bmatrix}
\]

\[x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z\]

Covariant forces

\[
\begin{align*}
F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\
F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\end{align*}
\]

Covariant kinetic metric

\[
\gamma_{\rho \rho} = m \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \rho} = m E_\rho \cdot E_\rho = m (\cos^2 \phi + \sin^2 \phi) = m \\
\gamma_{\phi \phi} = m \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} = m E_\rho \cdot E_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\
\gamma_{z z} = m \frac{\partial x}{\partial z} \frac{\partial x}{\partial z} = m E_z \cdot E_z = m
\]

Lagrangian

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

\[
p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho} = m \dot{\rho} \\
p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi \phi} \dot{\phi} = m \rho^2 \dot{\phi} \\
p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{z z} \dot{z} = m \dot{z}
\]

Contravariant kinetic metric

\[
\gamma^{\rho \rho} = 1 / m \\
\gamma^{\phi \phi} = 1 / (m \rho^2) \\
\gamma^{z z} = 1 / m
\]

Contravariant momenta

\[
p^\rho = \dot{\rho} \\
p^\phi = \dot{\phi} \\
p^z = \dot{z}
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}, \quad \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[\mathbf{e}_z = \mathbf{E}^z = F^\rho E^\rho + F^\phi E^\phi + F_z E^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z\]

\[E^\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho} \quad \gamma_{\rho \rho} = m \quad \gamma_{\phi \phi} = \frac{1}{m \rho^2} \quad \gamma_{zz} = \frac{1}{m} \]

\[p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho} \quad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi \phi} \dot{\phi} \quad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} \]

\[p^\rho = \dot{\rho} \quad p^\phi = \dot{\phi} \quad p^z = \dot{z}\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial \rho}{\partial \rho} = \cos \phi & \frac{\partial \rho}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial \phi}{\partial \rho} = \sin \phi & \frac{\partial \phi}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
, \quad
\begin{pmatrix}
\frac{\partial \rho}{\partial \phi} = \cos \phi & \frac{\partial \rho}{\partial \rho} = \sin \phi & 0 \\
\frac{\partial \phi}{\partial \phi} = -\sin \phi & \frac{\partial \phi}{\partial \rho} = \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}
\]

\[x = \rho \cos \phi \quad y = \rho \sin \phi \quad z = z\]

Covariant forces

\[F_\rho = f_x \frac{\partial}{\partial \rho} + f_y \frac{\partial}{\partial \phi} + f_z \frac{\partial}{\partial z} = f_x \cos \phi + f_y \sin \phi + 0 \]

\[F_\phi = f_x \frac{\partial}{\partial \phi} + f_y \frac{\partial}{\partial \phi} + f_z \frac{\partial}{\partial z} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \]

\[F_z = f_x \frac{\partial}{\partial z} + f_y \frac{\partial}{\partial z} + f_z \frac{\partial}{\partial z} = 0 + 0 + f_z \]

Lagrangian

\[T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2\]

Lagrange and the Riemann covariant force equations

\[F_\rho = \frac{d \rho}{dt} - \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n\]

Contravariant kinetic metric

\[\gamma^{\rho \rho} = 1 / \rho \]

\[\gamma^{\phi \phi} = 1 / (m \rho^2)\]

\[\gamma^{zz} = 1 / m\]

Contravariant momenta

\[p_\rho = \gamma_{\rho \rho} \dot{\rho} = m \rho \dot{\rho}\]

\[p_\phi = \gamma_{\phi \phi} \dot{\phi} = m \rho^2 \dot{\phi}\]

\[p_z = \gamma_{zz} \dot{z} = m \dot{z}\]

Only three non-zero Christoffel coefficients appear, and only two are independent.
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi \\
\frac{\partial y}{\partial \phi} = -\sin \phi \\
0
\end{pmatrix}
\]

Contravariant forces

\[
F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0
\]

\[
F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = f_x \sin \phi + f_y \rho \cos \phi + 0
\]

\[
F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z
\]

Covariant forces

\[
\{F\} = \begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi \\
\frac{\partial y}{\partial \rho} = \rho \cos \phi \\
0
\end{pmatrix}, \quad \{\kappa\} = \begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi \\
\frac{\partial \rho}{\partial y} = -\sin \phi \\
0
\end{pmatrix}
\]

Lagrangian

\[
T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2
\]

Lagrange and the Riemann covariant force equations

\[
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q_\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[
\Gamma_{\phi \rho;\phi} = -m \rho
\]

\[
\Gamma_{\rho \rho;\phi} = m \rho
\]

Contravariant kinetic metric

\[
\gamma^{\rho \rho} = 1 / m
\]

\[
\gamma^{\phi \phi} = 1 / \left( m \rho^2 \right)
\]

\[
\gamma^{zz} = 1 / m
\]

Contravariant momenta

\[
p^\rho = \dot{\rho}
\]

\[
p^\phi = \dot{\phi}
\]

\[
p^z = \dot{z}
\]

Covariant momenta

\[
p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho \rho} \dot{\rho} = m \dot{\rho}
\]

\[
p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi \phi} \dot{\phi} = m \rho \dot{\phi}
\]

\[
p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z}
\]
Lagrangian
\[ \lvert J \rvert = \begin{bmatrix}
\frac{\partial x}{\partial \rho} &= \cos \phi \\
\frac{\partial y}{\partial \rho} &= \sin \phi \\
0 &= 0 \\
0 &= 0
\end{bmatrix}, \quad \lvert K \rvert = \begin{bmatrix}
\frac{\partial \rho}{\partial x} &= \cos \phi \\
\frac{\partial \rho}{\partial y} &= \sin \phi \\
0 &= 0 \\
0 &= 0
\end{bmatrix} \]

\[ \begin{align*}
\uparrow & \quad \uparrow & \quad \uparrow \\
E_\rho & \quad E_\phi & \quad E_z
\end{align*} = \lvert J^{-1} \rvert \]

Covariant forces
\[ F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \]
\[ F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \]
\[ F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z \]

Lagrangian
\[ T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 \]

Lagrange and the Riemann covariant force equations
\[ F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q_\ell} = \gamma_{\ell n} \dot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n \]

Only three non-zero Christoffel coefficients appear, and only two are independent.
\[ \begin{align*}
F_\rho &= \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho \rho} \dot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n \\
F_\phi &= \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi \phi} \dot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n \\
\end{align*} \]

Contravariant equations are acceleration equations.
\[ F^\rho = \gamma^{\rho \rho} F_\rho = \ddot{q}^\rho + \Gamma_{mn}^{\rho} \dot{q}^m \dot{q}^n \]
\[ F^\phi = \gamma^{\phi \phi} F_\phi = \ddot{q}^\phi + \Gamma_{mn}^{\phi} \dot{q}^m \dot{q}^\phi \]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\begin{pmatrix}
\frac{\partial x}{\partial \rho} = \cos \phi, & \frac{\partial x}{\partial \phi} = -\rho \sin \phi, & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi, & \frac{\partial y}{\partial \phi} = \rho \cos \phi, & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1
\end{pmatrix}, \quad
\begin{pmatrix}
\frac{\partial \rho}{\partial x} = \cos \phi, & \frac{\partial \rho}{\partial y} = \sin \phi, & 0 \\
\frac{\partial \phi}{\partial x} = \sin \phi, & \frac{\partial \phi}{\partial y} = -\rho \cos \phi, & \rho \\
0 & 0 & \frac{\partial \gamma}{\partial z} = 1
\end{pmatrix}
\]

\[\left\langle \gamma \right\rangle = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\rho}^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2\\
\]

Lagrangian and the Riemann covariant force equations

\[F_\ell = \frac{dp_\ell}{dt} \frac{d}{dq_\ell} = \gamma_{\ell \kappa} \dot{q}^\kappa + \Gamma_{\ell \kappa \rho} \dot{q}^\kappa \dot{q}^\rho
\]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[F_\rho = \frac{dp_\rho}{dt} - \frac{d}{dq_\rho} = \gamma_{\rho \phi} \dot{q}^\phi + \Gamma_{\rho \phi \rho} \dot{q}^\phi \dot{q}^\rho
\]

\[= \frac{d}{dt} \left( m \dot{\rho}^2 \right) - m \dot{\rho} \dot{\phi}^2 \quad \text{so:} \quad \Gamma_{\phi \phi \rho} = -m \rho
\]

Contravariant equations are acceleration equations.

\[F^\rho = \gamma^{\rho \phi} F_\phi = \ddot{q}^\rho + \gamma_{\rho \phi} \dot{q}^\phi + \Gamma_{\rho \phi \rho} \dot{q}^\phi \dot{q}^\rho
\]

\[= \ddot{\rho} - \rho \dot{\phi}^2 \quad \text{so:} \quad \Gamma_{\phi \phi \rho} = -\rho \quad \gamma^{\rho \phi} = 1 / m
\]

\[F^\phi = \gamma^{\phi \phi} F_\rho = \ddot{q}^\phi + \gamma_{\phi \phi} \dot{q}^\phi + \Gamma_{\phi \phi \phi} \dot{q}^\phi \dot{q}^\phi
\]

\[= \ddot{\phi} + \rho \dot{\phi}^2 \rho \quad \text{so:} \quad \Gamma_{\phi \rho \phi} = 1 / \rho = \Gamma_{\phi \phi} \quad \gamma^{\phi \phi} = 1 / \left( m \rho^2 \right)
\]
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

\[
\left\{ \begin{array}{ccc}
\frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\
\frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1 \\
\end{array} \right. \]

\[\{ \kappa \} = \left[ \begin{array}{ccc}
\frac{\partial \rho}{\partial \rho} = \cos \phi & \frac{\partial \rho}{\partial \phi} = -\sin \phi & 0 \\
\frac{\partial \phi}{\partial \rho} = -\sin \phi & \frac{\partial \phi}{\partial \phi} = \cos \phi & 0 \\
0 & 0 & \frac{\partial z}{\partial z} = 1 \\
\end{array} \right] \]

\[\left( J^{-1} \right) \]

Covariant forces

\[F_\rho = f_x \frac{\partial}{\partial \rho} + f_y \frac{\partial}{\partial \phi} + f_z \frac{\partial}{\partial z} = f_x \cos \phi + f_y \sin \phi + 0 \]

\[F_\phi = f_x \frac{\partial}{\partial \rho} + f_y \frac{\partial}{\partial \phi} + f_z \frac{\partial}{\partial z} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \]

\[F_z = f_x \frac{\partial}{\partial \rho} + f_y \frac{\partial}{\partial \phi} + f_z \frac{\partial}{\partial z} = 0 + 0 + f_z \]

Lagrange and the Riemann covariant force equations

\[T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 \]

\[\rho \gamma_{\rho \rho} \dot{\rho} + \gamma_{\rho \phi} \dot{\phi} + \gamma_{\rho z} \dot{z} + \Gamma_{\rho \rho \rho} \dot{q}^\rho \dot{q}^\rho \dot{q}^\rho + \Gamma_{\rho \phi \phi} \dot{q}^\rho \dot{q}^\phi \dot{q}^\phi + \Gamma_{\rho z z} \dot{q}^\rho \dot{q}^z \dot{q}^z \]

Only three non-zero Christoffel coefficients appear, and only two are independent.

\[F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial q_\rho} = \gamma_{\rho \rho} \ddot{\rho} + \gamma_{\rho \phi} \ddot{\phi} + \gamma_{\rho z} \ddot{z} + \Gamma_{\rho \rho \rho} \dot{q}^\rho \dot{q}^\rho \dot{q}^\rho + \Gamma_{\rho \phi \phi} \dot{q}^\rho \dot{q}^\phi \dot{q}^\phi + \Gamma_{\rho z z} \dot{q}^\rho \dot{q}^z \dot{q}^z \]

Contravariant equations are acceleration equations. $F^k = \gamma^{ik} F_j = \dot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$

\[F^\rho = \gamma^{\rho \rho} F_\rho = \ddot{\rho} - \rho \dot{\phi}^2 \quad \text{so: } \Gamma^{\rho \rho}_\rho = -\rho \gamma^{\rho \rho} = 1 / m \]

\[\ddot{\rho} = F^\rho + \rho \dot{\phi}^2 \quad \text{(Centrifugal acceleration)} \]

\[\ddot{\phi} = F^\phi - 2 \dot{\rho} \dot{\phi} / \rho \quad \text{(Coriolis acceleration)} \]

Contravariant kinetic metric

\[\gamma^{\rho \rho} = 1 / m \]

\[\gamma^{\phi \phi} = 1 / (m \rho^2) \]

\[\gamma^{zz} = 1 / m \]

Contravariant momenta

\[p^\rho = \ddot{\rho} \]

\[p^\phi = \ddot{\phi} \]

\[p^z = \ddot{z} \]
Rewriting GCC Lagrange equations:

\[
\begin{align*}
\dot{p}_r & \equiv \frac{dp_r}{dt} = M \ddot{r} \\
& \quad \text{Centrifugal (center-fleeing) force equals total} \\
& \quad \text{Centripetal (center-pulling) force}
\end{align*}
\]

\[
\begin{align*}
= M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \\
& \quad \text{potential } U \text{ has no explicit } \phi \text{-dependence}
\end{align*}
\]

Conventional forms

radial force: \( M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \)

angular force or torque: \( M r^2 \ddot{\phi} = -2 M r \dot{r} \dot{\phi} - \frac{\partial U}{\partial \phi} \)

Field-free (U=0)

radial acceleration: \( \ddot{r} = r \dot{\phi}^2 \)

angular acceleration: \( \ddot{\phi} = -2 \frac{\dot{r} \dot{\phi}}{r} \)

Coriolis acceleration with \( \dot{\phi} > 0 \) and \( \dot{r} < 0 \)

Effect on Northern Hemisphere local weather

Inward flow to pressure Low \( \dot{r} < 0 \)

...makes wind turn to the right

Cyclonic flow around lows
Separation of GCC Equations: Effective Potentials

Small radial oscillations
Cycloid vs Pendulum
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!}) \]

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct}) \]
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \] (Numerically correct ONLY!)

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Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

\( H \) has no explicit \( \phi \)-dependence and the \( \phi \)-momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \]  

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p^2 + \frac{1}{2m \rho^2} p^2 + \frac{1}{2m} p_z^2 + V \]  

(Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( V = V(\rho) \))

\( H \) has no explicit \( \phi \)-dependence and the \( \phi \)-momenta is constant.

\[ m \rho^2 \phi = p_\phi = const. = \mu \]

If \( H \) has no explicit \( z \)-dependence then the \( z \)-momenta is constant, too.

\[ m \dot{z} = p_z = const. = k \]
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]  

(\text{Numerically correct ONLY!})

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p^2 + \frac{1}{2m} \rho^2 p_\phi^2 + \frac{1}{2m} z^2 + V \]  

(\text{Formally and Numerically correct})

Potential \( V \) is \textit{isotropic} (cylindrical) function of radius \( \rho \). (\( V = V(\rho) \))

\( H \) has no explicit \( \dot{\phi} \)–dependence and the \( \dot{\phi} \)–momenta is constant.

\[ m \dot{\rho}^2 \phi = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} \dot{z}^2 + V \]  
\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_{\rho}^2 + \frac{1}{2m} p_{\phi}^2 + \frac{1}{2m} p_z^2 + V \]  

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]  

(Let \( k = 0 \))
Separation of GCC Equations: Effective Potentials

\[
H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad \text{(Numerically correct ONLY!)}
\]

\[
= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad \text{(Formally and Numerically correct)}
\]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)

\( H \) has no explicit \( \phi \)–dependence and the \( \phi \)–momenta is constant.

\[
m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu
\]

\[
H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}
\]

Symmetry reduces problem to a one-dimensional form.

\[
H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.}
\]

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[
m \dot{z} = p_z = \text{const.} = k
\]

(Let \( k = 0 \))
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \]  
\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2} m p_\rho^2 + \frac{1}{2 m \rho^2} p_\phi^2 + \frac{1}{2} m p_z^2 + V \]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)
\( H \) has no explicit \( \phi \)–dependence and the \( \phi \)–momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2 m} p_\rho^2 + \frac{\mu^2}{2 m \rho^2} + \frac{k^2}{2 m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.
\[ H = \frac{1}{2 m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

An effective potential \( V^{\text{eff}}(\rho) \) has a centrifugal barrier.
\[ V^{\text{eff}}(\rho) = \frac{\mu^2}{2 m \rho^2} + V(\rho) \]

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.
\[ m \dot{z} = p_z = \text{const.} = k \]

(Let \( k = 0 \))
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]

( Numerically correct ONLY! )

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p^2 + \frac{1}{2m} \rho^2 \dot{\phi}^2 + \frac{1}{2m} p^2 + V \]

( Formally and Numerically correct)

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

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\[ H = \frac{1}{2m} p^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

An effective potential \( V^{\text{eff}}(\rho) \) has a centrifugal barrier.

\[ V^{\text{eff}}(\rho) = \frac{\mu^2}{2m \rho^2} + V(\rho) \]

Velocity relations:

\[ \dot{\phi} = \mu / (m \rho^2) \]

\[ \dot{\rho} = \frac{d \rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm \sqrt{\frac{2m}{m}(E - V^{\text{eff}}(\rho))} \]

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]

(Let \( k = 0 \) )
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \ddot{q}^m \ddot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]  

(\text{Numerically correct ONLY!})

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_{\rho}^2 + \frac{1}{2m} \rho^2 p_{\phi}^2 + \frac{1}{2m} p_z^2 + V \]  

(\text{Formally and Numerically correct})

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)

\( H \) has no explicit \( \phi \)–dependence and the \( \phi \)–momenta is constant.

\[ m \rho^2 \dot{\phi} = p_{\phi} = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_{\rho}^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p_{\rho}^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

An effective potential \( V^{\text{eff}}(\rho) \) has a centrifugal barrier.

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Velocity relations:

\[ \dot{\phi} = \frac{\mu}{m \rho^2} \]

\[ \dot{\rho} = \frac{dp_{\rho}}{dt} = \frac{\partial H}{\partial p_{\rho}} = \frac{p_{\rho}}{m} = \pm \sqrt{\frac{2}{m} \left( E - V^{\text{eff}}(\rho) \right)} \]

Equations solved by a quadrature integral for time versus radius.

\[ \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m} \left( E - V^{\text{eff}}(\rho) \right)}} = \left( \text{Travel time } \rho_0 \text{ to } \rho_1 \right) = t_1 - t_0 \]
Separation of GCC Equations: Effective Potentials

Small radial oscillations
Cycloid vs Pendulum
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\frac{dV^{\text{eff}}(\rho)}{d\rho} \bigg|_{\rho_0} = 0, \quad \text{with:} \quad \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0} > 0.$$  

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{\text{eff}}(\rho) = V^{\text{eff}}(\rho_0) + 0 + \frac{1}{2}(\rho - \rho_0)^2 \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0}$$  

Stable flat $\left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0} > 0$

Unstable flat $\left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_0} < 0$

Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point
**Small radial oscillations**

Stable minimal-energy radius will satisfy a zero-slope equation.

\[
\left. \frac{dV_{\text{eff}}(\rho)}{d\rho} \right|_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} > 0.
\]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[
V_{\text{eff}}(\rho) = V_{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}
\]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[
k_{\text{eff}} = \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}
\]

\[
\omega_{\rho_{\text{stable}}} = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V_{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}}
\]
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[ \frac{dV^{\text{eff}}(\rho)}{d\rho} \bigg|_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \frac{d^2V^{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} > 0. \]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[ V^{\text{eff}}(\rho) = V^{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \frac{d^2V^{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} \]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[ k^{\text{eff}} = \frac{d^2V^{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} \quad \omega_{\rho_{\text{stable}}} = \sqrt{\frac{k^{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \frac{d^2V^{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}}} \]

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

\[ \frac{\omega_{\rho_{\text{stable}}}}{\omega_{\phi}} = \frac{\omega_{\rho_{\text{stable}}}}{\phi'(\rho_{\text{stable}})} = \frac{n_{\rho}}{n_{\phi}} \iff \text{Orbit is closed-periodic} \]
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[
\left. \frac{dV^{\text{eff}}(\rho)}{d\rho} \right|_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} > 0.
\]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[
V^{\text{eff}}(\rho) = V^{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}.
\]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[
k^{\text{eff}} = \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}
\]

\[
\omega_{\rho_{\text{stable}}} = \sqrt{\frac{k^{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}}.
\]

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

\[
\frac{\omega_{\rho_{\text{stable}}}}{\omega_{\phi}} = \frac{\omega_{\rho_{\text{stable}}}}{\phi(\rho_{\text{stable}})} = \frac{n_{\rho}}{n_{\phi}} \Leftrightarrow \text{Orbit is closed-periodic}
\]

Some generic shapes resulting from various ratios \(n_{\rho} : n_{\phi}\)
<table>
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<tr>
<th>m:n</th>
<th>1:1</th>
<th>2:1</th>
<th>3:1</th>
<th>4:1</th>
<th>5:1</th>
<th>6:1</th>
</tr>
</thead>
<tbody>
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<td>3:2</td>
<td>4:2</td>
<td>5:2</td>
<td>6:2</td>
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</tr>
<tr>
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<td>2:3</td>
<td>3:3</td>
<td>4:3</td>
<td>5:3</td>
<td>6:3</td>
<td></td>
</tr>
<tr>
<td>1:4</td>
<td>2:4</td>
<td>3:4</td>
<td>4:4</td>
<td>5:4</td>
<td>6:4</td>
<td></td>
</tr>
<tr>
<td>1:5</td>
<td>2:5</td>
<td>3:5</td>
<td>4:5</td>
<td>5:5</td>
<td>6:5</td>
<td></td>
</tr>
</tbody>
</table>

(b) $\omega : \omega_\phi$ just below 1
prograde precession of nodes

(c) $\omega : \omega_\phi$ just below 2
prograde precession of nodes

$\omega_\phi = 1$
Separation of GCC Equations: Effective Potentials

Small radial oscillations

Cycloid vs Pendulum
time = 174.180
Θ = +1.384
ω = +1.000
E = +1.999
time = 53.940
Θ = -0.381
dΘ/dt = -1.933
E = +0.940