Lecture 19
Tue. 10.30.2012

Reimann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula
What's a tensor? What's not?

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Separation of GCC Equations: Effective Potentials

Small radial oscillations
Cycloid vs Pendulum

→ *Covariant derivative and Christoffel Coefficients* $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula
What's a tensor? What's not?

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}^{k}$

GCC q^m derivatives of vectors **U** are due to:

(1) changing U^m components   *(2)* curving GCC vectors $\mathbf{E_n}$.

$$\frac{\partial \mathbf{U}}{\partial q^i}=\frac{\partial}{\partial q^i}\left(U^j\mathbf{E}_j\right)=\boxed{\frac{\partial U^m}{\partial q^i}}\left(\mathbf{E}_m\right)+U^n\boxed{\frac{\partial \mathbf{E}_n}{\partial q^i}}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}^{k}$

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Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}^\ell$ or else $\mathbf{E}_m$

$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \Gamma_{in;\ell}\,\mathbf{E}^\ell$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}^{k}$

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$$\frac{\partial \mathbf{E}_\mathbf{n}}{\partial q^i} = \boxed{\Gamma_{in;\ell}}\,\mathbf{E}^\ell$$

Christoffel coefficients $\Gamma_{ij;k}$ *of the first kind*

defined by:

$$\boxed{\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}}$$

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i,n to n,i
symmetry
guaranteed here

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$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E_i}}{\partial q^n}$$

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$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E_i}}{\partial q^n}$$

Q: Do we need a third kind of Γ-coefficient or a Λ-coefficient?

(to differentiate <u>*contra*</u>variant-$\mathbf{E}^n$ or <u>*co*</u>variant U_n)

$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda_{im}^{n}\mathbf{E}^m \text{ , where: } \Lambda_{im}^{n} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}^{k}$

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$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda^n_{im}\mathbf{E}^m \text{ , where: } \Lambda^n_{im} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

A: NO! That Λ-coefficient is just a Γ-coefficient with a (-).

$$0 = \frac{\partial\left(\delta^n_m\right)}{\partial q^i} = \frac{\partial\left(\mathbf{E}^n \bullet \mathbf{E}_m\right)}{\partial q^i} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m + \mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

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$$\Gamma_{in;}^{\;\;\;m} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}^{\;\;\;m}$$

Any vector derivative can be expressed using $\Gamma_{ij;}^{k}$ in terms of $\mathbf{E}_m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^{\;\;\;m}\right)\mathbf{E}_m$$

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$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}\right)\mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^{n}\right)\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

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$$= \quad U^m{}_{;i}\ \mathbf{E}_m \quad = \quad U_{m;i}\ \mathbf{E}^m$$

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$$= U^m_{\ ;i}\,\mathbf{E}_m = U_{m;i}\,\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

Defining *covariant derivative* $U^m_{;i}$
of a contravariant component U^m

$$U^m_{\ ;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^{\ \ m}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}^{k}$

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Any vector derivative can be expressed using $\Gamma_{ij;}^{\;\;k}$ in terms of $\mathbf{E}_m$ or $\mathbf{E}^m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n\Gamma_{in;}^{\;\;m}\right)\mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n\Gamma_{im;}^{\;\;n}\right)\mathbf{E}^m$$

$$= U^m_{\;;i}\,\mathbf{E}_m = U_{m;i}\,\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i}\bullet\mathbf{E}_m = -\mathbf{E}^n\bullet\frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

Defining *covariant derivative* $U^m_{\;;i}$ *of a contravariant component* U^m

$$U^m_{\;;i} = \frac{\partial U^m}{\partial q^i} + U^n\Gamma_{in;}^{\;\;m}$$

...and *covariant derivative* $U_{m;i}$ *of a covariant component* U_m

$$U_{m;i} = \frac{\partial U_m}{\partial q^i} - U_n\Gamma_{im;}^{\;\;n}$$

Intrinsic derivatives:
(Mathematicians being cute)

Defining *intrinsic derivative of contravariant vector components*.

$$\frac{\delta V^k}{\delta t}=\frac{dV^k}{dt}+\Gamma^k_{mn}V^m\dot{q}^n=\frac{\partial V^k}{\partial q^n}\dot{q}^n+\Gamma^k_{mn}V^m\dot{q}^n=V^k_{;n}\dot{q}^n$$

$$F_k=\frac{\delta p_k}{\delta t}$$

Defining intrinsic derivative of covariant vector components.

$$\frac{\delta V_k}{\delta t}=\frac{dV_k}{dt}-\Gamma^m_{kn}V_m\dot{q}^n=\frac{\partial V_k}{\partial q^n}\dot{q}^n-\Gamma^m_{kn}V_m\dot{q}^n=V_{k\,;\,n}\dot{q}^n$$

$$F^k=\frac{\delta p^k}{\delta t}$$

Tensor chain rules.

$\frac{\delta V^k}{\delta t}=V^k_{;n}\dot{q}^n$, replaces: $\frac{dV^k}{dt}=\frac{\partial V^k}{\partial q^n}\dot{q}^n$ where: $V^k_{;n}=\frac{\partial V^k}{\partial q^n}+\Gamma^k_{mn}V^m$

$\frac{\delta V_k}{\delta t}=V_{k\,;\,n}\dot{q}^n$, replaces: $\frac{dV_k}{dt}=\frac{\partial V_k}{\partial q^n}\dot{q}^n$ where: $V_{k\,;\,n}=\frac{\partial V_k}{\partial q^n}-\Gamma^m_{kn}V_m$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

→ *Christoffel g-derivative formula*

What's a tensor? What's not?

Christoffel g-derivative formula

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\quad\Gamma_{im;n}\quad+\quad\Gamma_{in;m}$$

Christoffel g-derivative formula

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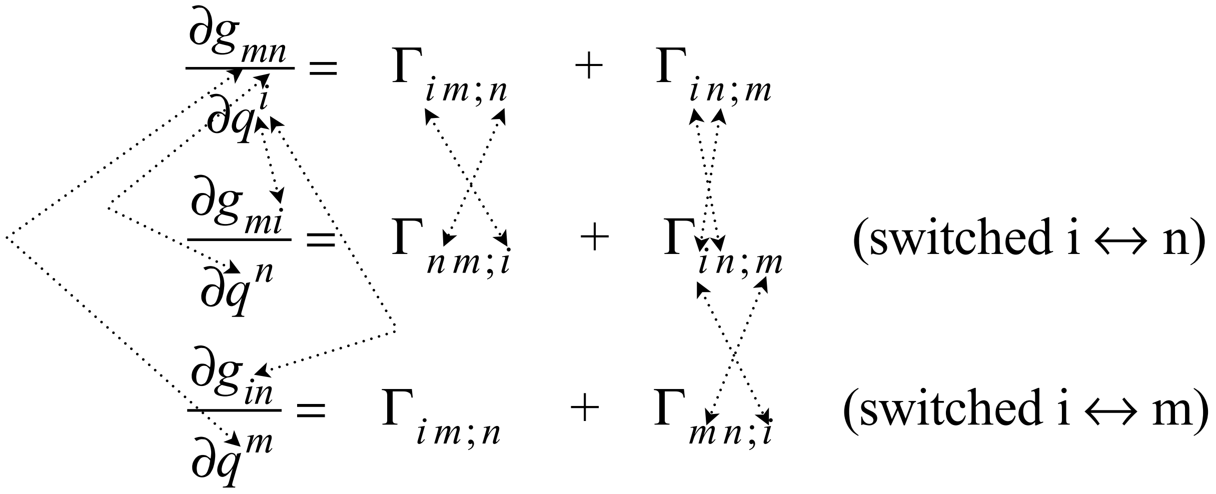

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$\frac{\partial g_{mi}}{\partial q^n}=\Gamma_{nm;i}+\Gamma_{in;m}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

Christoffel g-derivative formula

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}$$ (switched i ↔ n)

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}$$ (switched i ↔ m)

Christoffel g-derivative formula

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i} = \frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\cancel{\Gamma_{nm;i}} - \cancel{\Gamma_{in;m}} \quad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \cancel{\Gamma_{mn;i}} \quad \text{(switched i} \leftrightarrow \text{m)}$$

Christoffel g-derivative formula

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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$$-\frac{\partial g_{mi}}{\partial q^n}=-\cancel{\Gamma_{nm;i}}-\cancel{\Gamma_{in;m}}$$ (switched i ↔ n)

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}$$ (switched i ↔ m)

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula

→ *What’s a tensor? What’s not?*

What's a tensor? What's not?

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Gives the Christoffel formula

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

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What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

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What's a tensor? What's not?

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}\cancel{+\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=\cancel{-\Gamma_{nm;i}}\cancel{-\Gamma_{in;m}}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}\cancel{+\Gamma_{mn;i}}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar m}{}_{;\bar n}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar q^{\bar n}}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ of covariant derivative $U^m{}_{;n}=\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{E_m}$

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$$\boxed{\bar{U}^{\bar m}{}_{;\bar n}=\frac{\partial\bar q^{\bar m}}{\partial q^m}\frac{\partial q^n}{\partial\bar q^{\bar n}}U^m{}_{;n}}$$

The transformation of $U^m{}_{;n}=\frac{\partial U^m}{\partial q^n}+U^\ell\Gamma_{n\ell;}^{m}$ is that of general 2nd-rank tensor $T^m{}_n$

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Otherwise, $U^m{}_{,n}$ needs "correction" $U^\ell\Gamma_{n\ell;}^{m}$. And, that $U^\ell\Gamma_{n\ell;}^{m}$ *cannot* be a $T^m{}_n$-tensor either!

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \cancel{\left\{ \frac{\partial x^j}{\partial t} \right\}} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \cancel{\left\{ \frac{\partial x^k}{\partial t} \right\}} \right)$$

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Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

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$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *<u>all</u>* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

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Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\cancel{\frac{\partial x^j}{\partial t}}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\cancel{\frac{\partial x^k}{\partial t}}\right\}\right)$$

All explicit-t-dependent terms are *<u>zero</u>*

1st term involves ***covariant momentum*** $\boxed{p_\ell}$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *<u>all</u>* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Following is for <u>fixed</u> GCC only:

$$F_\ell = \boxed{\frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right)} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \boxed{\gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{q}^k$ to GCC $T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \cancel{\left\{ \frac{\partial x^j}{\partial t} \right\}} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \cancel{\left\{ \frac{\partial x^k}{\partial t} \right\}} \right)$$

All explicit-t-dependent terms are *zero*

$$F_\ell = \frac{d}{dt} \boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell}$$

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Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \boxed{\frac{d}{dt} \left(\gamma_{\ell n} \dot{q}^n \right)} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T=\frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m+\cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n+\cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are zero

1st term involves *covariant momentum* p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

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Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for <u>*all*</u> GCC, fixed or explicit in time *t*:

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Following is for <u>fixed</u> GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose
Christoffel coefficients:

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T=\frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m+\cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n+\cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are zero

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

1st term involves ***covariant momentum*** p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

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Christoffel coefficients:

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

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Lagrange equations for fixed GCC convert to tensor form

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$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are <u>zero</u>

1st term involves *covariant momentum* $\boxed{p_\ell}$.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

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$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\boxed{\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m}} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

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This gives *covariant Riemann equations*

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

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$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose
Christoffel coefficients:

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

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$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

This gives *covariant Riemann equations* and *contravariant Riemann equations*.

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

$$F^k = \ddot{q}^k + \Gamma^k_{mn}\,\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

→ *Example of Riemann-Christoffel forms in cylindrical polar OCC* ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

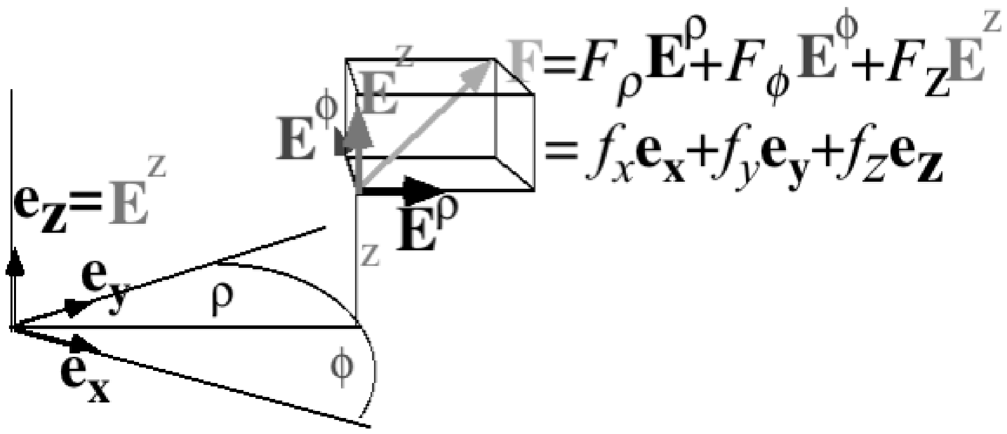
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array} , \quad \langle K\rangle = \begin{array}{c} \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array} \\ =\langle J^{-1}\rangle \end{array}$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1=\rho$, $q^2=\phi$, $q^3=z$)

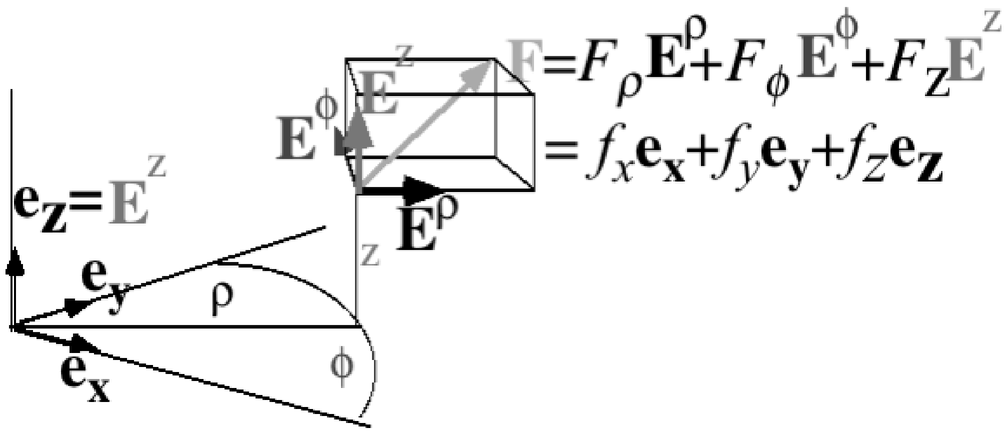
$$\langle J\rangle = \begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \\ \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} , \quad \langle K\rangle = \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \begin{array}{l} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{array}$$

$$=\langle J^{-1}\rangle$$

$x=\rho\cos\phi$

$y=\rho\sin\phi$

$z=z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho}= \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi}=-f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z}= \quad 0 \quad + \quad 0 \quad +f_z$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{ccc} \left(\begin{array}{ccc} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \left(\begin{array}{ccc} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$= \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

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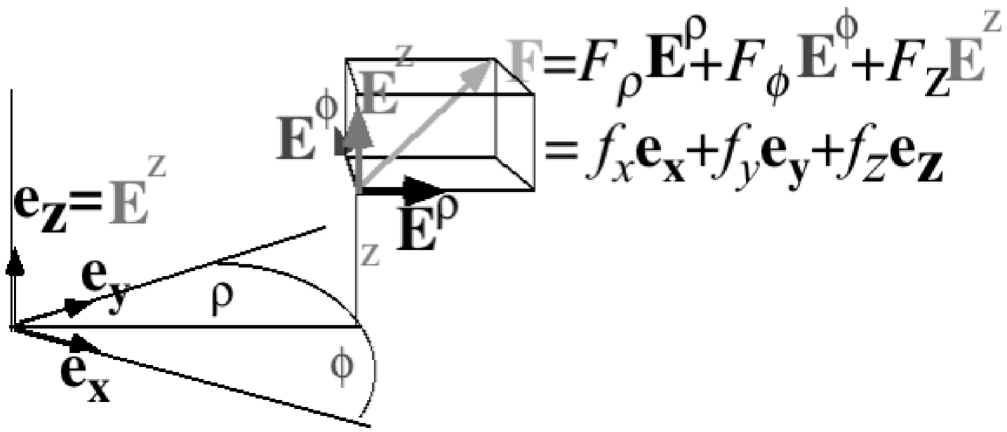

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2 \cos^2\phi + \rho^2 \sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \bullet \mathbf{E}_z = m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \left(\begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array} , \quad \langle K\rangle = \begin{array}{c} \begin{array}{cc} \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) & \begin{array}{l} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{array} \end{array} \\ =\langle J^{-1}\rangle \end{array}$$

$x = \rho \cos \phi$

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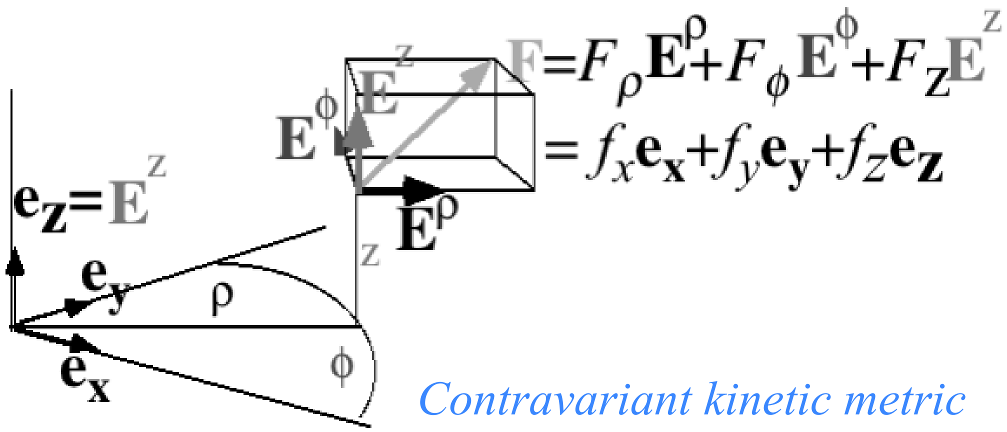

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

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Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\rho} \\ \leftarrow \mathbf{E}^{\phi} \\ \leftarrow \mathbf{E}^{z} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_{\rho} & \mathbf{E}_{\phi} & \mathbf{E}_{z} \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

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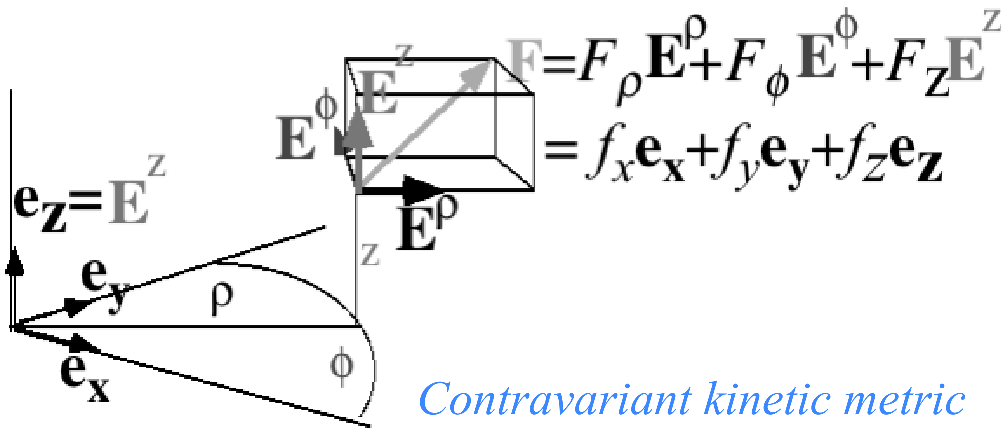

Covariant forces

$$F_{\rho} = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_{\phi} = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

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Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

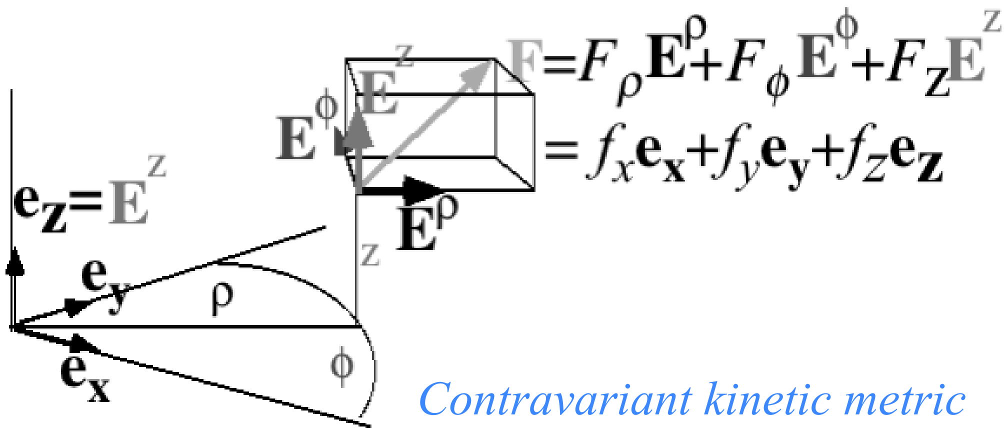
$$\langle J\rangle = \begin{array}{ccc} \left(\begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \begin{array}{l} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{array}$$

$$=\langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right)=m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right)=m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

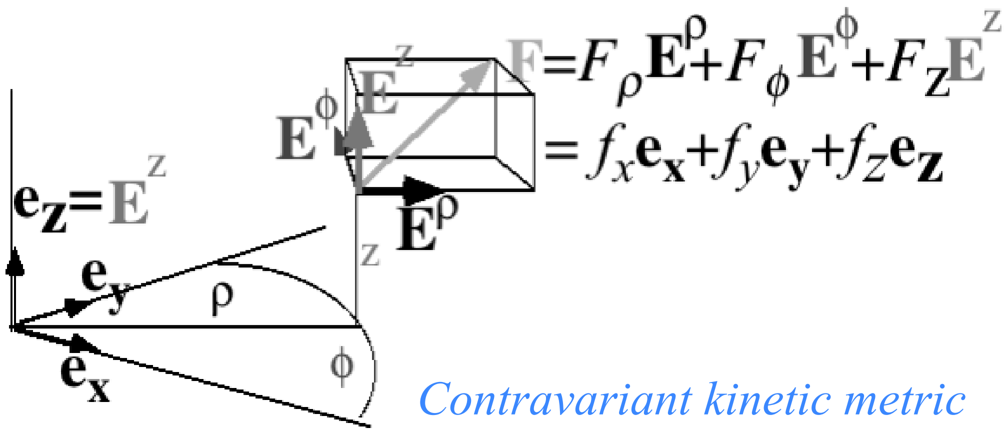
Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \\ \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} = \langle J^{-1}\rangle$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$



Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z}$$

$$= m\dot{\rho} \qquad = m\rho^2\dot{\phi} \qquad = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

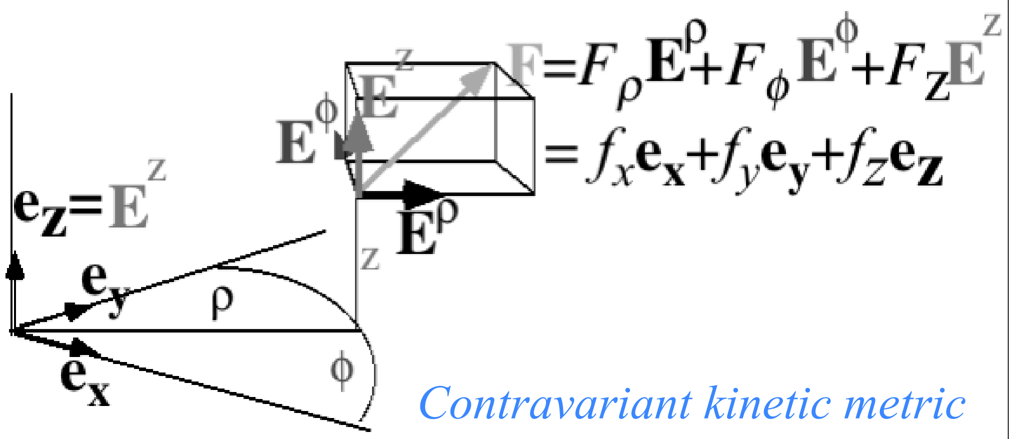
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \left(\begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \quad\quad\uparrow & \quad\quad\uparrow \\ \mathbf{E}_\rho & \quad\quad\mathbf{E}_\phi & \quad\quad\mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{c} \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \begin{array}{l} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{array} \\ =\langle J^{-1}\rangle \end{array}$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z}$$

$$= m\dot{\rho} \qquad\qquad = m\rho^2\dot{\phi} \qquad\qquad = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

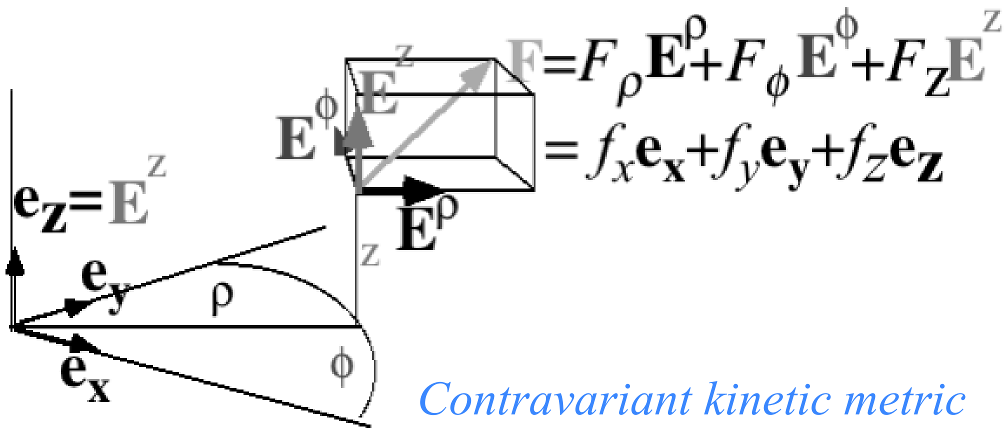
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \\ \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{matrix}$$

$$=\left\langle J^{-1}\right\rangle$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt}-\frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n+\Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt}-\frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot{\rho}+\Gamma_{mn;\rho}\,\dot{q}^m\dot{q}^n \qquad F_\phi = \frac{dp_\phi}{dt}-\frac{\partial T}{\partial\phi} = \gamma_{\phi\phi}\ddot{\phi}+\Gamma_{mn;\phi}\,\dot{q}^m\dot{q}^n$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

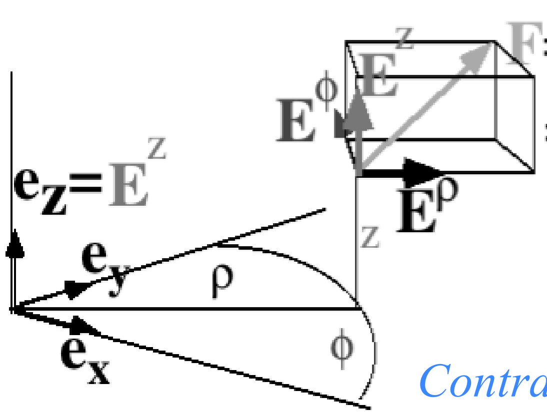
$$\langle J\rangle = \begin{array}{ccc} \left(\begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \begin{array}{l} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{array}$$

$$=\left\langle J^{-1}\right\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



$$\mathbf{F}=F_\rho\mathbf{E}^\rho+F_\phi\mathbf{E}^\phi+F_z\mathbf{E}^z$$
$$=f_x\mathbf{e}_x+f_y\mathbf{e}_y+f_z\mathbf{e}_z$$

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho}= \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi}=-f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z}= \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right)=m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right)=m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n=\frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho=\frac{\partial T}{\partial\dot{\rho}}=\gamma_{\rho\rho}\dot{\rho} \qquad p_\phi=\frac{\partial T}{\partial\dot{\phi}}=\gamma_{\phi\phi}\dot{\phi} \qquad p_z=\frac{\partial T}{\partial\dot{z}}=\gamma_{zz}\dot{z}$$

$$=m\dot{\rho} \qquad =m\rho^2\dot{\phi} \qquad =m\dot{z}$$

Contravariant momenta

$$p^\rho=\dot{\rho}$$

$$p^\phi=\dot{\phi}$$

$$p^z=\dot{z}$$

Lagrange and the Riemann *covariant force* equations

$$F_\ell=\frac{dp_\ell}{dt}-\frac{\partial T}{\partial q^\ell}=\gamma_{\ell n}\ddot{q}^n+\Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho=\frac{dp_\rho}{dt}-\frac{\partial T}{\partial\rho}=\gamma_{\rho\rho}\ddot{\rho}+\Gamma_{mn;\rho}\,\dot{q}^m\dot{q}^n$$

$$=\frac{d\left(m\dot{\rho}\right)}{dt}-\frac{\partial}{\partial\rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right)=m\ddot{\rho}-m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho}=-m\rho$$

$$F_\phi=\frac{dp_\phi}{dt}-\frac{\partial T}{\partial\phi}=\gamma_{\phi\phi}\ddot{\phi}+\Gamma_{mn;\phi}\,\dot{q}^m\dot{q}^n$$

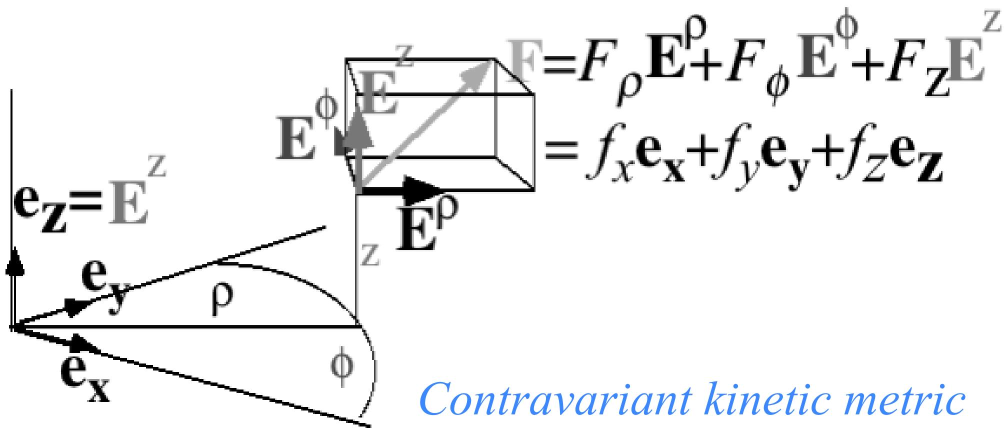
$$=\frac{d\left(m\rho^2\dot{\phi}\right)}{dt}-0=m\rho^2\ddot{\phi}+2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi}=m\rho=\Gamma_{\phi\rho;\phi}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \left(\begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$=\langle J^{-1}\rangle$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2+\frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z}$$

$$= m\dot{\rho} \qquad = m\rho^2\dot{\phi} \qquad = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt}-\frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n+\Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt}-\frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot{\rho}+\Gamma_{mn;\rho}\,\dot{q}^m\dot{q}^n$$

$$= \frac{d\left(m\dot{\rho}\right)}{dt}-\frac{\partial}{\partial\rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right) = m\ddot{\rho}-m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

$$F_\phi = \frac{dp_\phi}{dt}-\frac{\partial T}{\partial\phi} = \gamma_{\phi\phi}\ddot{\phi}+\Gamma_{mn;\phi}\,\dot{q}^m\dot{q}^n$$

$$= \frac{d\left(m\rho^2\dot{\phi}\right)}{dt}-0 = m\rho^2\ddot{\phi}+2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

*Contra*variant equations are *acceleration* equations. $F^k = \gamma^{jk}F_j = \ddot{q}^k+\Gamma^k_{mn}\dot{q}^m\dot{q}^n$

$$F^\rho = \gamma^{\rho\rho}F_\rho = \ddot{q}^\rho+\Gamma^\rho_{mn}\dot{q}^m\dot{q}^n \qquad F^\phi = \gamma^{\phi\phi}F_\phi = \ddot{q}^\rho+\Gamma^\rho_{mn}\dot{q}^m\dot{q}^\phi$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

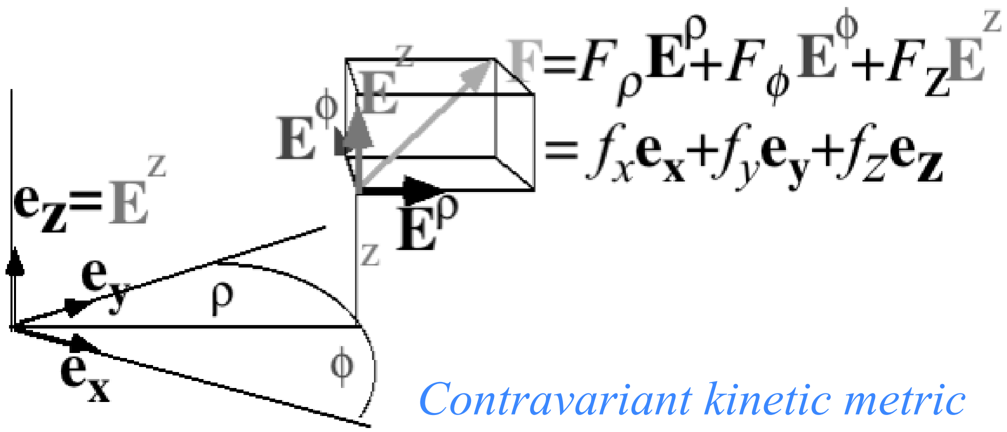
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad\qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi+f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi+f_y\rho\cos\phi+0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad +f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right)=m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right)=m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot q^m\dot q^n = \frac{1}{2}m\dot\rho^2+\frac{1}{2}m\rho^2\dot\phi^2+\frac{1}{2}m\dot z^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot\rho} = \gamma_{\rho\rho}\dot\rho \qquad p_\phi = \frac{\partial T}{\partial\dot\phi} = \gamma_{\phi\phi}\dot\phi \qquad p_z = \frac{\partial T}{\partial\dot z} = \gamma_{zz}\dot z$$

$$= m\dot\rho \qquad\qquad = m\rho^2\dot\phi \qquad\qquad = m\dot z$$

Contravariant momenta

$$p^\rho = \dot\rho$$

$$p^\phi = \dot\phi$$

$$p^z = \dot z$$

Lagrange and the Riemann *covariant force* equations

$$F_\ell = \frac{dp_\ell}{dt}-\frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot q^n+\Gamma_{mn;\ell}\dot q^m\dot q^n$$

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt}-\frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot\rho+\Gamma_{mn;\rho}\dot q^m\dot q^n$$

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*Contra*variant equations are *acceleration* equations. $F^k=\gamma^{jk}F_j=\ddot q^k+\Gamma^k_{mn}\dot q^m\dot q^n$

$$F^\rho = \gamma^{\rho\rho}F_\rho = \ddot q^\rho+\Gamma^\rho_{mn}\dot q^m\dot q^n$$

$$= \ddot\rho-\rho\dot\phi^2 \quad \text{so: } \Gamma^\rho_{\phi\phi} = -\rho \qquad \gamma^{\rho\rho} = 1/m$$

$$F^\phi = \gamma^{\phi\phi}F_\phi = \ddot q^\rho+\Gamma^\rho_{mn}\dot q^m\dot q^n$$

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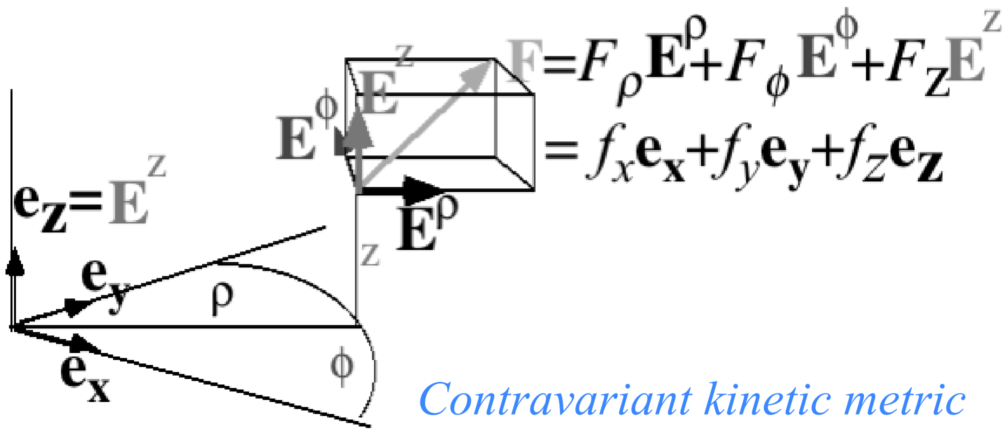

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial\rho} + f_y \frac{\partial y}{\partial\rho} + f_z \frac{\partial z}{\partial\rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

$$F_\phi = f_x \frac{\partial x}{\partial\phi} + f_y \frac{\partial y}{\partial\phi} + f_z \frac{\partial z}{\partial\phi} = -f_x \rho\sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

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$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

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$$T = \frac{1}{2}\gamma_{mn}\dot q^m \dot q^n = \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\phi^2 + \frac{1}{2}m\dot z^2$$

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$$= \frac{d\left(m\rho^2\dot\phi\right)}{dt} - 0 = m\rho^2\ddot\phi + 2m\rho\dot\rho\dot\phi \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

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$$F^\rho = \gamma^{\rho\rho}F_\rho = \ddot q^\rho + \Gamma^\rho_{mn}\dot q^m \dot q^n$$

$$= \ddot\rho - \rho\dot\phi^2 \quad \text{so: } \Gamma^\rho_{\phi\phi} = -\rho \qquad \gamma^{\rho\rho} = 1/m$$

$$F^\phi = \gamma^{\phi\phi}F_\phi = \ddot q^\rho + \Gamma^\rho_{mn}\dot q^m \dot q^n$$

$$= \ddot\phi + 2\dot\rho\dot\phi/\rho \quad \text{so: } \Gamma^\phi_{\rho\phi} = 1/\rho = \Gamma^\phi_{\phi\rho} \qquad \gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\ddot\rho = F^\rho + \rho\dot\phi^2 \qquad \textit{(Centrifugal acceleration)}$$

$$\ddot\phi = F^\phi - 2\dot\rho\dot\phi/\rho \qquad \textit{(Coriolis acceleration)}$$

Rewriting GCC Lagrange equations :

(Review of Lecture 11)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r}$$
$$= M\,r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$
$$= 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if
potential U has no explicit ϕ-dependence

Conventional forms

radial force: $M\ddot{r} = M\,r\dot{\phi}^2 - \frac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

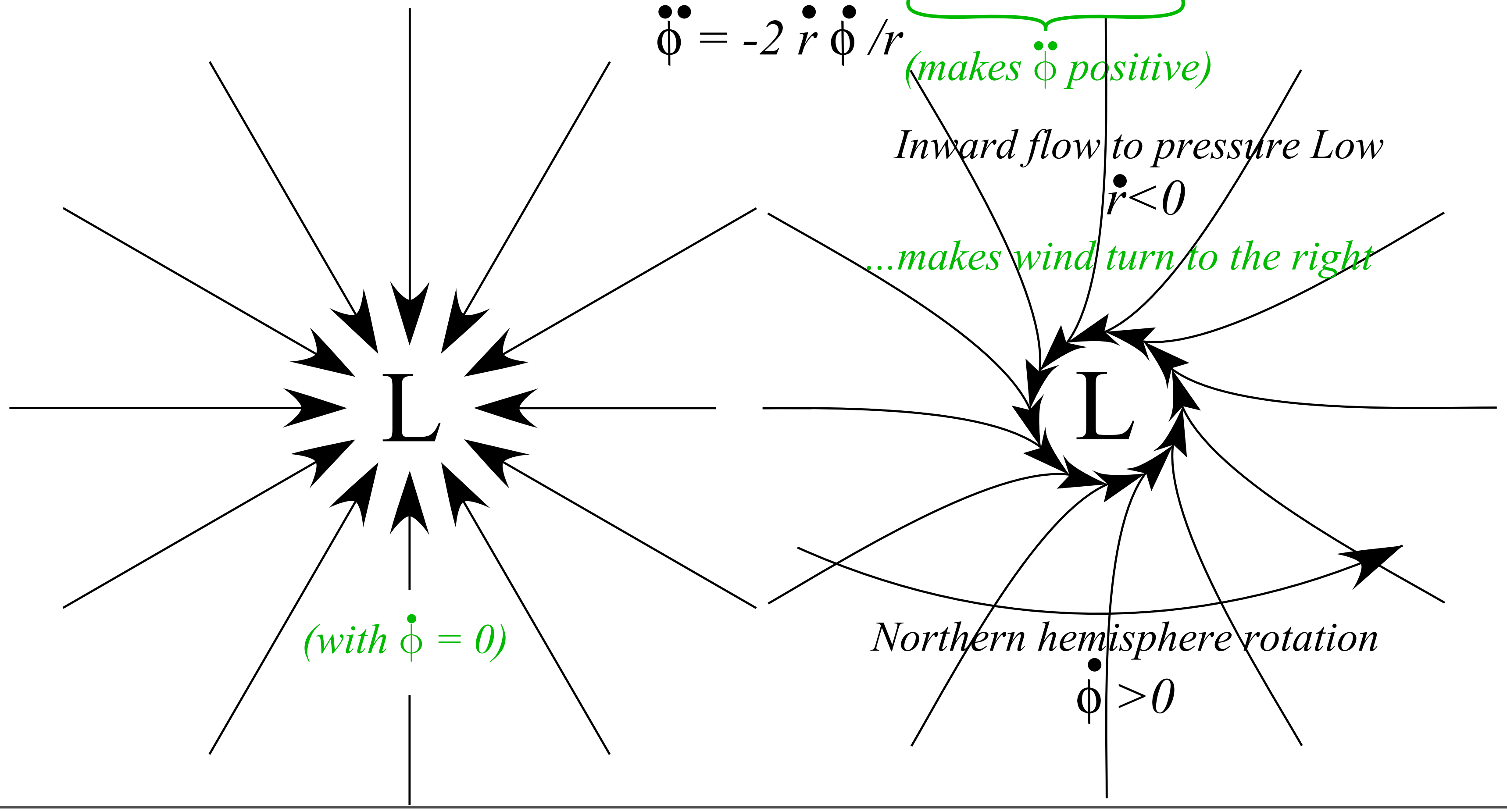
Field-free ($U=0$)

radial acceleration: $\ddot{r} = r\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\underbrace{\dot{\phi} > 0 \text{ and } \dot{r} < 0}_{\text{(makes } \ddot{\phi} \text{ positive)}}$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$



(with $\dot{\phi} = 0$)

Inward flow to pressure Low
$\dot{r} < 0$

...makes wind turn to the right

Northern hemisphere rotation
$\dot{\phi} > 0$

Effect on
Northern
Hemisphere
local weather

Cyclonic flow
around lows

→ *Separation of GCC Equations: Effective Potentials*

Small radial oscillations
Cycloid vs Pendulum

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

(Numerically correct ONLY!)

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

(Formally **and** Numerically correct)

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

(Numerically correct ONLY!)

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(Formally **and** Numerically correct)

Potential *V* is *isotropic* (cylindrical) function of radius ρ. (*V* = *V*(ρ))

H has no explicit ϕ–dependence and the ϕ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

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If *H* has no explicit *z*–dependence then the *z*–momenta is constant, too.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

$$m\dot{z} = p_z = const. = k$$

Separation of GCC Equations: Effective Potentials

$$H=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n+V=\frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2+V$$

(Numerically correct ONLY!)

$$=\frac{1}{2}\gamma^{mn}p_m p_n+V=\frac{1}{2m}p_\rho^2+\frac{1}{2m\rho^2}p_\phi^2+\frac{1}{2m}p_z^2+V$$

(Formally **and** Numerically correct)

Potential V is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)
H has no explicit ϕ–dependence and the ϕ–momenta is constant.

If H has no explicit z–dependence then the z–momenta is constant, too.

$$m\rho^2\dot{\phi}=p_\phi=const.=\mu$$

$$m\dot{z}=p_z=const.=k$$

$$H=\frac{1}{2m}p_\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{k^2}{2m}+V\left(\rho\right)=E=const.$$

Separation of GCC Equations: Effective Potentials

$$H=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n+V=\frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2+V$$

(Numerically correct ONLY!)

$$=\frac{1}{2}\gamma^{mn}p_m p_n+V=\frac{1}{2m}p_\rho^2+\frac{1}{2m\rho^2}p_\phi^2+\frac{1}{2m}p_z^2+V$$

(Formally **and** Numerically correct)

Potential *V* is ***isotropic*** (cylindrical) function of radius ρ. (*V = V(*ρ*)*)

H has no explicit ϕ–dependence and the ϕ–momenta is constant.

If *H* has no explicit *z*–dependence then the *z*–momenta is constant, too.

$$m\rho^2\dot{\phi}=p_\phi=const.=\mu$$

$$m\dot{z}=p_z=const.=k$$

$$H=\frac{1}{2m}p_\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{k^2}{2m}+V\left(\rho\right)=E=const.$$

(Let k =0)

Separation of GCC Equations: Effective Potentials

$$H=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n+V=\frac{1}{2}m\dot{\rho}^2\ +\ \frac{1}{2}m\rho^2\dot{\phi}^2\ +\ \frac{1}{2}m\dot{z}^2+V$$

(Numerically correct ONLY!)

$$=\frac{1}{2}\gamma^{mn}p_mp_n+V=\frac{1}{2m}p_\rho^2+\frac{1}{2m\rho^2}p_\phi^2+\frac{1}{2m}p_z^2+V$$

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$$H=\frac{1}{2m}p_\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{k^2}{2m}+V(\rho)=E=const.$$

(Let k =0)

Symmetry reduces problem to a one-dimensional form.

$$H=\frac{1}{2m}p_\rho^2+V^{eff}(\rho)=E=const.$$

Separation of GCC Equations: Effective Potentials

$$H=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n+V=\frac{1}{2}m\dot{\rho}^2 \;+\; \frac{1}{2}m\rho^2\dot{\phi}^2 \;+\; \frac{1}{2}m\dot{z}^2+V$$

(Numerically
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(Formally **and** Numerically
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$$H=\frac{1}{2m}p_\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{k^2}{2m}+V(\rho)=E=const.$$

(Let $k=0$)

Symmetry reduces problem to a one-dimensional form.

$$H=\frac{1}{2m}p_\rho^2+V^{eff}(\rho)=E=const.$$

An *effective potential* $V^{eff}(\rho)$ has a *centrifugal barrier*.

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+V(\rho)$$

Separation of GCC Equations: Effective Potentials

$$H=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n+V=\frac{1}{2}m\dot{\rho}^2 \;+\; \frac{1}{2}m\rho^2\dot{\phi}^2 \;+\; \frac{1}{2}m\dot{z}^2+V$$

(Numerically correct ONLY!)

$$=\frac{1}{2}\gamma^{mn}p_m p_n+V=\frac{1}{2m}p_\rho^2+\frac{1}{2m\rho^2}p_\phi^2+\frac{1}{2m}p_z^2+V$$

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$$H=\frac{1}{2m}p_\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{k^2}{2m}+V(\rho)=E=const.$$

(Let k =0)

Symmetry reduces problem to a one-dimensional form.

$$H=\frac{1}{2m}p_\rho^2+V^{eff}(\rho)=E=const.$$

An ***effective potential*** *V*eff(ρ) has a ***centrifugal barrier***.

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+V(\rho)$$

Velocity relations:

$$\dot{\phi}=\mu/\left(m\rho^2\right)$$

$$\dot{\rho}=\frac{d\rho}{dt}=\frac{\partial H}{\partial p_\rho}=\frac{p_\rho}{m}=\pm\sqrt{\frac{2}{m}\left(E-V^{eff}(\rho)\right)}$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \;+\; \frac{1}{2}m\rho^2\dot{\phi}^2 \;+\; \frac{1}{2}m\dot{z}^2 + V$$

(Numerically correct ONLY!)

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

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$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

$$m\dot{z} = p_z = const. = k$$

$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const.$$

(Let k =0)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m}p_\rho^2 + V^{eff}(\rho) = E = const.$$

An *effective potential Veff(*ρ*)* has a *centrifugal barrier*.

$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$

Velocity relations:

$$\dot{\phi} = \mu / \left(m\rho^2\right)$$

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}$$

Equations solved by a *quadrature integral* for time versus radius.

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}} = \left(\text{Travel time } \rho_0 \text{ to } \rho_1\right) = t_1 - t_0$$

Separation of GCC Equations: Effective Potentials

→ *Small radial oscillations*
Cycloid vs Pendulum

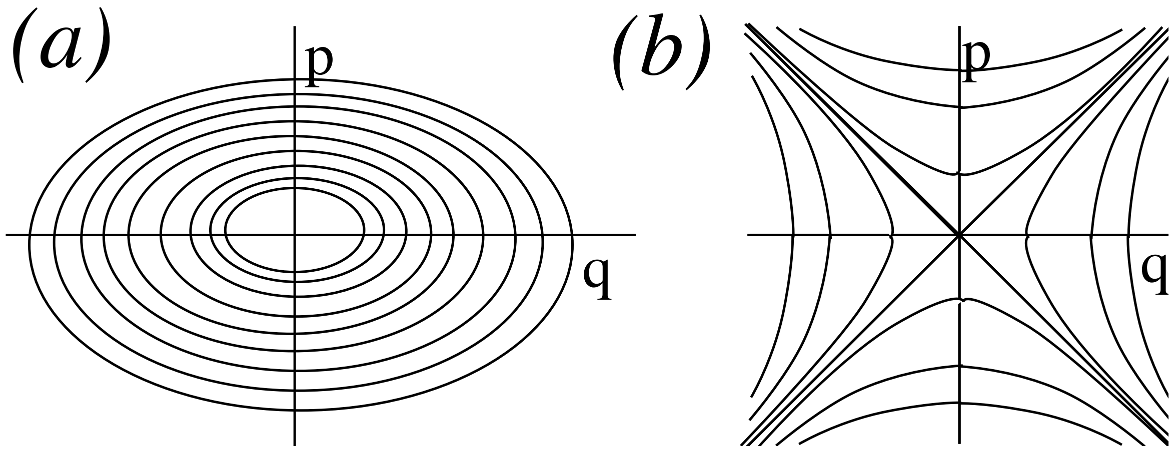
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_0} = 0\ , \quad \text{with:} \quad \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0\ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_0) + 0 + \frac{1}{2}(\rho-\rho_0)^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0}$$



Stable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0$

Unstable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} < 0$

Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0\ , \quad \text{with:} \quad \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0\ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \qquad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}$$

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

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Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_\phi} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_\rho}{n_\phi} \Leftrightarrow \text{Orbit is closed-periodic}$$

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0\,, \quad \text{with: } \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0\,.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

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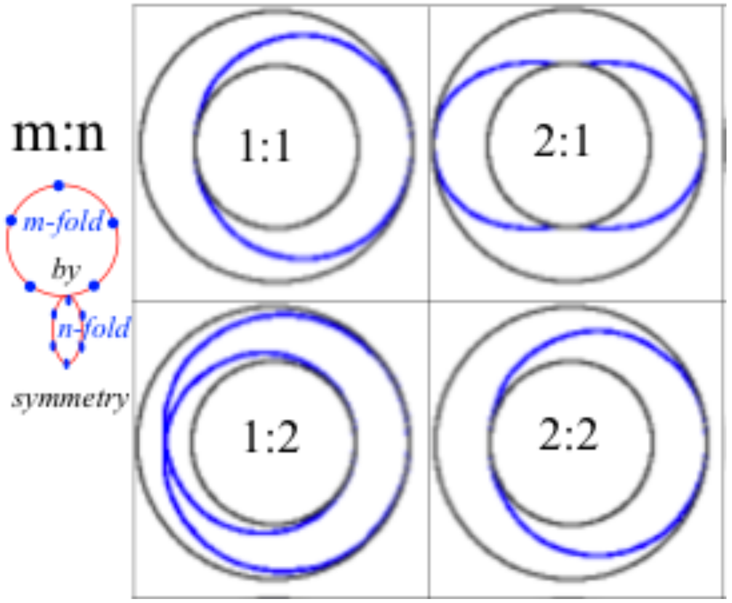
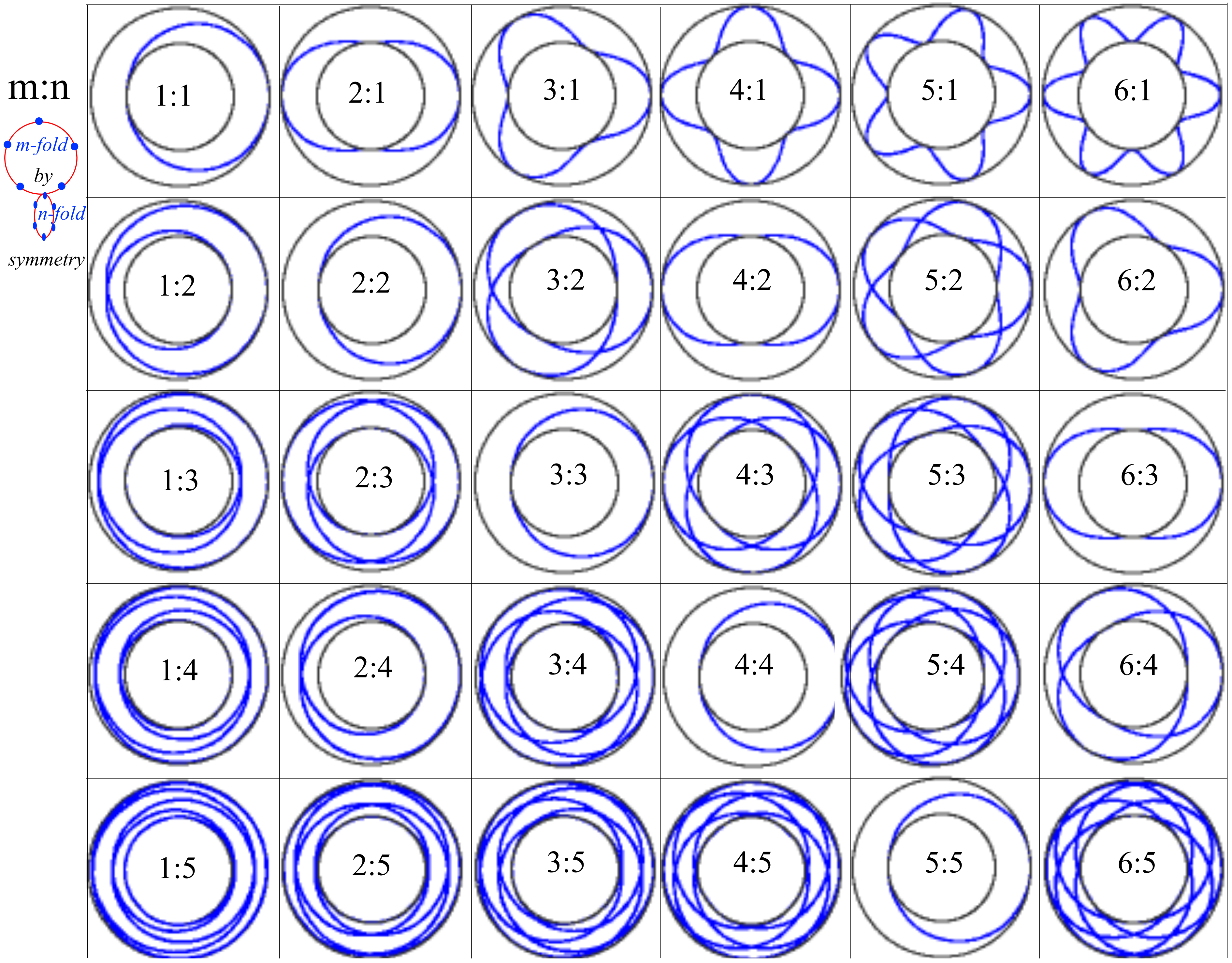
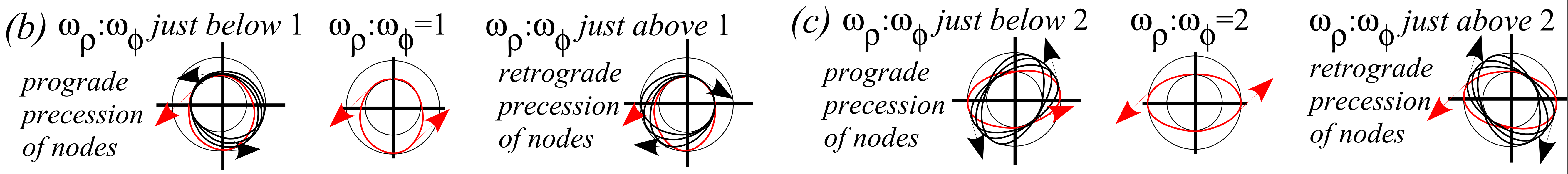
Some generic shapes resulting from various ratios $n\rho : n\phi$






(b) $\omega_\rho:\omega_\phi$ *just below* 1

prograde precession of nodes

$\omega_\rho:\omega_\phi=1$

$\omega_\rho:\omega_\phi$ *just above* 1

retrograde precession of nodes

(c) $\omega_\rho:\omega_\phi$ *just below* 2

prograde precession of nodes

$\omega_\rho:\omega_\phi=2$

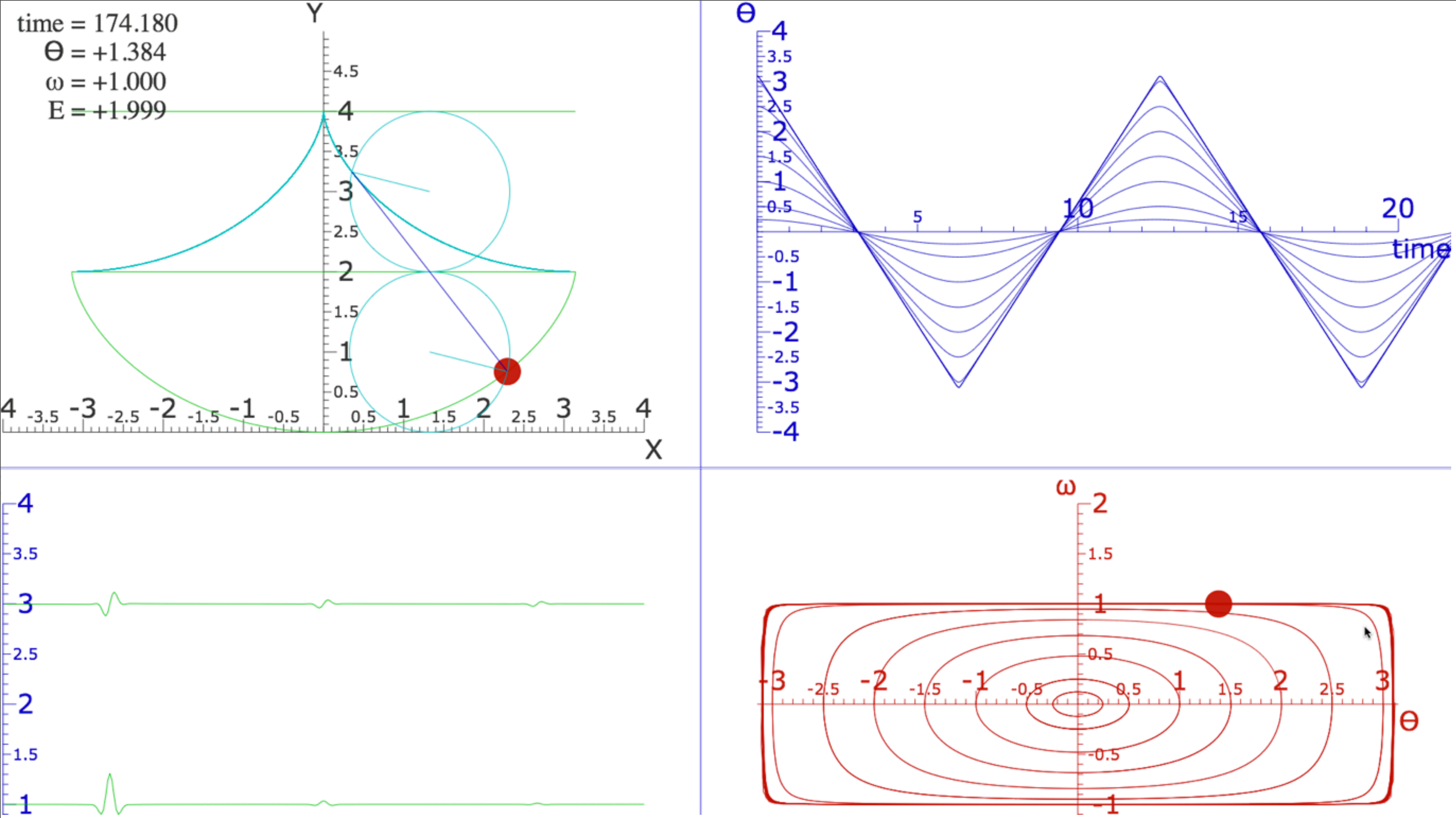
$\omega_\rho:\omega_\phi$ *just above* 2

retrograde precession of nodes

Separation of GCC Equations: Effective Potentials

Small radial oscillations

→ *Cycloid vs Pendulum*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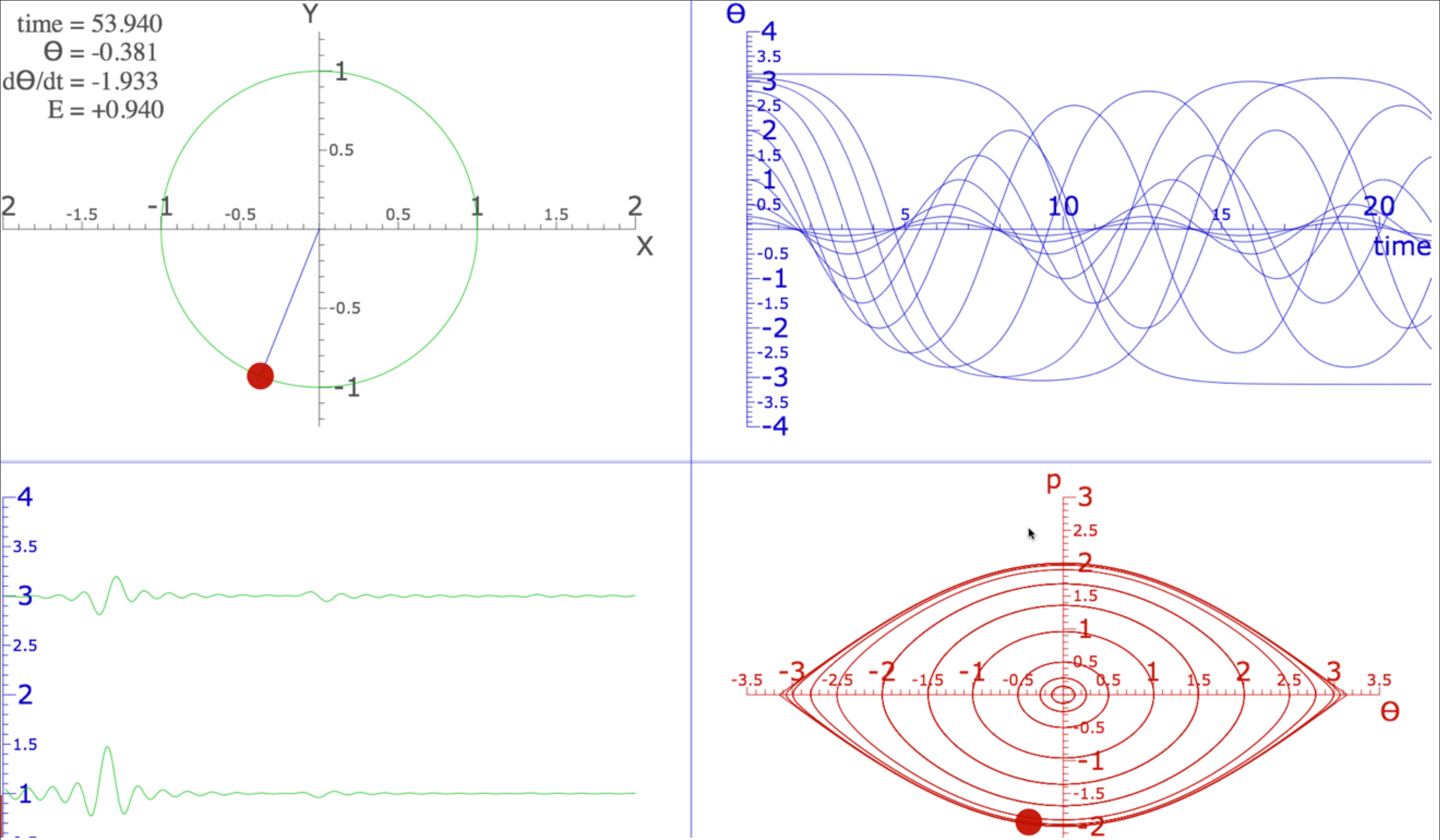
Θ = +1.384
ω = +1.000
E = +1.999
Y
X
4.5
4
3.5
3
2.5
2
1.5
1
0.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
3.5
4
θ
5
10
15
20
time
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
ω
2
1.5
1
0.5
-0.5
-1
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
θ



time = 53.940
Θ = -0.381
dΘ/dt = -1.933
E = +0.940