Complex Variables, Series, and Field Coordinates I.
(Ch. 10 of Unit 1)

1. The Story of $e$ (A Tale of Great $Interest$)
   How good are those power series?
   Taylor-Maclaurin series, imaginary interest, and complex exponentials

2. What good are complex exponentials?
   Easy trig
   Easy 2D vector analysis
   Easy oscillator phase analysis
   Easy rotation and “dot” or “cross” products

3. Easy 2D vector calculus
   Easy 2D vector derivatives
   Easy 2D source-free field theory
   Easy 2D vector field-potential theory

4. Riemann-Cauchy relations (What’s analytic? What’s not?)
   Easy 2D curvilinear coordinate discovery
   Easy 2D circulation and flux integrals
   Easy 2D monopole, dipole, and $2^n$-pole analysis

5. Non-analytic 2D source field analysis
   Easy $2^n$-multipole field and potential expansion
   Easy stereo-projection visualization

Lecture 14 Tue. 10.09.
starts here

1. Complex numbers provide “automatic trigonometry”
2. Complex numbers add like vectors.
3. Complex exponentials $Ae^{i\omega t}$ track position and velocity using Phasor Clock.
4. Complex products provide 2D rotation operations.
5. Complex products provide 2D “dot”($\cdot$) and “cross”($\times$) products.

6. Complex derivative contains “divergence”($\nabla \cdot F$) and “curl”($\nabla \times F$) of 2D vector field
7. Invent source-free 2D vector fields [$\nabla \cdot F=0$ and $\nabla \times F=0$]
8. Complex potential $\phi$ contains “scalar”($F=\nabla \phi$) and “vector”($F=\nabla \times A$) potentials
   The $half$-$n$-half results: (Riemann-Cauchy Derivative Relations)
9. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field
10. Complex integrals $\int f(z)dz$ count 2D “circulation”($\int F \cdot dr$) and “flux”($\int F \times dr$)
11. Complex integrals define 2D monopole fields and potentials
12. Complex derivatives give 2D dipole fields

Lecture 15 Thur. 10.11.
starts here

13. More derivatives give 2D $2^n$-pole fields…
14. …and $2^n$-pole multipole expansions of fields and potentials…
15. …and Laurent Series…
16. …and non-analytic source analysis.
The Story of e (A Tale of Great $Interest$)

Simple interest at some rate $r$ based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time $t$ later they would pay you $p(t)=(1+r\cdot t)p(0)$.

$1.00$ at rate $r=1$ (like Israel and Brazil that once had 100% interest.) gives $2.00$ at $t=1$ year.
The Story of e (A Tale of Great $Interest$)

Simple interest at some rate $r$ based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time $t$ later they would pay you $p(t) = (1+r \cdot t)p(0)$. $1.00$ at rate $r=1$ (like Israel and Brazil that once had 100% interest.) gives $2.00$ at $t=1$ year.

Semester compounded interest gives $p(\frac{t}{2}) = (1+r\cdot \frac{t}{2})p(0)$ at the half-period $\frac{t}{2}$ and then use $p(\frac{t}{2})$ during the last half to figure final payment. Now $1.00$ at rate $r=1$ earns $2.25$.

$$p^{\frac{1}{2}}(t) = (1 + r\cdot \frac{t}{2})p(\frac{t}{2}) = (1 + r\cdot \frac{t}{2}) \cdot (1 + r\cdot \frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = \frac{9}{4} = 2.25$$
Semester compounded interest gives at the half-period and then use during the last half to figure final payment. Now $1.00 at rate $r=1$ earns $2.25$.

Simple interest at some rate $r$ based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time $t$ later they would pay you $p(t)=(1+r t)p(0)$. $1.00 at rate $r=1$ (like Israel and Brazil that once had 100% interest.) gives $2.00 at $t=1$ year.

$\frac{1}{2}$

Semester compounded interest gives $p(\frac{t}{2})=(1+r \frac{t}{2})p(0)$ at the half-period $\frac{t}{2}$ and then use $p(\frac{t}{2})$ during the last half to figure final payment. Now $1.00 at rate $r=1$ earns $2.25$.

$$p^{\frac{1}{2}}(t) = (1 + r \frac{t}{2})p(\frac{t}{2}) = (1 + r \frac{t}{2})\cdot(1 + r \frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = \frac{9}{4} = 2.25$$

Trimester compounded interest gives $p(\frac{t}{3})=(1+r \frac{t}{3})p(0)$ at the 1/3rd-period 1st trimester and then use that to figure the 2nd trimester and so on. Now $1.00 at rate $r=1$ earns $2.37$.

$$p^{\frac{1}{3}}(t) = (1 + r \frac{t}{3})p(\frac{t}{3}) = (1 + r \frac{t}{3})\cdot(1 + r \frac{t}{3})p(\frac{t}{3}) = (1 + r \frac{t}{3})\cdot(1 + r \frac{t}{3})\cdot(1 + r \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$
The Story of e (A Tale of Great $Interest$)

Simple interest at some rate $r$ based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time $t$ later they would pay you $p(t) = (1 + r \cdot t)p(0)$.

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$$p^{\frac{1}{2}}(t) = (1 + r \cdot \frac{t}{2})p(\frac{t}{2}) = (1 + r \cdot \frac{t}{2}) \cdot (1 + r \cdot \frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = 2.25$$

Trimester compounded interest gives $p(\frac{t}{3}) = (1 + r \cdot \frac{t}{3})p(0)$ at the $1/3$rd-period $\frac{t}{3}$ or $1^{st}$ trimester and then use that to figure the $2^{nd}$ trimester and so on. Now $1.00$ at rate $r=1$ earns $2.37$.

$$p^{\frac{1}{3}}(t) = (1 + r \cdot \frac{t}{3})p(\frac{2t}{3}) = (1 + r \cdot \frac{t}{3}) \cdot (1 + r \cdot \frac{t}{3})p(\frac{t}{3}) = (1 + r \cdot \frac{t}{3}) \cdot (1 + r \cdot \frac{t}{3}) \cdot (1 + r \cdot \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$

So if you compound interest more and more frequently, do you approach INFININTEREST?

∞
The Story of e (A Tale of Great $Interest$)

Simple interest at some rate \( r \) based on a 1 year period.

You gave a principal \( p(0) \) to the bank and some time \( t \) later they would pay you \( p(t) = (1 + r \cdot t)p(0) \).

$1.00 at rate \( r=1 \) (like Israel and Brazil that once had 100% interest.) gives $2.00 at \( t=1 \) year.

\[
p(t) = (1 + r \cdot t)p(0)
\]

\[
p\left( \frac{1}{2} \right) = (1 + r \cdot \frac{1}{2})p(0)
\]

\[
p\left( \frac{1}{2} \right) = (1 + r \cdot \frac{1}{2}) \cdot (1 + r \cdot \frac{1}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = \frac{9}{4} = 2.25
\]

Semester compounded interest gives \( p\left( \frac{1}{2} \right) = (1 + r \cdot \frac{1}{2})p(0) \) at the half-period \( \frac{1}{2} \) and then use \( p\left( \frac{1}{2} \right) \) during the last half to figure final payment. Now $1.00 at rate \( r=1 \) earns $2.25.

\[
p\left( \frac{1}{3} \right) = (1 + r \cdot \frac{1}{3})p(0)
\]

\[
p\left( \frac{1}{3} \right) = (1 + r \cdot \frac{1}{3}) \cdot (1 + r \cdot \frac{1}{3}) \cdot (1 + r \cdot \frac{1}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37
\]

Trimester compounded interest gives \( p\left( \frac{1}{3} \right) = (1 + r \cdot \frac{1}{3})p(0) \) at the \( 1/3^{rd} \)-period \( \frac{1}{3} \) or 1st trimester and then use that to figure the 2nd trimester and so on. Now $1.00 at rate \( r=1 \) earns $2.37.

\[
p\left( \frac{1}{4} \right) = (1 + r \cdot \frac{1}{4})p(0)
\]

\[
p\left( \frac{1}{4} \right) = (1 + r \cdot \frac{1}{4}) \cdot (1 + r \cdot \frac{1}{4}) \cdot (1 + r \cdot \frac{1}{4}) \cdot (1 + r \cdot \frac{1}{4})p(0) = \frac{5}{4} \cdot \frac{5}{4} \cdot \frac{5}{4} \cdot \frac{5}{4} \cdot 1 = \frac{625}{256} = 2.44140625
\]

So if you compound interest more and more frequently, do you approach INFININTEREST?

NOT!!
The Story of e (A Tale of Great $\text{Interest}$)

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Semester compound interest gives $p(t)=(1+r\cdot \frac{t}{2})p(0)$ at the half-period $\frac{t}{2}$ and then use $p\left(\frac{t}{2}\right)$ during the last half to figure final payment. Now $1.00$ at rate $r=1$ earns $2.25$.

$$p^{\frac{1}{2}}(t) = (1+r\cdot \frac{t}{2})p\left(\frac{t}{2}\right) = (1+r\cdot \frac{t}{2})\cdot (1+r\cdot \frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = 2.25$$

Trimester compound interest gives $p(t)=(1+r\cdot \frac{t}{3})p(0)$ at the $1/3^{rd}$-period $\frac{t}{3}$ or 1st trimester and then use that to figure the 2nd trimester and so on. Now $1.00$ at rate $r=1$ earns $2.37$.

$$p^{\frac{1}{3}}(t) = (1+r\cdot \frac{t}{3})p\left(\frac{2t}{3}\right) = (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})p\left(\frac{t}{3}\right) = (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$

So if you compound interest more and more frequently, do you approach INFININTEREST?

$$p^{\frac{1}{1}}(t) = (1+r\cdot \frac{t}{1})^1 p(0) = \left(\frac{2}{1}\right)^1 \cdot 1 = 2.00$$

$+25\epsilon$

$$p^{\frac{1}{2}}(t) = (1+r\cdot \frac{t}{2})^2 p(0) = \left(\frac{3}{2}\right)^2 \cdot 1 = 2.25$$

$+12\epsilon$

$$p^{\frac{1}{3}}(t) = (1+r\cdot \frac{t}{3})^3 p(0) = \left(\frac{4}{3}\right)^3 \cdot 1 = \frac{64}{27} = 2.37$$

$+7\epsilon$

$$p^{\frac{1}{4}}(t) = (1+r\cdot \frac{t}{4})^4 p(0) = \left(\frac{5}{4}\right)^4 \cdot 1 = \frac{625}{256} = 2.44$$
The Story of e (A Tale of Great $Interest$)

Simple interest at some rate \( r \) based on a 1 year period.

You gave a principal \( p(0) \) to the bank and some time \( t \) later they would pay you \( p(t) = (1+r\cdot t)p(0) \). $1.00 at rate \( r=1 \) (like Israel and Brazil that once had 100\% interest.) gives $2.00 at \( t=1 \) year.

*Semester compounded* interest gives \( p\left(\frac{t}{2}\right) = (1+r\cdot \frac{t}{2})p(0) \) at the half-period \( \frac{t}{2} \) and then use \( p\left(\frac{t}{2}\right) \) during the last half to figure final payment. Now $1.00 at rate \( r=1 \) earns $2.25.

\[
p\left(\frac{1}{2}\right) = (1+r\cdot \frac{1}{2})p(0) = (1+r\cdot \frac{1}{2})\cdot (1+r\cdot \frac{1}{2})p(0) = \frac{3}{2}\cdot \frac{3}{2}\cdot 1 = \frac{9}{4} = 2.25
\]

*Trimester compounded* interest gives \( p\left(\frac{t}{3}\right) = (1+r\cdot \frac{t}{3})p(0) \) at the \( 1/3rd \)-period \( \frac{t}{3} \) or 1st trimester and then use that to figure the 2nd trimester and so on. Now $1.00 at rate \( r=1 \) earns $2.37.

\[
p\left(\frac{1}{3}\right) = (1+r\cdot \frac{1}{3})p(2\frac{1}{3}) = (1+r\cdot \frac{1}{3})\cdot (1+r\cdot \frac{1}{3})p(\frac{1}{3}) = (1+r\cdot \frac{1}{3})\cdot (1+r\cdot \frac{1}{3})\cdot (1+r\cdot \frac{1}{3})p(0) = \frac{4}{3}\cdot \frac{4}{3}\cdot \frac{4}{3}\cdot 1 = \frac{64}{27} = 2.37
\]

So if you compound interest more and more frequently, do you approach \textbf{INFININTEREST}?

\[
p\left(\frac{1}{4}\right) = (1+r\cdot \frac{1}{4})p(0) = \left(\frac{2}{3}\right)\cdot 1 = \frac{2}{4} = 2.00
\]

\[
p\left(\frac{1}{2}\right) = (1+r\cdot \frac{1}{2})^2 p(0) = \left(\frac{3}{2}\right)^2 \cdot 1 = \frac{9}{4} = 2.25
\]

\[
p\left(\frac{1}{3}\right) = (1+r\cdot \frac{1}{3})^3 p(0) = \left(\frac{4}{3}\right)^3 \cdot 1 = \frac{64}{27} = 2.37
\]

\[
p\left(\frac{1}{4}\right) = (1+r\cdot \frac{1}{4})^4 p(0) = \left(\frac{5}{4}\right)^4 \cdot 1 = \frac{625}{256} = 2.44
\]

\[
\text{Monthly: } p\left(\frac{1}{12}\right) = (1+r\cdot \frac{1}{12})^{12} p(0) = \left(\frac{13}{12}\right)^{12} \cdot 1 = 2.613
\]

\[
\text{Weekly: } p\left(\frac{1}{52}\right) = (1+r\cdot \frac{1}{52})^{52} p(0) = \left(\frac{53}{52}\right)^{52} \cdot 1 = 2.693
\]

\[
\text{Daily: } p\left(\frac{1}{365}\right) = (1+r\cdot \frac{1}{365})^{365} p(0) = \left(\frac{366}{365}\right)^{365} \cdot 1 = 2.7145
\]

\[
\text{Hrly: } p\left(\frac{1}{8760}\right) = (1+r\cdot \frac{1}{8760})^{8760} p(0) = \left(\frac{8761}{8760}\right)^{8760} \cdot 1 = 2.7181
\]
Let: $m \cdot r \cdot t = n$
or: $1/m = r \cdot t / n$

$$p^{1/m}(1) = (1 + \frac{1}{m})^m \quad \xrightarrow{m \to \infty} \quad 2.718281828459\ldots = e^{r \cdot t}$$

For $m = 1,000$

$$p^{1/m}(1) = 2.7169239322$$

For $m = 10,000$

$$p^{1/m}(1) = 2.7181459268$$

For $m = 100,000$

$$p^{1/m}(1) = 2.7182682372$$

For $m = 1,000,000$

$$p^{1/m}(1) = 2.7182804693$$

For $m = 10,000,000$

$$p^{1/m}(1) = 2.7182816925$$

For $m = 100,000,000$

$$p^{1/m}(1) = 2.7182818149$$

For $m = 1,000,000,000$

$$p^{1/m}(1) = 2.7182818271$$

**Interest product formula is really inefficient:** $10^6$ products for 6-figures! .. $10^9$ products for 9 ...
Interest product formula is really inefficient: $10^6$ products for 6-figures! .. $10^9$ products for 9 ...

\[ p^{1/m}(1) = (1 + \frac{1}{m})^m \rightarrow 2.718281828459 \ldots \]

\[ e \]

Let: \( m \cdot r \cdot t = n \)
or: \( \frac{1}{m} = r \cdot t / n \)

\[ (1 + \frac{1}{m})^{m \cdot r \cdot t} \rightarrow e^{r \cdot t} \]

Can improve computational efficiency using binomial theorem:

\[
(x + y)^n = x^n + n \cdot x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \ldots + n \cdot xy^{n-1} + y^n
\]

Define: Factorials (!):

\[ 0! = 1, 1! = 1, 2! = 1 \cdot 2, 3! = 1 \cdot 2 \cdot 3, \ldots \]
Interest product formula is really inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...

\[ p^{1/m}(1) = (1 + \frac{1}{m})^m \rightarrow e \quad m \rightarrow \infty \]

\[ e \approx 2.718281828459 \]

Let: \( m \cdot r \cdot t = n \)
or: \( \frac{1}{m} = \frac{r \cdot t}{n} \)

Can improve efficiency using binomial theorem:

\[(x + y)^n = x^n + n \cdot x^{n-1}y + \frac{n(n-1)}{2!} x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3}y^3 + ... + n \cdot xy^{n-1} + y^n \]

\[(1 + \frac{r \cdot t}{n})^n = 1 + n \cdot \left( \frac{r \cdot t}{n} \right) + \frac{n(n-1)}{2!} \left( \frac{r \cdot t}{n} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left( \frac{r \cdot t}{n} \right)^3 + ... \]

Define: Factorials(!):

\[ 0! = 1 = 1!, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3, ... \]

As \( n \rightarrow \infty \) let:

\[ n(n-1) \rightarrow n^2, \]

\[ n(n-1)(n-2) \rightarrow n^3, \text{ etc.} \]
Interest product formula is really inefficient: $10^6$ products for 6-figures! .. $10^9$ products for 9 ...

\[ p^{1/m}(1) = (1 + \frac{1}{m})^m \xrightarrow{m \to \infty} 2.718281828459\ldots = e \]

Let: \( m \cdot r \cdot t = n \)

or: \( \frac{1}{m} = \frac{r \cdot t}{n} \)

\[ (1 + \frac{1}{m})^m \cdot r \cdot t \xrightarrow{m \to \infty} e^{r \cdot t} \]

Can improve efficiency using binomial theorem:

\[
(x + y)^n = x^n + n \cdot x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \ldots + n \cdot x y^{n-1} + y^n
\]

\[
(1 + \frac{r \cdot t}{n})^n = 1 + n \cdot \left( \frac{r \cdot t}{n} \right) + \frac{n(n-1)}{2!} \left( \frac{r \cdot t}{n} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left( \frac{r \cdot t}{n} \right)^3 + \ldots
\]

Define: Factorials (!):

\[
0! = 1 = 1!, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3, \ldots
\]

As \( n \to \infty \) let:

\[
n(n-1) \to n^2,
\]

\[
n(n-1)(n-2) \to n^3, \text{ etc.}
\]

Precision order:

- \((o=1)\)-e-series = 2.00000 = 1 + 1
- \((o=2)\)-e-series = 2.50000 = 1 + 1 + 1/2
- \((o=3)\)-e-series = 2.66667 = 1 + 1 + 1/2 + 1/6
- \((o=4)\)-e-series = 2.70833 = 1 + 1 + 1/2 + 1/6 + 1/24
- \((o=5)\)-e-series = 2.71667 = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120
- \((o=6)\)-e-series = 2.71805 = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 + 1/720
- \((o=7)\)-e-series = 2.71825
- \((o=8)\)-e-series = 2.71828

About 12 summed quotients for 6-figure precision (A lot better!)
Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, etc.$

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Set $t=0$ to get $c_0 = x(0)$. 
Start with a general power series with constant coefficients $c_0, c_1, etc.$

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Rate of change of position $x(t)$ is *velocity* $v(t)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 t + 2 c_2 t^2 + 3 c_3 t^3 + 4 c_4 t^4 + 5 c_5 t^5 + ... + n c_n t^{n-1} +$$

Set $t=0$ to get $c_0 = x(0)$.  Set $t=0$ to get $c_1 = v(0)$.  

*Power Series Good!  Need general power series development*
Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0$, $c_1$, etc. Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

Rate of change of position $x(t)$ is velocity $v(t)$. Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is acceleration $a(t)$. Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$
Power Series Good!  Need general power series development

Start with a general power series with constant coefficients $c_0$, $c_1$, etc.  

Set $t=0$ to get $c_0 = x(0)$.  

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Rate of change of position $x(t)$ is velocity $v(t)$.  

Set $t=0$ to get $c_1 = v(0)$.  

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + n c_n t^{n-1} +$$

Change of velocity $v(t)$ is acceleration $a(t)$.  

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.  

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3 c_3 t + 3\cdot 4 c_4 t^2 + 4\cdot 5 c_5 t^3 + ... + n(n-1) c_n t^{n-2} +$$

Change of acceleration $a(t)$ is jerk $j(t)$.  (Jerk is NASA term.)  

Set $t=0$ to get $c_3 = \frac{1}{6} j(0)$.  

$$j(t) = \frac{d}{dt} a(t) = 0 + 2\cdot 3 c_3 + 2\cdot 3\cdot 4 c_4 t + 3\cdot 4\cdot 5 c_5 t^2 + ... + n(n-1)(n-2) c_n t^{n-3} +$$

Change of jerk $j(t)$ is inauguration $i(t)$.  (Be silly like NASA!)  

Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.  

$$i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot 3\cdot 4 c_4 + 2\cdot 3\cdot 4\cdot 5 c_5 t + ... + n(n-1)(n-2)(n-3) c_n t^{n-4} +$$
Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, etc.$

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Rate of change of position $x(t)$ is velocity $v(t)$. Set $t=0$ to get $c_0 = x(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is acceleration $a(t)$. Set $t=0$ to get $c_1 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + ... + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is jerk $j(t)$. (Jerk is NASA term.) Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

$$j(t) = \frac{d}{dt} a(t) = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 t + 3\cdot 4\cdot 5c_5 t^2 + ... + n(n-1)(n-2)c_n t^{n-3} +$$

Change of jerk $j(t)$ is inauguration $i(t)$. (Be silly like NASA!) Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.

$$i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5 t + ... + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0) t + \frac{1}{2!} a(0) t^2 + \frac{1}{3!} j(0) t^3 + \frac{1}{4!} i(0) t^4 + \frac{1}{5!} r(0) t^5 + ... + \frac{1}{n!} x^{(n)} t^n +$$
Power Series Good!  Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, etc.$  

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Rate of change of position $x(t)$ is velocity $v(t)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is acceleration $a(t)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + ... + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is jerk $j(t)$. (Jerk is NASA term.)

$$j(t) = \frac{d}{dt} a(t) = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 t + 3\cdot 4\cdot 5c_5 t^2 + ... + n(n-1)(n-2)c_n t^{n-3} +$$

Change of jerk $j(t)$ is inauguration $i(t)$. (Be silly like NASA!)

$$i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5 t + ... + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!} a(0) t^2 + \frac{1}{3!} j(0) t^3 + \frac{1}{4!} i(0) t^4 + \frac{1}{5!} r(0) t^5 + ... + \frac{1}{n!} x^{(n)} t^n +$$

Good old UP I formula!
Power Series Good! Need general power series development

Start with a general power series with constant coefficients \( c_0, c_1, \text{ etc.} \)

\[ x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n + \]

Rate of change of position \( x(t) \) is velocity \( v(t) \).

\[ v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} + \]

Change of velocity \( v(t) \) is acceleration \( a(t) \).

\[ a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} + \]

Change of acceleration \( a(t) \) is jerk \( j(t) \). (Jerk is NASA term.)

\[ j(t) = \frac{d}{dt} a(t) = 0 + 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4 t + 3 \cdot 4 \cdot 5c_5 t^2 + \ldots + n(n-1)(n-2)c_n t^{n-3} + \]

Change of jerk \( j(t) \) is inauguration \( i(t) \). (Be silly like NASA!)

\[ i(t) = \frac{d}{dt} j(t) = 0 + 2 \cdot 3 \cdot 4c_4 + 2 \cdot 3 \cdot 4 \cdot 5c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} + \]

Gives Maclaurin (or Taylor) power series

\[ x(t) = x(0) + v(0) t + \frac{1}{2}! a(0) t^2 + \frac{1}{3}! j(0) t^3 + \frac{1}{4}! i(0) t^4 + \frac{1}{5}! r(0) t^5 + \ldots + \frac{1}{n}! x^{(n)} t^n + \]

Setting all initial values to \( 1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots \)
gives exponential:

\[ e^t = 1 + t + \frac{1}{2}! t^2 + \frac{1}{3}! t^3 + \frac{1}{4}! t^4 + \frac{1}{5}! t^5 + \ldots + \frac{1}{n}! t^n + \]

Good old UP I formula!
But, how good are power series?

Gives Maclaurin (or Taylor) power series

\[ x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)}t^n + \]

Setting all initial values to \( I = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots \)

gives exponential:

\[ e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} t^5 + \ldots + \frac{1}{n!} t^n + \]
How good are power series? Depends...

\[ x(t) = \cos t = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \ldots \]

\[ x(t) = \sin t = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \ldots \]
1. The Story of e (A Tale of Great $\text{Interest}$)

How good are those power series?
Taylor-Maclaurin series,
imaginary interest, and complex exponentials
Suppose the fancy bankers really went bonkers and made interest rate $r$ an **imaginary number** $r = i\theta$.

Imaginary number $i = \sqrt{-1}$ powers have repeat-after-4-pattern: $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1,$ etc...

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ...$$  (From exponential series)

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - ...$$  

$(i = \sqrt{-1}$ imples: $i^1 = i, i^2 = -1, i^3 = -i, i^4 = +1, i^5 = i,...)$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - ...\right)$$
Suppose the fancy bankers really went bonkers and made interest rate \( r \) an *imaginary number* \( r = i \theta \).

Imaginary number \( i = \sqrt{-1} \) powers have *repeat-after-4-pattern*: \( i^0 = 1, \ i^1 = i, \ i^2 = -1, \ i^3 = -i, \ i^4 = 1, etc... \)

\[
e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots
\]

(From exponential series)

\[
e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots \quad \text{(} i = \sqrt{-1} \text{ imples: } i^1 = i, \ i^2 = -1, \ i^3 = -i, \ i^4 = +1, \ i^5 = i, \ldots \text{)}
\]

\[
e^{i\theta} = \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \right) + \left( i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \ldots \right)
\]

To match series for

\[
\begin{align*}
cos \theta : \cos x & = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sin \theta : \sin x & = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots 
\end{align*}
\]

\[
e^{i\theta} = \cos \theta + i \sin \theta
\]

*Euler-DeMoivre Theorem*
Suppose the fancy bankers really went bonkers and made interest rate $r$ an imaginary number $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have repeat-after-4-pattern: $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$, etc...

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ...$$

From exponential series

$$= 1 + i\theta - \frac{\theta^2}{2!} - i \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} - ...$$

($i = \sqrt{-1}$ imples: $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=+1$, $i^5=i$, ...)

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + \left(i\theta - i \frac{\theta^3}{3!} + i \frac{\theta^5}{5!} - ...ight)$$

To match series for

\[
\begin{align*}
\text{cosine: } \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\text{sine: } \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{align*}
\]

**Euler-DeMoivre Theorem**

Imaginary axis

(i axis)

$$z = re^{i\theta} = x + iy$$

$$(1 \text{ axis})$$

$x = r \cos \theta$

$y = r \sin \theta$

$re^{i\theta} = r \cos \theta + i \sin \theta$

Unit 1

Fig. 10.3
2. What Good Are Complex Exponentials?

- Easy trig
- Easy 2D vector analysis
- Easy oscillator phase analysis
- Easy rotation and "dot" or "cross" products
What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is \( \cos(a+b) \) or \( \sin(a+b) \)? Just factor \( e^{i(a+b)} = e^{ia}e^{ib} \) ...

\[
\begin{align*}
e^{i(a+b)} &= e^{ia}e^{ib} \\
cos(a+b) + i \sin(a+b) &= (\cos a + i \sin a) (\cos b + i \sin b) \\
cos(a+b) + i \sin(a+b) &= [\cos a \cos b - \sin a \sin b] + i[\sin a \cos b + \cos a \sin b]
\end{align*}
\]
What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is \( \cos(a+b) \) or \( \sin(a+b) \)? Just factor \( e^{i(a+b)} = e^{ia} e^{ib} \)

\[
\begin{align*}
\cos(a+b) + i \sin(a+b) &= (\cos a + i \sin a) (\cos b + i \sin b) \\
&= [\cos a \cos b - \sin a \sin b] + i[\sin a \cos b + \cos a \sin b]
\end{align*}
\]

2. Complex numbers add like vectors.

\( z_{\text{sum}} = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y') \)

\( z_{\text{diff}} = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y') \)

Unit 1
Fig. 10.6

Unit 1
Fig. 10.6
What Good Are Complex Exponentials? (contd.)

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors

(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos \omega t - iA\sin \omega t = x + iy$

Unit 1
Fig. 10.5
What Good Are Complex Exponentials? (contd.)

3. Complex exponentials $A e^{-j\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors

(b) Quantum Phasor Clock $\psi = A e^{-j\omega t} = A \cos \omega t - j A \sin \omega t = x + iy$

Some Rect-vs-Polar relations worth remembering

Cartesian ($x, y$) form

\[
\begin{align*}
\psi_x &= \text{Re } \psi(t) = x(t) = A \cos \omega t = \frac{\psi + \psi^*}{2} \\
\psi_y &= \text{Im } \psi(t) = \frac{\psi(t)}{\omega} = -A \sin \omega t = \frac{\psi - \psi^*}{2i} \\
\psi &= re^{i\theta} = re^{-i\omega t} = r(\cos \omega t - i \sin \omega t) \\
\psi^* &= re^{-i\theta} = re^{i\omega t} = r(\cos \omega t + i \sin \omega t)
\end{align*}
\]

Polar ($r, \theta$) form

\[
\begin{align*}
r &= A = |\psi| = \sqrt{\psi_x^2 + \psi_y^2} = \sqrt{\psi^* \psi} \\
\theta &= -\omega t = \arctan(\psi_y / \psi_x) \\
\cos \theta &= \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \\
\text{Re } \psi &= \frac{\psi + \psi^*}{2} \\
\sin \theta &= \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \\
\text{Im } \psi &= \frac{\psi - \psi^*}{2i}
\end{align*}
\]
2. What Good Are Complex Exponentials?

   Easy trig
   Easy 2D vector analysis
   Easy oscillator phase analysis
   Easy rotation and “dot” or “cross” products
4. **Complex products provide 2D rotation operations.**

\[
e^{i\phi} \cdot z = (\cos\phi + i \sin\phi) \cdot (x + iy) = x \cos\phi - y \sin\phi + i (x \sin\phi + y \cos\phi)
\]

\[
\mathbf{R}_{+\phi} \cdot \mathbf{r} = (x \cos\phi - y \sin\phi)\hat{e}_x + (x \sin\phi + y \cos\phi)\hat{e}_y
\]

\[
\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
x \cos \phi - y \sin \phi \\
x \sin \phi + y \cos \phi
\end{pmatrix}
\]
4. Complex products provide 2D rotation operations.

\[ e^{i\phi} \cdot z = (\cos\phi + i \sin\phi) \cdot (x + iy) = x \cos\phi - y \sin\phi + i (x \sin\phi + y \cos\phi) \]

\[ \mathbf{R} \cdot \mathbf{r} = (x \cos \phi - y \sin \phi) \hat{\mathbf{e}}_x + (x \sin \phi + y \cos \phi) \hat{\mathbf{e}}_y \]

\[ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \phi - y \sin \phi \\ x \sin \phi + y \cos \phi \end{pmatrix} \]

e^{i\phi} \text{ acts on this: } z = re^{i\theta}

to give this: \( e^{i\phi} e^{i\phi} z = re^{i\phi} e^{i\theta} \)
4. Complex products provide 2D rotation operations.

\[ e^{i\phi}z = (\cos\phi + i \sin\phi) \cdot (x + iy) = x \cos\phi - y \sin\phi + i \ (x \sin\phi + y \cos\phi) \]

\[ \mathbf{R}_{\phi} \cdot \mathbf{r} = (x \cos \phi - y \sin \phi) \hat{e}_x + (x \sin \phi + y \cos \phi) \hat{e}_y \]

\[
\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix} =
\begin{pmatrix}
x \cos\phi - y \sin\phi \\
x \sin\phi + y \cos\phi \\
\end{pmatrix}
\]

5. Complex products provide 2D “dot”(•) and “cross”(×) products.

Two complex numbers \( A = A_x + iA_y \) and \( B = B_x + iB_y \) and their “star” (*)-product \( A*B \).

\[ A * B = (A_x + iA_y) \ast (B_x + iB_y) = (A_x - iA_y)(B_x + iB_y) \]

\[ = (A_x B_x + A_y B_y) + i(A_x B_y - A_y B_x) = A \cdot B + i \ |A \times B| \mathbf{Z} \perp (x,y) \]

Real part is scalar or “dot”(•) product \( A \cdot B \).

Imaginary part is vector or “cross”(×) product, but just the Z-component normal to xy-plane.

Rewrite \( A * B \) in polar form.

\[ A * B = |A| e^{i\theta_A} \ast |B| e^{i\theta_B} = |A| e^{-i\theta_A} |B| e^{i\theta_B} = |A| |B| e^{i(\theta_B - \theta_A)} \]

\[ = |A| |B| \cos(\theta_B - \theta_A) + i |A| |B| \sin(\theta_B - \theta_A) = A \cdot B + i |A \times B| \mathbf{Z} \perp (x,y) \]
What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

\[ e^{i\phi} \cdot z = (\cos \phi + i \sin \phi) \cdot (x + iy) = x \cos \phi - y \sin \phi + i (x \sin \phi + y \cos \phi) \]

\[ \mathbf{R} + \phi \cdot \mathbf{r} = (x \cos \phi - y \sin \phi) \hat{e}_x + (x \sin \phi + y \cos \phi) \hat{e}_y \]

5. Complex products provide 2D “dot”(•) and “cross”(×) products.

Two complex numbers \( A = A_x + iA_y \) and \( B = B_x + iB_y \) and their “star” (*)-product \( A * B \).

\[ A * B = (A_x + iA_y)^* (B_x + iB_y) = (A_x - iA_y)(B_x + iB_y) \]

\[ = (A_x B_x + A_y B_y) + i(A_x B_y - A_y B_x) = A \cdot B + i |A \times B| \perp (x,y) \]

Real part is scalar or “dot”(•) product \( A \cdot B \).

Imaginary part is vector or “cross”(×) product, but just the Z-component normal to xy-plane.

Rewrite \( A * B \) in polar form.

\[ A * B = (|A| e^{i\theta_A})^* (|B| e^{i\theta_B}) = |A| e^{-i\theta_A} |B| e^{i\theta_B} = |A||B| e^{i(\theta_B - \theta_A)} \]

\[ = |A||B| \cos(\theta_B - \theta_A) + i |A||B| \sin(\theta_B - \theta_A) = A \cdot B + i |A \times B| \perp (x,y) \]

\[ A \cdot B = |A||B| \cos(\theta_B - \theta_A) \]

\[ = |A| \cos \theta_A |B| \cos \theta_B + |A| \sin \theta_A |B| \sin \theta_B \]

\[ = A_x B_x + A_y B_y \]

\[ |A \times B| = |A||B| \sin(\theta_B - \theta_A) \]

\[ = |A| \cos \theta_A |B| \sin \theta_B - |A| \sin \theta_A |B| \cos \theta_B \]

\[ = A_x B_y - A_y B_x \]
What Good are complex variables?

Easy 2D vector calculus
Easy 2D vector derivatives
Easy 2D source-free field theory
Easy 2D vector field-potential theory
What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence”(\(\nabla \cdot \mathbf{F}\)) and “curl”(\(\nabla \times \mathbf{F}\)) of 2D vector field

Relation of \((z, z^*)\) to \((x=\text{Re}z, y=\text{Im}z)\) defines a \(z\)-derivative \(\frac{df}{dz}\) and “star” \(z^*\)-derivative. \(\frac{df}{dz^*}\)

\[
z = x + iy \quad \quad x = \frac{1}{2} (z + z^*)
\]
\[
z^* = x - iy \quad \quad y = \frac{1}{2i} (z - z^*)
\]

}\[\frac{df}{dz} = \frac{\partial x}{\partial z} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{i}{2} \frac{\partial f}{\partial y}\]

\[\frac{df}{dz^*} = \frac{\partial x}{\partial z^*} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*} \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y}\]
6. Complex derivative contains “divergence” ($\nabla \cdot \mathbf{F}$) and “curl” ($\nabla \times \mathbf{F}$) of 2D vector field

Relation of $(z, z^*)$ to $(x=\text{Re}z, y=\text{Im}z)$ defines a $z$-derivative $\frac{df}{dz}$ and “star” $z^*$-derivative. $\frac{df}{dz^*}$

$$z = x + iy \quad x = \frac{1}{2} (z + z^*)$$
$$z^* = x - iy \quad y = \frac{1}{2}i (z - z^*)$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla \cdot \mathbf{f}$ and imaginary part has curl $\nabla \times \mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz} (f_x + if_y) = \frac{1}{2} (\frac{\partial f_x}{\partial x} - i \frac{\partial f_y}{\partial y}) (f_x + if_y) = \frac{1}{2} (\frac{df_x}{dx} + \frac{df_y}{dy}) + i \frac{1}{2} (\frac{df_y}{dx} - \frac{df_x}{dy}) = \frac{1}{2} \nabla \cdot \mathbf{f} + i \frac{1}{2} |\nabla \times \mathbf{f}|_{Z \perp (x, y)}$$
6. Complex derivative contains “divergence”\((\nabla \cdot \mathbf{F})\) and “curl”\((\nabla \times \mathbf{F})\) of 2D vector field

Relation of \((z,z^*)\) to \((x=\text{Re}z, y=\text{Im}z)\) defines a \(z\)-derivative \(\frac{df}{dz}\) and “star” \(z^*\)-derivative. \(\frac{df}{dz^*}\)

\[
z = x + iy \quad x = \frac{1}{2} (z + z^*)
\]

\[
z^* = x - iy \quad y = \frac{1}{2i} (z - z^*)
\]

Derivative chain-rule shows real part of \(\frac{df}{dz}\) has 2D divergence \(\nabla \cdot \mathbf{f}\) and imaginary part has curl \(\nabla \times \mathbf{f}\).

\[
\frac{df}{dz} = \frac{dz}{dz} (f_x + i f_y) = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) + i \frac{1}{2} \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) = \frac{1}{2} \nabla \cdot \mathbf{f} + i \frac{1}{2} |\nabla \times \mathbf{f}| Z \perp (x,y)
\]

7. Invent source-free 2D vector fields \([\nabla \cdot \mathbf{F}=0 \text{ and } \nabla \times \mathbf{F}=0]\)

We can invent source-free 2D vector fields that are both zero-divergence and zero-curl. Take any function \(f(z)\), conjugate it (change all \(i\)’s to \(-i\)) to give \(f^*(z^*)\) for which \(\frac{df^*}{dz} = 0\).
6. Complex derivative contains “divergence” (\(\nabla \cdot \mathbf{F}\)) and “curl” (\(\nabla \times \mathbf{F}\)) of 2D vector field

Relation of \((z, z^*)\) to \((x = \text{Re} z, y = \text{Im} z)\) defines a \(z\)-derivative \(\frac{df}{dz}\) and “star” \(z^*\)-derivative. \(\frac{df}{dz^*}\)

\[
z = x + iy \quad x = \frac{1}{2} (z + z^*) \quad \frac{df}{dz} = \frac{\partial x}{\partial z} \frac{df}{dx} + \frac{\partial y}{\partial z} \frac{df}{dy} = \frac{1}{2} \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}
\]

\[
z^* = x - iy \quad y = \frac{1}{2i} (z - z^*) \quad \frac{df}{dz^*} = \frac{\partial x}{\partial z^*} \frac{df}{dx} + \frac{\partial y}{\partial z^*} \frac{df}{dy} = \frac{1}{2} \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}
\]

Derivative chain-rule shows real part of \(\frac{df}{dz}\) has 2D divergence \(\nabla \cdot \mathbf{f}\) and imaginary part has curl \(\nabla \times \mathbf{f}\).

\[
\frac{df}{dz} = \frac{d}{dz} (f_x + i f_y) = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) (f_x + i f_y) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) + i \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) = \frac{1}{2} \nabla \cdot \mathbf{f} + i \frac{1}{2} |\nabla \times \mathbf{f}|_{Z \perp (x,y)}
\]

7. Invent source-free 2D vector fields [\(\nabla \cdot \mathbf{F} = 0\) and \(\nabla \times \mathbf{F} = 0\)]

We can invent source-free 2D vector fields that are both zero-divergence and zero-curl. Take any function \(f(z)\), conjugate it (change all \(i\)’s to \(-i\)) to give \(f^*(z^*)\) for which \(\frac{df}{dz} = 0\)

For example: if \(f(z) = a \cdot z\) then \(f^*(z^*) = a \cdot z^* = a(x-iy)\) is not function of \(z\) so it has zero \(z\)-derivative.

\(\mathbf{F} = (F_x, F_y) = (f^*_x, f^*_y) = (a \cdot x, -a \cdot y)\) has zero divergence: \(\nabla \cdot \mathbf{F} = 0\) and has zero curl: \(|\nabla \times \mathbf{F}| = 0\).

\[
\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial (ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0
\]

\[
|\nabla \times \mathbf{F}|_{Z \perp (x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial (-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0
\]

A DFL field \(\mathbf{F}\) (Divergence-Free-Laminar)
7. Invent source-free 2D vector fields \([\nabla \cdot \mathbf{F} = 0 \text{ and } \nabla \times \mathbf{F} = 0]\)

We can invent source-free 2D vector fields that are both zero-divergence and zero-curl. Take any function \(f(z)\), conjugate it (change all \(i\)'s to \(-i\)) to give \(f^*(z^*)\) for which

For example: if \(f(z) = a \cdot z\) then \(f^*(z^*) = a \cdot z^* = a(x-iy)\) is not a function of \(z\) so it has zero \(z\)-derivative.

\[\mathbf{F} = (F_x, F_y) = (f^*_x, f^*_y) = (a \cdot x, -a \cdot y)\] has zero divergence: \(\nabla \cdot \mathbf{F} = 0\) and has zero curl: \(\nabla \times \mathbf{F} = 0\).

\[
\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial (ax)}{\partial x} + \frac{\partial (-ay)}{\partial y} = 0
\]

\[
|\nabla \times \mathbf{F}|_{z \perp (x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial (-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0
\]

\[\mathbf{F} = (f^*_x, f^*_y) = (a \cdot x, -a \cdot y)\] is a divergence-free laminar (DFL) field.
What Good are complex variables?
- Easy 2D vector calculus
- Easy 2D vector derivatives
- Easy 2D source-free field theory
- Easy 2D vector field-potential theory
8. Complex potential $\phi$ contains “scalar” ($F = \nabla \Phi$) and “vector” ($F = \nabla \times A$) potentials

Any DFL field $F$ is a gradient of a scalar potential field $\Phi$ or a curl of a vector potential field $A$.

$F = \nabla \Phi$  \hspace{1cm} $F = \nabla \times A$

A complex potential $\phi(z) = \Phi(x,y) + iA(x,y)$ exists whose $z$-derivative is $f(z) = d\phi/dz$.

Its complex conjugate $\phi^*(z^*) = \Phi(x,y) - iA(x,y)$ has $z^*$-derivative $f^*(z^*) = d\phi^*/dz^*$ giving DFL field $F$. 

Tuesday, October 9, 2012
8. Complex potential \( \phi \) contains “scalar”\((F=\nabla \Phi)\) and “vector”\((F=\nabla \times \mathbf{A})\) potentials

Any DFL field \( \mathbf{F} \) is a gradient of a scalar potential field \( \Phi \) or a curl of a vector potential field \( \mathbf{A} \).

\[
\mathbf{F} = \nabla \Phi \quad \quad \mathbf{F} = \nabla \times \mathbf{A}
\]

A complex potential \( \phi(z) = \Phi(x,y) + iA(x,y) \) exists whose \( z \)-derivative is \( f(z) = d\phi/dz \).

Its complex conjugate \( \phi^*(z^*) = \Phi(x,y) - iA(x,y) \) has \( z^* \)-derivative \( f^*(z^*) = d\phi^*/dz^* \) giving DFL field \( \mathbf{F} \).

To find \( \phi = \Phi + iA \) integrate \( f(z) = a\cdot z \) to get \( \phi \) and isolate real \((\text{Re} \ \phi = \Phi)\) and imaginary \((\text{Im} \ \phi = A)\) parts.
What Good Are Complex Exponentials? (contd.)

8. Complex potential \( \phi \) contains “scalar”(\( \mathbf{F} = \nabla \Phi \)) and “vector”(\( \mathbf{F} = \nabla \times \mathbf{A} \)) potentials

Any DFL field \( \mathbf{F} \) is a gradient of a scalar potential field \( \Phi \) or a curl of a vector potential field \( \mathbf{A} \).

\[
\mathbf{F} = \nabla \Phi \quad \mathbf{F} = \nabla \times \mathbf{A}
\]

A complex potential \( \phi(z) = \Phi(x,y) + iA(x,y) \) exists whose \( z \)-derivative is \( f(z) = d\phi/dz \).

Its complex conjugate \( \phi^*(z^*) = \Phi(x,y) - iA(x,y) \) has \( z^* \)-derivative \( f^*(z^*) = d\phi^*/dz^* \) giving DFL field \( \mathbf{F} \).

To find \( \phi = \Phi + iA \) integrate \( f(z) = a \cdot z \) to get \( \phi \) and isolate real (\( \Re \phi = \Phi \)) and imaginary (\( \Im \phi = A \)) parts.

\[
f(z) = \frac{d\phi}{dz} \Rightarrow \quad \phi = \Phi + iA = \int f \cdot dz = \int a \cdot dz = \frac{1}{2} a z^2
\]
8. Complex potential $\phi$ contains “scalar”($\mathbf{F} = \nabla \Phi$) and “vector”($\mathbf{F} = \nabla \times \mathbf{A}$) potentials

Any DFL field $\mathbf{F}$ is a gradient of a scalar potential field $\Phi$ or a curl of a vector potential field $\mathbf{A}$.

$$\mathbf{F} = \nabla \Phi \quad \mathbf{F} = \nabla \times \mathbf{A}$$

A complex potential $\phi(z) = \Phi(x,y) + iA(x,y)$ exists whose $z$-derivative is $f(z) = d\phi/dz$.

Its complex conjugate $\phi^*(z^*) = \Phi(x,y) - iA(x,y)$ has $z^*$-derivative $f^*(z^*) = d\phi^*/dz^*$ giving DFL field $\mathbf{F}$.

To find $\phi = \Phi + iA$ integrate $f(z) = a \cdot z$ to get $\phi$ and isolate real ($\text{Re} \phi = \Phi$) and imaginary ($\text{Im} \phi = A$) parts.

$$f(z) = \frac{d\phi}{dz} \Rightarrow \phi = \Phi + i \int f \cdot dz = \int a z \cdot dz = \frac{1}{2} a z^2 = \frac{1}{2} a (x + iy)^2$$

$$= \frac{1}{2} a (x^2 - y^2) + i ax y$$
8. Complex potential $\phi$ contains “scalar”($\mathbf{F}=\nabla \Phi$) and “vector”($\mathbf{F}=\nabla \times \mathbf{A}$) potentials

Any DFL field $\mathbf{F}$ is a gradient of a scalar potential field $\Phi$ or a curl of a vector potential field $\mathbf{A}$.

$$\mathbf{F} = \nabla \Phi \quad \mathbf{F} = \nabla \times \mathbf{A}$$

A complex potential $\phi(z) = \Phi(x,y) + i\mathbf{A}(x,y)$ exists whose $z$-derivative is $f(z) = d \phi/dz$.

Its complex conjugate $\phi^*(z^*) = \Phi(x,y) - i\mathbf{A}(x,y)$ has $z^*$-derivative $f^*(z^*) = d \phi^*/dz^*$ giving DFL field $\mathbf{F}$.

To find $\phi = \Phi + i\mathbf{A}$ integrate $f(z) = a\cdot z$ to get $\phi$ and isolate real ($\text{Re } \phi = \Phi$) and imaginary ($\text{Im } \phi = \mathbf{A}$) parts.

$$f(z) = \frac{d\phi}{dz} \Rightarrow \phi = \Phi + i \int f \cdot dz = \int az \cdot dz = \frac{1}{2} az^2 = \frac{1}{2} a(x + iy)^2$$

$$= \frac{1}{2} a(x^2 - y^2) + i axy$$
8. Complex potential $\phi$ contains “scalar”($F = \nabla \Phi$) and “vector”($F = \nabla \times A$) potentials

Any **DFL** field $F$ is a gradient of a *scalar potential field* $\Phi$ or a curl of a *vector potential field* $A$.

$$F = \nabla \Phi \quad F = \nabla \times A$$

A **complex potential** $\phi(z) = \Phi(x,y) + iA(x,y)$ exists whose $z$-derivative is $f(z) = d\phi/dz$.

Its complex conjugate $\phi^*(z^*) = \Phi(x,y) - iA(x,y)$ has $z^*$-derivative $f^*(z^*) = d\phi^*/dz^*$ giving **DFL** field $F$.

To find $\phi = \Phi + iA$ integrate $f(z) = a \cdot z$ to get $\phi$ and isolate real ($\text{Re } \phi = \Phi$) and imaginary ($\text{Im } \phi = A$) parts.

$$f(z) = a\frac{dz}{dz} \Rightarrow \phi = \Phi + iA = \int f \cdot dz = \int az \cdot dz = \frac{1}{2} a z^2 = \frac{1}{2} a(x + iy)^2$$

**BONUS!**

Get a free coordinate system!

The $(\Phi, A)$ grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = \text{const.}$

$q^2 = A = (xy) = \text{const.}$

*Actually it’s OCC.
What Good are complex variables?

Easy 2D vector calculus
Easy 2D vector derivatives
Easy 2D source-free field theory
Easy 2D vector field-potential theory

The half-n‘half results: (Riemann-Cauchy Derivative Relations)
What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential $\phi$ contains “scalar”($\mathbf{F}=\nabla \Phi$) and “vector”($\mathbf{F}=\nabla \times \mathbf{A}$) potentials

...and either one (or half-n’-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix}$ of scalar $\Phi$ and curl $\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix}$ of vector $\mathbf{A}$ (and they’re equal!)

$f(z) = \frac{\phi}{dz} \Rightarrow$

$$\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - i \mathbf{A}) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)(\Phi - i \mathbf{A}) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times \mathbf{A}$$
What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential $\phi$ contains "scalar"($F=\nabla \Phi$) and "vector"($F=\nabla \times A$) potentials

...and either one (or half-n’-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix}$ of scalar $\Phi$ and curl $\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix}$ of vector $A$ (and they’re equal!)

$f(z) = \frac{d\phi}{dz} \Rightarrow \frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - iA) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\Phi - iA) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times A$

Note, mathematician definition of force field $F=+\nabla \Phi$ replaces usual physicist’s definition $F=-\nabla \Phi$
Derivative $\frac{d\phi}{dz^*}$ has 2D gradient $\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)$ of scalar $\phi$ and curl $\nabla \times A = \left( \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x} \right)$ of vector $A$ (and they’re equal!)

...and either one (or half-n’-half!) works just as well.

Note, mathematician definition of force field $F=+\nabla \Phi$ replaces usual physicist’s definition $F=-\nabla \Phi$
What Good Are Complex Exponentials? (contd.)

8. (contd.) **Complex potential** \( \phi \) contains “**scalar**” \( \mathbf{F} = \nabla \phi \) and “**vector**” \( \mathbf{F} = \nabla \times \mathbf{A} \) potentials

...and either one (or half-n’-half!) works just as well.

Derivative \( \frac{d\phi^*}{dz^*} \) has 2D gradient \( \nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix} \) of scalar \( \phi \) and curl \( \nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} \) of vector \( \mathbf{A} \) (and they’re equal!)

\[
\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\phi - iA) = \frac{1}{2} \left( \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right) (\phi - iA) = \frac{1}{2} \left( \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \phi + \frac{1}{2} \nabla \times \mathbf{A}
\]

Note, mathematician definition of force field \( \mathbf{F} = \nabla \phi \) replaces usual physicist’s definition \( \mathbf{F} = -\nabla \phi \)

Scalar **static potential lines** \( \phi = \text{const.} \) and vector **flux potential lines** \( \mathbf{A} = \text{const.} \) define **DFL field-net**.
What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential $\phi$ contains "scalar"($F=\nabla \Phi$) and "vector"($F=\nabla \times \mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix}$ of scalar $\Phi$ and curl $\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix}$ of vector $\mathbf{A}$ (and they’re equal!)

\[
\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - i \mathbf{A}) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\Phi - i \mathbf{A}) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times \mathbf{A}
\]

Note, mathematician definition of force field $F=+\nabla \Phi$ replaces usual physicist’s definition $F=-\nabla \Phi$

Given $\phi$: find:

\[
\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}
\]

or find:

\[
\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix} = \mathbf{F}
\]

Scalar static potential lines $\Phi=\text{const.}$ and vector flux potential lines $\mathbf{A}=\text{const.}$ define DFL field-net.

The half-n'-half results are called Riemann-Cauchy Derivative Relations

\[
\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y} \quad \text{is:} \quad \frac{\partial \text{Re}f(z)}{\partial x} = \frac{\partial \text{Im}f(z)}{\partial y}
\]

\[
\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x} \quad \text{is:} \quad \frac{\partial \text{Re}f(z)}{\partial y} = -\frac{\partial \text{Im}f(z)}{\partial x}
\]
4. Riemann-Cauchy conditions What’s analytic? (…and what’s not?)
Review \((z,z^*)\) to \((x,y)\) transformation relations

\[
\begin{align*}
z &= x + iy \\
z^* &= x - iy
\end{align*}
\]

\[
\begin{align*}
x &= \frac{1}{2} (z + z^*) \\
y &= \frac{1}{2 i} (z - z^*)
\end{align*}
\]

\[
\begin{align*}
\frac{df}{dz} &= \frac{\partial f}{\partial z} + \frac{1}{2} \left( \frac{\partial f}{\partial x} - \frac{i}{2} \frac{\partial f}{\partial y} \right) \\
\frac{df}{dz^*} &= \frac{\partial f}{\partial z^*} + \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y} \right)
\end{align*}
\]

Criteria for a field function \(f = f_x(x,y) + i f_y(x,y)\) to be an \textbf{analytic function} \(f(z)\) of \(z = x + iy\):

First, \(f(z)\) must \textbf{not} be a function of \(z^* = x - iy\), that is: \(\frac{df}{dz^*} = 0\)

This implies \(f(z)\) satisfies differential equations known as the \textbf{Riemann-Cauchy conditions}

\[
\begin{align*}
\frac{\partial f_x}{\partial x} &= \frac{\partial f_y}{\partial y} \\
\frac{\partial f_y}{\partial x} &= -\frac{\partial f_x}{\partial y}
\end{align*}
\]
Review \((z,z^*)\) to \((x,y)\) transformation relations

\[
\begin{align*}
z &= x + iy \\
z^* &= x - iy
\end{align*}
\]

\[
\begin{align*}
x &= \frac{1}{2} (z + z^*) \\
y &= \frac{1}{2i} (z - z^*)
\end{align*}
\]

\[
\begin{align*}
df \over dz &= \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \\
df \over dz^* &= \frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = \frac{1}{2} \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}
\end{align*}
\]

Criteria for a field function \(f = f(x,y) + if_y(x,y)\) to be an analytic function \(f(z)\) of \(z = x + iy\):
First, \(f(z)\) must not be a function of \(z^* = x - iy\), that is: \(df \over dz^* = 0\)

This implies \(f(z)\) satisfies differential equations known as the

\[\text{Riemann-Cauchy conditions}\]

\[
\begin{align*}
\frac{\partial f_x}{\partial x} &= \frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}
\end{align*}
\]

Criteria for a field function \(f = f_*(x,y) + i f_y(x,y)\) to be an analytic function \(f(z)\) of \(z = x + iy\):
First, \(f(z)\) must not be a function of \(z^* = x - iy\), that is: \(df \over dz^* = 0\)

This implies \(f(z)\) satisfies differential equations we call \(\text{Anti-Riemann-Cauchy conditions}\)

\[
\begin{align*}
\frac{\partial f_x}{\partial x} &= -\frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}
\end{align*}
\]
What’s analytic? (...and what’s not?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z = z + iy$?
What’s analytic? (...and what’s not?)

Example: Q: Is \( f(x,y) = 2x + i4y \) an analytic function of \( z = z + iy \)?

Well, test it using definitions:

\[
\begin{align*}
  z &= x + iy \\
  z^* &= x - iy \\
  or: \quad x &= (z+z^*)/2 \\
  and: \quad y &= -i(z-z^*)/2
\end{align*}
\]
What’s analytic? (...and what’s not?)

Example: Q: Is \( f(x,y) = 2x + i4y \) an analytic function of \( z = z + iy \)?

Well, test it using definitions: 
\[
\begin{align*}
z &= x + iy & \text{and:} & \quad z^* &= x - iy \\
or: & \quad x = (z+z^*)/2 & & \text{and:} & \quad y = -i(z-z^*)/2
\end{align*}
\]

\[
f(x,y) = 2x + i4y = 2 \frac{(z+z^*)}{2} + i4\left(-i\frac{(z-z^*)}{2}\right)
\]
What’s analytic? (...and what’s not?)

Example: Q: Is \( f(x,y) = 2x + i4y \) an analytic function of \( z = z + iy \)?

Well, test it using definitions:

\[
\begin{align*}
z &= x + iy \\
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or: & \quad x = (z+z^*)/2 \\
\text{and:} & \quad y = -i(z-z^*)/2
\end{align*}
\]

\[
f(x,y) = 2x + i4y = 2 \frac{(z+z^*)}{2} + i4 \frac{-i(z-z^*)}{2}
\]

\[
= z + z^* + (2z - 2z^*)
\]
What’s analytic? (…and what’s not?)

Example: Q: Is \( f(x,y) = 2x + 4iy \) an analytic function of \( z = x + iy \)?

Well, test it using definitions:

\[
\begin{align*}
z &= x + iy & \text{and:} & & z^* &= x - iy \\
\text{or:} & & x &= \frac{z+z^*}{2} & \text{and:} & & y &= -\frac{i(z-z^*)}{2}
\end{align*}
\]

\[
\begin{align*}
f(x,y) &= 2x + 4iy = 2 \frac{z+z^*}{2} + 4i \frac{-i(z-z^*)}{2} \\
&= z + z^* + (2z - 2z^*) \\
&= 3z - z^*
\end{align*}
\]
**What’s analytic? (...and what’s not?)**

Example: Q: Is \( f(x,y) = 2x + i4y \) an analytic function of \( z = z + iy \)?

Well, test it using definitions:
- \( z = x + iy \)
- \( z^* = x - iy \)
- \( x = (z + z^*)/2 \)
- \( y = -i(z - z^*)/2 \)

Then,
\[
f(x,y) = 2x + i4y = 2 \frac{(z + z^*)}{2} + i4 \frac{-i(z - z^*)}{2}
\]
\[
= z + z^* + 2z - 2z^*
\]
\[
= 3z - z^*
\]

A: \( \textbf{NO!} \)  It’s a function of \( z \) and \( z^* \) so not analytic for either.
What’s analytic? (...and what’s not?)

Example: Q: Is \( f(x,y) = 2x + i4y \) an analytic function of \( z=z+iy \)?

Well, test it using definitions: \( z = x + iy \) and: \( z^* = x - iy \)

or: \( x = (z+z^*)/2 \) and: \( y =-i(z-z^*)/2 \)

\[
f(x,y) = 2x + i4y = 2 \left( \frac{z+z^*}{2} \right) + i4 \left( -\frac{i(z-z^*)}{2} \right) \\
= \frac{z+z^*}{2} + (2z-2z^*) \\
= \frac{3z-z^*}{2}
\]

A: **NO!** It’s a function of \( z \) and \( z^* \) so not analytic for either.

Example 2: Q: Is \( r(x,y) = x^2 + y^2 \) an analytic function of \( z=z+iy \)?

A: **NO!** \( r(xy) = z^*z \) is a function of \( z \) and \( z^* \) so not analytic for either.
What’s analytic? (...and what’s not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z = z + iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y =-i(z-z^*)/2$

\[
f(x,y) = 2x + i4y = 2 \left( \frac{z+z^*}{2} \right) + i4 \left( \frac{-i(z-z^*)}{2} \right)
= \frac{z+z^*}{2} + \frac{2z-2z^*}{2}
= \frac{3z-z^*}{2}
\]

A: **NO!** It’s a function of $z$ and $z^*$ so not analytic for either.

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z = z + iy$?

A: **NO!** $r(xy) = z^*z$ is a function of $z$ and $z^*$ so not analytic for either.

Example 3: Q: Is $s(x,y) = x^2 - y^2 + 2ixy$ an analytic function of $z = z + iy$?

A: **YES!** $s(xy) = (x+iy)^2 = z^2$ is analytic function of $z$. (Yay!)
4. **Riemann-Cauchy conditions** What’s analytic? (...and what’s not?)

- Easy 2D circulation and flux integrals
- Easy 2D curvilinear coordinate discovery
- Easy 2D monopole, dipole, and $2^n$-pole analysis
- Easy $2^n$-multipole field and potential expansion
- Easy stereo-projection visualization
What Good Are Complex Exponentials? (contd.)

9. Complex integrals \( \int f(z)dz \) count 2D “circulation” \( \int \mathbf{F} \cdot d\mathbf{r} \) and “flux” \( \int \mathbf{F} \times d\mathbf{r} \)

Integral of \( f(z) \) between point \( z_1 \) and point \( z_2 \) is potential difference \( \Delta \phi = \phi(z_2) - \phi(z_1) \)

\[
\Delta \phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \Phi(x_2, y_2) - \Phi(x_1, y_1) + i[A(x_2, y_2) - A(x_1, y_1)]
\]

\[\Delta \phi = \Delta \Phi + i \Delta A\]

In DFL-field \( \mathbf{F} \), \( \Delta \phi \) is independent of the integration path \( z(t) \) connecting \( z_1 \) and \( z_2 \).
9. Complex integrals \( \int f(z)dz \) count 2D “circulation” (\( \int F \cdot dr \)) and “flux” (\( \int F \times dr \))

Integral of \( f(z) \) between point \( z_1 \) and point \( z_2 \) is potential difference \( \Delta \phi = \phi(z_2) - \phi(z_1) \)

\[
\Delta \phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \Phi(x_2, y_2) - \Phi(x_1, y_1) + i[A(x_2, y_2) - A(x_1, y_1)]
\]

In DFL-field \( \mathbf{F} \), \( \Delta \phi \) is independent of the integration path \( z(t) \) connecting \( z_1 \) and \( z_2 \).

\[
\int f(z)dz = \int \left( f^*(z^*) \right)^* dz = \int \left( f^*(z^*) \right)^* \left( dx + i dy \right) = \int \left( f_x^* + i f_y^* \right)^* \left( dx + i dy \right) = \int \left( f_x^* - i f_y^* \right) \left( dx + i dy \right)
\]

\[
= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx)
\]

\[
= \int F \cdot dr + i \int F \times dr \cdot \hat{e}_Z
\]

\[
= \int F \cdot dr + i \int F \cdot dS \quad \text{where:} \quad dS = dr \times \hat{e}_Z
\]
9. Complex integrals $\int f(z)dz$ count 2D “circulation”($\int \mathbf{F} \cdot d\mathbf{r}$) and “flux”($\int \mathbf{F} \times d\mathbf{S}$)

Integral of $f(z)$ between point $z_1$ and point $z_2$ is potential difference $\Delta \phi = \phi(z_2) - \phi(z_1)$

$$\Delta \phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \Phi(x_2, y_2) - \Phi(x_1, y_1) + i[A(x_2, y_2) - A(x_1, y_1)]$$

$$\Delta \phi = \Delta \Phi + i \Delta A$$

In DFL-field $\mathbf{F}$, $\Delta \phi$ is independent of the integration path $z(t)$ connecting $z_1$ and $z_2$.

$$\int f(z)dz = \int \left( f^*(z^*) \right)^* dz = \int \left( f^* (z^*) \right)^* (dx + i dy) = \int \left( f_x^* + i f_y^* \right) (dx + i dy) = \int \left( f_x^* - i f_y^* \right) (dx + i dy)$$

$$= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx)$$

$$= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \times d\mathbf{r} \cdot \hat{e}_Z$$

$$= \int \mathbf{F} \cdot d\mathbf{r} + i \int \mathbf{F} \cdot d\mathbf{S} \quad \text{where:} \quad d\mathbf{S} = d\mathbf{r} \times \hat{e}_Z$$

Real part $\int \mathbf{F} \cdot d\mathbf{r} = \Delta \Phi$

sums $\mathbf{F}$ projections along path $d\mathbf{r}$ that is, circulation on path to get $\Delta \Phi$.

Imaginary part $\int \mathbf{F} \times d\mathbf{S} = \Delta A$

sums $\mathbf{F}$ projection across path $d\mathbf{r}$ that is, flux thru surface elements $d\mathbf{S} = d\mathbf{r} \times \hat{e}_Z$ normal to $d\mathbf{r}$ to get $\Delta A$. 
Here the scalar potential $\Phi = (x^2 - y^2)/2$ is stereo-plotted vs. $(x,y)$

The $\Phi = (x^2 - y^2)/2 = \text{const.}$ curves are topography lines

The $A = (xy) = \text{const.}$ curves are streamlines normal to topography lines
4. Riemann-Cauchy conditions What’s analytic? (…and what’s not?)

Easy 2D circulation and flux integrals
Easy 2D curvilinear coordinate discovery
Easy 2D monopole, dipole, and $2^n$-pole analysis
Easy $2^n$-multipole field and potential expansion
Easy stereo-projection visualization
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,Α) grid is a GCC coordinate system*:

\[ q^1 = Φ = \frac{(x^2-y^2)}{2} = \text{const.} \]
\[ q^2 = Α = (xy) = \text{const.} \]

*Actually it’s OCC.

**Kajobian**

\[
\begin{bmatrix}
\frac{∂q^1}{∂x} & \frac{∂q^1}{∂y} \\
\frac{∂q^2}{∂x} & \frac{∂q^2}{∂y}
\end{bmatrix}
\begin{bmatrix}
\frac{∂Φ}{∂x} & \frac{∂Φ}{∂y} \\
\frac{∂Α}{∂x} & \frac{∂Α}{∂y}
\end{bmatrix}
= \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \leftrightarrow E^Φ
\]

**Jacobian**

\[
\begin{bmatrix}
\frac{∂x}{∂q^1} & \frac{∂x}{∂q^2} \\
\frac{∂y}{∂q^1} & \frac{∂y}{∂q^2}
\end{bmatrix}
\begin{bmatrix}
\frac{∂x}{∂Α} & \frac{∂x}{∂Φ} \\
\frac{∂y}{∂Α} & \frac{∂y}{∂Φ}
\end{bmatrix}
= \frac{1}{r^2} \begin{bmatrix} x & y \\ -y & x \end{bmatrix}
\]

**Metric tensor**

\[
\begin{pmatrix}
g_{ΦΦ} & g_{ΦΑ} \\
g_{ΑΦ} & g_{ΑΑ}
\end{pmatrix}
= \begin{pmatrix}
E_Φ \cdot E_Φ & E_Φ \cdot E_Α \\
E_Α \cdot E_Φ & E_Α \cdot E_Α
\end{pmatrix}
= \begin{pmatrix} r^2 & 0 \\
0 & r^2 \end{pmatrix}
\]

where: \( r^2 = x^2+y^2 \)
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The \((\Phi, A)\) grid is a GCC coordinate system*:

- \(q^1 = \Phi = (x^2 - y^2)/2 \quad \text{const.}\)
- \(q^2 = A = (xy) = \text{const.}\)

*Actually it’s OCC.

\[
\begin{align*}
\text{Kajobian} &= \begin{pmatrix}
\frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\
\frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\
\frac{\partial A}{\partial x} & \frac{\partial A}{\partial y}
\end{pmatrix} = \begin{pmatrix}
x & -y \\
y & x
\end{pmatrix} \leftarrow E^\Phi
\end{align*}
\]

\[
\begin{align*}
\text{Jacobian} &= \begin{pmatrix}
\frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\
\frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\
\frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A}
\end{pmatrix} = \begin{pmatrix}
x & y \\
y & x
\end{pmatrix} \quad \text{where: } r^2 = x^2 + y^2
\end{align*}
\]

\[
\begin{align*}
\text{Metric tensor} &= \begin{pmatrix}
g_{\Phi\Phi} & g_{\Phi A} \\
g_{A\Phi} & g_{AA}
\end{pmatrix} = \begin{pmatrix}
E_{\Phi} \cdot E_{\Phi} & E_{\Phi} \cdot E_{A} \\
E_{A} \cdot E_{\Phi} & E_{A} \cdot E_{A}
\end{pmatrix} = \begin{pmatrix}
r^2 & 0 \\
0 & r^2
\end{pmatrix}
\end{align*}
\]

**Riemann-Cauchy Derivative Relations make coordinates orthogonal**

\[
\nabla \Phi = \begin{pmatrix}
\frac{\partial \Phi}{\partial x} \\
\frac{\partial \Phi}{\partial y}
\end{pmatrix} = \begin{pmatrix}
ax \\
-ay
\end{pmatrix} = F
\]

\[
The \text{half-}n\text{-half results assure}
\begin{align*}
E_{\Phi} \cdot E_{A} &= \frac{\partial \Phi}{\partial x} \frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial A}{\partial y} \\
&= -\frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y} \frac{\partial \Phi}{\partial x} = 0
\end{align*}
\]

\[
\nabla \times A = \begin{pmatrix}
\frac{\partial A}{\partial y} \\
-\frac{\partial A}{\partial x}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial y} axy \\
-\frac{\partial}{\partial x} axy
\end{pmatrix} = \begin{pmatrix}
ax \\
-ay
\end{pmatrix} = F
\]
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The \((\Phi,A)\) grid is a GCC coordinate system*:

\[ q^1 = \Phi = (x^2-y^2)/2 = \text{const.} \]
\[ q^2 = A = (xy) = \text{const.} \]

*Actually it’s OCC.

\[
\nabla \Phi = \begin{pmatrix}
\frac{\partial \Phi}{\partial x} \\
\frac{\partial \Phi}{\partial y}
\end{pmatrix} = \begin{pmatrix}
ax \\
-ay
\end{pmatrix} = F
\]

\[
\nabla \times A = \begin{pmatrix}
\frac{\partial A}{\partial y} \\
-\frac{\partial A}{\partial x}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial y} axy \\
-\frac{\partial}{\partial x} axy
\end{pmatrix} = \begin{pmatrix}
ax \\
-ay
\end{pmatrix} = F
\]

\[
\text{The half-n’-half results assure}
\]

or Riemann-Cauchy

Zero divergence requirement: \(0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0\)

and so does \(A\)

potential \(\Phi\) obeys Laplace equation

\[
\Phi \text{ obeys Laplace equation}
\]
4. Riemann-Cauchy conditions  What’s analytic? (...and what’s not?)

Easy 2D circulation and flux integrals
Easy 2D curvilinear coordinate discovery
Easy 2D monopole, dipole, and \(2^n\)-pole analysis
Easy \(2^n\)-multipole field and potential expansion
Easy stereo-projection visualization
11. Complex integrals define 2D monopole fields and potentials
Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z) = \frac{1}{z} = z^{-1}$

It has a logarithmic potential $\phi(z)=a \cdot \ln(z) = a \cdot \ln(x+iy)$. 
11. **Complex integrals define 2D monopole fields and potentials**

Of all power-law fields \( f(z) = az^n \) one lacks a power-law potential \( \phi(z) = \frac{a}{n+1}z^{n+1} \). It is the \( n = -1 \) case.

**Unit monopole field:** \( f(z) = \frac{1}{z} = z^{-1} \)

\( f(z) = \frac{a}{z} = az^{-1} \) **Source-\( a \) monopole**

It has a **logarithmic potential** \( \phi(z) = a \cdot \ln(z) = a \cdot \ln(x+iy) \).

\[
\phi(z) = \Phi + iA = \int f(z)dz = \int \frac{a}{z}dz = a \ln(z)
\]
11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields \( f(z) = az^n \) one lacks a power-law potential \( \phi(z) = \frac{a}{n+1} z^{n+1} \). It is the \( n = -1 \) case.

Unit monopole field: \( f(z) = \frac{1}{z} \)

\[ f(z) = \frac{a}{z} = az^{-1} \] Source-\( a \) monopole

It has a logarithmic potential \( \phi(z) = a \ln(z) = a \ln(x + iy) \). Note: \( \ln(ab) = \ln(a) + \ln(b) \), \( \ln(e^{i\theta}) = i\theta \), and \( z = re^{i\theta} \).

\[
\phi(z) = \Phi + iA = \int f(z) dz = \int \frac{a}{z} dz = a \ln(z) = a \ln(re^{i\theta})
\]

\[
= a \ln(r) + i a \theta
\]
11. Complex integrals define 2D monopole fields and potentials
Of all power-law fields \( f(z) = az^n \) one lacks a power-law potential \( \phi(z) = \frac{a}{n+1}z^{n+1} \). It is the \( n = -1 \) case.

Unit monopole field: \( f(z) = \frac{1}{z} = z^{-1} \)

It has a logarithmic potential \( \phi(z) = a \cdot \ln(z) = a \cdot \ln(x + iy) \). Note: \( \ln(ab) = \ln(a) + \ln(b) \), \( \ln(e^{i\theta}) = i\theta \), and \( z = re^{i\theta} \).

\[
\phi(z) = \Phi + iA = \int f(z)\,dz = \int a\cdot\frac{dz}{z} = a\ln(z) = a\ln(re^{i\theta})
\]

(a) Unit Z-line-flux field \( f(z) = 1/z \)

Field:
\( f^*(z*) = 1/z^* = e^{i\theta}/r \)
\( F(x,y) = (x,y)/r^2 \)

Potential:
\( \phi(z) = \ln z \)
\[
= \ln r + i\theta \\
= \Phi + iA
\]
11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields \( f(z) = az^n \) one lacks a power-law potential \( \phi(z) = \frac{a}{n+1}z^{n+1} \). It is the \( n = -1 \) case.

Unit monopole field: \( f(z) = \frac{1}{z} = z^{-1} \)

It has a logarithmic potential \( \phi(z) = a \cdot \ln(z) = a \cdot \ln(x+iy) \). Note: \( \ln(ab) = \ln(a) + \ln(b) \), \( \ln(e^{i\theta}) = i\theta \), and \( z = re^{i\theta} \).

\[
\phi(z) = \Phi + iA = \int f(z)\,dz = \int \frac{a}{z}\,dz = a\ln(z) = a\ln(re^{i\theta})
\]

\[
= a\ln(r) + ia\theta
\]

(a) Unit Z-line-flux field \( f(z) = 1/z \)

(b) Unit Z-line-vortex field \( f(z) = i/z \)
What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials
Of all power-law fields \( f(z) = az^n \) one lacks a power-law potential \( \phi(z) = \frac{a}{n+1} z^{n+1} \). It is the \( n = -1 \) case.

Unit monopole field: \( f(z) = \frac{1}{z} = z^{-1} \)

It has a logarithmic potential \( \phi(z) = a \cdot \ln(z) = a \cdot \ln(x + iy) \). Note: \( \ln(a \cdot b) = \ln(a) + \ln(b) \), \( \ln(e^{i \theta}) = i \theta \), and \( z = re^{i \theta} \).

\[
\phi(z) = \Phi + iA = \int f(z) \, dz = \int \frac{a}{z} \, dz = a \ln(z) = a \ln(re^{i \theta})
\]

\[
= a \ln(r) + i a \theta
\]

A monopole field is the only power-law field whose integral (potential) depends on path of integration.

\[
\Delta \phi = \oint f(z) \, dz = a \oint \frac{dz}{z} = a \left. \int_{\theta=0}^{2\pi N} d\left(\frac{Re^{i \theta}}{Re^{i \theta}}\right) = a \int_{\theta=0}^{2\pi N} id\theta = ai \theta \right|_{0}^{2\pi N} = 2a\pi i N
\]
\[
\phi(z) = \Phi + iA = \int f(z)dz = \int \frac{a}{z}dz = a \ln(re^{i\theta}) \\
= \ln(r) + i\theta
\]

Each turn around origin adds 2\pi i to vector potential \(iA\)
(a) Unit Z-line-flux field $f(z) = 1/z$

Field:
\[ f(z^*) = \frac{1}{z^*} = \frac{e^{i\theta}}{r} \]
\[ F(x,y) = (x, y)^2 \]
Potential:
\[ \phi(z) = \ln z = \ln r + i\theta = \Phi + iA \]

(b) Unit Z-line-vortex field $f(z) = i/z$

Field:
\[ f(z^*) = -i/z^* = -i(e^{i\theta}/r) \]
\[ F(x,y) = (y-x)^2 \]
Potential:
\[ \phi(z) = i \ln z = 0 + i \ln r = \Phi + iA \]
What Good Are Complex Exponentials? (contd.)

\[ f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2} \]

"Vortex" \[ f(z) = (0.75 + i0.25)/z = e^{i18°}/z\sqrt{n} \]

Tuesday, October 9, 2012
4. Riemann-Cauchy conditions

What’s analytic? (…and what’s not?)

Easy 2D circulation and flux integrals
Easy 2D curvilinear coordinate discovery
Easy 2D monopole, dipole, and 2^n-pole analysis
Easy 2^n-multipole field and potential expansion
Easy stereo-projection visualization
Now let these two line-sources of equal but opposite source constants \( +a \) and \(-a \) be located at \( z=\pm \Delta/2 \) separated by a small interval \( \Delta \). This sum (actually difference) of \( f^{1\text{-pole}} \)-fields is called a dipole field.

\[
\begin{align*}
\phi^{\text{dipole}}(z) &= a \ln(z - \frac{\Delta}{2}) - a \ln(z + \frac{\Delta}{2}) = a \ln \frac{z - \frac{\Delta}{2}}{z + \frac{\Delta}{2}} \\
&= a \ln z - a \ln z - a \Delta^2/4 \\
&= -a \Delta^2/4
\end{align*}
\]

12. Complex derivatives give 2D dipole fields

Start with \( f(z)=az^{-1} \): 2D line monopole field and is its monopole potential \( \phi(z)=a \ln z \) of source strength \( a \).

\[
\begin{align*}
f^{1\text{-pole}}(z) &= \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz} \\
\phi^{1\text{-pole}}(z) &= a \ln z
\end{align*}
\]
12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line monopole field and is its monopole potential $\phi(z) = a \ln z$ of source strength $a$. 

$$f^{1\text{-pole}}(z) = \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz} \quad \phi^{1\text{-pole}}(z) = a \ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval $\Delta$. This sum (actually difference) of $f^{1\text{-pole}}$-fields is called a dipole field.

$$f^{\text{dipole}}(z) = \frac{a}{z+\frac{\Delta}{2}} - \frac{a}{z-\frac{\Delta}{2}} = \frac{-a \cdot \Delta}{z^2 - \frac{\Delta^2}{4}} \quad \phi^{\text{dipole}}(z) = a \ln(z - \frac{\Delta}{2}) - a \ln(z + \frac{\Delta}{2}) = a \ln \frac{z - \frac{\Delta}{2}}{z + \frac{\Delta}{2}} $$

If interval $\Delta$ is tiny and is divided out we get a point-dipole field $f^{2\text{-pole}}$ that is the $z$-derivative of $f^{1\text{-pole}}$.

$$f^{2\text{-pole}} = \frac{-a}{z^2} = \frac{df^{1\text{-pole}}}{dz} = \frac{d\phi^{2\text{-pole}}}{dz} \quad \phi^{2\text{-pole}} = \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz}$$
12. Complex derivatives give 2D dipole fields

Start with $f(z) = az^{-1}$: 2D line monopole field and is its monopole potential $\phi(z) = a \ln z$ of source strength $a$.

$$f^{1\text{-pole}}(z) = \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz} \quad \phi^{1\text{-pole}}(z) = a \ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z = \pm \Delta/2$ separated by a small interval $\Delta$. This sum (actually difference) of $f^{1\text{-pole}}$-fields is called a dipole field.

$$f^{\text{dipole}}(z) = \frac{a}{z + \frac{\Delta}{2}} - \frac{a}{z - \frac{\Delta}{2}} = \frac{-a \cdot \Delta}{z^2 - \frac{\Delta^2}{4}} \quad \phi^{\text{dipole}}(z) = a \ln(z - \frac{\Delta}{2}) - a \ln(z + \frac{\Delta}{2}) = a \ln \frac{z - \frac{\Delta}{2}}{z + \frac{\Delta}{2}}$$

If interval $\Delta$ is tiny and is divided out we get a point-dipole field $f^{2\text{-pole}}$ that is the $z$-derivative of $f^{1\text{-pole}}$.

$$f^{2\text{-pole}} = -\frac{a}{z^2} = \frac{df^{1\text{-pole}}}{dz} = \frac{d\phi^{2\text{-pole}}}{dz} \quad \phi^{2\text{-pole}} = \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz}$$
12. Complex derivatives give 2D dipole fields

Start with \( f(z) = az^{-1} \): 2D line monopole field and is its monopole potential \( \phi(z) = a \ln z \) of source strength \( a \).

\[
f^{1\text{-pole}}(z) = \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz} \quad \phi^{1\text{-pole}}(z) = a \ln z
\]

Now let these two line-sources of equal but opposite source constants \(+a\) and \(−a\) be located at \(z = \pm \Delta/2\) separated by a small interval \(\Delta\). This sum (actually difference) of \(f^{1\text{-pole}}\)-fields is called a dipole field.

\[
f^{\text{dipole}}(z) = \frac{a}{z + \frac{\Delta}{2}} - \frac{a}{z - \frac{\Delta}{2}} = \frac{-a \cdot \Delta}{z^2 - \frac{\Delta^2}{4}} \quad \phi^{\text{dipole}}(z) = a \ln(z - \frac{\Delta}{2}) - a \ln(z + \frac{\Delta}{2}) = a \ln \left( \frac{z - \frac{\Delta}{2}}{z + \frac{\Delta}{2}} \right)
\]

If interval \(\Delta\) is tiny and is divided out we get a point-dipole field \(f^{2\text{-pole}}\) that is the \(z\)-derivative of \(f^{1\text{-pole}}\).

\[
f^{2\text{-pole}} = \frac{-a}{z^2} = \frac{df^{1\text{-pole}}}{dz} = \frac{d\phi^{2\text{-pole}}}{dz} \quad \phi^{2\text{-pole}} = \frac{a}{z} = \frac{d\phi^{1\text{-pole}}}{dz}
\]

A point-dipole potential \(\phi^{2\text{-pole}}\) (whose \(z\)-derivative is \(f^{2\text{-pole}}\)) is a \(z\)-derivative of \(\phi^{1\text{-pole}}\).

\[
\phi^{2\text{-pole}} = \frac{a}{z} = \frac{a}{x + iy} = \frac{a}{x + iy} \frac{x - iy}{x - iy} = \frac{ax}{x^2 + y^2} + i \frac{-ay}{x^2 + y^2} = a \frac{\cos \theta - i \frac{a}{r} \sin \theta}{r}
\]

\[
= \Phi^{2\text{-pole}} + i \Delta^{2\text{-pole}}
\]
A **point-dipole potential** $\phi^{2\text{-pole}}$ (whose $z$-derivative is $f^{2\text{-pole}}$) is a $z$-derivative of $\phi^{1\text{-pole}}$.

\[
\phi^{2\text{-pole}} = \frac{a}{z} = \frac{a}{x + iy} = \frac{a}{x + iy} \frac{x - iy}{x + iy \cdot x - iy} = \frac{ax}{x^2 + y^2} + i \frac{-ay}{x^2 + y^2} = \frac{a}{r} \cos \theta - i \frac{a}{r} \sin \theta \\
= \Phi^{2\text{-pole}} + i A^{2\text{-pole}}
\]
$2^n$-pole analysis \textit{(quadrupole:} $2^2$=4-pole, octapole:} $2^3$=8-pole,\ldots, pole dancer,

What if we put a (-)copy of a 2-pole near its original?

Well, the result is 4-pole or \textit{quadrupole} field $f^{4\text{-pole}}$ and potential $\phi^{4\text{-pole}}$.

Each a z-derivative of $f^{2\text{-pole}}$ and $\phi^{2\text{-pole}}$.

\[
\begin{align*}
    f^{4\text{-pole}} &= \frac{a}{z^3} = \frac{1}{2} \frac{df^{2\text{-pole}}}{dz} = \frac{d\phi^{4\text{-pole}}}{dz} \\
    \phi^{4\text{-pole}} &= -\frac{a}{2z^2} = \frac{1}{2} \frac{d\phi^{2\text{-pole}}}{dz}
\end{align*}
\]
2\(n\)-pole analysis (quadrupole: \(2^2=4\)-pole, octapole: \(2^3=8\)-pole, ..., pole dancer,

What if we put a (-)copy of a 2-pole near its original?

Well, the result is 4-pole or quadrupole field \(f^{4\text{-pole}}\) and potential \(\phi^{4\text{-pole}}\).

Each a \(z\)-derivative of \(f^{2\text{-pole}}\) and \(\phi^{2\text{-pole}}\).

\[
f^{4\text{-pole}} = \frac{a}{z^3} = \frac{1}{2} \frac{df^{2\text{-pole}}}{dz} = \frac{d\phi^{4\text{-pole}}}{dz}
\]

\[
\phi^{4\text{-pole}} = -\frac{a}{2z^2} = \frac{1}{2} \frac{d\phi^{2\text{-pole}}}{dz}
\]
4. Riemann-Cauchy conditions  What’s analytic? (…and what’s not?)

Easy 2D circulation and flux integrals
  Easy 2D curvilinear coordinate discovery
  Easy 2D monopole, dipole, and $2^n$-pole analysis
  Easy $2^n$-multipole field and potential expansion
  Easy stereo-projection visualization
Laurent series or multipole expansion of a given complex field function \( f(z) \) around \( z=0 \).

\[
f(z) = ... a_{-3} z^{-3} + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + ... \\
\cdots 2^2\text{-pole} \quad 2^1\text{-pole} \quad 2^0\text{-pole} \quad 2^1\text{-pole} \quad 2^2\text{-pole} \quad 2^3\text{-pole} \quad 2^4\text{-pole} \quad 2^5\text{-pole} \quad 2^6\text{-pole} \cdots \\
\text{at } z=0 \quad \text{at } z=0 \quad \text{at } z=0 \quad \text{at } z=\infty \quad \text{at } z=\infty \quad \text{at } z=\infty \quad \text{at } z=\infty \quad \text{at } z=\infty \quad \text{at } z=\infty
\]

\[
\phi(z) = ... \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-2}}{-1} z^{-1} + \frac{a_{-1}}{-1} \ln z + a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \frac{a_3}{4} z^4 + \frac{a_4}{5} z^5 + \frac{a_5}{6} z^6 + ... \\
\]

All field terms \( a_{m-1} z^{m-1} \) except 1-pole \( \frac{a_{-1}}{z} \) have potential term \( a_{m-1} z^{m}/m \) of a \( 2^m\)-pole.

These are located at \( z=0 \) for \( m<0 \) and at \( z=\infty \) for \( m>0 \).

\[
\phi(z) = ... \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-2}}{-1} z^{-1} + \frac{a_{-1}}{-1} \ln z + a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + ... \\
\phi(w) = ... \frac{a_{-3}}{-2} w^{-2} + \frac{a_{-2}}{-1} w^{-1} + \frac{a_{-1}}{-1} \ln w + a_0 w + \frac{a_1}{2} w^2 + \frac{a_2}{3} w^3 + ... \\
\quad (\text{with } z=w^{-1})
\]

\[
\quad = ... \frac{a_2}{3} z^{-2} + \frac{a_1}{2} z^{-2} + a_0 z^{-1} - a_{-1} \ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^2 + \frac{a_{-3}}{-2} z^3 + ... \\
\quad (\text{with } w=z^{-1})
\]
\[ z = x + iy \]

\[ w = u + iv \]

\[ |z| = \frac{\tan \theta}{2} = |w|^{-1} \]

\[ |w| = \frac{\cot \theta}{2} = |z|^{-1} \]

\[ \phi(z) = \ldots + a_{-3} z^{-2} + a_{-2} z^{-1} + a_{-1} \ln z + a_{0} z + \frac{a_{1}}{2} z^{2} + \frac{a_{2}}{3} z^{3} + \ldots \]

\[ \phi(w) = \ldots + a_{-3} w^{-2} + a_{-2} w^{-1} + a_{-1} \ln w + a_{0} w + \frac{a_{1}}{2} w^{2} + \frac{a_{2}}{3} w^{3} + \ldots \]

\[ = \ldots + a_{2} z^{-2} + \frac{a_{1}}{2} z^{-1} - a_{-1} \ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^{2} + \frac{a_{-3}}{-2} z^{3} + \ldots \]

(a) (+) monopole field at North Pole

(b) (-) monopole field near South Pole
dipole field centered at North Pole

As constant field near South Pole
Of all $2^m$-pole field terms $a_m i z^{m-1}$, only the $m=0$ monopole $a_1 z^{-1}$ has a non-zero loop integral (10.39).

\[ \oint f(z) dz = \oint a_1 z^{-1} dz = 2\pi i a_1 \]

\[ a_1 = \frac{1}{2\pi i} \oint f(z) dz \]

This $m=1$-pole constant-$a_1$ formula is just the first in a series of Laurent coefficient expressions.

\[ \cdots a_3 = \frac{1}{2\pi i} \oint z^2 f(z) dz \quad a_2 = \frac{1}{2\pi i} \oint z f(z) dz \quad a_1 = \frac{1}{2\pi i} \oint f(z) dz \quad a_0 = \frac{1}{2\pi i} \oint \frac{f(z)}{z} dz \quad a_1 = \frac{1}{2\pi i} \oint \frac{f(z)}{z^2} dz \quad \cdots \]

Source analysis starts with 1-pole loop integrals $\oint z^{-1} dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1} dz = 2\pi i$.

They hold for any loop about point-$a$. Function $f(z)$ is just $f(a)$ on a tiny circle around point-$a$.

\[ \oint \frac{f(z)}{z-a} dz = \oint \frac{f(a)}{z-a} dz = f(a) \oint \frac{1}{z-a} dz = 2\pi i f(a) \]

\[ f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz \]

The $f(a)$ result is called a **Cauchy integral**. Then repeated $a$-derivatives gives a sequence of them.

\[ \frac{df(a)}{da} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^2} dz \quad \frac{d^2 f(a)}{da^2} = \frac{2}{2\pi i} \oint \frac{f(z)}{(z-a)^3} dz \quad \frac{d^3 f(a)}{da^3} = \frac{3!}{2\pi i} \oint \frac{f(z)}{(z-a)^4} dz \quad \cdots \quad \frac{d^n f(a)}{da^n} = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}} dz \]

This leads to a general **Taylor-Laurent** power series expansion of function $f(z)$ around point-$a$.

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{where : } a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}} dz \left( = \frac{1}{n!} \frac{d^n f(a)}{da^n} \right) \quad \text{for : } n \geq 0 \]