Lecture 12
Tue. 10.2.2012

Hamiltonian vs. Lagrange mechanics in Generalized Curvilinear Coordinates (GCC)

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lectures 9-11 procedures:

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Deriving Hamilton's equations from Lagrange's equations
Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m
Polar-coordinate example of Hamilton's equations
Hamilton's equations in Runga-Kutta (computer solution) form

Examples of Hamiltonian mechanics in effective potentials

Isotropic Harmonic Oscillator in polar coordinates and effective potential (Simulation)
Coulomb orbits in polar coordinates and effective potential (Simulation)

Parabolic and 2D-IHO orbital envelopes

Clues for take-home assignment 7 (Simulation)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulation)
1D-HO phase-space control (Simulation)

Quick Review of Lagrange Relations in Lectures 9-11

→ *0^{th} and 1^{st} equations of Lagrange and Hamilton and their geometric relations*

Quick Review of Lagrange Relations in Lectures 9-11

0^{th} and 1^{st} equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian** and **Estrangian** have no explicit dependence on **momentum p

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian** and **Estrangian** have no explicit dependence on **velocity v

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian** and **Hamiltonian** have no explicit dependence on **speedinum V

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \bullet \mathbf{M} \bullet \mathbf{v}}{2} = \mathbf{M} \bullet \mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1^{st} equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \bullet \mathbf{M}^{-1} \bullet \mathbf{p}}{2} = \mathbf{M}^{-1} \bullet \mathbf{p} = \mathbf{v}$$

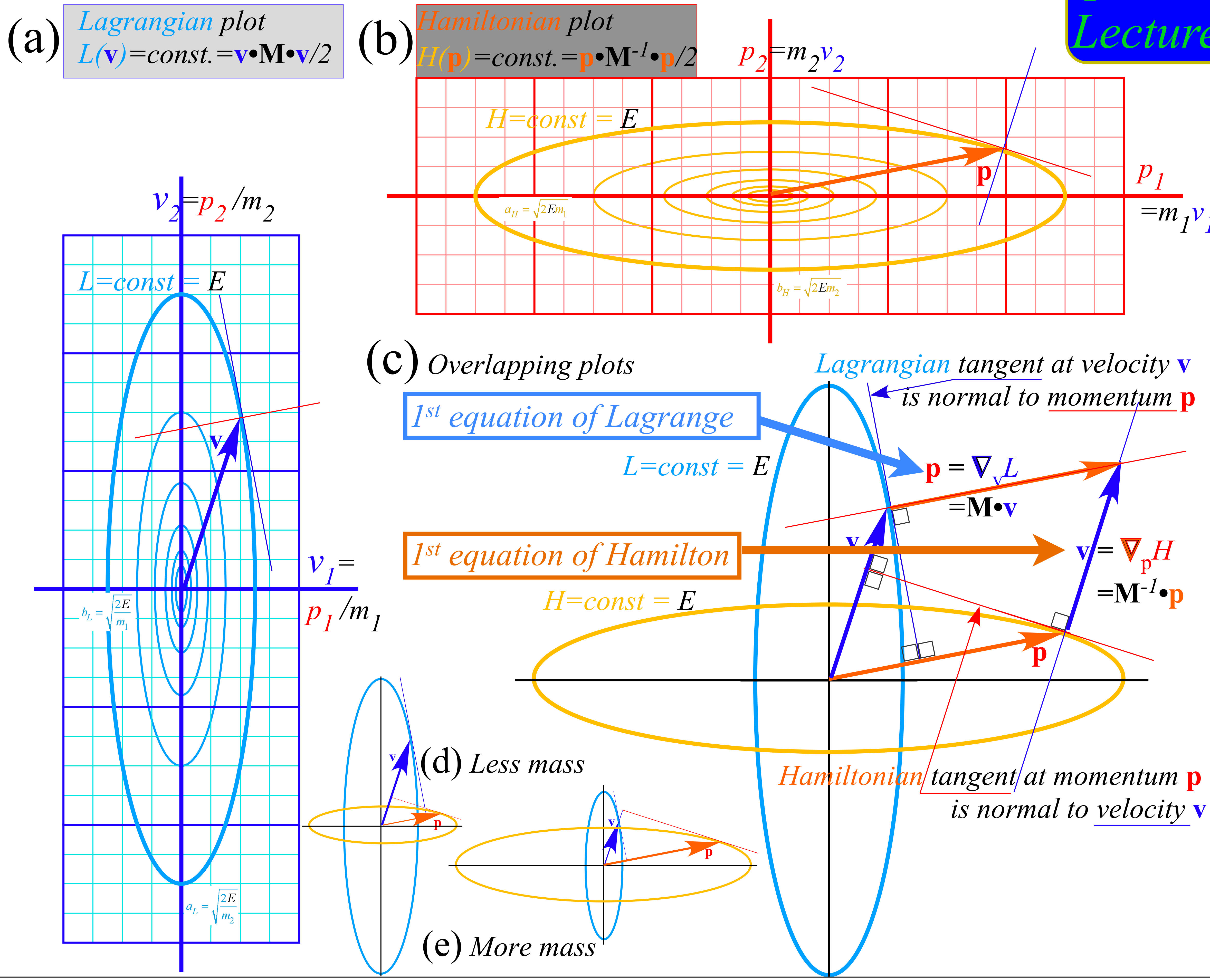
$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1^{st} equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

(Forget Estrangian for now)

Unit 1
Fig. 12.2



(Review of Lecture 11)

Review of Lagrange Equations in Lecture 11

Lagrange prefers Covariant g_{mn} *with* Contravariant *velocity* $\dot{q}^m$

GCC Lagrangian definition

GCC "canonical" momentum p_m *definition*

GCC "canonical" force F_m *definition*

Coriolis "fictitious" forces (... and weather effects)

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. *(Review of Lecture 11)*

$$L(\mathbf{v}) = \tfrac{1}{2} M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2} M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2} M(\mathbf{E}_m \dot{q}^m)\bullet(\mathbf{E}_n \dot{q}^n) - U = \tfrac{1}{2} M(g_{mn}\dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2} M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

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GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising; radial momentum p_r has the usual linear $M\cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi = Mr^2\omega$.

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2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

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Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$

Centrifugal (center-fleeing) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 11)**

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$$\dot{p}_r\equiv\frac{dp_r}{dt}=M\,\ddot{r}$$
$$=M\,r\dot{\phi}^2-\frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi\equiv\frac{dp_\phi}{dt}=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}$$
$$=0-\frac{\partial U}{\partial\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

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Rewriting GCC Lagrange equations :

(Review of Lecture 11)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

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$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Conventional forms

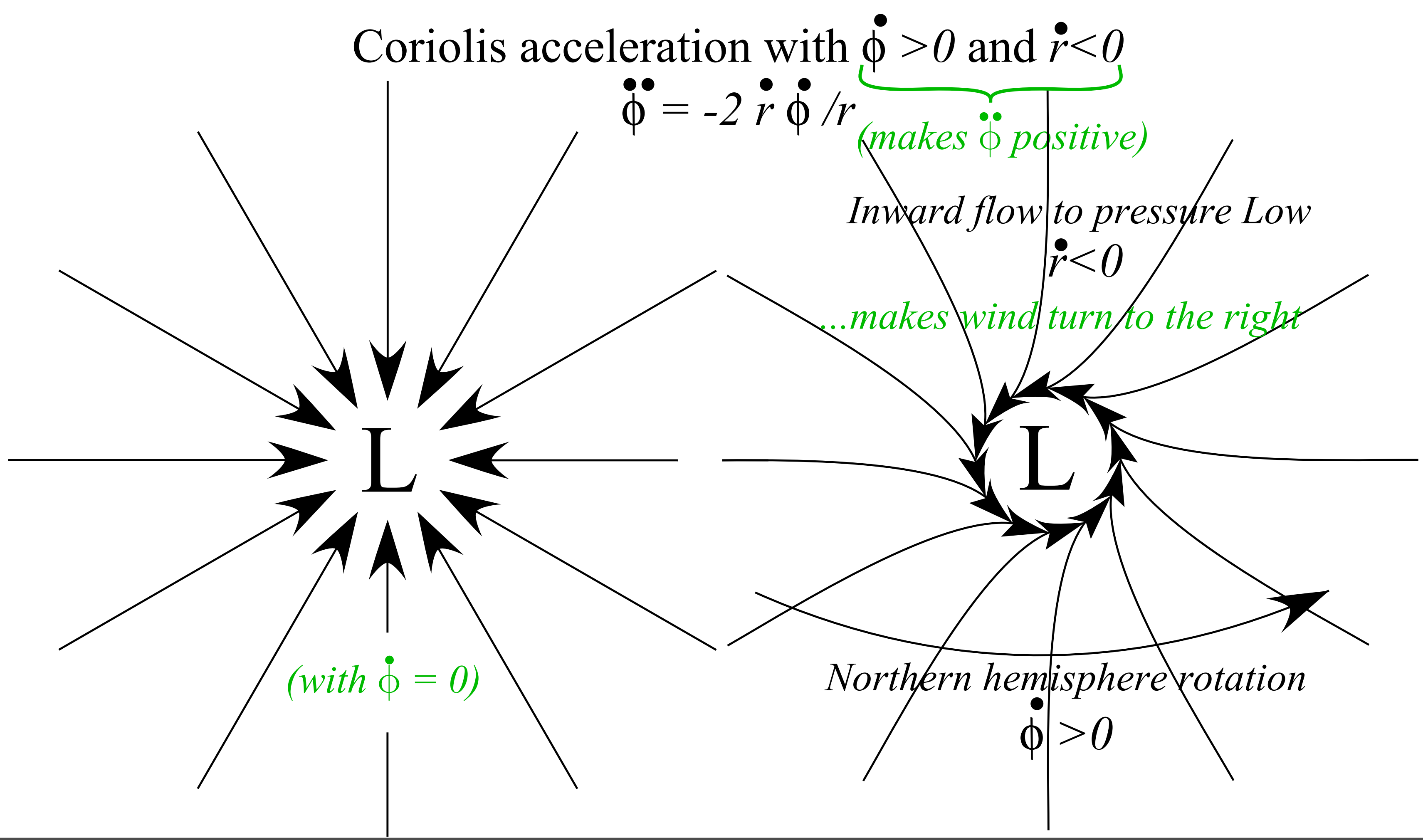
radial force: $M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free (U=0)

radial acceleration: $\ddot{r} = r\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$



Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Hamilton prefers Contravariant g^{mn} *with* Covariant ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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...of coordinates and velocity and <u>time</u>, too. (*You can safely drop last chain-rule factor* [*1=dt/dt*])

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}\frac{dt}{dt}$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
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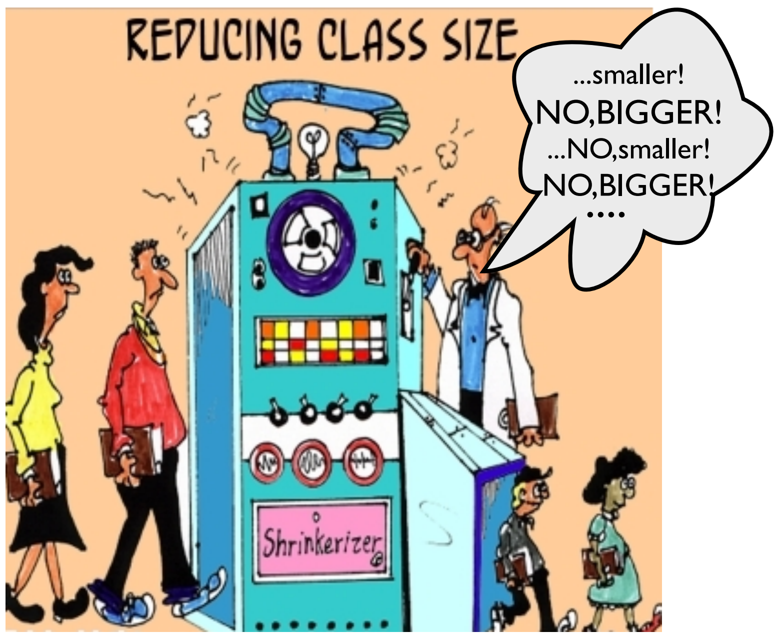
...of coordinates and velocity and ***time****, too. (Imagine Mad Scientist turning U(t)-dial.)*

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$



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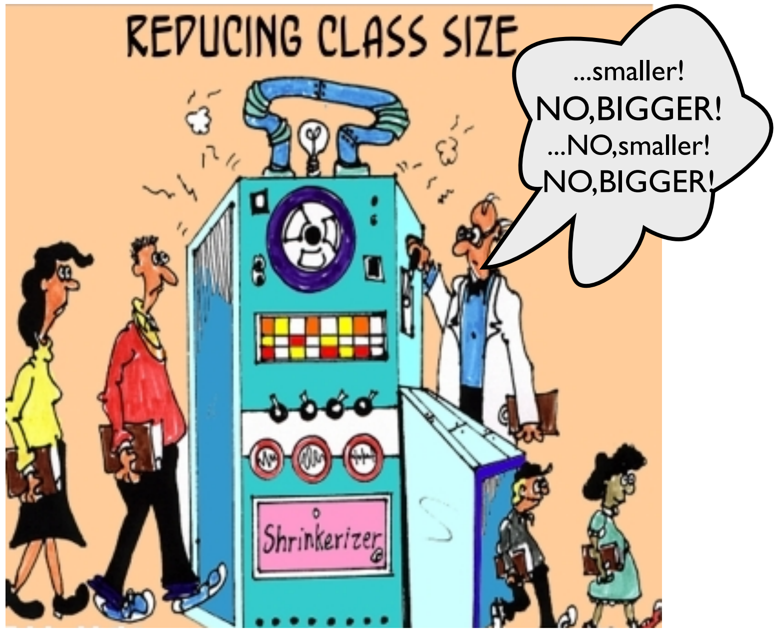

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Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

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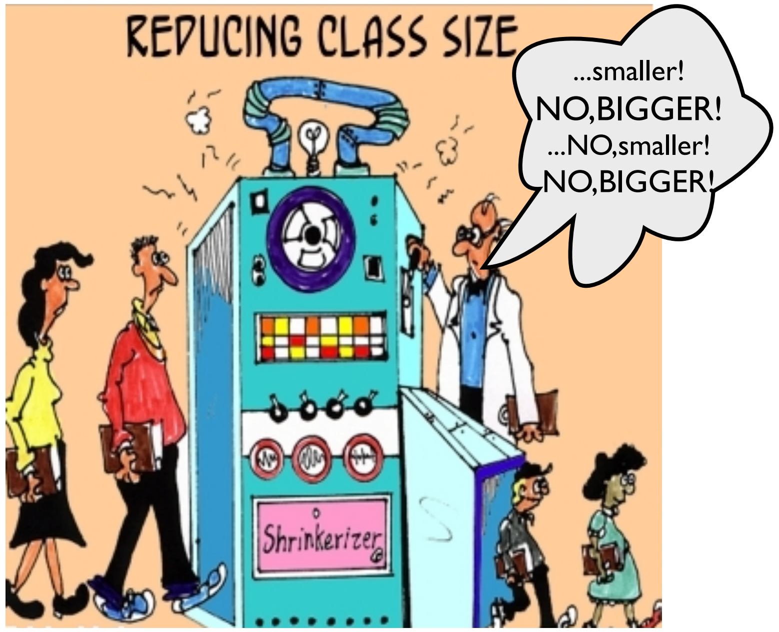

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$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Use product rule:
$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ *...*

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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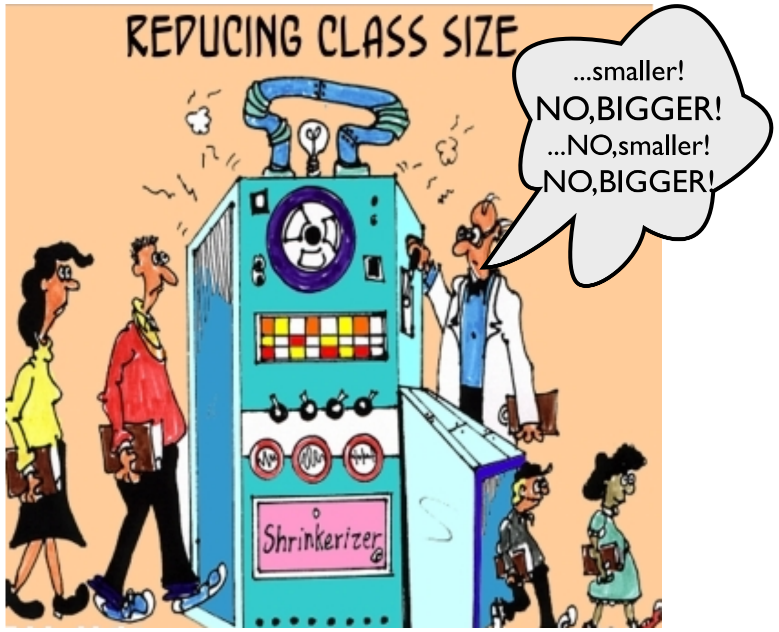

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Use product rule:

$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

and switch the dL/dt and $\partial L/\partial t$ *to define the Hamiltonian function* $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

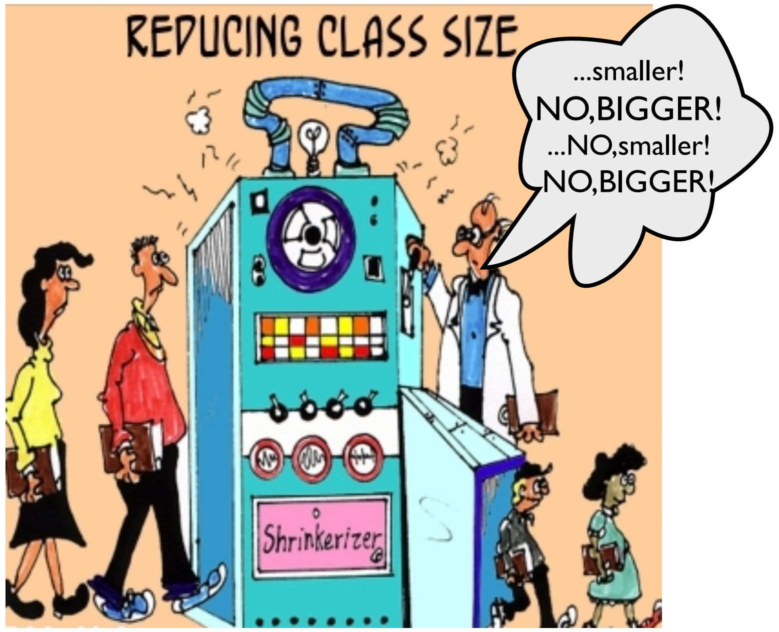
$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}=\frac{dH}{dt}\qquad where: H=p_m\dot{q}^m-L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and <u>time</u>, too. (Imagine Mad Scientist turning U-dial.)



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$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$ (That's the old Legendre transform)

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}=\frac{dH}{dt}$$

where: $H=p_m\dot{q}^m-L$ (Recall $\frac{\partial L}{\partial p_m}\equiv 0$)

Hamilton's 1st GCC equation

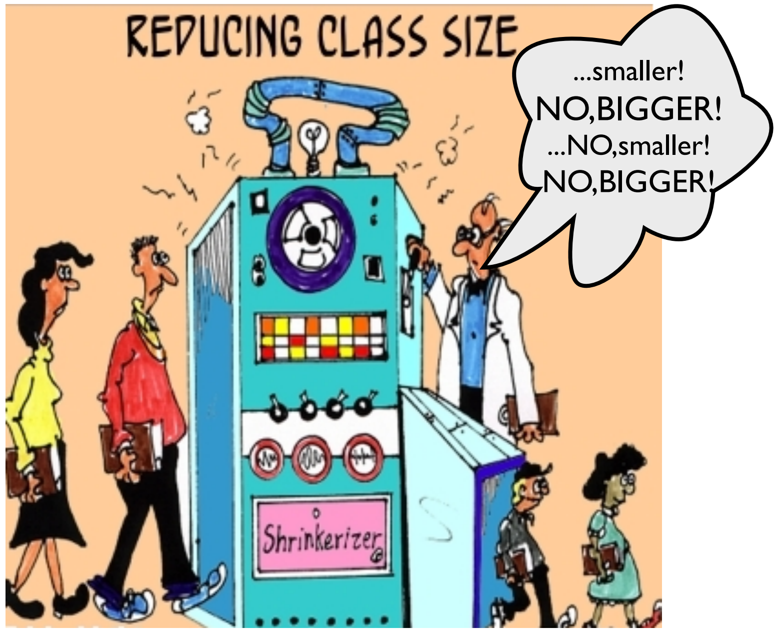
$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},\;)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)



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Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Use product rule:
$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$ *(That's the old Legendre transform)*

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}=\frac{dH}{dt}\quad where: H=p_m\dot{q}^m-L$$

(Recall: $\frac{\partial L}{\partial p_m}\equiv 0$ *and:* $\frac{\partial H}{\partial \dot{q}^m}\equiv 0$ *)*

Hamilton's 1st GCC equation
$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

$$\frac{\partial H}{\partial q^m}=0\cdot 0-\frac{\partial L}{\partial q^m}=-\dot{p}_m$$

Hamilton's 2nd GCC equation
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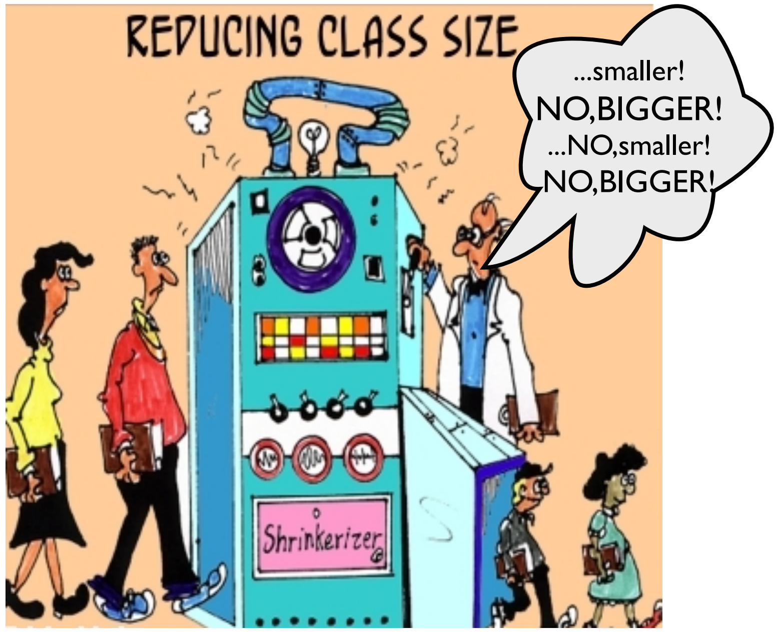

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a most **peculiar** relation involving partial vs total

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

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Hamilton prefers Contravariant g^{mn} *with* Covariant ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m

Polar-coordinate example of Hamilton's equations

Hamilton's equations in Runga-Kutta (computer solution) form

Hamilton prefers Contra*variant* g^{mn} *with* Co*variant momentum* p_m

Using Legendre transform of Lagrangian $L=T-U$ *with covariant metric definitions of* L *and* p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \dfrac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:

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This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$*.)*

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H=p_m\dot{q}^m-L$ *and:* $L(\dot{q})=\frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n-U$ *and:* $p_m=\frac{\partial L}{\partial\dot{q}^m}=Mg_{mn}\dot{q}^n$

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An inverse metric relation $\dot{q}^m=\frac{1}{M}g^{mn}p_n$ *gives correct form that depends on momentum p_m.*

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(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \quad \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \quad \frac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

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Polar coordinate Hamiltonian is given here:

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→ *Polar-coordinate example of Hamilton's equations*

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$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + U(r,\phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$$

Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$ *in 2D-polar coordinates satifies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m} = \dot{q}^m$ *Hamilton's 2nd equations:* $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

Polar coordinate example of Hamilton's equations

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Polar coordinate example of Hamilton's equations

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$$\frac{\partial H}{\partial p_r}=\dot{r} \qquad \frac{\partial H}{\partial p_r}=\frac{p_r}{M} \qquad p_r=M\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi} \qquad \frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2} \qquad p_\phi=Mr^2\dot{\phi}$$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial r}=-\dot{p}_r \qquad \frac{\partial H}{\partial r}=-2\frac{p_\phi^2}{2Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

$$\dot{p}_r=M\ddot{r}=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi \qquad \frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi} \qquad \dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

Hamiltonian $H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$ *in 2D-polar coordinates satifies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

$$p_r=M\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

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$$\frac{\partial H}{\partial r}=-2\frac{p_\phi^2}{2Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

$$\dot{p}_r=M\ddot{r}=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$=Mr\dot{\phi}^2-\partial_r U(r,\phi)$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

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$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

$$\dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

$$\dot{p}_\phi=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}=-\partial_\phi U(r,\phi)$$

Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations

→ *Hamilton's equations in Runga-Kutta (computer solution) form*

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r} \qquad \dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$= Mr\dot{\phi}^2 - \partial_r U(r,\phi)$$

$$p_\phi = Mr^2\dot{\phi} \qquad \dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi$$

Runga-Kutta form:

$$\dot{r} = \dot{r}(r, p_r, \phi, p_\phi) \quad = \frac{p_r}{M}$$

$$\dot{p}_r = \dot{p}_r(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi^2}{Mr^3} - \partial_r U(r,\phi)$$

$$\dot{\phi} = \dot{\phi}(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi}{Mr^2}$$

$$\dot{p}_\phi = \dot{p}_\phi(r, p_r, \phi, p_\phi) \quad = -\partial_\phi U(r,\phi)$$

Examples of Hamiltonian mechanics in effective potentials

→ *Isotropic Harmonic Oscillator in polar coordinates and effective potential (Simulation)*

Coulomb orbits in polar coordinates and effective potential (Simulation)

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$:

$$H(p_r,p_\phi,r,\phi) = \frac{1}{2M}(g^{rr}p_r{}^2 + g^{\phi\phi}p_\phi{}^2) + k\cdot r^2/2 = \frac{1}{2M}(p_r{}^2 + \frac{1}{r^2}\cdot p_\phi{}^2) + \frac{k\cdot r^2}{2} = E = const.$$

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H is not explicit function of ϕ, and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

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Same applies to any radial potential U(r)

$$E = \frac{p_r{}^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{U(r)}^{\text{“real” PE}}}^{\text{“effective” PE}}$$

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}$$

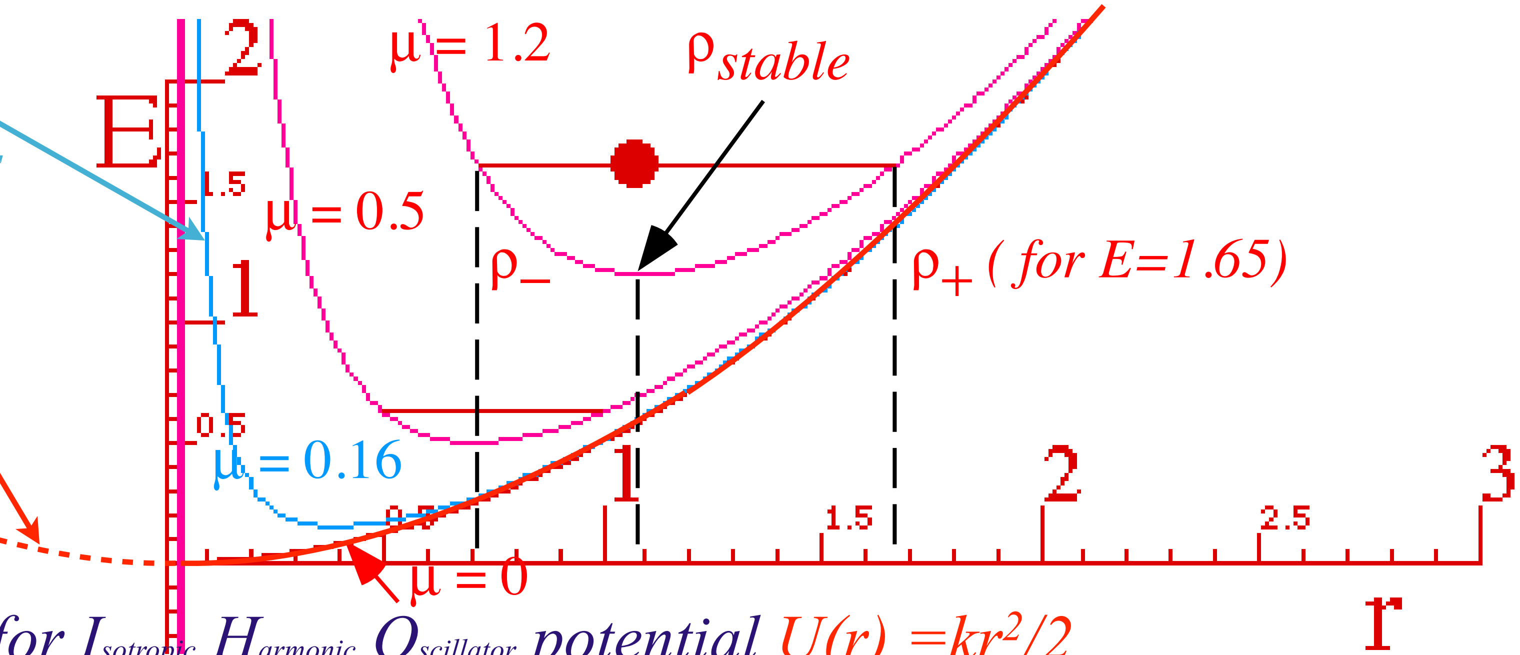
Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{k}{M}\cdot r^2}}$

Called the "quadrature" or 1/4-cycle solution if $r_< = 0$ and $r_> =$ max amplitude

Time vs r for any radial U(r):

$$t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{2U(r)}{M}}}$$

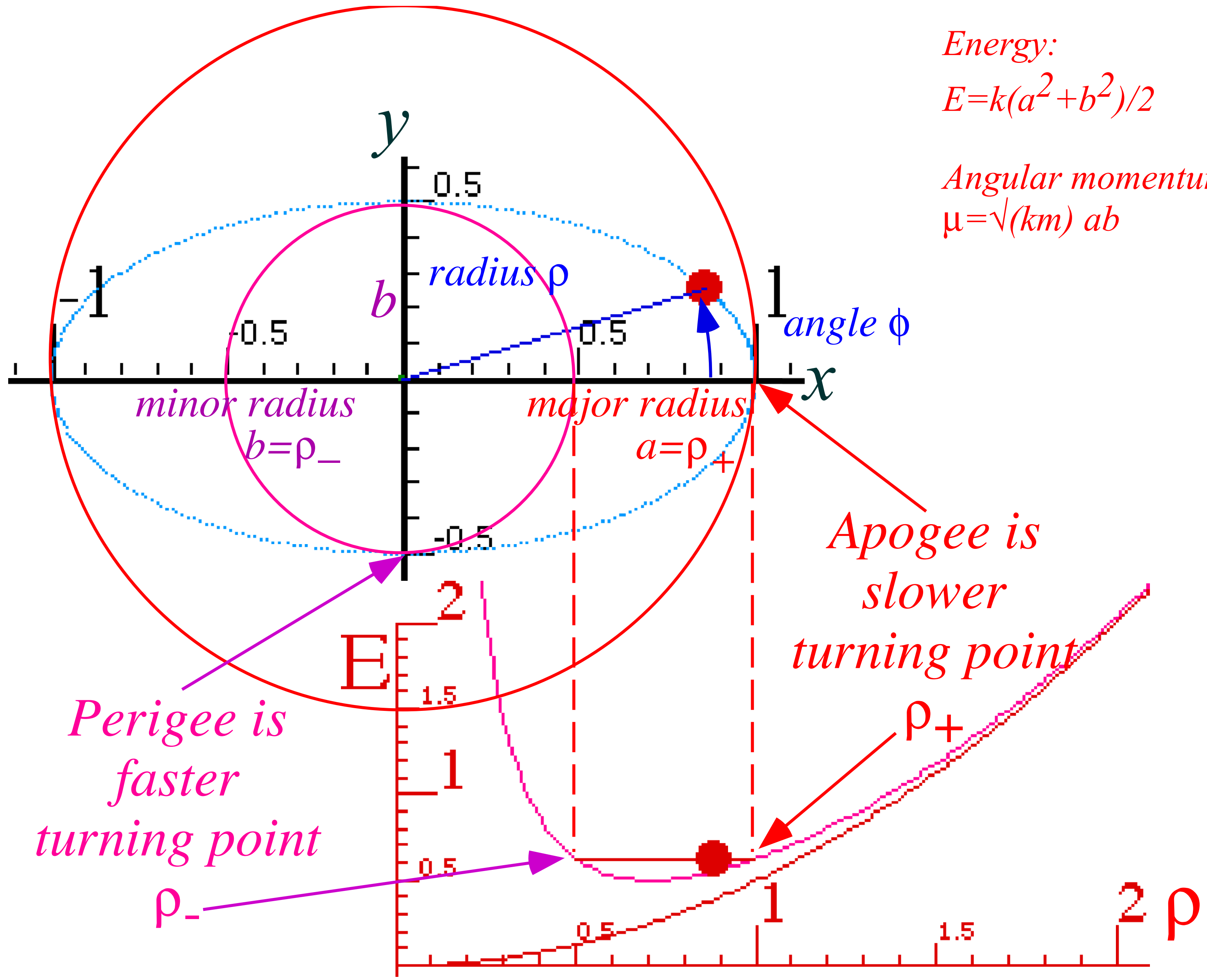
$$E = \frac{p_r^{\;2}}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential U(r) =kr²/2

Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\, ab$



Examples of Hamiltonian mechanics in effective potentials

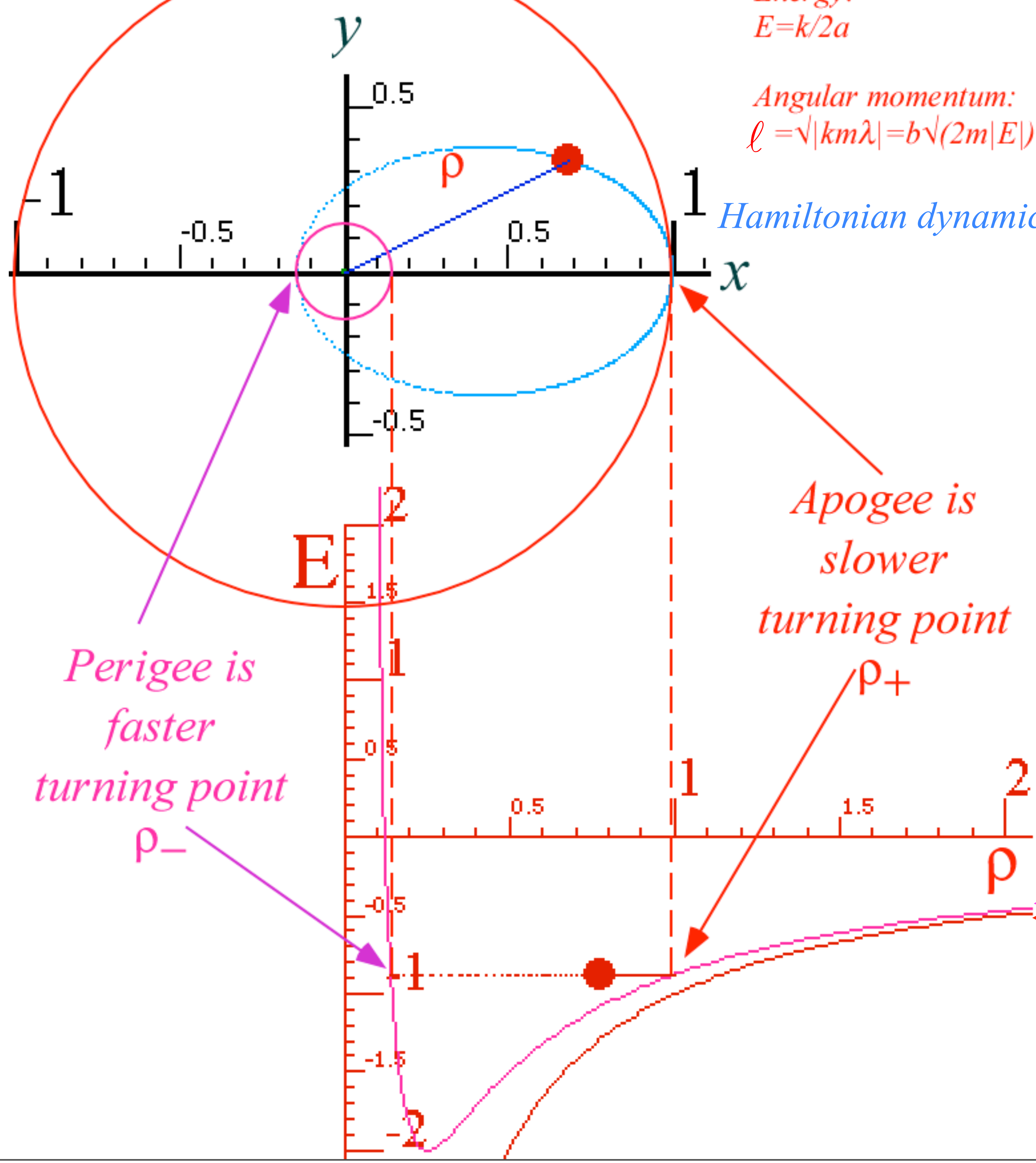
Isotropic Harmonic Oscillator in polar coordinates and effective potential (Simulation)

Coulomb orbits in polar coordinates and effective potential (Simulation)

Energy:
E=k/2a

Angular momentum:
$\ell = \sqrt{|km\lambda|} = b\sqrt{(2m|E|)}$

Hamiltonian dynamics for Coulomb potential U(r) =-k/r

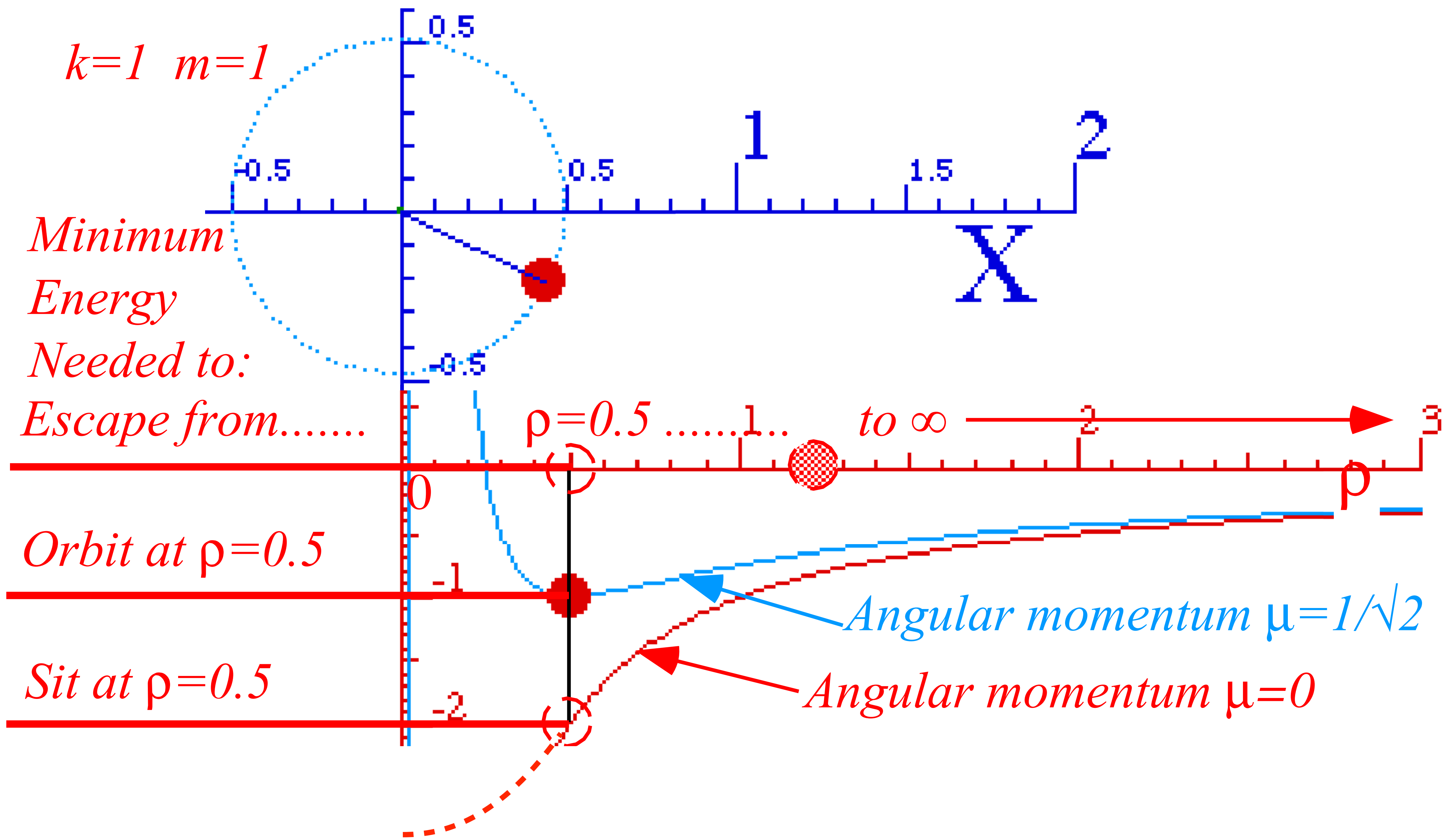


"effective" PE

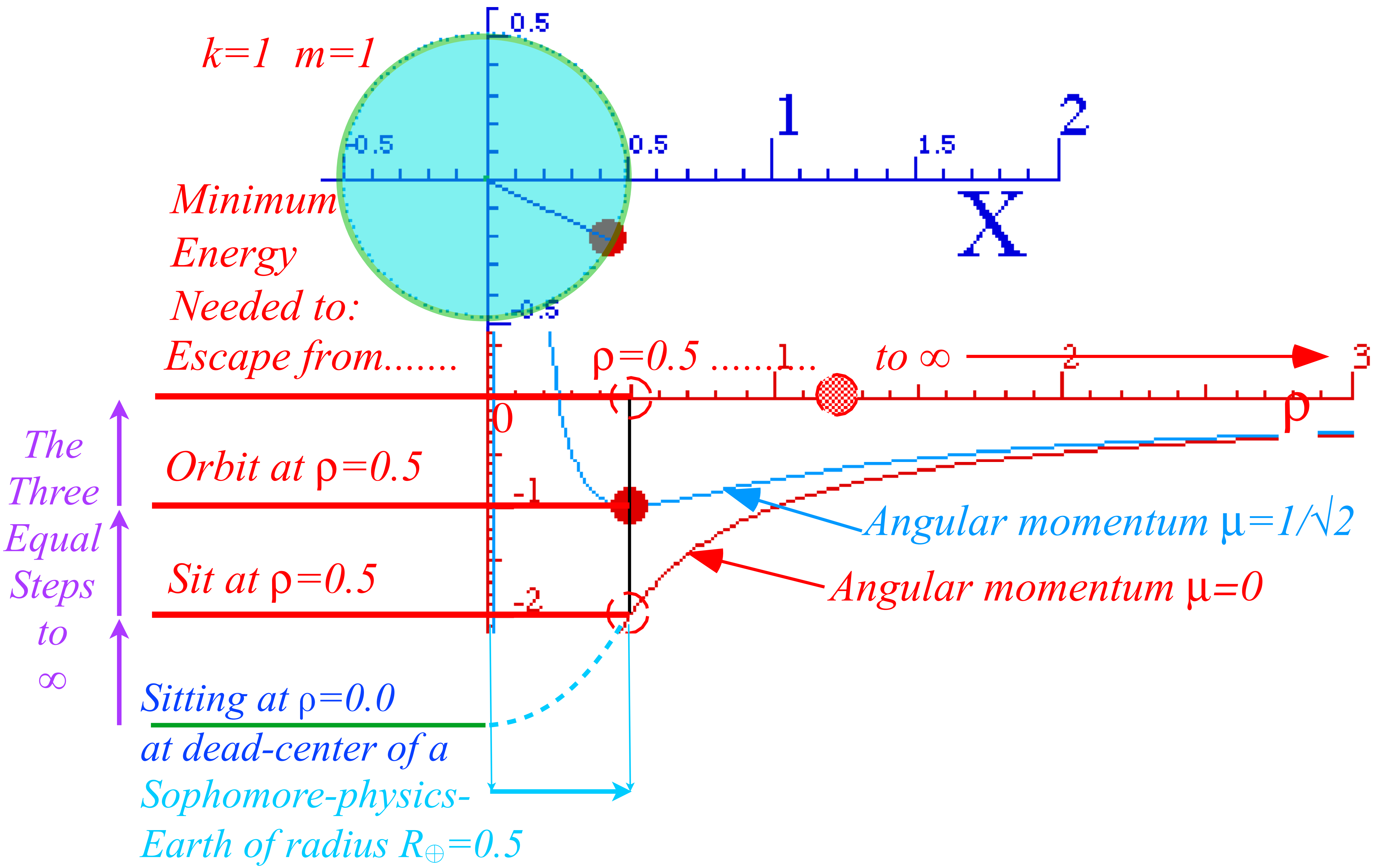
"real" PE

$$E = \frac{p_r^{\,2}}{2M} + \underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} - \overbrace{\frac{k}{r}}^{\text{"real" PE}}$$

"centifugal-barrier" PE



k=1 m=1
Minimum
Energy
Needed to:
Escape from.......
ρ=0.5 ......1...
to ∞
Orbit at ρ=0.5
Sit at ρ=0.5
Angular momentum μ=1/√2
Angular momentum μ=0
X
ρ

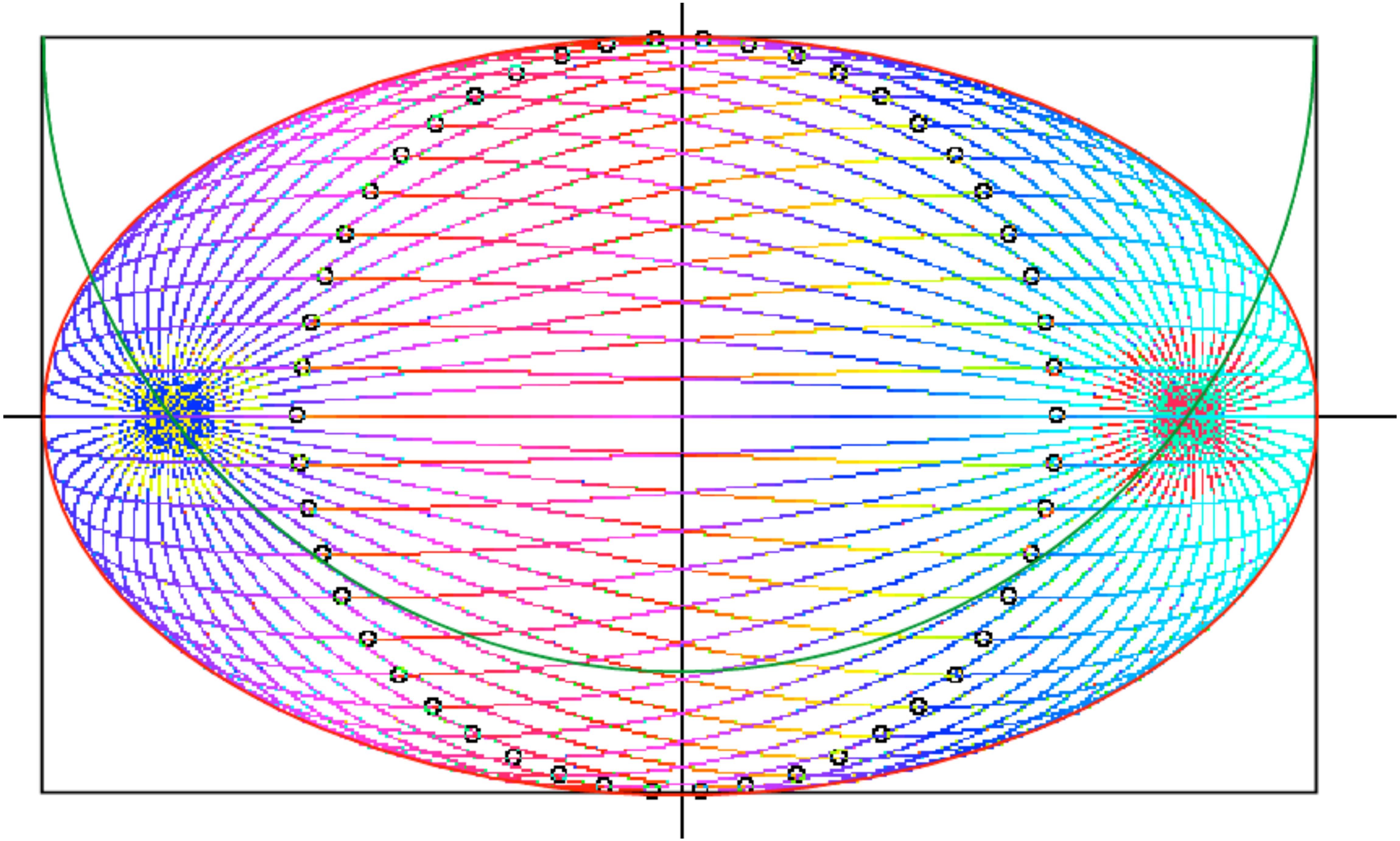


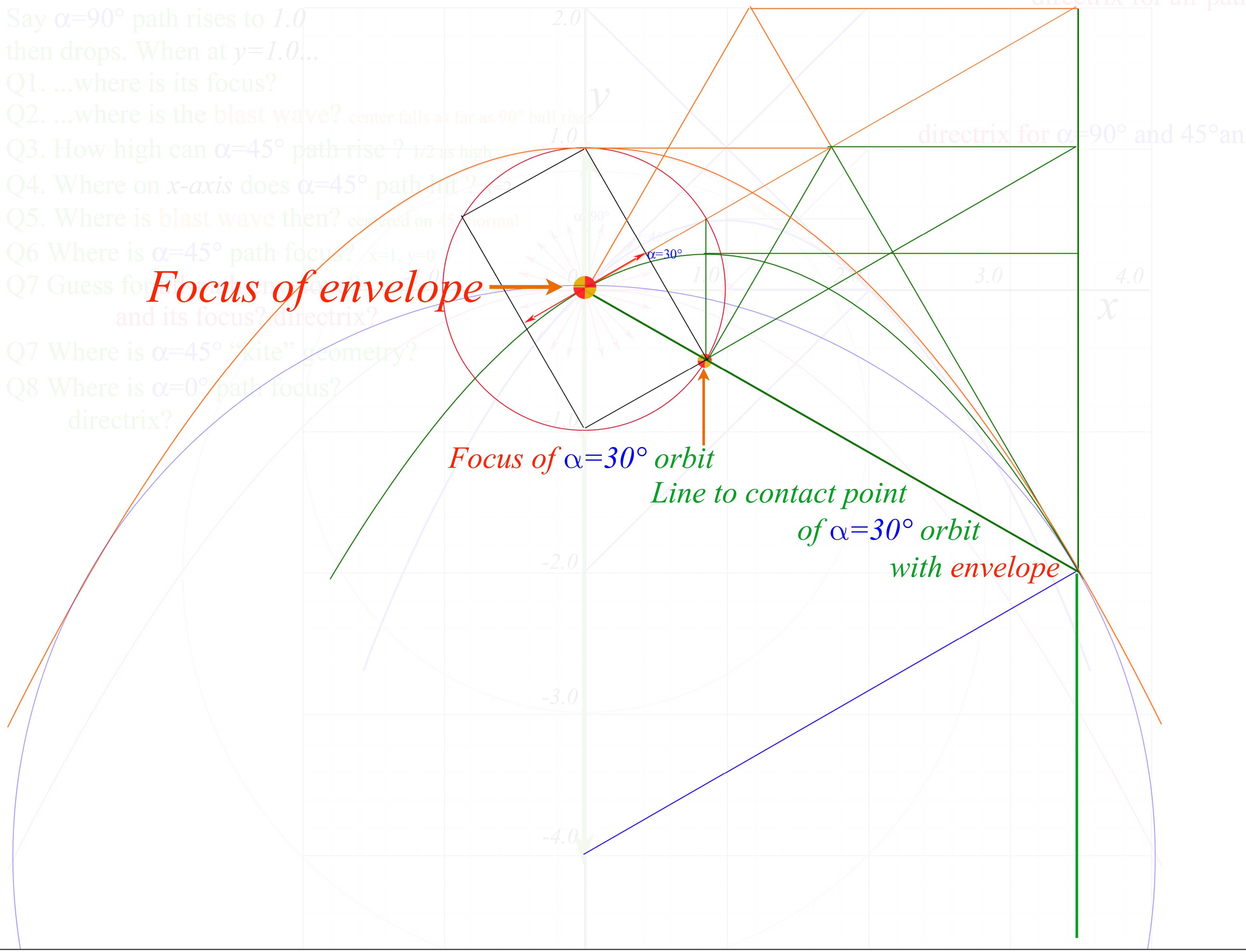
k=1 m=1
Minimum Energy Needed to:
Escape from.......
ρ=0.5 ......1.. to ∞
Orbit at ρ=0.5
Sit at ρ=0.5
The Three Equal Steps to ∞
Angular momentum μ=1/√2
Angular momentum μ=0
Sitting at ρ=0.0 at dead-center of a Sophomore-physics-Earth of radius R⊕=0.5
X
ρ

Parabolic and 2D-IHO elliptic orbital envelopes

Some clues for take-home assignment 7.2 (Simulation)

Exploding-starlet elliptical envelope and contacting elliptical trajectories





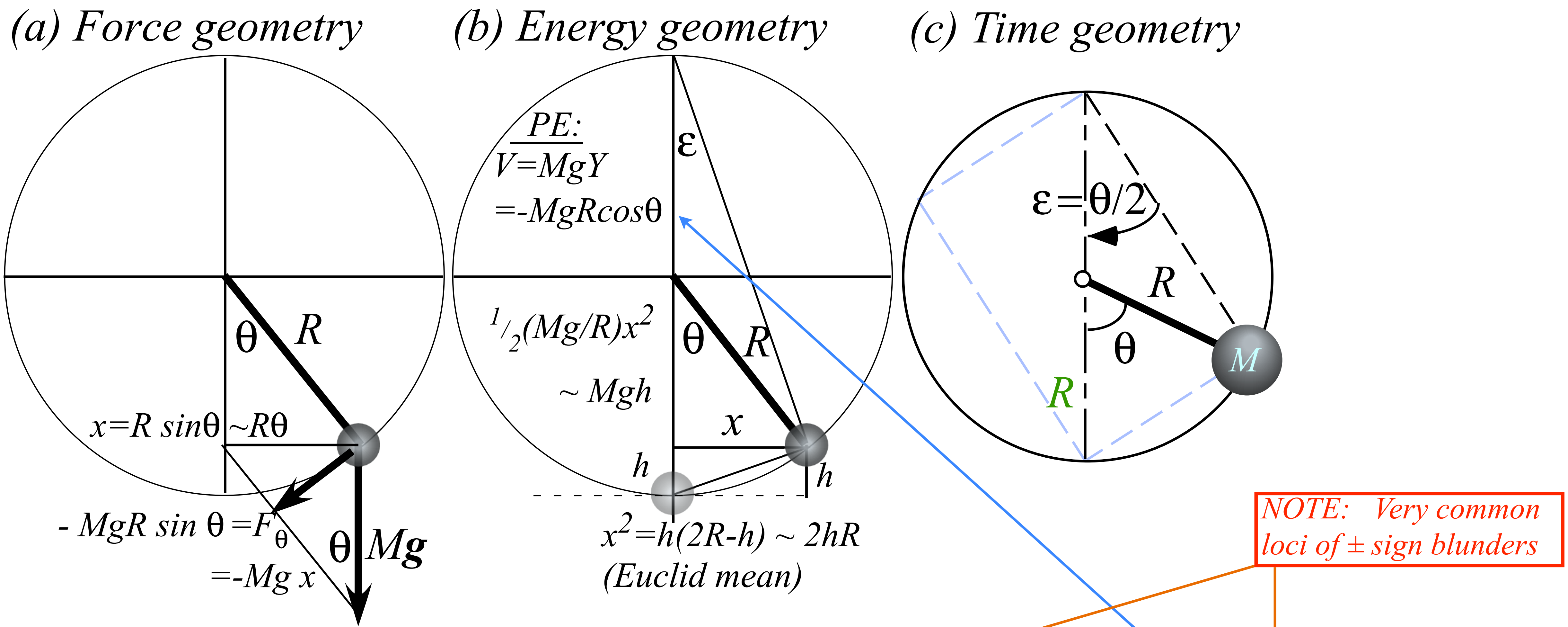
Say α=90° path rises to 1.0
then drops. When at y=1.0...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can α=45° path rise ? 1/2 as high
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is α=45° path focus? x=1, y=0
Q7 Where is α=45° "kite" geometry?
Q8 Where is α=0° path focus?
directrix?
directrix for all-path
directrix for α=90° and 45°and
Focus of envelope
α=30°
Focus of α=30° orbit
Line to contact point
of α=30° orbit
with envelope
x
y
2.0
1.0
-1.0
-2.0
-3.0
-4.0
3.0
4.0

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulation)

1D-HO phase-space control (Simulation)

1D Pendulum and phase plot

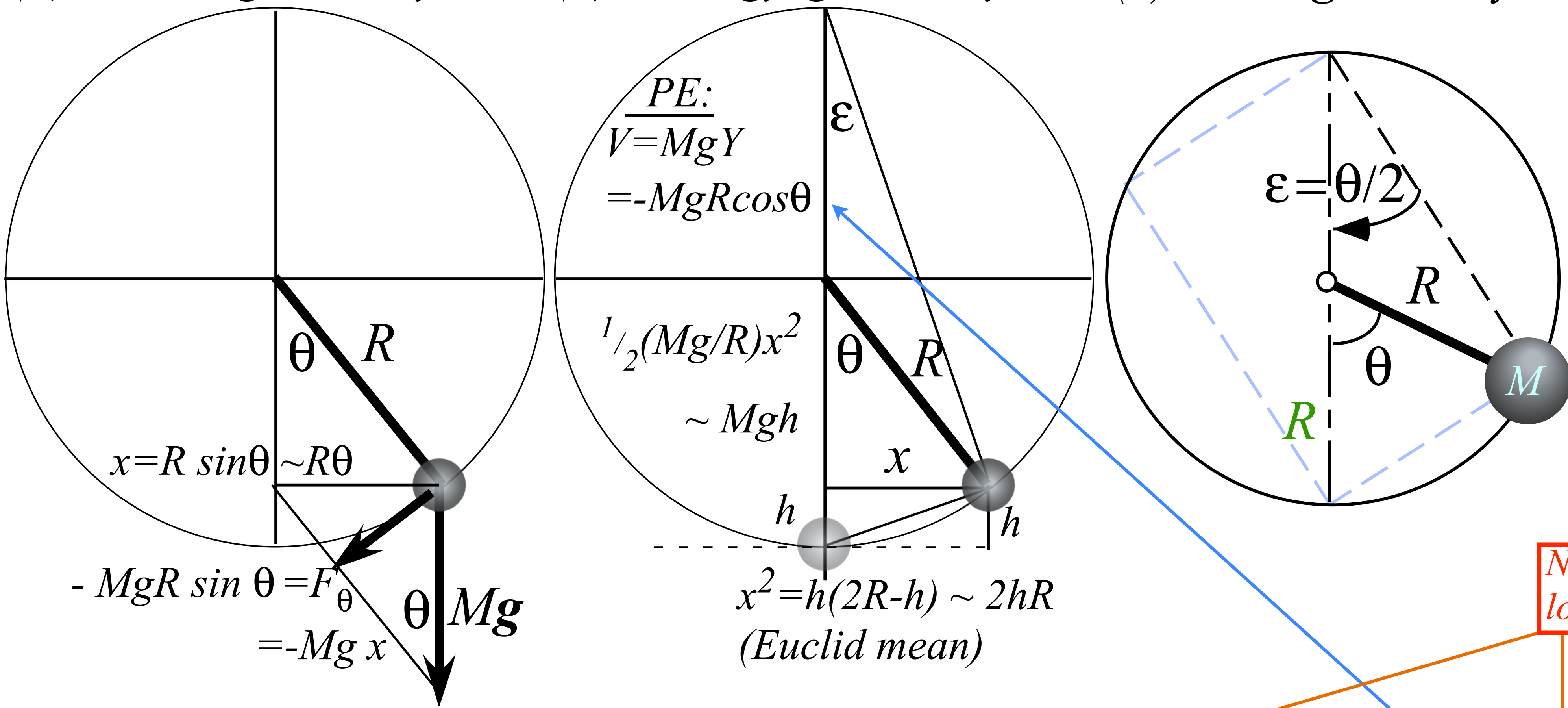


Lagrangian function L= KE - PE = T - U where potential energy is $U(\theta) = -MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2} I\dot{\theta}^2 - U(\theta) = \frac{1}{2} I\dot{\theta}^2 + MgR\cos\theta$$

1D Pendulum and phase plot

(a) Force geometry *(b) Energy geometry* *(c) Time geometry*



NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

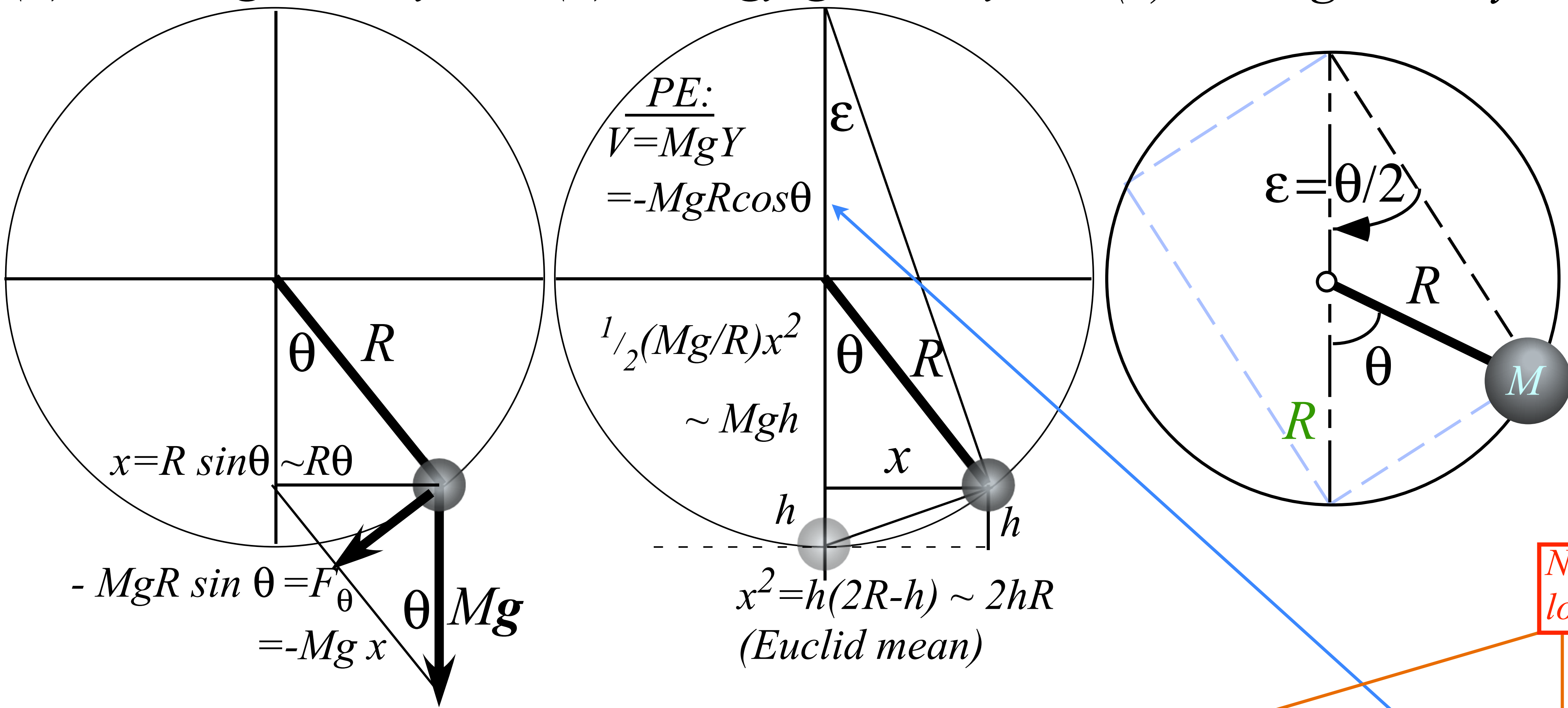
$$L(\dot{\theta},\theta) = \frac{1}{2} I\dot{\theta}^2 - U(\theta) = \frac{1}{2} I\dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I} p_\theta{}^2 + U(\theta) = \frac{1}{2I} p_\theta{}^2 - MgR\cos\theta \;=E \quad =const.$$

1D Pendulum and phase plot

(a) Force geometry (b) Energy geometry (c) Time geometry



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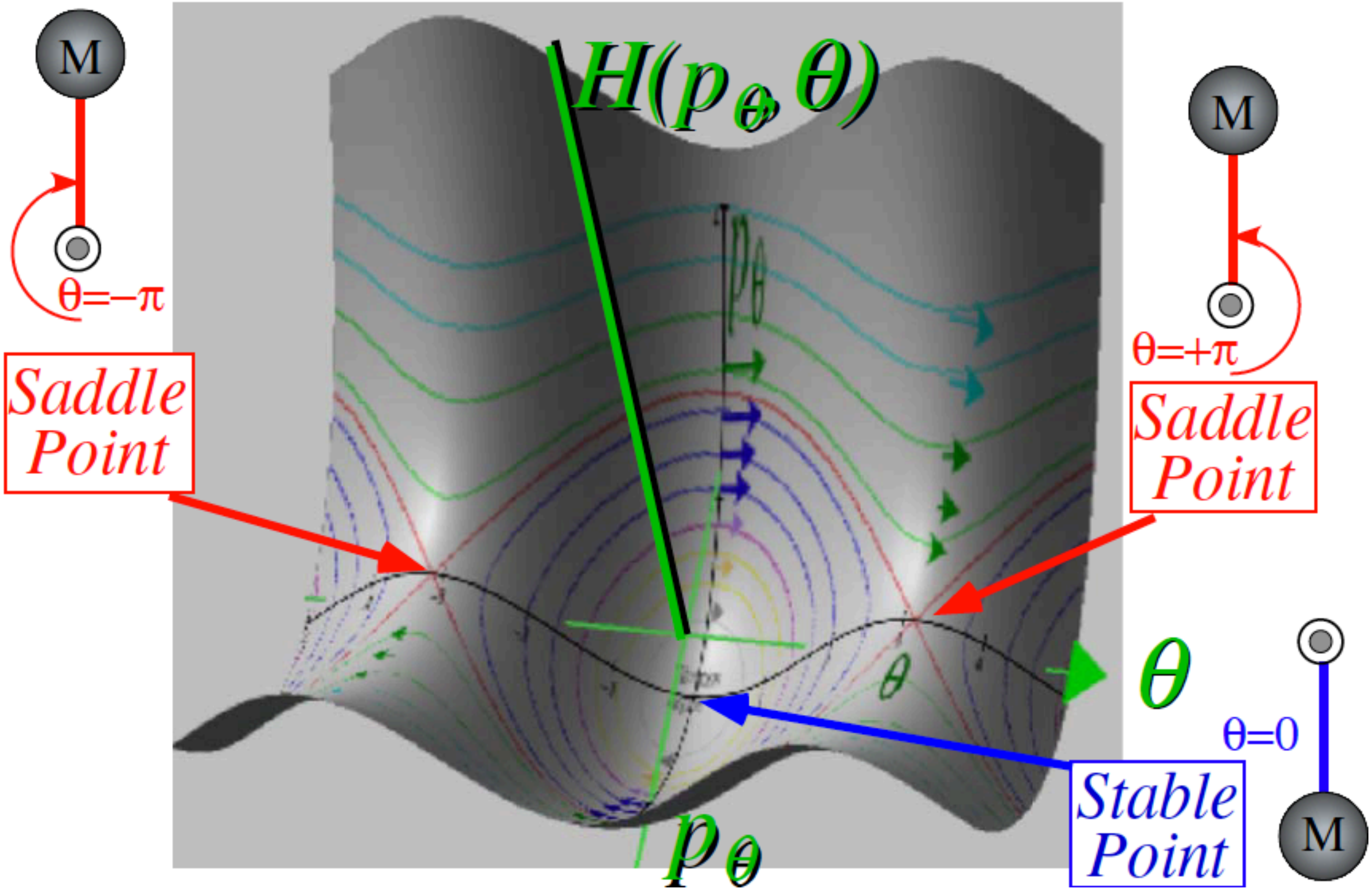
Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

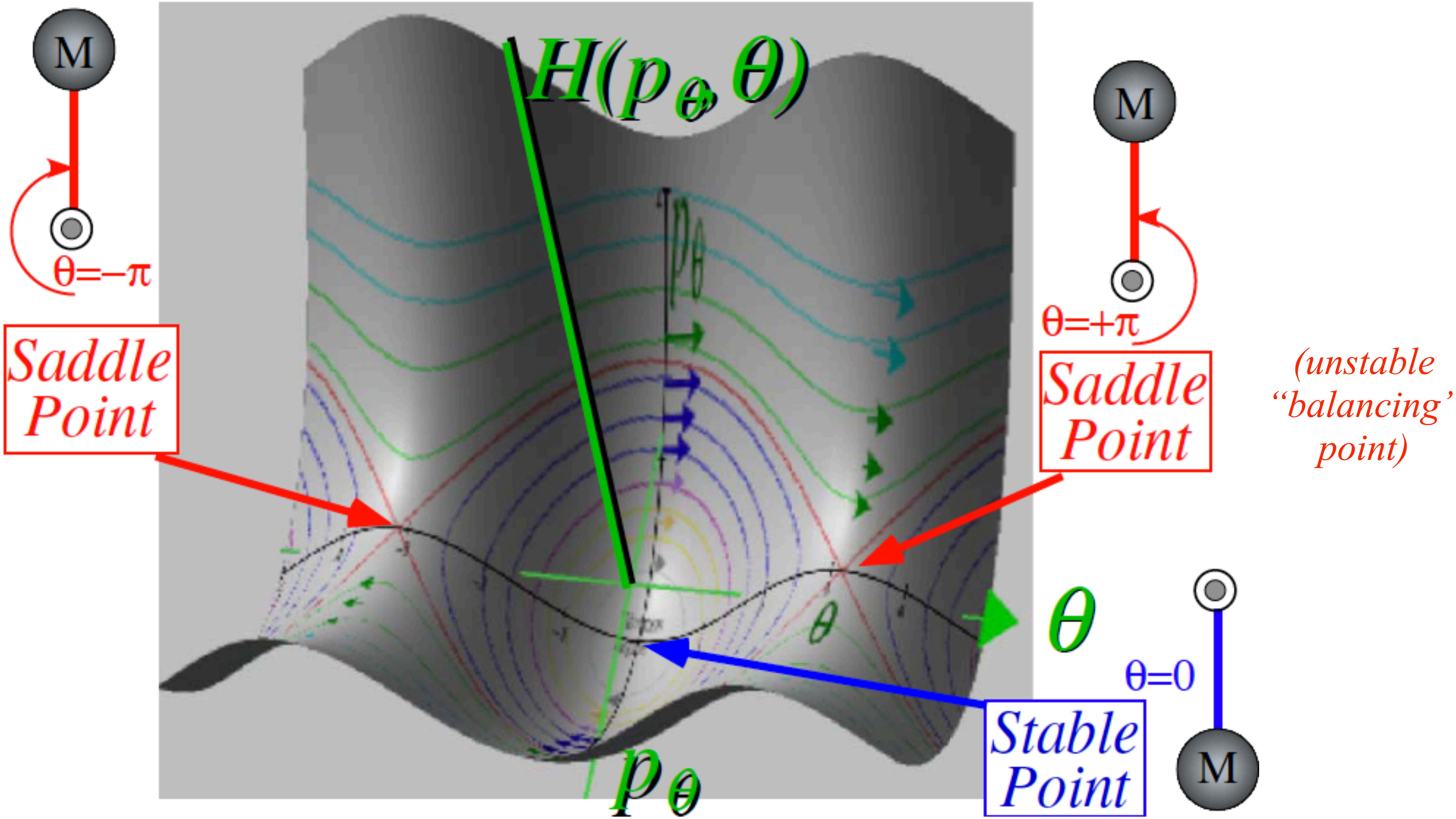
$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\;=E\quad =\text{const.}$$

implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta) = E = \frac{1}{2I}p_\theta^{\ 2} - MgR\cos\theta \ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,pθ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

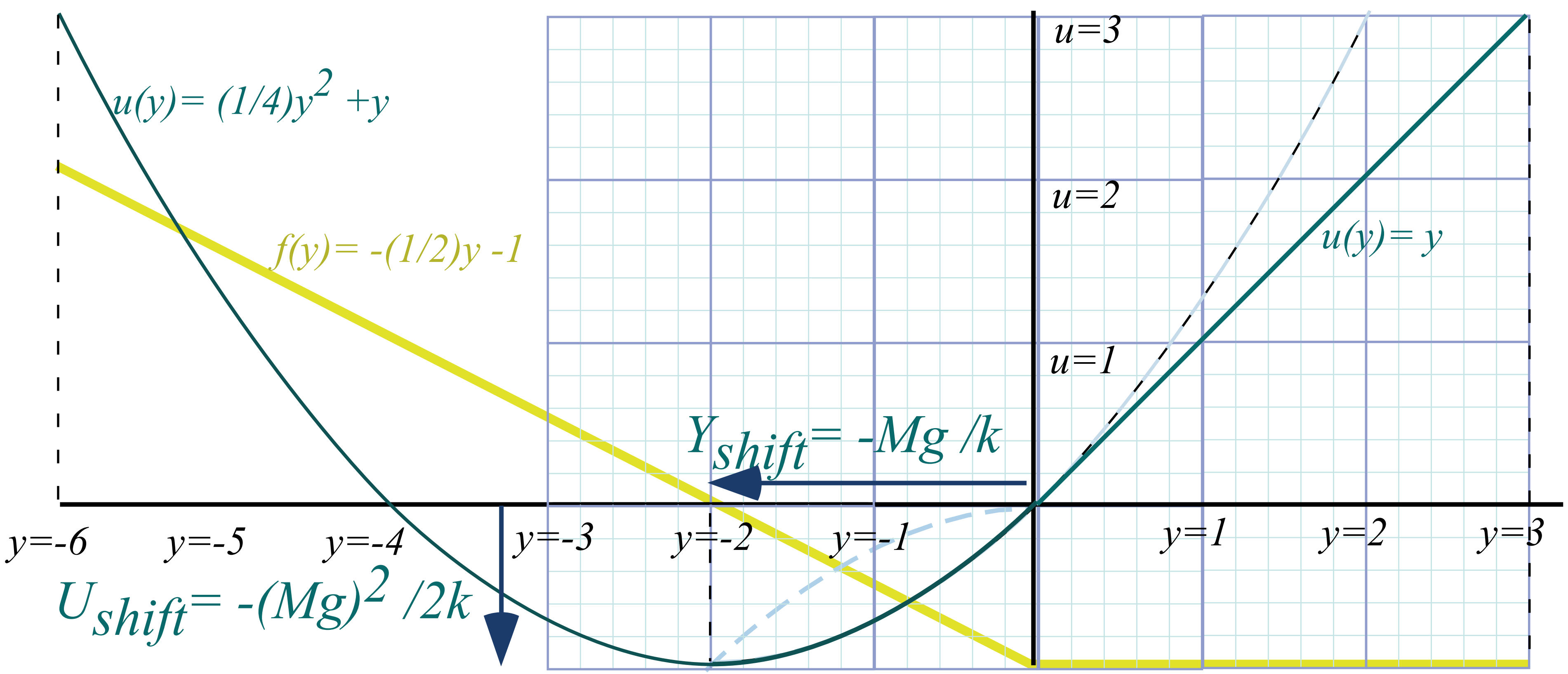
$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix}=\begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix}=\mathbf{e_H}\times(-\nabla H)=(\text{H-axis})\times(\text{fall line}),\ \text{where:}\begin{cases}(\text{H-axis})=\mathbf{e_H}=\mathbf{e_q}\times\mathbf{e_p}\\ (\text{fall line})=-\nabla H\end{cases}$$

2. Examples of Hamiltonian dynamics and phase plots

1D Pendulum and phase plot (Simulation)

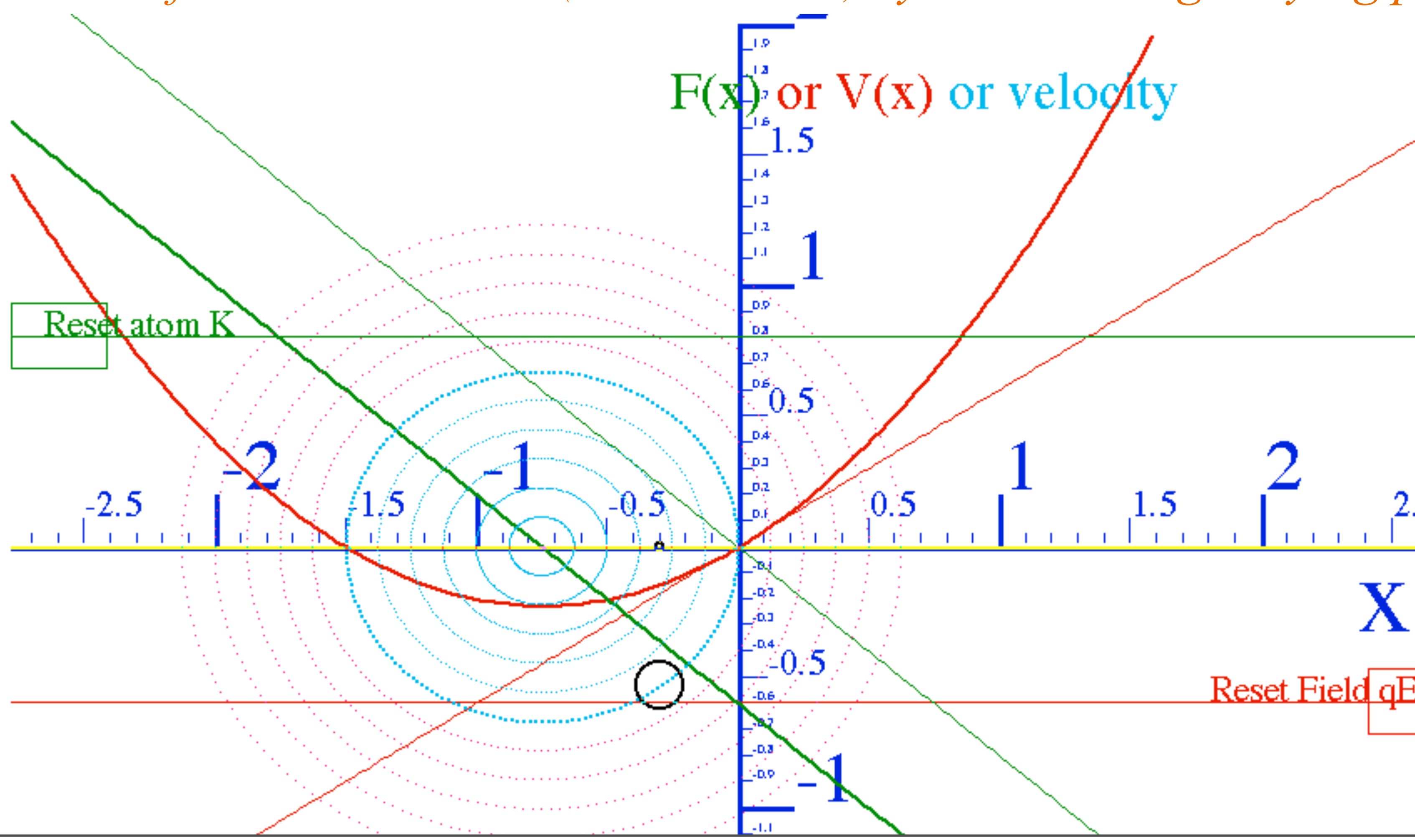
→ ***Phase control*** *(Simulation)*

$F(Y)=-kY-Mg$ $\qquad$ $U(Y)=(1/2)kY^2+Mg\,Y$



Unit 1
Fig. 7.4

Simulation of atomic classical (or semi-classical) dynamics using varying phase control



Lecture 12 ends here
Thur. 10.2.2012