Quadratic form geometry and development of mechanics of Lagrange and Hamilton
(Ch. 12 of Unit 1 and Ch. 4-5 of Unit 7)

Scaling transformation between Lagrangian and Hamiltonian views of KE (Review of Lecture 9)
Introducing 1st Lagrange and Hamilton differential equations of mechanics (Review Of Lecture 9)

Introducing the Poincare’ and Legendre contact transformations
Geometry of Legendre contact transformation
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

A general contact transformation from sophomore physics
Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”
Intuitive-geometric development of ” and ” and ”
Scaling transformation between Lagrangian and Hamiltonian views of KE (Review of Lecture 9)
Introducing the (partial) differential equations of mechanics (Review Of Lecture 9)
1st equations of Lagrange and Hamilton
Introducing the (partial $\frac{\partial}{\partial \varphi}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and Estrangian have no explicit dependence on momentum $\mathbf{p}$

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian and Estrangian have no explicit dependence on velocity $\mathbf{v}$

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian and Hamiltonian have no explicit dependence on speed $\mathbf{v}$

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}}{2} = \mathbf{M} \cdot \mathbf{v} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p}}{2} = \mathbf{M}^{-1} \cdot \mathbf{p} = \mathbf{v}$$

(Lagrange’s 1st equation(s))

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

(Hamilton’s 1st equation(s))

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

(Forget Estrangian for now)
Unit 1
Fig. 12.2

(a) Lagrangian plot
\[ L(v) = \text{const.} = v \cdot M \cdot v / 2 \]

(b) Hamiltonian plot
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2 \]

(c) Overlapping plots

Lagrangian tangent at velocity \( v \) is normal to momentum \( p \)

Hamiltonian tangent at momentum \( p \) is normal to velocity \( v \)

(d) Less mass

(e) More mass

Thursday, September 20, 2012
Introducing the Poincaré and Legendre contact transformations

Geometry of Legendre contact transformation
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
Introducing the Poincaré’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

Q-forms \( L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \) or \( H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \) to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\( Q \)-forms \( L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \) or \( H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \) to be \( H = (1/2) \mathbf{p} \cdot \mathbf{v} \) or equivalently \( L = (1/2) \mathbf{v} \cdot \mathbf{p} \).

Numerically-CORRECT, but Differentially-WRONG!
Introducing the Poincare’ and Legendre contact transformations

Given matrix relation: \( \mathbf{p} = \mathbf{M} \cdot \mathbf{v} \) or its inverse: \( \mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{p} \) you might be tempted to rewrite

\[ Q \text{-forms} \quad L(\mathbf{v}..) = (1/2) \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \quad \text{or} \quad H(\mathbf{p}..) = (1/2) \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \quad \text{to be} \quad H = (1/2) \mathbf{p} \cdot \mathbf{v} \quad \text{or equivalently} \quad L = (1/2) \mathbf{v} \cdot \mathbf{p}. \]

Numerically-CORRECT, but Differentially-WRONG!

Instead try: \( H(\mathbf{p}..) = \mathbf{p} \cdot \mathbf{v} - (1/2) \mathbf{v} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}..) \) or else: \( L(\mathbf{v}..) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}..) \)
Introducing the Poincaré and Legendre contact transformations

Given matrix relation: \( p = M \cdot v \) or its inverse: \( v = M^{-1} \cdot p \) you might be tempted to rewrite

Q-forms \( L(v) = (1/2)v \cdot M \cdot v \) or \( H(p) = (1/2)p \cdot M^{-1} \cdot p \) to be \( H = (1/2)p \cdot v \) or equivalently \( L = (1/2)v \cdot p \).

Numerically-CORRECT, but Differentially-WRONG!

Instead try: \( H(p) = p \cdot v - (1/2)v \cdot p = p \cdot v - L(v) \) or else: \( L(v) = p \cdot v - H(p) \)

**Legendre contact transformation**

\[
L(v) = p \cdot v - H(p) \quad \quad \quad H(p) = p \cdot v - L(v)
\]

Now explicit dependency (non)-relations give the right derivatives

\[
\frac{\partial L(v)}{\partial p} = \frac{\partial}{\partial p} (p \cdot v) - \frac{\partial H(p)}{\partial p} \quad \quad \quad \frac{\partial H(p)}{\partial v} = \frac{\partial}{\partial v} (p \cdot v) - \frac{\partial L(v)}{\partial v} \]

\[
0 = v - \frac{\partial H(p)}{\partial p} \quad \quad \quad 0 = p - \frac{\partial L(v)}{\partial v}
\]

That is Hamilton’s 1\textsuperscript{st} equation(s) and Lagrange’s 1\textsuperscript{st} equation(s)
Introducing the Poincare’ and Legendre contact transformations

Geometry of Legendre contact transformation
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
(a) Lagrangian plot
\[ L(v) = v \cdot p - H(p) \]

(b) Hamiltonian plot
\[ H(p) = p \cdot v - L(v) \]
**How Legendre contact transformations work...** (to make $\frac{\partial H}{\partial v} = 0$ or $\frac{\partial L}{\partial p} = 0$)

Secant lines $L(v) = p \cdot v - H$ of fixed slope $p = \frac{\partial L}{\partial v}$

and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > ...$

for increasing velocity $v_{-2} > v_{-1} > ... > v_0$

lead to unique tangent to $L(v)$-curve at the tangent contact point $v = v_0$ that has max $H(p, v_0)$

Thus $\frac{\partial H}{\partial v} = 0$

(Similarly...)

Unit 1
Fig. 12.4

(a) Secant lines: $L(v) = p \cdot v - H$

for fixed slope $p$ and varying $H$

Tangent line points to extreme value $-H(v_0)$ of intercept $-H$ thus:

$\frac{dH(v)}{dv} = 0$

(b) Secant lines: $H(p) = p \cdot v - L(v)$

for fixed slope $v$ and varying $L$

Tangent line points to extreme value $-L(p_0)$ of intercept $-L$ thus:

$\frac{dL(p)}{dp} = 0$
Introducing the Poincaré and Legendre contact transformations
Geometry of Legendre contact transformation
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
Example of Legendre contact transformation in thermodynamics

Internal energy $U(S,V)$ is defined as a function of entropy $S$ and volume $V$.

A new function *enthalpy* $H(S,P)$ depends on entropy and *pressure* $P$.

It is a Legendre transform $H(S,P)=P \cdot V + U$ of energy $U(S,V)$ to new variable $P = -(\frac{\partial U}{\partial V})_S$.

Except for ± signs, it’s our Hamiltonian $H(p) = p \cdot v - L(v)$ going from Lagrangian $L(v)$ to use new variable momentum $p = (\frac{\partial L}{\partial v})_x$. 
Introducing the Poincare’ and Legendre contact transformations

Geometry of Legendre contact transformation

Example from thermodynamics

Legendre transform: special case of General Contact Transformation (lights, camera, ACTION!)
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function: \( S(x,y:X,Y) = \text{const.} \) does mapping. Y(X) is mapped from y(x) as an envelope of contacting \( S = \text{const.} \) curves.

The Legendre transformation does it with contacting straight line tangents.

Unit 1
Fig. 12.7

Unit 1
Fig. 12.9
Legendre transform: special case of General Contact Transformation

**Active-Contact-Transformation Generator or Action function**: $S(x,y:X,Y) = \text{const.}$ does mapping.

$Y(X)$ is mapped from $y(x)$ as an envelope of contacting $S=\text{const.}$ curves.  

...And, Visa-Versa !...

The Legendre transformation does it with contacting straight line tangents.

---

**Legendre Transform**

- **Legendre transform**
- **Legendre transformation**
- **Legendre function**

**Legendre transform** is a mathematical operation that transforms a function of one variable into a function of another variable. It is a special case of the more general **contact transformation**.

**Legendre transformation** is widely used in various fields of physics and engineering, particularly in the study of classical mechanics and thermodynamics. It is particularly useful for transforming Hamilton's equations into the more manageable **Hamilton's equations of motion**.

**Legendre transform** is defined as:

$$L(q) = \int_{q_0}^{q} f(p) \, dp$$

And its inverse,

$$H(p) = \int_{p_0}^{p} f(q) \, dq$$

where $f(q)$ and $f(p)$ are the functions to be transformed.

**Legendre function** refers to a type of special function that appears in the solutions of differential equations, particularly in the study of wave phenomena.

**Legendre transform** is a key concept in the study of systems governed by **Legendre's equation**, which arises in the study of potential theory and spherical harmonics.
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or Action function: $S(x, y; X, Y) = \text{const.}$ does mapping.

$Y(X)$ is mapped from $y(x)$ as an envelope of contacting $S = \text{const.}$ curves.

...And, Visa-Versa !...

The Legendre transformation does it with contacting straight line tangents.

Poincaré’s differential action

$$dS = L\, dt = p \cdot \dot{q} \, dt - H \cdot dt$$

$$= p \cdot dq - H \cdot dt$$

(Quantum phase differential)

Unit 1
Fig. 12.7

Unit 1
Fig. 12.9
A general contact transformation from sophomore physics

Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”

Intuitive-geometric development of “ ” and “ ”
\[
\alpha = 45^\circ
\]

(a) Volcanic plumes on Jupiter’s moon \textit{Io}

(b) Atomic clock controls expanding balls of Cesium atoms rising and falling in Earth gravity

\textit{(NIST Boulder Labs)}

(c) Trajectory family for fixed \(g\) and \(v_0\)

\textit{Atom ball expands at constant rate} \(v_0\) \textit{as center falls at constantly increasing rate} \(g\) \textit{and it maintains two contact points with the envelope after reaching its highest point.}
UP-1 formulas for trajectories in constant gravity $g$

$$x(t) = (v_0 \cos \alpha) t$$
$$y(t) = (v_0 \sin \alpha) t - \frac{1}{2} gt^2$$

$$\dot{x}(0) = v_x(0) = v_0 \cos \alpha$$
$$\dot{y}(0) = v_y(0) = v_0 \sin \alpha$$

Substitute time $t = x/(v_0 \cos \alpha)$ into $y(t)$

$$y(x) = \frac{v_0 \sin \alpha}{v_0 \cos \alpha} x - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$
Convert $y(x)$ solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x \tan \alpha - \frac{g x^2}{2 v_0^2 \cos^2 \alpha} \quad \text{becomes:} \quad S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{g x^2}{2 v_0^2 \cos^2 \alpha} = 0$$

Unit 1

Fig. 12.6
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha: x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha: x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

Envelopes of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha: x, y)}{\partial \alpha} = 0
\]
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha: x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha: x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

**Envelopes** of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha: x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^{-2} \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2 \sin \alpha}{\cos^3 \alpha}
\]
Convert $y(x)$ solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{becomes:} \quad S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0$$

Envelopes of the $v_0$-trajectory region contain extremal contact points with each trajectory where:

$$\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0$$

$$x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^2 \alpha}{\partial \alpha} = 0 = \frac{x \cos^2 \alpha - g v_0^2 \sin \alpha}{v_0^2 \cos^2 \alpha} \quad \text{gives:} \quad \tan \alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g \tan \alpha}.$$
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{becomes:} \quad S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

\( v_0 \)

\( \alpha = 45^\circ \)

\( y_{\text{env}}(x) = x \tan \alpha - \frac{gx^2}{2v_0^2} \left( 1 + \tan^2 \alpha \right) \Rightarrow y_{\text{env}}(x) = \frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2} \left( 1 + \frac{v_0^4}{g^2x^2} \right) \)

\( x = \frac{v_0^2}{g \tan \alpha} \)

\( \alpha = 45^\circ \)

\( \alpha = 60^\circ \)

\( \alpha = 75^\circ \)

Envelopes of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^2 \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2 \sin \alpha}{\cos^3 \alpha}
\]

\[
\tan \alpha = \frac{v_0^2}{gx} \quad \text{or} \quad x = \frac{v_0^2}{g \tan \alpha}
\]

Unit 1

Fig. 12.6
Convert \( y(x) \) solution into Active Contact Transformation Generator \( S(v_0, \alpha : x, y) \)

\[
y(x) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}
\]

becomes:

\[
S(v_0, \alpha : x, y) = -y + x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = 0
\]

Envelopes of the \( v_0 \)-trajectory region contain extremal contact points with each trajectory where:

\[
\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0
\]

\[
x \frac{\partial \tan \alpha}{\partial \alpha} - \frac{gx^2}{2v_0^2} \frac{\partial \cos^2 \alpha}{\partial \alpha} = 0 = \frac{x}{\cos^2 \alpha} - \frac{gx^2}{2v_0^2} \frac{2 \sin \alpha}{\cos^3 \alpha}
\]

\[
\tan \alpha = \frac{v_0^2}{gx} \quad \text{or} \quad x = \frac{v_0^2}{g \tan \alpha}
\]

\[
y_{env}(x) = x \tan \alpha - \frac{gx^2}{2v_0^2} \left(1 + \tan^2 \alpha\right) \Rightarrow y_{env}(x) = x \frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2} \left(1 + \frac{v_0^4}{g^2x^2}\right)
\]

\[
y_{env}(x) = \frac{v_0^2}{g} - \frac{gx^2}{2v_0^2} - \frac{g^2}{2v_0^2} \frac{v_0^4}{g^2x^2} = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}
\]

Envelope function
The Plumes of Prometheus
NASA-Galileo Project
Io fly-by on August 18, 1997

Pretty bad sketch of plumes (LasVegas model of planetary ejecta?)

Do these guys need a geometry lesson?

Go fly a kite?

---

IO'S ALIEN VOLCANOES

Scientists are eager for a closer look at the solar system's strangest and most active volcanoes when Galileo flies by Io on October 11.

October 4, 1999: Thirty years ago, before the Voyager probes visited Jupiter, if you had described Io to a literary critic it would have been declared overwrought science fiction. Jupiter's strange moon is literally bursting with volcanoes. Dozens of active vents pepper the landscape which also includes gigantic frosty plains, towering mountains and volcanic rings the size of California. The volcanoes themselves are the hottest spots in the solar system with temperatures exceeding 1800 K (1527 C). The plumes which rise 300 km into space are so large they can be seen from Earth by the Hubble Space Telescope. Confounding common sense, these high-rising ejecta seem to be made up of, not blisteringly hot lava, but frozen sulfur dioxide. And to top it all off, Io bears a striking resemblance to a pepperoni pizza. Simply unbelievable.

Right: Digital Radiance simulation of Pillan Patera just before the Galileo flyby. [click for animation]
...conventional parabolic geometry...carried to extremes...

Recall Lecture 6 p.29

Unit 1
Fig. 9.4
A general contact transformation from sophomore physics

Algebra-calculus development of “The Volcanoes of Io” and “The Atoms of NIST”

Intuitive-geometric development of ” ” ” and ” ” ” ”
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?

Q2. ...where is the blast wave? center falls as far as 90° ball rises.

Q3. How high can $\alpha=45^\circ$ path rise?

Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises.
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high.
Q4. Where on x-axis does $\alpha=45^\circ$ path hit? $x=2$.
Q5. Where is blast wave then?
Q6 Where is $\alpha=45^\circ$ path focus?
Q7 Guess for all-path envelope? and its focus? directrix?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1$, $y=0$
Q7 Guess for all-path envelope?
    and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rising
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on x-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45° normal
Q6. Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?

Where is $\alpha=30^\circ$ path?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave? Center falls as far as $90^\circ$ ball rises.
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high.
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? Centered on $45^\circ$ normal.
Q6. Where is $\alpha=45^\circ$ path focus? $x=1$, $y=0$
Q7. Guess for all-path envelope and its focus? Directrix?
Q7. Where is $\alpha=45^\circ$ “kite” geometry?
Q8. Where is $\alpha=0^\circ$ path focus? Directrix?

Where is $\alpha=30^\circ$ path? ...and kite structure?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can $\alpha=45^\circ$ path rise? 1/2 as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1$, $y=0$
Q7 Guess for all-path envelope and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?

Where is $\alpha=30^\circ$ path? ...and kite structure?
Where is $\alpha=60^\circ$ path? 
...and kite structure?

For $\alpha=60^\circ$ parabolic trajectory
contact-parabolic envelope,
timing ($\alpha=0^\circ$)-parabola,
($\alpha=90^\circ$)-blast-wave-circle,
($\alpha=60^\circ$)-blast-wave-circle.