Lecture 28

Multi-particle and Rotational Dynamics

(Ch. 2-7 of Unit 6   12.06.12)

2-Particle orbits

Copernican view

Ptolemetric view

2-Particle scattering Lab-vs.-Body frame views

Ruler & compass construction

Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

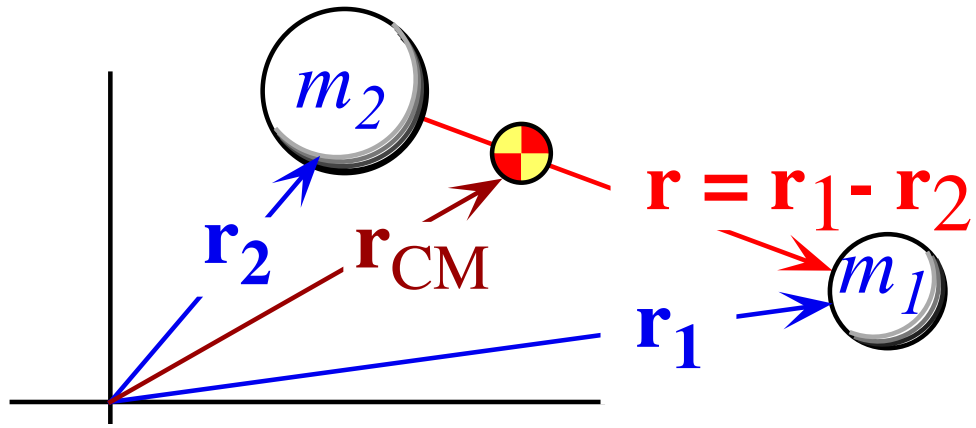
angular velocity $\boldsymbol{\omega}$-ellipsoid vs. angular momentum **L**-*ellipsoid*

Lagrangian $\boldsymbol{\omega}$-equations vs. Hamiltonian momentum **L**-*equation*

Symmetric-top dynamics (Constant **L***)*

BOD-frame cone rolling on LAB frame cone

2-Particle orbits and center-of-mass (CM) coordinate frame



$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Defining *relative coordinate vector*

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and *mass-weighted-average* or *center-of-mass coordinate vector* $\mathbf{r}_{CM}$

$$\bar{\mathbf{r}} = \mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

The inverse coordinate transformation.

$$\mathbf{r}_1 = \mathbf{r}_{CM} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_{CM} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\mu = \frac{m_2 m_1}{m_1 + m_2}$ $\quad \ddot{\mathbf{r}}_\mathbf{CM} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_\mathbf{1} \quad] - [\quad m_2\ddot{\mathbf{r}}_\mathbf{2} \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[m_2\ddot{\mathbf{r}}_\mathbf{CM} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}...\right) \quad (m_1 >> m_2)$$

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}...\right) \quad (m_2 >> m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

each particle keeps it original mass *m1* or *m2*, but feels

coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = F(\frac{m_1}{\mu}r_1)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = F(\frac{m_2}{\mu}r_2)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

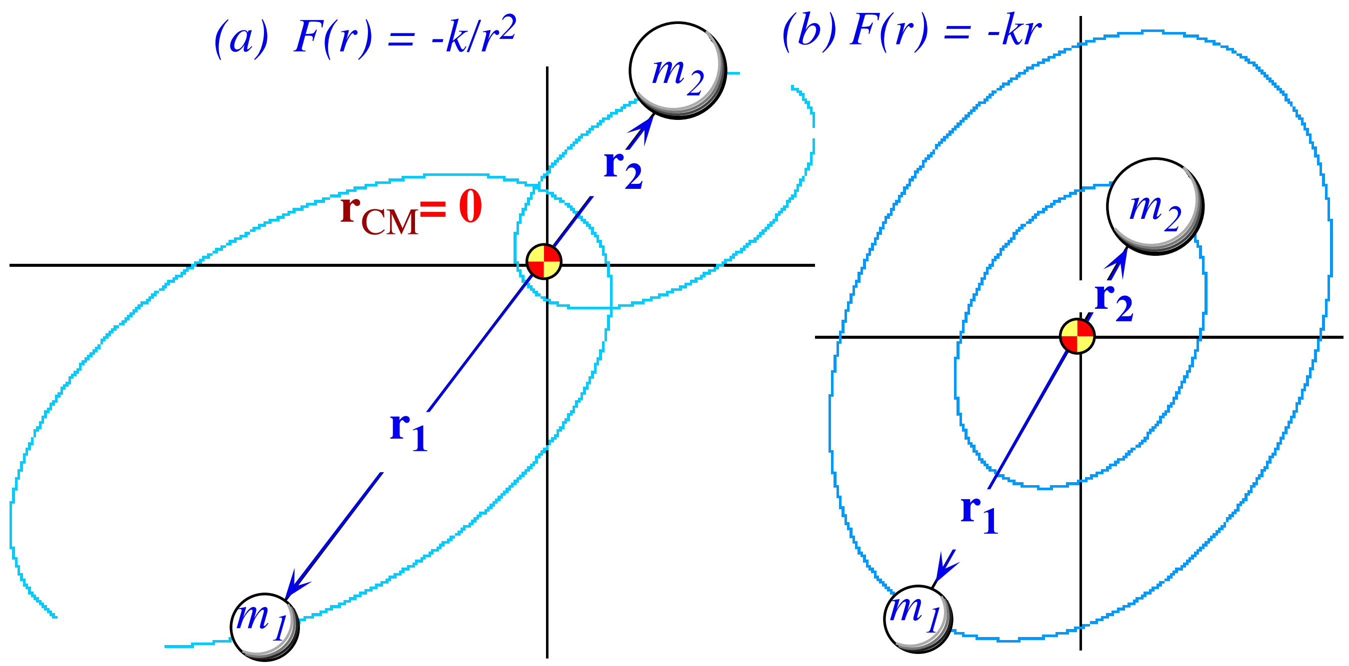
$F(r) = \frac{k}{r^2}$ becomes: $F(\frac{m_1}{\mu}r_1) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$,

$k \to k_1 = k\mu^2/m_1^2, \quad k \to k_2 = k\mu^2/m_2^2$

$F(r) = -k\,r$ becomes: $F(\frac{m_1}{\mu}r_1) = -\frac{m_1}{\mu}k\,r_1$,

$k \to k_1 = k\,m_1/\mu, \quad k \to k_2 = k\,m_2/\mu$

Examples of Coulomb and harmonic oscillator 2-particle "Ptolemetric" orbits.



Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.
Orbits differ in size of axes (a_1 , b_1) and (a_2 , b_2)
Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).
Orbit axial dimensions (a_k , b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

Harmonic oscillator periods and Coulomb orbit periods and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}} \qquad T_{Coul} = 2\pi\sqrt{\frac{\mu\, a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1{}^3}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2{}^3}{k_2}} \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon$$

Three Coulomb orbit energy values satisfy the same proportion relation as their axes

$$E_1 m_1 = E_2 m_2 = E\mu \ , \quad \text{where:} \quad |E_1| = \frac{|k_1|}{2a_1} \ , \quad |E_2| = \frac{|k_2|}{2a_2} \ , \quad |E| = \frac{|k|}{2a} \ .$$

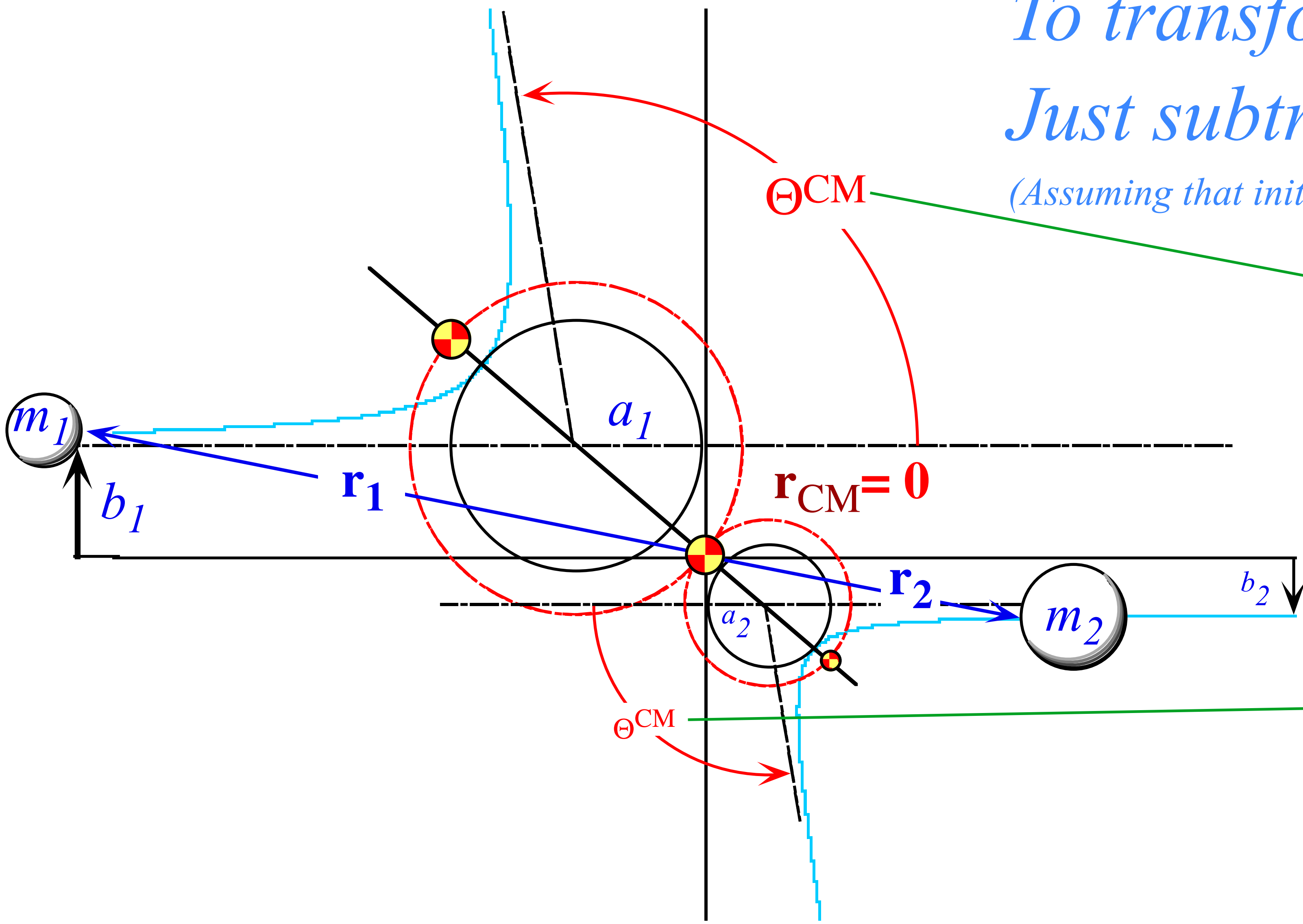
Energy values and axes satisfy similar sum relations

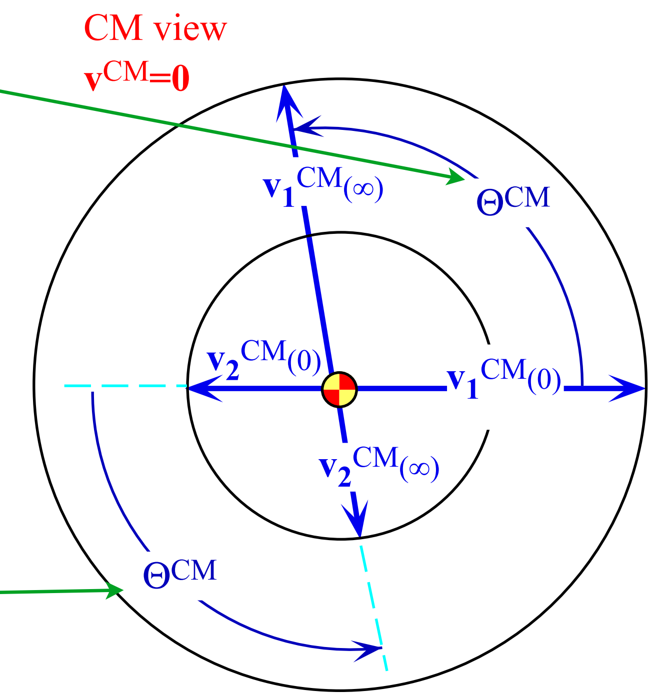
$$E_1 + E_2 = \frac{m_1}{\mu}E + \frac{m_2}{\mu}E = E \ , \quad \text{and:} \quad a_1 + a_2 = \frac{m_1}{\mu}a + \frac{m_2}{\mu}a = a$$

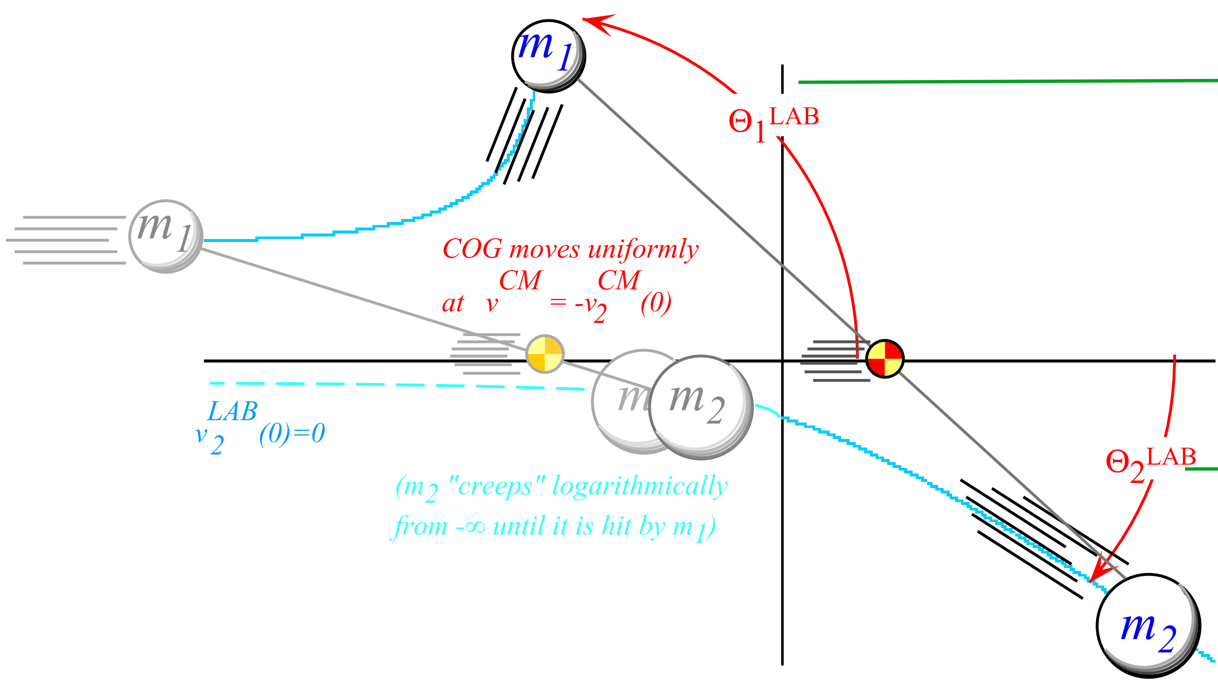
To transform CM to LAB frame

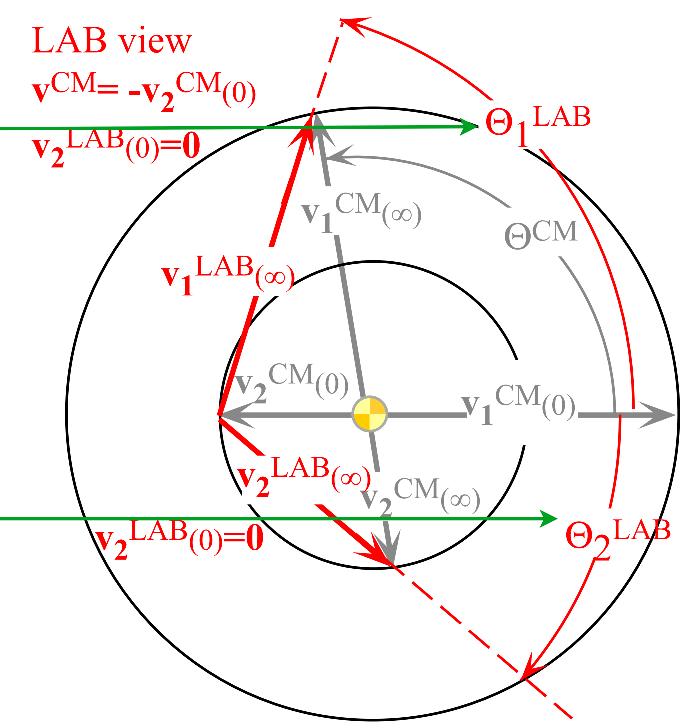
Just subtract $\mathbf{v}_2^{CM}(0)$ *from all*

(Assuming that initial $\mathbf{v}_2^{LAB}(0)$ *is zero so* $\mathbf{v}_2^{CM}(0)$ *is CM velocity in LAB)*

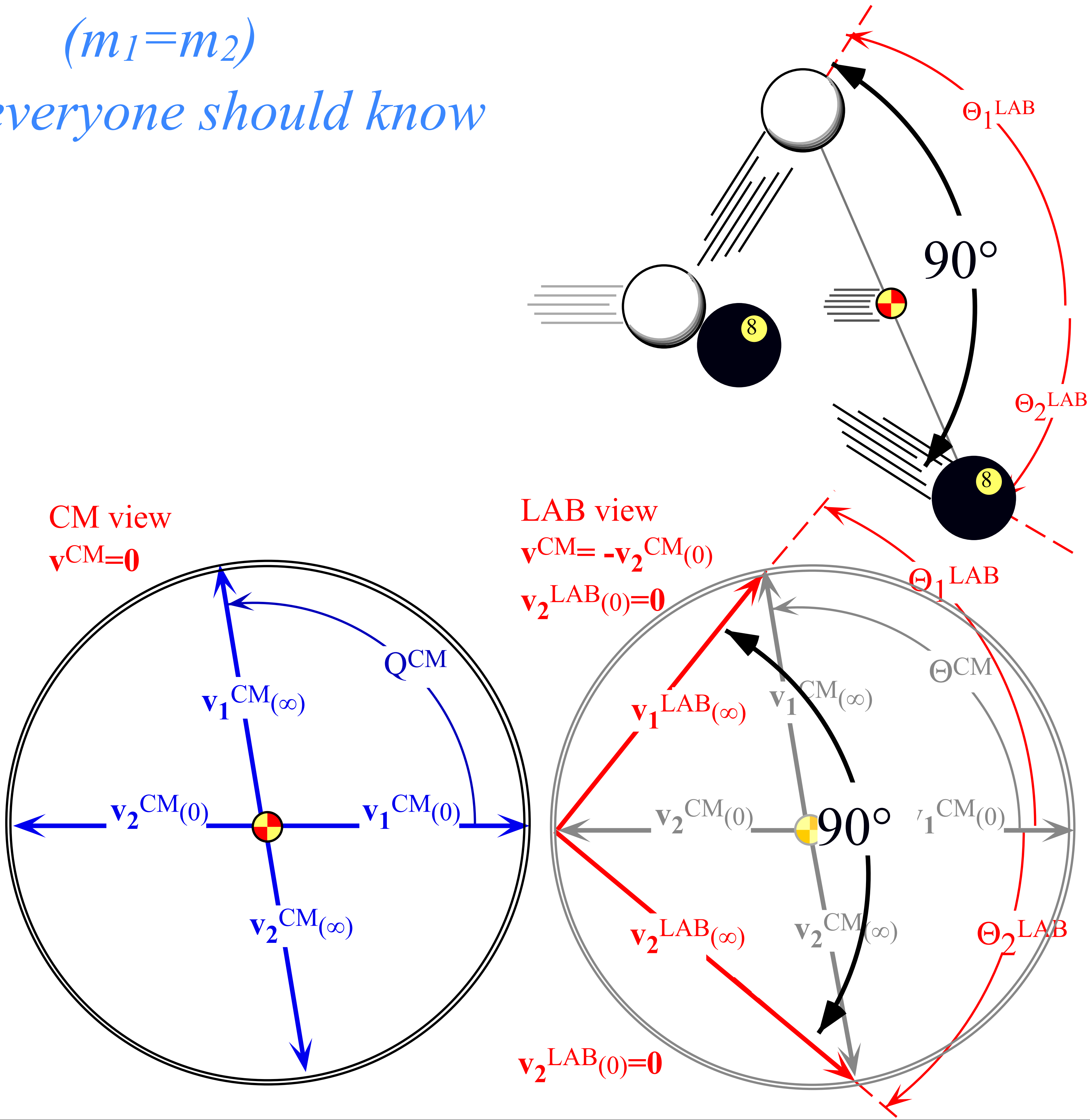








A common type of scattering
($m_1=m_2$)
...that everyone should know



Inertia tensors

angular velocity and angular momentum relation

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \qquad \mathbf{L} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^{3} m_j \mathbf{r}_j \times \left(\boldsymbol{\omega}\times\mathbf{r}_j\right)$$

Levi-Civita analysis

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right) = \left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B} - \left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$$

$$\mathbf{L} = \sum_{j=1}^{3} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega} - \left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right] = \sum_{j=1}^{3} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

the *rotational inertia tensor* **I**

$$\overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{3} \overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{3} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j\right]$$

matrix form the ω-to-**L** relation

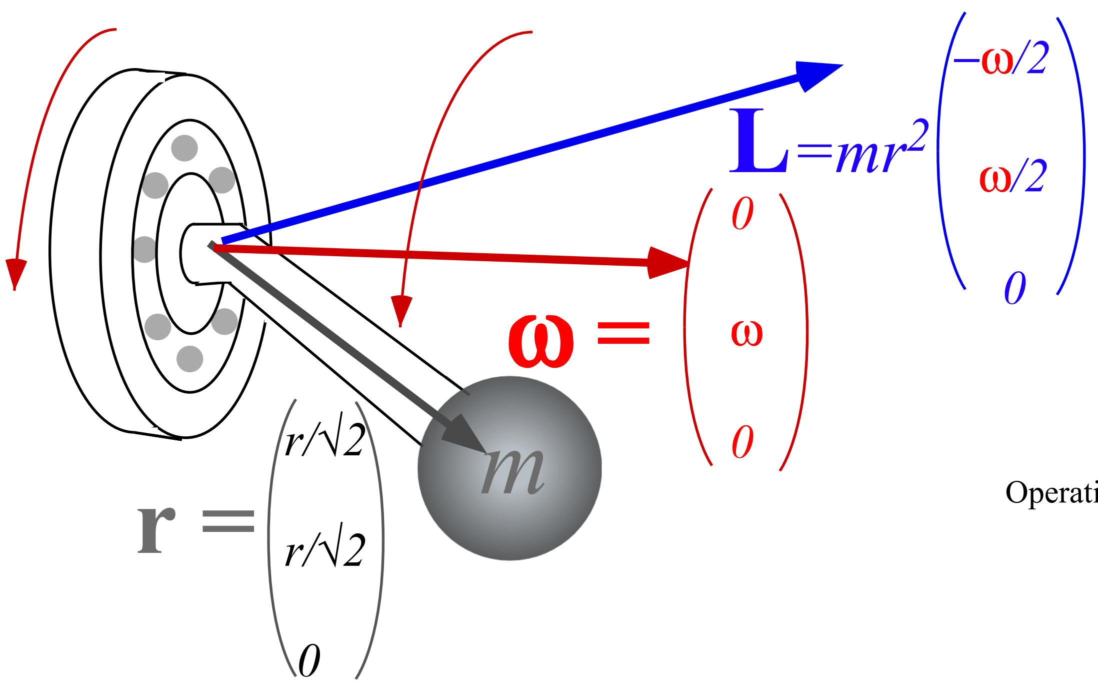
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{3} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

the *inertia matrix* <**I**>

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = \sum_{j=1}^{3}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle = \sum_{j=1}^{3} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}$$

mass m at the end of a bent axle that is rotating around a fixed bearing instantaneously at

$$\mathbf{r}_m = (x_m, y_m, z_m) = r\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$



$$\overleftrightarrow{\mathbf{I}} = mr^2 \begin{pmatrix} \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0 \\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0 \\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2 \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Operating on the angular velocity gives the angular momentum

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2 \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \omega$$

Kinetic energy in terms of velocity ω and rotational Lagrangian

Kinetic energy T of a rotating rigid body can be expressed in terms of the inertia matrix $\mathbf{I}$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j\dot{\mathbf{r}}_j\bullet\dot{\mathbf{r}}_j=\frac{1}{2}\sum_{j=1}^{3} m_j\left(\boldsymbol{\omega}\times\mathbf{r}_j\right)\bullet\left(\boldsymbol{\omega}\times\mathbf{r}_j\right)$$

Levi-Civita identity

$$\left(\mathbf{A}\times\mathbf{B}\right)\times\left(\mathbf{C}\times\mathbf{D}\right)=\left(\mathbf{A}\bullet\mathbf{C}\right)\left(\mathbf{B}\bullet\mathbf{D}\right)-\left(\mathbf{A}\bullet\mathbf{D}\right)\left(\mathbf{B}\bullet\mathbf{C}\right)$$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j\left[\left(\boldsymbol{\omega}\bullet\boldsymbol{\omega}\right)\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)-\left(\boldsymbol{\omega}\bullet\mathbf{r}_j\right)\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\right]$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\sum_{j=1}^{3} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\left(\mathbf{r}_j\right)\left(\mathbf{r}_j\right)\right]\bullet\boldsymbol{\omega}$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\vec{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Kinetic energy is a *quadratic form*

$$T=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\langle\omega|x\rangle & \langle\omega|y\rangle & \langle\omega|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\mathbf{I}|x\rangle & \langle x|\mathbf{I}|y\rangle & \langle x|\mathbf{I}|z\rangle\\ \langle y|\mathbf{I}|x\rangle & \langle y|\mathbf{I}|y\rangle & \langle y|\mathbf{I}|z\rangle\\ \langle z|\mathbf{I}|x\rangle & \langle z|\mathbf{I}|y\rangle & \langle z|\mathbf{I}|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\omega\rangle\\ \langle y|\omega\rangle\\ \langle z|\omega\rangle\end{pmatrix}\quad\text{(Dirac notation)}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\sum_{j=1}^{3} m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

Simplifies in *principle inertial axes {X,Y,Z}* or *body eigen-axes*

$$T=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & I_{XY} & I_{XZ}\\ I_{YX} & I_{YY} & I_{YZ}\\ I_{ZX} & I_{ZY} & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & 0 & 0\\ 0 & I_{YY} & 0\\ 0 & 0 & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}=\frac{I_{XX}\omega_X^2}{2}+\frac{I_{YY}\omega_Y^2}{2}+\frac{I_{ZZ}\omega_Z^2}{2}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity $\boldsymbol{\omega}$, momentum $\mathbf{L}$, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$,   generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

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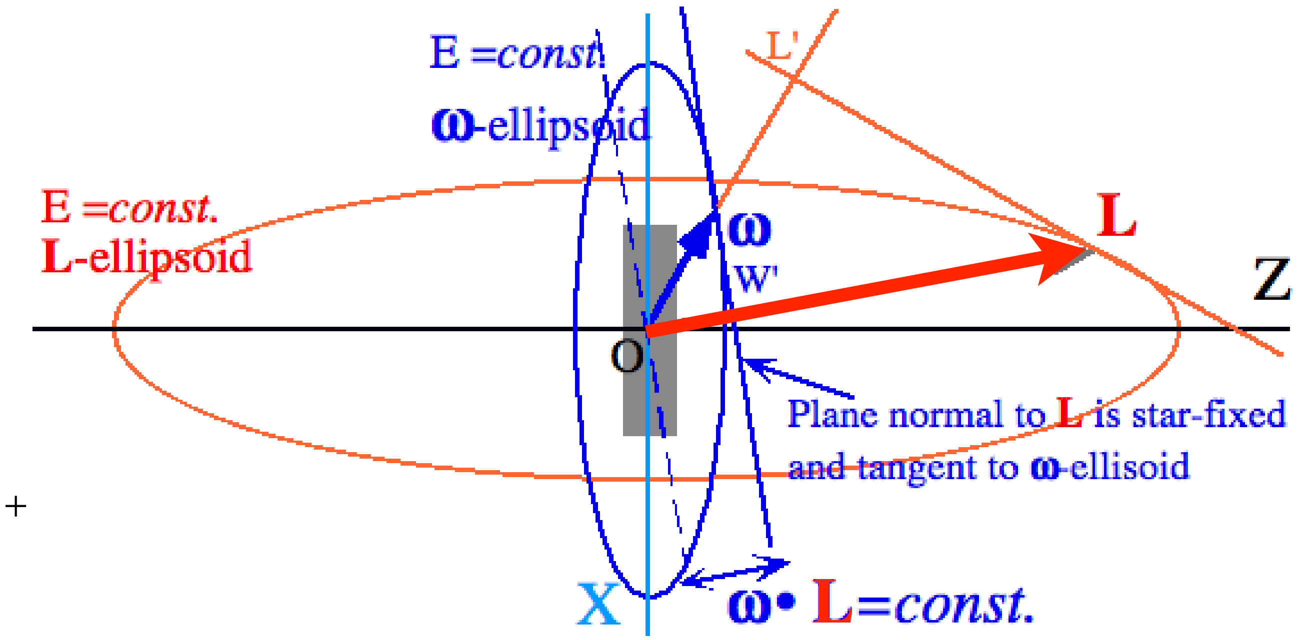
$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} +$$



Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

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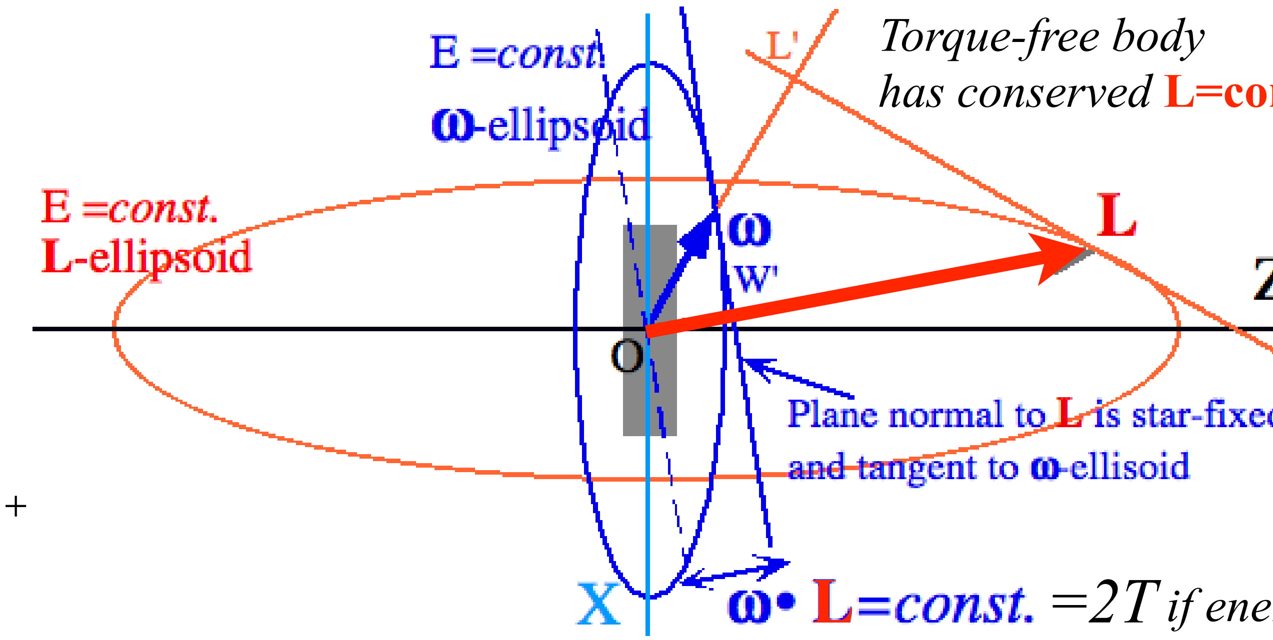
$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} +$$

Torque-free body has conserved **L=const.**



ω• L=*const.* =$2T$ *if energy is not dissipated internally*

Hamiltonian form is the equation of the *angular momentum or* **L***-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*

Kinetic energy in terms of momentum L and rotational Hamiltonian

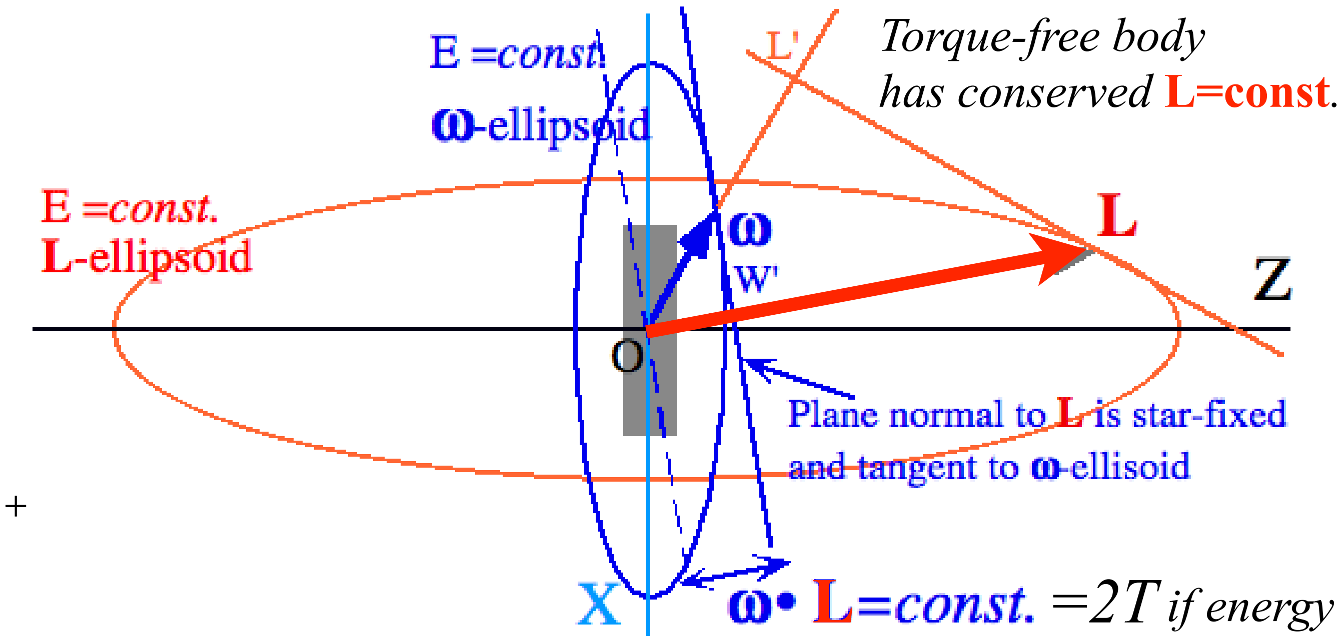
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} +$$



Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid* *ω is generally not conserved unless it is aligned to* **L** *or body has symmetry*

Canonical momentum: $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}}\frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

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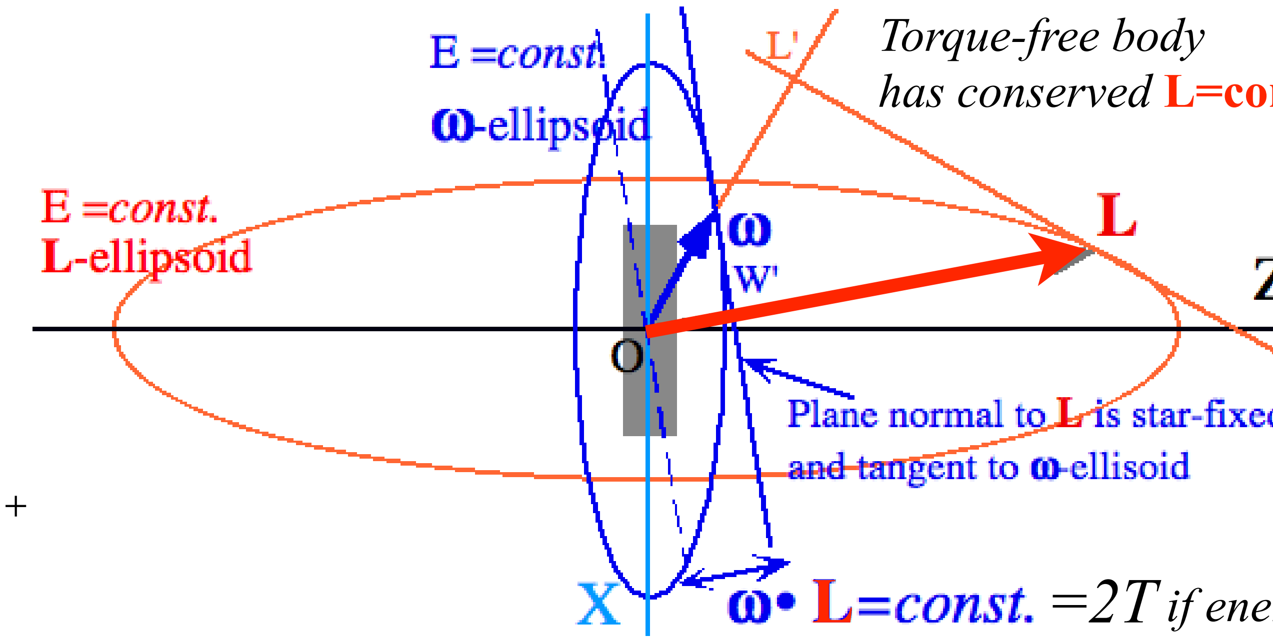
Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} +$$



Torque-free body has conserved **L=const**.

$\boldsymbol{\omega} \bullet \mathbf{L}$*=const.* $=2T$ *if energy is not dissipated internally*

Hamiltonian form is the equation of the *angular momentum or* ***L****-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

ω is generally not conserved unless it is aligned to **L** *or body has symmetry*

Canonical momentum: $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega}$$

Hamilton's 1st equations : $\dot{q}^\mu = \dfrac{\partial H}{\partial p_\mu}$ (where: $H = T$)

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Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum $\mathbf{L}$, or both at once. once

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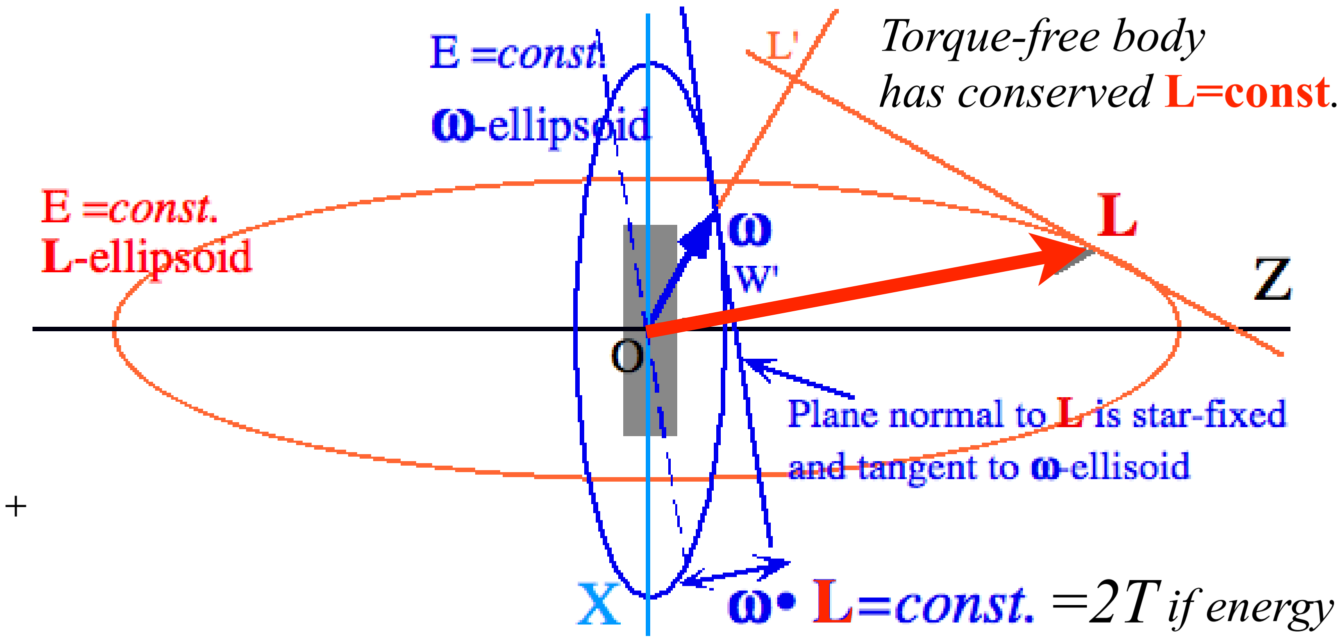

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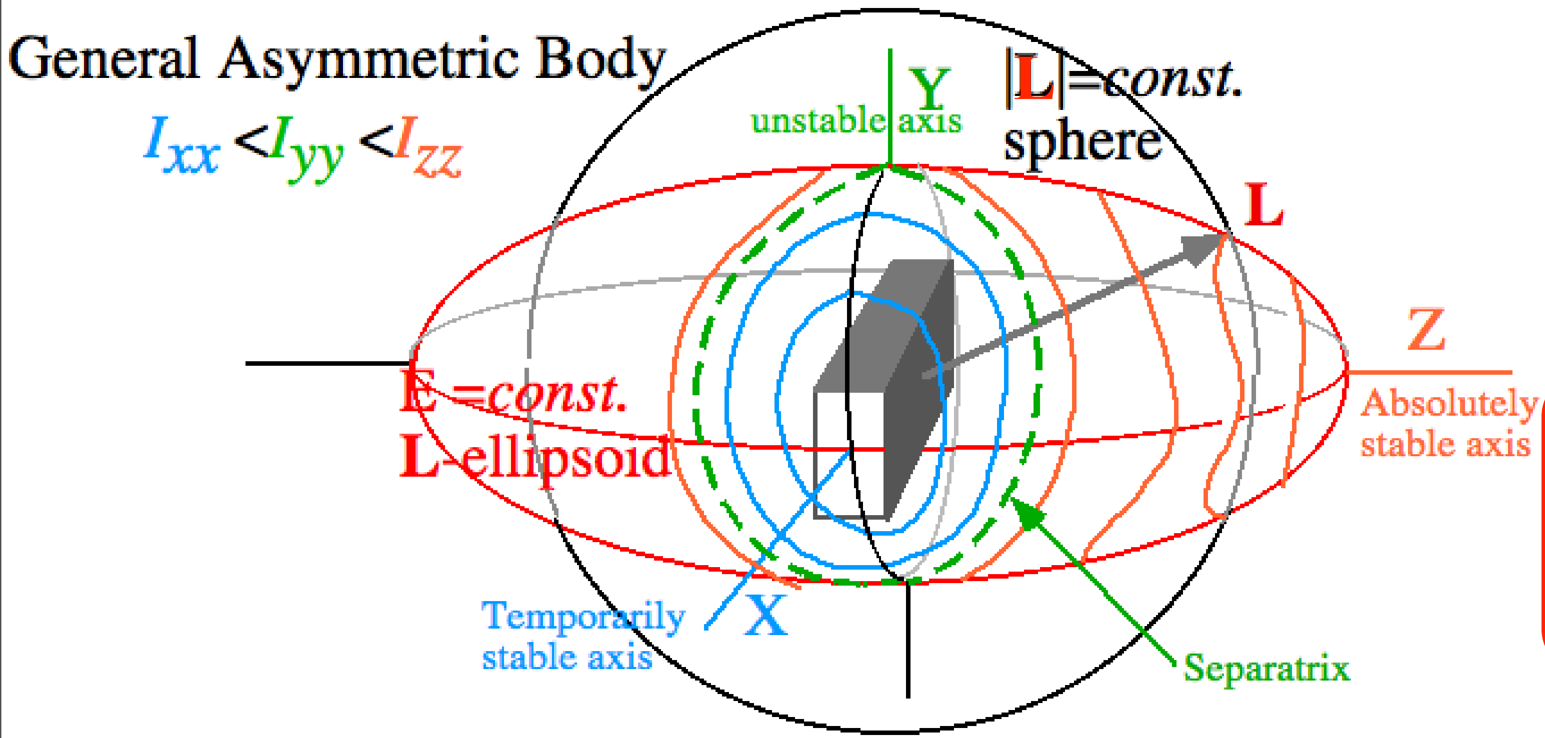

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Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overset{\leftrightarrow}{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overset{\leftrightarrow}{\mathbf{I}}^{-1} \bullet \mathbf{L}$

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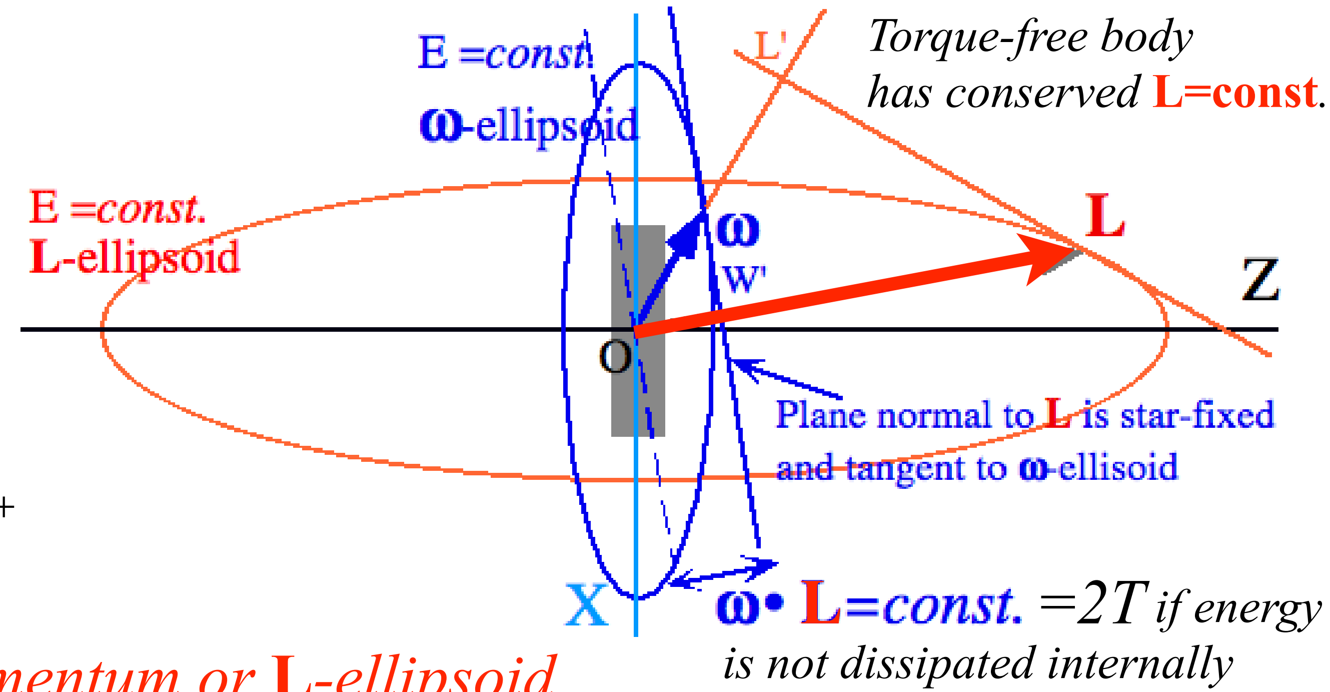

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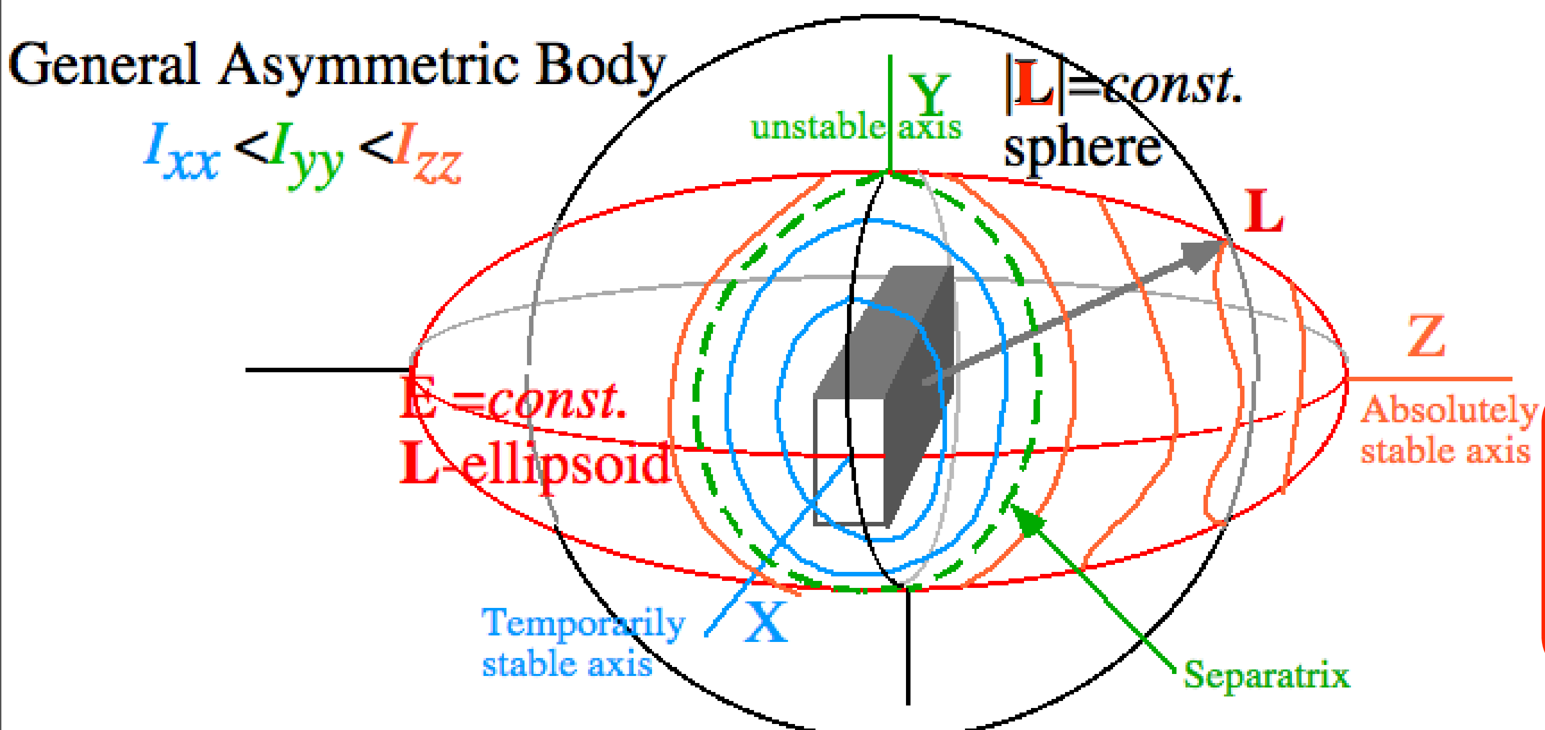

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In body frame momentum ***L*** *moves along intersection of* ***L****-ellipsoid and* ***L****-sphere (Length |****L****| is constant in any classical frame.)*

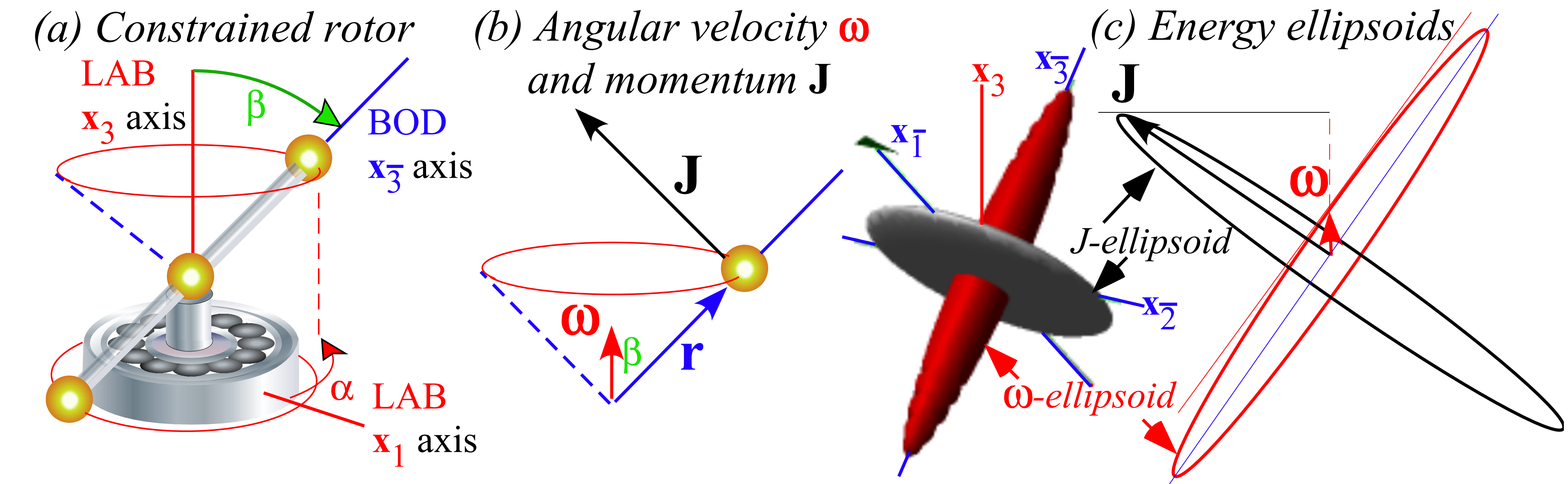


Fig. 6.7.1 Elementary ω-constrained rotor and angular velocity-momentum geometry.

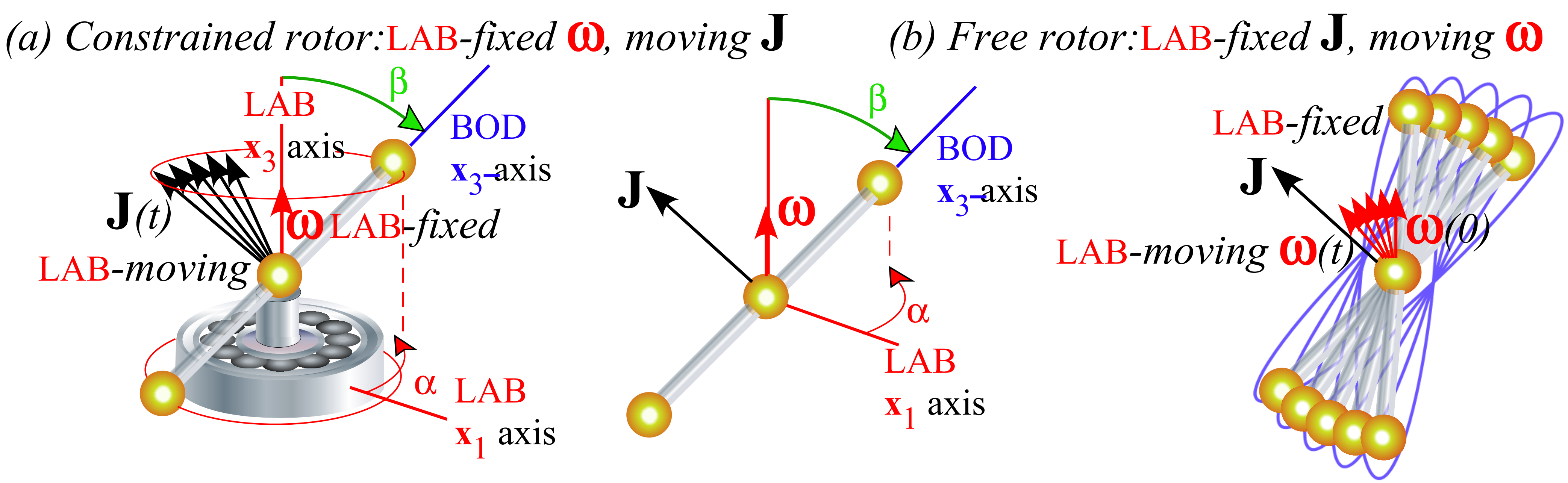
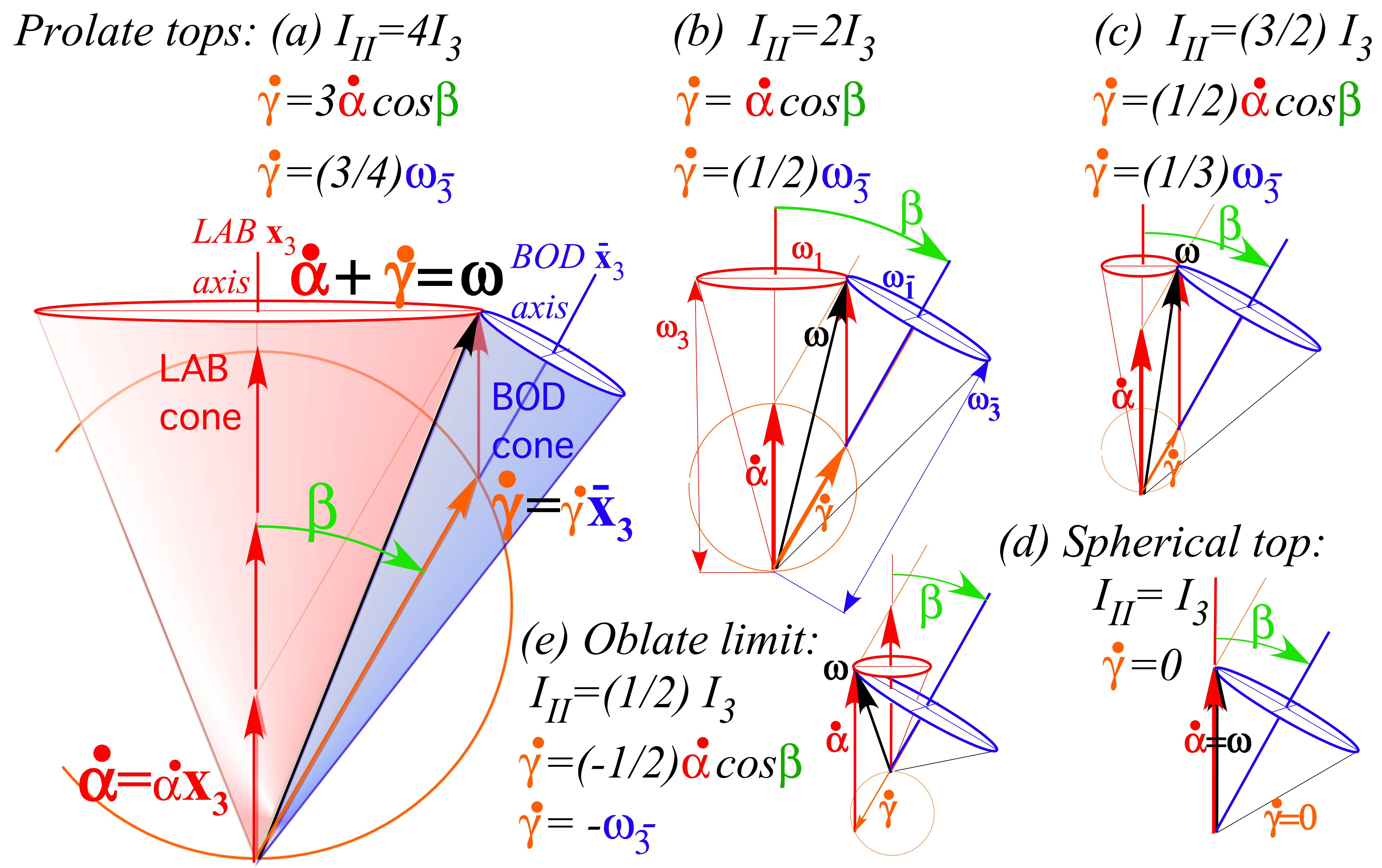


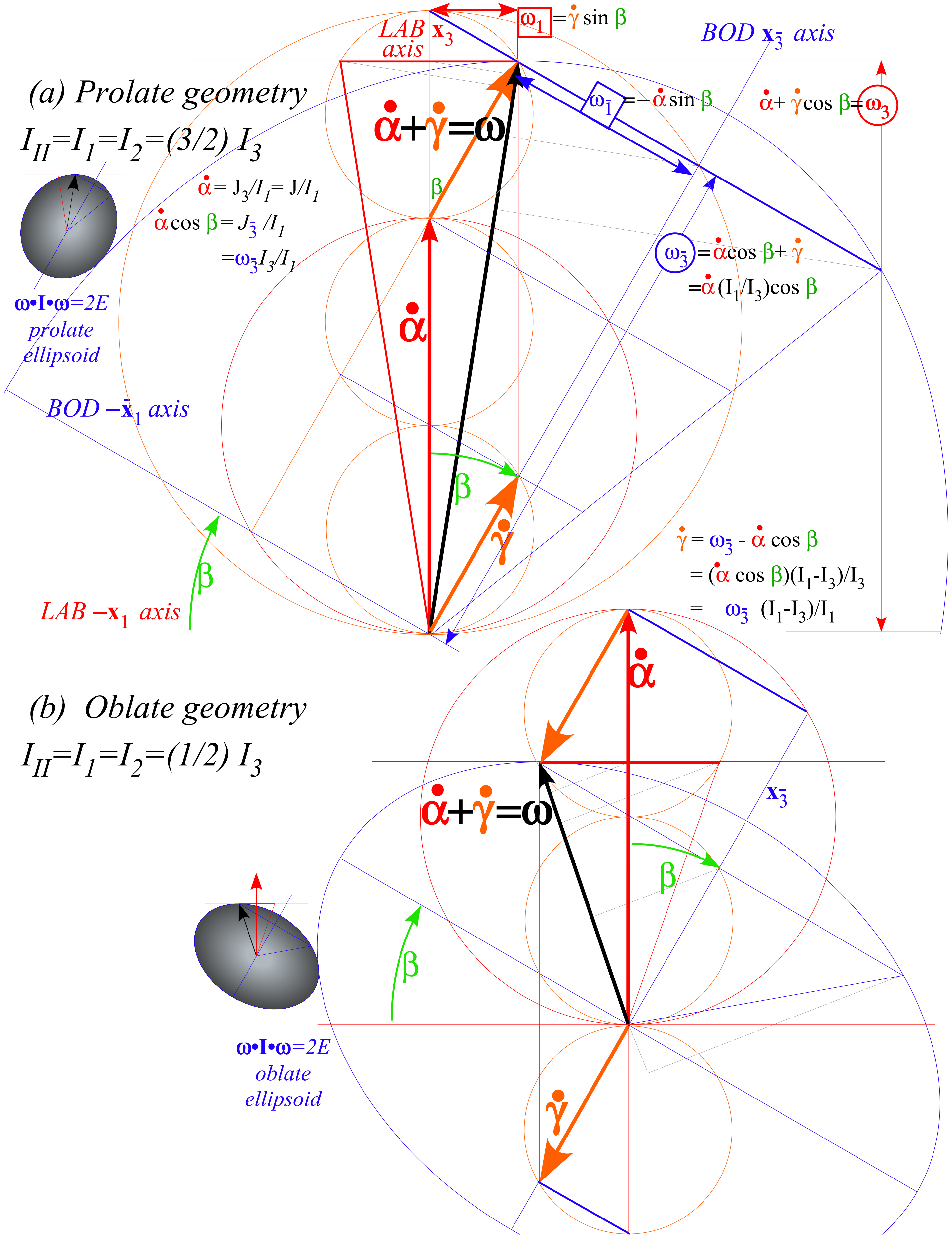
Fig. 6.7.2 Free rotor cut loose from LAB-constraining ω-axis changes dynamics accordingly.

..this was the kind of dynamics that started me dropping superballs...



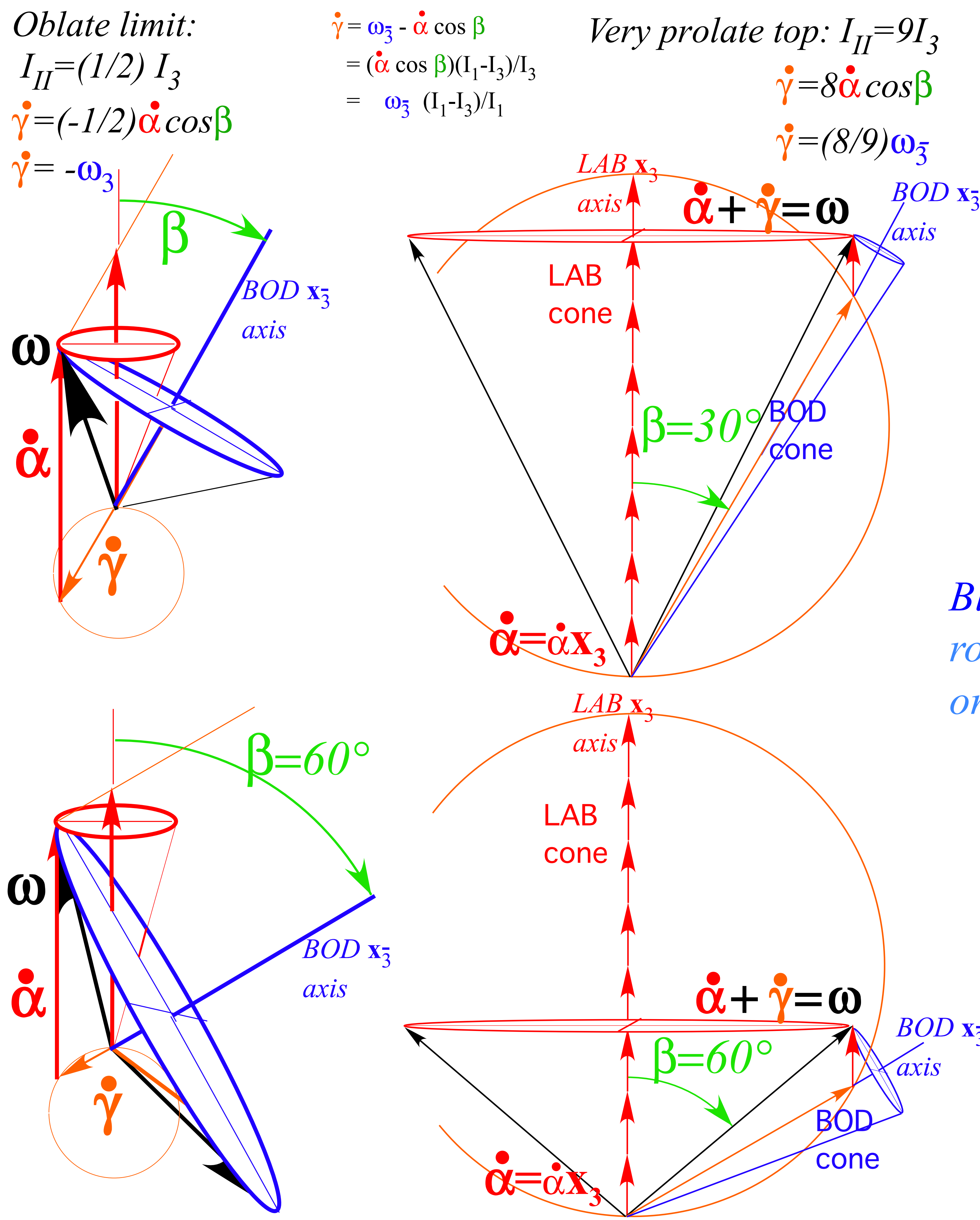
Blue BOD-frame cones roll (around ω-sticking axis) without slipping on red LAB-frame cone

Fig. 6.7.3 Symmetric top ω-cones for β=30° and inertial ratios: (a) $\frac{I_{II}-I_3}{I_3}=3$, (b) 1, (c) $\frac{1}{2}$, (d) 0, (e) $-\frac{1}{2}$.



Blue BOD-frame cones roll without slipping on red LAB-frame cone

Fig. 6.7.4 Detailed geometry of symmetric top kinetics. (a) Prolate case. (b) Most-oblate case



Blue BOD-frame cones
roll without slipping
on red LAB-frame cone

Fig. 6.7.5 Extreme cases (Oblate vs. Prolate) of symmetric-top geometry.