

Lecture 23

Introduction to coupled oscillation and eigenmodes

(Ch. 2-4 of Unit 4 11.13.12)

Review: *Green's Solution* for the *FDHO* (*Forced-Damped-Harmonic Oscillator*)

Beat, lifetimes, and quality factor $q = \omega_0/2\Gamma$ and $Q = \nu_0/2\Gamma = q/2\pi$

Review: Approximate Lorentz-*Green's Function* for high quality *FDHO* (Quantum propagator)

Common Lorentzian (a.k.a. Witch of Agnesi) and geometry

2D harmonic oscillator equations

Lagrangian and matrix forms

Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues \Rightarrow eigenvectors)

Spectral decomposition and P-operator expansions (how projectors \Rightarrow eigensolutions)

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry

Initial state projection, mixed mode beat dynamics with fluid phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)

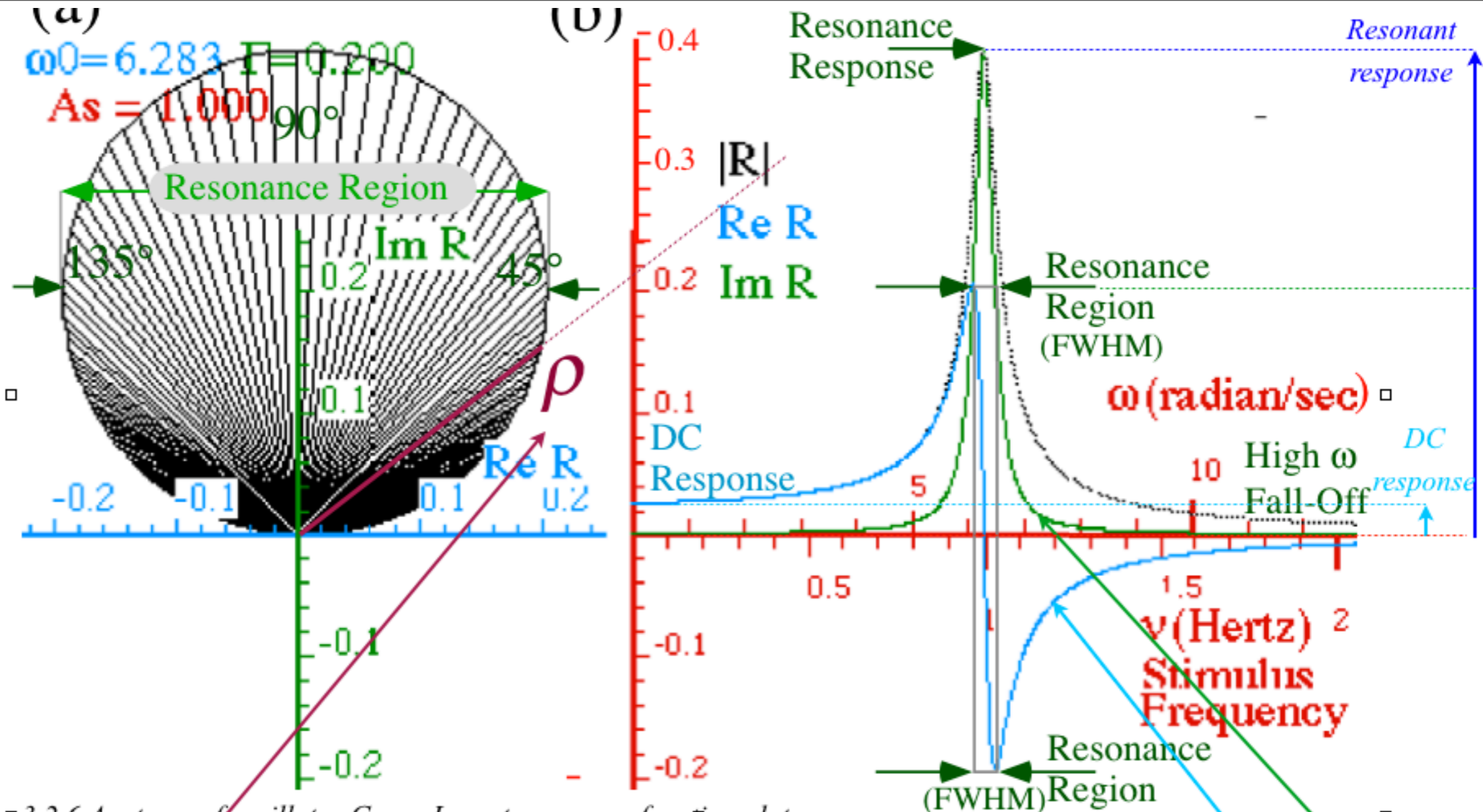


Fig 3.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1} \left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2} \right)$$

$$\text{Re } G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Real part

$$\text{Im } G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0}(\omega_s = \omega_0)|}{|G_{\omega_0}(0)|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\text{angular quality factor})$$

Review of Approximate Lorentz-Green's Function for high quality *FDHO* (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow{\omega_s \rightarrow \omega_0} \frac{1}{2\omega_s} \frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0} \frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

Define *complex detuning-decay* $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \text{Re } L + i \text{Im } L = \frac{\Delta}{\Delta^2 + \Gamma^2} + i \frac{\Gamma}{\Delta^2 + \Gamma^2} = |L|^2 \Delta + i |L|^2 \Gamma$$

$$= |L| e^{i\rho} = |L| \cos \rho + i |L| \sin \rho = \frac{\cos \rho}{\sqrt{\Delta^2 + \Gamma^2}} + i \frac{\sin \rho}{\sqrt{\Delta^2 + \Gamma^2}} \text{ where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

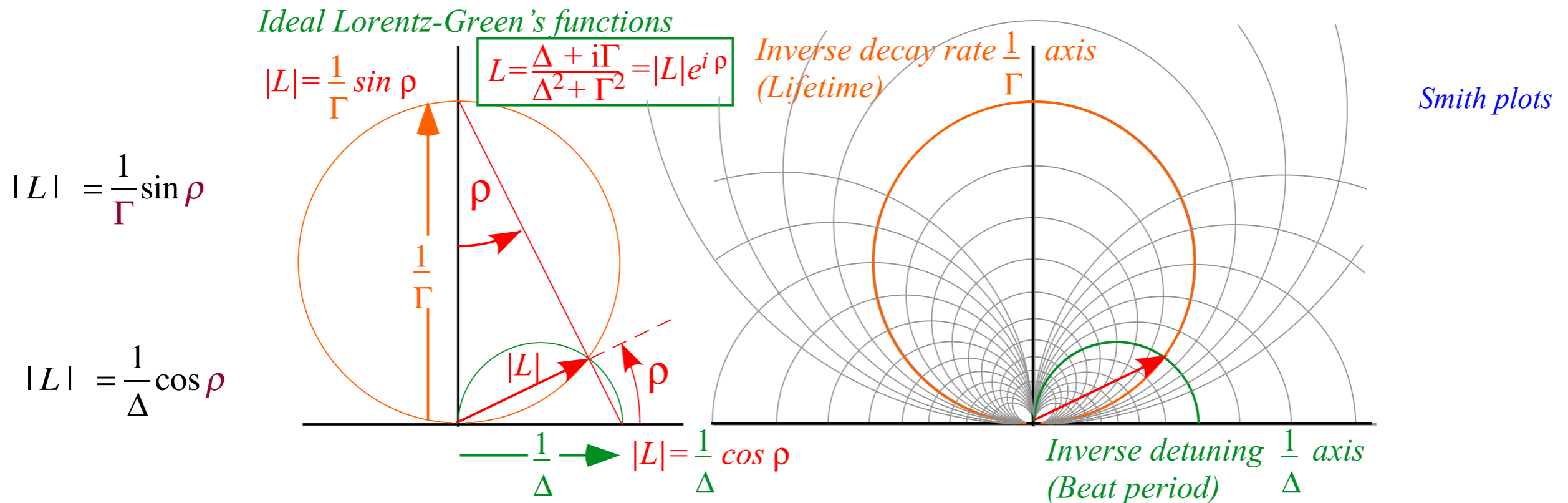


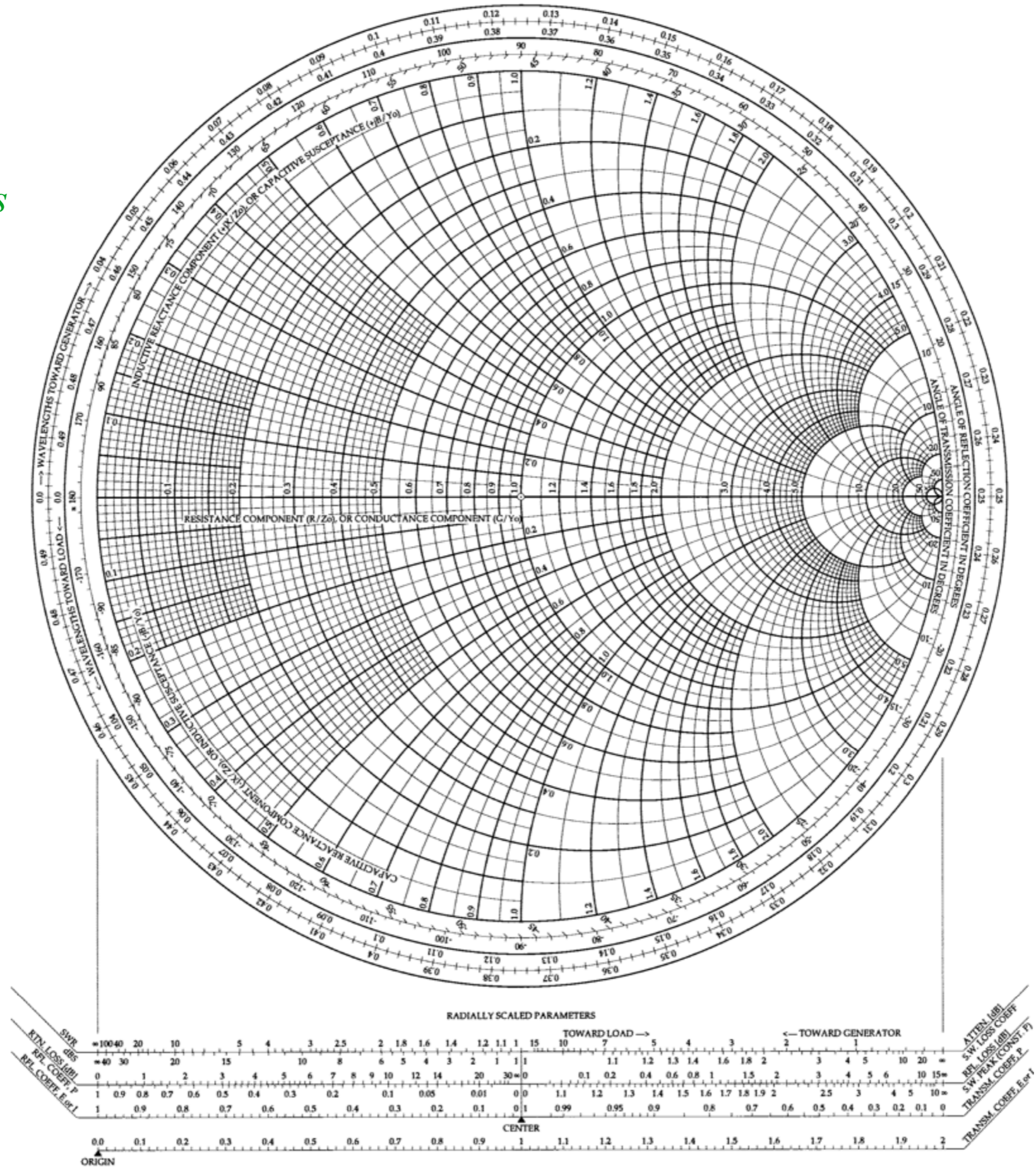
Fig. 3.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ vs. beat-period $1/\Delta$ coordinates)

Constant Δ and Γ curves in Fig. 3.2.13 are orthogonal circles of $1/z$ -dipolar coordinates. Recall Fig. 1.10.11.

An FDHO Green's
Function
Slide rule

A plot of
 $f(z) = 1/z$

For wavy
"Ohm's Laws"
 $V = I \cdot Z$
 $I = V/Z$

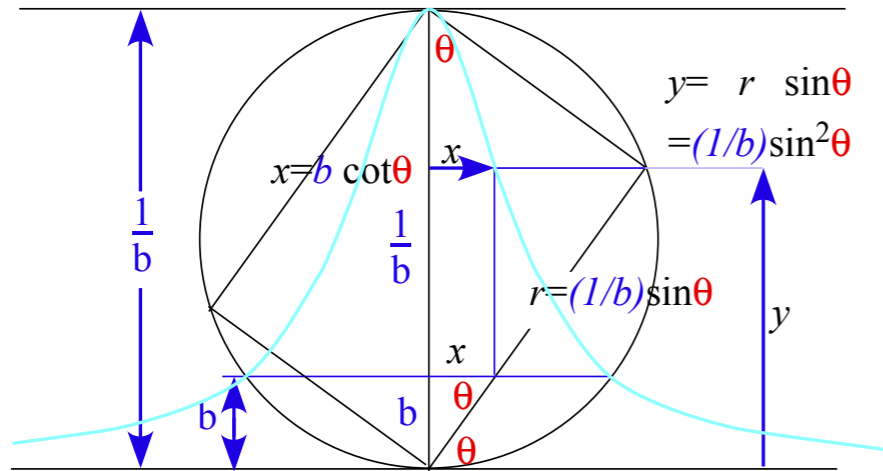


The Common Lorentzian (a.k.a. The Witch of Agnesi)

Maria Gaetana Agnesi



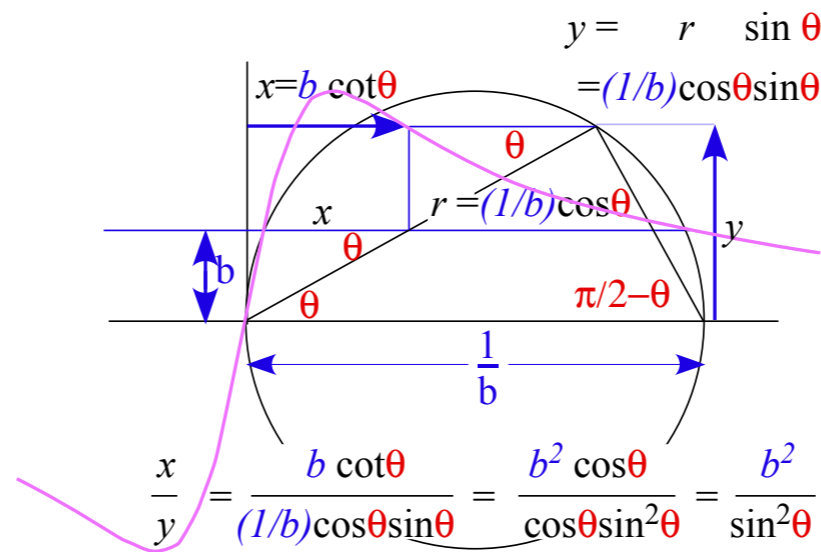
Born May 16, 1718
 Died January 9, 1799 (aged 80)
 Residence Italy
 Nationality Italy
 Fields Mathematics



$$x^2 = b^2 \cot^2 \theta = b^2 \frac{\cos^2 \theta}{\sin^2 \theta} = b^2 \frac{1 - \sin^2 \theta}{\sin^2 \theta} = \frac{b^2}{\sin^2 \theta} b^2$$

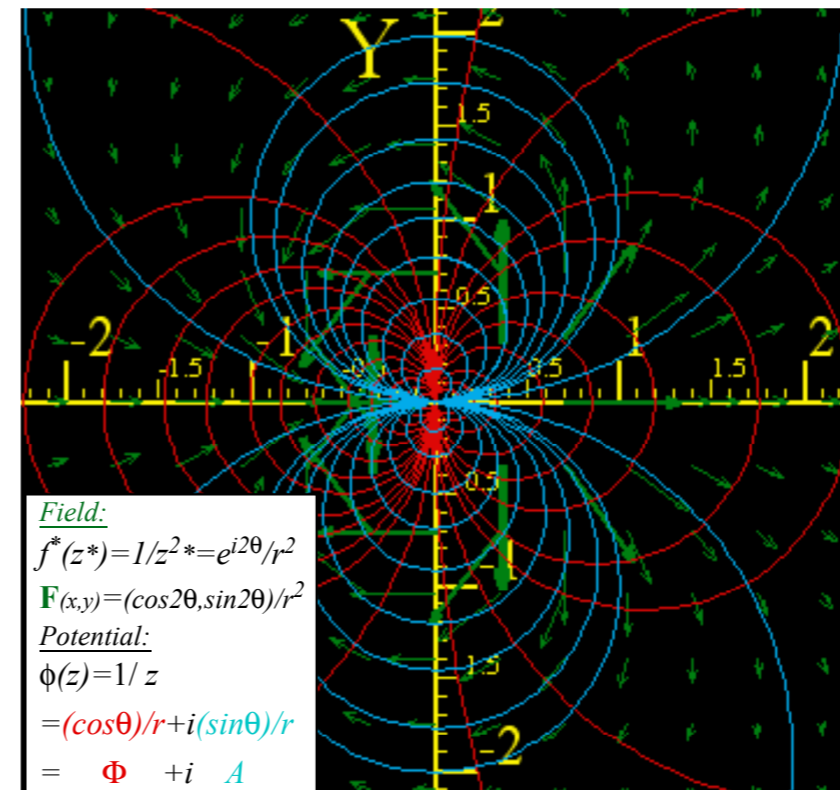
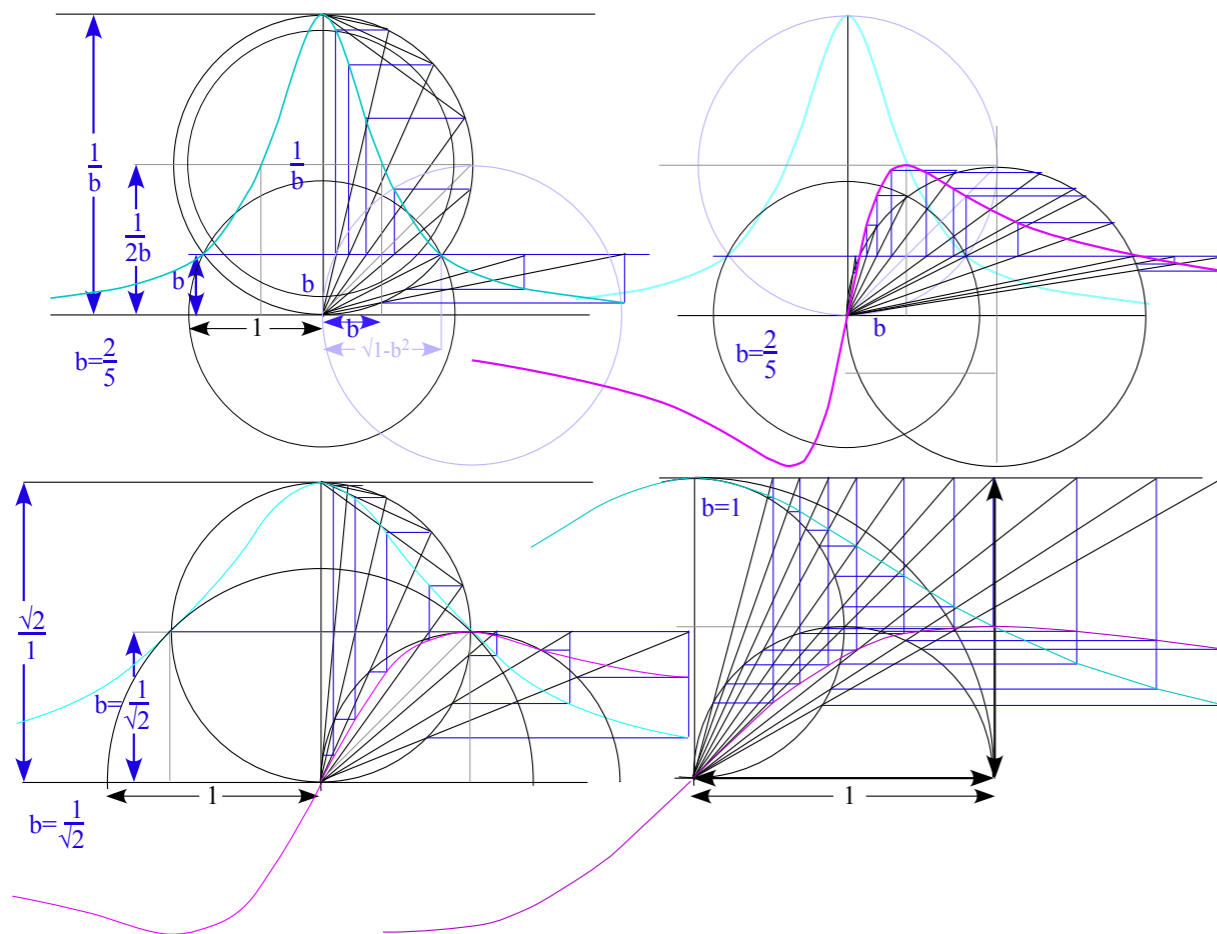
$$x^2 + b^2 = \frac{b^2}{\sin^2 \theta} = \frac{b}{y} \quad \text{Common Lorentzian function I. (imaginary "absorbptive" part)}$$

$$y = \frac{b}{x^2 + b^2}$$



$$x^2 + b^2 = \frac{b^2}{\sin^2 \theta} = \frac{x}{y} \quad \text{Common Lorentzian function II. (real "refractory" part)}$$

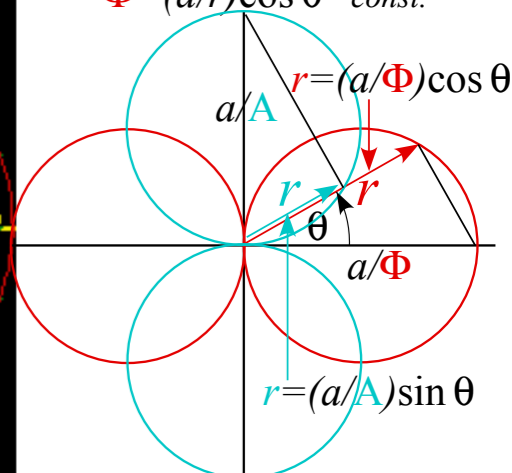
$$y = \frac{x}{x^2 + b^2}$$



Field:
 $f^*(z^*) = 1/z^{*2} = e^{i2\theta}/r^2$
 $\mathbf{F}(x,y) = (\cos 2\theta, \sin 2\theta)/r^2$
 Potential:
 $\phi(z) = 1/z$
 $= (\cos \theta)/r + i(\sin \theta)/r$
 $= \Phi + i A$

Scalar potentials

$$\Phi = (a/r) \cos \theta = \text{const.}$$



Vector potentials

$$A = (a/r) \sin \theta = \text{const.}$$

Fig. 10.11 Dipole \mathbf{F} -field $f(z) = 1/z^2$ and scalar potential ($\Phi = \text{const.}$)-circles orthogonal to ($A = \text{const.}$)-circles.

➔ *2D harmonic oscillator equations*
Lagrangian and matrix forms
Reciprocity symmetry

2D harmonic oscillators

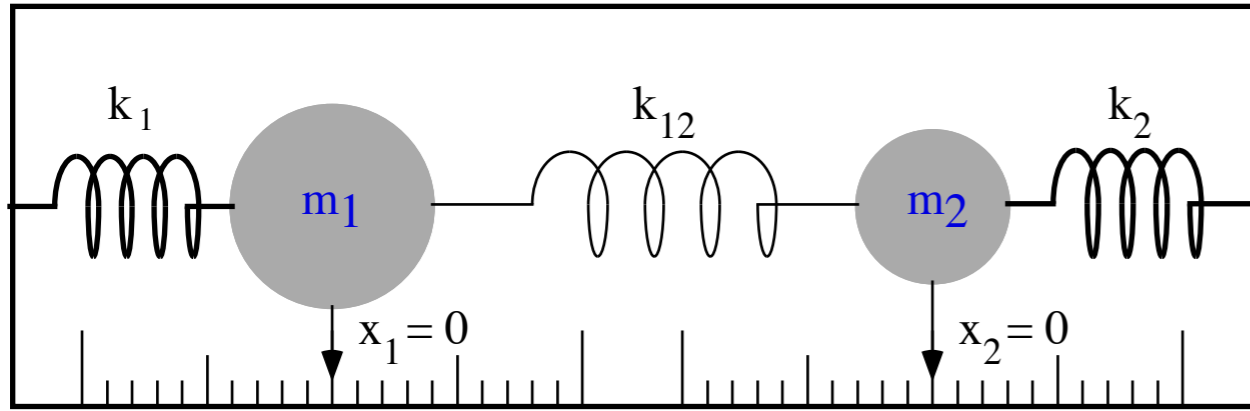


Fig. 3.3.1 Two 1-dimensional coupled oscillators

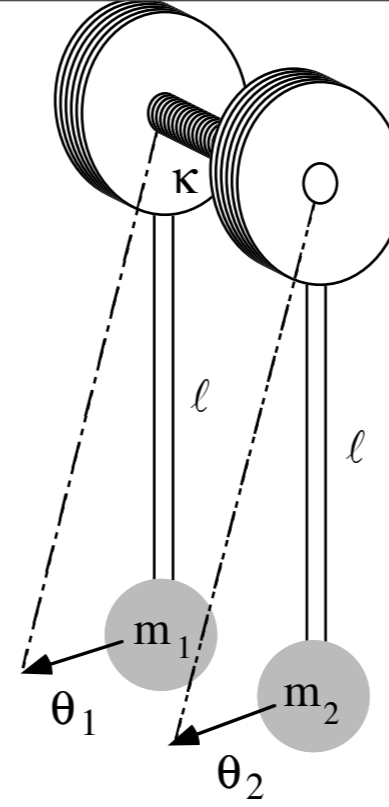
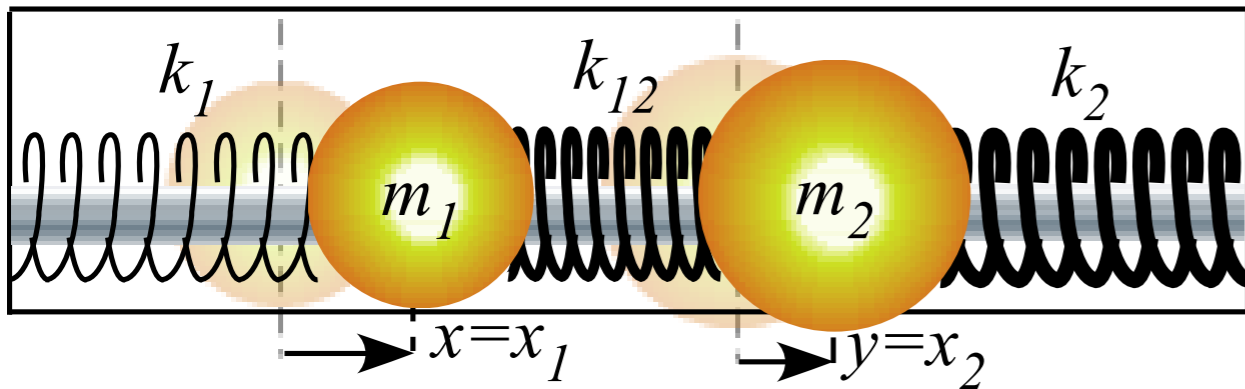


Fig. 3.3.2 Coupled pendulums

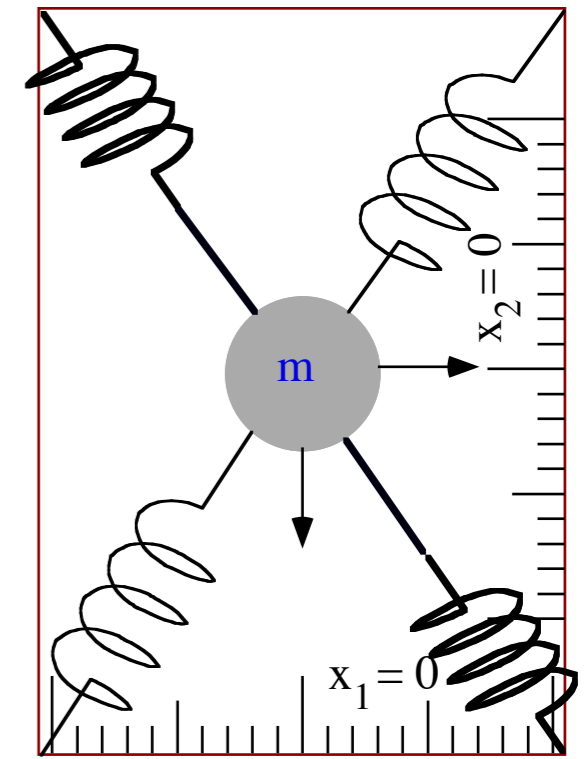


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D harmonic oscillator energy

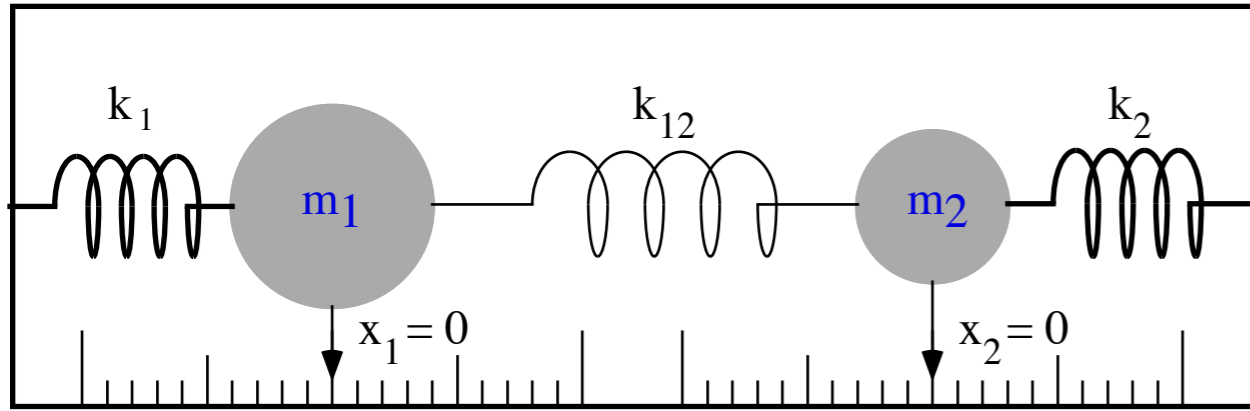
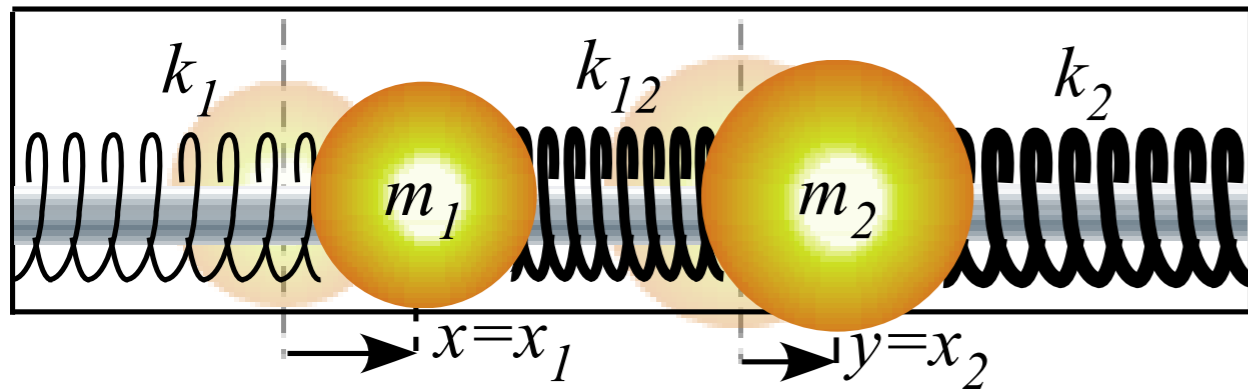


Fig. 3.3.1 Two 1-dimensional coupled oscillators



2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

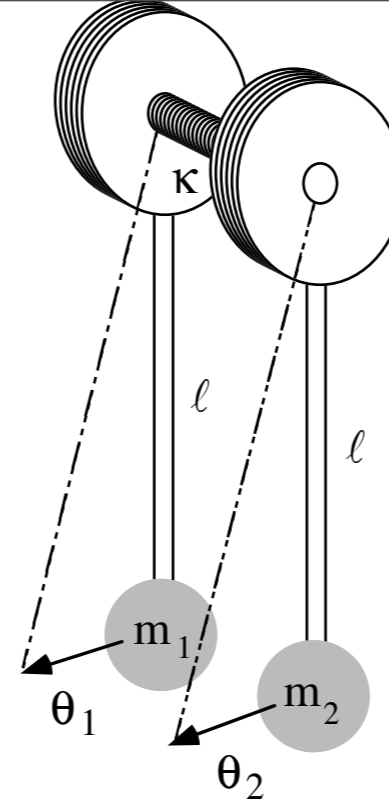


Fig. 3.3.2 Coupled pendulums

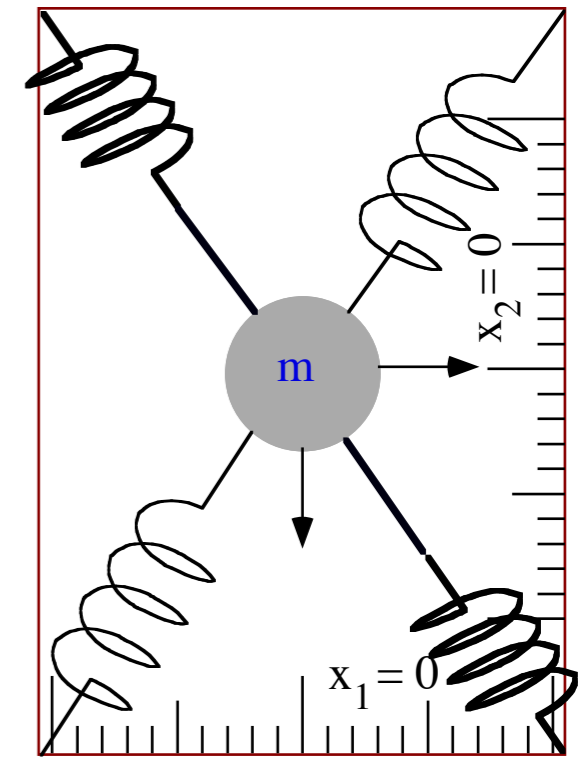


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D harmonic oscillator energy

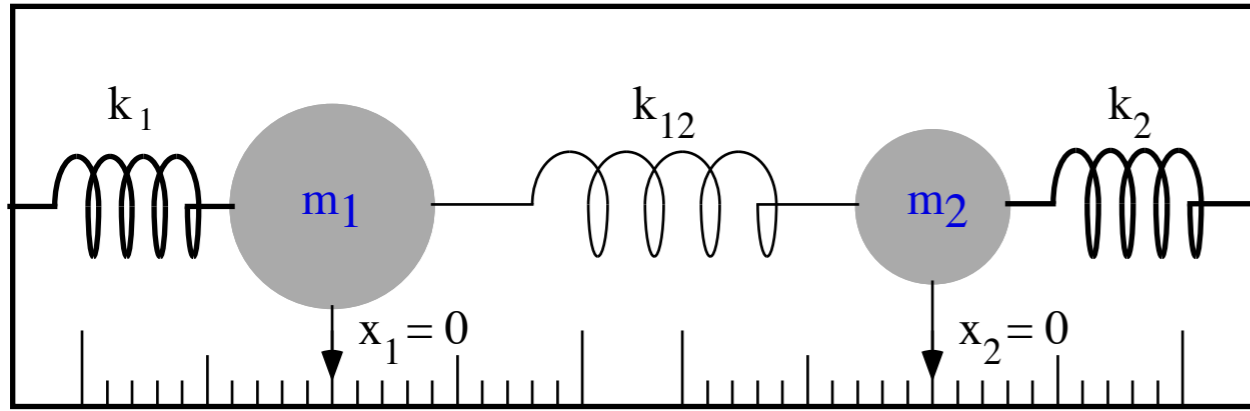


Fig. 3.3.1 Two 1-dimensional coupled oscillators

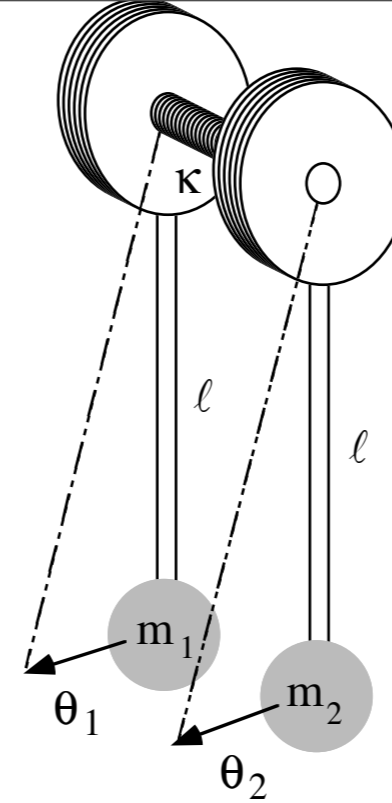
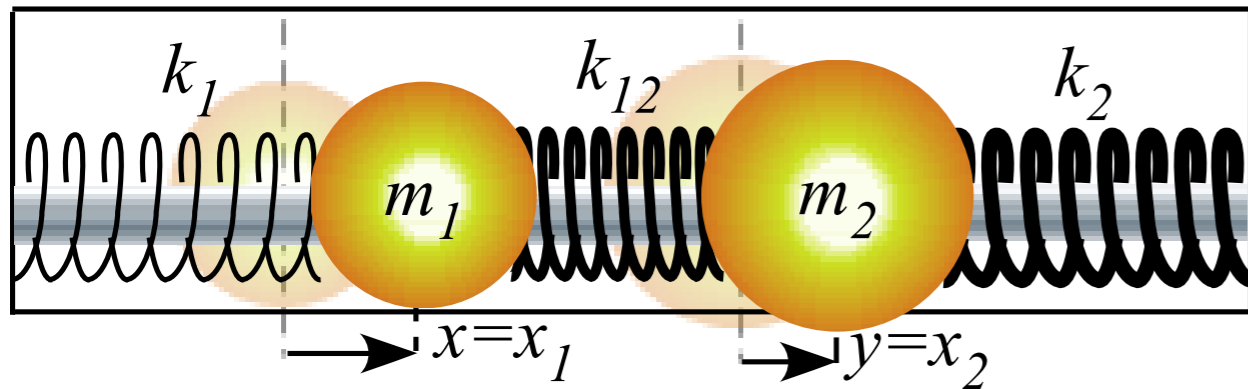


Fig. 3.3.2 Coupled pendulums

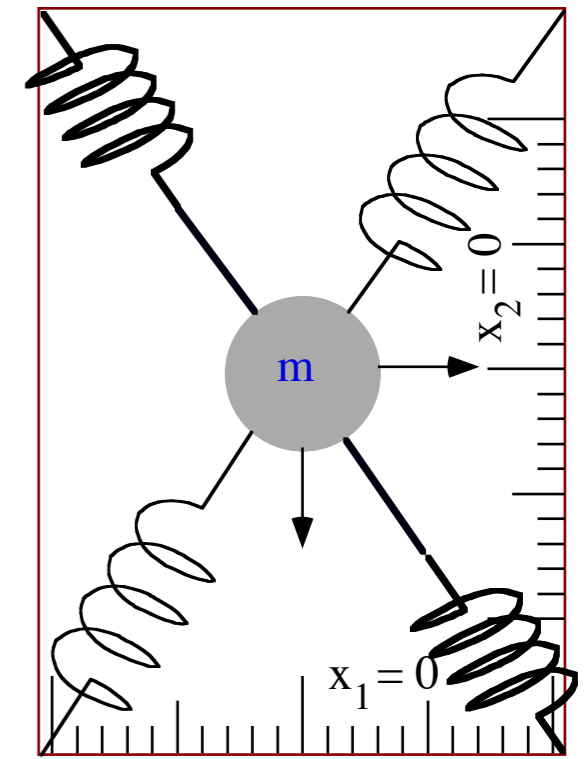


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$\begin{aligned} V &= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2 \\ &= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2 \end{aligned}$$

Lagrangian $L=T-V$

2D harmonic oscillator equations

➔ *Lagrangian and matrix forms*

Reciprocity symmetry

2D harmonic oscillator equations

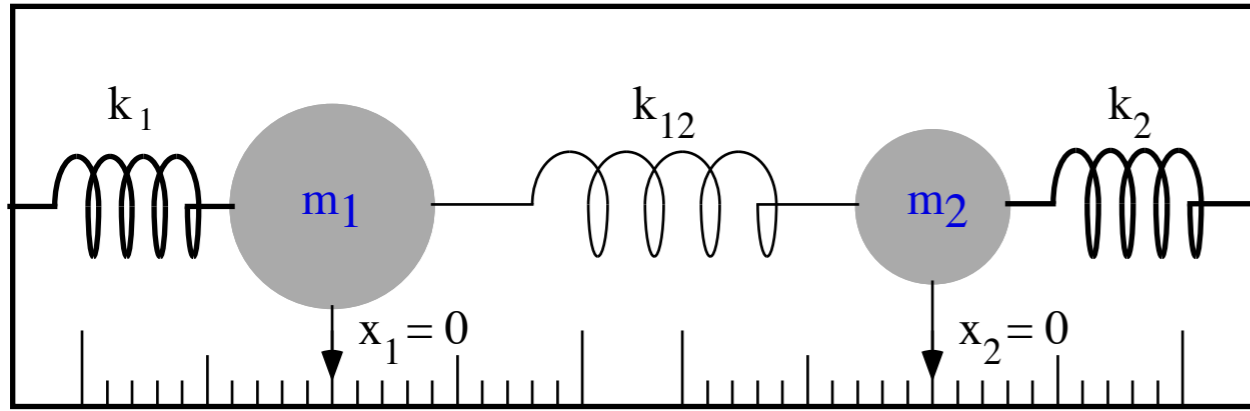


Fig. 3.3.1 Two 1-dimensional coupled oscillators

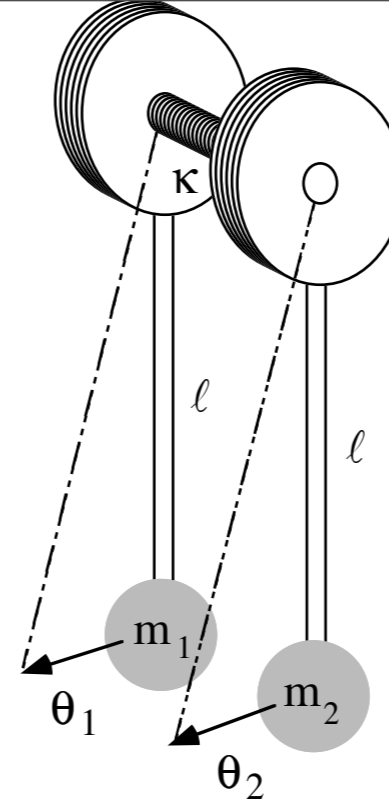
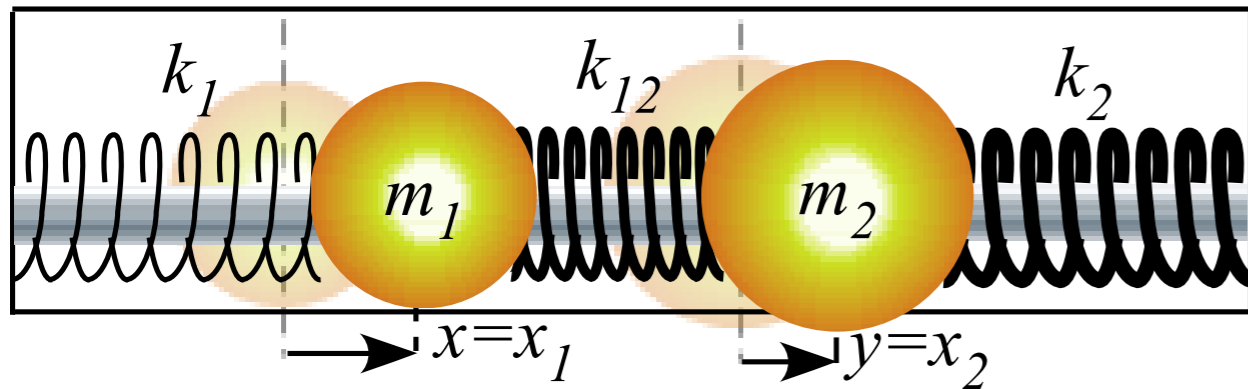


Fig. 3.3.2 Coupled pendulums

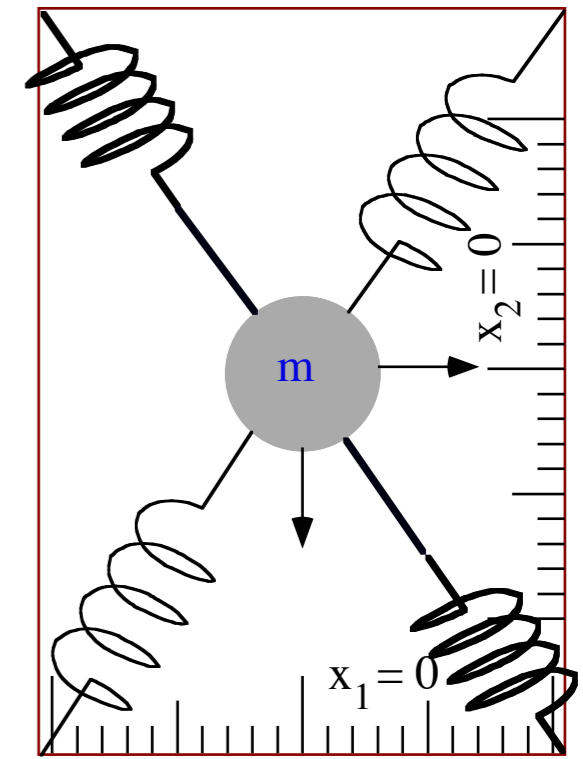


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

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Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = - \frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D harmonic oscillator equations

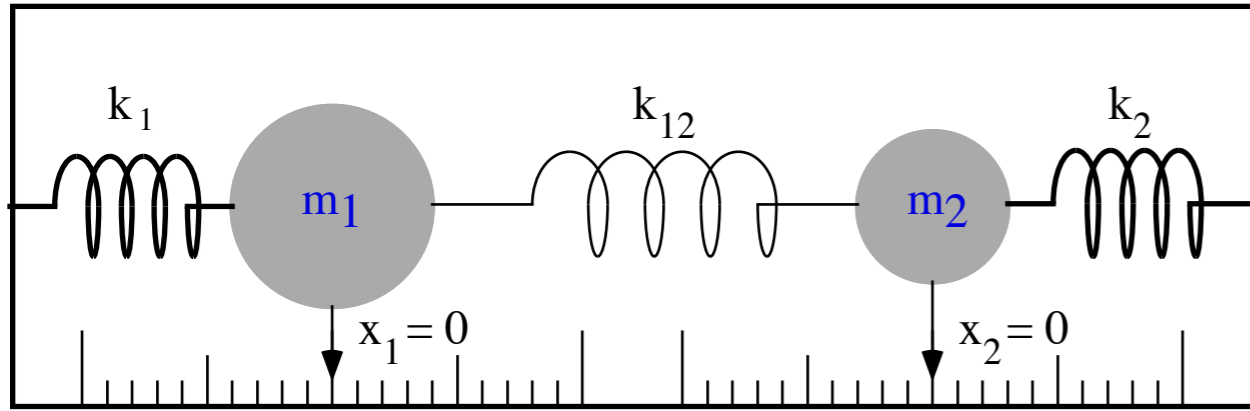


Fig. 3.3.1 Two 1-dimensional coupled oscillators

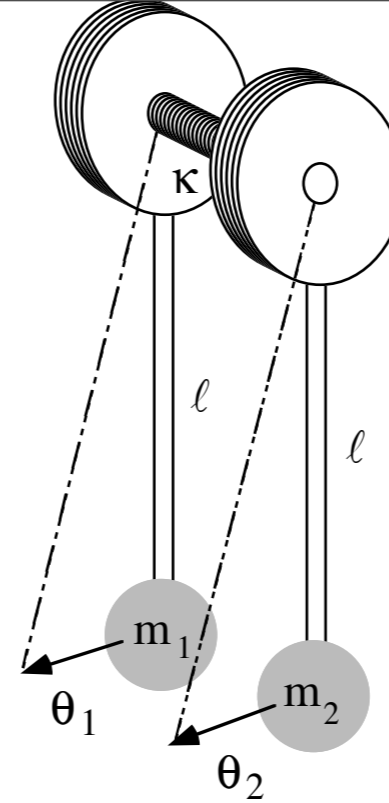
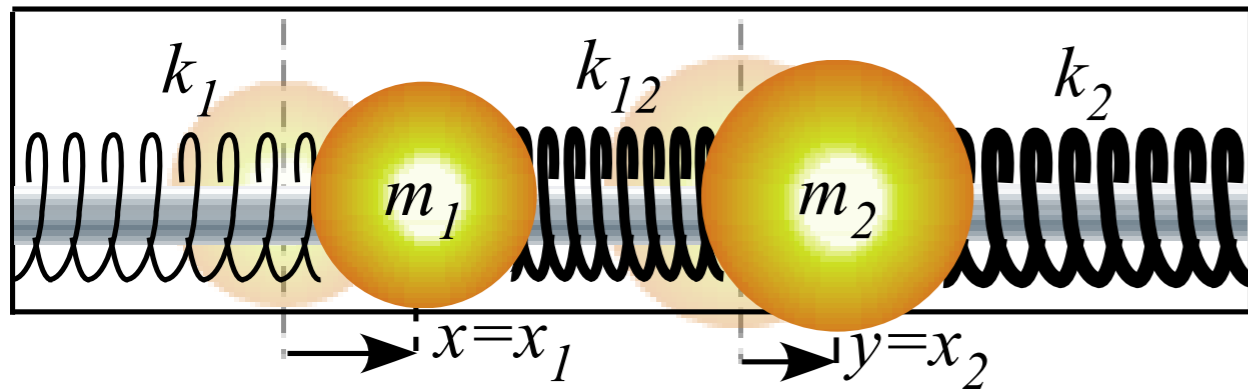


Fig. 3.3.2 Coupled pendulums

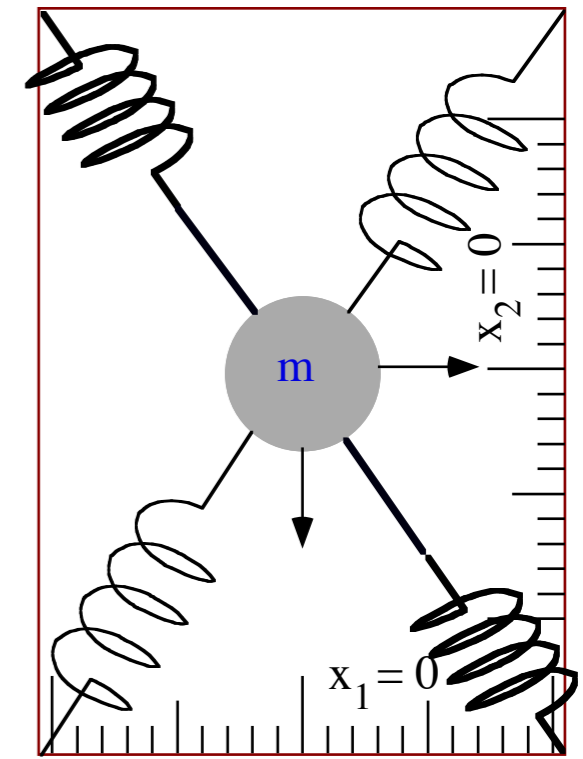


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2$$

$$= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = - \frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations

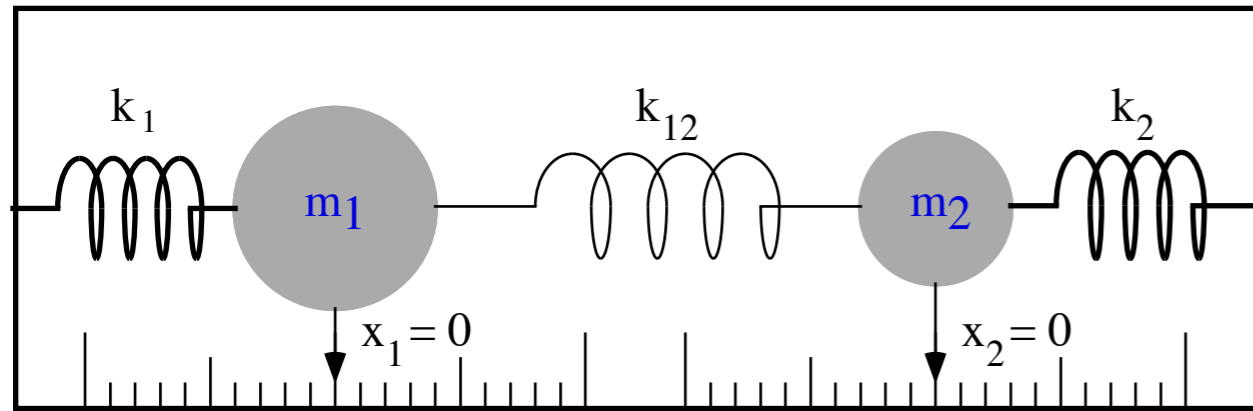


Fig. 3.3.1 Two 1-dimensional coupled oscillators

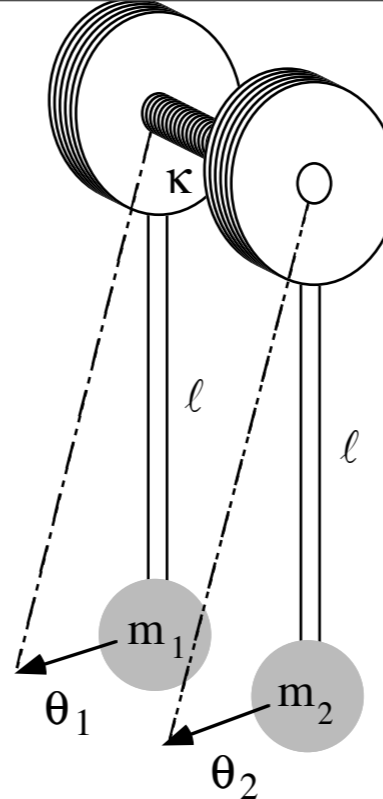
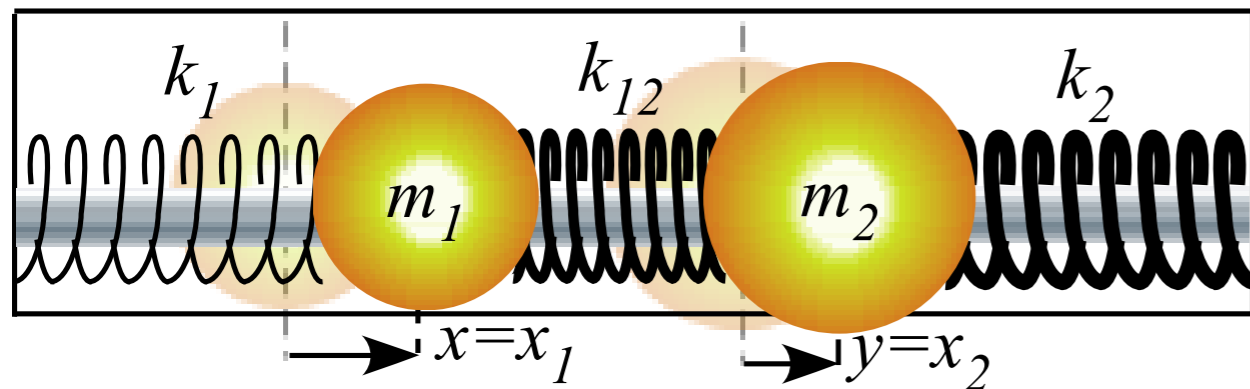


Fig. 3.3.2 Coupled pendulums

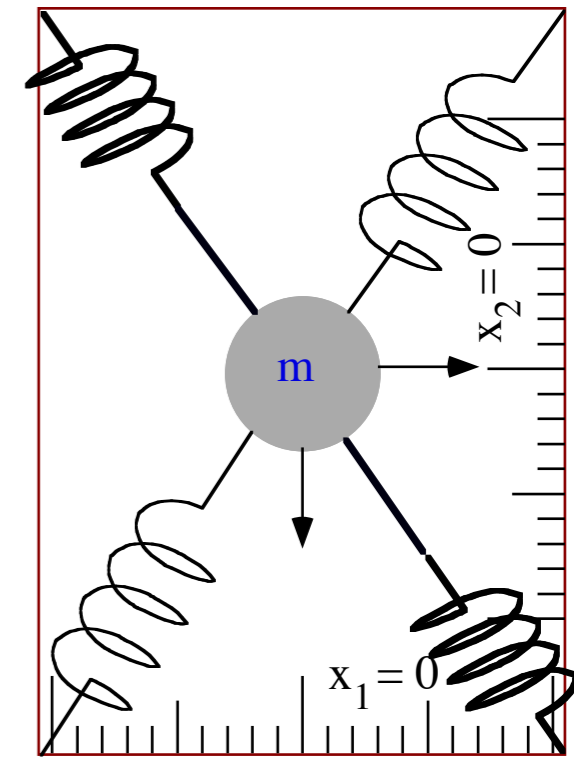


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2$$

$$= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = - \frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \cdot |\ddot{\mathbf{x}}\rangle = - \mathbf{K} \cdot |\mathbf{x}\rangle$$

2D harmonic oscillator equations

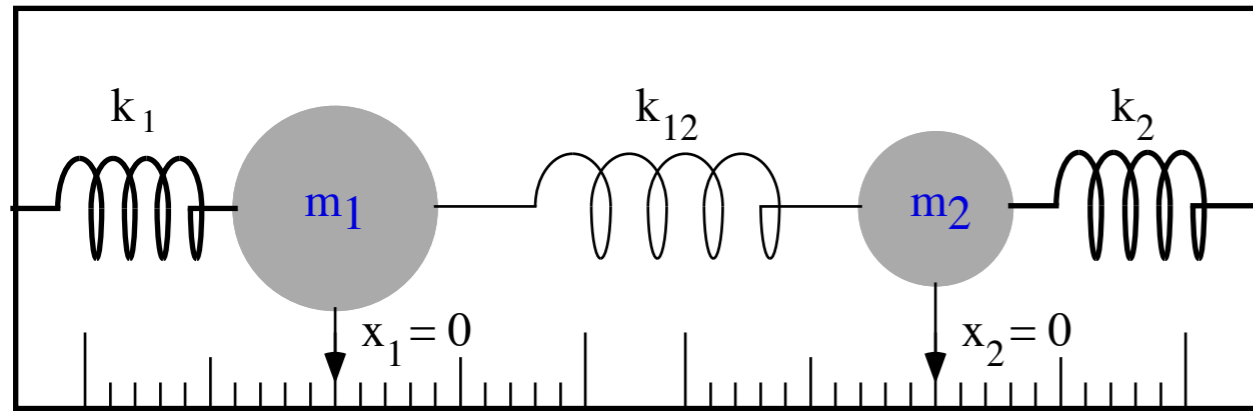


Fig. 3.3.1 Two 1-dimensional coupled oscillators

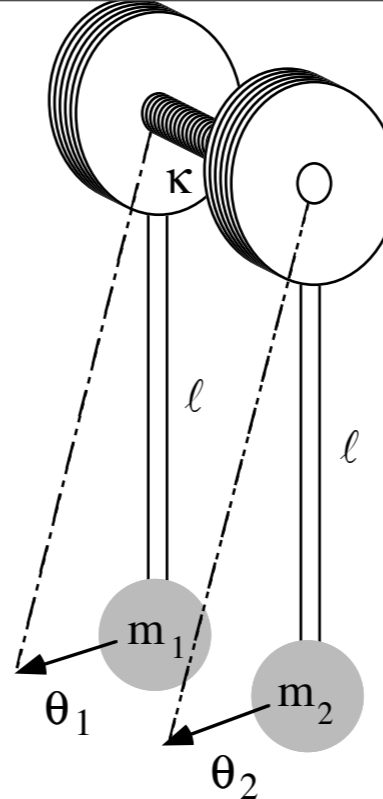
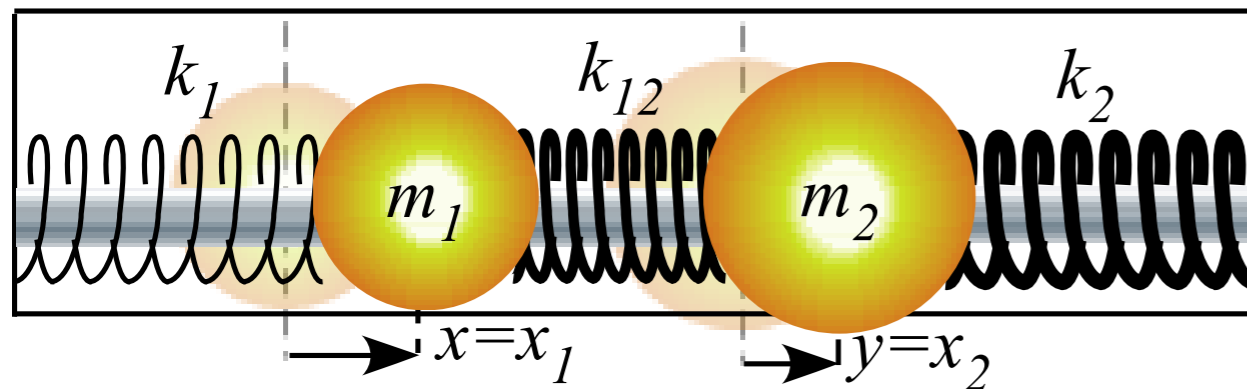


Fig. 3.3.2 Coupled pendulums

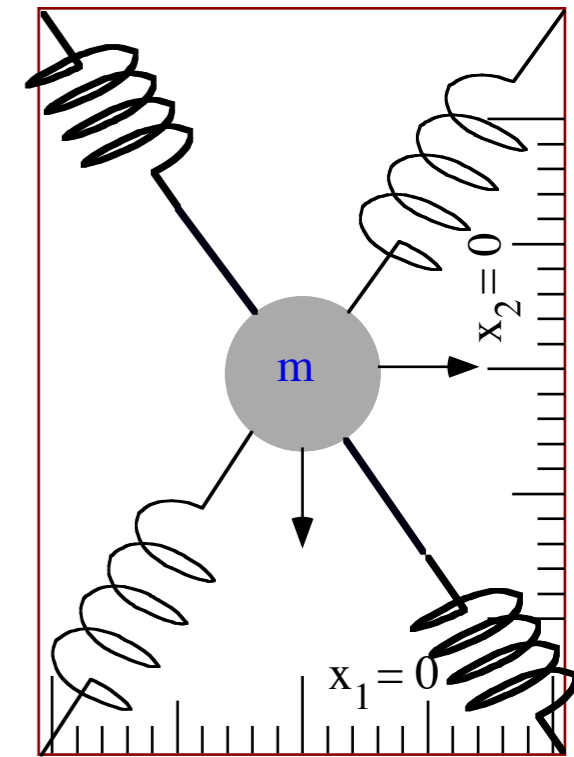


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \cdot | \ddot{\mathbf{x}} \rangle = - \mathbf{K} \cdot | \mathbf{x} \rangle$$

2D harmonic oscillator equations
Lagrangian and matrix forms
➔ *Reciprocity symmetry*

Matrix equations and reciprocity symmetry

General form of 2D-HO equation of motion has force matrix components: $\kappa_{11} = k_1 + k_{12}$, $\kappa_{22} = k_2 + k_{12}$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} m_1 \ddot{x}_1 \\ m_2 \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Off-diagonal force constants satisfy *Reciprocity Relations*: $-\kappa_{12} = k_{12} = \frac{\partial F_1}{\partial x_2} = -\frac{\partial^2 V}{\partial x_2 \partial x_1} = -\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial F_2}{\partial x_1} = k_{21} = -\kappa_{21}$

Rescaling and symmetrization

Each coordinate (x_1, x_2) is rescaled $(q_1 = s_1 x_1, q_2 = s_2 x_2)$ to symmetrize mass factors on \ddot{q}_j -terms.

$$\begin{aligned} -\frac{m_1}{s_1} \ddot{q}_1 &= \kappa_{11} \frac{q_1}{s_1} + \kappa_{12} \frac{q_2}{s_2} & -\ddot{q}_1 &= \frac{\kappa_{11}}{m_1} q_1 + \frac{\kappa_{12} s_1}{m_1 s_2} q_2 \equiv \mathbf{K}_{11} q_1 + \mathbf{K}_{12} q_2 \\ -\frac{m_2}{s_2} \ddot{q}_2 &= \kappa_{12} \frac{q_1}{s_1} + \kappa_{22} \frac{q_2}{s_2} & -\ddot{q}_2 &= \frac{\kappa_{12} s_2}{m_2 s_1} q_1 + \frac{\kappa_{22}}{m_2} q_2 \equiv \mathbf{K}_{21} q_1 + \mathbf{K}_{22} q_2 \end{aligned}$$

New constants K_{ij} have pseudo-reciprocity symmetry for a special scale factor ratio: $\frac{s_2}{s_1} = \sqrt{\frac{m_2}{m_1}}$

$$\mathbf{K}_{21} = \frac{\kappa_{12} s_2}{m_2 s_1} = \mathbf{K}_{12} = \frac{\kappa_{12} s_1}{m_1 s_2} = \frac{-k_{12}}{\sqrt{m_1 m_2}}$$

Diagonal constants K_{jj} are not affected by scaling. To be equal requires: $\frac{\kappa_{11}}{m_1} = \frac{\kappa_{22}}{m_2}$ or: $\frac{\kappa_{11}}{\kappa_{22}} = \frac{m_1}{m_2}$

$$\mathbf{K}_{11} = \frac{\kappa_{11}}{m_1} = \frac{k_1 + k_{12}}{m_1} \quad \mathbf{K}_{22} = \frac{\kappa_{22}}{m_2} = \frac{k_2 + k_{12}}{m_2}$$

Caution is advised since such forced symmetry may give modes with imaginary frequency.

→ *2D harmonic oscillator equation eigensolutions*

Geometric method

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular eq., Hamilton-Cayley eq., Idempotent projectors, (how eigenvalues \Rightarrow eigenvectors)

Spectral decomposition and P-operator expansions (how projectors \Rightarrow eigensolutions)

2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M} \cdot |\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle$ in acceleration matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1} \cdot \mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} \frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \dots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

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and ω_n is an *eigenfrequency*

To introduce eigensolutions we take a simple case of unit masses ($m_1=1=m_2$)

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

2D harmonic oscillator equation eigensolutions

➔ *Geometric method*

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

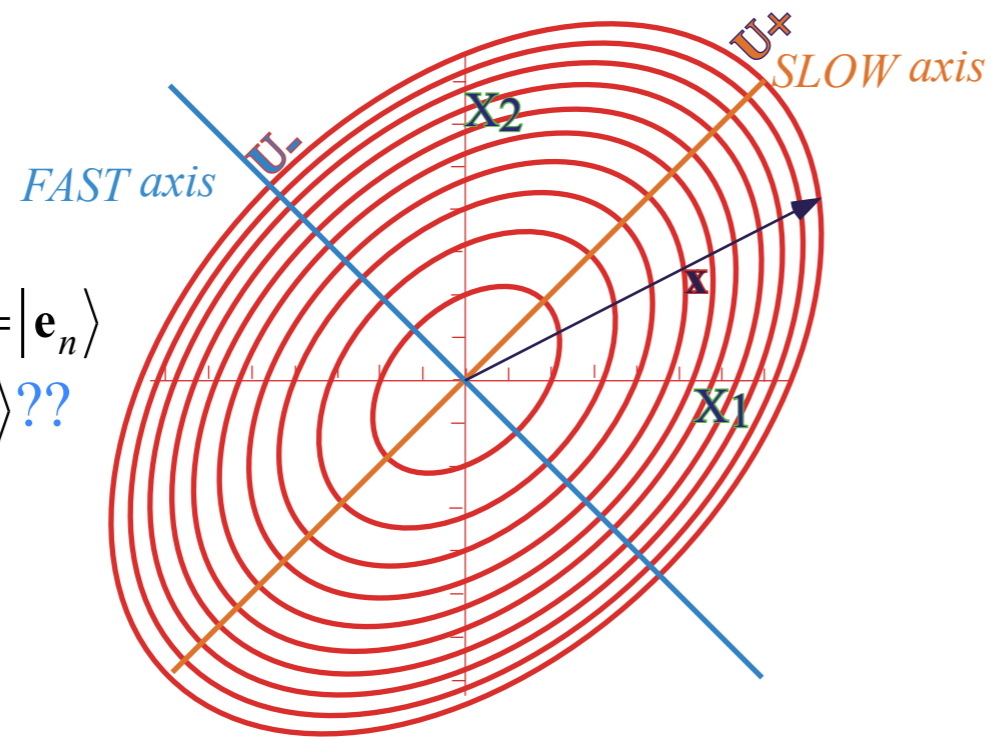
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Spectral decomposition and P-operator expansions (how projectors \Rightarrow eigensolutions)

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V -contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle = \frac{1}{2}\mathbf{x} \cdot \mathbf{K} \cdot \mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$
is the same as $\mathbf{K}|\mathbf{x}\rangle??$

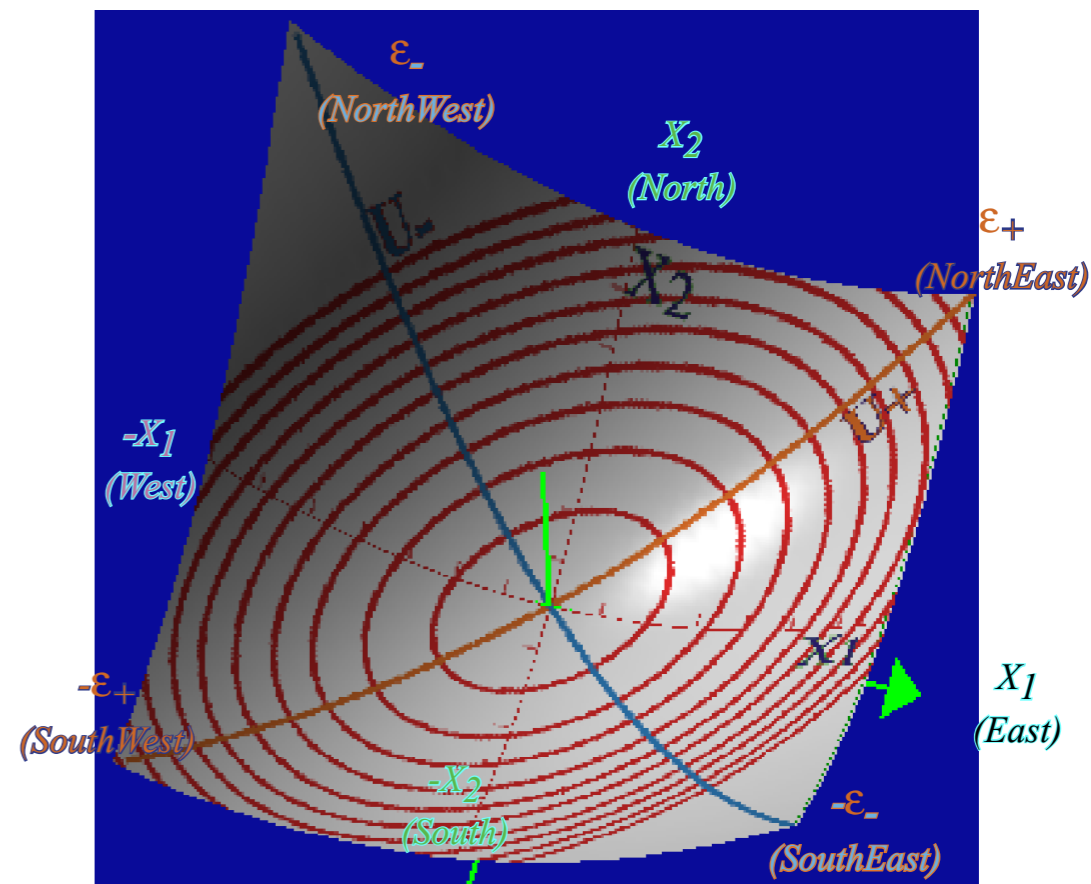


Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2) = \text{const.}$ level curves.

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(a) PE Contours

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is the same as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!

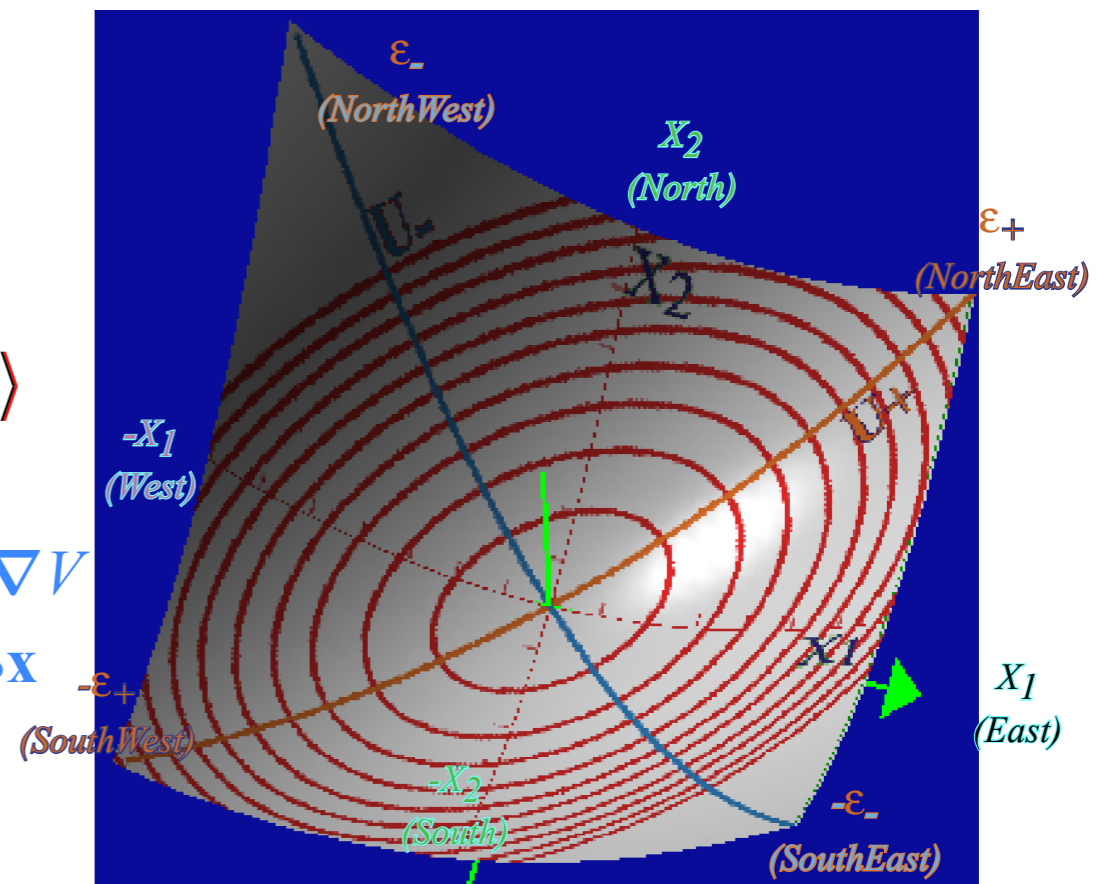
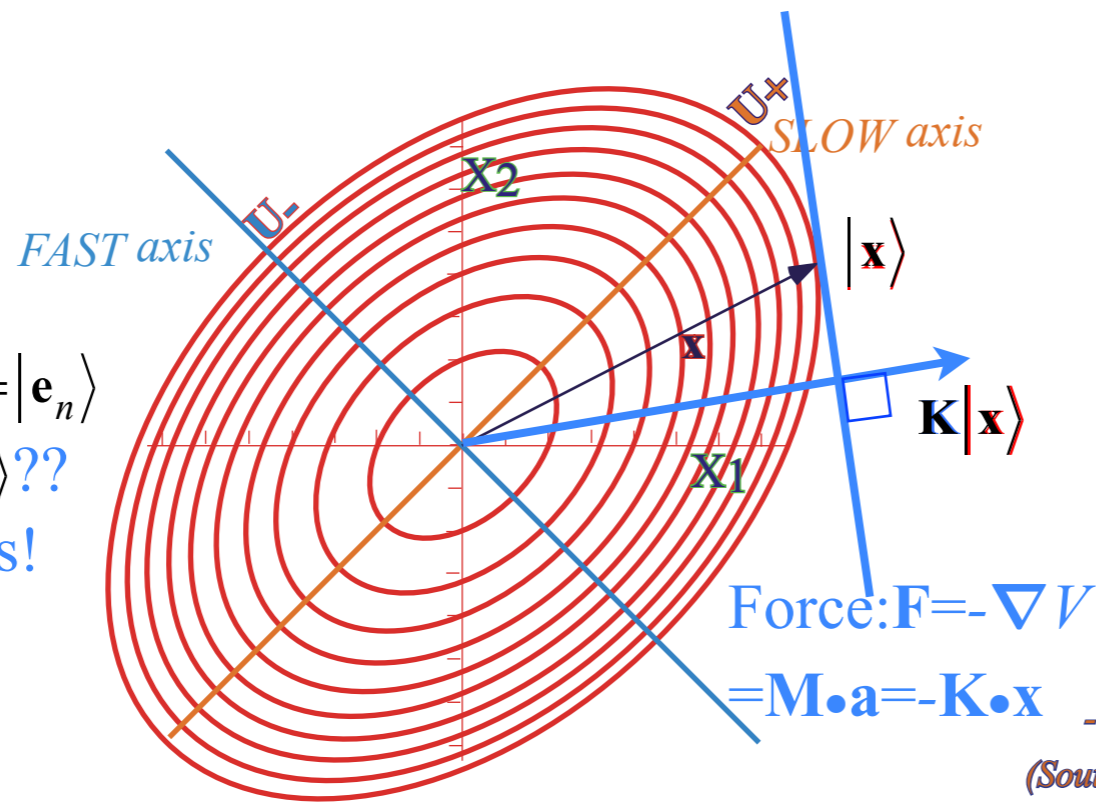
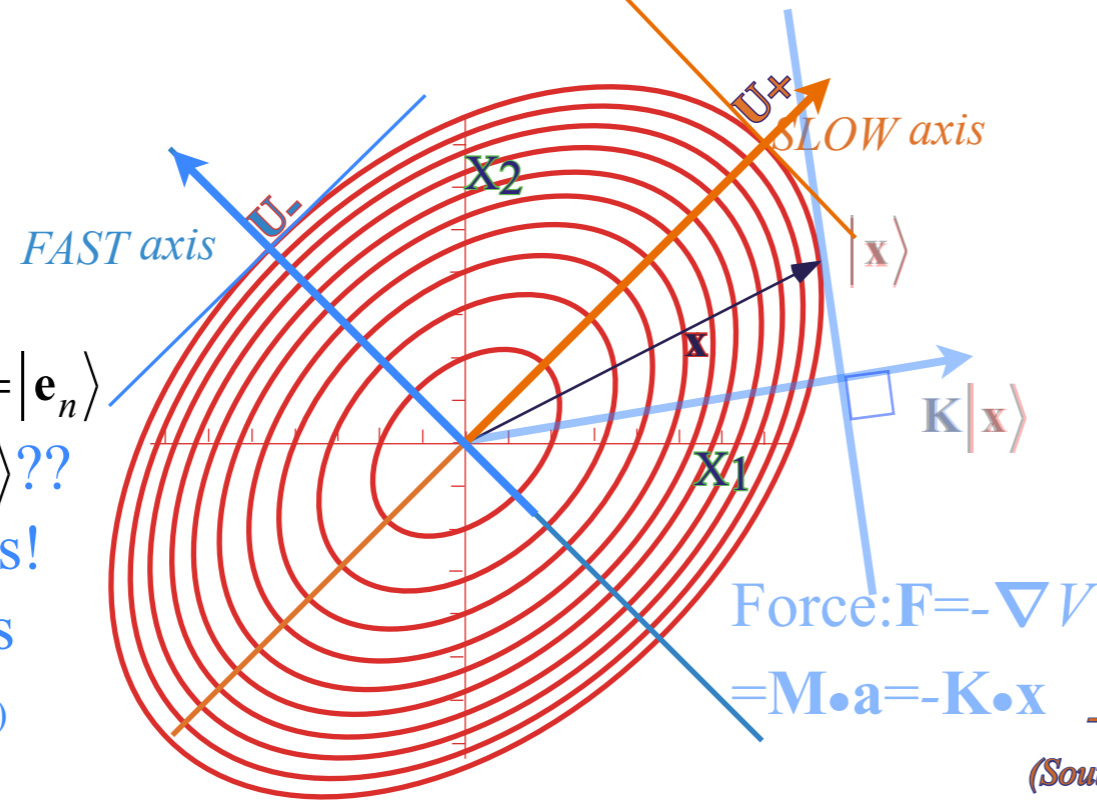


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(a) PE Contours



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the same as $\mathbf{K}|\mathbf{x}\rangle$??
 Not most directions!
 Only extremal axes work. (major or minor axes)

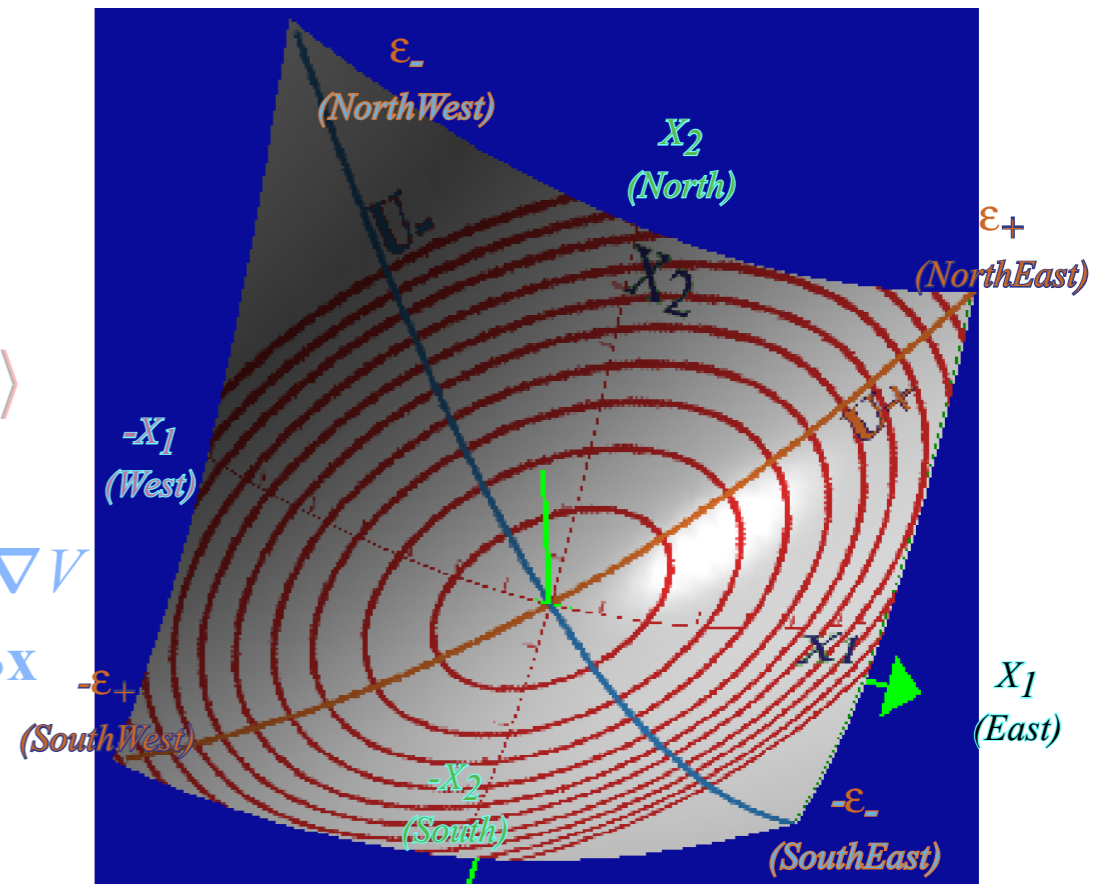
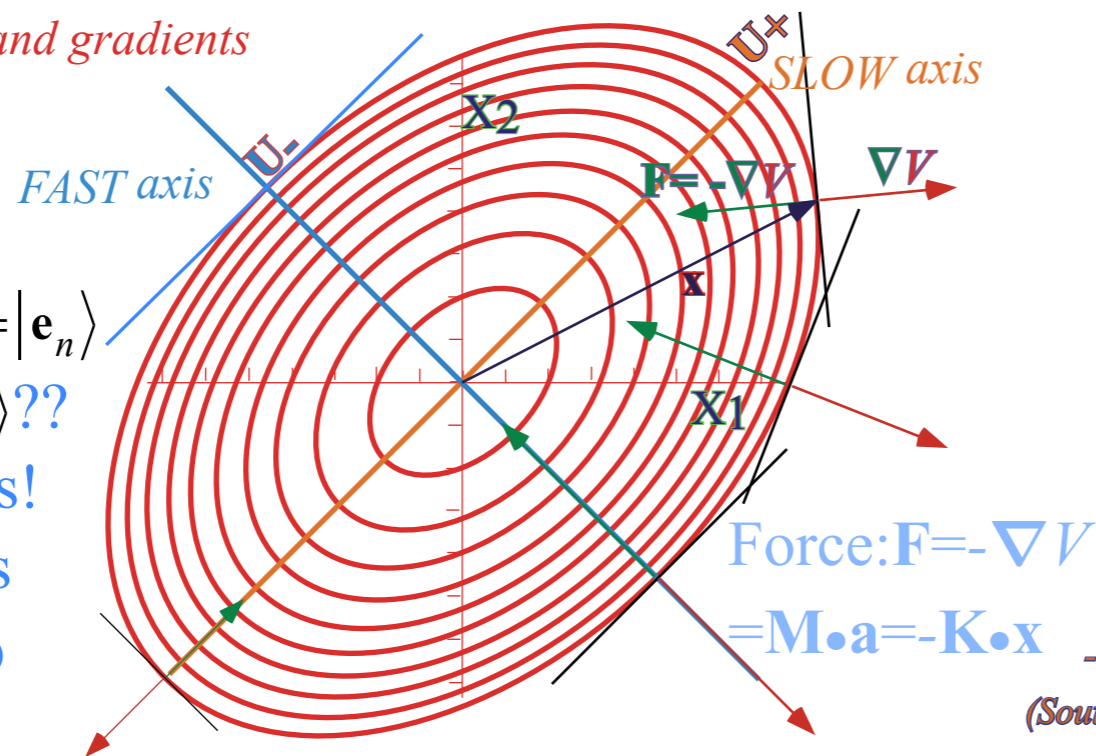


Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2) = \text{const.}$ level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V -contours (Here: $k_1 = k = k_2$)

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(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the same as $\mathbf{K}|\mathbf{x}\rangle$??
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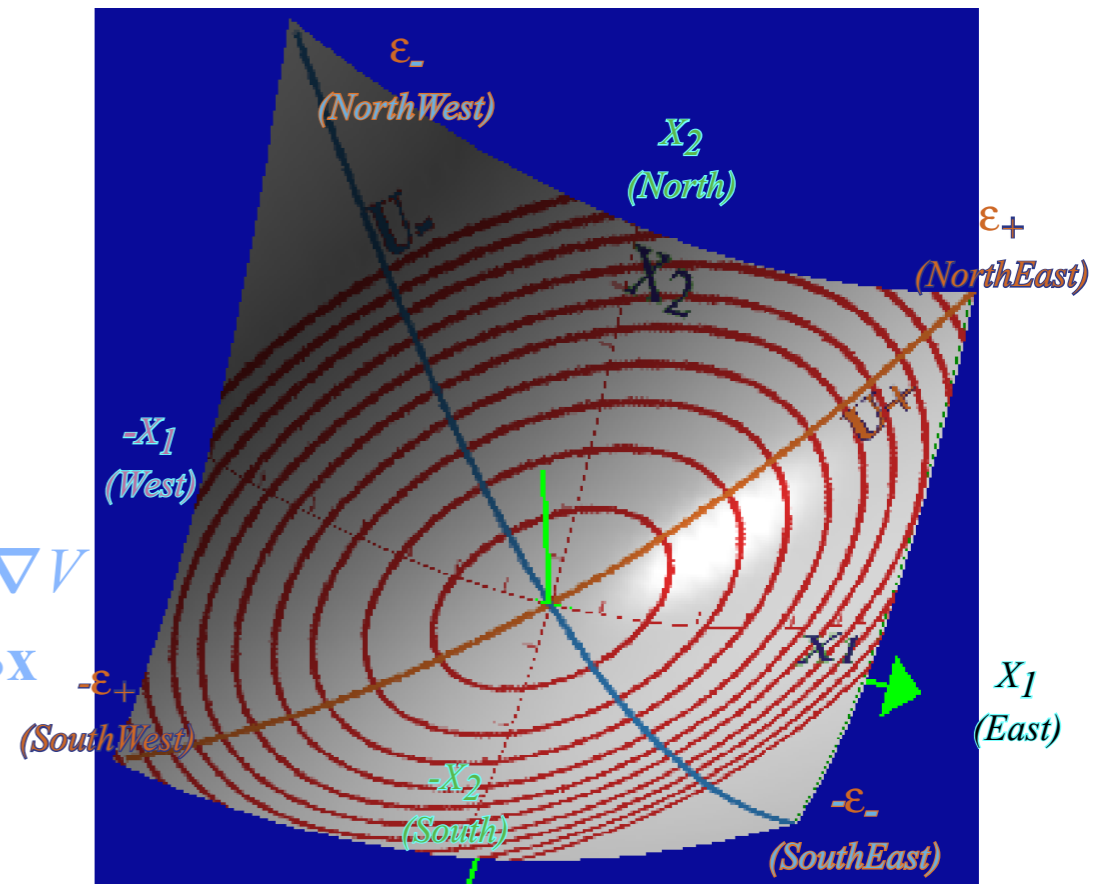
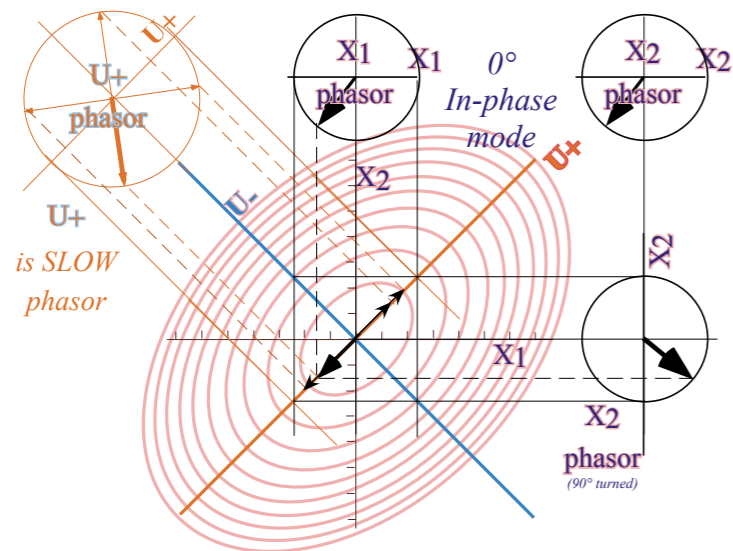
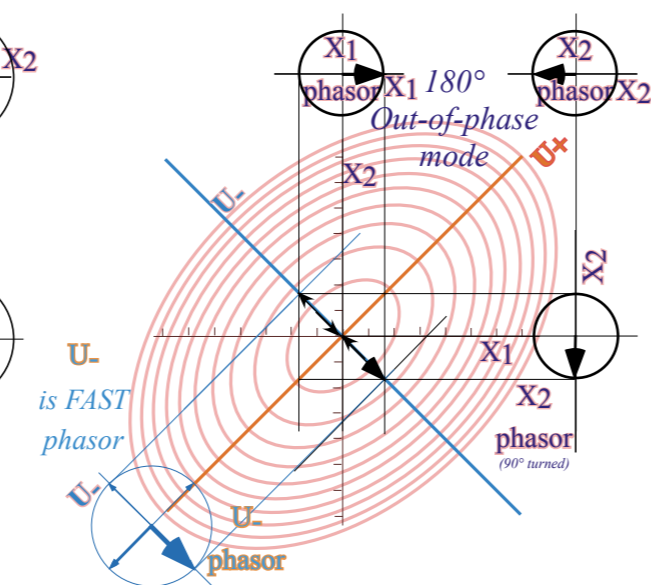


Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2) = \text{const.}$ level curves.

(b) Symmetric $U+$ Coordinate SLOW Mode



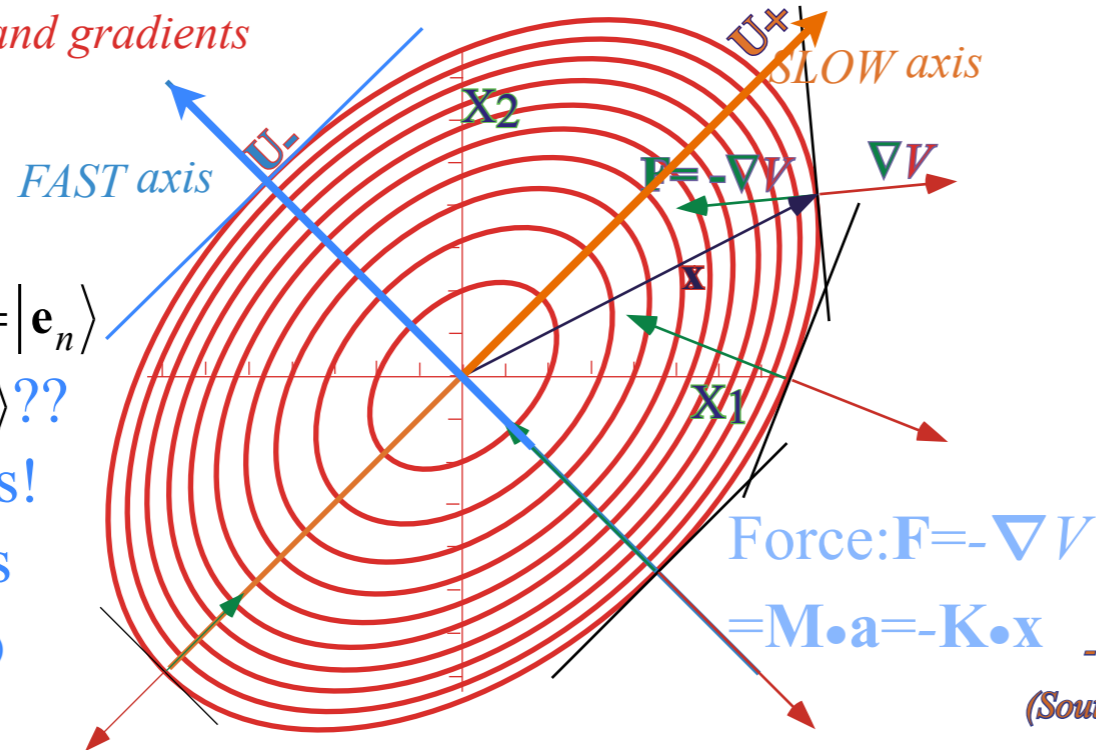
(c) Anti-symmetric $U-$ Coordinate FAST Mode



2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V -contours (Here: $k_1 = k = k_2$)

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(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the same as $\mathbf{K}|\mathbf{x}\rangle$?
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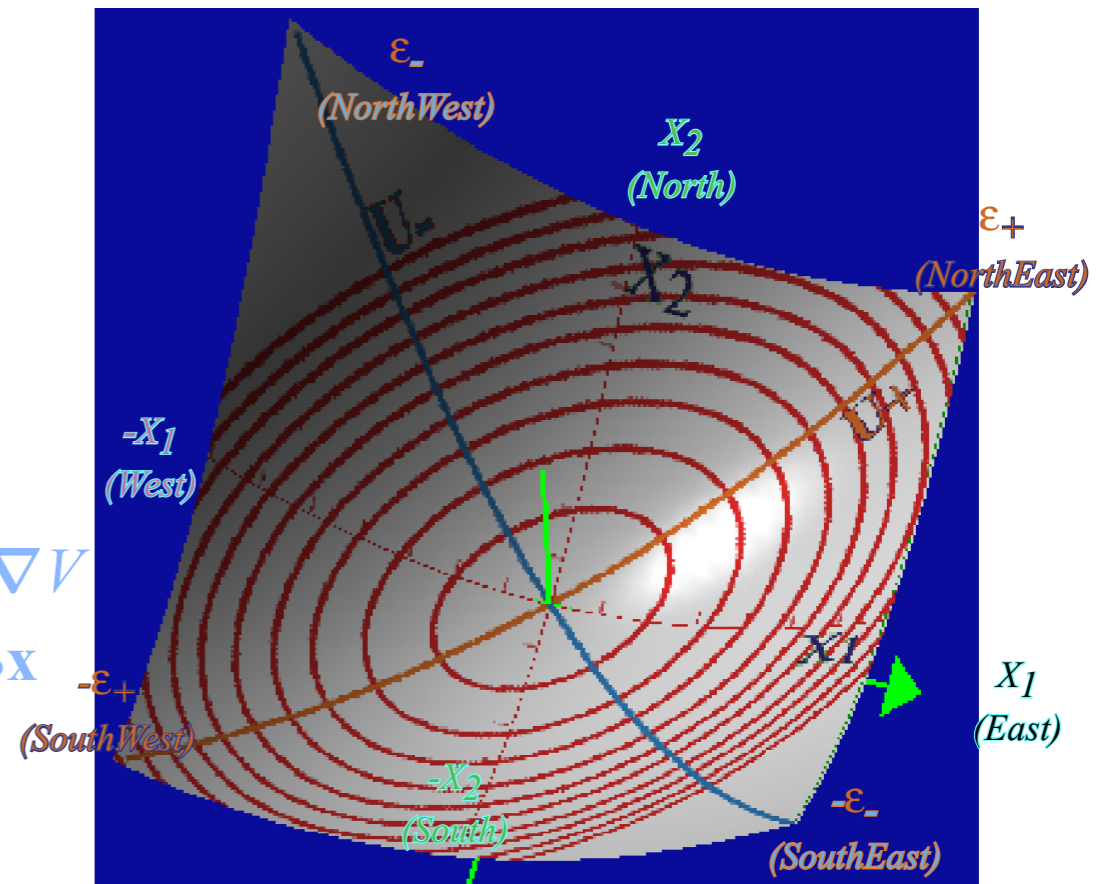
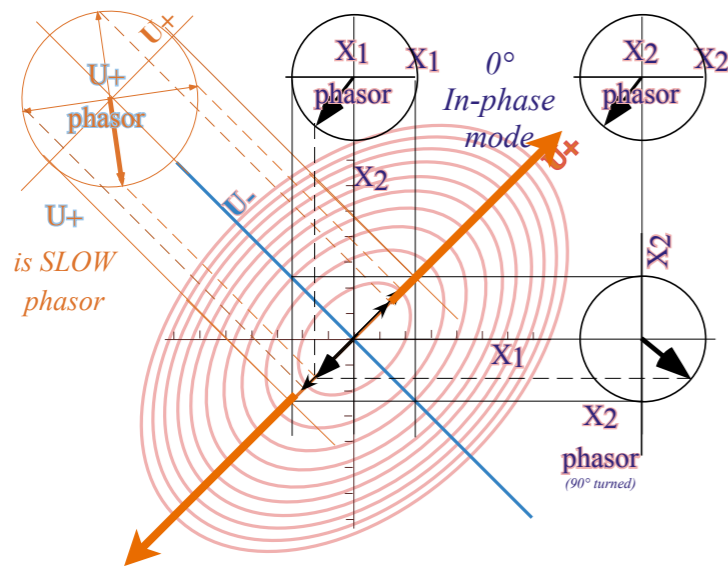
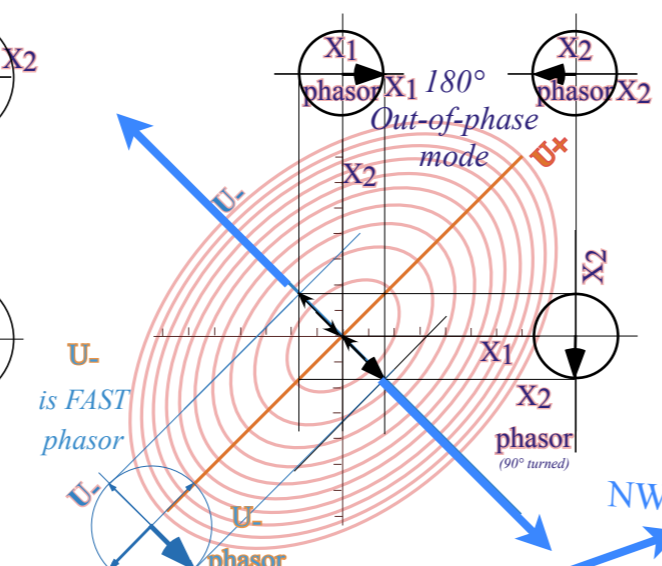


Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2) = \text{const.}$ level curves.

(b) Symmetric $U+$ Coordinate SLOW Mode



(c) Anti-symmetric $U-$ Coordinate FAST Mode



With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at $\pm 45^\circ$

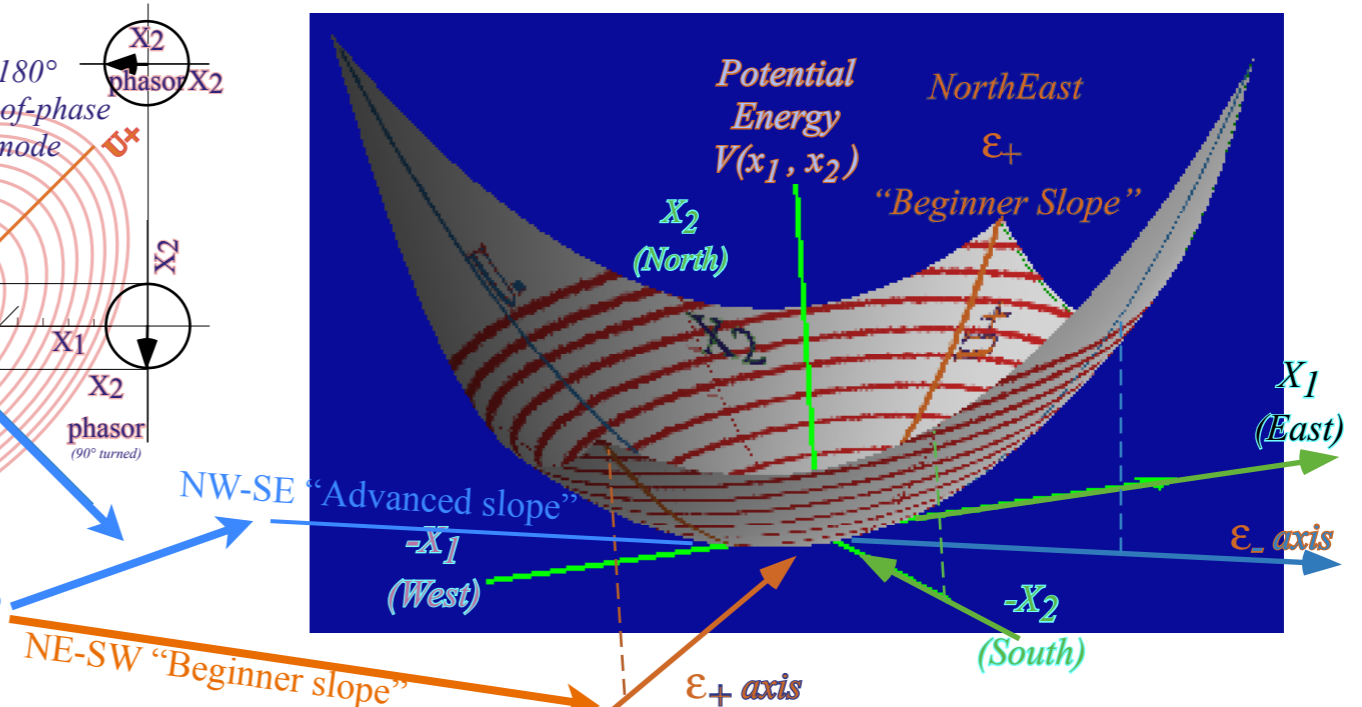


Fig. 3.3.5 Topography lines of potential function $V(x_1, x_2)$ and orthogonal ϵ_+ and ϵ_- normal mode slopes

2D harmonic oscillator equation eigensolutions

Geometric method

➔ *Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$*

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Matrix-algebraic method for finding eigenvector and eigenvalues With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An *eigenvector* $|\epsilon_k\rangle$ of \mathbf{M} is in a direction that is left unchanged by \mathbf{M} .

$$\mathbf{M}|\epsilon_k\rangle = \epsilon_k|\epsilon_k\rangle, \text{ or: } (\mathbf{M} - \epsilon_k \mathbf{1})|\epsilon_k\rangle = \mathbf{0}$$

$$\mathbf{M}|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ϵ_k is *eigenvalue* associated with eigenvector $|\epsilon_k\rangle$ direction.

A change of basis to $\{|\epsilon_1\rangle, |\epsilon_2\rangle, \dots, |\epsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle \epsilon_1 | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_1 | \mathbf{M} | \epsilon_2 \rangle & \cdots & \langle \epsilon_1 | \mathbf{M} | \epsilon_n \rangle \\ \langle \epsilon_2 | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_2 | \mathbf{M} | \epsilon_2 \rangle & \cdots & \langle \epsilon_2 | \mathbf{M} | \epsilon_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \epsilon_n | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_n | \mathbf{M} | \epsilon_2 \rangle & \cdots & \langle \epsilon_n | \mathbf{M} | \epsilon_n \rangle \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_n \end{pmatrix}$$

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Trying to solve by Kramer's inversion:

$$x = \frac{\det \begin{pmatrix} 0 & 1 \\ 0 & 2-\varepsilon \end{pmatrix}}{\det \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}} \quad \text{and} \quad y = \frac{\det \begin{pmatrix} 4-\varepsilon & 0 \\ 3 & 0 \end{pmatrix}}{\det \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}}$$

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First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \dots + a_{n-1} \epsilon + a_n)$$

where:

$$a_1 = -\text{Trace} \mathbf{M}, \dots, a_k = (-1)^k \sum \text{diagonal k-by-k minors of } \mathbf{M}, \dots, a_n = (-1)^n \det|\mathbf{M}|$$

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Only possible non-zero $\{x, y\}$ if denominator is zero, too!

$$0 = \det|\mathbf{M} - \epsilon \cdot \mathbf{1}| = \det \left[\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \det \begin{pmatrix} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{pmatrix}$$

$$0 = (4-\epsilon)(2-\epsilon) - 1 \cdot 3 = 8 - 6\epsilon + \epsilon^2 - 3 = \epsilon^2 - 6\epsilon + 5$$

$$0 = \epsilon^2 - \text{Trace}(\mathbf{M})\epsilon + \det(\mathbf{M})$$

2D harmonic oscillator equation eigensolutions

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$$\begin{pmatrix} \langle \epsilon_1 | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_1 | \mathbf{M} | \epsilon_2 \rangle & \dots & \langle \epsilon_1 | \mathbf{M} | \epsilon_n \rangle \\ \langle \epsilon_2 | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_2 | \mathbf{M} | \epsilon_2 \rangle & \dots & \langle \epsilon_2 | \mathbf{M} | \epsilon_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \epsilon_n | \mathbf{M} | \epsilon_1 \rangle & \langle \epsilon_n | \mathbf{M} | \epsilon_2 \rangle & \dots & \langle \epsilon_n | \mathbf{M} | \epsilon_n \rangle \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon_n \end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \dots + a_{n-1} \epsilon + a_n)$$

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$$\det|\mathbf{M} - \epsilon \mathbf{1}| = 0 = (-1)^n (\epsilon - \epsilon_1)(\epsilon - \epsilon_2) \dots (\epsilon - \epsilon_n)$$

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Trying to solve by Kramer's inversion:

$$x = \frac{\det \begin{pmatrix} 0 & 1 \\ 0 & 2 - \epsilon \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}} \quad \text{and} \quad y = \frac{\det \begin{pmatrix} 4 - \epsilon & 0 \\ 3 & 0 \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}}$$

Only possible non-zero $\{x, y\}$ if denominator is zero, too!

$$0 = \det|\mathbf{M} - \epsilon \cdot \mathbf{1}| = \det \left[\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}$$

$$0 = (4 - \epsilon)(2 - \epsilon) - 1 \cdot 3 = 8 - 6\epsilon + \epsilon^2 - 3 = \epsilon^2 - 6\epsilon + 5$$

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$$0 = (\epsilon - 1)(\epsilon - 5) \text{ so let: } \epsilon_1 = 1 \text{ and: } \epsilon_2 = 5$$

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$$\mathbf{p}_1 = (\mathbf{1})(\mathbf{M} - \epsilon_2 \mathbf{1}) \dots (\mathbf{M} - \epsilon_n \mathbf{1})$$

$$\mathbf{p}_2 = (\mathbf{M} - \epsilon_1 \mathbf{1})(\mathbf{1}) \dots (\mathbf{M} - \epsilon_n \mathbf{1})$$

\vdots

$$\mathbf{p}_n = (\mathbf{M} - \epsilon_1 \mathbf{1})(\mathbf{M} - \epsilon_2 \mathbf{1}) \dots (\mathbf{1})$$

(Assume distinct e-values here: *Non-degeneracy clause*)

$$\epsilon_j \neq \epsilon_k \neq \dots$$

$$\mathbf{M}|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\begin{aligned} \mathbf{p}_1 &= (\mathbf{1})(\mathbf{M} - \epsilon_2\mathbf{1}) \dots (\mathbf{M} - \epsilon_n\mathbf{1}) \\ \mathbf{p}_2 &= (\mathbf{M} - \epsilon_1\mathbf{1})(\mathbf{1}) \dots (\mathbf{M} - \epsilon_n\mathbf{1}) \\ &\vdots \\ \mathbf{p}_n &= (\mathbf{M} - \epsilon_1\mathbf{1})(\mathbf{M} - \epsilon_2\mathbf{1}) \dots (\mathbf{1}) \end{aligned} \quad \begin{array}{l} \text{(Assume distinct e-values here: } \\ \text{Non-degeneracy clause)} \\ \epsilon_j \neq \epsilon_k \neq \dots \end{array}$$

Each \mathbf{p}_k contains *eigen-bra-kets* since: $(\mathbf{M} - \epsilon_k\mathbf{1})\mathbf{p}_k = \mathbf{0}$ or: $\mathbf{M}\mathbf{p}_k = \epsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$.

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$$\mathbf{M}\mathbf{p}_2 = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = 5 \cdot \mathbf{p}_2$$

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

➔ *Secular eq., Hamilton-Cayley eq., **Idempotent** projectors, (how eigenvalues \Rightarrow eigenvectors)*

Spectral decomposition and P-operator expansions (how projectors \Rightarrow eigensolutions)

*Idempotent means: **P·P=P***

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

$$\mathbf{p}_1 = (\mathbf{M} - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_1 \mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Multiplication properties of \mathbf{p}_j :

$$\mathbf{p}_j \mathbf{p}_k = \prod_{m \neq k} (\varepsilon_j \mathbf{p}_j - \varepsilon_m \mathbf{p}_j) = \mathbf{p}_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{p}_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases}$$

$$\mathbf{p}_2 = (\mathbf{M} - 1 \cdot \mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

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Last step is to make *Idempotent Projectors*:
$$\mathbf{P}_k = \frac{\mathbf{p}_k}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} = \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)}$$

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1})$$

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$$\mathbf{P}_j \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases}$$

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Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \epsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \epsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

$$\mathbf{p}_1 = (\mathbf{M} - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

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Multiplication properties of \mathbf{p}_j :

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Factoring bra-kets into "Ket-Bras:

Last step is to make *Idempotent Projectors*: $\mathbf{P}_k = \frac{\mathbf{p}_k}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)} = \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)}$

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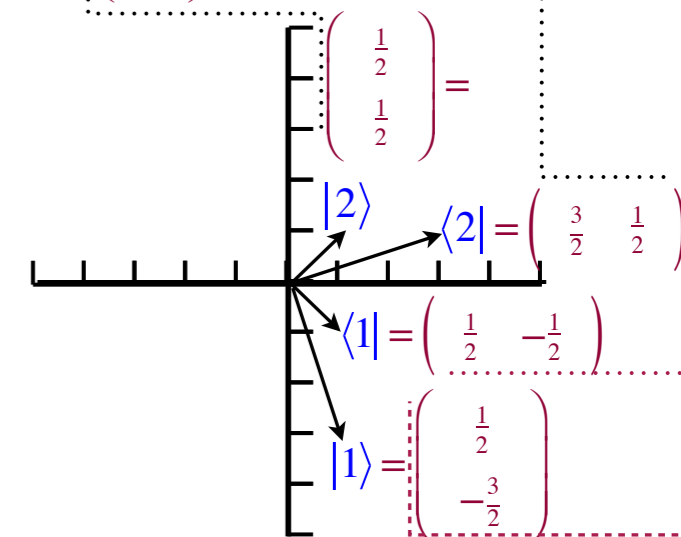
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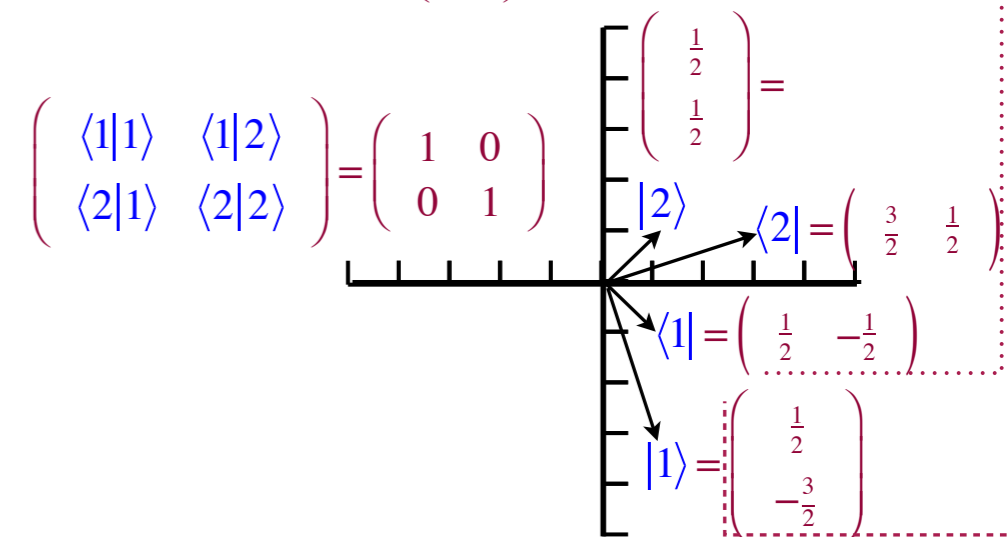
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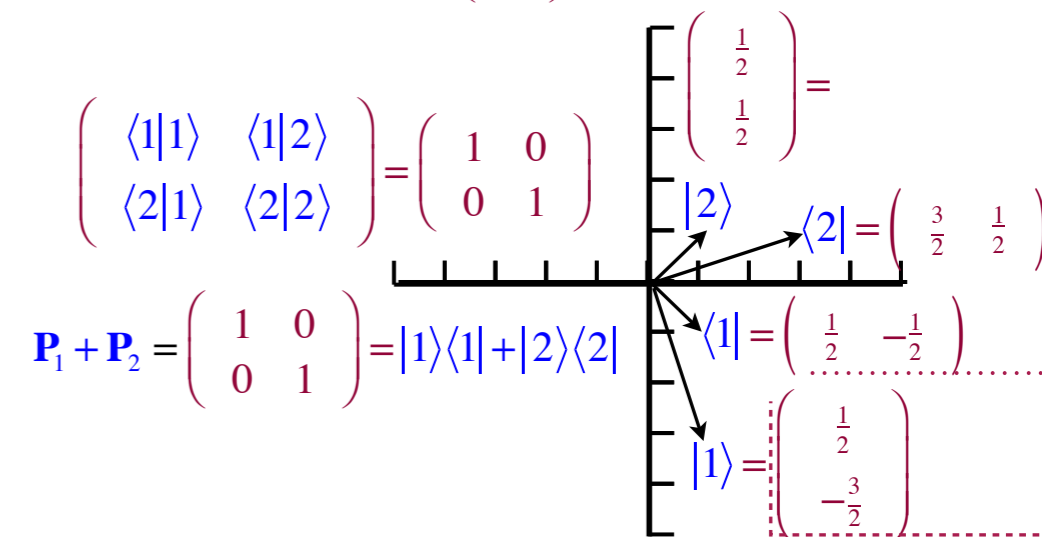
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$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| + |2\rangle\langle 2|$$

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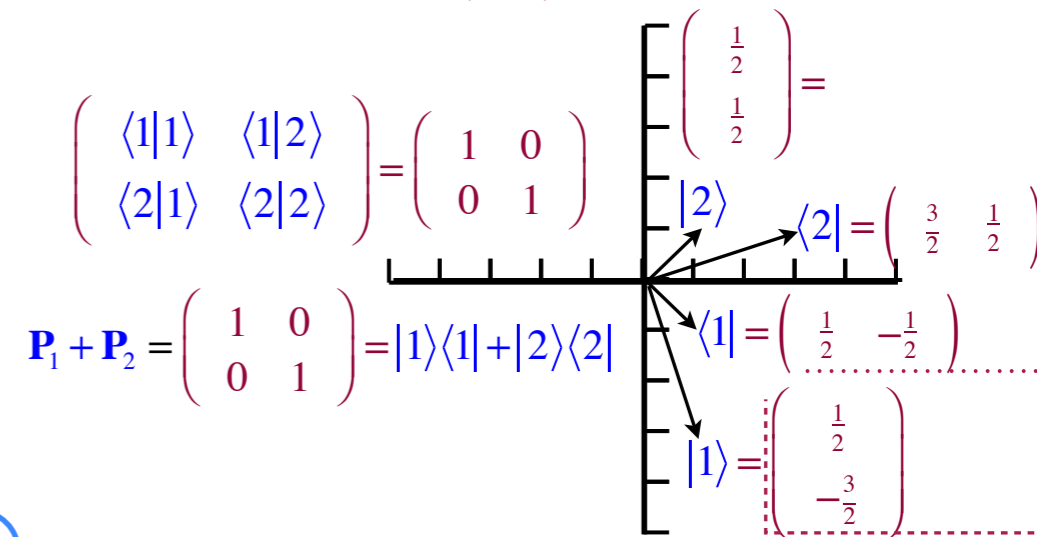
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$\{|x\rangle, |y\rangle\}$ -orthonormality with $\{|1\rangle, |2\rangle\}$ -completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|1\rangle\langle 1|y\rangle + \langle x|2\rangle\langle 2|y\rangle$$

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Vector-tensor analysis
vs.
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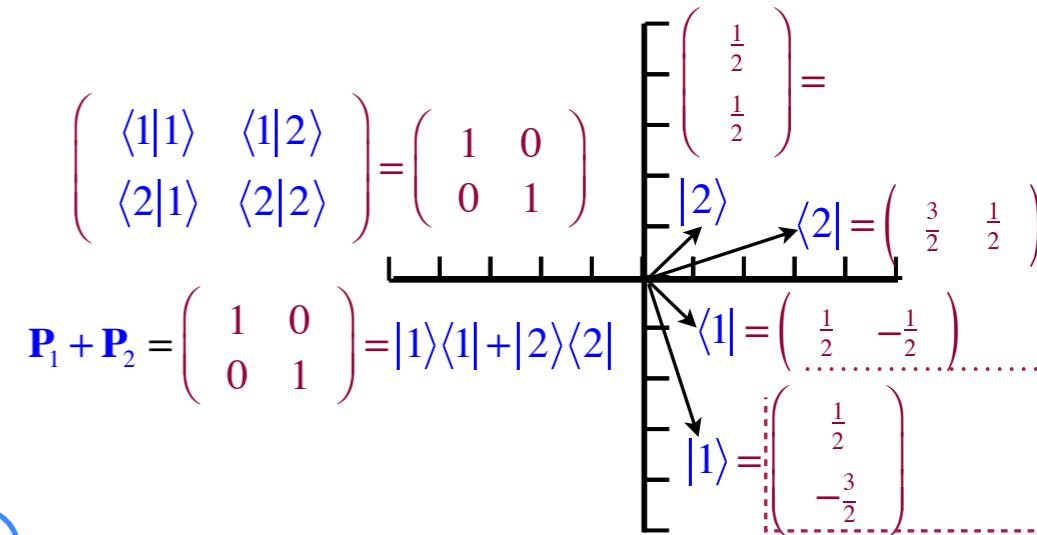
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wave-functional analysis

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic method with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

*Secular eq., Hamilton-Cayley eq., **Idempotent** projectors, (how eigenvalues \Rightarrow eigenvectors)*

 *Spectral decomposition and **P-operator** expansions (how projectors \Rightarrow eigensolutions)*

*Idempotent means: **P·P=P***

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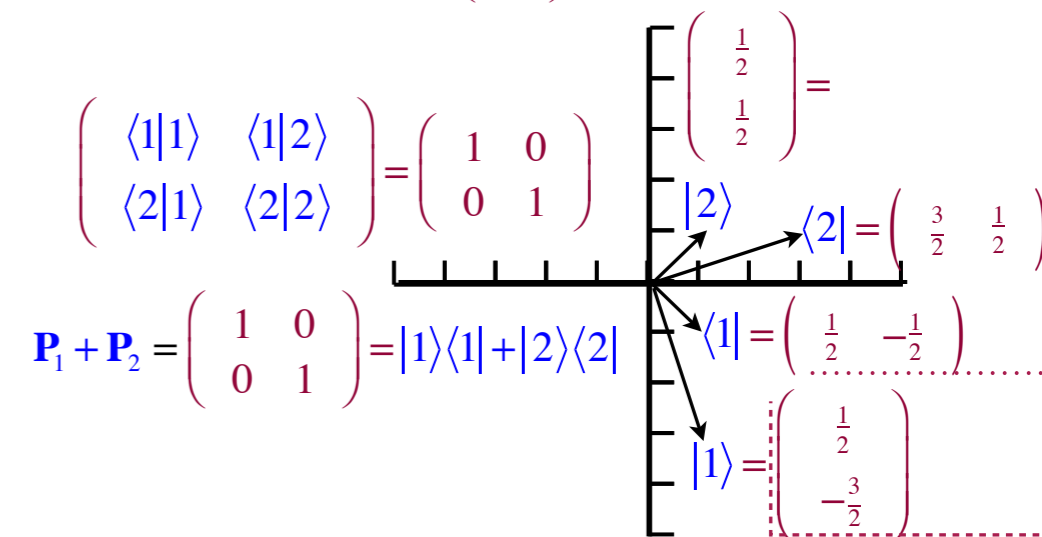
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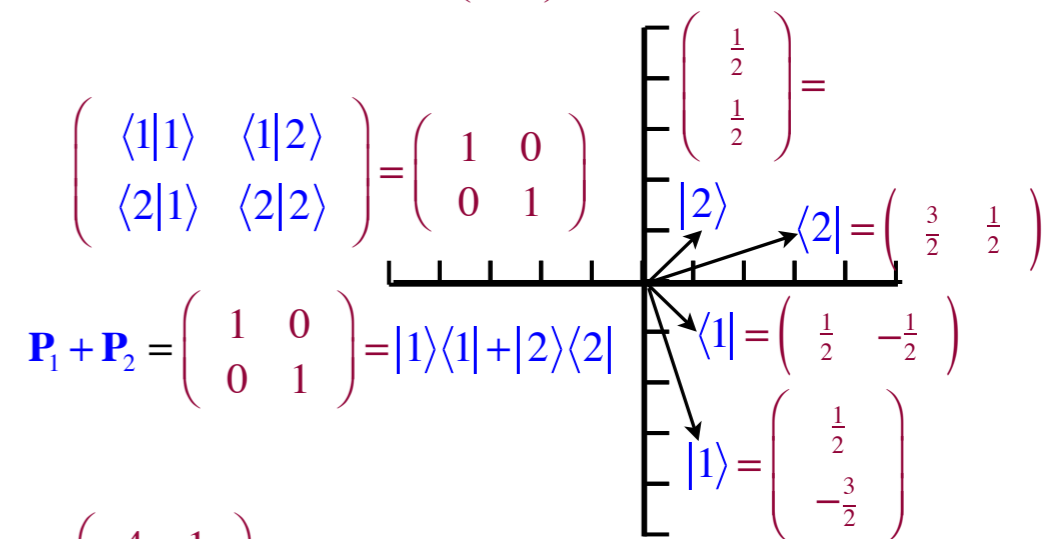
$$\mathbf{P}_j \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases} \quad \mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k$$

The \mathbf{P}_j are *Mutually Ortho-Normal* as are bra-ket $\langle j|$ and $|j\rangle$ vectors inside \mathbf{P}_j 's and satisfy a *Completeness Relation*: $\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = |1\rangle\langle 1| + |2\rangle\langle 2| + \dots + |n\rangle\langle n|$

Factoring bra-kets into "Ket-Bras":

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = |1\rangle\langle 1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} = |2\rangle\langle 2|$$



$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2|$$

$$= 1 \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5 \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k$ then give *Spectral Decomposition* of operator \mathbf{M}

$$\mathbf{M} = \mathbf{M} \mathbf{P}_1 + \mathbf{M} \mathbf{P}_2 + \dots + \mathbf{M} \mathbf{P}_n = \varepsilon_1 \mathbf{P}_1 + \varepsilon_2 \mathbf{P}_2 + \dots + \varepsilon_n \mathbf{P}_n$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

$$\mathbf{p}_1 = (\mathbf{M} - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \quad \mathbf{p}_1 \mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M} - 1 \cdot \mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Multiplication properties of \mathbf{p}_j :

$$\mathbf{p}_j \mathbf{p}_k = \prod_{m \neq k} (\varepsilon_j \mathbf{p}_j - \varepsilon_m \mathbf{p}_j) = \mathbf{p}_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{p}_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases}$$

Last step is to make *Idempotent Projectors*: $\mathbf{P}_k = \frac{\mathbf{p}_k}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} = \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)}$

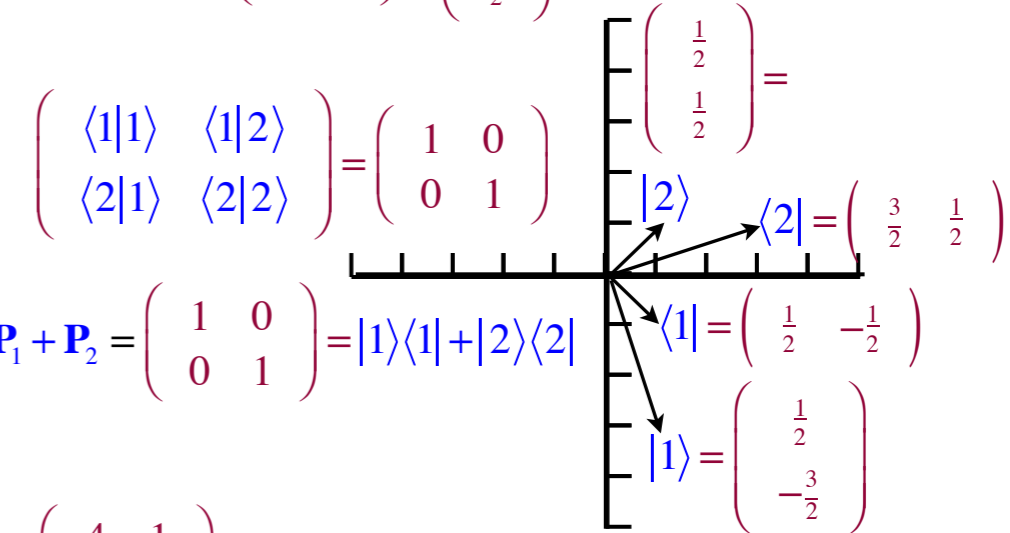
$$\mathbf{P}_j \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases} \quad \mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k$$

The \mathbf{P}_j are *Mutually Ortho-Normal* as are bra-ket $\langle j|$ and $|j\rangle$ vectors inside \mathbf{P}_j 's and satisfy a *Completeness Relation*: $\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = |1\rangle\langle 1| + |2\rangle\langle 2| + \dots + |n\rangle\langle n|$

Factoring bra-kets into "Ket-Bras":

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = |1\rangle\langle 1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} = |2\rangle\langle 2|$$



$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| + |2\rangle\langle 2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2|$$

$$= 1 \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5 \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Examples with $\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm \sqrt{1} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm \sqrt{5} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k$ then give *Spectral Decomposition* of operator \mathbf{M}

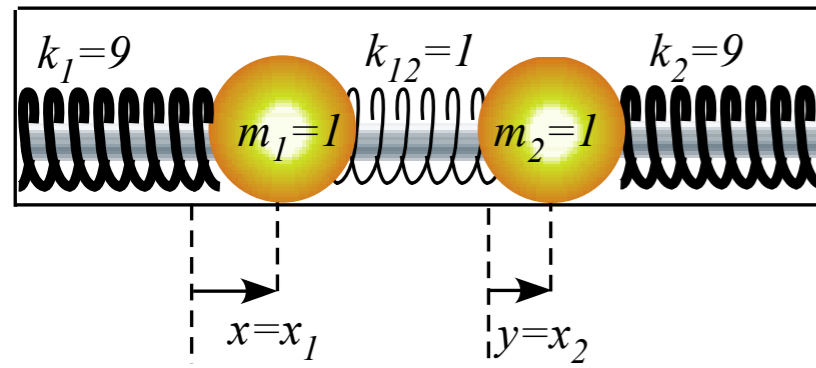
$$\mathbf{M} = \mathbf{M} \mathbf{P}_1 + \mathbf{M} \mathbf{P}_2 + \dots + \mathbf{M} \mathbf{P}_n = \varepsilon_1 \mathbf{P}_1 + \varepsilon_2 \mathbf{P}_2 + \dots + \varepsilon_n \mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function $f(\mathbf{M})$ of \mathbf{M}

$$f(\mathbf{M}) = f(\varepsilon_1) \mathbf{P}_1 + f(\varepsilon_2) \mathbf{P}_2 + \dots + f(\varepsilon_n) \mathbf{P}_n$$

→ *2D-HO eigensolution example with bilateral (B-Type) symmetry*
Mixed mode beat dynamics and fixed $\pi/2$ phase

Analyzing 2D-HO beats and mixed mode eigen-solutions



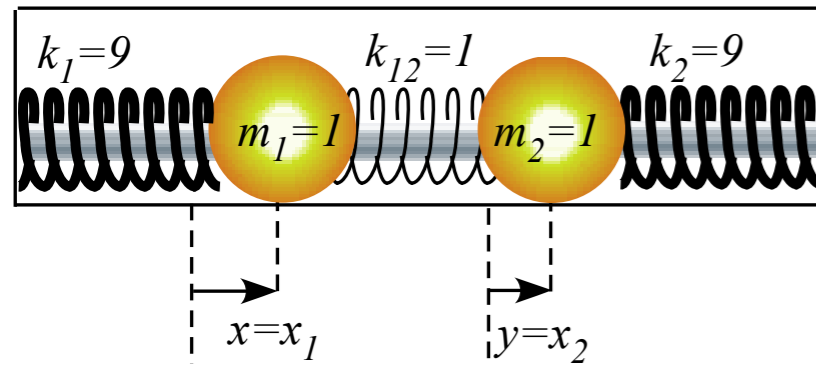
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$
 $Trace(\mathbf{K}) = 10 + 10 = 20$

The \mathbf{K} secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$
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The \mathbf{K} secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

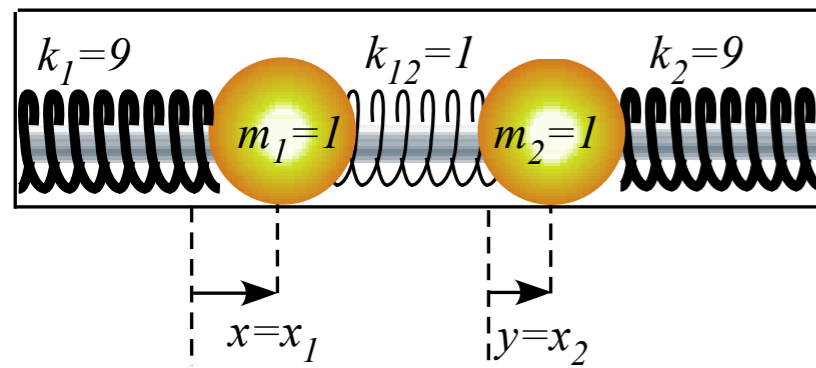
$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

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The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

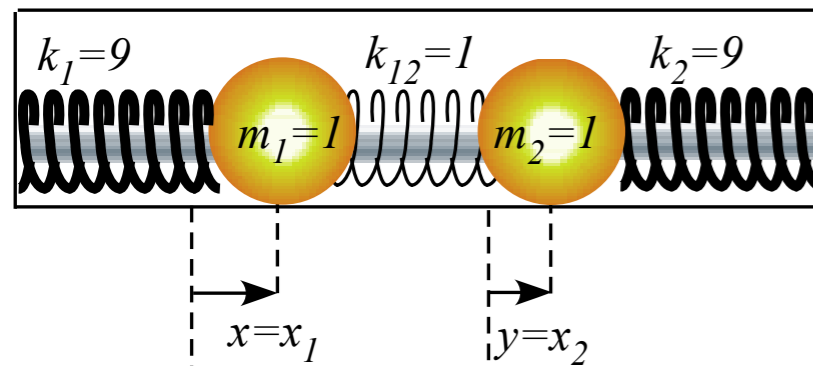
Eigen-projectors \mathbf{P}_k

$$\begin{aligned} \mathbf{P}_1 &= \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2} \\ &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1| \end{aligned}$$

$$\begin{aligned} \mathbf{P}_2 &= \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2} \\ &= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2| \end{aligned}$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right), \quad \langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

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$$\begin{aligned} \mathbf{P}_1 &= \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2} \\ &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1| \end{aligned}$$

$$\begin{aligned} \mathbf{P}_2 &= \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2} \\ &= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2| \end{aligned}$$

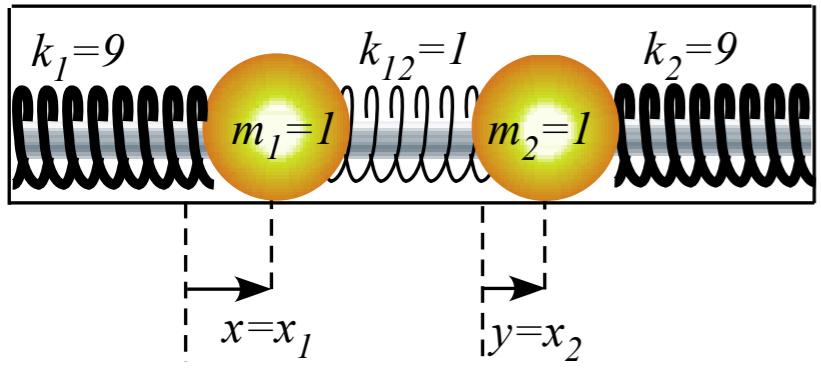
Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right)$, $\langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

$$\text{Trace}(\mathbf{K}) = 10 + 10 = 20$$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right), \quad \langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

100% modulation (SWR=0) $\frac{e^{ia} + e^{ib}}{2} = e^{\frac{i(a+b)}{2}} \frac{e^{\frac{i(a-b)}{2}} + e^{-\frac{i(a-b)}{2}}}{2}$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\omega_1 t} + e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t} - e^{-i\omega_2 t}}{2} \end{pmatrix} = \frac{e^{-\frac{i(\omega_1 + \omega_2)t}}{2}} \begin{pmatrix} e^{-\frac{i(\omega_1 - \omega_2)t}{2}} + e^{\frac{i(\omega_1 - \omega_2)t}{2}} \\ e^{-\frac{i(\omega_1 - \omega_2)t}{2}} - e^{\frac{i(\omega_1 - \omega_2)t}{2}} \end{pmatrix}$$

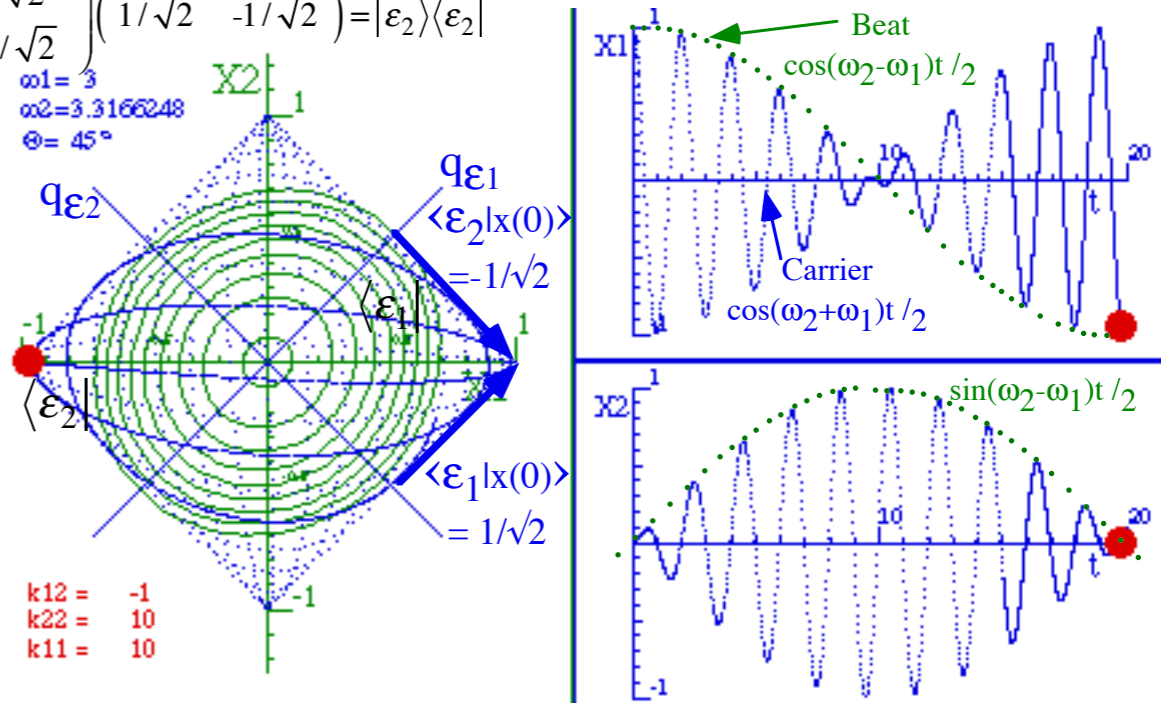
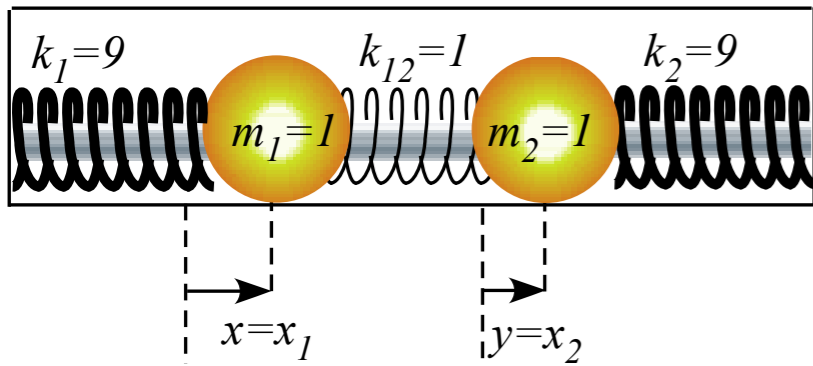


Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 10 \cdot 10 - 1 = 99$$

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Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$

$$K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$$

Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10 - 11 & -1 \\ -1 & 10 - 11 \end{pmatrix}}{9 - 11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10 - 9 & -1 \\ -1 & 10 - 9 \end{pmatrix}}{11 - 9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right)$, $\langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

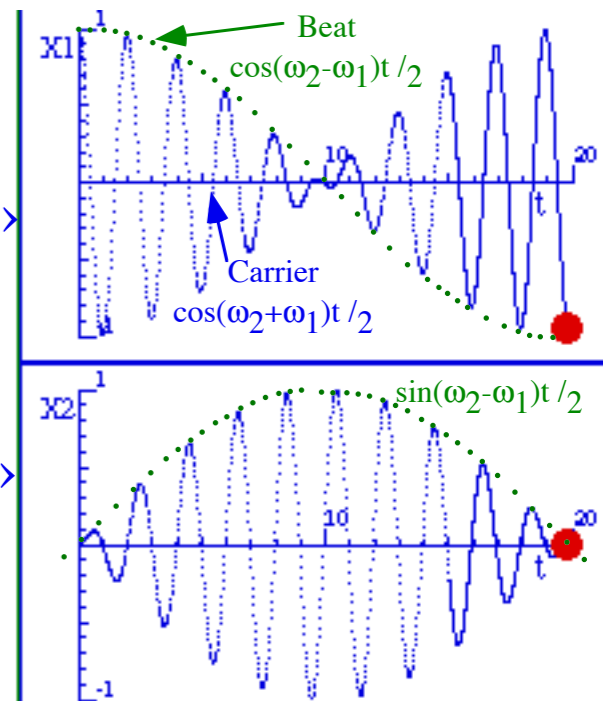
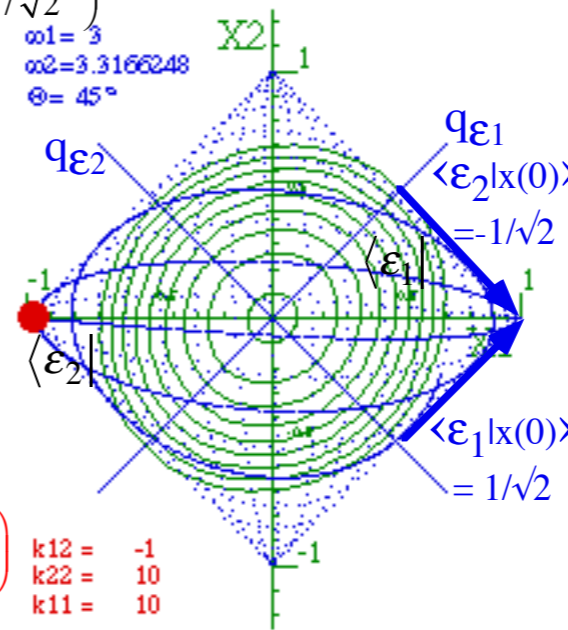
Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

100% modulation (SWR=0) $\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$

$$\begin{matrix} k_{12} = & -1 \\ k_{22} = & 10 \\ k_{11} = & 10 \end{matrix}$$



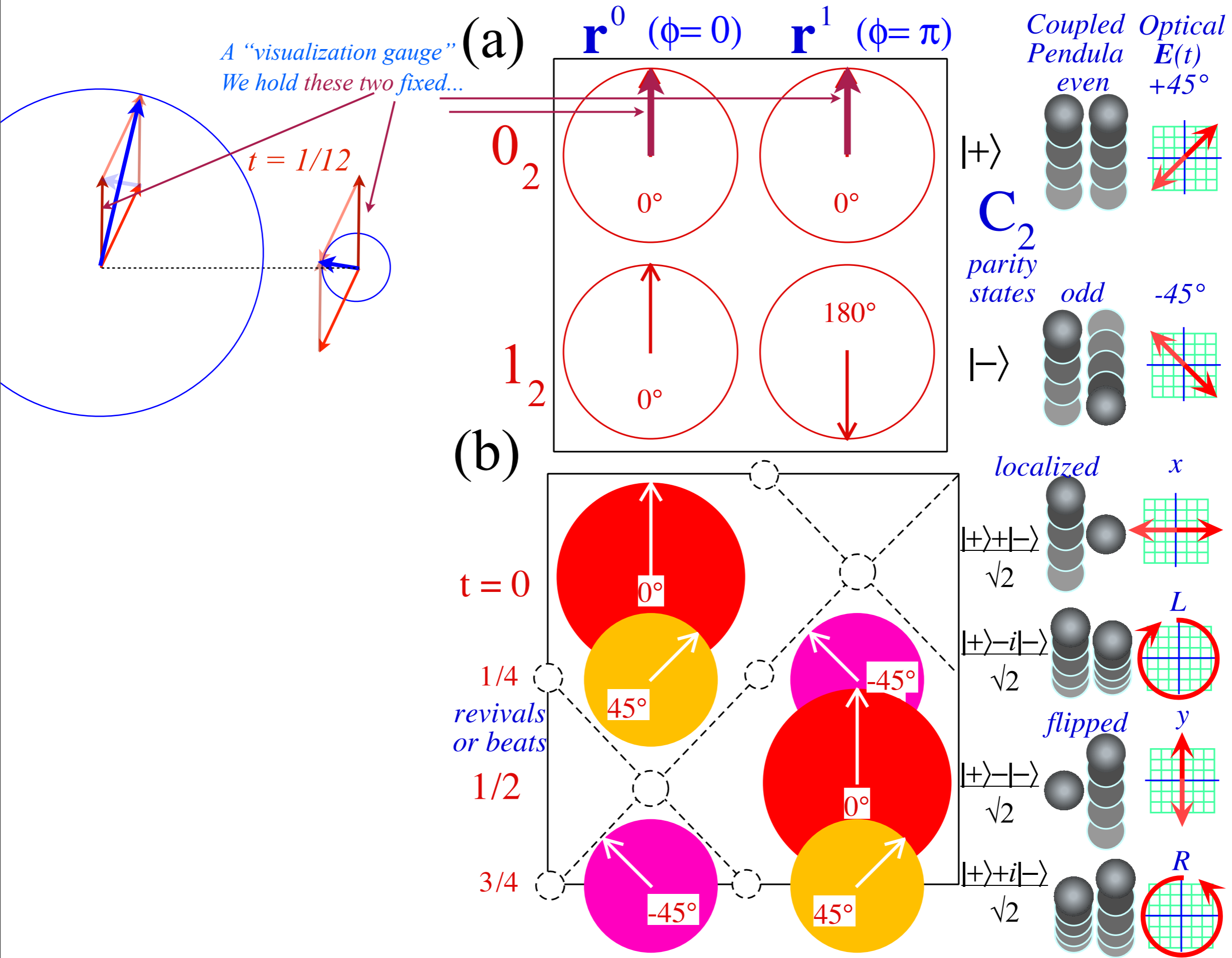
Note the i phase

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

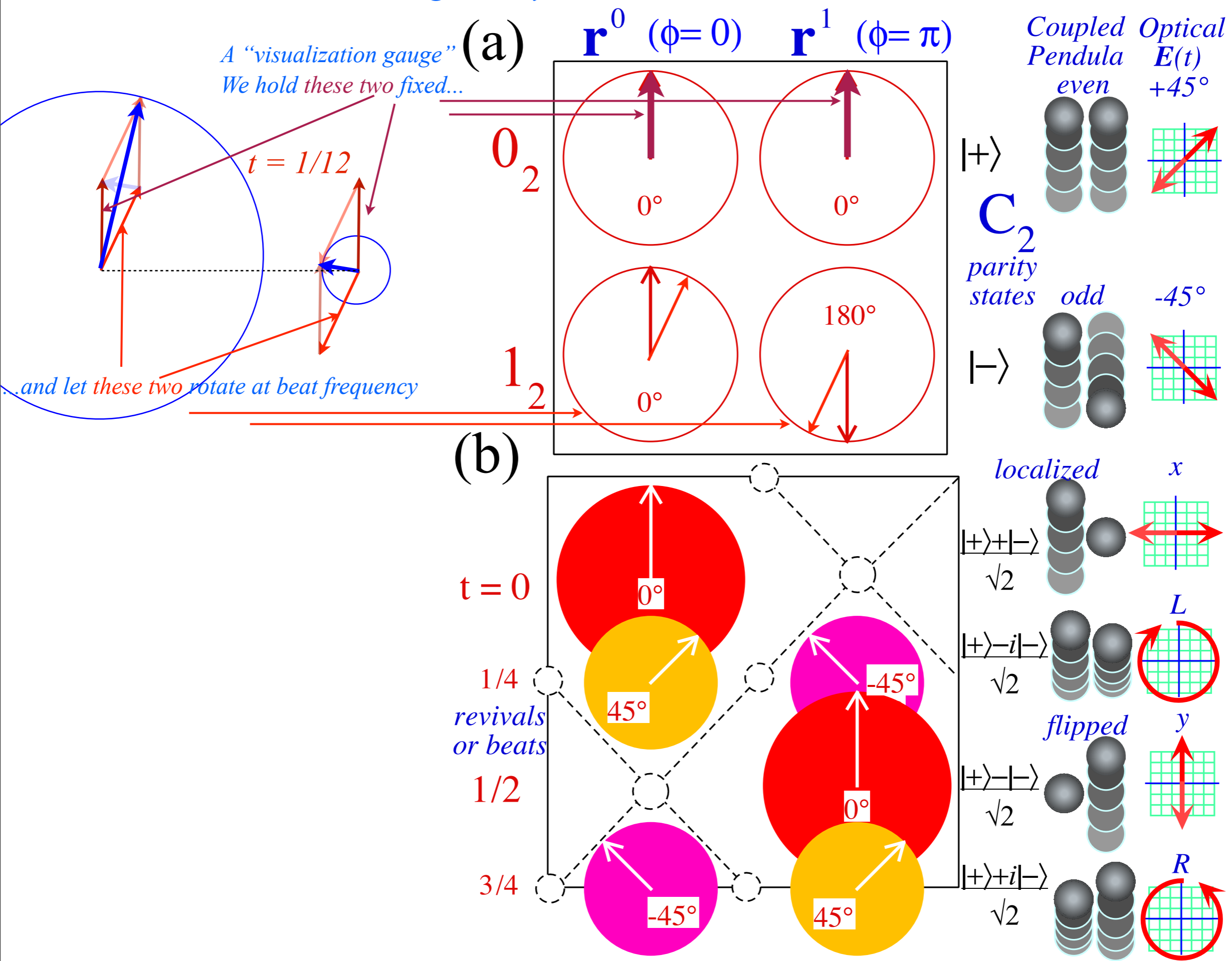
2D-HO eigensolution example with bilateral (B-Type) symmetry

➔ *Mixed mode beat dynamics and fixed $\pi/2$ phase*

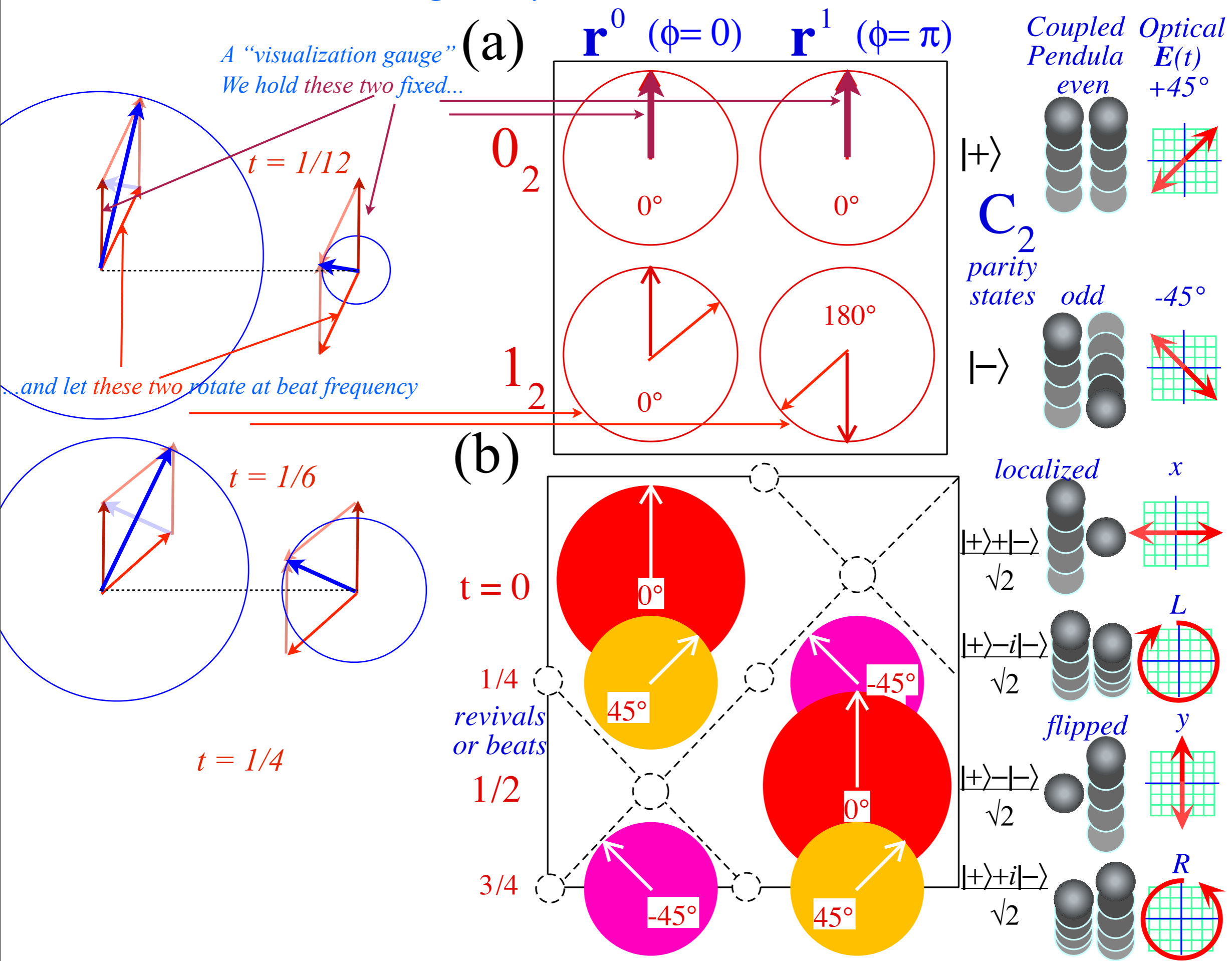
2D-HO beats and mixed mode geometry



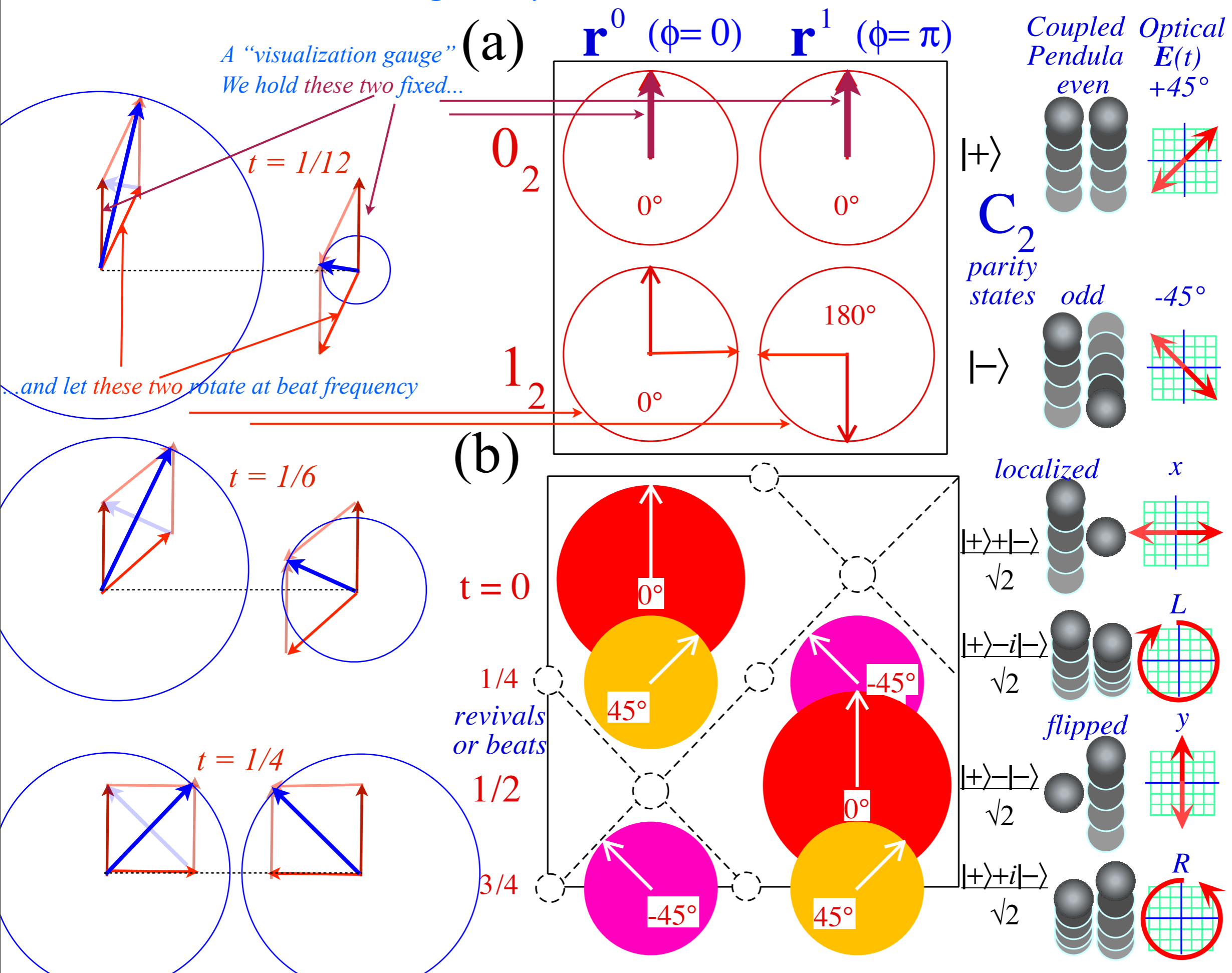
2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry

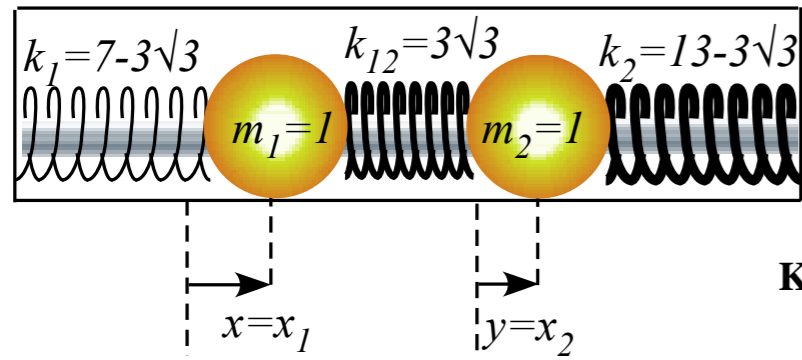


2D-HO beats and mixed mode geometry



→ *2D-HO eigensolution example with asymmetric (A-Type) symmetry*
Initial state projection, mixed mode beat dynamics with fluid phase

Spectral decomposition of 2D-HO mode dynamics for lower symmetry

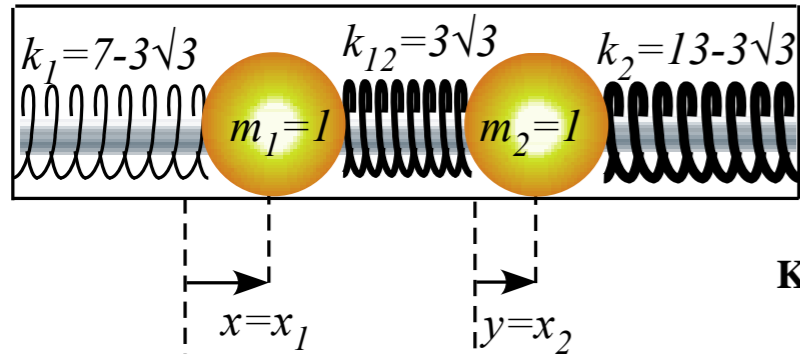


$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

$$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$$

$$\text{Trace}(\mathbf{K}) = 7 + 13 = 20$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



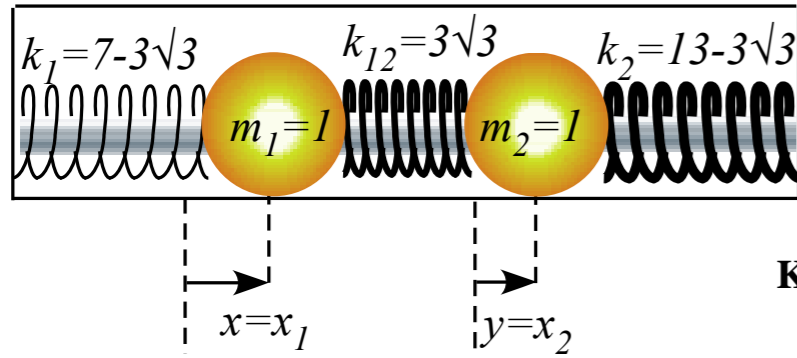
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The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0$

$$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$$

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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

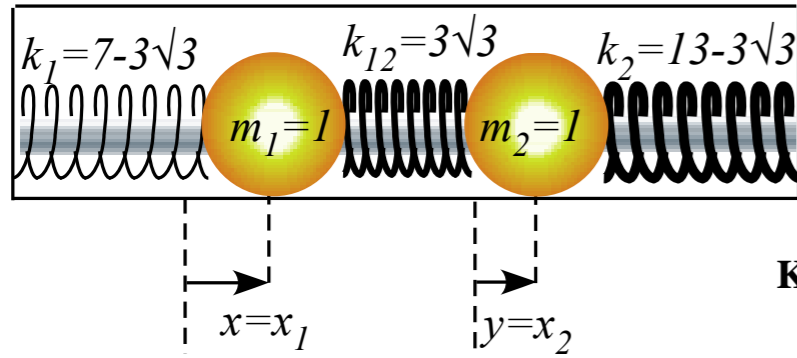
The \mathbf{K} secular equation

$$K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K - 4)(K - 16)$$

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Spectral decomposition of 2D-HO mode dynamics for lower symmetry

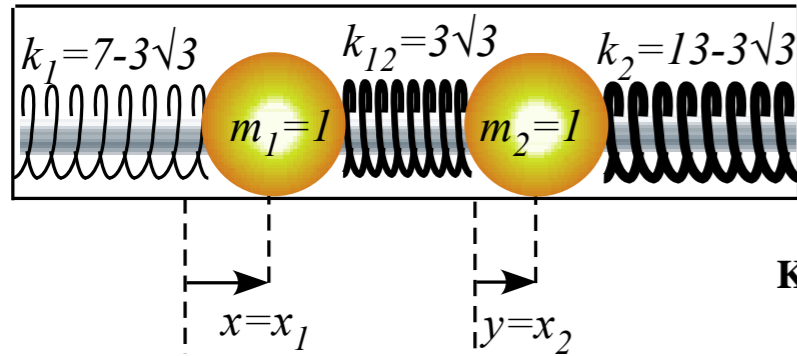


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Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 4, \quad K_2 = \omega_0^2(\epsilon_2) = 16,$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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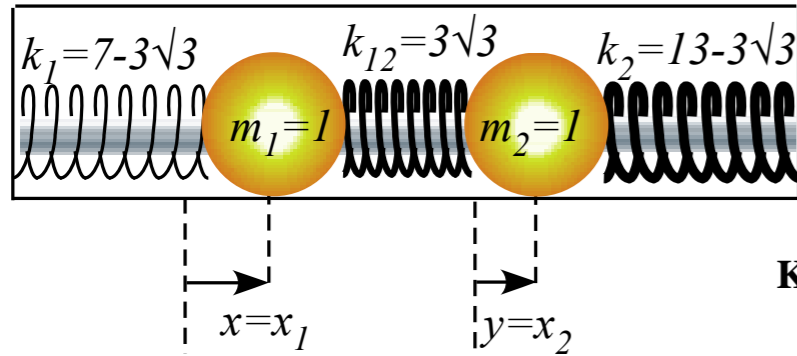
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Eigen-projectors \mathbf{P}_k

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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Eigen-projectors \mathbf{P}_k

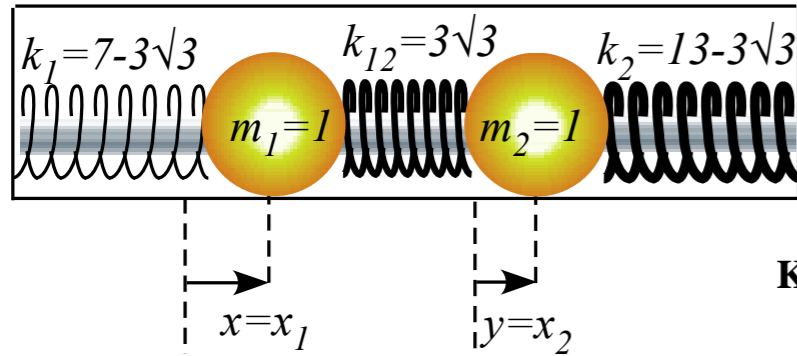
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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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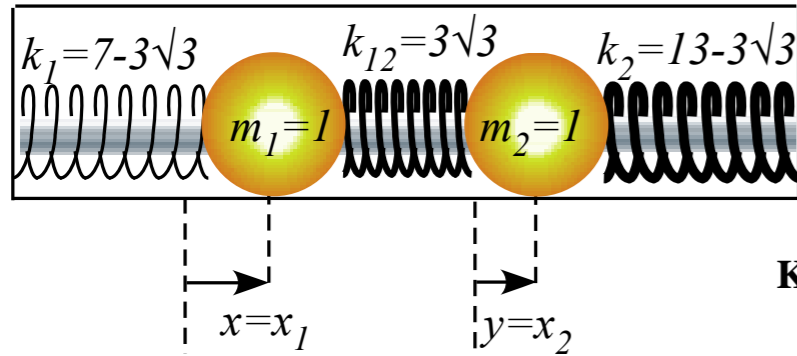
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Eigenbra vectors: $\langle\epsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\epsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

2D-HO eigensolution example with asymmetric (A-Type) symmetry

➔ *Initial state projection, mixed mode beat dynamics with fluid phase*

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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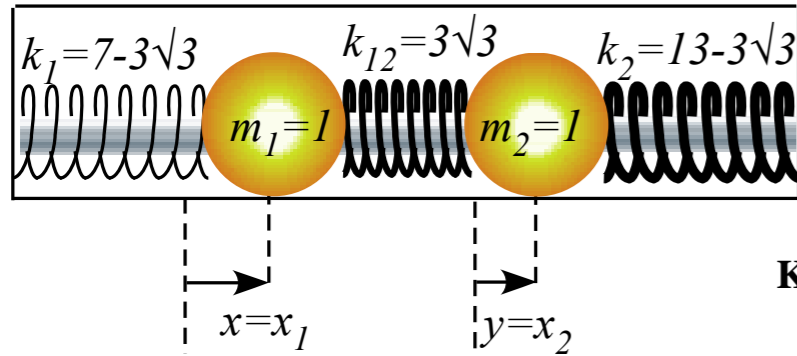
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Spectral decomposition of initial state $\mathbf{x}(0) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_1 + \mathbf{P}_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3} & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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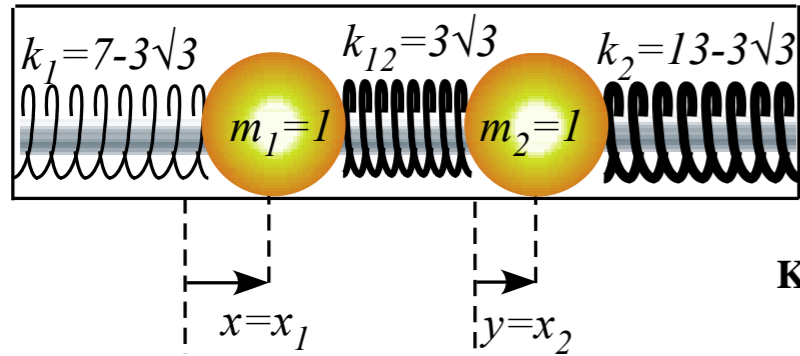
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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3} \end{pmatrix}$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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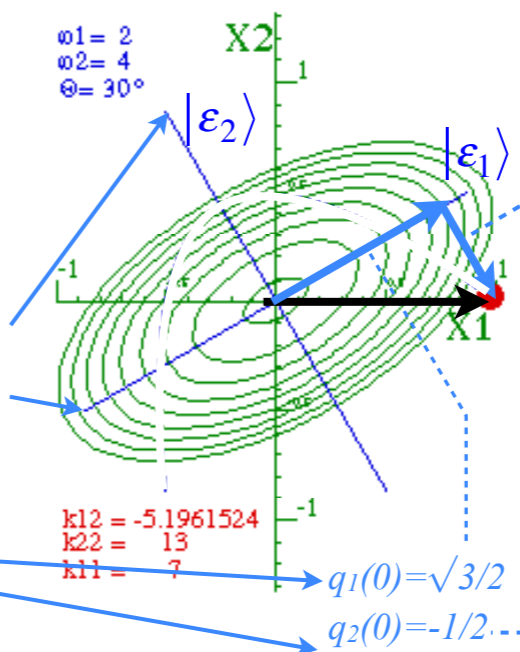
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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

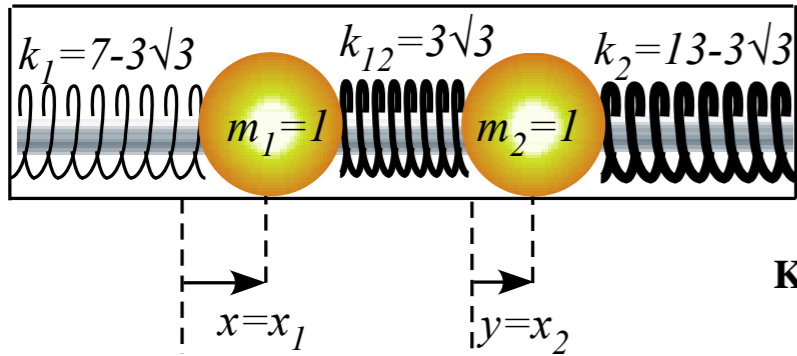


(Note projection of $\mathbf{x}(0)$ onto eigen-axes)

$k_{12} = -5.1961524$
 $k_{22} = 13$
 $k_{11} = 7$

$q_1(0) = \sqrt{3}/2$
 $q_2(0) = -1/2$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 & \\ & 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 7 - 4 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 4 \end{pmatrix}}{16 - 4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4} = \begin{pmatrix} -1/2 & \\ & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\epsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_1 + \mathbf{P}_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

(Note projection of $\mathbf{x}(0)$ onto eigen-axes)

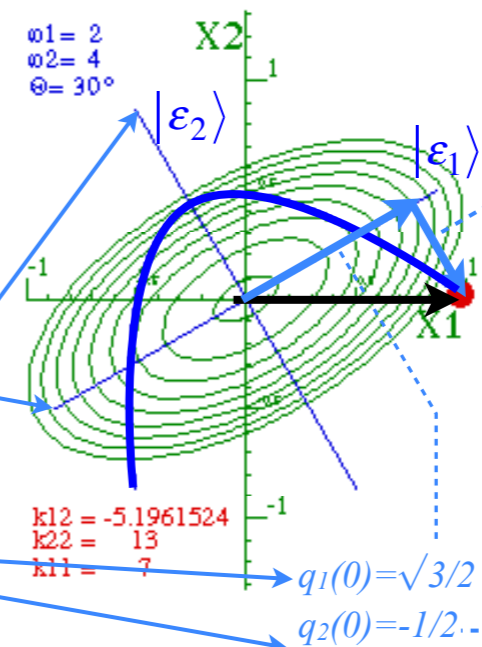
$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

Using $\cos 4t = 2 \cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2(t) = -\frac{1}{2} (2 \cos^2 2t - 1) = -\frac{4}{3} [q_1(t)]^2 + \frac{1}{2}$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example of a Tschebycheff Polynomial order 2



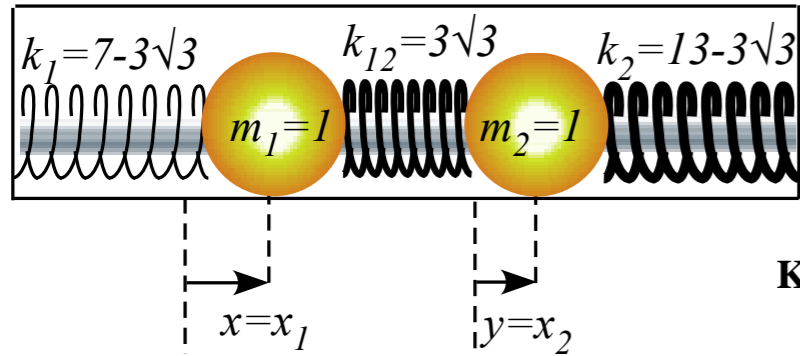
Pafnuty Chebyshev



Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshev, Tchebychev or Tchebycheff, or Tschebyshev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk
 Died: December 8, 1894, Saint Petersburg

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K - 4)(K - 16)$

$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$
 $\text{Trace}(\mathbf{K}) = 7 + 13 = 20$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 4$, $K_2 = \omega_0^2(\epsilon_2) = 16$,

Eigen-projectors \mathbf{P}_k

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$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 & \\ & 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

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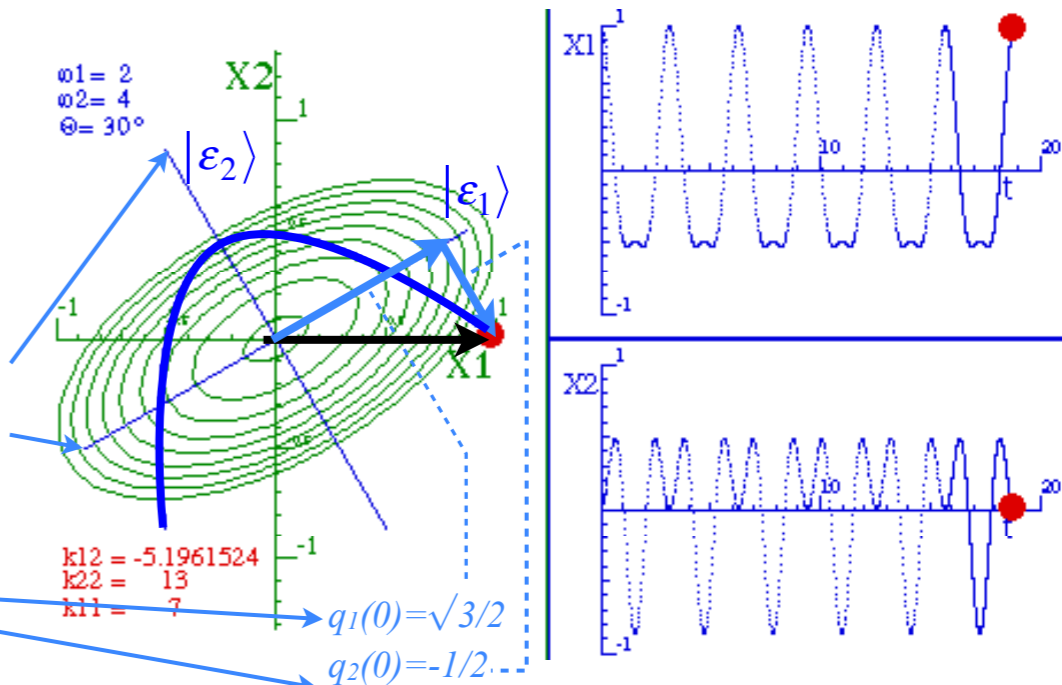


Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential ($V=\text{const.}$) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\epsilon_1) = 2.0$, $\omega_0(\epsilon_2) = 4.0$) and zero initial velocity.

➔ *ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$*
Hamilton-Pauli spinor symmetry (ABCD-Types)

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First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

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$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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Then start with classical Hamiltonian.

$$H_c = \frac{A}{2}(p_1^2 + x_1^2) + B(x_1x_2 + p_1p_2) + C(x_1p_2 - x_2p_1) + \frac{D}{2}(p_2^2 + x_2^2)$$

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*QM vs. Classical
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$$\begin{aligned} \ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A(Ax_1 + Bx_2 + Cp_2) - B(Bx_1 + Dx_2 - Cp_1) - C(Bp_1 + Dp_2 + Cx_1) \\ &= -(A^2 + B^2 + C^2)x_1 - (AB + BD)x_2 - C(A + D)p_2 \end{aligned}$$

$$\begin{aligned} \ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B(Ax_1 + Bx_2 + Cp_2) - D(Bx_1 + Dx_2 - Cp_1) + C(Ap_1 + Bp_2 - Cx_2) \\ &= -(AB + BD)x_1 - (B^2 + D^2 + C^2)x_2 + C(A + D)p_1 \end{aligned}$$

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$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For $C=0$ Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$H_c = \frac{A}{2}(p_1^2 + x_1^2) + B(x_1x_2 + p_1p_2) + C(x_1p_2 - x_2p_1) + \frac{D}{2}(p_2^2 + x_2^2)$$

Then Hamilton's equations of motion are the following.

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus **Classical 2DHO:** $\partial^2_t \mathbf{x} = -\mathbf{K} \cdot \mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\begin{aligned} \dot{x}_1 &= Ap_1 + Bp_2 - Cx_2 & \dot{p}_1 &= -Ax_1 - Bx_2 - Cp_2 \\ \dot{x}_2 &= Bp_1 + Dp_2 + Cx_1 & \dot{p}_2 &= -Bx_1 - Dx_2 + Cp_1 \end{aligned}$$

QM vs. Classical Equations are identical

$$\begin{aligned} \dot{x}_1 &= \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 & \dot{p}_1 &= -\frac{\partial H_c}{\partial x_1} = -(Ax_1 + Bx_2 + Cp_2) \\ \dot{x}_2 &= \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 & \dot{p}_2 &= -\frac{\partial H_c}{\partial x_2} = -(Bx_1 + Dx_2 - Cp_1) \end{aligned}$$

Finally a 2nd time derivative (Assume constant A, B, D , and **let $C=0$**) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle$

$$\begin{aligned} \ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A(Ax_1 + Bx_2 + Cp_2) - B(Bx_1 + Dx_2 - Cp_1) - C(Bp_1 + Dp_2 + Cx_1) \\ &= -(A^2 + B^2 + C^2)x_1 - (AB + BD)x_2 - C(A + D)p_2 \end{aligned}$$

$$\begin{aligned} \ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B(Ax_1 + Bx_2 + Cp_2) - D(Bx_1 + Dx_2 - Cp_1) + C(Ap_1 + Bp_2 - Cx_2) \\ &= -(AB + BD)x_1 - (B^2 + D^2 + C^2)x_2 + C(A + D)p_1 \end{aligned}$$

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For $C=0$ Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

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Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like “square-root” of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle}$