AMOP Lecture 12

Thur 3.22 - Fri 3.23 2014

Based on QTCA Lectures 7, 23-25
Group Theory in Quantum Mechanics

Spectral Analysis of U(2) Operators

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22, PSDS - Ch. 8)

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver & three famous 2-state systems

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic
Spinor vector algebra like complex vector algebra
Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem vs Lorentz)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space
2D Spinor vs 3D vector rotation
NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*
How probability ψ-waves and flux ψ-waves evolved
*Properties of amplitude ψ*ψ-squares*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$
Spin-1 (3D-real vector) case
Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

Fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

From QTCA Lectures 8-9,

➡ *Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver & three famous 2-state systems* ⬅

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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2D Spinor vs 3D vector rotation

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Polarization ellipse and spinor state dynamics

Review: How symmetry groups become eigen-solvers

Suppose you need to diagonalize a complicated operator **K** and knew that **K** commutes with some other operators **G** and **H** for which irreducible projectors are more easily found.

$$\mathbf{KG} = \mathbf{GK} \text{ or } \mathbf{G}^\dagger\mathbf{KG} = \mathbf{K} \text{ or } \mathbf{GKG}^\dagger = \mathbf{K}$$

$$\mathbf{KH} = \mathbf{HK} \text{ or } \mathbf{H}^\dagger\mathbf{KH} = \mathbf{K} \text{ or } \mathbf{HKH}^\dagger = \mathbf{K}$$

(Here assuming *unitary* $\mathbf{G}^\dagger = \mathbf{G}^{-1}$ and $\mathbf{H}^\dagger = \mathbf{H}^{-1}$.)

This means **K** is *invariant* to the transformation by **G** and **H** and all their products $\mathbf{GH}$, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

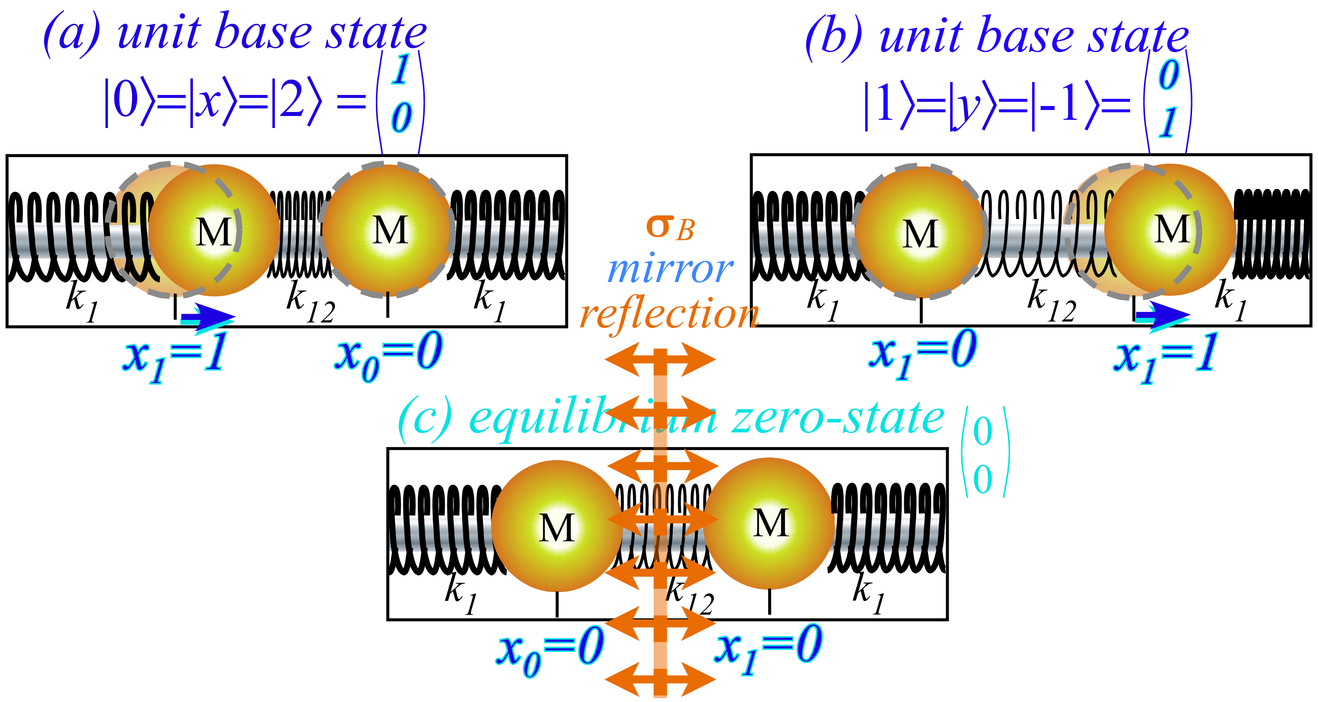
The group $\mathscr{G}_{\mathbf{K}} = \{\mathbf{1}, \mathbf{G}, \mathbf{H}, ..\}$ so formed by such operators is called a *symmetry group* for **K**.

In certain ideal cases a **K**-matrix $\langle\mathbf{K}\rangle$ is a linear combination of matrices $\langle\mathbf{1}\rangle$,$\langle\mathbf{G}\rangle$,$\langle\mathbf{H}\rangle$,... from $\mathscr{G}_{\mathbf{K}}$.
Then spectral resolution of $\{\langle\mathbf{1}\rangle,\langle\mathbf{G}\rangle,\langle\mathbf{H}\rangle,..\}$ also resolves $\langle\mathbf{K}\rangle$.

We will study ideal cases first. More general cases are built from this idea.

Review: C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0, x_1\}$



2D HO Matrix operator equations

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M} \\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\;|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0, x_1\} \rightarrow \{x_1, x_2\}$

C_2 (Bilateral σ_B reflection) symmetry conditions:

$K_{11} \equiv K \equiv K_{22}$ and: $K_{12} \equiv k \equiv K_{12} = -k_{12}$ (Let: $M = 1$)

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} K & k \\ k & K \end{pmatrix} = K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{K} = K\cdot\mathbf{1} + k\cdot\sigma_B$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Symmetry *product table* gives C_2 group representations in *group basis* $\{|0\rangle = \mathbf{1}|0\rangle \equiv |\mathbf{1}\rangle,\ |1\rangle = \sigma_B|0\rangle \equiv |\sigma_B\rangle\}$

$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^{\pm}$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Review: C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Diagonalizing transformation (D-tran) of K-matrix:

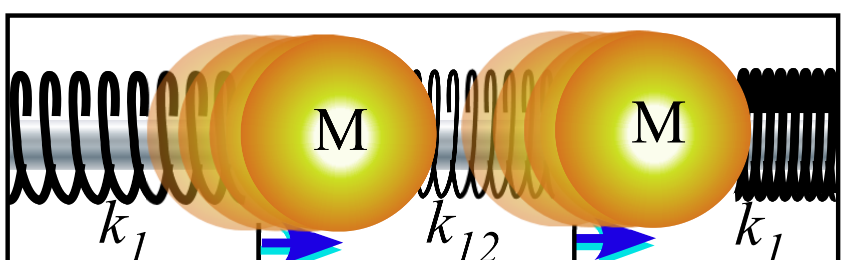
$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K-k_{12}$, $\quad\varepsilon^-(\mathbf{K}) = K+k_{12}$

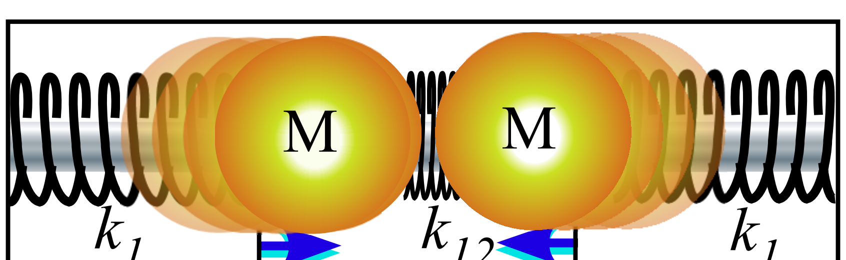
$= k_1$ $\quad\quad = k_1+2k_{12}$

Even mode $|+\rangle=|0_2\rangle = \begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=1/\sqrt{2}$

Odd mode $|-\rangle=|1_2\rangle = \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$

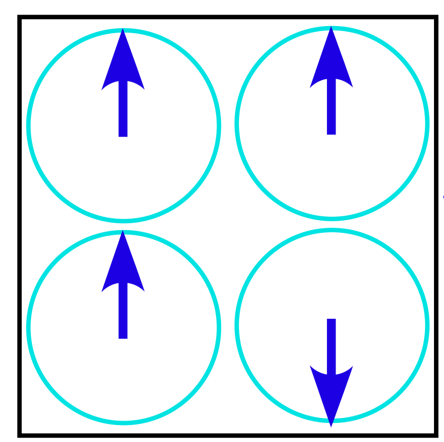


$x_0=1/\sqrt{2}$ $\quad x_1=-1/\sqrt{2}$

C_2 *mode phase character tables*

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number or "momentum"

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

Review: C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are* ***un****coupled in* $\{+,-\}$-basis

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

$$\begin{pmatrix}\langle x_1|\ddot{\mathbf{x}}\rangle\\ \langle x_2|\ddot{\mathbf{x}}\rangle\end{pmatrix}=-\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle\\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{x}\rangle\\ \langle x_2|\mathbf{x}\rangle\end{pmatrix}$$

$$\begin{pmatrix}\ddot{x}_+\\ \ddot{x}_-\end{pmatrix}=-\begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}\begin{pmatrix}x_+\\ x_-\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

$$\begin{pmatrix}\langle +|\ddot{\mathbf{x}}\rangle\\ \langle -|\ddot{\mathbf{x}}\rangle\end{pmatrix}=-\begin{pmatrix}\langle +|\mathbf{K}|+\rangle & \cancel{\langle +|\mathbf{K}|-\rangle}\\ \cancel{\langle -|\mathbf{K}|+\rangle} & \langle -|\mathbf{K}|-\rangle\end{pmatrix}\begin{pmatrix}\langle +|\mathbf{x}\rangle\\ \langle -|\mathbf{x}\rangle\end{pmatrix}$$

C_2 Symmetric 2DHO ***un****coupled dynamics*

Eigenbra vectors: $\langle +| = \left(\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right),\quad \langle -| = \left(\frac{1}{\sqrt{2}}\ \ \frac{-1}{\sqrt{2}}\right)$

Each mode runs independently

$$\begin{pmatrix}M\ddot{x}_+ + (k_1)x_+\\ M\ddot{x}_- + (k_1+2k_{12})x_-\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$(+)$*-mode at frequency* $\omega_+=\sqrt{(k_1/M)}$

$(-)$*-mode at frequency* $\omega_-=\sqrt{(k_1+2k_{12})/M}$

Eigenket vectors: $|+\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix},\quad |-\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}$

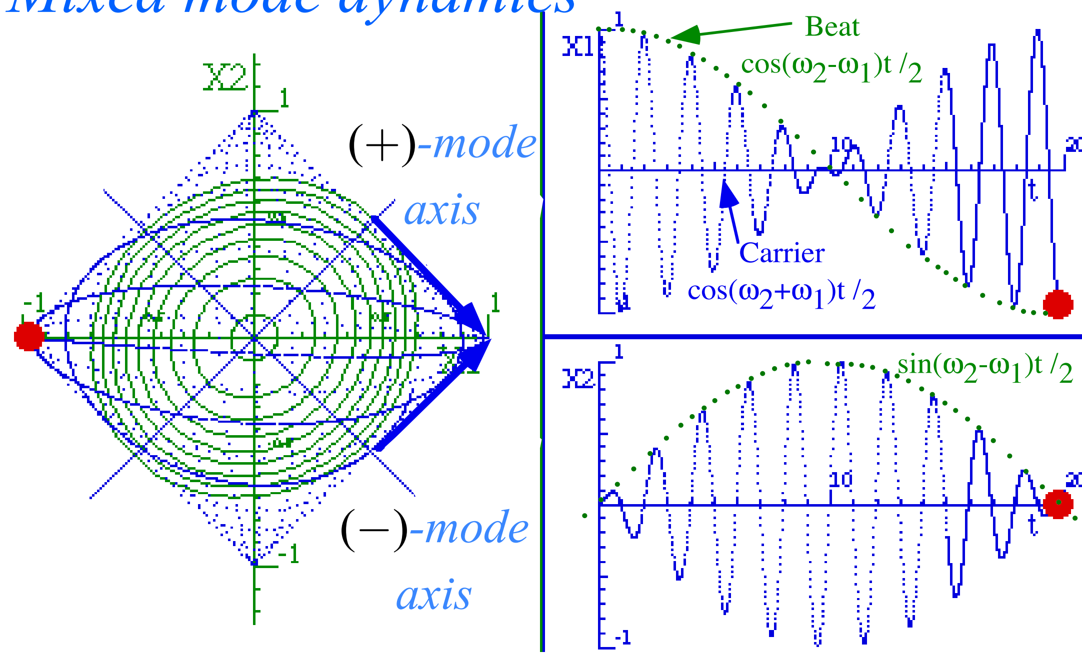
Spectral decomposition of initial state $\mathbf{x}(0)=(x_1\ \ x_2)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_+ + \mathbf{P}_-)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\left(\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right)\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\left(\frac{1}{\sqrt{2}}\ \ \frac{-1}{\sqrt{2}}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$\mathbf{x}(0)=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\left(\frac{1}{\sqrt{2}}\right)+\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}|+\rangle+\frac{1}{\sqrt{2}}|-\rangle=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}+\begin{pmatrix}\frac{1}{2}\\ \frac{-1}{2}\end{pmatrix}$$

$$\text{so: } \mathbf{x}(t)=e^{-i\omega_+ t}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}+e^{-i\omega_- t}\begin{pmatrix}\frac{1}{2}\\ \frac{-1}{2}\end{pmatrix}$$

Mixed mode dynamics



100% AM modulation results

$$\frac{e^{ia}+e^{ib}}{2}=e^{i\frac{a+b}{2}}\,\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}=e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

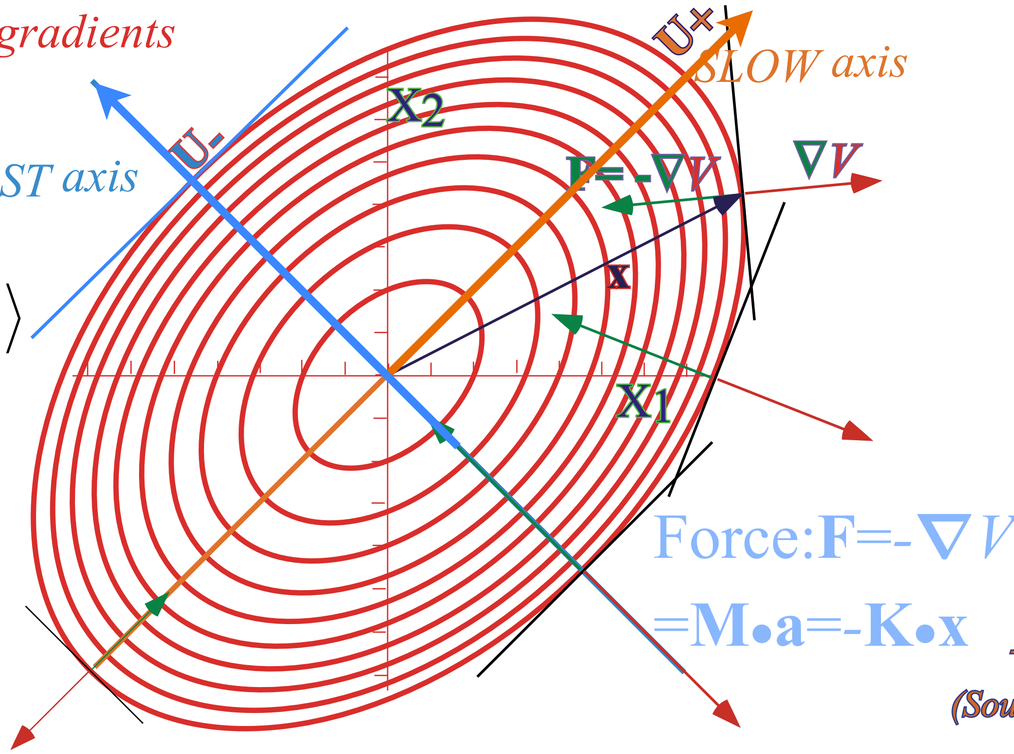
$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}\dfrac{e^{-i\omega_+ t}+e^{-i\omega_- t}}{2}\\ \dfrac{e^{-i\omega_+ t}-e^{-i\omega_- t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_+ +\omega_-)}{2}t}}{2}\begin{pmatrix}e^{-i\frac{(\omega_+-\omega_-)}{2}t}+e^{i\frac{(\omega_+-\omega_-)}{2}t}\\ e^{-i\frac{(\omega_+-\omega_-)}{2}t}-e^{i\frac{(\omega_+-\omega_-)}{2}t}\end{pmatrix}=e^{-i\frac{(\omega_+ +\omega_-)}{2}t}\begin{pmatrix}\cos\dfrac{(\omega_- -\omega_+)t}{2}\\ i\sin\dfrac{(\omega_- -\omega_+)t}{2}\end{pmatrix}$$

Note the i *phase*

Review: 2D HO potential energy $V(x_1, x_2)$ form defines elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k+k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

Not <u>most</u> directions!

Only extremal axes work. (major or minor axes)

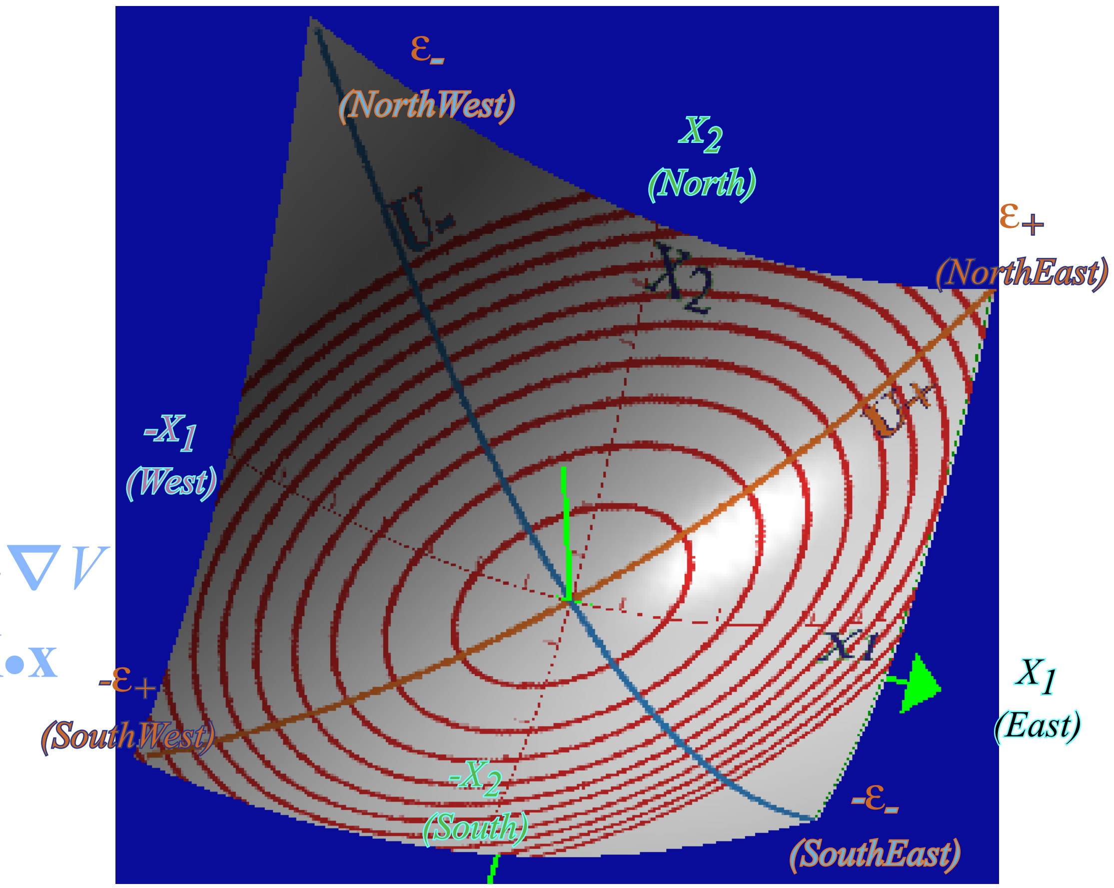
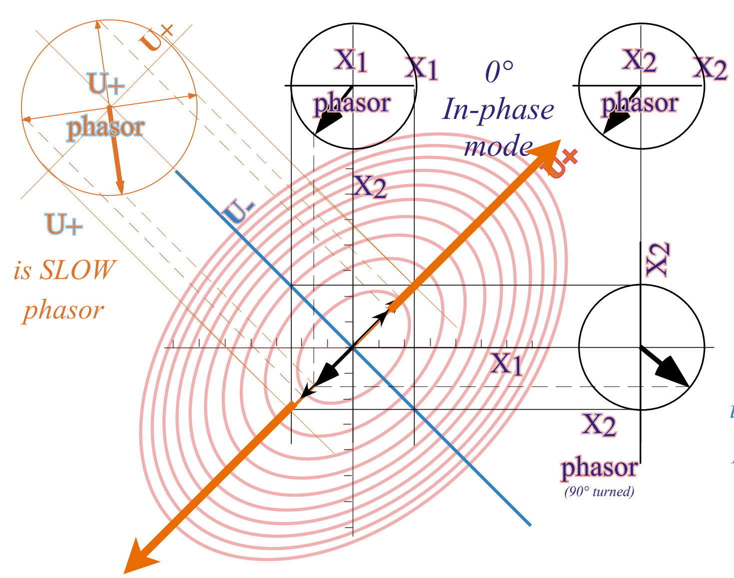
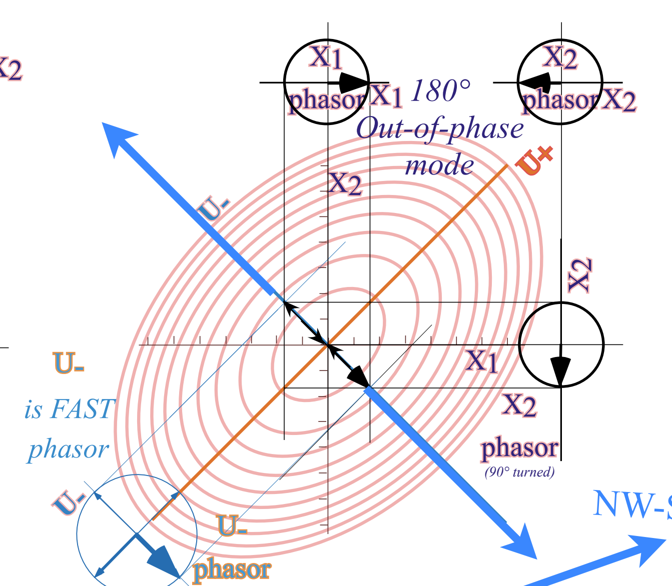


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode



With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

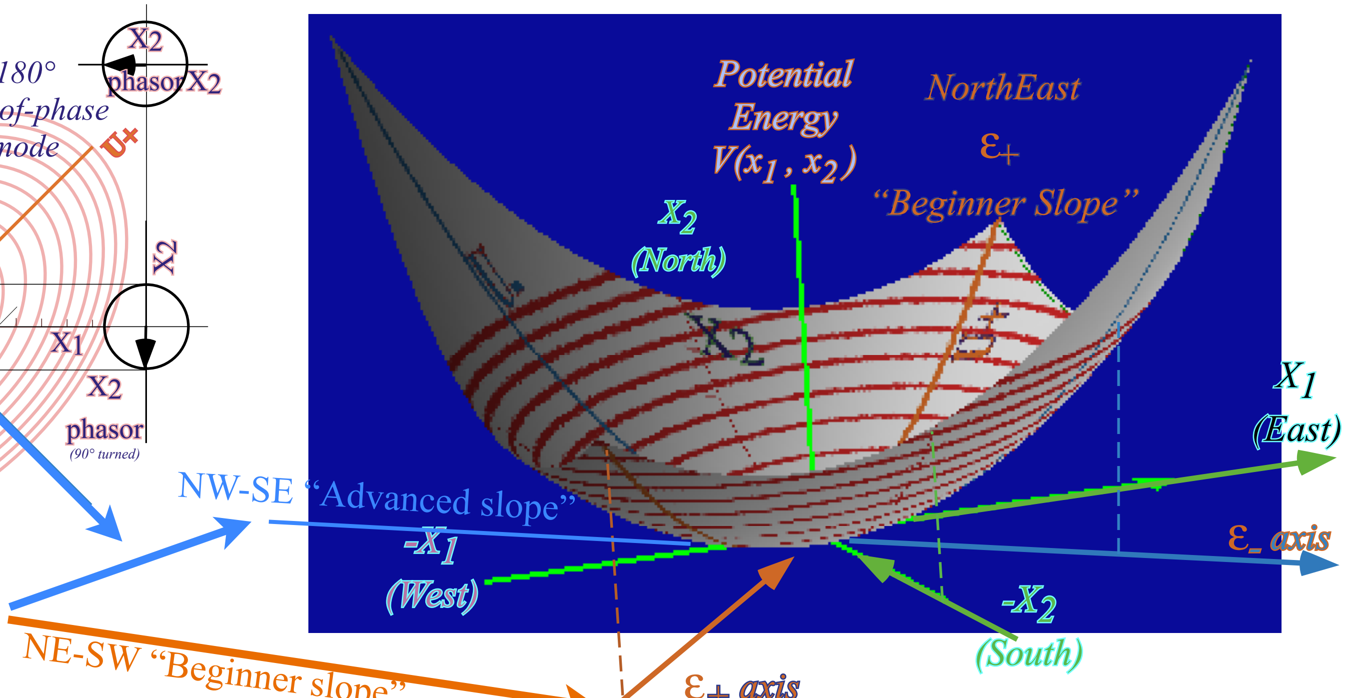
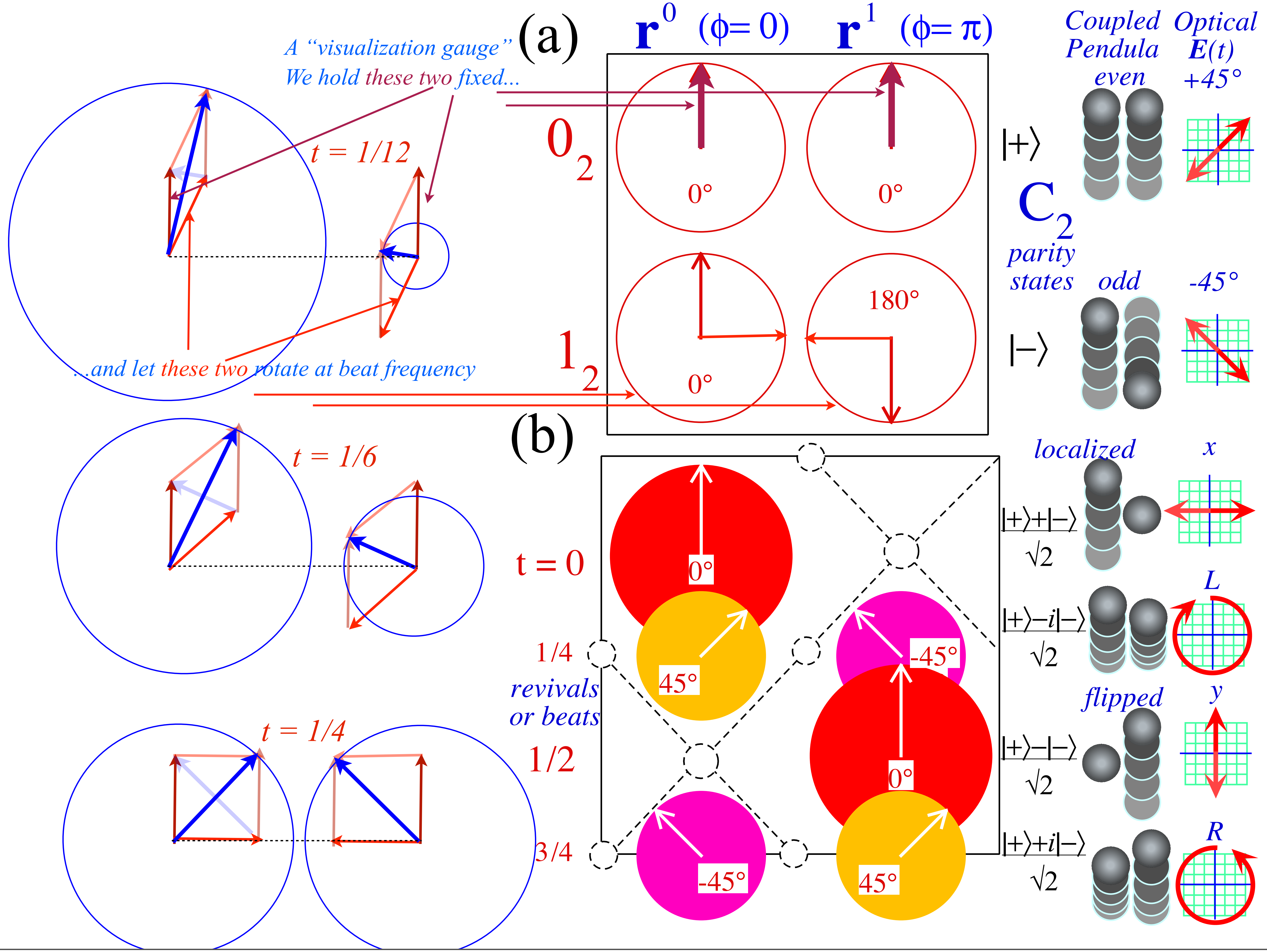


Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

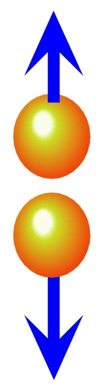
Review: 2D-HO beats and mixed mode geometry



Review: *Some of the most famous 2-state systems and their two-complex-component coordinates.*

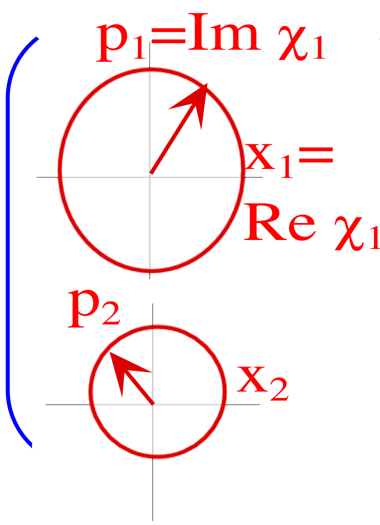
(a) Electron Spin-1/2-Polarization

Spin-up $|1\rangle = |\uparrow\rangle$



Spin-dn $|2\rangle = |\downarrow\rangle$

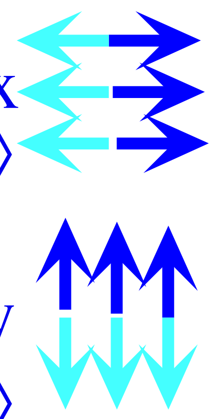
$$|\chi\rangle = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} =$$



$$= |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

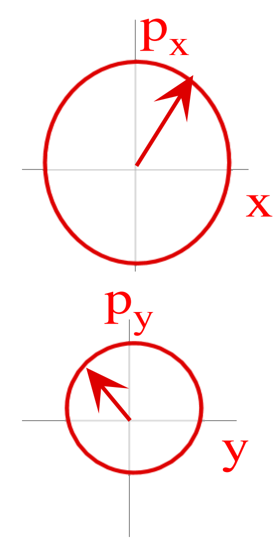
(b) Photon Spin-1-Polarization

Plane-x $|1\rangle = |x\rangle$



Plane-y $|2\rangle = |y\rangle$

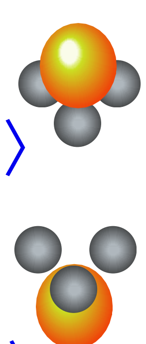
$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} =$$



$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

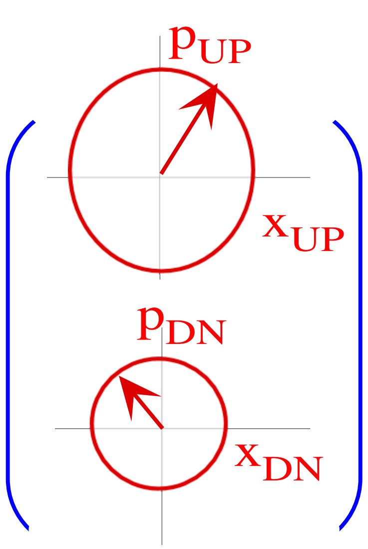
(c) Ammonia (NH_3) Inversion States

N-UP $|1\rangle = |UP\rangle$



N-DN $|2\rangle = |DN\rangle$

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} =$$



$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Fig. 10.5.1
QTCA Unit 3 Chapter 10

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem vs Lorentz)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

*NMR Hamiltonian: 3D Spin Moment **m** in **B** field*

How probability ψ-waves and flux ψ-waves evolved

Properties of amplitude $\psi^*\psi$-squares

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$

Polarization ellipse and spinor state dynamics

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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First start with 2-by-2 Hermitian (self-conjugate) matrix

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Both have 4 parameters *($2^2 = 2+2$)*

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For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\equiv\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}=-\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

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Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of ***real*** 1st-order differential equations.

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1 x_2 + p_1 p_2\right) + C\left(x_1 p_2 - x_2 p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot x_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot p_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

$$\dot x_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot p_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

$$\dot x_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot p_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot x_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot p_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Finally a 2nd time derivative (Assume *constant* A, B, D, and *let* $C=0$) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\ddot x_1 = A\dot p_1 + B\dot p_2 - C\dot x_2$$
$$= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right)$$
$$= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2$$

$$\ddot x_2 = B\dot p_1 + D\dot p_2 + C\dot x_1$$
$$= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right)$$
$$= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1$$

$$\begin{pmatrix} \ddot x_1 \\ \ddot x_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For $C=0$ Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot x_1 \\ \ddot x_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

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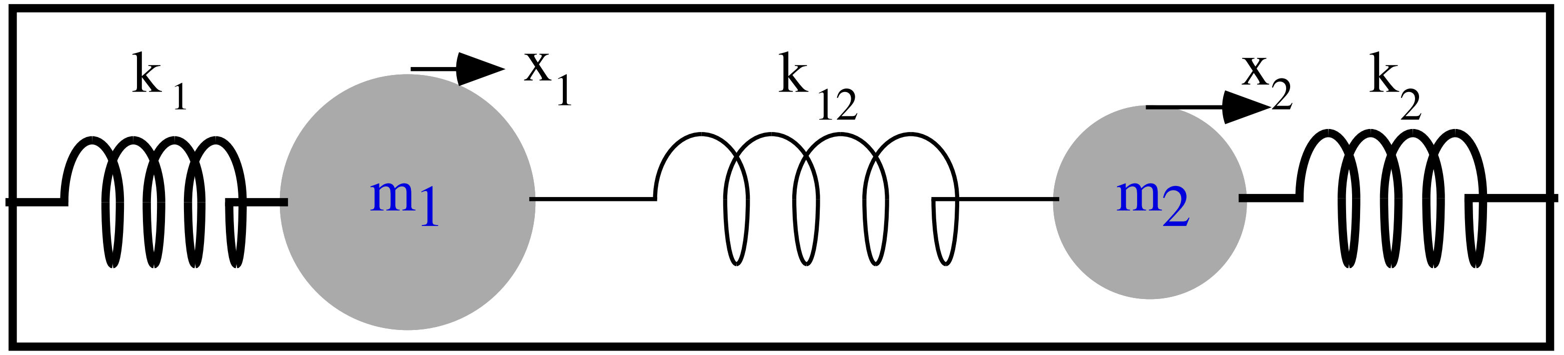
$$-\ddot{x}_1 = K_{11}x_1 + K_{12}x_2$$
$$-\ddot{x}_2 = K_{21}x_1 + K_{22}x_2$$

(a)



$$m_1K_{11} = A^2 + B^2 = k_1 + k_{12}, \quad m_1K_{12} = AB + BD = -k_{12},$$
$$m_2K_{21} = AB + BD = -k_{12}, \quad m_2K_{22} = B^2 + D^2 = k_2 + k_{12}.$$

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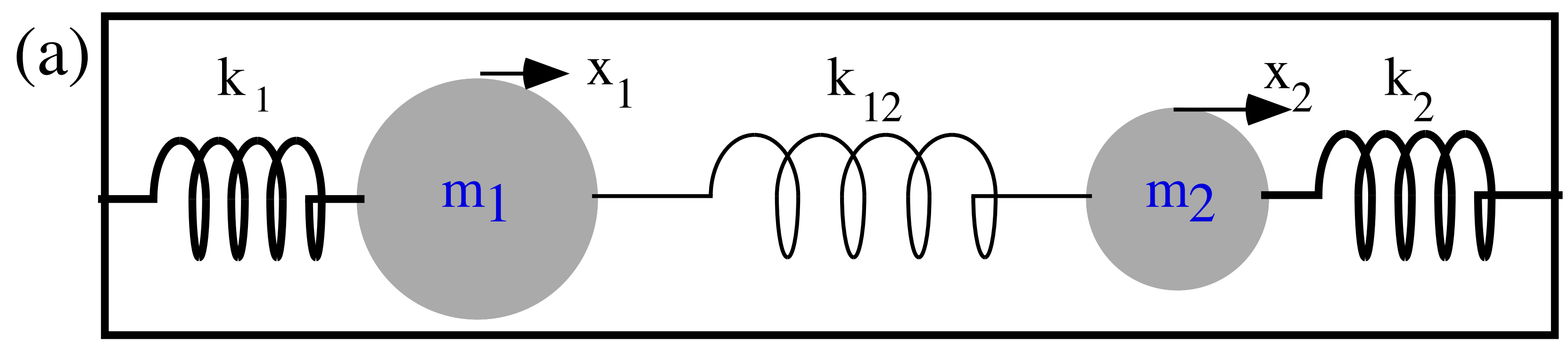
General case for $C\neq 0$

$$-\ddot{x}_1 = K_{11}x_1 + K_{12}x_2$$
$$-\ddot{x}_2 = K_{21}x_1 + K_{22}x_2$$



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Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem vs Lorentz)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ *in* $\mathbf{B}$ *field*

How probability ψ-waves and flux ψ-waves evolved

*Properties of amplitude ψ*ψ-squares*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*

(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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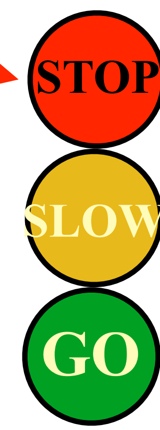
Motivation for coloring scheme:
The Traffic Signal



Standing waves

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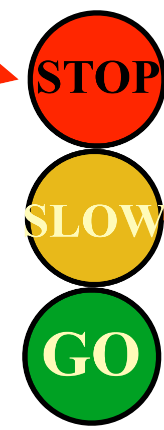

Standing waves *C=0*

C≠0: Moving waves or "Galloping" waves

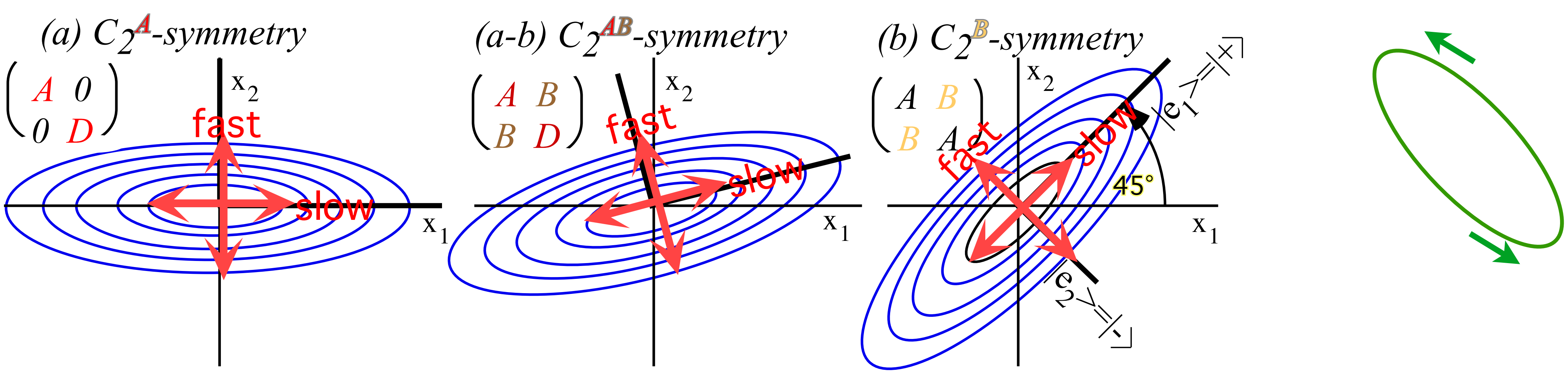

Fig. 10.1.2 Potentials for (a) $C_2{}^A$-asymmetric-diagonal, (ab) $C_2{}^{AB}$-mixed , (b) $C_2{}^B$-bilateral (c) $C_2{}^C$-cicular U(2)system.

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Standing waves *C=0*

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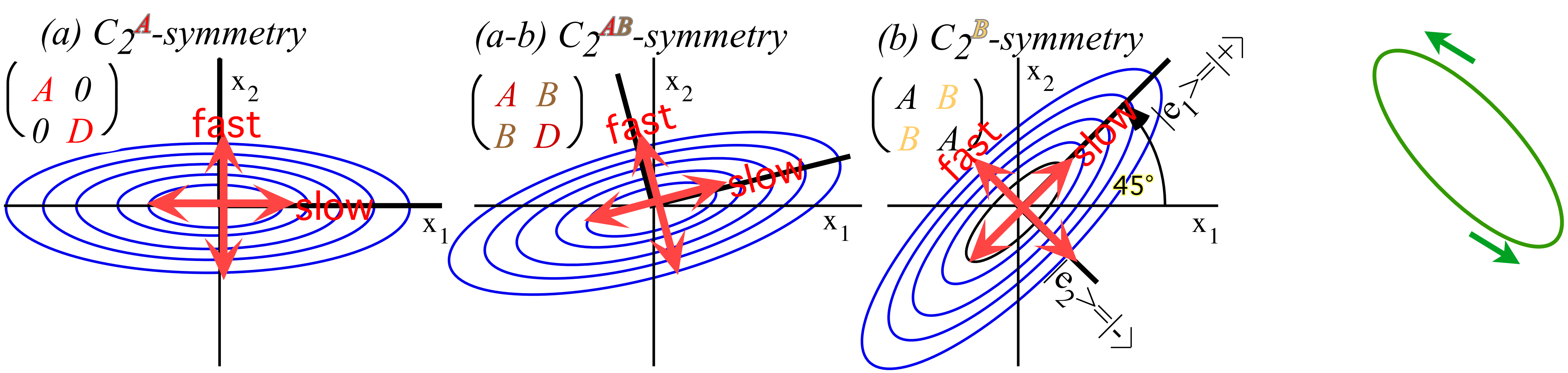

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Standing waves $C=0$

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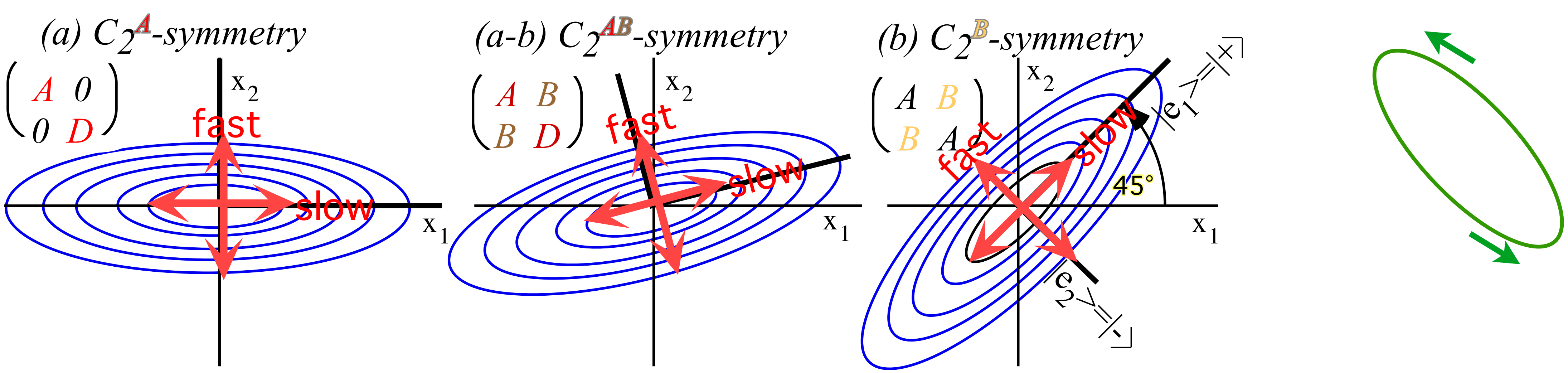

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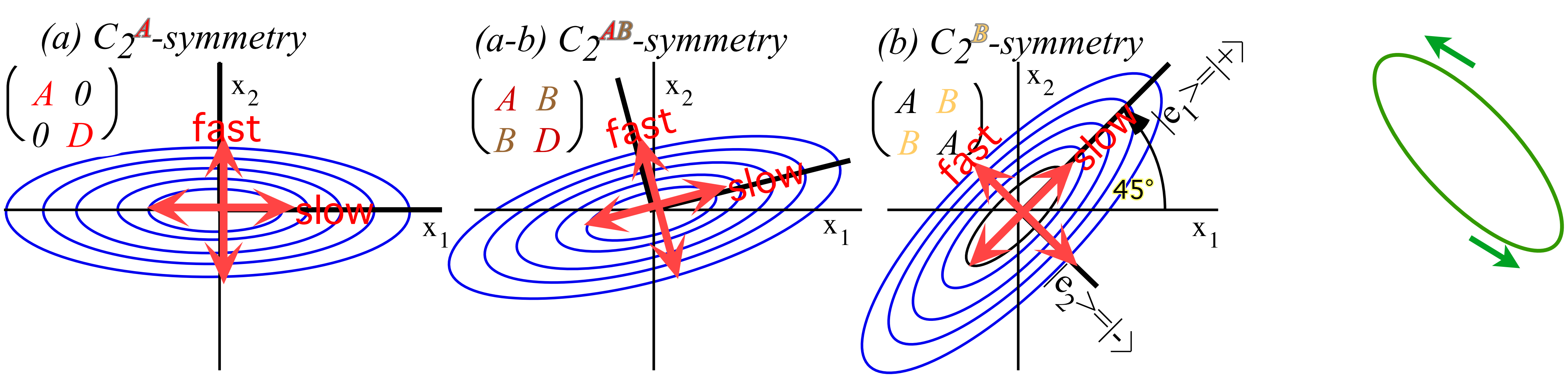

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Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X^2=\sigma_Y^2=\sigma_Z^2=+\mathbf{1}$) (Each makes a cyclic C_2 group $C_2^A=\{\mathbf{1},\sigma_A\}$, $C_2^B=\{\mathbf{1},\sigma_B\}$, or $C_2^C=\{\mathbf{1},\sigma_C\}$.)

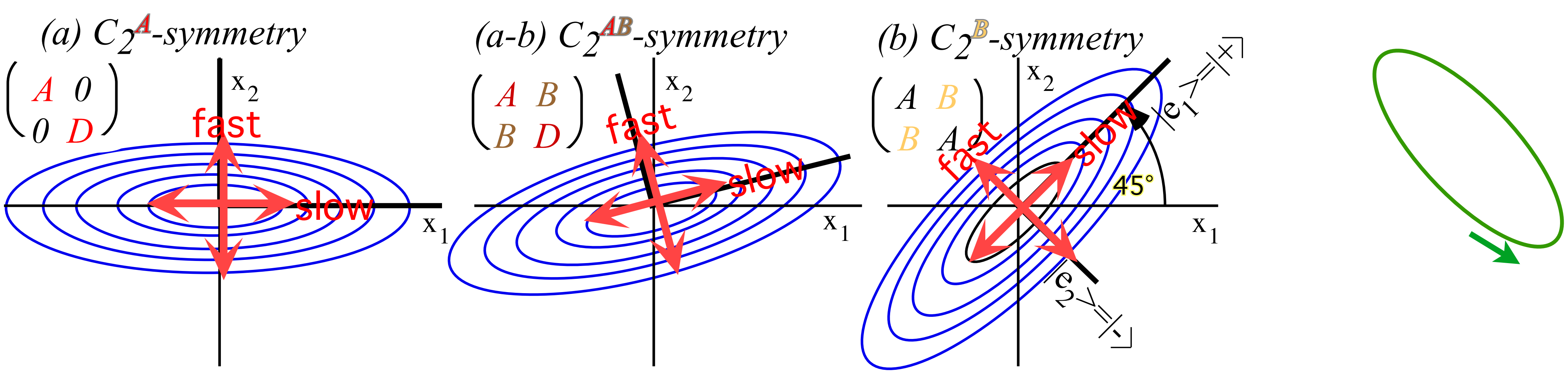

Fig. 10.1.2 Potentials for (a) $C_2{}^A$-asymmetric-diagonal, (ab) $C_2{}^{AB}$-mixed , (b) $C_2{}^B$-bilateral (c) $C_2{}^C$-cicular U(2)system.

Review: How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

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- Spinor arithmetic like complex arithmetic
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Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ in $\mathbf{B}$ field

How probability ψ-waves and flux ψ-waves evolved

Properties of amplitude $\psi^*\psi$-squares

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

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3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

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OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

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$$= \begin{matrix} a_X\sigma_X a_X\sigma_X & +a_X\sigma_X a_Y\sigma_Y & +a_X\sigma_X a_Z\sigma_Z \\ +a_Y\sigma_Y a_X\sigma_X & +a_Y\sigma_Y a_Y\sigma_Y & +a_Y\sigma_Y a_Z\sigma_Z \\ +a_Z\sigma_Z a_X\sigma_X & +a_Z\sigma_Z a_Y\sigma_Y & +a_Z\sigma_Z a_Z\sigma_Z \end{matrix} = \begin{matrix} a_X a_X\sigma_X\sigma_X & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ +a_Y a_X\sigma_Y\sigma_X & +a_Y a_Y\sigma_Y\sigma_Y & +a_Y a_Z\sigma_Y\sigma_Z \\ +a_Z a_X\sigma_Z\sigma_X & +a_Z a_Y\sigma_Z\sigma_Y & +a_Z a_Z\sigma_Z\sigma_Z \end{matrix}$$

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$$\sigma_Z \cdot \sigma_X: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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To finish we need another symmetry property called *anti-commutation*: $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$, *etc.*

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So: $\sigma_a^2 = \mathbf{1}$

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σ-products do dot • *and* cross × products by symmetries: $\sigma_X\sigma_Y = i\sigma_Z = -\sigma_Y\sigma_X$, $\quad\sigma_Z\sigma_X = i\sigma_Y = -\sigma_X\sigma_Z$, $\quad\sigma_Y\sigma_Z = i\sigma_X = -\sigma_Z\sigma_Y$

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Write the product in Gibbs notation. (This is where Gibbs *got* his {**i,j,k**} notation!)

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(Recall complex variable result.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
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Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ *in* $\mathbf{B}$ *field*

How probability ψ-waves and flux ψ-waves evolved

Properties of amplitude $\psi^*\psi$*-squares*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, *Balance* $S_B = S_X$, *and Chirality* $S_C = S_Y$

Polarization ellipse and spinor state dynamics

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Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

Unit spinor vector

$$\sigma_\varphi = \frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi} = (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$$

$$= \frac{\varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}}$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

ABCD Time evolution operator

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\quad \sigma_B = \sigma_X$ $\quad \sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 \cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0 \cdot t} \text{ where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

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Unit spinor vector

$$\sigma_\varphi = \frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi} = (\boldsymbol{\sigma}\bullet\hat{\varphi}) = \frac{\varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}}$$

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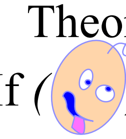
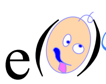
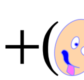
The Crazy Thing Theorem:

If ()^2= -**1**

Then:

e$^{(}$  $^{)\varphi}$ =**1**cosφ +()sinφ

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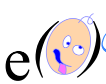
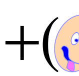
The Crazy Thing Theorem:

If ()^2= -**1**

Then:

e()$^\varphi$ =**1**cosφ +()sinφ

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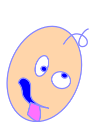
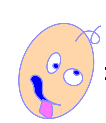
$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \ - \ i\ \sigma_\varphi \sin\varphi$

Here:  = -*i*

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem vs Lorentz)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ *in* $\mathbf{B}$ *field*

How probability ψ-waves and flux ψ-waves evolved

*Properties of amplitude ψ*ψ-squares*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

"Crazy-Thing"-Theorem vs Lorentz

Use projectors to derive regular rotations and Lorentz rotations

Symmetry ***product table*** gives C_2 group representations in ***group basis*** $\{|0\rangle = \mathbf{1}|0\rangle \equiv |\mathbf{1}\rangle\ ,\ |1\rangle = \boldsymbol{\sigma}_B|0\rangle \equiv |\boldsymbol{\sigma}_B\rangle\}$

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or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:

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Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

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Regular rotation $R_B(\varphi) = e^{-i\varphi\boldsymbol{\sigma}_B}$

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with eigenvalues:

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+,\mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^\pm$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Regular rotation $R_B(\varphi)=e^{-i\varphi\sigma_B}$

$$\begin{aligned} R_B(\varphi) = e^{-i\varphi\sigma_B} &= e^{-i\varphi\chi^+(\sigma_B)}\ \mathbf{P}^+ + e^{-i\varphi\chi^-(\sigma_B)}\ \mathbf{P}^- \\ &= e^{-i\varphi(+1)}\ \mathbf{P}^+ + e^{-i\varphi(-1)}\ \mathbf{P}^- \\ &= e^{-i\varphi}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{+i\varphi}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{-i\varphi}+e^{+i\varphi} & e^{-i\varphi}-e^{+i\varphi} \\ e^{-i\varphi}-e^{+i\varphi} & e^{-i\varphi}+e^{+i\varphi} \end{pmatrix} \end{aligned}$$

Calculation agrees with "Crazy-thing" Theorem

$$= \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} = \mathbf{1}\cos\varphi - i\boldsymbol{\sigma}_B\sin\varphi$$

"Crazy-Thing"-Theorem vs Lorentz

Use projectors to derive regular rotations and Lorentz rotations

Symmetry *product table* gives C_2 group representations in *group basis* $\{|0\rangle = \mathbf{1}|0\rangle \equiv |\mathbf{1}\rangle\ ,\ |1\rangle = \sigma_B|0\rangle \equiv |\sigma_B\rangle\}$

$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

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$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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Regular rotation $R_B(\varphi) = e^{-i\varphi\sigma_B}$

$$\begin{aligned} R_B(\varphi) = e^{-i\varphi\sigma_B} &= e^{-i\varphi\chi^+(\sigma_B)}\ \mathbf{P}^+ + e^{-i\varphi\chi^-(\sigma_B)}\ \mathbf{P}^- \\ &= e^{-i\varphi(+1)}\ \mathbf{P}^+ + e^{-i\varphi(-1)}\ \mathbf{P}^- \\ &= e^{-i\varphi}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{+i\varphi}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{-i\varphi}+e^{+i\varphi} & e^{-i\varphi}-e^{+i\varphi} \\ e^{-i\varphi}-e^{+i\varphi} & e^{-i\varphi}+e^{+i\varphi} \end{pmatrix} \end{aligned}$$

Calculation agrees with "Crazy-thing" Theorem

$$= \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} = \mathbf{1}\cos\varphi - i\sigma_B\sin\varphi$$

Lorentz rotation $L_B(\rho) = e^{-\rho\sigma_B}$

$$\begin{aligned} L_B(\rho) = e^{-\rho\sigma_B} &= e^{-\rho\chi^+(\sigma_B)}\ \mathbf{P}^+ + e^{-\rho\chi^-(\sigma_B)}\ \mathbf{P}^- \\ &= e^{-\rho(+1)}\ \mathbf{P}^+ + e^{-\rho(-1)}\ \mathbf{P}^- \\ &= e^{-\rho}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{+\rho}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{-\rho}+e^{+\rho} & e^{-\rho}-e^{+\rho} \\ e^{-\rho}-e^{+\rho} & e^{-\rho}+e^{+\rho} \end{pmatrix} \\ &= \begin{pmatrix} \cosh\rho & -\sinh\rho \\ -\sinh\rho & \cosh\rho \end{pmatrix} = \mathbf{1}\cosh\rho - \sigma_B\sinh\rho \end{aligned}$$

Comparing Lorentz rotations

Lorentz rotation $L_A(\rho)=e^{-\rho\sigma_A}$

$$L_A(\rho)=e^{-\rho\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$=\begin{pmatrix}e^{-\rho}&0\\0&e^{+\rho}\end{pmatrix}$$

$$=\mathbf{1}\cosh\rho-\sigma_A\sinh\rho$$

Lorentz rotation $L_B(\rho)=e^{-\rho\sigma_B}$

$$L_B(\rho)=e^{-\rho\begin{pmatrix}0&1\\1&0\end{pmatrix}}$$

$$=\begin{pmatrix}\cosh\rho&-\sinh\rho\\-\sinh\rho&\cosh\rho\end{pmatrix}$$

Lorentz rotation $L_C(\rho)=e^{-\rho\sigma_C}$

$$L_C(\rho)=e^{-\rho\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$=\begin{pmatrix}\cosh\rho&+i\sinh\rho\\-i\sinh\rho&\cosh\rho\end{pmatrix}$$

$$=\mathbf{1}\cosh\rho-\sigma_C\sinh\rho$$

Comparing regular rotations

Regular rotation $R_A(\varphi)=e^{-i\varphi\sigma_A}$

$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A$$

$$=\begin{pmatrix}\cos\varphi_A-i\sin\varphi_A&0\\0&\cos\varphi_A+i\sin\varphi_A\end{pmatrix}$$

$$=\begin{pmatrix}e^{-i\varphi_A}&0\\0&e^{i\varphi_A}\end{pmatrix}$$

Example A:
A or Z
rotation

Regular rotation $R_B(\varphi)=e^{-i\varphi\sigma_B}$

$$e^{-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\varphi_B}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_B-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\varphi_B$$

$$=\begin{pmatrix}\cos\varphi_B&-i\sin\varphi_B\\-i\sin\varphi_B&\cos\varphi_B\end{pmatrix}$$

Example B:
B or X
rotation

Regular rotation $R_C(\varphi)=e^{-i\varphi\sigma_C}$

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C$$

$$=\begin{pmatrix}\cos\varphi_C&-\sin\varphi_C\\\sin\varphi_C&\cos\varphi_C\end{pmatrix}$$

Example C:
C or Y
rotation

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic	*like*	*complex arithmetic*
Spinor vector algebra	*like*	*complex vector algebra*
Spinor exponentials	*like*	*complex exponentials (“Crazy-Thing”-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

How probability ψ*-waves and flux* ψ*-waves evolved*

Properties of amplitude $\psi^*\psi$*-squares*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$,*and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

ABCD Time evolution operator

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega t - i\sigma_{\varphi}\sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\ \cos\varphi \quad - \quad i\ \sin\varphi$$

generalizes to:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi \quad - \quad i\ \sigma_\varphi \sin\varphi$$

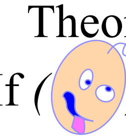
where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\varphi_A$$

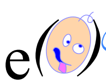
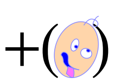
$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A + i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\text{face})\varphi}$  = **1**cosφ + ()sinφ

ABCD Time evolution operator

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$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\quad$ $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\quad$ $\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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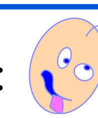
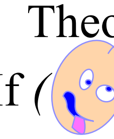
Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

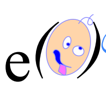
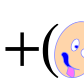
$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A + i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

Example 1: A or Z rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\text{})\varphi}$ = **1** cos φ + () sin φ

Unit spinor vector

$$\sigma_\varphi = \frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi} = (\boldsymbol{\sigma}\bullet\hat{\varphi})$$

$$= \frac{\varphi_A\boldsymbol{\sigma}_A + \varphi_B\boldsymbol{\sigma}_B + \varphi_C\boldsymbol{\sigma}_C}{\sqrt{\varphi_A^{\ 2} + \varphi_B^{\ 2} + \varphi_C^{\ 2}}}$$

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$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\quad\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\quad\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

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Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

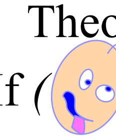
Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

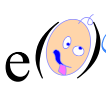
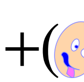
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\text{})\varphi}$ = **1**$\cos\varphi$ + ()$\sin\varphi$

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$$= \frac{\varphi_A\boldsymbol{\sigma}_A + \varphi_B\boldsymbol{\sigma}_B + \varphi_C\boldsymbol{\sigma}_C}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}}$$

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$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$$

generalizes to:

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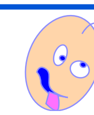
Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

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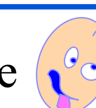
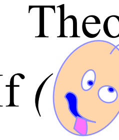
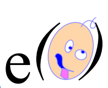
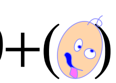
$$= \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix}e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1:
A or Z
rotation

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2:
C or Y
rotation

The 
Crazy Thing
Theorem:
If ()^2= -**1**
Then:
e()θ =**1**cosθ+()sinθ

Unit spinor vector

$$\sigma_\varphi = \frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi} = (\boldsymbol{\sigma}\bullet\hat{\varphi})$$

$$= \frac{\varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C}{\sqrt{\varphi_A^{\ 2} + \varphi_B^{\ 2} + \varphi_C^{\ 2}}}$$

Let: $\vec{\varphi} = \vec{\omega}\cdot t$

Example 3:
Any φ=ωt-axial rotation

$$e^{-i(\sigma\bullet\vec{\varphi})} = e^{-i(\sigma\bullet\hat{\varphi})\varphi} = e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi = \mathbf{1}\cos\varphi - i\ (\sigma\bullet\hat{\varphi})\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\sigma_A\hat{\varphi}_A\sin\varphi \quad - i\sigma_B\hat{\varphi}_B\sin\varphi \quad - i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix}\cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi\\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi\end{pmatrix}$$

ABCD Time evolution operator

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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generalizes to:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi - i\,\sigma_\varphi \sin\varphi$$

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Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

We test these operators by making them rotate each other....

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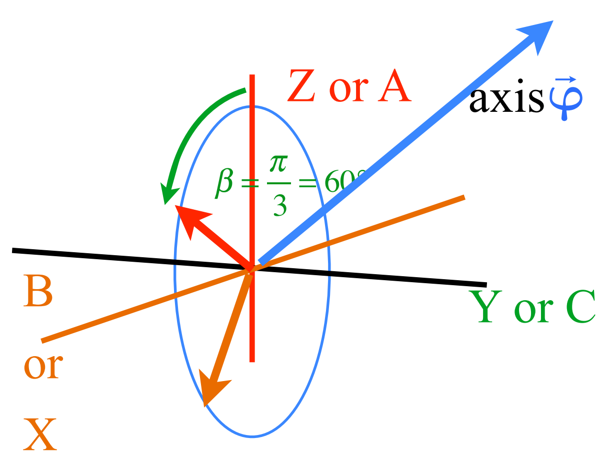

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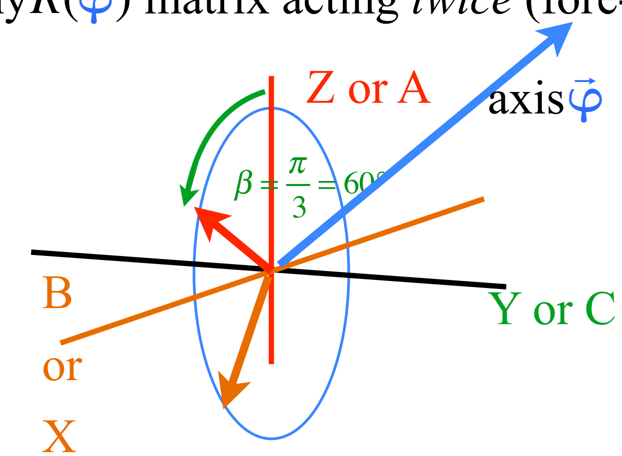

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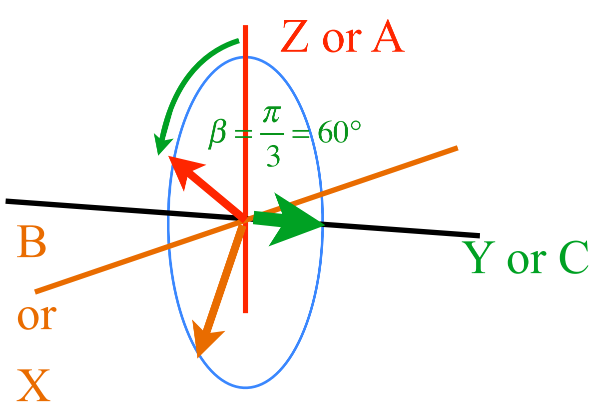

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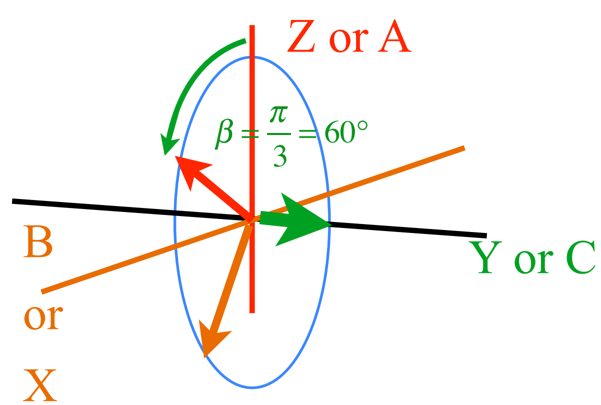

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3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_{\varphi} = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_{\varphi}\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_{\varphi}\sin\varphi$ about axis $\vec{\varphi}$.

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$$\mathbf{R}(\varphi_C)\ \cdot\ \boldsymbol{\sigma}_A\ \cdot\ \mathbf{R}^{-1}(\varphi_C)$$
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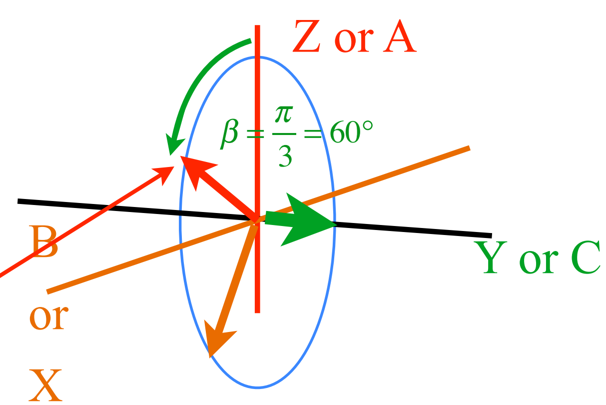

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

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$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$
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Example 2: C or Y rotation

3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$ about axis $\vec{\varphi}$.

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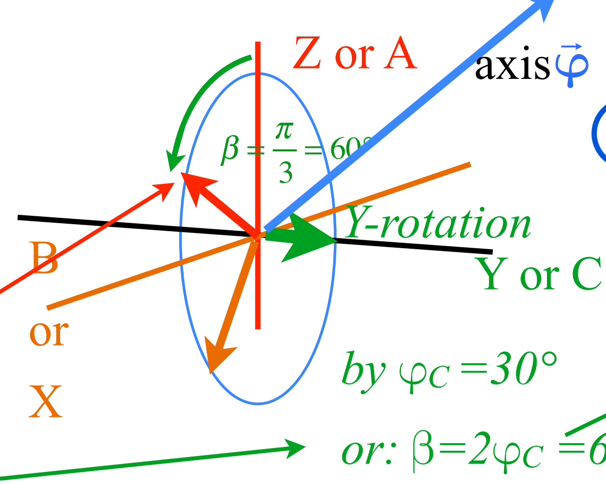

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

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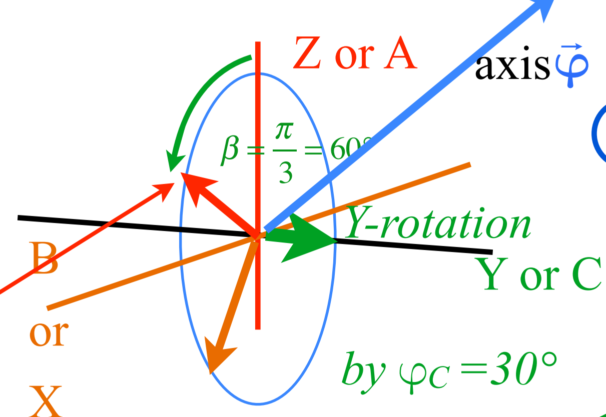

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

$\vec{\varphi}=\vec{\omega}\cdot t$ equal to $\vec{\omega}$ only at $t=1$ but $\hat{\varphi}=\hat{\omega}$ always.

$$\hat{\varphi} = \begin{pmatrix}\hat{\varphi}_A\\ \hat{\varphi}_B\\ \hat{\varphi}_C\end{pmatrix} = \begin{pmatrix}\varphi_A\\ \varphi_B\\ \varphi_C\end{pmatrix}\frac{1}{\sqrt{\varphi_A^2+\varphi_B^2+\varphi_C^2}} = \begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}\frac{1}{\sqrt{\omega_A^2+\omega_B^2+\omega_C^2}}$$

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$$\mathbf{R}(\varphi_C)\ \cdot\ \boldsymbol{\sigma}_A\ \cdot\ \mathbf{R}^{-1}(\varphi_C)$$

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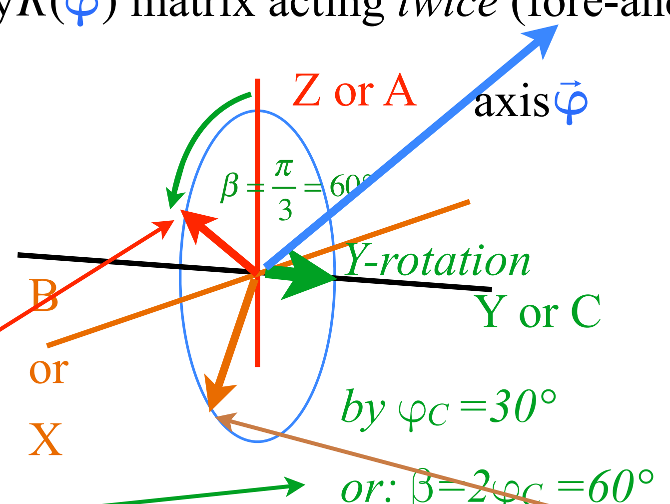

$$\mathbf{R}(\varphi_C)\ \cdot\ \boldsymbol{\sigma}_B\ \cdot\ \mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C\\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$

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Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

*NMR Hamiltonian: 3D Spin Moment **m** in **B** field*

How probability ψ-waves and flux ψ-waves evolved

Properties of amplitude $\psi^*\psi$-squares

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$,and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$

Polarization ellipse and spinor state dynamics

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$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\,\sigma_0+\omega_A\,\sigma_A+\omega_B\,\sigma_B+\omega_C\,\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

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Notation for 3D Vector space

0th component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

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$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

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(Often labeled $\{J_X, J_Y, J_Z\}$)

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$$= \omega_0 \quad \sigma_0 \quad + \quad \omega_A \quad \sigma_A \quad + \omega_B \quad \sigma_B \quad + \omega_C \quad \sigma_C \quad = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0\mathbf{1} + \omega\boldsymbol{\sigma}_\omega$$

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(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t} = e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged — *components A, B, C switch 1/2-factor from ω-velocity to S-momentum*

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

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$$= e^{-i(\Omega_0\mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t} = e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}} = e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2} - i\boldsymbol{\sigma}_\omega \sin\frac{\Omega\cdot t}{2}\right)$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0 = \frac{A+D}{2}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Notation for 2D Spinor space

$$=\omega_0\ \sigma_0 + \omega_A\ \sigma_A + \omega_B\ \sigma_B + \omega_C\ \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0\mathbf{1}+\omega\sigma_\omega$$

$$=\Omega_0\ \mathbf{1} + \Omega_A\ \mathbf{S}_A + \Omega_B\ \mathbf{S}_B + \Omega_C\ \mathbf{S}_C = \Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

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(Often labeled $\{\mathbf{J}_X, \mathbf{J}_Y, \mathbf{J}_Z\}$)

"Crank" vector

$$\vec{\varphi}=\begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$$

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\sigma_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\sigma_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

"Crank" vector

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

➡ *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*

➡ *2D Spinor vs 3D vector rotation*

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

How probability ψ-waves and flux ψ-waves evolved

*Properties of amplitude ψ*ψ-squares*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

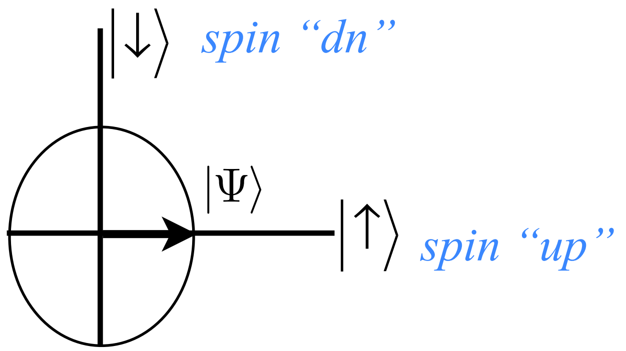
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor $\{|\uparrow\rangle,|\downarrow\rangle\}$*-space (complex)*

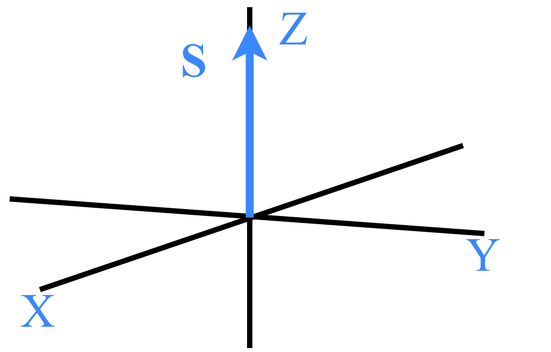
R(3):3D Spin Vector $\{S_X, S_Y, S_Z\}$*-space (real)*

State vector $|\Psi\rangle=|\uparrow\rangle\langle\uparrow|\Psi\rangle+|\downarrow\rangle\langle\downarrow|\Psi\rangle$

Spin vector $\mathbf{S}=|X\rangle\langle X|\mathbf{S}\rangle+|Y\rangle\langle Y|\mathbf{S}\rangle+|Z\rangle\langle Z|\mathbf{S}\rangle$



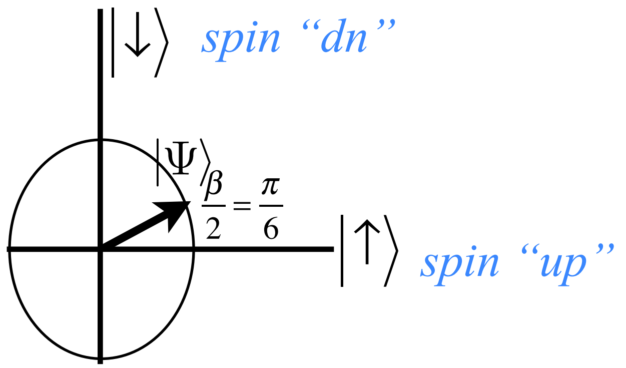
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$



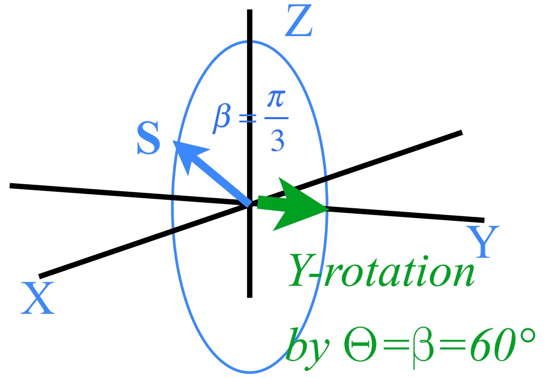
$$\begin{pmatrix}S_X\\S_Y\\S_Z\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

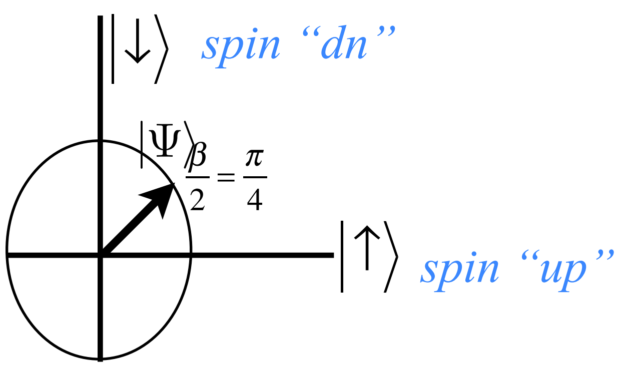
$$\begin{pmatrix}S_X\\S_Y\\S_Z\end{pmatrix}=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$



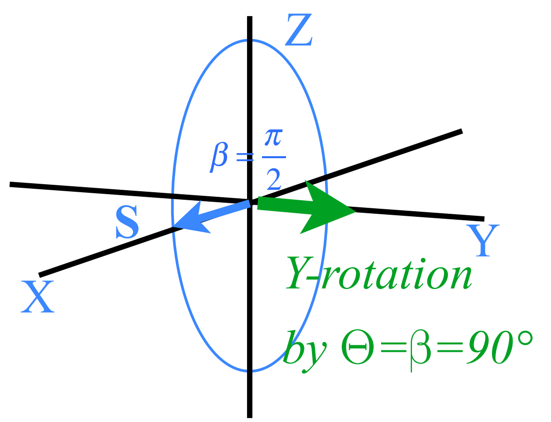
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



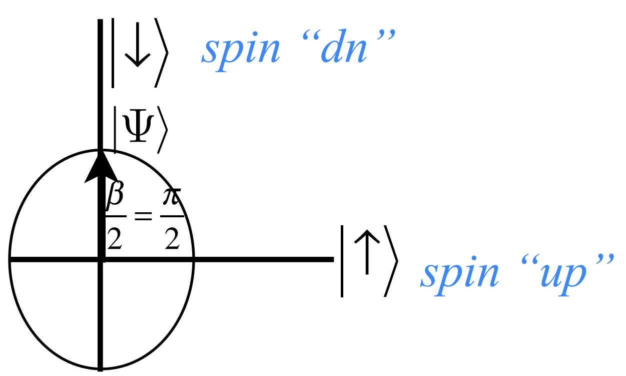
$$\begin{pmatrix}S_X\\S_Y\\S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}\sqrt{3}/2\\0\\1/2\end{pmatrix}$$



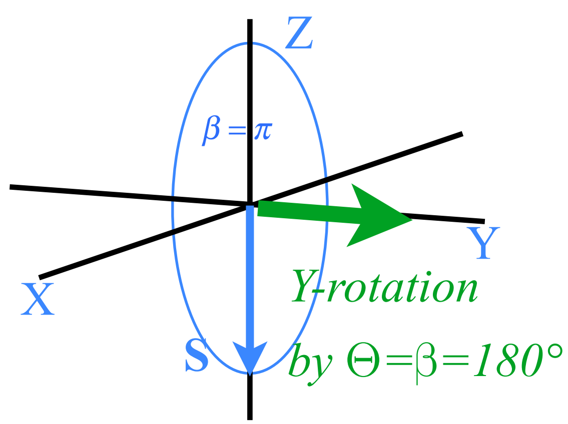
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\S_Y\\S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$



$$\begin{pmatrix}S_X\\S_Y\\S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast"

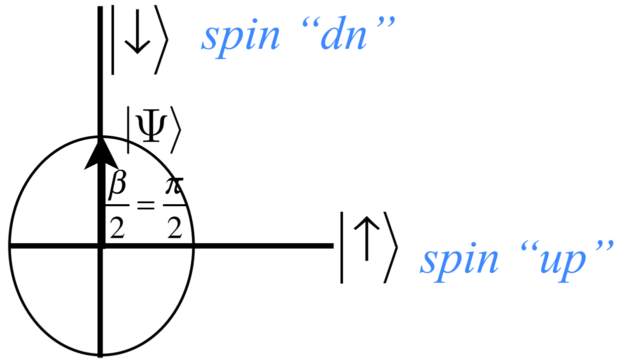
"Spinor" means spin S=1/2

"Vector" means spin S=1

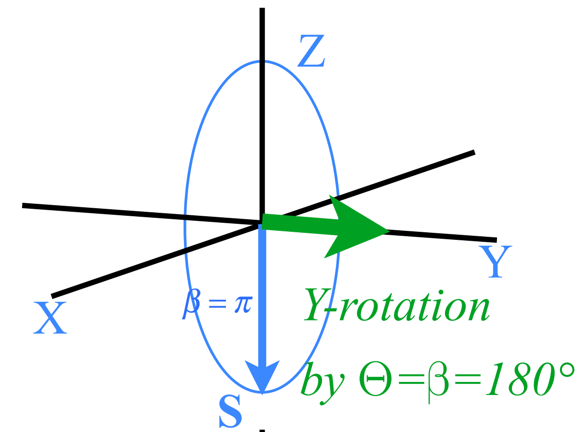
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex) *R(3):3D Spin Vector {S_X, S_Y, S_Z}-space (real)*

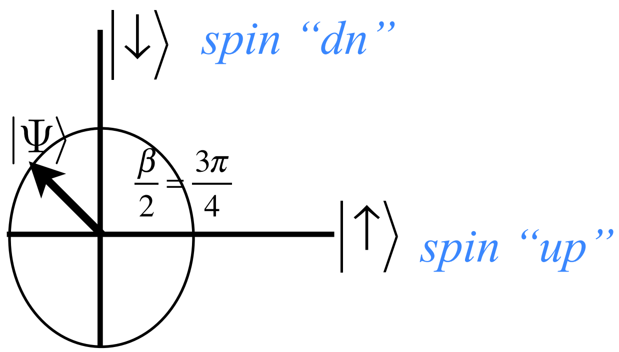
State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$ *Spin vector* $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



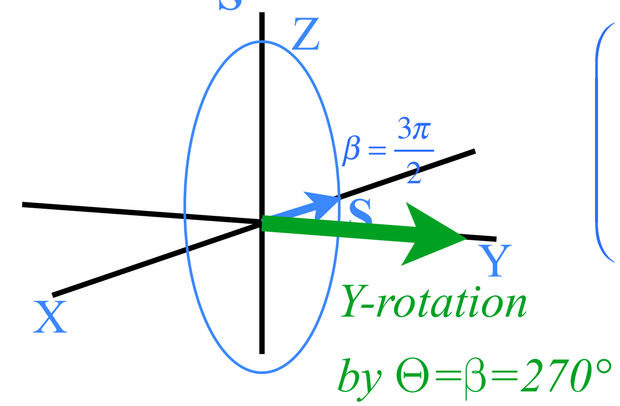
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$



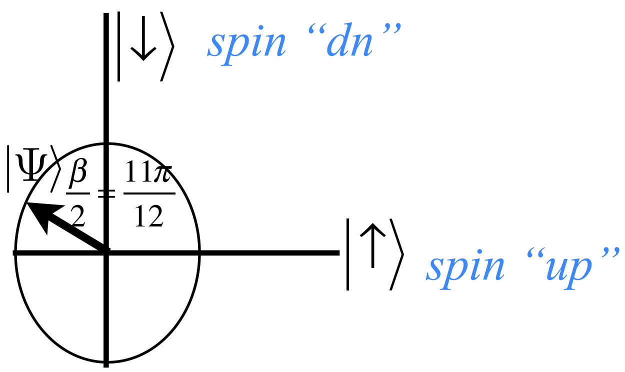
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$



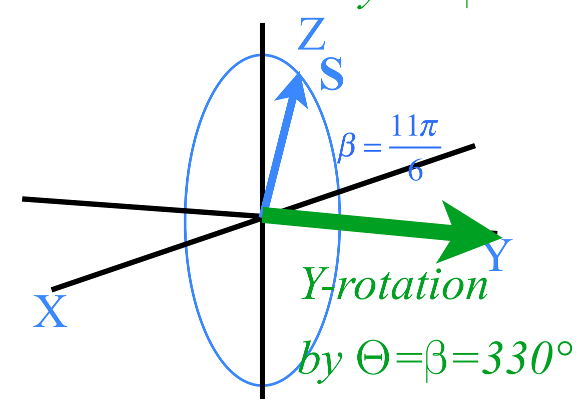
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



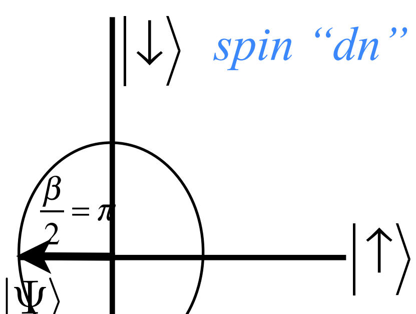
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



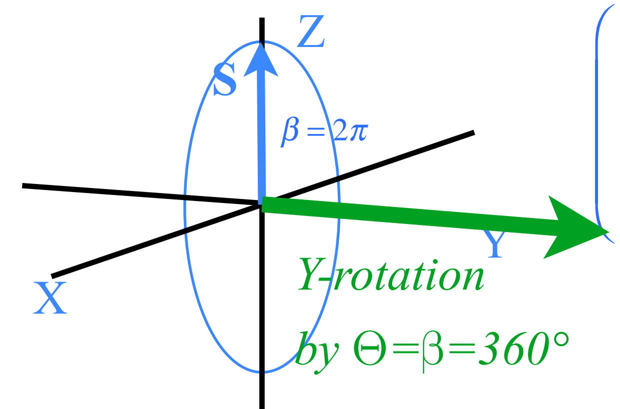
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} =$$



-spin "up"
with π-phase

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}-1\\ 0\end{pmatrix}$$

(Only "half-way" home after 2π =360° rotation)



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast" and needs Θ=4π=720° to return to original state

"Spinor" means spin S=1/2 *"Vector" means spin S=1*

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

→ *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

→ *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

How probability ψ-waves and flux ψ-waves evolved

*Properties of amplitude ψ*ψ-squares*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for

2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z\ \sigma_A \quad + \quad gB_X\ \sigma_X \quad + gB_Y\ \sigma_Y \quad =\vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for 2D Spinor space

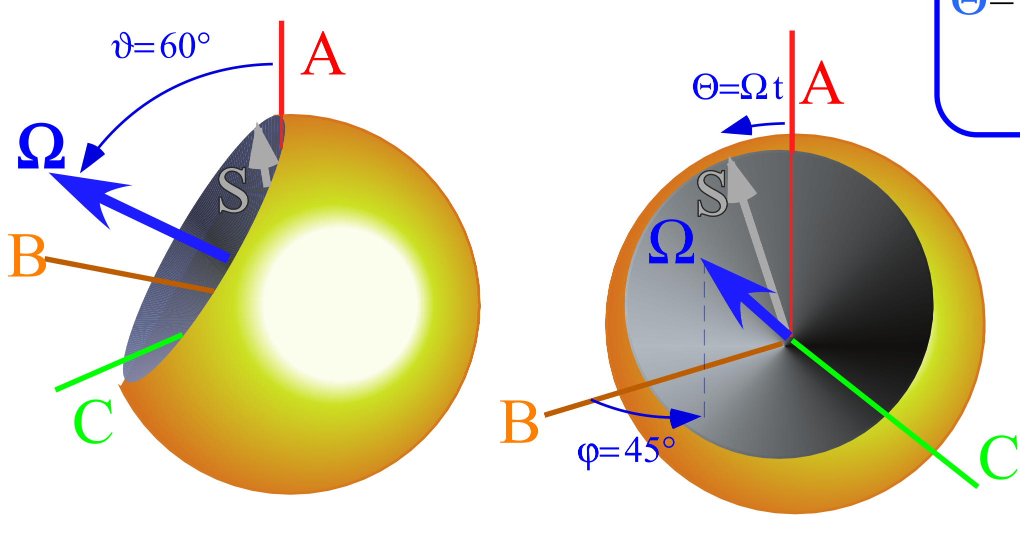
Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

The driving $\Theta=\Omega t$ *crank vector defined by ABCD of Hamiltonian* $\mathbf{H}$.

Notation for 3D Vector space

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$



Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ *whirling Stokes state vector* S *in ABC-space.*

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ in field $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z\ \sigma_A \quad + \quad gB_X\ \sigma_X \quad + gB_Y\ \sigma_Y \quad =\vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

The driving $\Theta=\Omega t$ *crank vector defined by ABCD of Hamiltonian* **H** .

Notation for 3D Vector space

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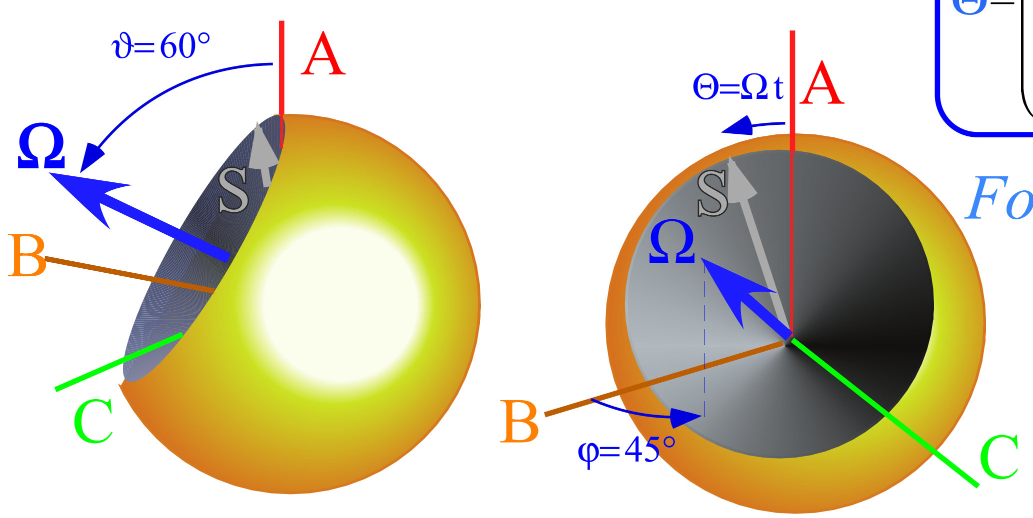
For fermion spin that $\mathbf{\Omega}$ *is the* g**B***-field!*



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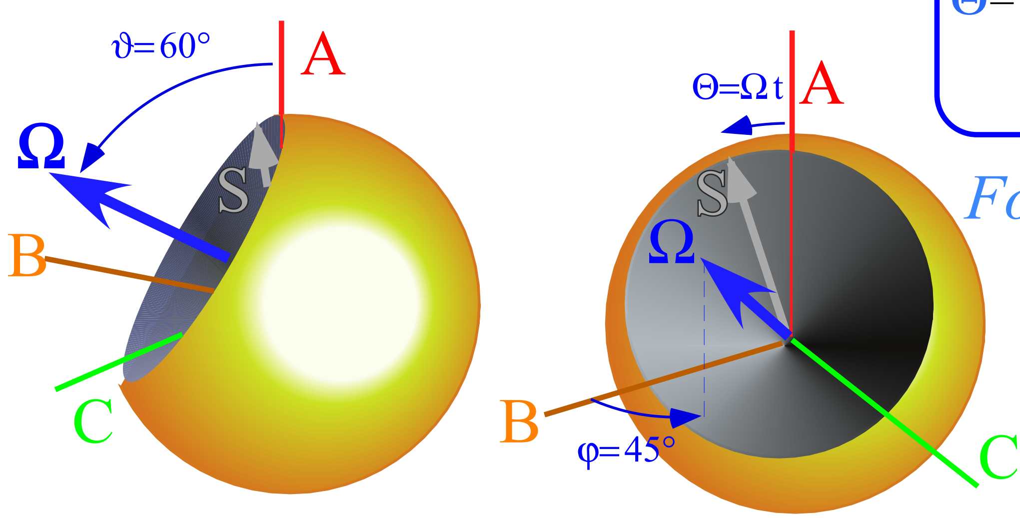

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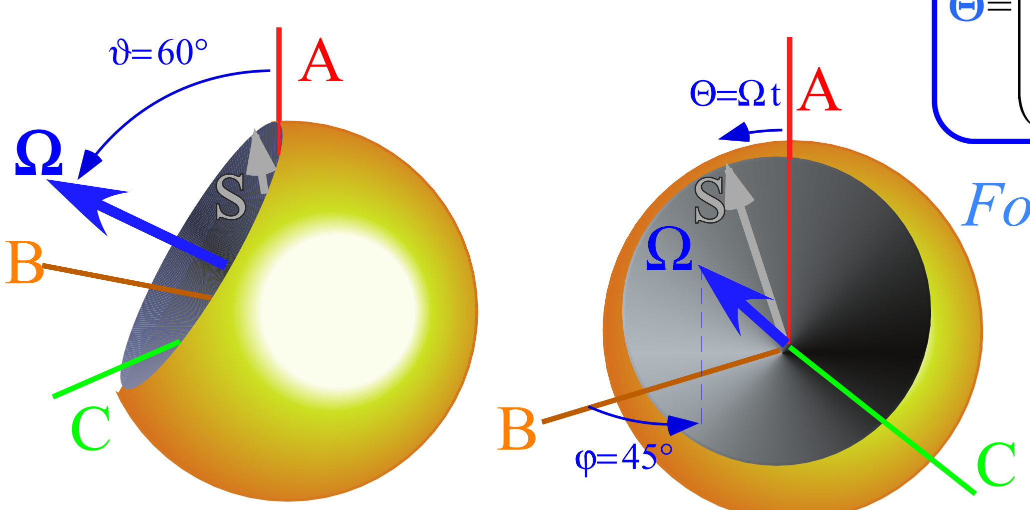

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A: By U(2) group operator $|\psi(t)\rangle=\mathbf{R}[\mathbf{\Theta}]|\psi(0)\rangle$.

See Euler angles $\alpha\beta\gamma$ *(p. 107-117) or Spin Asymmetry* S_A, *Balance* S_B, *Chirality* S_C *(p.119-122)*

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2D Spinor vs 3D vector rotation

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How Probability Amplitudes ψ or ψ Come About *(An optical view)*

Maxwell-Planck-Poynting flux $S=cU=c\varepsilon_0|\mathsf{E}|^2=c\varepsilon_0\mathsf{E}^*\mathsf{E}=nh\upsilon$ has count rate $n=Nc/V(m^{-2}s^{-1})$

If each E-field amplitude factor is scaled by a factor $\sqrt{\frac{c\varepsilon_0}{h\upsilon}}=\sqrt{\frac{\varepsilon_0}{h\kappa}}$ the result is a *flux probability amplitude* $\psi=\mathsf{E}\sqrt{\frac{c\varepsilon_0}{h\upsilon}}$ whose square equals flux count rate $n(m^{-2}s^{-1})$.

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$$Energy \sim |A|^2\upsilon^2 = \left|A\sqrt{\upsilon}\right|^2\upsilon = \left|\frac{E}{2\pi\upsilon}\sqrt{\upsilon}\right|^2\upsilon = \left|\frac{E}{2\pi\sqrt{\upsilon}}\right|^2\upsilon \sim \left|E\sqrt{\frac{c\varepsilon_0}{h\upsilon}}\right|^2 = nh\upsilon$$

Both (n) and (υ) in $nh\upsilon$ are *frequencies* and will transform as such.

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Probability Waves $\psi(x,t)$ *(More optical views)*

Optical E-field amplitudes like $\mathsf{E}(x,t)=\mathsf{E}_0e^{i(kx-\omega t)}$ vary with space x and time t. So do scaled $\psi(x,t)$ ampliudes whose sum-Σ (integral-$\int$) over cells ΔV (or dV) must be particle number N. For 1-particle systems *(N=1)* this is the *unit norm* rule.

$$\Sigma_j\psi(x_j,t)^*\psi(x_j,t)\Delta V_j=N \qquad \text{or:} \qquad \int\psi(x,t)^*\psi(x,t)dV=N$$

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Born interpreted $\psi(x,t)^*\psi(x,t)$ as *probable expectation* of particle count. Schrodinger objected to the *probability wave* interpretation that is now accepted and called the Schrodinger theory. A relativistic wave view lends merit to his objections.

Doppler Transformation of 2-CW Modes

Doppler shift of *opposite-k 1-CW* beams. As derived before phases are invariant: $(k'x'-\omega't'=kx-\omega t)$

E-*wave:* $E=E_{\rightarrow}e^{i(k_{\rightarrow}x-\omega_{\rightarrow}t)}+E_{\leftarrow}e^{i(k_{\leftarrow}x-\omega_{\leftarrow}t)}$

Ψ-*wave:* $\Psi=\psi_{\rightarrow}e^{i(k_{\rightarrow}x-\omega_{\rightarrow}t)}+\psi_{\leftarrow}e^{i(k_{\leftarrow}x-\omega_{\leftarrow}t)}$

blue shift

$$E'_{\rightarrow}= b\, E_{\rightarrow} = e^{+\rho}E_{\rightarrow}$$

red shift

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Parameters related to *relative velocity* u:

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Transformation of *SWR* (or *SWQ*) and u_{GROUP} (or u_{PHASE}) is a *non-linear* transformation

$$SWR'=\frac{E'_{\rightarrow}-E'_{\leftarrow}}{E'_{\rightarrow}+E'_{\leftarrow}}=\frac{b^2E_{\rightarrow}-E_{\leftarrow}}{b^2E_{\rightarrow}+E_{\leftarrow}}=\frac{(1+\beta)E_{\rightarrow}-(1-\beta)E_{\leftarrow}}{(1+\beta)E_{\rightarrow}+(1-\beta)E_{\leftarrow}}=\frac{(E_{\rightarrow}-E_{\leftarrow})+\beta(E_{\rightarrow}+E_{\leftarrow})}{(E_{\rightarrow}+E_{\leftarrow})+\beta(E_{\rightarrow}-E_{\leftarrow})}=\frac{SWR+\beta}{1+\beta\cdot SWR}$$

SWR (or *SWQ*) *Transformation*

$$SWR'=\frac{SWR+\beta}{1+SWR\cdot\beta}=\frac{SWR+u/c}{1+SWR\cdot u/c}$$

u_{GROUP} (or u_{PHASE}) *Transformation*

$$u'_{GROUP}/c=\frac{u_{GROUP}/c+\beta}{1+u_{GROUP}\cdot\beta/c}=\frac{(u_{GROUP}+u)/c}{1+u_{GROUP}\cdot u/c^2}$$

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- *Spinor vector algebra like complex vector algebra*
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→ *Euler's state definition using rotations* $\mathbf{R}(\alpha,0,0)$*,* $\mathbf{R}(0,\beta,0)$*,and* $\mathbf{R}(0,0,\gamma)$

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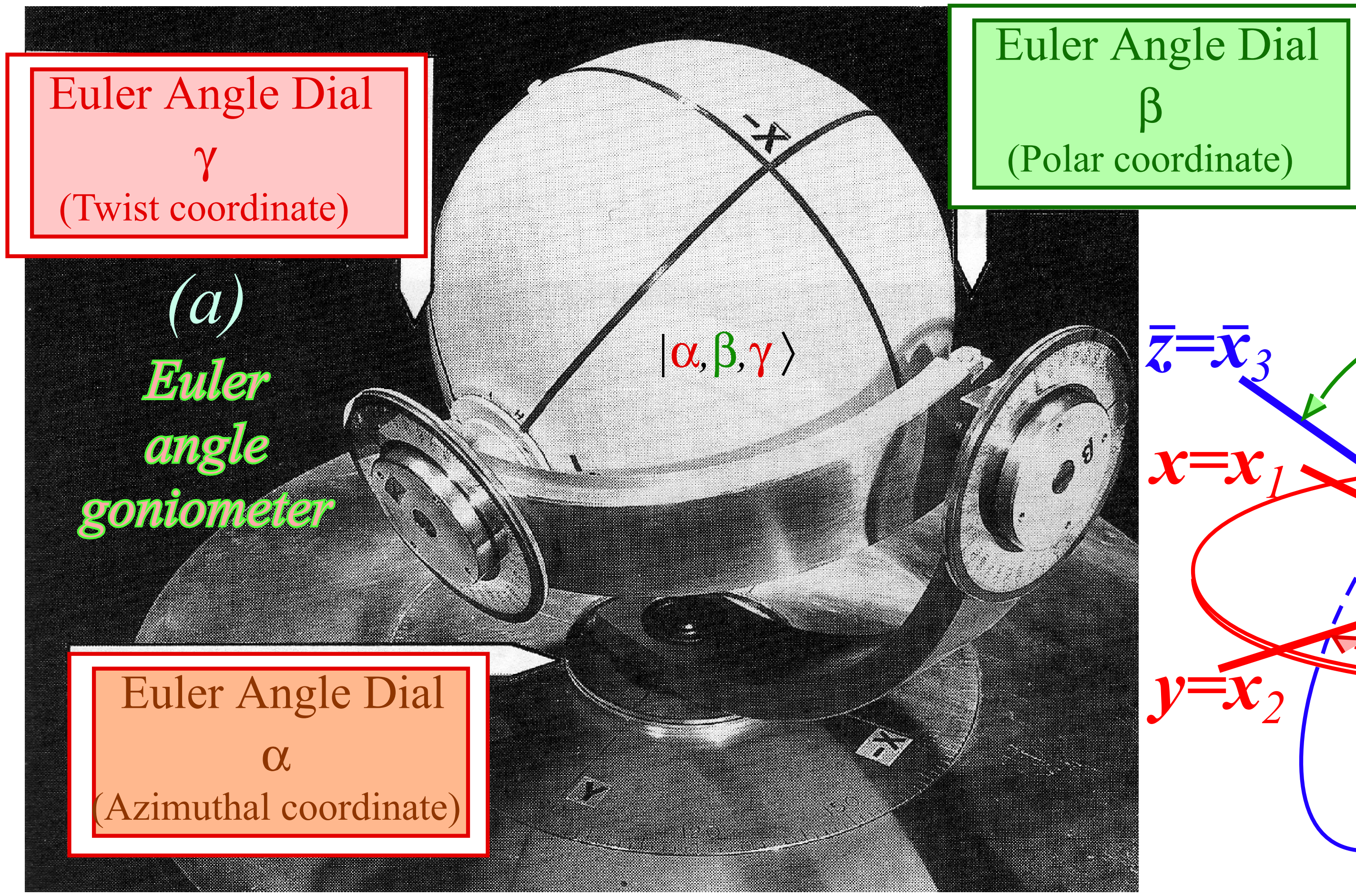
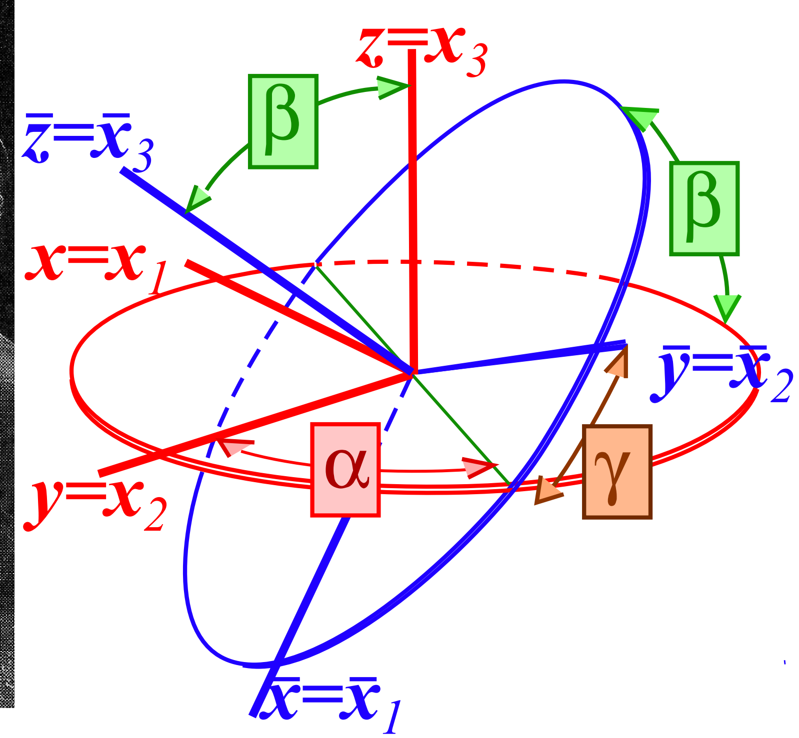


An astronomer's diagram



Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

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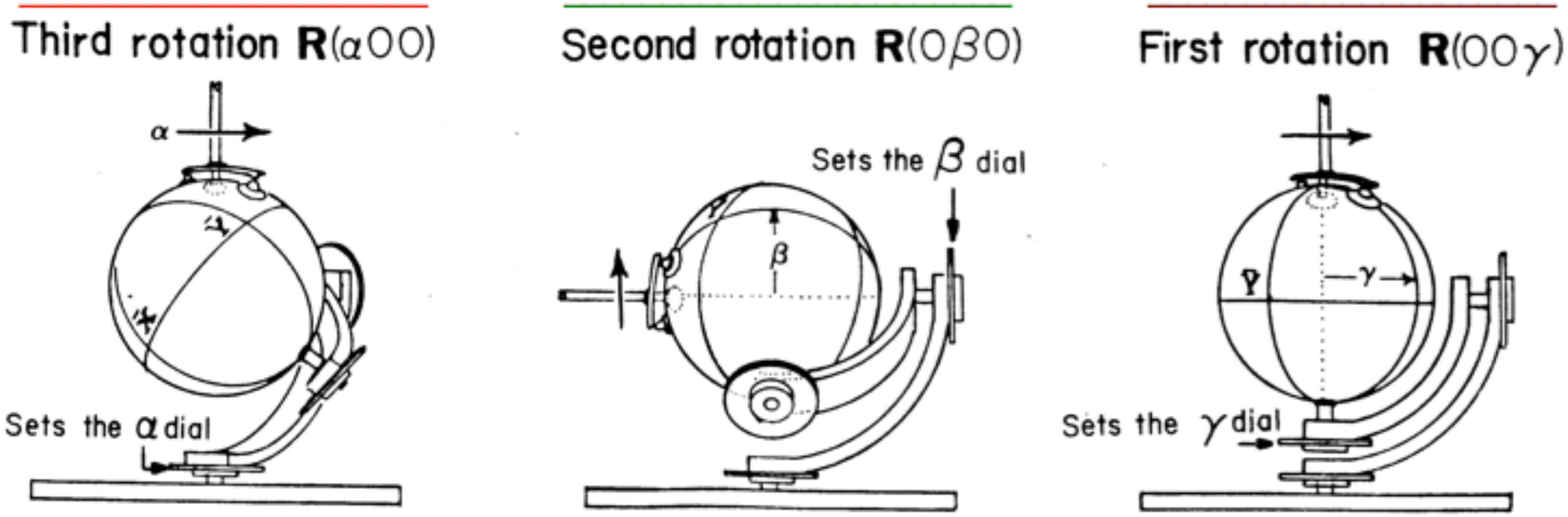

$$\left\langle R(\alpha\beta\gamma)\right\rangle = \left\langle R(\alpha 00)\right\rangle \qquad \left\langle R(0\beta 0)\right\rangle \qquad \left\langle R(00\gamma)\right\rangle$$

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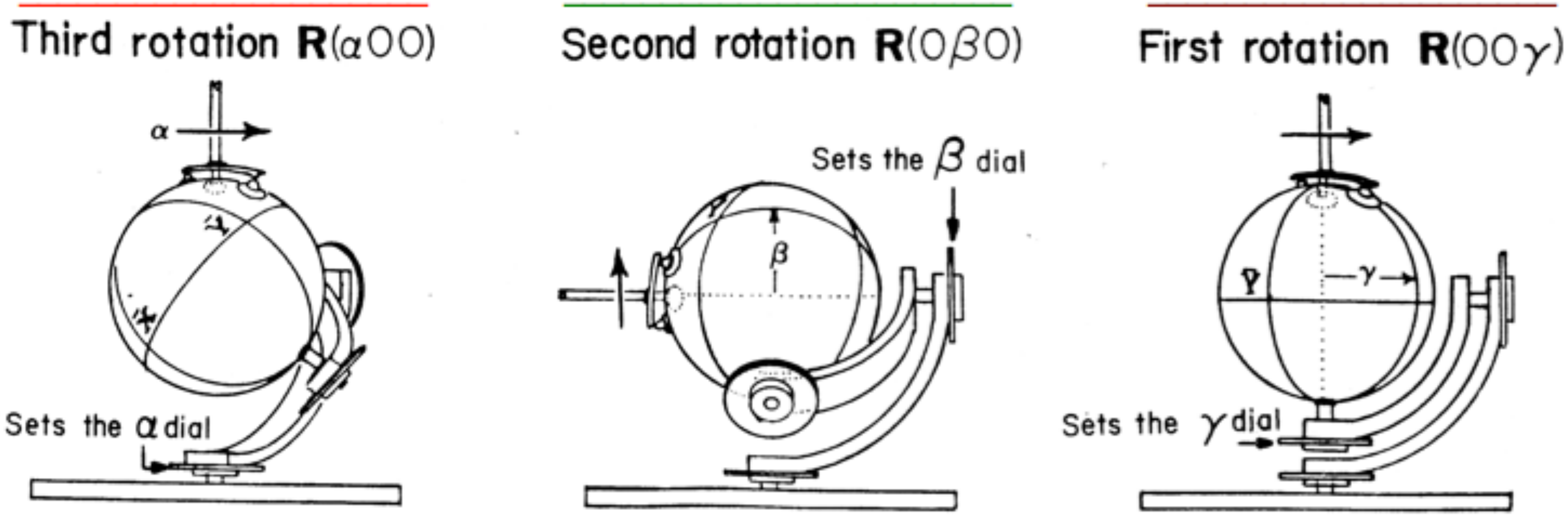


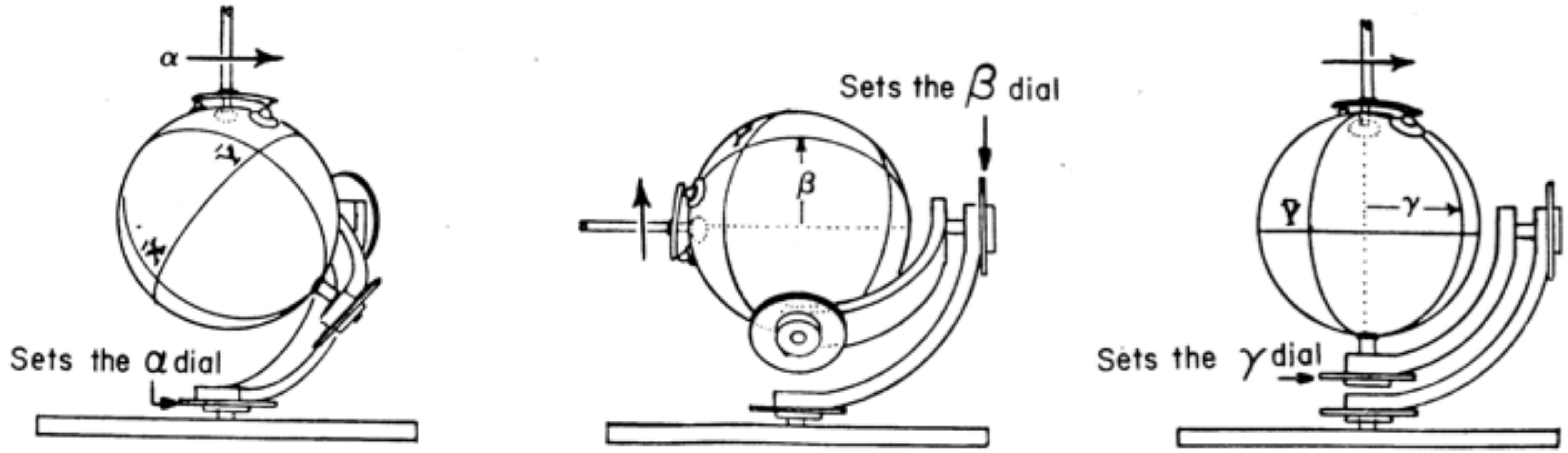
$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

Third rotation **R**$(\alpha 00)$ — Second rotation **R**$(0\beta 0)$ — First rotation **R**(00γ)



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \langle R(0\beta 0)\rangle \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

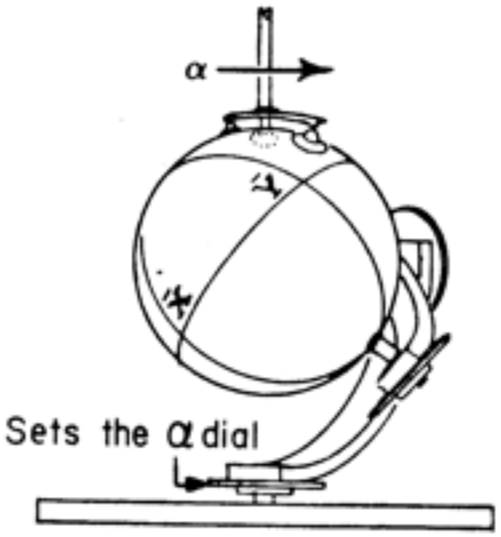
$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X | \\ \langle \mathbf{e}_Y | \\ \langle \mathbf{e}_Z | \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab frame polar coordinates

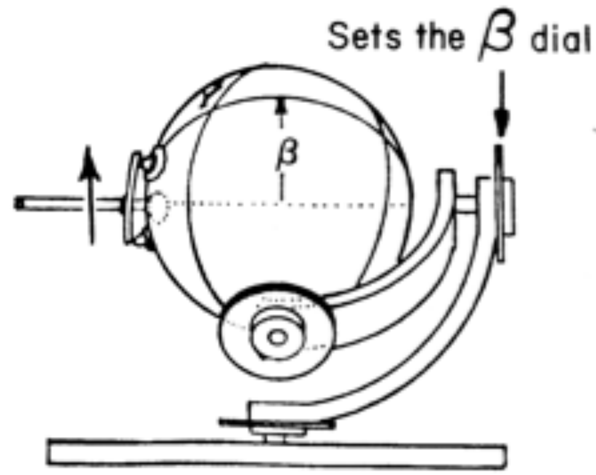
Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

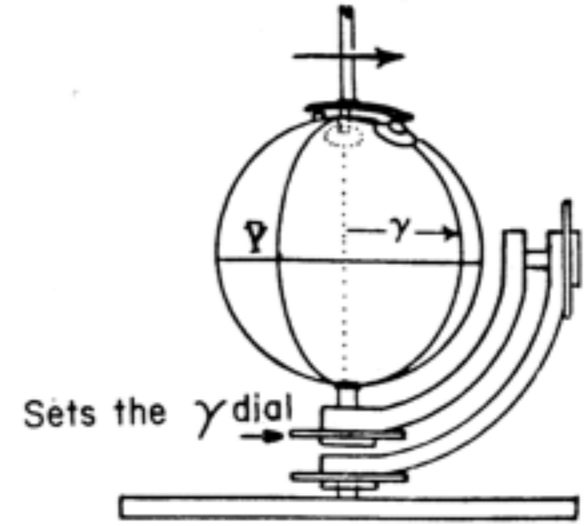
Third rotation $\mathbf{R}(\alpha 00)$



Second rotation $\mathbf{R}(0\beta 0)$



First rotation $\mathbf{R}(00\gamma)$



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \langle R(0\beta 0)\rangle \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & \text{-}\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ \text{-}\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \text{-}\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X | \\ \langle \mathbf{e}_Y | \\ \langle \mathbf{e}_Z | \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \text{-}\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & \text{-}\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab-frame polar coordinates of Z(body)

...and body-frame polar coordinates of Z(lab)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

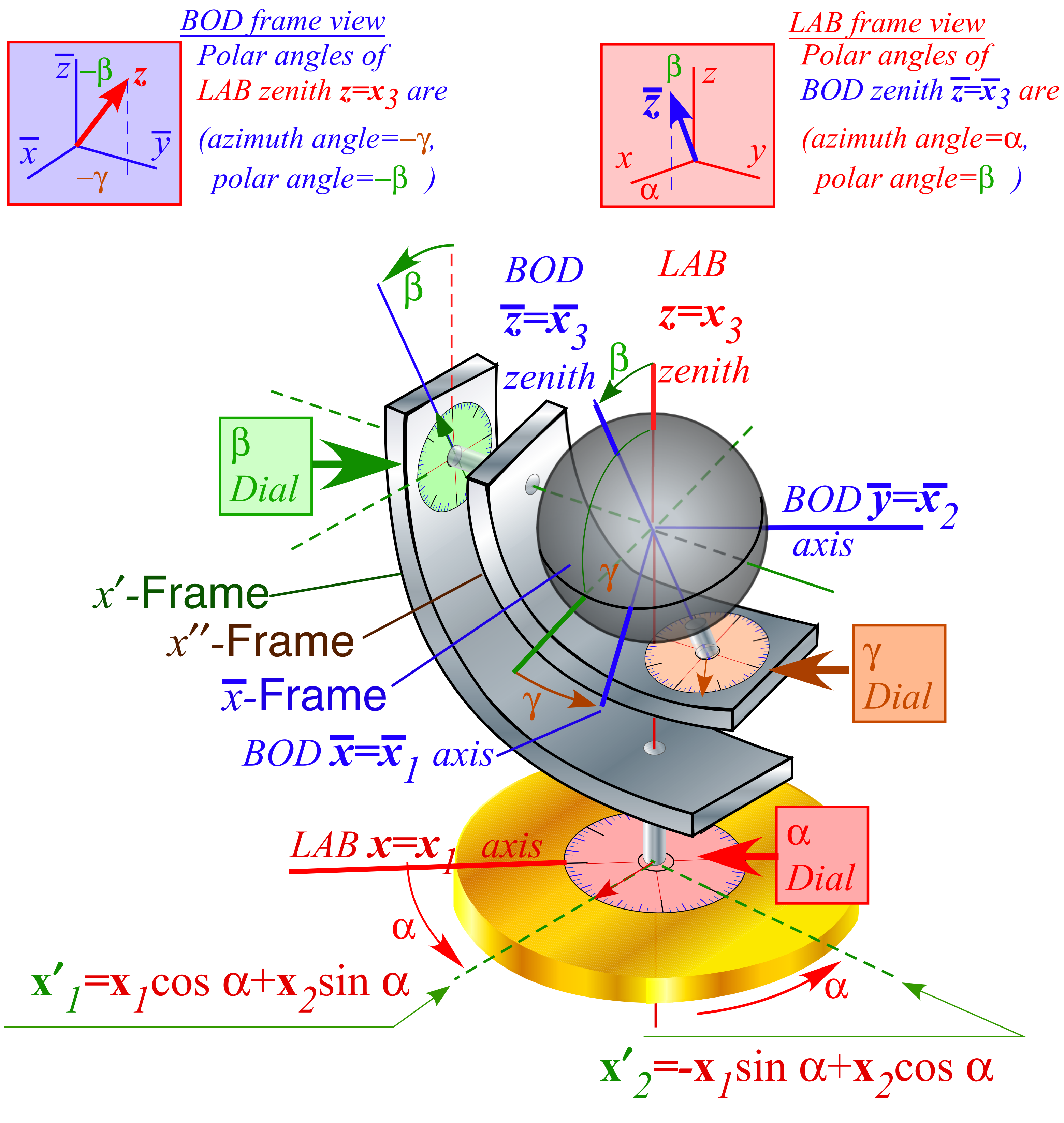
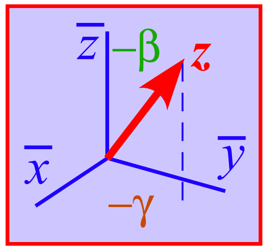


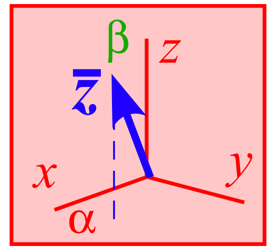
Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

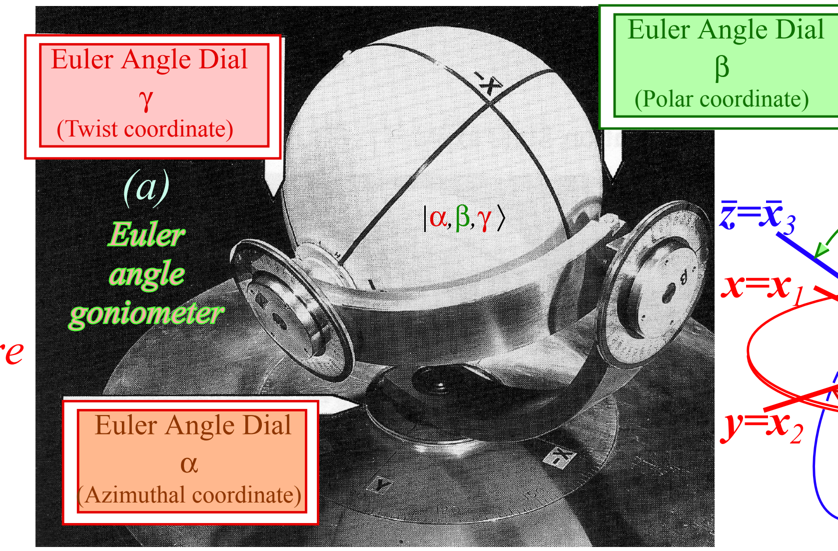
Spin-1 (3D-real vector) case



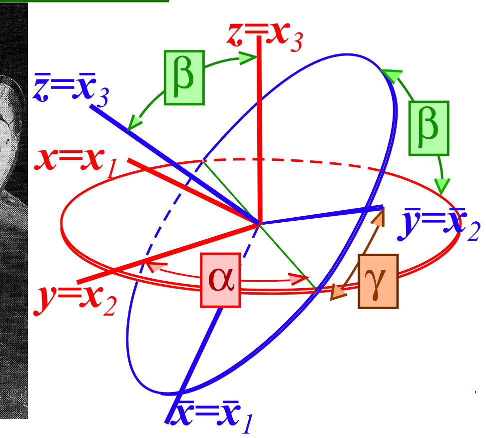
BOD frame view
Polar angles of LAB zenith $z=x_3$ *are*
(azimuth angle=$-\gamma$, polar angle=$-\beta$)



LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are*
(azimuth angle=α, polar angle=β)



An astronomer's diagram



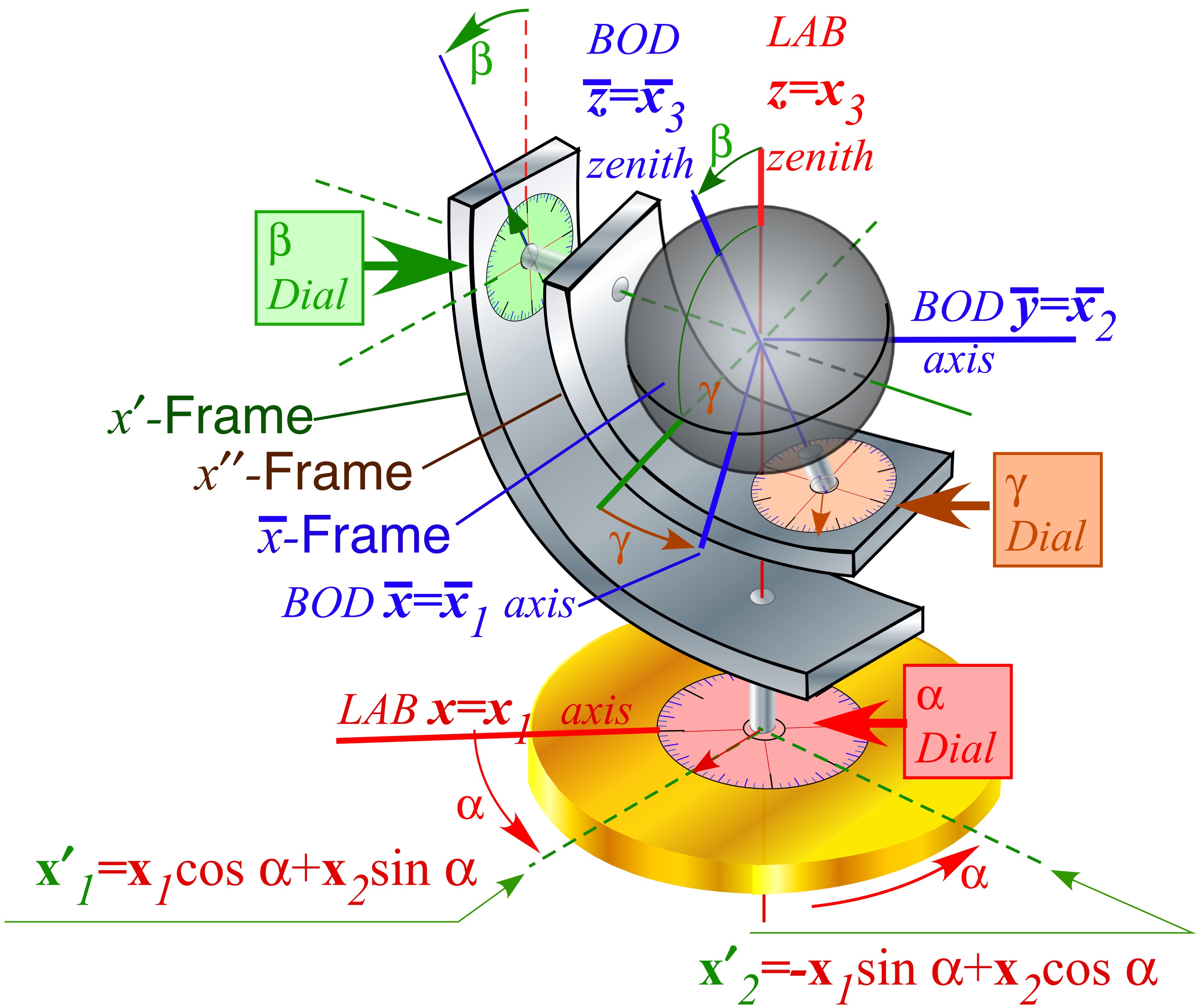
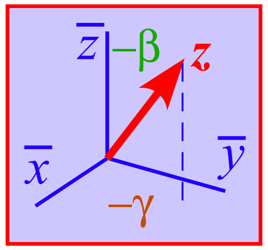


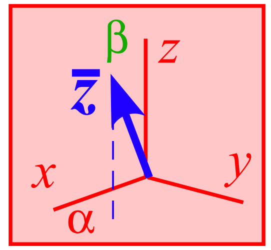
Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

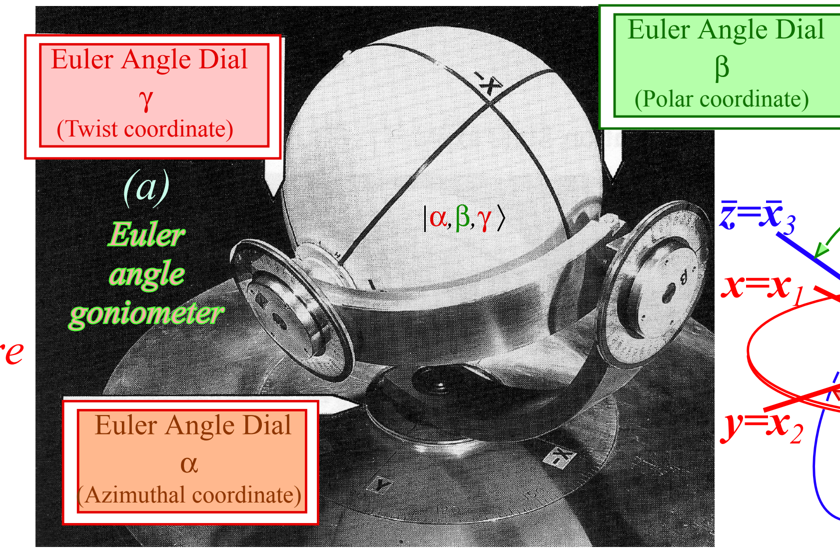
Spin-1 (3D-real vector) case



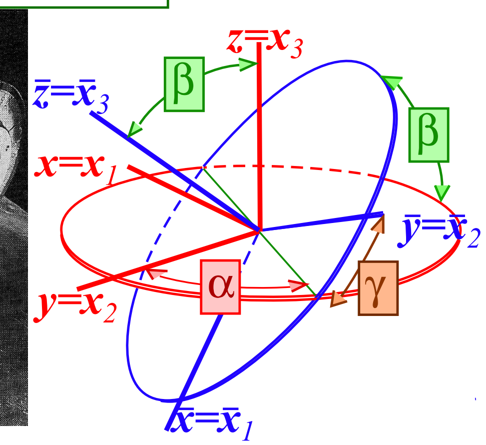
BOD frame view
Polar angles of LAB zenith $z=x_3$ are (azimuth angle$=-\gamma$, polar angle$=-\beta$)

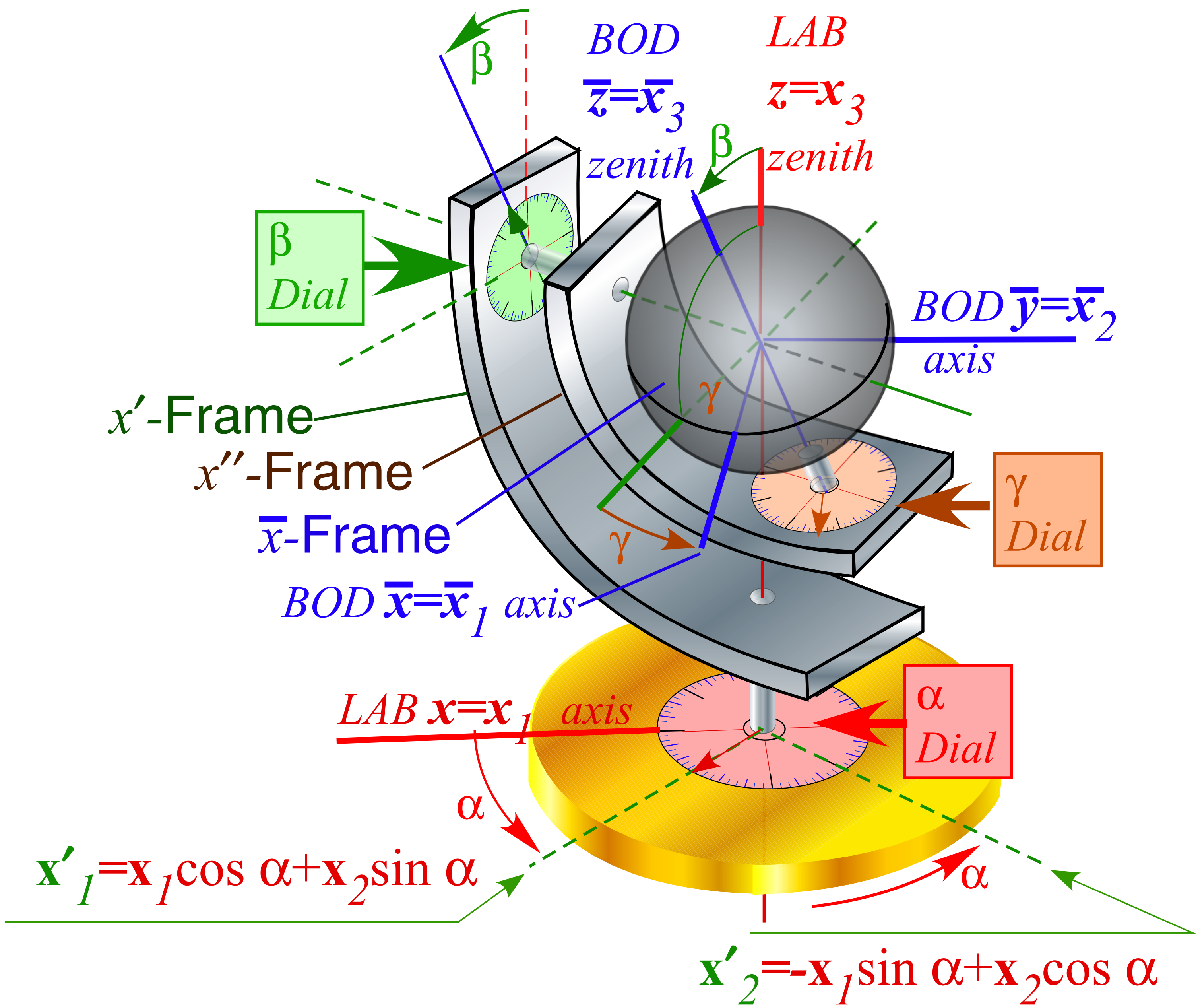


LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are (azimuth angle$=\alpha$, polar angle$=\beta$)



An astronomer's diagram





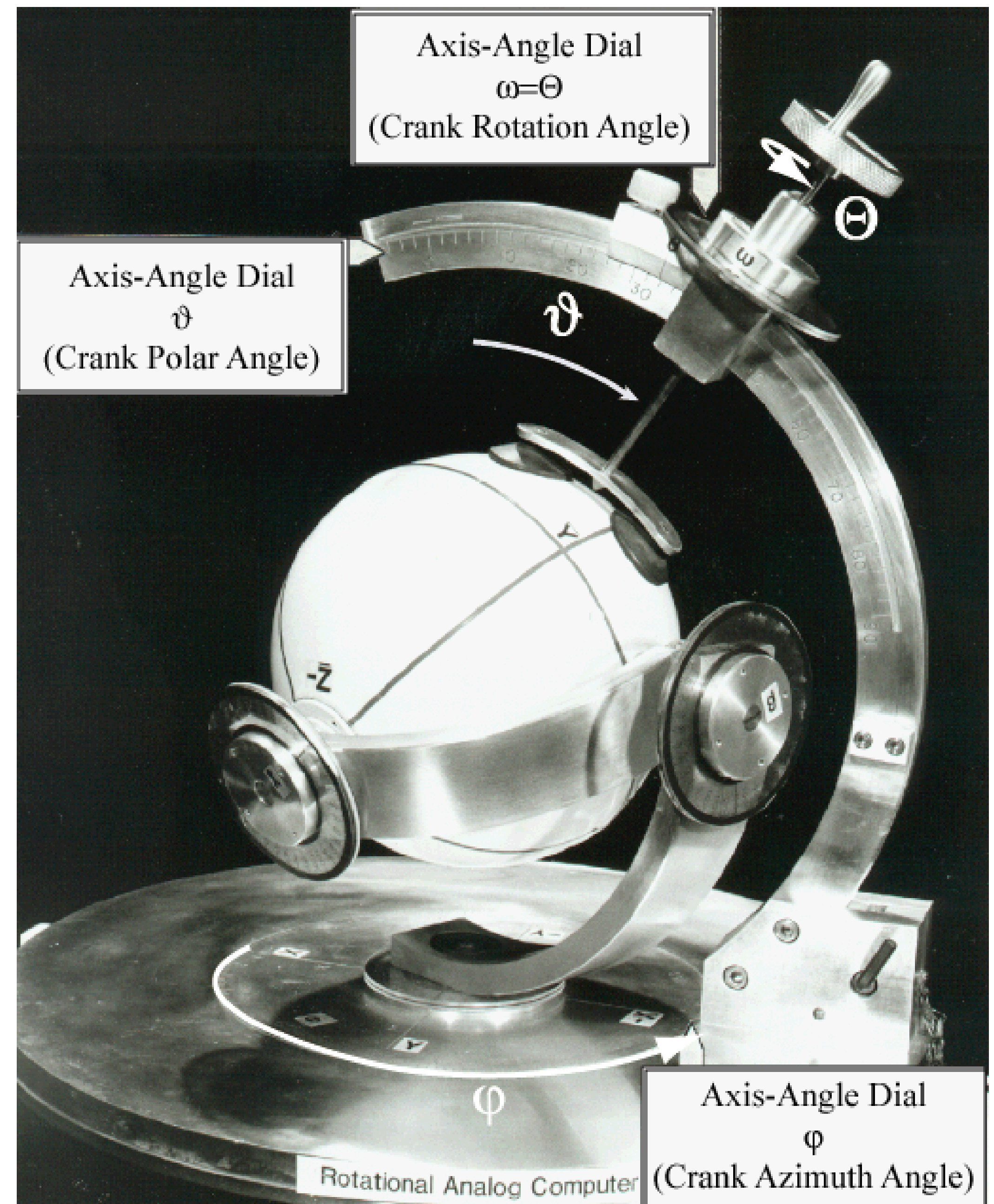


Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

- *Hamilton-Pauli spinor symmetry (* $\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$
- *Deriving* $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - *Spinor arithmetic like complex arithmetic*
  - *Spinor vector algebra like complex vector algebra*
  - *Spinor exponentials like complex exponentials* *("Crazy-Thing"-Theorem)*
- *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
    - *2D Spinor vs 3D vector rotation*
    - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*
- *Euler's state definition using rotations* $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$*,and* $\mathbf{R}(0,0,\gamma)$
  - *Spin-1 (3D-real vector) case*
  - *Spin-1/2 (2D-complex spinor) case*
- *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*
  - *Asymmetry* $S_A=S_Z$*, Balance* $S_B=S_X$*, and Chirality* $S_C=S_Y$
  - *Polarization ellipse and spinor state dynamics*

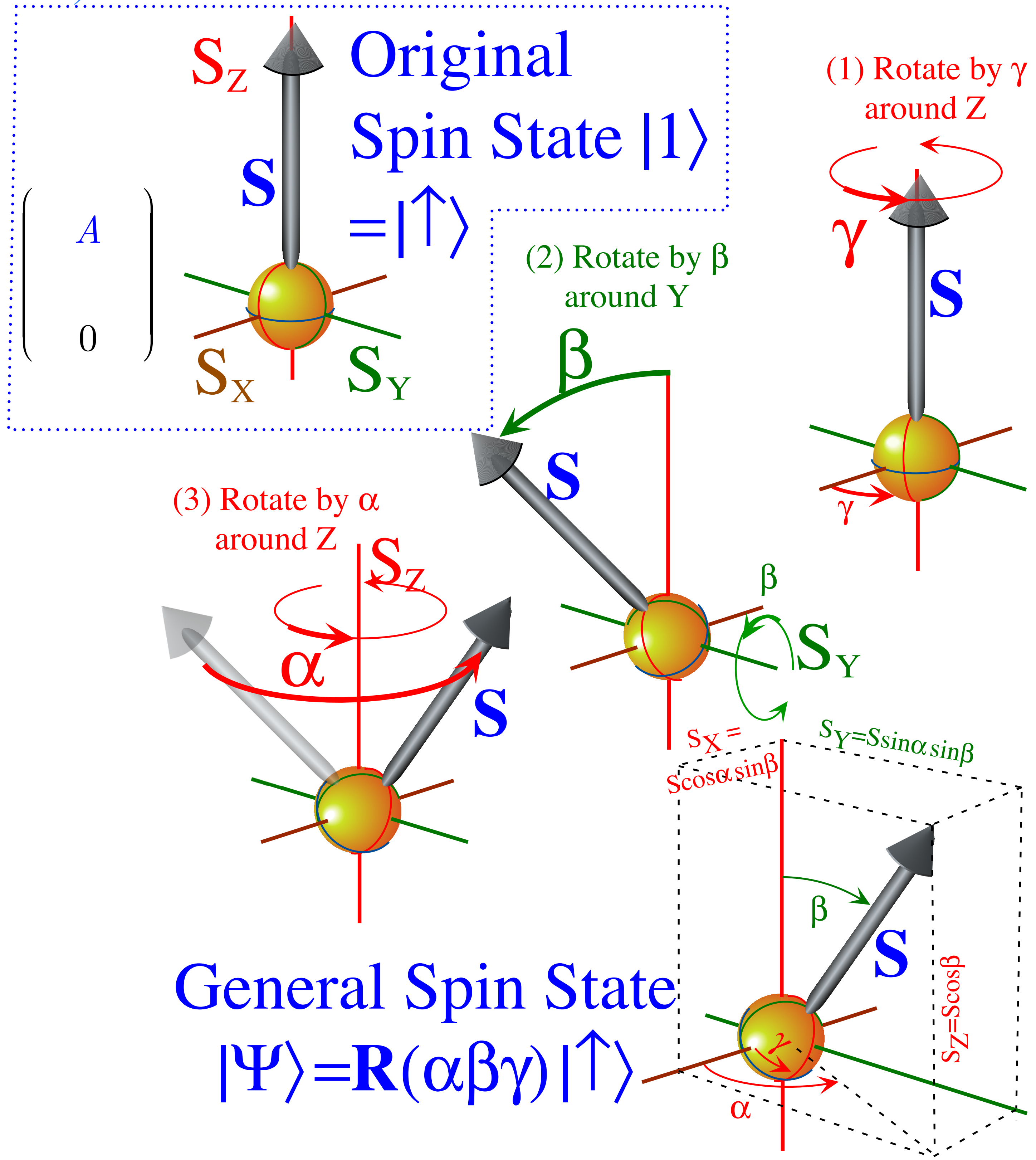
Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$|a\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]|\uparrow\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$



Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

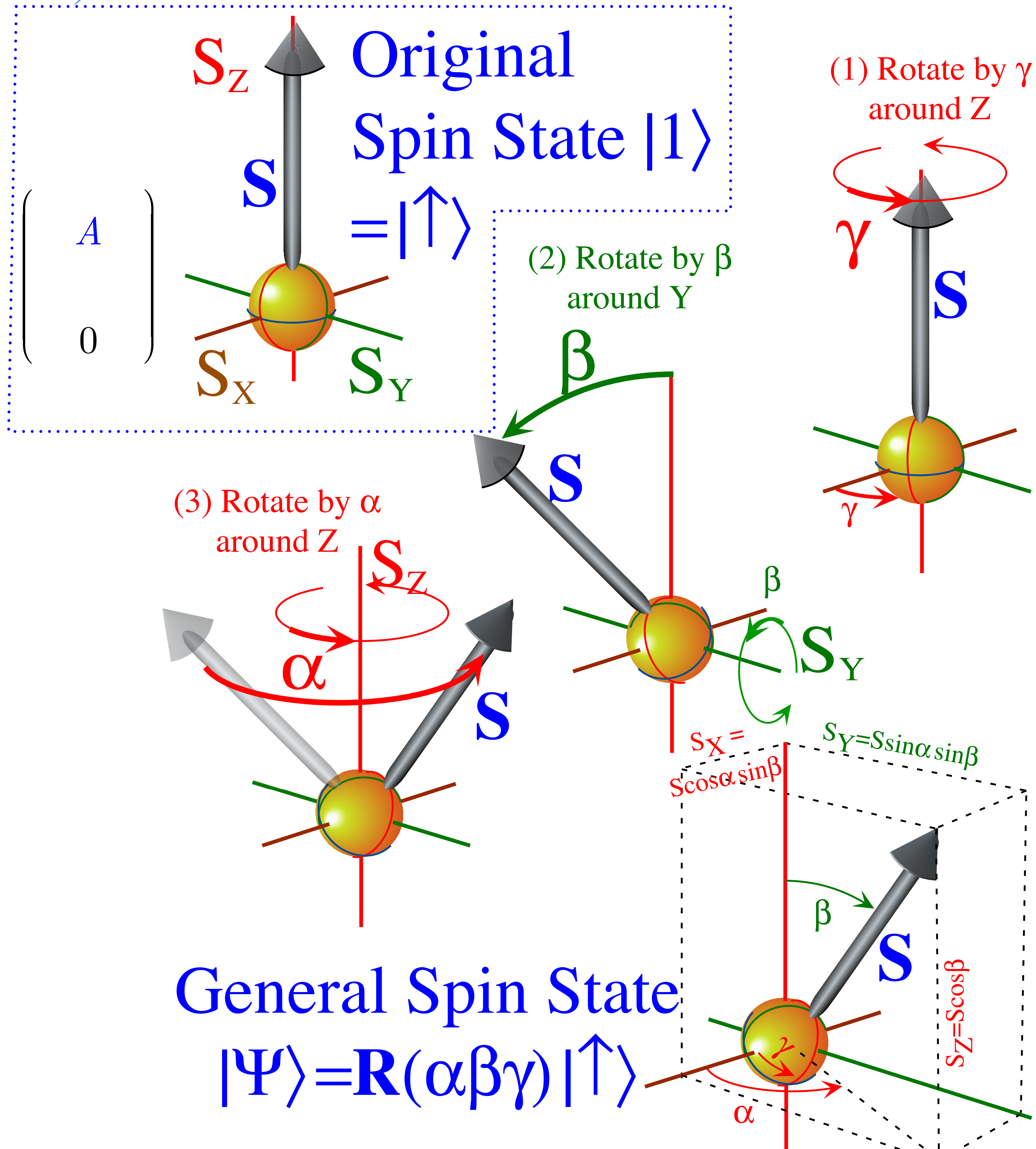
$$\left|a\right\rangle = \mathbf{R}\left(\alpha\beta\gamma\right)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



Review: How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

- *Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$*
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    - *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*
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  - *Spin-1 (3D-real vector) case*
  - *Spin-1/2 (2D-complex spinor) case*
- ➡ *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*
  - ➡ *Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$*
  - *Polarization ellipse and spinor state dynamics*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\textit{Asymmetry } S_A = \frac{1}{2}\left(a\middle|\sigma_A\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$$

$$\textit{Balance } S_B = \frac{1}{2}\left(a\middle|\sigma_B\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$$

$$\textit{Chirality } S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$$

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

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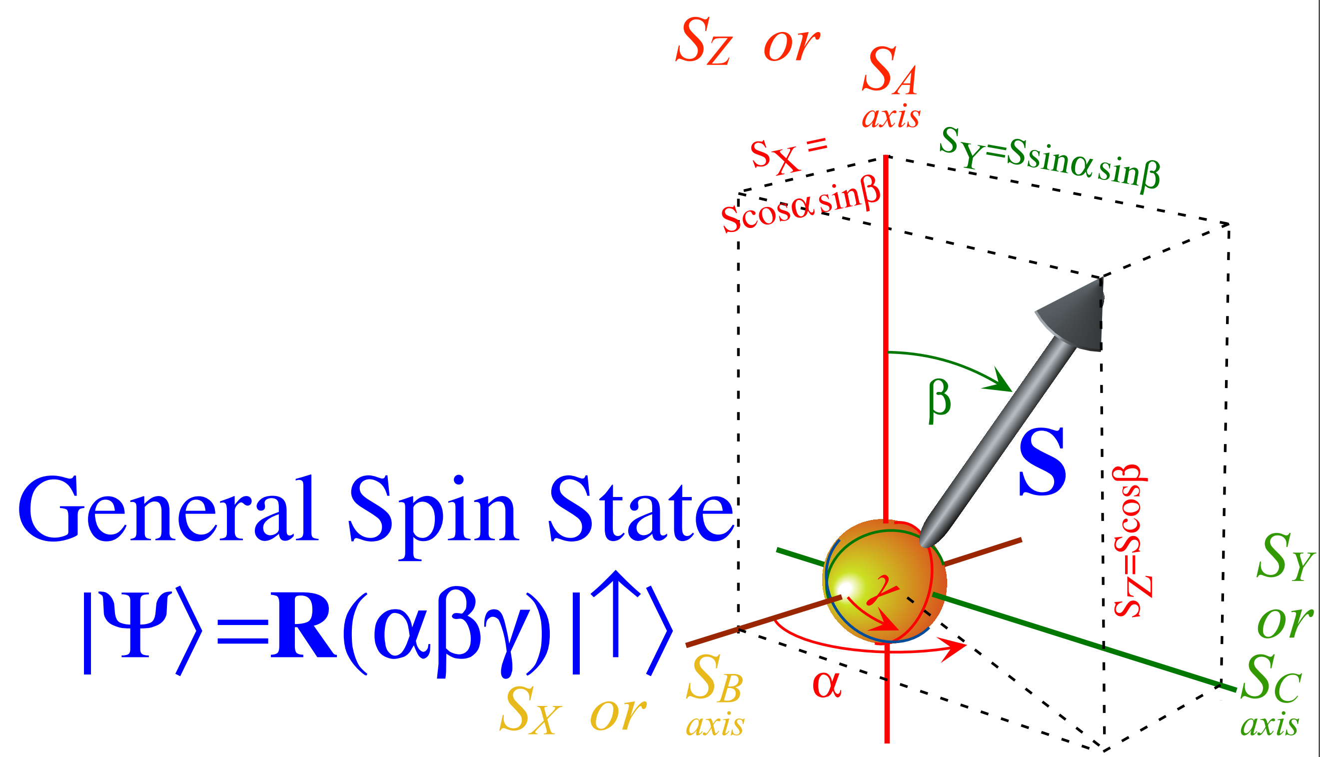
$$\text{Balance } S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

$$\text{Chirality } S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot - \sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

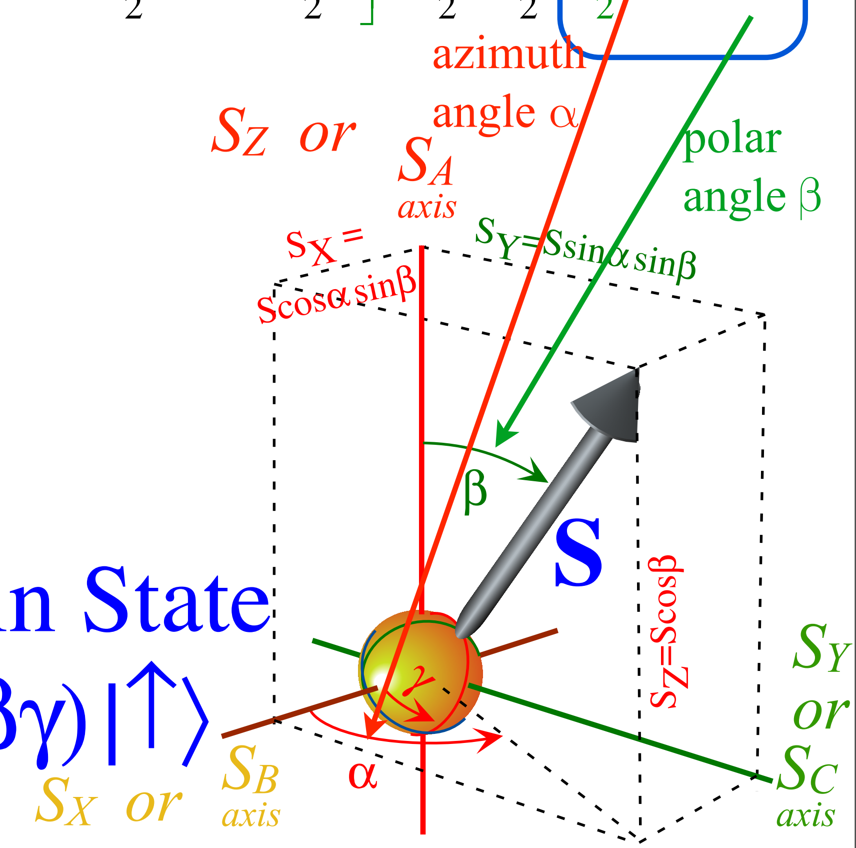
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

Asymmetry $$S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$$

Balance $$S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

Chirality $$S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$

General Spin State
$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

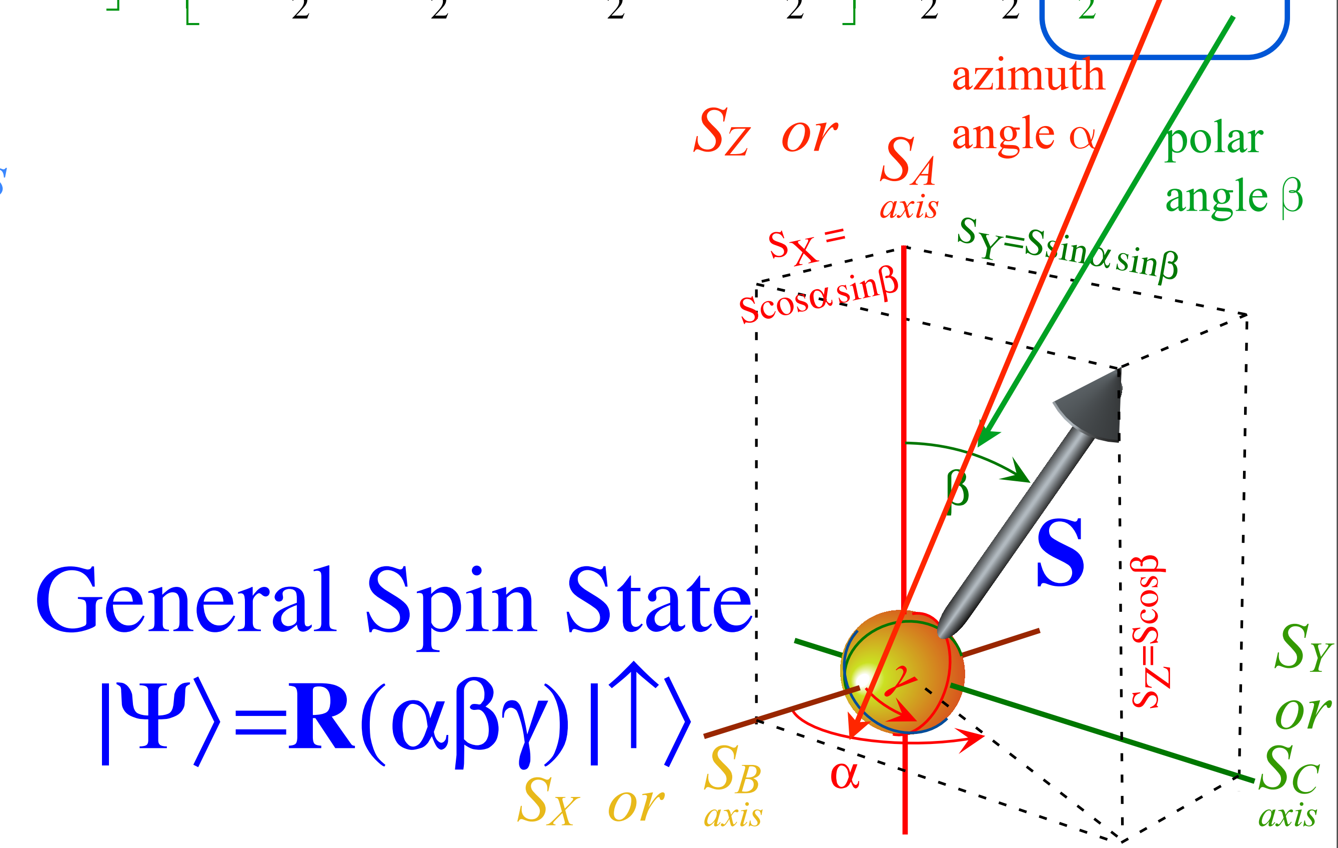
Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\text{Asymmetry } S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$$

$$\text{Balance } S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

$$\text{Chirality } S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$



3-Ways to view spin-1/2 or 2-level system states

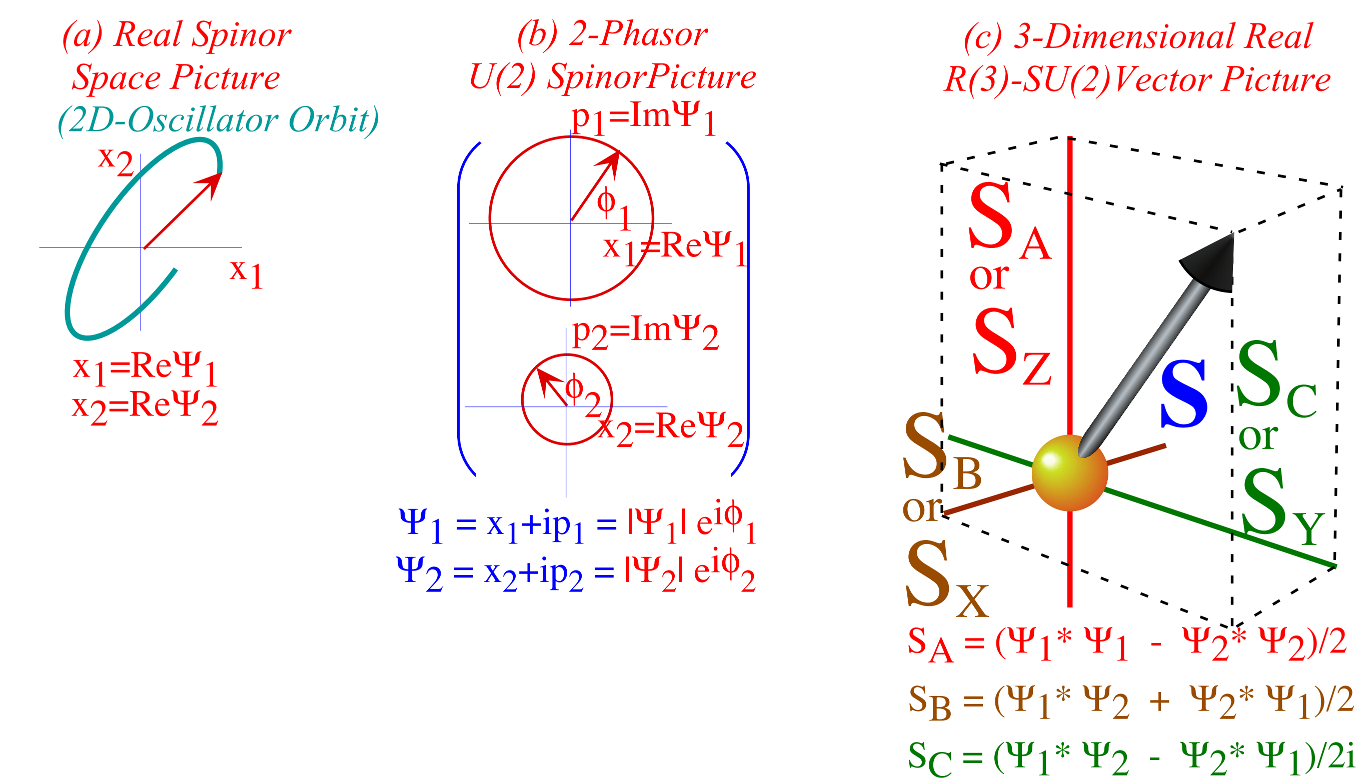


Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$,*and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

➡ *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

➡ *Polarization ellipse and spinor state dynamics*

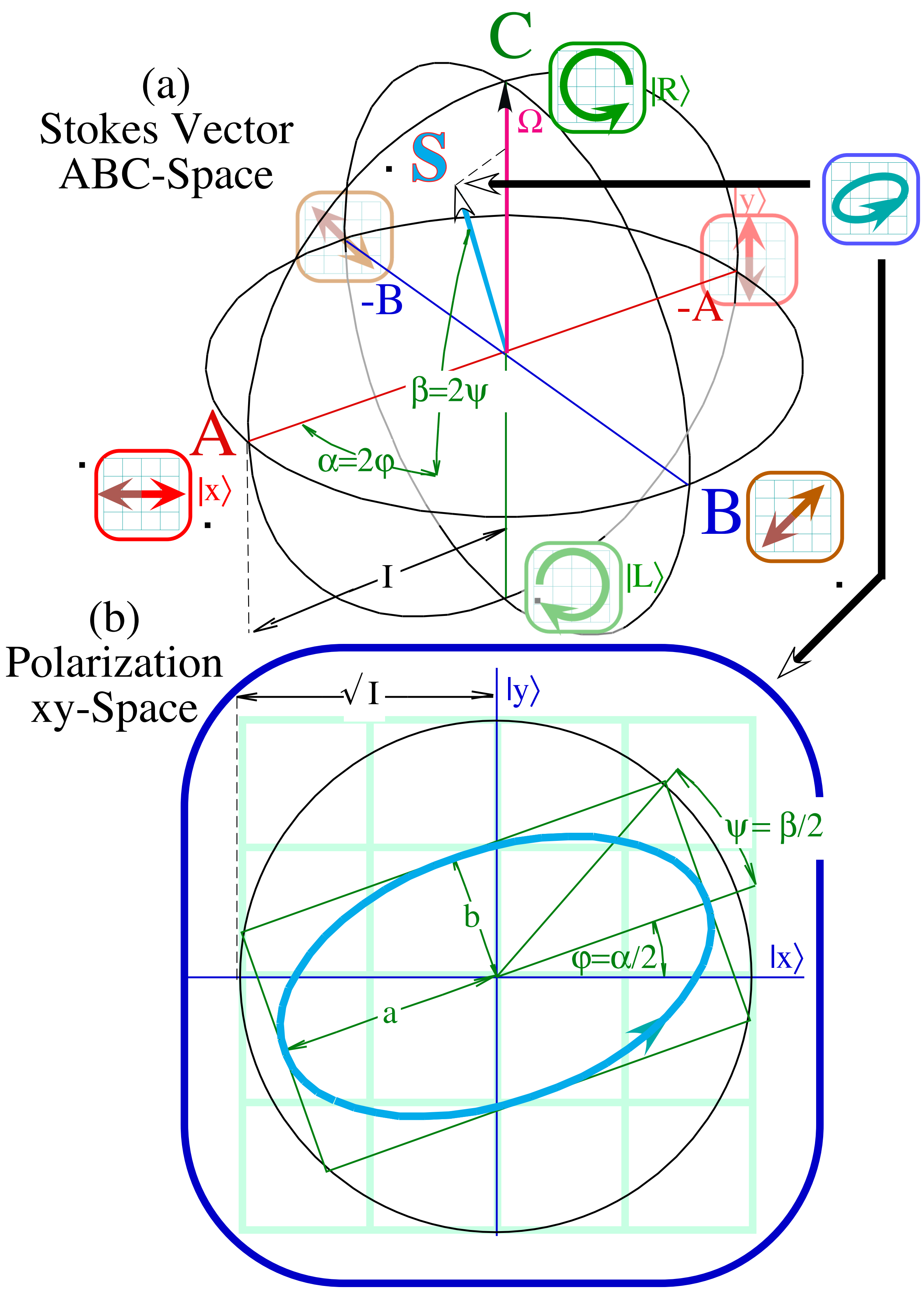
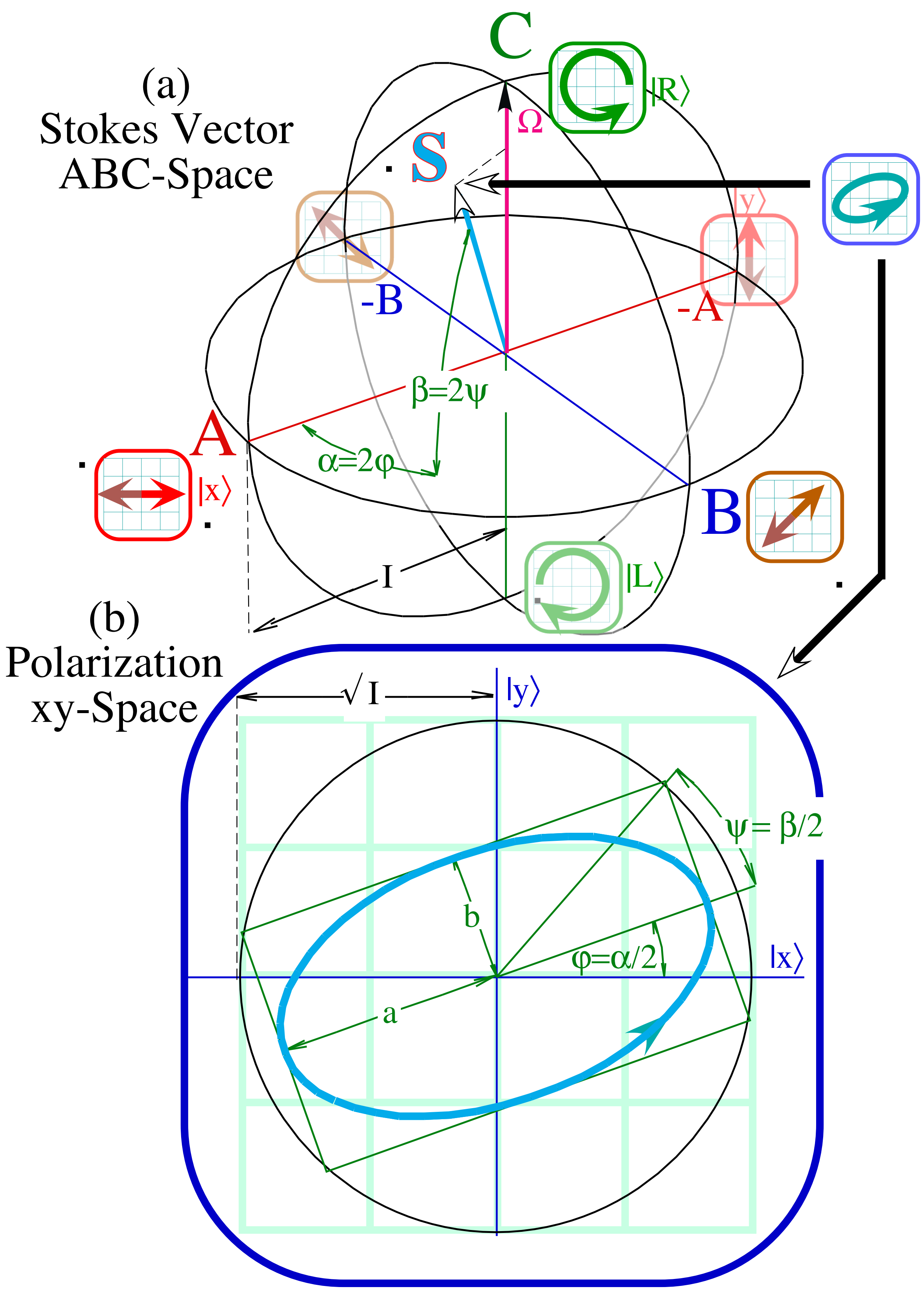


Fig. 10.B.3 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Polarization ellipse and spinor state dynamics



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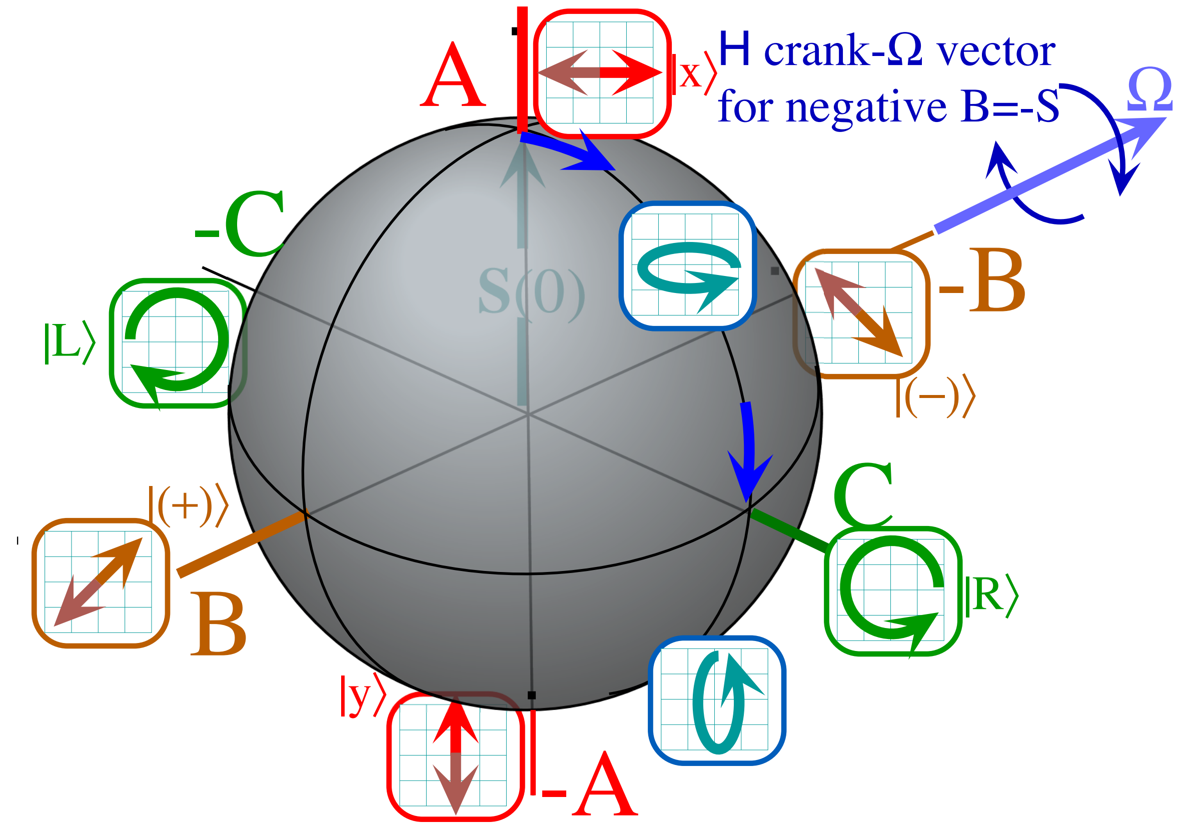


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Polarization ellipse and spinor state dynamics

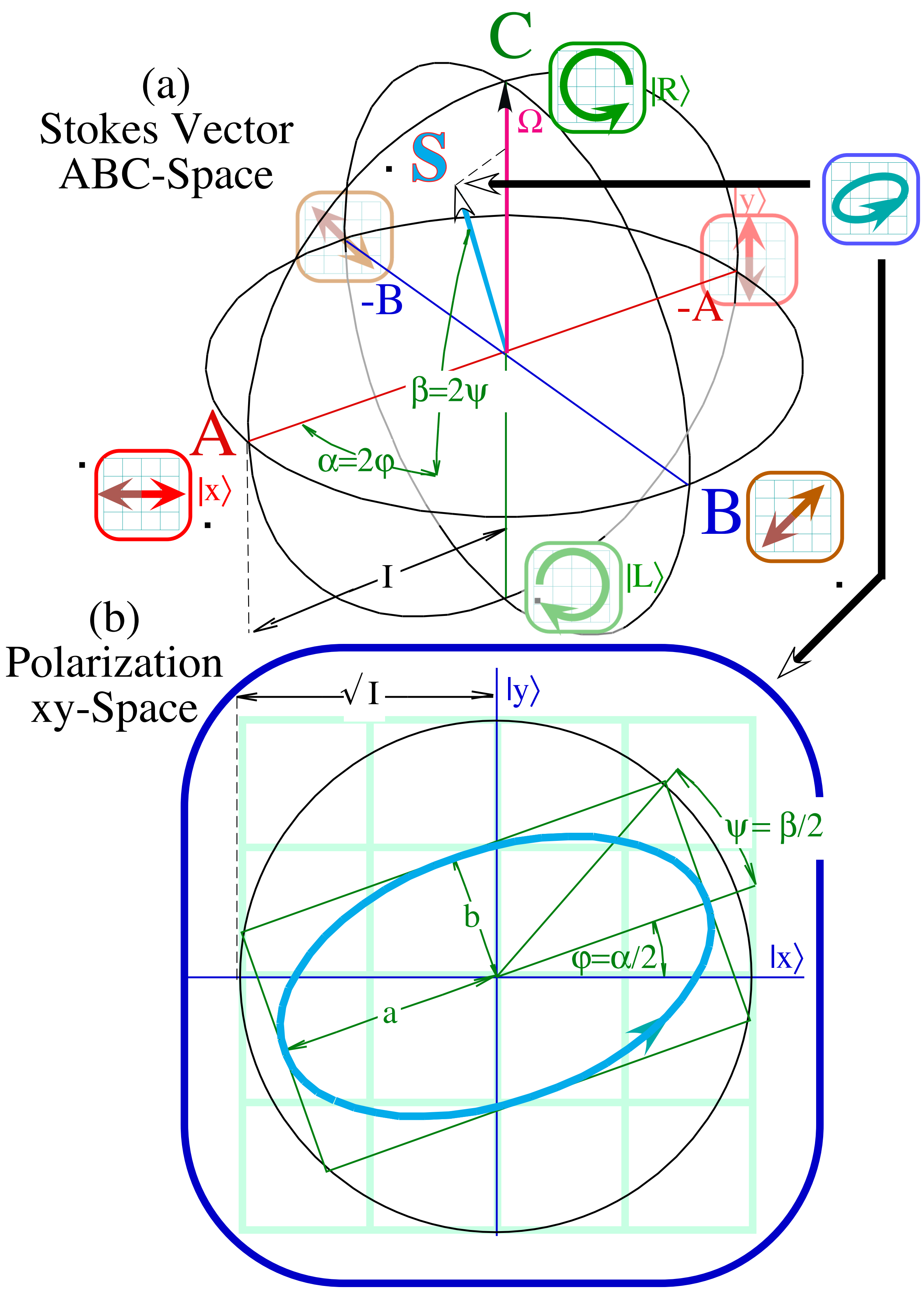


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

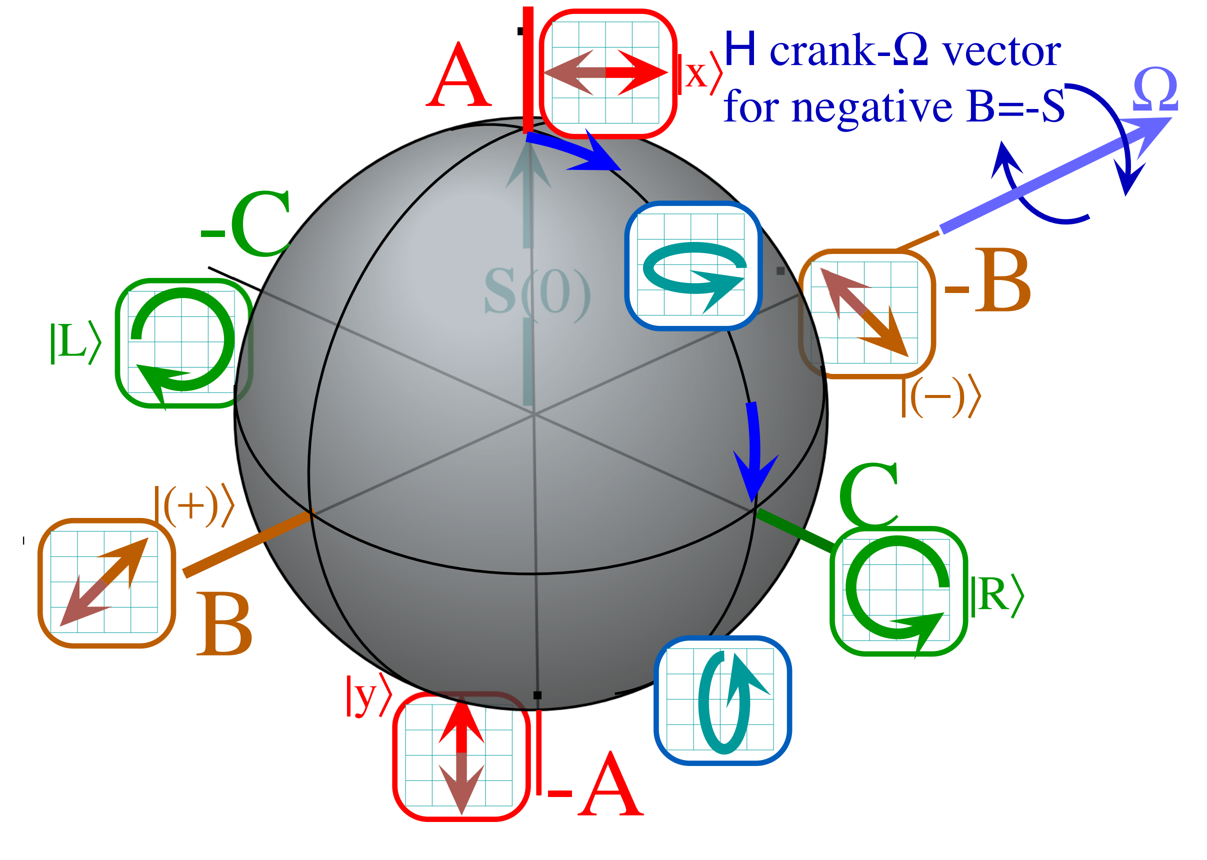
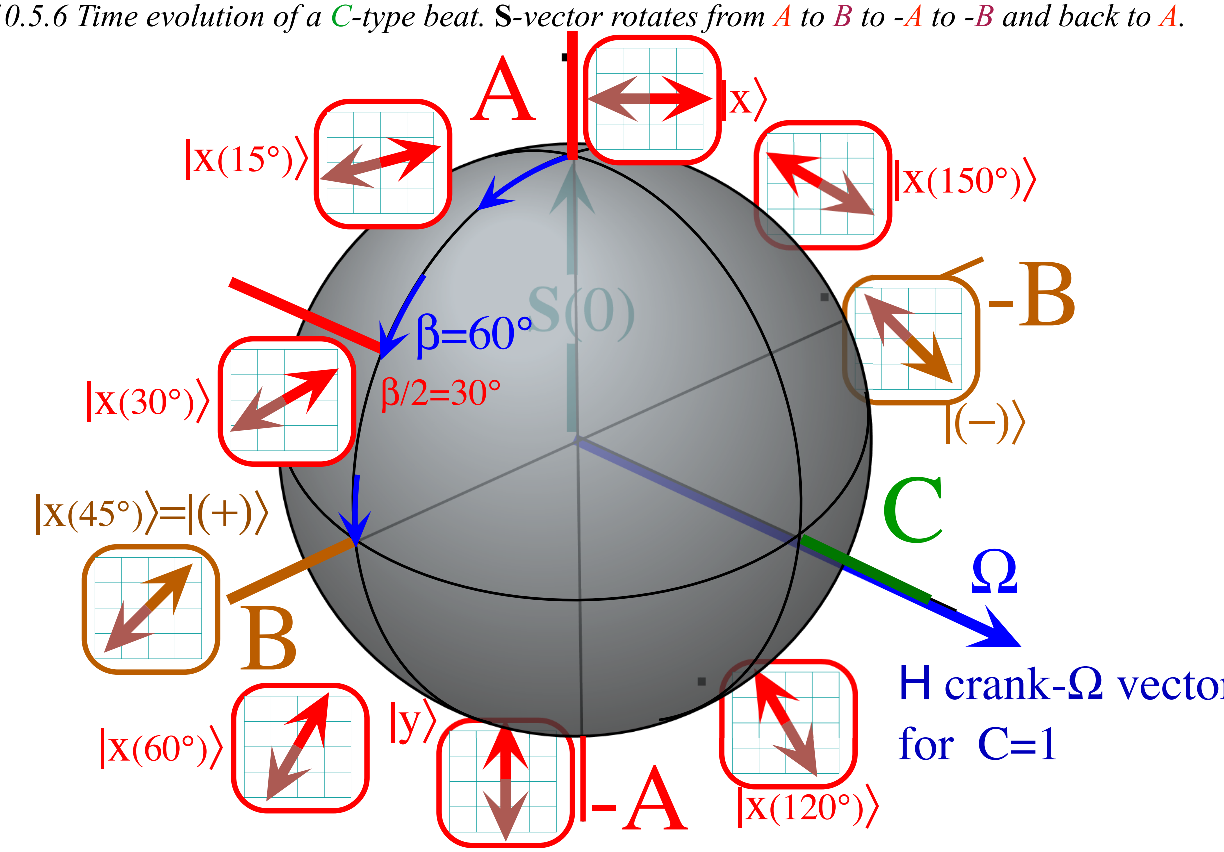


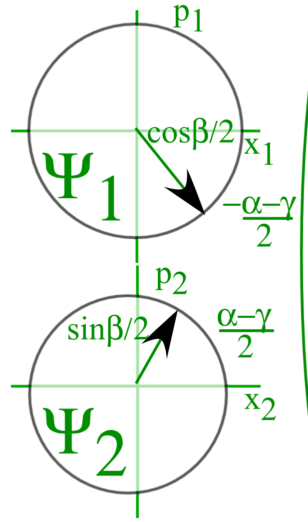
Fig. 10.5.5 Time evolution of a B-type beat. **S**-vector rotates from A to C to -A to -C and back to A.

Fig. 10.5.6 Time evolution of a C-type beat. **S**-vector rotates from A to B to -A to -B and back to A.

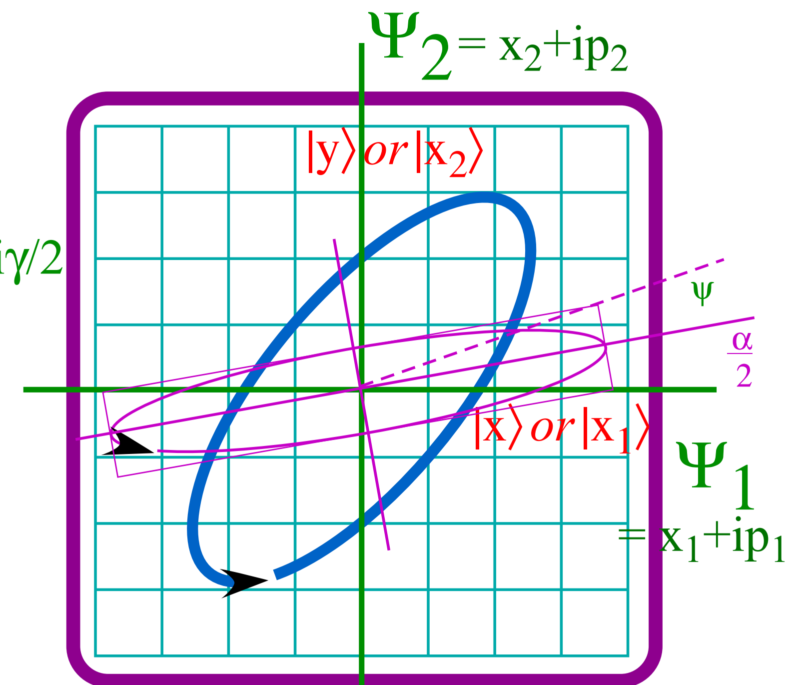


U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle=$



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2\\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

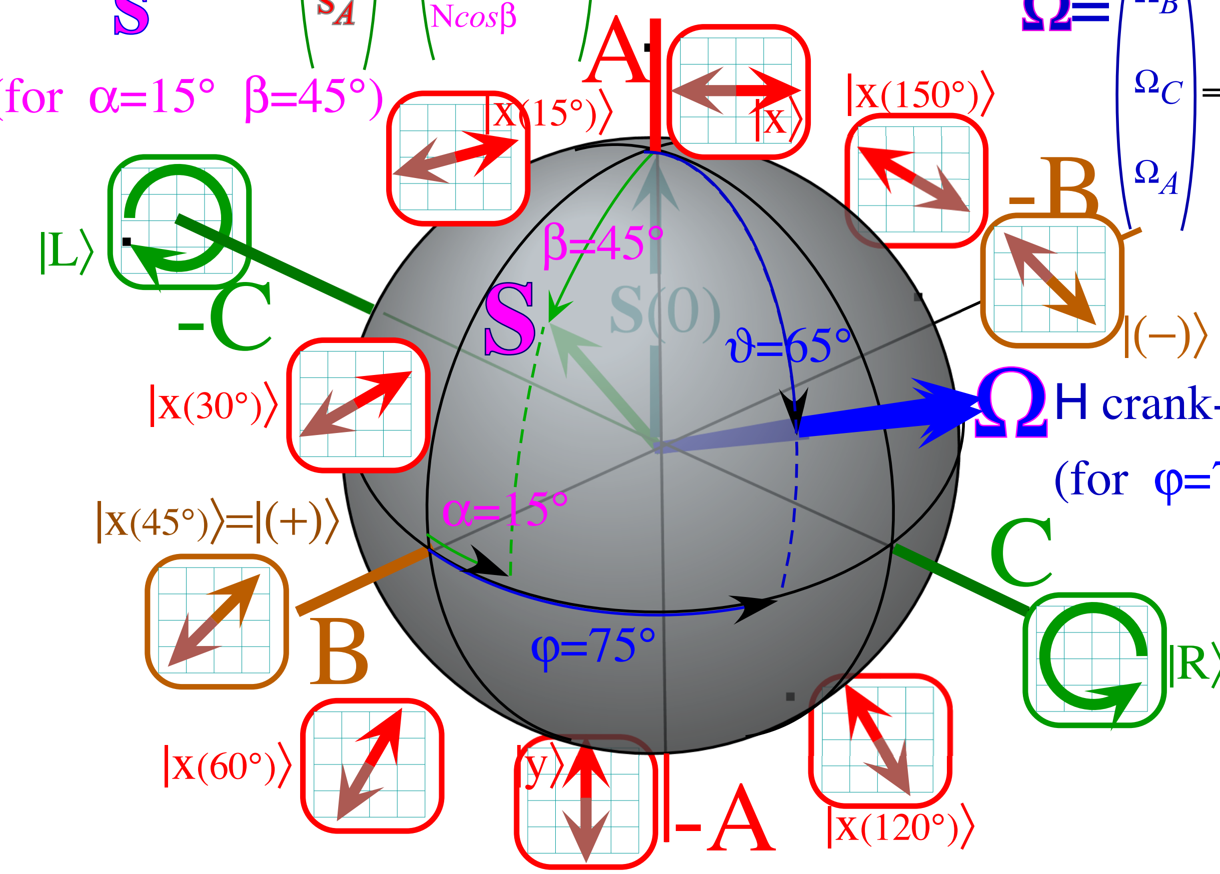
$$\begin{pmatrix}s_B\\ s_C\\ s_A\end{pmatrix}=\begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

(for α=15° β=45°)

H-Operator $\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\boldsymbol{\Omega}=\begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix}=\begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix}=\begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$



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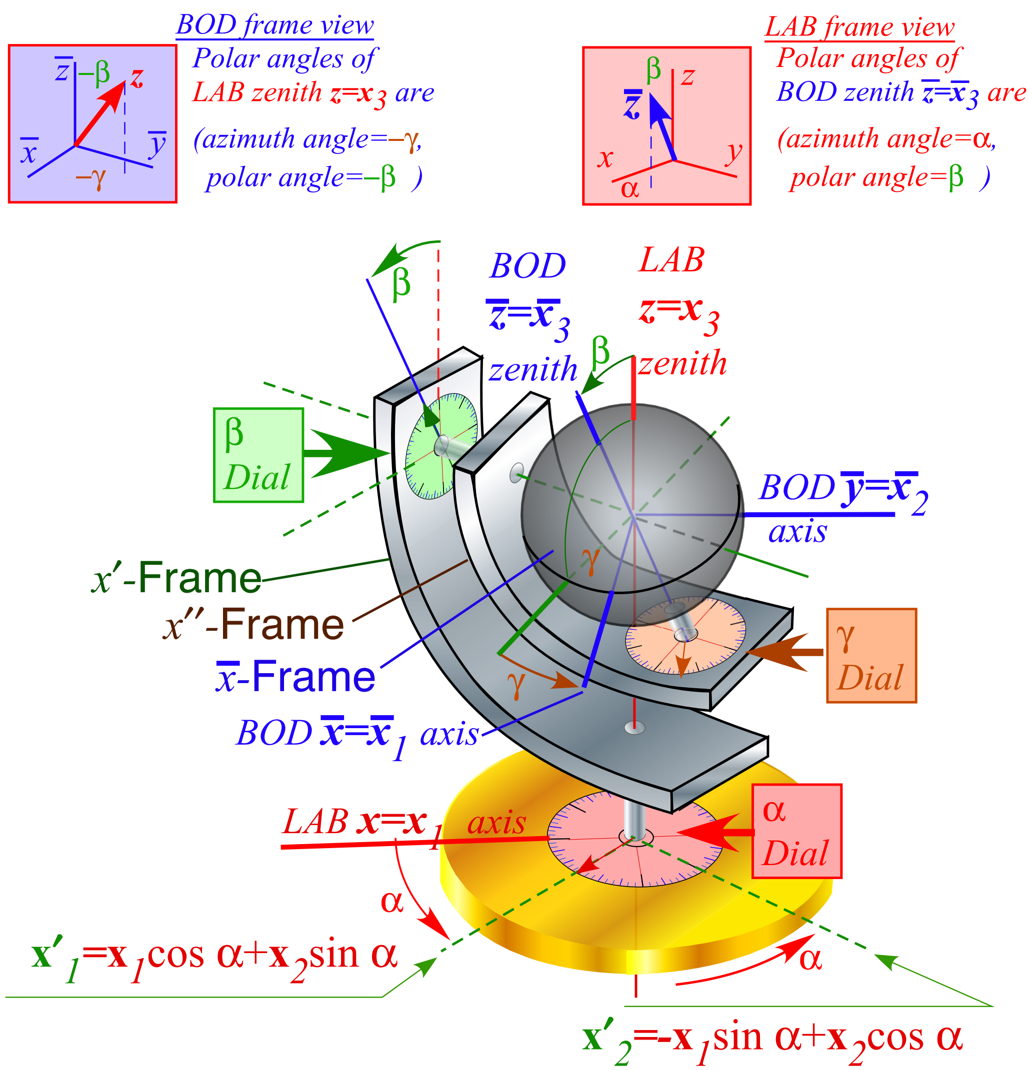
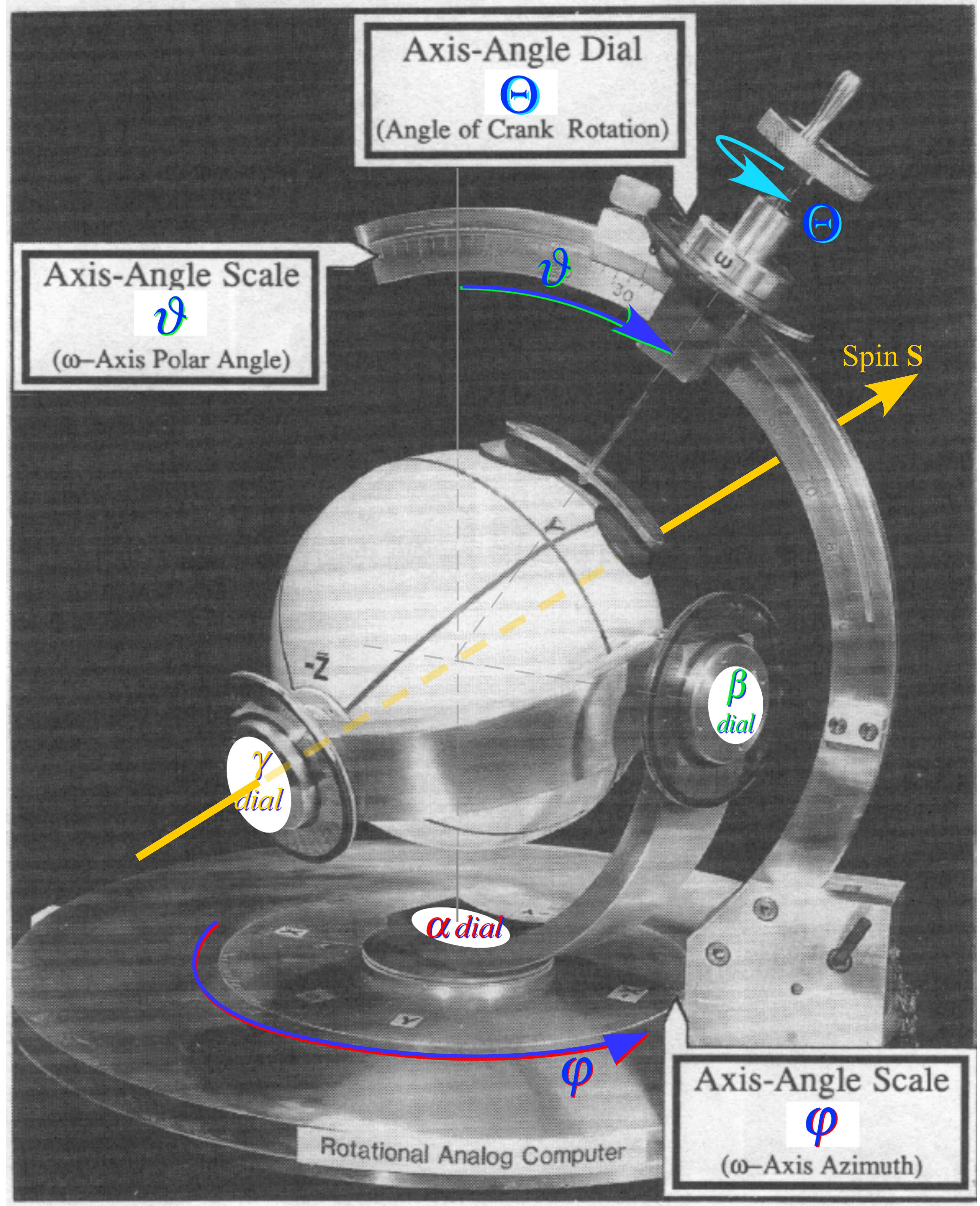
Polarization ellipse and spinor state dynamics

➡ *Fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ and Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ representations of U(2) and R(3)*

From QTCA Lectures 8-9,

Here spin-rotor S-polar coordinates are Euler angles

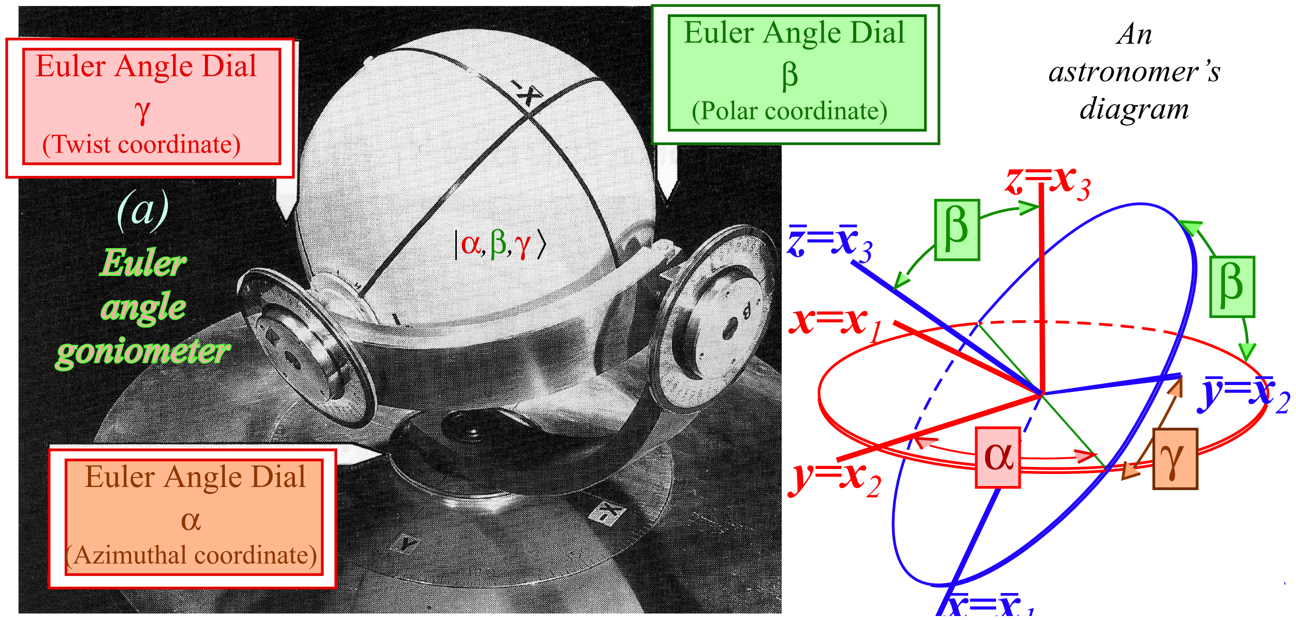


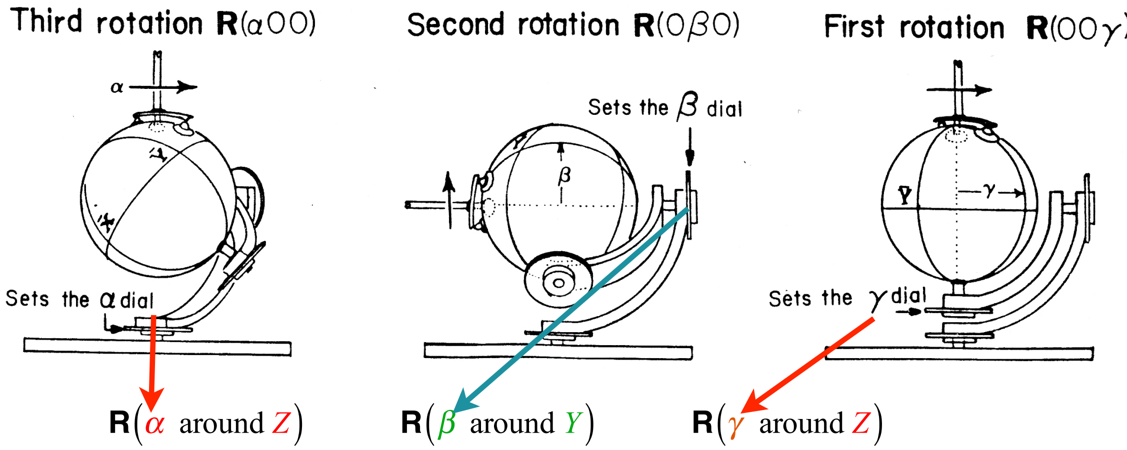


Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]

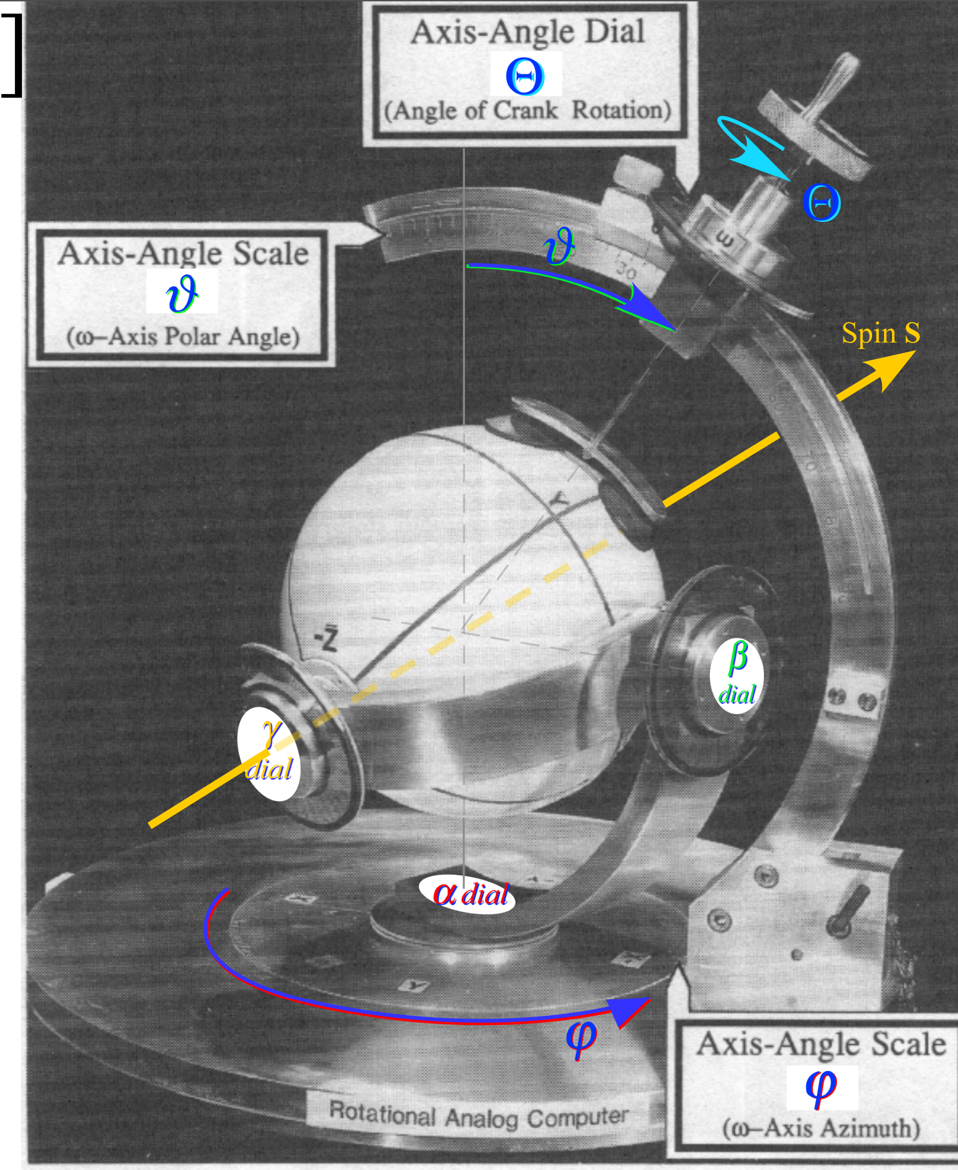




$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

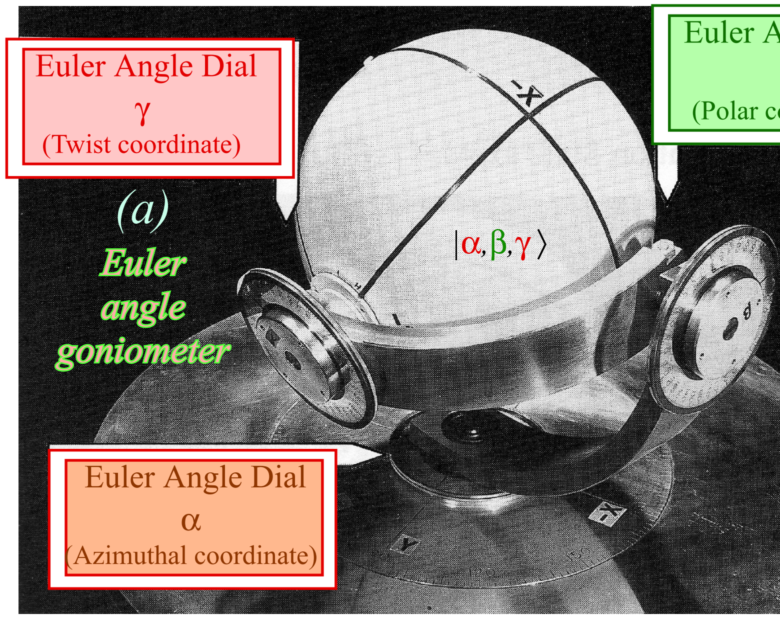


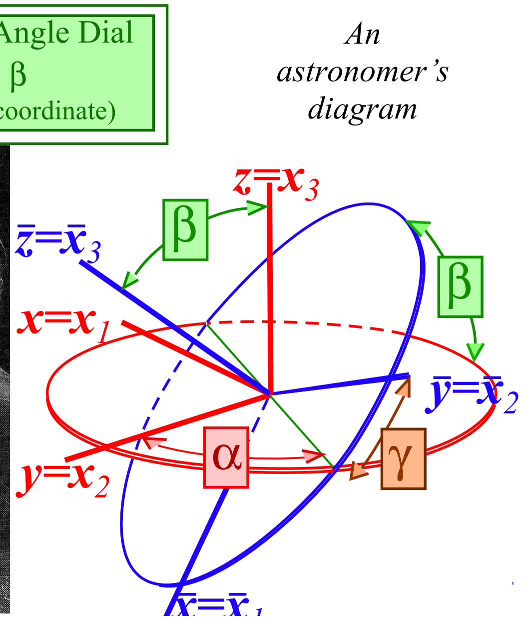
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

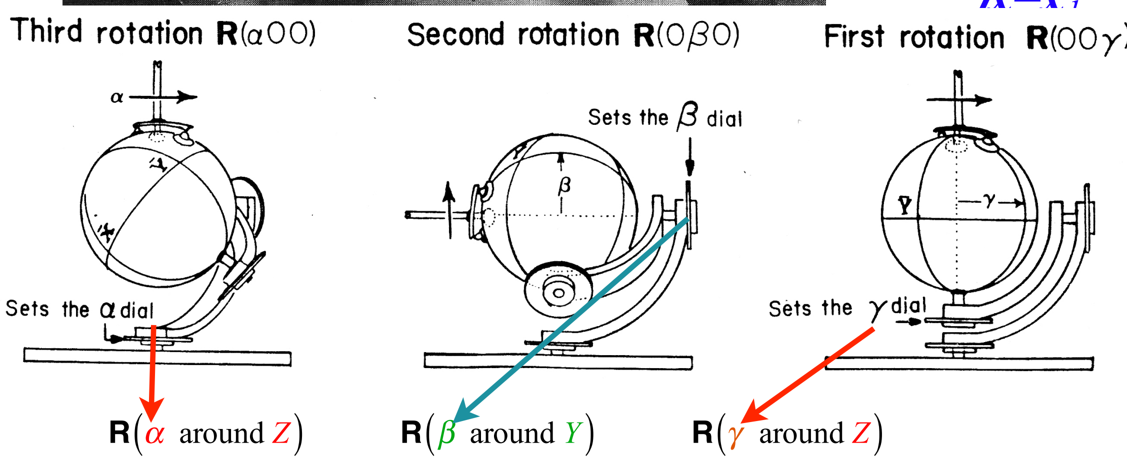
$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

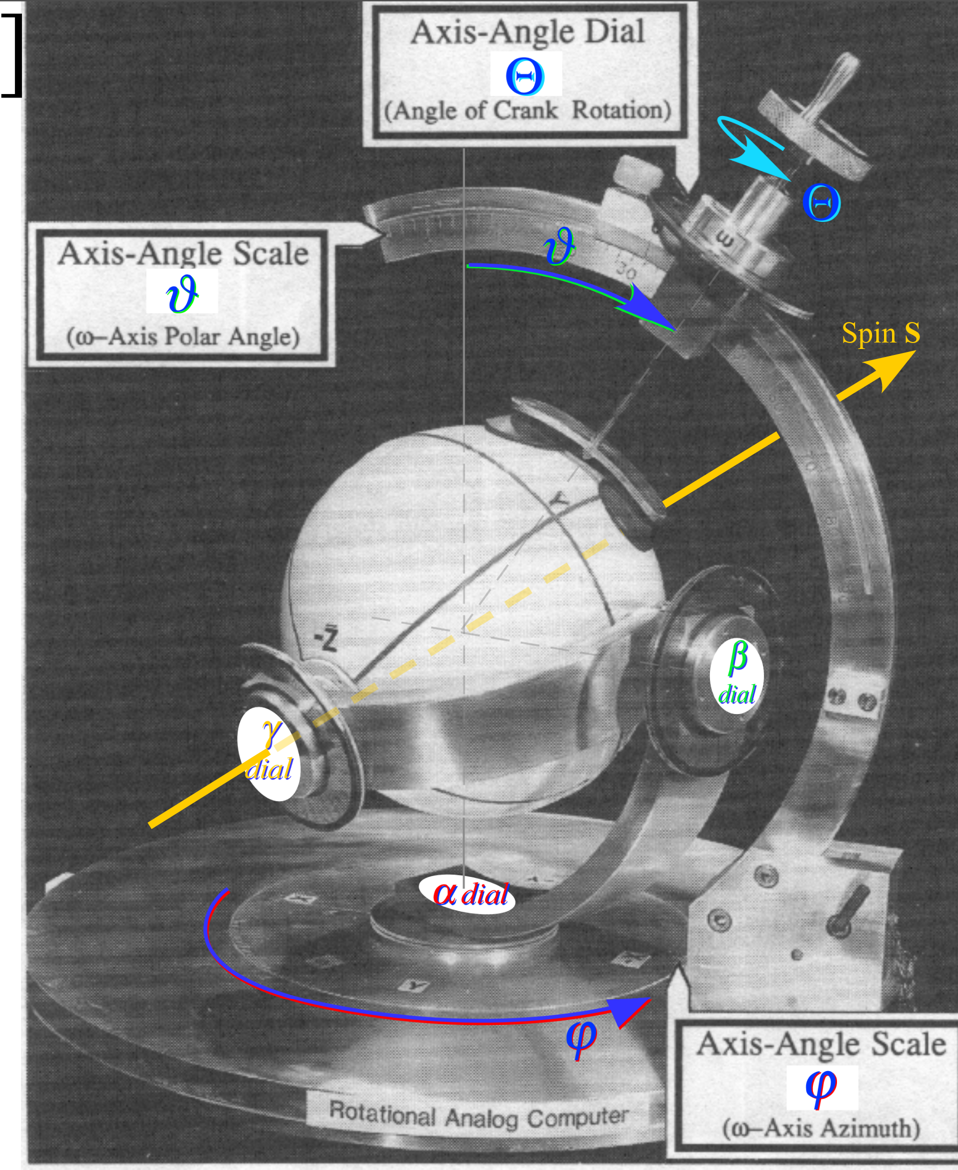
$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]









$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\hat{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y)\\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta)\\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) to R[φϑΘ] ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

$$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2=\cos\Theta/2$$

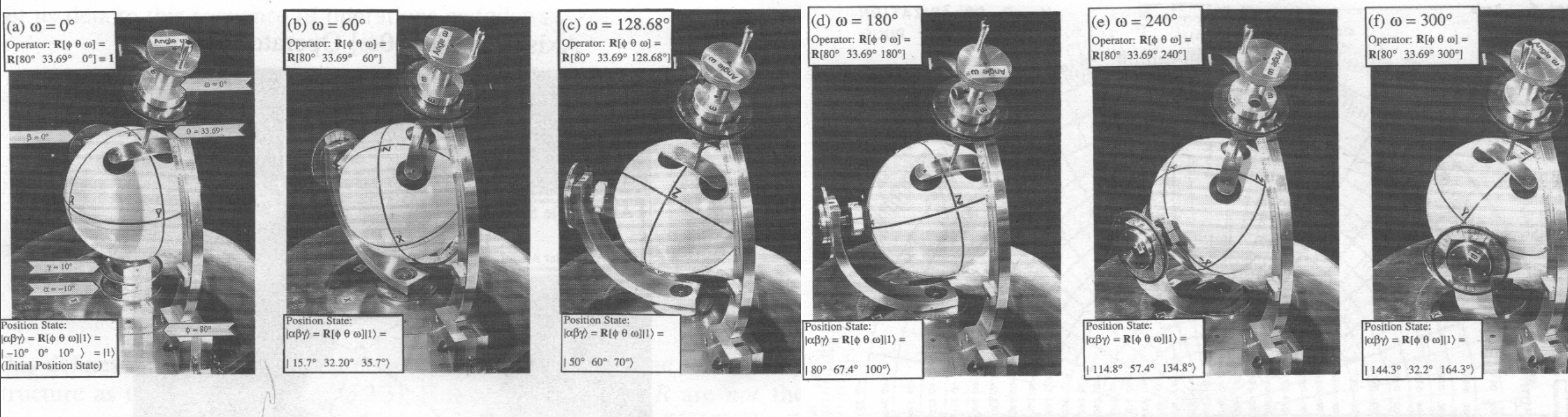
$$-p_2=\sin[(\gamma-\alpha)/2]\sin\beta/2=\hat{\Theta}_X\sin\Theta/2=\cos\varphi\ \sin\vartheta\ \sin\Theta/2$$

$$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2=\hat{\Theta}_Y\sin\Theta/2=\sin\varphi\ \sin\vartheta\ \sin\Theta/2$$

$$-p_1=\sin[(\gamma+\alpha)/2]\cos\beta/2=\hat{\Theta}_Z\sin\Theta/2=\cos\vartheta\ \sin\Theta/2$$

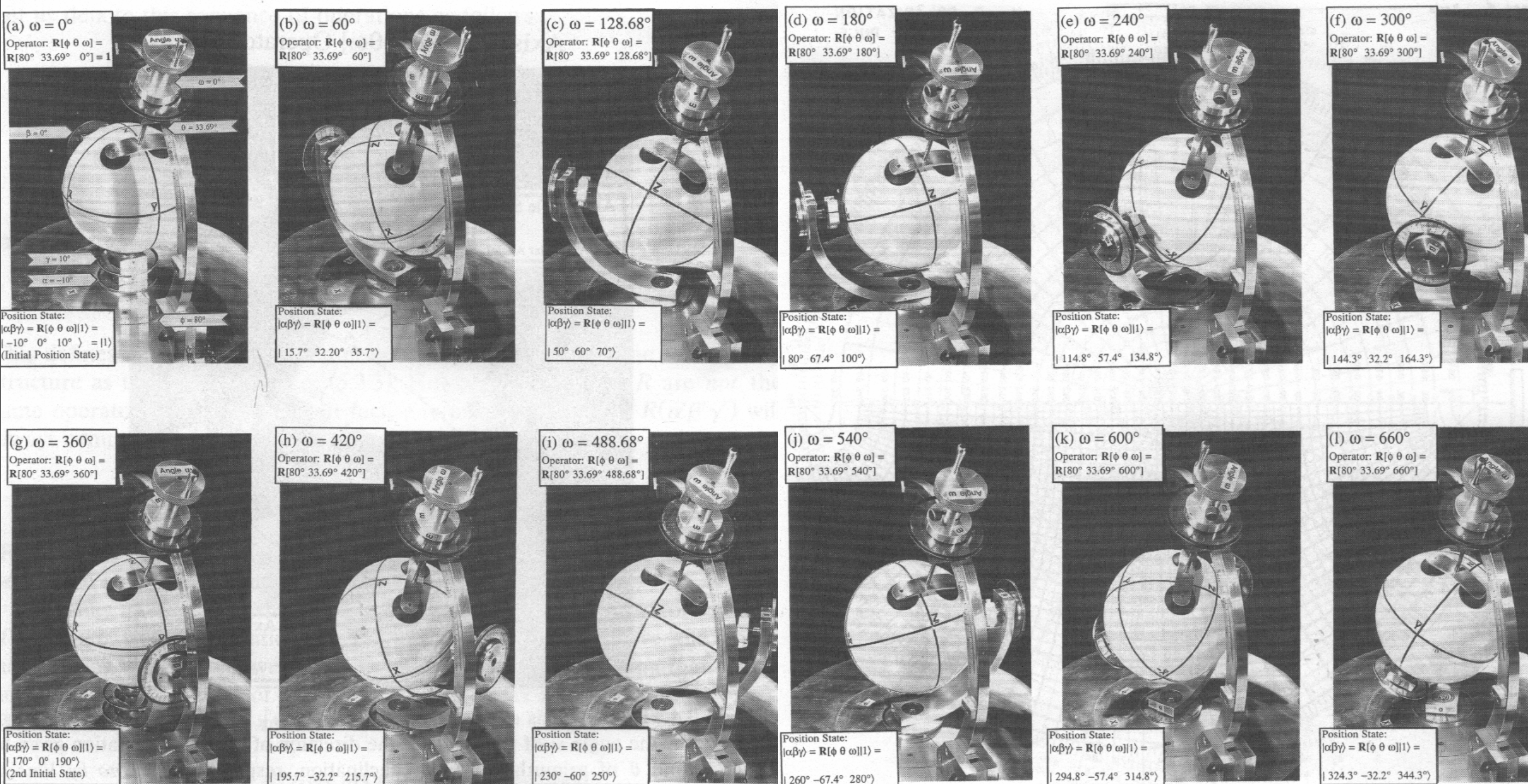
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−*4π-sequence* [φϑ] *fixed*

$\Theta=0°$ | $\Theta=60°$ | $\Theta=128.7°$ | $\Theta=180°$ | $\Theta=240°$ | $\Theta=300°$



Euler **R**($\alpha\beta\gamma$) *rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

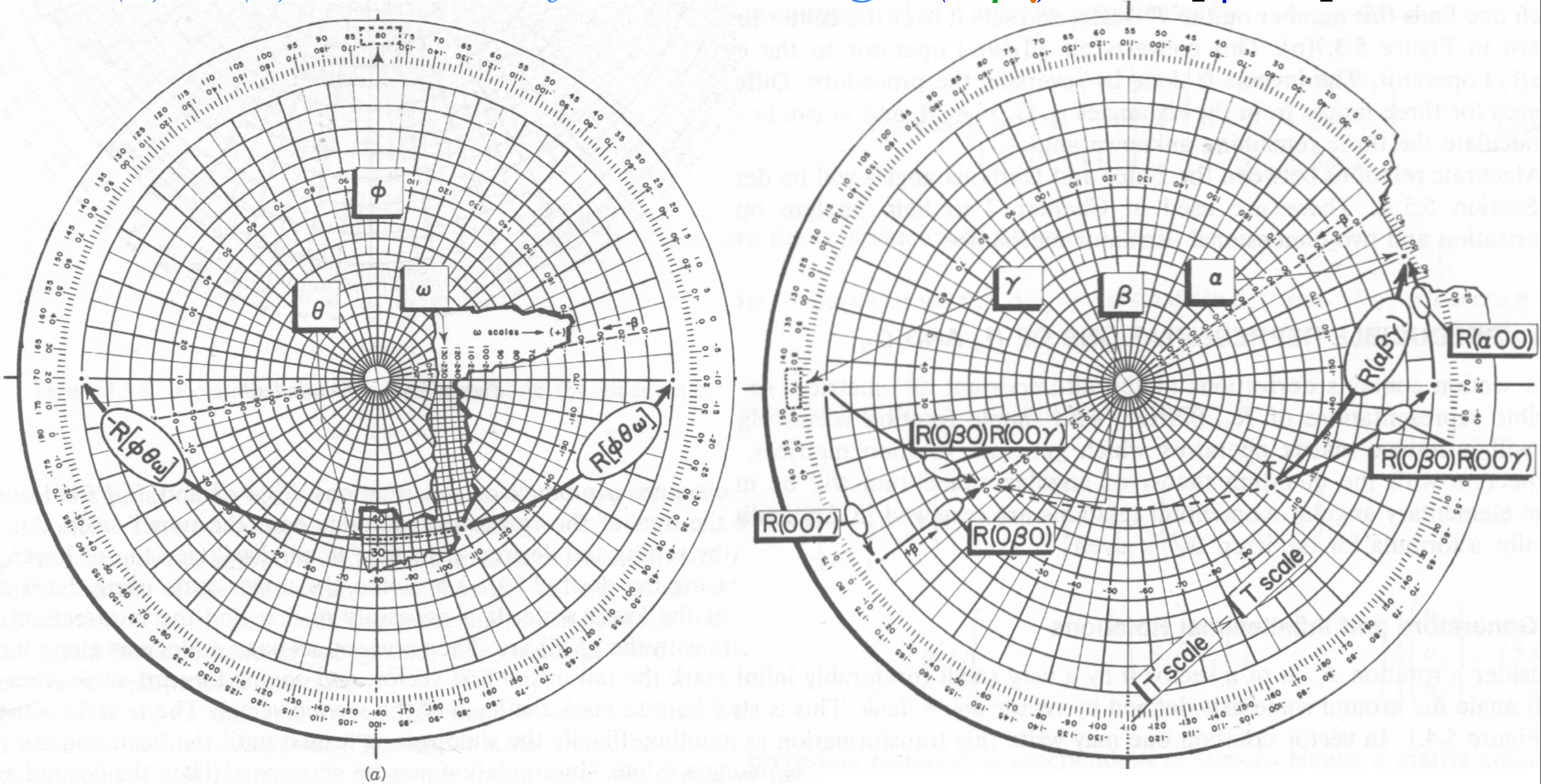
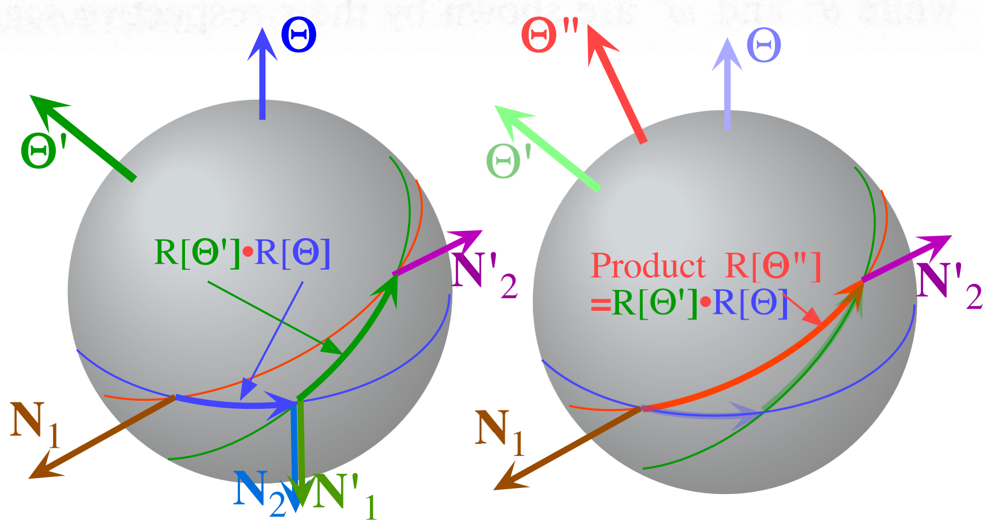
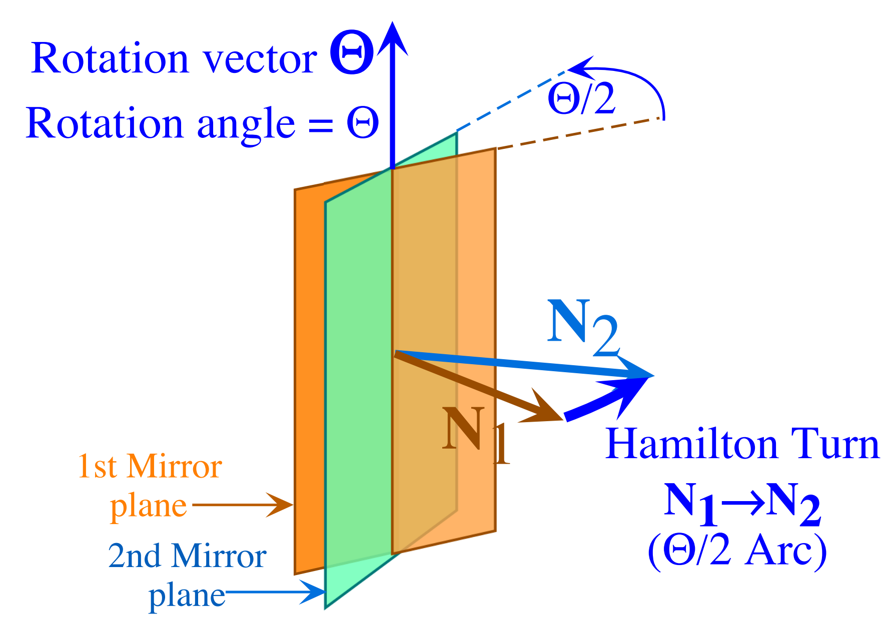
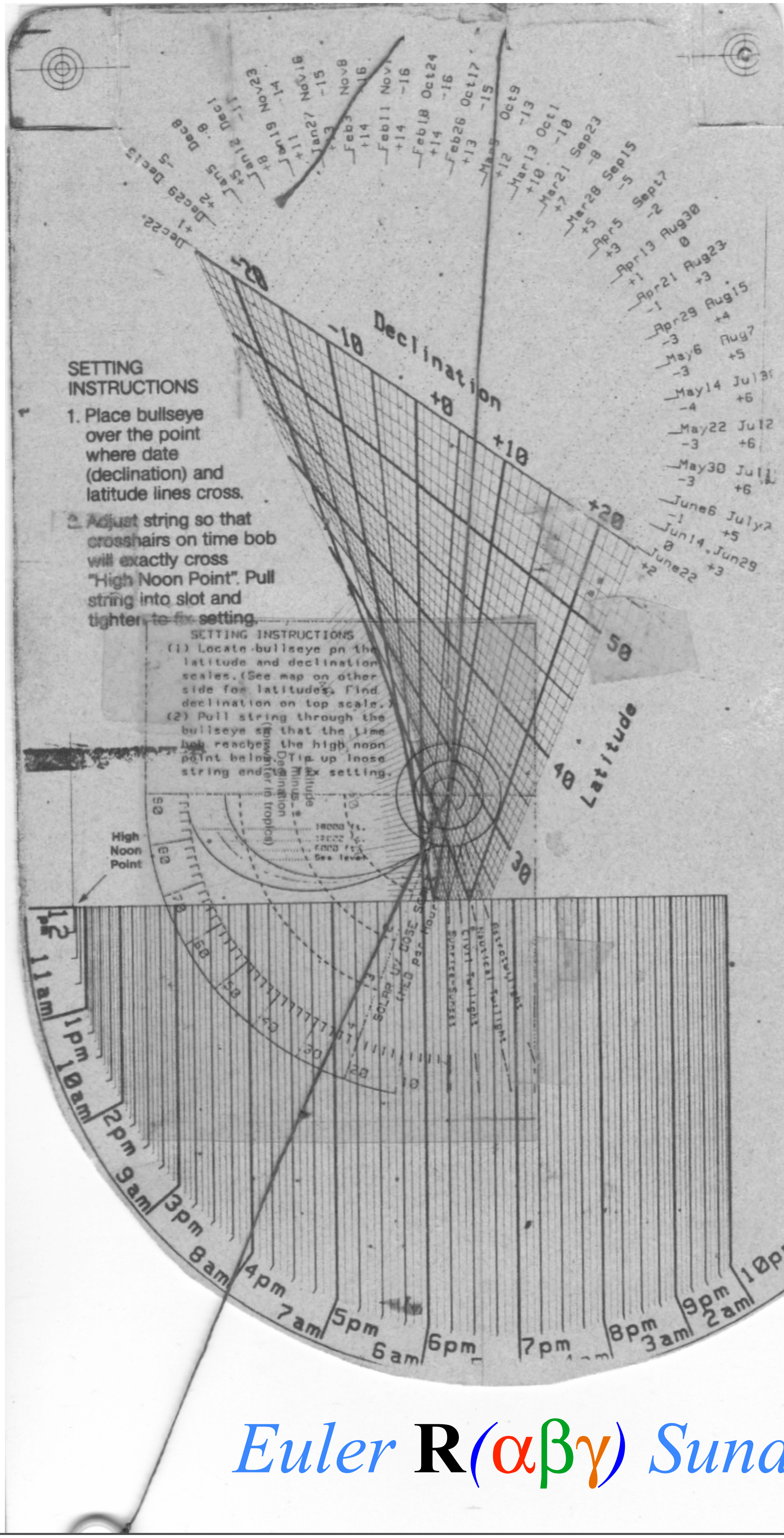
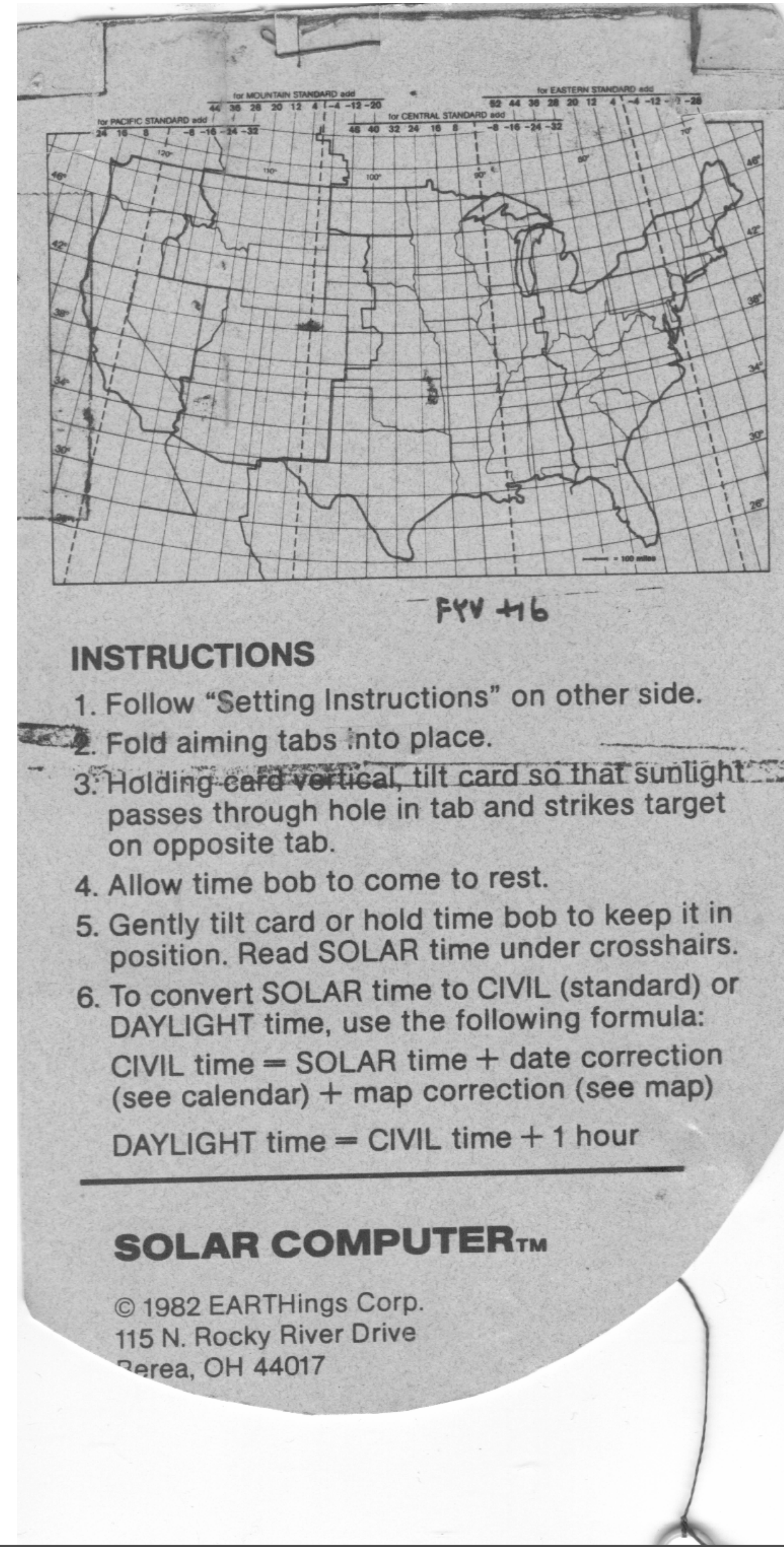


Figure 5.3.7 Setting the rotational slide rule. (a) Darboux or axis angles. (b) Euler angles.









Euler **R**(αβγ) *Sundial*